Metric Fixed Point Theorems for Locally and Globally Contractions

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***DOCTOR OF PHILOSOPHY IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD

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We accept this thesis as conforming to the required standard

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2016

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DEDICATED TO....

My parents, teachers, uncle, friends and family for supporting and encouraging me.

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3 M. Arshad and A. Hussain, Fixed point results for generalized rational α-Geraghty contraction, Miskolc Math. Notes (in press)

4 A Hussain, M Arshad and S.U Khan, τ-Generalization of fixed point results for F-contractions, Bangmod Int J Math Comp Sci 1(1) (2015), 136-146

5 M A Kutbi, M Arshad and A Hussain, Fixed point results for Ćirić type α-η-GF-contractions, J. Comput Anal. Appl 21(3) (2016), 466-481

6 A Hussain, M Arshad and M Nazam, Connection of Ćirić type F-contraction involving fixed point on closed ball, Gazi Uni J Sci (in press)

7 M. A Kutbi, M Arshad and A. Hussain, On modified (α-η)- contractive mappings, Abstr. Appl Anal 2014 (2014), Art ID 657858

CONTENTS

Preface 1

1 Preliminaries 5

1 1 Some basic concepts 5

1 2 Single-valued and multivalued commuting mappings 10

1 3 Single-valued and multivalued α-admissible mappings 11

1 4 Single-valued and multivalued F-contraction mappings 16

2 Fixed Points of Geraghty Contractive Mappings in Metric Spaces 21

2 1 Introduction 21

2 2 Modified Geraghty contraction involving fixed points in metric spaces 22

2 3 Modified α-Geraghty Ćirić type theorems 34

2 4 Fixed point results for rational α-Geraghty contraction 45

3 Single-valued and Multivalued Theorems for F-Contraction in Metric Spaces 57

3 1 Introduction 57

3 2 Fixed point results for Ćirić type α-η-GF-contractions 58

3 3 Fixed point results for multivalued Ćirić type α-η-GF-contractions 68

3 4 Generalization of fixed point results for F-contraction 78

3 5 Modified fixed point results for F-contraction 83

4 Dynamic Process for Generalized (f, L)-Almost F-Contraction 88

4 1 Introduction 89

4 2 Fixed point results for generalized (f, L)-almost F-contraction 94

4 3 Applications 98

5 Fixed Point Theorems for Local F-Contraction on a Closed Ball in Metric Spaces 105

5 1 Introduction 105

5 2 Fixed point theorems for Ciric type F-contraction on a closed ball 106

5 3 Fixed point theorems for Ciric type GF-contraction on a closed ball 111

Bibliography 115

Preface

Fixed point theorems deal with the assurance that the functional equation $x = Tx$ has one or more solutions. Such solutions are known as fixed points of the mapping T. A large variety of problems of analysis and applied mathematics relate to find solutions of nonlinear functional equations which can be formulated in terms of finding the fixed points of nonlinear mappings. In fact, fixed point theorems are extremely substantial tools to determine the existence and uniqueness of solutions of various mathematical models exhibiting phenomena arising in a broad spectrum of fields, such as steady state temperature distribution, chemical equations, neutron transport theory, economic theories, financial analysis, epidemics, biomedical research and flow of fluids etc. These results are also used to study the problems of optimal control related to these systems.

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the "Banach Contraction Principle" which is one of the most important result of analysis and is considered to be the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics because it only requires the structure of a complete metric space with a contractive condition on the map which is easy to test in this setting. The Banach Contraction Principle has been generalized in many different directions. In fact, there is a vast amount of literature dealing with extensions/generalizations of this remarkable theorem.

A multivalued function is a function which takes on set values. In the last forty years, the theory of multivalued functions has advanced in a variety of ways. In 1969, the systematic study of Banach type fixed theorems of multivalued mappings was started with the work of Nadler [106], who proved that a multivalued contractive mapping of a complete metric space X into the family of closed bounded subsets of X has a fixed point.

In 2012, Samet et al. [114] introduced the concept of α-admissible mappings and suggested a very interesting class of mapping (α-ψ-contraction mappings) to investigate the existence and uniqueness of a fixed point. Further Mohammadi et al. [99] extended some results on fixed points of α-ψ-Ciric generalized multifunctions. Asl et al. [57] introduced the notion of α^*-ψ-contractive multifunctions and established fixed point result for multifunctions. Recently

Hussain et al. [59] established certain new fixed point results for multi-valued as well as single-valued mappings satisfying an α-ψ-contractive conditions in a complete metric space. The notion of an α-admissible mapping has been characterized in many direction. For details see [16, 30, 31, 73, 99, 108, 109, 112, 118, 121].

In 2012, Wardowski [123] introduced a new type of contraction called an F-contraction and established a new fixed point theorem concerning F-contractions. Wardowski et al. [124] introduced the notion of an F-weak contraction which generalizes some known results from the literature. He gave an interesting generalization of the Banach contraction principle. Afterwards Secelean [115] proved fixed point theorems consisting of F-contractions by iterated function systems. Piri et al. [107] extended the result of Wardowski by applying some weaker conditions on the selfmap in a complete metric space. Cosentino and Vetro [49] presented some fixed point results of Hardy-Rogers-type for self-mappings on complete metric spaces and complete ordered metric spaces.

Abbas et al. [4] further generalized the concept of F-contractions and proved certain fixed and common fixed point results. Hussain and Salimi [67] introduced an α-GF-contraction with respect to a general family of functions G and established Wardowski type fixed point results in metric and ordered metric spaces. Altun et al. [10] extended multivalued theorems to mappings with a δ-distance and established fixed point results in complete metric spaces. Acar et al. [11] introduced the concept of generalized multivalued F-contraction mappings and established a fixed point result which is a proper generalization of some multivalued fixed point theorems including Nadler's result in [106]. Recently Mınak [97] proved some fixed point results for Ćirić type generalized $F-$ contractions on complete metric spaces. Sgroi and Vetro [117] proved results for obtaining fixed points of multivalued mappings which generalize Nadler's theorem [106]. Recently, Abbas et al. [2] introduced the concept of multivalued f-almost F-contractions which generalizes the class of multivalued almost contraction mappings and obtained coincidence point results. Naturally many authors have started to investigate the existence and uniqueness of a fixed point theorem via F-contraction mappings and variations of the concept of F-contractive type mappings. For more details see [19, 45, 86].

Azam et al. [32] proved a significant result concerning the existence of fixed points of a mapping satisfying contractive conditions on a closed ball of a complete metric space. Shoaib et al. [27] exploited this concept of a closed ball in a dislocated metric space to approximate the unique solution of nonlinear functional equations. He also establised fixed point and common fixed point theorems for a pair of contractive dominated mappings on a closed ball in an ordered dislocated metric space. For more details see [24, 28].

This dissertation consists of five chapters. Each chapter begins with a brief introduction which acts as a summary to the material therein.

Chapter 1 is a survey aimed at clarifying the terminology to be used and recalls basic definitions and facts.

Chapter 2 is devoted to the study of the existence of fixed and common fixed points of mappings satisfying generalized contractive conditions. The aim of this chapter is to improve the notion of a Geraghty contraction and to establish some fixed point theorems for α-admissible mappings with respect to η, satisfying a modified $(\alpha-\eta)$-contractive condition in the framework of complete metric spaces. We prove new fixed point theorems for α-Geraghty contractions and rational α-Geraghty contraction type mappings in complete metric spaces.

Chapter 3 deals with single-valued and multivalued F-contraction mappings. We introduce the concept of Ćiric type α-η-GF-contractions and establish some new fixed point theorems for single-valued and multivalued mappings in the setting of complete metric spaces. We extend the concept of a multivalued and an α_*-τ-F-contraction and α_*-η-τF-contraction and obtain some new Wardowski type fixed point theorems in the framework of complete metric spaces.

Chapter 4 introduces the notion of a generalized dynamic process for generalized (f, L)-almost F-contraction mappings and obtains coincidence and common fixed point results for such processes. We discuss applications of our theorem and obtain the existence and uniqueness of common solution of system of functional equations in dynamical programming and the existence and uniqueness of common solution of system of Volterra type integral equations.

Chapter 5 deals with locally F-contractions and introduces the concept of an F-contraction on a closed ball. We establish some fixed point theorems for F-contractions and GF-contractions on a closed ball in complete metric spaces.

Chapter 1

Preliminaries

The aim of this chapter is to present some basic concepts and to explain the terminology used throughout this dissertation. Some previously known results are given without proof. Section 1.1 is concerned with the introduction of single-valued and multivalued contractions. Section 1.2 is devoted to some introductory material on the notions of commuting single-valued and multivalued mappings. In Section 1.3, we present the concept of α-admissible mappings of single-valued and multivalued mappings for an α-ψ-contraction. Section 1.4 introduce the basic concepts of single-valued and multivalued F-contraction mappings.

1.1 Some basic concepts

Contraction mappings are a special type of uniformly continuous functions defined on a metric space. Fixed point results for such mappings play an important role in analysis and applied mathematics.

1.1.1 Definition [5]

Let X be a nonempty set, $S, T : X \longrightarrow X$. Then $x \in X$ is called a

(i) fixed point if the image Tx coincides with x (i.e., $Tx = x$).

(ii) common fixed point of the pair (S, T) if $Sx = Tx = x$.

(iii) coincidence point of the pair (S, T) if $Sx = Tx$,

(iv) point of coincidence of the pair (S, T) for some $y \in X$ such that $x = Sy = Ty$.

1.1.2 Definition

Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called

(i) a Banach contraction, if there is a positive real number $0 < \lambda < 1$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \lambda d(x, y)$$

(ii) an Edelstein contraction

$$d(Tx, Ty) < d(x, y) \text{ for each } x \neq y, \; x, y \in X$$

(iii) non-expansive if

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all } x, y \in X$$

(iv) expansive if

$$d(Tx, Ty) \geq \eta d(x, y) \quad \text{for all } x, y \in X \text{ where } \eta > 1$$

(v) Ciric-type if

$$d(Tx, Ty) \leq M(x, y),$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$$

Berinde [39] introduced the following concept of a weak contraction mapping

1.1.3 Definition [39]

Let (X, d) be a metric space. A self mapping f on X is called a weak contraction if there exist constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$d(fx, fy) \leq \theta d(x, y) + L d(y, fx)$$

holds for each x, y in X

For more details on weak contraction mappings, we refer to [12, 43] and the references therein

1.1.4 Definition

Let X be a nonempty set and 2^X denote collection of all nonempty subset of X. Then T : $X \longrightarrow 2^X$ is called a multivalued mapping. A point $x \in X$ is said to be a

(i) fixed point of T if $x \in Tx$,

(ii) coincidence point of a pair of multivalued mappings (T, S) if $Tx \cap Sx \neq \emptyset$

(iii) common fixed point of the pair (T, S) if $x \in Tx \cap Sx$

Let (X, d) be a metric space and

$CB(X) = \{A : A$ is nonempty closed and bounded subset of $X\}$

$CL(X) =$ the class of all nonempty closed subsets of X

$K(X) =$ the family of all nonempty compact subsets of X

In order to make the family $CB(X)$ into a metric space, we need to have a measure of "distance" between two sets A and B of $CB(X)$. One such notion of distance is

$$d(A, B) = \inf\{d(x, y) : x \in A, \; y \in B\}$$

This definition fails to discriminate sufficiently between sets. We would like the distance between two sets to be zero only if the two sets are the same both in shape and position. For this purpose, the following concept is useful (cf., [89])

1.1.5 Definition

Let (X, d) be a metric space. For $A, B \in CB(X)$ and $\varepsilon > 0$ the sets $N(\varepsilon, A)$ and $E_{A,B}$ are defined as follows

$$N(\varepsilon, A) = \{x \in X : d(x, A) < \varepsilon\}$$

$$E_{A,B} = \{\varepsilon : A \subseteq N(\varepsilon, B), \; B \subseteq N(\varepsilon, A)\}$$

where $d(x, A) = \inf\{d(x, y) : y \in A\}$. The distance function H on $CB(X)$ induced by d is defined by

$$H(A, B) = \inf E_{A,B}$$

which is known as Hausdorff metric on X.

1.1.6 Lemma [106]

Let (X,d) be a metric space, if $A,B \in CB(X)$. For $\lambda > 0$, $a \in A$ there exists a $b \in B$ such that $d(a,b) \leq H(A,B) + \lambda$.

1.1.7 Definition [106]

A mapping $T : X \longrightarrow CB(X)$ is said to be a multivalued contraction if there exists a constant α, $0 \leq \alpha < 1$, such that, for all $x, y \in X$,

$$H(Tx, Ty) \leq \alpha d(x,y)$$

Nadler [106] generalized the Banach contraction principle to multivalued mappings and proved the following important fixed point result for multivalued contractions.

1.1.8 Theorem [106]

Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ a multivalued contraction. Then T has a fixed point.

Berinde and Berinde [10] extended the notion of weak contraction mappings as follows.

1.1.9 Definition[40, 41]

A mapping $T : X \rightarrow CL(X)$ is called a multivalued weak contraction if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx)$$

holds for each x, y in X.

The following definition of a generalized multivalued $(\theta, L)-$strict almost contraction mapping is due to Berinde and Păcurar [41].

1.1.10 Definition [41]

A mapping $T : X \to CL(X)$ is called a generalized multivalued $(\theta, L)-$strict almost contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x,y) + L \min\{d(y,Tx), d(x,Ty), d(x, Tx), d(y,Ty)\}$$

holds for each x, y in X.

The following fixed point theorem appears in [41].

1.1.11 Theorem

Let (X,d) be a complete metric space and $T : X \to CL(X)$ a generalized multivalued $(\theta, L)-$strict almost contraction mapping. Then $F(T) \neq \emptyset$. Moreover, for any $p \in F(T)$, T is continuous at p.

Kamran [81] extended the notion of a multivalued weak contraction mapping to a hybrid pair $\{f, T\}$ of a single-valued mapping f and a multivalued mapping T. For more discussion on multivalued mappings we refer the reader to [13, 61] and the references therein.

1.1.12 Definition

Let (X,d) be a metric space and f a selfmap of X. A multivalued mapping $T : X \to CL(X)$ is called a generalized multivalued $(f, \theta, L)-$weak contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(fx, fy) + Ld(fy, Tx)$$

holds for each x, y in X.

Abbas [1] extended the above definition as follows.

1.1.13 Definition [1]

Let (X,d) be a metric space and f a selfmap on X. A multivalued mapping $T : X \to CL(X)$ is called a generalized multivalued $(f, \theta, L)-$almost contraction mapping if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta M(x, y) + L N(x, y)$$

holds for all x, y in X, where

$$\begin{aligned} M(x, y) &= \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}, \\ N(x, y) &= \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}. \end{aligned}$$

1.2 Single-valued and multivalued commuting mappings

Sessa [116] generalized the concept of commuting mappings as follows.

1.2.1 Definition

Let (X, d) be a metric space. Then two mappings $f, g : X \longrightarrow X$ are said to be weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.

1.2.2 Remark

Note that commuting mappings are weakly commuting but the converse is not true in general (see [116]). Many authors have obtained nice fixed point theorems utilizing this concept. However, since elementary functions such as $fx = x^3$, $gx = 2x^3$ are not weakly commutative, Jungck [76] introduced a less restrictive concept of compatible mappings. He also pointed out in [77, 78] the potential of compatible mappings for proving generalized fixed point theorems.

1.2.3 Definition [76]

Mappings $f, g : X \longrightarrow X$ are said to be compatible if whenever there is a sequence $\{x_n\} \subset X$ satisfying $\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_n = u$, then $\lim_{n \longrightarrow \infty} d(fgx_n, gfx_n) = 0$.

1.2.4 Definition [79]

A pair (f, T) of self-mappings on X are said to be weakly compatible if they commute at each coincidence point (i.e. $fTx = Tfx$ whenever $fx = Tx$).

Junck [76] improved the Banach contraction principle for commuting mappings as follows

1.2.5 Theorem [76]

Let (X, d) be a complete metric space and $f, g : X \longrightarrow X$ be two commuting mappings. If there exists a constant α, $0 \leq \alpha < 1$, such that $gX \subseteq fX$, $d(gx, gy) \leq \alpha d(fx, fy)$, then f and g have a unique common fixed point

1.2.6 Definition

Let $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$ a multivalued mapping. The pair (f, T) is called

(i) commuting if $Tfx = fTx$ for all $x \in X$,

(ii) weakly compatible if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f, T)$ ([79])

A map f is called T- weakly commuting at $x \in X$ if $f^2x \in Tfx$. If a hybrid pair (f, T) is weakly compatible at $x \in C(f, T)$ then f is $T-$weakly commuting at x and hence $f^n(x) \in C(f, T)$. However, the converse is not true in general. For a detailed discussion on the above mentioned notions and their implications we refer the reader to [17], [72], [75, 76, 77, 78] and the references therein

1.3 Single-valued and multivalued α-admissible mappings

Samet et al. [114] introduced the notions of an α-admissible mapping

1.3.1 Definition [114]

Let $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that S is α-admissible if $x, y \in X$, and $\alpha(x, y) \geq 1$ imply that $\alpha(Sx, Sy) \geq 1$

1.3.2 Example [92]

Consider $X = [0, \infty)$, and define $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = 2x$ for all $x, y \in X$, and

$$\alpha(x, y) = \begin{cases} e^{\frac{y}{x}} & \text{if } x \geq y,\ x \neq 0 \\ 0 & \text{if } x < y \end{cases}$$

Then S is $\alpha-$admissible.

We define Ψ to be the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ and $\psi(0) = 0$ for each $t > 0$, where ψ^n is the n-th power of ψ.

1.3.3 Lemma [113]

If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.

1.3.4 Definition [114]

Let (X, d) be a metric space and $S : X \to X$ be a given mapping. We say that S is an $\alpha - \psi -$ contractive mapping if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y)d(Sx, Sy) \leq \psi(d(x, y))$$

for all $x, y \in X$.

1.3.5 Definition [113]

Let $S : X \to X$ and let $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that S is an α-admissible mapping with respect to η if for each $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Sx, Sy) \geq \eta(Sx, Sy)$. Note that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 1.3.1. Also, if we take $\alpha(x, y) = 1$, then S is an η-subadmissible mapping.

1.3.6 Example

Let $X = [0, \infty)$ and $S : X \to X$ be defined by $Sx = x/2$. Define also $\alpha, \eta : X \times X \to [0, +\infty)$ by $\alpha(x, y) = 3$ and $\eta(x, y) = 1$ for all $x, y \in X$. Then S is an α-admissible mapping with respect to η.

1.3.7 Definition [9]

Let $S, T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that the pair (S, T) is α-admissible if for each $x, y \in X$ such that $\alpha(x, y) \geq 1$, we have $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$.

1.3.8 Example

Let $X = [0, \infty)$, and define a pair of self-mapping $S, T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = 2x$, $Tx = x^2$ for all $x, y \in X$ and

$$\alpha(x, y) = \begin{cases} e^{xy} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then the pair (S, T) is α-admissible.

1.3.9 Definition [87]

Let $S : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that S is triangular α-admissible if $x, y, z \in X$, $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ implies that $\alpha(x, y) \geq 1$.

1.3.10 Example [87]

Let $X = [0, \infty)$, $Sx = x^2 + e^x$ and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Then S is a triangular α-admissible mapping.

1.3.11 Definition [87]

Let $S : X \to X$ and $\alpha : X \times X \to \mathbf{R}$. We say that S is a triangular α-admissible mapping if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Sy) \geq 1$, for each $x, y \in X$.

(T2) $\alpha(x, z) \geq 1$, $\alpha(z, y) \geq 1$ implies that $\alpha(x, y) \geq 1$ for each $x, y, z \in X$.

1.3.12 Example [87]

Let $X = \mathbb{R}$, $Sx = \sqrt[3]{x}$ and $\alpha(x, y) = e^{x-y}$ then S is a triangular α-admissible mapping. Indeed, if $\alpha(x, y) = e^{x-y} \geq 1$, then $x \geq y$, which implies that $Sx \geq Sy$. That is, $\alpha(Sx, Sy) = e^{Sx - Sy} \geq 1$. Also, if $\alpha(x, z) \geq 1$, $\alpha(z, y) \geq 1$, then $x - z \geq 0$, $z - y \geq 0$, that is, $x - y \geq 0$ and so $\alpha(x, y) = e^{x-y} \geq 1$.

1.3.13 Definition [9]

Let $S, T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$. We say that a pair (S, T) is triangular α-admissible if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$, for each $x, y \in X$.

(T2) $\alpha(x, z) \geq 1$, $\alpha(z, y) \geq 1$, implies that $\alpha(x, y) \geq 1$, for each $x, y, z \in X$.

Note that if we take $S = T$, then this definition reduces to Definition 1.3.11.

1.3.14 Lemma [46]

Let $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

1.3.15 Lemma

Let $S, T : X \to X$ be a pair of triangular α-admissible. Assume that there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define the sequence $\{x_n\}$ with $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.

1.3.16 Theorem [53]

Let (X, d) be a metric space and $S : X \to X$ a self-mapping. Suppose that there exists a $\beta \in \Omega$ such that, for all $x, y \in X$,

$$d(Sx, Sy) \leq \beta(d(x, y))\, d(x, y).$$

Then S has a fixed unique point $p \in X$ and $\{S^n x\}$ converges to p for each $x \in X$.

1.3.17 Definition [67]

Let (X,d) be a metric space and $T: X \to X$ a self mapping. Suppose that $\alpha, \eta: X \times X \to [0, +\infty)$ are two functions. We say that T is an $\alpha - \eta$-continuous mapping on (X, d) if for a given $x \in X$ and sequence $\{x_n\}$ with

$$x_n \to x \text{ as } n \to \infty,\ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow Tx_n \to Tx.$$

In 1962 Edelstein proved the following version of the Banach contraction principle.

1.3.18 Theorem [50]

Let (X,d) be a compact metric space and $T: X \to X$ a self mapping. Assume that

$$d(Tx, Ty) < d(x,y) \text{ holds for each } x, y \in X \text{ with } x \neq y.$$

Then T has a unique fixed point in X.

Hussain et al. [69] modified the notions of α_*-admissible and α_*-ψ-contractive mappings as follows.

1.3.19 Definition [69]

Let $T: X \to 2^X$ be a multifunction $\alpha, \eta: X \times X \to [0, +\infty)$ two functions where η is bounded. We say that T is an α_*-admissible mapping with respect to η if $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty)$ for each $x, y \in X$, where $\alpha_*(A, B) = \inf\{\alpha(x, y) : x \in A, y \in B\}$ and $\eta_*(A, B) = \sup\{\eta(x,y) : x \in A, y \in B\}$.

If $\eta(x,y) = 1$ for all $x, y \in X$, then this definition reduces to Definition 4.1[69]. In Definition 1.3.19, if $\alpha(x,y) = 1$ for all $x, y \in X$, then T is called an η_*-subadmissible mapping.

1.3.20 Definition [94]

Let (X,d) be a metric space. Suppose $T: X \to CL(X)$ and $\alpha: X \times X \to [0, +\infty)$ is a function. We say that T is an α-continuous multivalued mapping on $(CL(X), H)$ if for given $x \in X$ and sequence $\{x_n\}$ with $\lim_{n \to \infty} d(x_n, x) = 0$, $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ implies that

$\lim_{n\longrightarrow\infty} H(Tx_n, Tx) = 0$

1.3.21 Definition

Let (X, d) be a metric space Suppose that $T: X \to CB(X)$ and $\alpha, \eta: X \times X \to [0, +\infty)$ are two functions We say that T is an $\alpha - \eta$-continuous multivalued mapping on $(CB(X), H)$ if for a given $x \in X$, and sequence $\{x_n\}$ with $x_n \xrightarrow{d} x$ as $n \to \infty$, $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ implies that $Tx_n \xrightarrow{H} Tx$, that is $\lim_{n\longrightarrow\infty} d(x_n, x) = 0$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ implies that $\lim_{n\longrightarrow\infty} H(Tx_n, Tx) = 0$

1.4 Single-valued and multivalued F-contraction mappings

Wordowski et al [123] defined F-contraction as follows

1.4.1 Definition [123]

Let (X, d) be a metric space A mapping $T: X \to X$ is said to be an F-contraction if there exists a $\tau > 0$ such that

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.1}$$

where $F: \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions

(F1) F is strictly increasing, i e , for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

(F3) There exists a $k \in (0, 1)$ such that $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$

We denote by F the set of all functions satisfying conditions (F1)-(F3)

1.4.2 Example [123]

Let $F: \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$ It is clear that F satisfies (F1)-(F2)-(F3) for any $k \in (0, 1)$ Note that (1.1) reduces to the following∂

$$d(Tx, Ty) \leq e^{-\tau} d(x, y),\ \text{for all } x, y \in X,\ Tx \neq Ty$$

It is clear that, for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$ also holds. Hence T is a Banach contraction.

1.4.3 Example [123]

If $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$ then F satisfies (F1)-(F3) and condition (1.1) is of the form

$$\frac{d(Tx, Ty)}{d(x, y)} \leq e^{d(Tx, Ty) - d(x, y)} \leq e^{-\tau} \quad \text{for all } x, y \in X, \; Tx \neq Ty$$

1.4.4 Remark

From (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

Wardowski [123] stated a modified version of the Banach contraction principle as follows.

1.4.5 Theorem [123]

Let (X, d) be a complete metric space and $T : X \to X$ an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to x^*.

Hussain et al. [67] introduced the following family of functions.

Let Δ_G denote the set of all functions $G : \mathbb{R}^{+4} \to \mathbb{R}^+$ satisfying

(G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 t_2 t_3 t_4 = 0$ there exists a $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

1.4.6 Definition [67]

Let (X, d) be a metric space and T be a self-mapping of X. Let $\alpha, \eta : X \times X \to [0, +\infty)$ be two function. We say that T is an α-η-GF-contraction if, for $x, y \in X$, with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(d(x, y))$$

where $G \in \Delta_G$ and $F \in \mathcal{F}$.

Acar et al. [11] introduced the concept of generalized multivalued F-contraction mappings and established a fixed point result, which was a proper generalization of some multivalued fixed point theorems, including Nadler's

1.4.7 Definition [11]

Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction if $F \in \mathcal{F}$ and there exists a $\tau > 0$ such that

$$x, y \in X, \ H(Tx, Ty) > 0 \Longrightarrow \tau + F(H(Tx, Ty)) \le F(M(x, y))$$

where

$$M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}[D(x, Ty) + D(y, Tx)]\}$$

1.4.8 Theorem [11]

Let (X, d) be a complete metric space and $T : X \longrightarrow K(X)$ be a generalized multivalued F-contraction. If T or F is continuous, then T has a fixed point in X.

Sgroi and Vetro [117] proved the following as a generalization of Nadler's Theorem 1.0.6.

1.4.9 Theorem [117]

Let (X, d) be a complete metric space and $T : X \to CL(X)$ a multivalued mapping. Assume that there exists an $F \in \mathcal{F}$ and $\tau \in \mathbb{R}_+$ such that

$$2\tau + F(H(Tx, Ty)) \le F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \delta d(x, Ty) + L d(y, Tx))$$

for all $x, y \in X$, with $Tx \neq Ty$, where $\alpha, \beta, \gamma, \delta, L \ge 0$, $\alpha + \beta + \gamma + 2\delta = 1$ and $\gamma \neq 1$. Then T has a fixed point.

1.4.10 Definition [74]

Let (X, d) be a metric space and $T : X \to CL(X)$ be a multivalued operator. T is called a multivalued weakly Picard operator (briefly MWP operator) if and only if or all $x \in X$ and all $y \in Tx$, there exists a sequence $\{x_n\}$ such that

(i) $x_0 = x$, $x_1 = y$,

(ii) $x_{n+1} \in Tx_n$ for all $n \in N \cup \{0\}$

(iii) the sequence $\{x_n\}$ is convergent and its limit is a fixed point of T

Recently Altun et al. [18] proved the following result

1.4.11 Theorem [18]

Let (X, d) be a complete metric space and $T : X \to CB(X)$. Assume that there exist an $F \in F$ and $\tau, \lambda \in \mathbb{R}_+$ such that for any $x, y \in X$ we have

$$H(Tx, Ty) > 0 \text{ implies that } \tau + F(H(Tx, Ty)) \leq F(d(x, y) + \lambda d(y, Tx))$$

Then T is a multivalued weakly Picard operator

For the definition of a multivalued weakly Picard operator and related results, we refer to [40]

Abbas et al. [2] gives the following definition

1.4.12 Definition

Let f be a selfmap of a metric space X and $T : X \to CL(X)$ be a multivalued mapping. Then T is called generalized multivalued $(f, L)-$almost $F-$contraction mapping if there exist $F \in F$ and $\tau \in \mathbb{R}_+$ and $L \geq 0$ such that

$$2\tau + F(H(Tx, Ty)) \leq F(M(x, y) + LN(x, y))$$

for each x, y in X with $Tx \neq Ty$ and

$$\begin{aligned} M(x, y) &= \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\} \\ N(x, y) &= \min\{d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\} \end{aligned}$$

1.4.13 Remark

Take $F(x) = \ln x$ in the Definition (1.4.12). Then we have

$$2\tau + \ln(H(Tx, Ty)) \leq \ln(M(x, y) + LN(x, y))$$

that is,

$$\begin{aligned} H(Tx,Ty)) &\leq e^{-2\tau} M(x,y) + e^{-2\tau} L N(x,y) \\ &= \theta_1 M(x,y) + L_1 N(x,y) \end{aligned}$$

where $\theta_1 = e^{-2\tau} \in (0,1)$ and $L_1 = e^{-2\tau} L \geq 0$. Thus we have a generalized multivalued $(f, \theta_1, L_1)-$almost contraction mapping [1].

1.4.14 Remark

Take $\alpha = \beta = \gamma = 1/4$, $\delta = 1/8 = L$. Note that $\alpha + \beta + \gamma + 2\delta = 1$. Then the contraction condition of Theorem 1.4.8 becomes

$$\begin{aligned} 2\tau + F(H(Tx,Ty)) &\leq F\left(\frac{1}{4}\left(d(x,y) + (d(x,Tx) + d(y,Ty)) + \frac{d(x,Ty) + d(y,Tx)}{2}\right)\right) \\ &\leq F\left(\frac{1}{4}(4M(x,y))\right) = F((M(x,y) + 0N(x,y))) \end{aligned}$$

for all $x, y \in X$, with $Tx \neq Ty$. Thus, for $L = 0$ and $f = I_X$ in

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty) + d(fy,Tx)}{2}\} \\ N(x,y) &= \min\{d(fx,fy), d(fx,Tx), d(fy,Ty)\} \end{aligned}$$

Thus, T an $(f,0)-$almost $F-$contraction, an is a special case of a generalized multivalued $(f,L)-$almost $F-$contraction (for $L = 0$ and $\tau = 2\tau_1$)

Chapter 2

Fixed Points of Geraghty Contractive Mappings in Metric Spaces

2.1 Introduction

In 1973 Geraghty [53] introduced one of the first generalizations of the Banach theorem. This extension was of significant importance and established some useful fixed point theorems for the Geraghty contraction.

In 2012 Samet et al. [111] introduced the concept of α-ψ-contractive type mappings and established various fixed point theorems for admissible mappings in the setting of complete metric spaces. Afterwards Karapinar et al. [88] refined the notions and obtained various fixed point results. Hussain et al. [65], extended the concept of α-admissible mappings and obtained useful fixed point theorems. Subsequently, Abdeljawad [9] introduced pairs of α-admissible mappings satisfying new sufficient contractive conditions different from those in [65, 111], and proved fixed point and common fixed point theorems. Recently Salimi et al. [113] modified the concept of α-ψ-contractive mappings and established fixed point results.

Hussain et al. [68] proved some fixed point results for single-valued and set-valued α-η-ψ-contractive mappings in the setting of a complete metric space. Mohammadi et al. [98]

introduced a new notion of α-ϕ-contractive mappings and showed that this is a real generalization of some previous results. Hussain et al. [59] derived generalized fixed point theorems for multi-valued α-ψ-contractive mappings. Since then, many papers have been published on Geraghty contractive type mappings in various spaces. For more detail see [80, 83, 84, 85] and the references therein.

In this chapter it is impossible to cover all of the known extensions/generalizations of the Banach Contraction Principle. However, an effort has been made to present some extensions of the Banach Contraction Principle and to explore fixed point and common fixed point results in complete metric spaces.

In this chapter we continue these investigations and explore fixed point and common fixed point results in complete metric spaces. In Section 2.2, we deal with Geraghty contractions in metric spaces and prove the existence and uniqeness of fixed points of α-admissible mappings with respect to η satisfying an $(\alpha - \eta)$-contractive type condition. Section 2.3 and 2.4 deal with Geraghty contractions in metric spaces. In Section 2.3, we improve the notion of a Geraghty contraction and establish some common fixed point theorems for a pair of α-admissible mappings under the improved notion of α-Geraghty contractive type condition in a complete metric space. In Section 2.4, we introduce the concept of α-Geraghty contraction type mappings and establish some common fixed point theorems for a pair of α-admissible mappings under the new approach of generalized rational α-Geraghty contractive type condition in a complete metric spaces.

2.2 Modified Geraghty contraction involving fixed points in metric spaces

Results given in this section have been published in [92].

In this section, Theorem 2.2.1 and Theorem 2.2.5 prove fixed point theorems for α-admissible mappings with respect to η, satisfying a modified $(\alpha - \eta)$-contractive condition in a complete metric space.

We denote by Ω the family of all functions $\beta : [0, +\infty) \to [0, 1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \to 1$ implies $t_n \to 0$.

2.2.1 Theorem

Let (X, d) be a complete metric space and S an α-admissible mapping with respect to η. Assume that there exists a function $\beta \in \Omega$ such that

$$(d(Sx, Sy) + l)^{\alpha(x, Sx)\alpha(y, Sy)} \leq (\beta(d(x, y))\, d(x, y) + l)^{\eta(x, Sx)\eta(y, Sy)} \tag{2.1}$$

for all $x, y \in X$, where $l \geq 1$. Also suppose that one of the following holds

(i) S is continuous,

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p, Sp) \geq \eta(p, Sp)$$

If there exist $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$, then S has a unique fixed point.

Proof Let $x_0 \in X$ and define

$$x_{n+1} = Sx_n, \text{ for all } n \geq 0$$

We shall assume that $x_n \neq x_{n+1}$ for each n. Otherwise, there exists an n such that $x_n = x_{n+1}$. Then $x_n = Sx_n$ and x_n is a fixed point of S. Since $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and S is an α-admissible mapping with respect to η, we have

$$\alpha(x_1, x_2) = \alpha(Sx_0, Sx_1) \geq \eta(Sx_0, Sx_1) = \eta(x_1, x_2)$$

Continuing in this way, we have

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \tag{2.2}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (2.2), we have

$$\alpha(x_{n-1}, x_n)\alpha(x_n, x_{n+1}) \geq \eta(x_{n-1}, x_n)\eta(x_n, x_{n+1})$$

Thus applying inequality (2.1), with $x = x_{k-1}$ and $y = x_k$ we obtain

$$\begin{aligned}
& (d(x_k, x_{k+1}) + l)^{\eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)} \\
= \; & (d(Sx_{k-1}, Sx_k) + l)^{\eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)}
\end{aligned}$$

$$\begin{aligned}
\leq \; & (d(Sx_{k-1}, Sx_k) + l)^{\alpha(x_{k-1}, Sx_{k-1})\alpha(x_k, Sx_k)} \\
\leq \; & (\beta(d(x_{k-1}, x_k))\, d(x_{k-1}, x_k) + l)^{\eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)},
\end{aligned}$$

which implies that

$$d(x_k, x_{k+1}) \leq \beta(d(x_{k-1}, x_k))\, d(x_{k-1}, x_k) \tag{2.3}$$

We obtain

$$d(x_k, x_{k+1}) \leq d(x_{k-1}, x_k)$$

Then we prove that $d(x_{k-1}, x_k) \to 0$. It is clear that $\{d(x_{k-1}, x_k)\}$ is a decreasing sequence. Therefore there exists some nonnegative number ϱ such that $\lim_{n\to\infty} d(x_k, x_{k+1}) = \varrho$. Now we shall prove that $\varrho = 0$. From (2.3), we have

$$\frac{d(x_k, x_{k+1})}{d(x_{k-1}, x_k)} \leq \beta(d(x_{k-1}, x_k)) \leq 1$$

Now by taking limit $k \to \infty$ we have

$$1 = \frac{d}{d} = \frac{\lim_{k\to\infty} d(x_k, x_{k+1})}{\lim_{k\to\infty} d(x_{k-1}, x_k)} \leq \beta(d(x_{k-1}, x_k)) \leq 1$$

$$\lim_{k\to\infty} \beta(d(x_{k-1}, x_k)) = 1$$

By definition of β function we have $\lim_{k\to\infty} d(x_{k-1}, x_k) = 0$. Thus

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = 0 \tag{2.4}$$

Now we prove that the sequence $\{x_n\}$ is Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that for all positive integers k, we have $n_k > m_k > k$

$$d(x_{m_k}, x_{n_k}) \geq \epsilon$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}), \end{aligned} \tag{2.5}$$

for all $k \in \mathbf{N}$ Now taking the limit as $k \to +\infty$ in (2.5) and using (2.4) we have

$$\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = \epsilon \tag{2.6}$$

Using triangle inequality we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})$$

Taking the limit as $k \to +\infty$ and using (2.4) and (2.6) we obtain

$$\lim_{k \to +\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon \tag{2.7}$$

Using (2.1), (2.6) and (2.7), we have

$$
\begin{aligned}
& \left(d(x_{m_{k+1}}, x_{n_{k+1}}) + l\right)^{\eta(x_{m_k}, Sx_{m_k})\eta(x_{n_k}, Sx_{n_k})} \\
\leq\ & \left(d(x_{m_{k+1}}, x_{n_{k+1}}) + l\right)^{\alpha(x_{m_k}, Sx_{m_k})\alpha(x_{n_k}, Sx_{n_k})} \\
\leq\ & \left(d(Sx_{m_k}, Tx_{n_k}) + l\right)^{\alpha(x_{m_k}, Sx_{m_k})\alpha(x_{n_k}, Sx_{n_k})} \\
\leq\ & \left(\beta(d(x_{m_k}, x_{n_k}))d(x_{m_k}, x_{n_k}) + l\right)^{\eta(x_{m_k}, Sx_{m_k})\eta(x_{n_k}, Sx_{n_k})}
\end{aligned}
$$

which implies that

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq \beta(d(x_{m_k}, x_{n_k}))d(x_{m_k}, x_{n_k})$$

Therefore we have

$$\frac{d(x_{m_{k+1}}, x_{n_{k+1}})}{d(x_{m_k}, x_{n_k})} \leq \beta(d(x_{m_k}, x_{n_k})) \leq 1 \tag{2.8}$$

Now taking the limit as $k \to +\infty$ in (2.8), we get

$$\lim_{n \to \infty} \beta(d(x_{m_k}, x_{n_k})) = 1$$

Hence $\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = 0 < \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a $p \in X$ such that $x_n \to p$. We now prove that $p = Sp$. Suppose that (i) holds, that is, S is continuous. Then we get

$$Sp = S \lim_{n \to \infty} x_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} x_{n+1} = p$$

and $p = Sp$. Now we suppose that (ii) holds

$$\alpha(x_n, x_{n-1}) \geq \eta(x_n, x_{n+1}),$$

for all $n \in \mathbb{N} \cup \{0\}$, by the hypotheses of (ii), we have

$$\alpha(p, Sp)\alpha(x_k, Sx_k) \geq \eta(p, Sp)\eta(x_k, Sx_k)$$

Using the triangle inequality and (2.1), we have

$$\begin{aligned}(d(Sp, x_{k+1}) + l)^{\eta(p,Sp)\eta(x_k, Sx_k)} &= (d(Sp, Sx_k) + l)^{\eta(p, Sp)\eta(x_k, Sx_k)} \\ &\leq (d(Sp, Sx_k) + l)^{\alpha(p, Sp)\alpha(x_k, Sx_k)} \\ &\leq (\beta(d(p, x_k))d(p, x_k) + l)^{\eta(p, Sp)\eta(x_k, Sx_k)},\end{aligned}$$

which implies that

$$d(Sp, x_{k+1}) \leq \beta(d(p, x_k))d(p, x_k)$$

Letting $k \to \infty$ we have $d(p, Sp) = 0$. Thus $p = Sp$. Suppose that q is another fixed point of S

$$\begin{aligned}(d(p, q) + l)^{\eta(p, Sp)\eta(q, Sq)} &= (d(Sp, Sq) + l)^{\eta(p, Sp)\eta(q, Sq)} \\ &\leq (d(Sp, Sq) + l)^{\alpha(p, Sp)\alpha(q, Sq)} \\ &\leq (\beta(d(p, q))d(p, q) + l)^{\eta(p, Sp)\eta(q, Sq)}\end{aligned}$$

which implies that

$$d(p, q) + l \leq \beta(d(p, q))d(p, q) + l$$

By definition of β function, $\beta(d(p, q)) = 1$, implies $d(p, q) = 0$. Then we have $p = q$. Hence S has a unique fixed point ■

If $\eta(x, y) = 1$ in Theorem 2.2.1, we get the following corollary

2.2.2 Corollary [65]

Let (X, d) be a complete metric space and S an α-admissible mapping. Assume that there exists a function $\beta \in \Omega$ such that

$$(d(Sx, Sy) + l)^{\alpha(x,Sx)\alpha(y, Sy)} \leq \beta(d(x, y))\, d(x, y) + l$$

for all $x, y \in X$, where $l \geq 1$. Also suppose that either

(i) S is continuous, or

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$

as $n \to +\infty$, then

$$\alpha(p, Sp) \geq 1$$

If there exist $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq 1$, then S has a fixed point.

If $\alpha(x, y) = 1$ in the Theorem 2.2.1, we get the following corollary.

2.2.3 Corollary [92]

Let (X, d) be a complete metric space and S η-subadmissible mapping. Assume that there exists a function $\beta \in \Omega$ such that

$$(d(Sx, Sy) + l) \leq (\beta(d(x, y))\, d(x, y) + l)^{\eta(x, Sx)\eta(y, Sy)}$$

for all $x, y \in X$ where $l \geq 1$. Also suppose that one of the following holds:

(i) S is continuous,

(ii) if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\eta(p, Sp) \leq 1$$

If there exists $x_0, x_1 \in X$ such that $\eta(x_0, x_1) \leq 1$, then S has a fixed point.

2.2.4 Example

Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and $S : X \to X$, $\alpha : X \times X \to [0, \infty)$ and $\beta : [0, +\infty) \to [0, 1]$ for all $x, y \in X$ be defined by

$$Sx = \left\{ \begin{array}{l} 0 \text{ if } x \in [0, 1] \\ \sqrt{x} \text{ if } x \in (1, 5] \end{array} \right\}, \quad \alpha(x, y) = \left\{ \begin{array}{ll} 1 & \text{if } x \geq y \\ 0 & \text{if } x < y \end{array} \right\}$$

$$\text{and } \beta(t) = \frac{1}{\sqrt{t}}, \quad \beta(0) \in [0, 1]$$

We prove that Corollary 2.2.2 can be applied to S. Let $x, y \in X$. Clearly $Sx \leq x$ and $Sy \leq y$, so that S is an α-admissible mapping $\alpha(x, y) \geq 1$, and $\alpha(x, Sx) \geq 1$, $\alpha(y, Sy) \geq 1$ and

$\alpha(x, Sx)\alpha(y, Sy) \geq 1$ implies that

$$(d(Sx, Sy) + l)^{\alpha(x, Sx)\alpha(y, Sy)} = Sx - Sy + l = \sqrt{x} - \sqrt{y} + l \leq \frac{x - y}{\sqrt{x} + \sqrt{y}} + l$$
$$\leq \frac{2(x - y)}{3\sqrt{x - y}} + l = \beta(d(x, y))(d(x, y)) + l$$

If $\alpha(x, Sx)\alpha(y, Sy) = 0$ then we have

$$(d(Sx, Sy) + l)^{\alpha(x, Sx)\alpha(y, Sy)} = 1 \leq \beta(d(x, y))(d(x, y)) + l$$

Let $x = 5$ and $y = 2$. Then

$$d(S5, S2)^{\alpha(5, S5)\alpha(3, S3)} = 0.8218 \leq \beta(d(5, 3))(d(5, 3)) = 1.4142$$

2.2.5 Theorem

Let (X, d) be a complete metric space and S an α-admissible mapping with respect to η. Assume that there exists a function $\beta \in \Omega$ such that

$$\alpha(x, Sx)\alpha(y, Sy)d(Sx, Sy) \leq \eta(x, Sx)\eta(y, Sy)\beta(d(x, y))\, d(x, y) \tag{2.9}$$

for all $x, y \in X$. Also suppose that one of the following holds

(i) S is continuous,

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p, Sp) \geq \eta(p, Sp)$$

If there exists $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$, then S has a unique fixed (need to prove uniqueness) point.

Proof. Let $x_0 \in X$ and define

$$x_{n+1} = Sx_n, \text{ for all } n \geq 0$$

We shall assume that $x_n \neq x_{n+1}$ for each n. Otherwise there exists an n such that $x_n =$

x_{n+1}. Then $x_n = Sx_n$ and x_n is a fixed point of S. Since $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and S is $\alpha-$admissible mapping with respect to η, we have

$$\alpha(x_1, x_2) = \alpha(Sx_0, Sx_1) \geq \eta(Sx_0, Sx_1) = \eta(x_1, x_2).$$

Continuing in this way we have

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \tag{2.10}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (2.10), we have

$$\alpha(x_{n-1}, x_n)\alpha(x_n, x_{n+1}) \geq \eta(x_{n-1}, x_n)\eta(x_n, x_{n+1}).$$

Thus applying inequality (2.9), with $x = x_{k-1}$ and $y = x_k$, we obtain

$$\begin{aligned} & \eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)(d(x_k, x_{k+1})) \\ = \;& \eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)d(Sx_{k-1}, Sx_k) \\ \leq \;& \alpha(x_{k-1}, Sx_{k-1})\alpha(x_k, Sx_k)d(Sx_{k-1}, Sx_k) \\ \leq \;& \eta(x_{k-1}, Sx_{k-1})\eta(x_k, Sx_k)\beta\left(d(x_{k-1}, x_k)\right) d(x_{k-1}, x_k) \end{aligned}$$

which implies that

$$d(x_k, x_{k+1}) \leq \beta\left(d(x_{k-1}, x_k)\right) d(x_{k-1}, x_k).$$

We suppose that

$$d(x_k, x_{k+1}) \leq d(x_{k-1}, x_k). \tag{2.11}$$

Then we prove that $d(x_{k-1}, x_k) \to 0$. It is clear that $\{d(x_{k-1}, x_k)\}$ is a decreasing sequence. Therefore, there exists some positive number ϱ such that $\lim_{n\to\infty} d(x_k, x_{k-1}) = \varrho$. Now we shall prove that $\varrho = 0$. From (2.11), we have

$$\frac{d(x_k, x_{k+1})}{d(x_{k-1}, x_k)} \leq \beta(d(x_{k-1}, x_k)) \leq 1.$$

Now, by taking the limit as $k \to \infty$, we have

$$1 = \frac{d}{d} = \frac{\lim_{k\to\infty} d(x_k, x_{k+1})}{\lim_{k\to\infty} d(x_{k-1}, x_k)} \leq \beta(d(x_{k-1}, x_k)) \leq 1$$

$$\lim_{k\to\infty} \beta(d(x_{k-1}, x_k)) = 1$$

Using definition of β, we have $\lim_{k\to\infty} d(x_{k-1}, x_k) = 0$. Thus

$$\lim_{k\to\infty} d(x_k, x_{k+1}) = 0 \tag{2.12}$$

Now we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sequences $\{m_k\}$ and $\{n_k\}$ such that, for all positive integers k, we have $n_k > m_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \epsilon$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}) \end{aligned} \tag{2.13}$$

for all $k \in \mathbb{N}$. Taking the limit as $k \to +\infty$ in (2.13), and using (2.12), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon \tag{2.14}$$

Again, using the triangle inequality, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}})$$

Taking the limit as $k \to +\infty$ and using (2.12) and (2.14), we obtain

$$\lim_{k \to +\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon \tag{2.15}$$

By using (2.9), (2.14) and (2.15), we have

$$\begin{aligned}
& \eta(x_{m_k}, Sx_{m_k})\eta(x_{n_k}, Sx_{n_k})d(x_{m_{k+1}}, x_{n_{k+1}}) \\
\leq\ & \alpha(x_{m_k}, Sx_{m_k})\alpha(x_{n_k}, Sx_{n_k})d(x_{m_{k+1}}, x_{n_{k+1}}) \\
\leq\ & \alpha(x_{m_k}, Sx_{m_k})\alpha(x_{n_k}, Sx_{n_k})d(Sx_{m_k}, Tx_{n_k}) \\
\leq\ & \eta(x_{m_k}, Sx_{m_k})\eta(x_{n_k}, Sx_{n_k})\beta(d(x_{m_k}, x_{n_k}))d(x_{m_k}, x_{n_k})
\end{aligned}$$

which implies that

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq \beta(d(x_{m_k}, x_{n_k}))d(x_{m_k}, x_{n_k})$$

Therefore we have

$$\frac{d(x_{m_{k+1}}, x_{n_{k+1}})}{d(x_{m_k}, x_{n_k})} \leq \beta(d(x_{m_k}, x_{n_k})) \leq 1 \tag{2.16}$$

Now, taking the limit as $k \to +\infty$ in (2.16), we get

$$\lim_{n \to \infty} \beta(d(x_{m_k}, x_{n_k})) = 1$$

Hence $\lim_{k \to \infty} d(x_{m_k}, x_{n_k}) = 0 < \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a $p \in X$ such that $x_n \to p$. Now we prove that $p = Sp$. Suppose (i) hold that is, S is continuous we get

$$Sp = S \lim_{n \to \infty} x_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} x_{n+1} = p$$

and $p = Sp$. Suppose that (ii) holds that, so we get. Then

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$$

for all $n \in \mathbf{N} \cup \{0\}$, by the hypotheses of (ii), we have

$$\alpha(p, Sp)\alpha(x_k, Sx_k) \geq \eta(p, Sp)\eta(x_k, Sx_k)$$

Using the triangle inequality and (2.9), we have

$$\begin{aligned}\eta(p, Sp)\eta(x_k, Sx_k)d(Sp, x_{k+1}) &= \eta(p, Sp)\eta(x_k, Sx_k)d(Sp, Sx_k) \\ &\leq \alpha(p, Sp)\alpha(x_k, Sx_k)d(Sp, Sx_k) \\ &\leq \eta(p, Sp)\eta(x_k, Sx_k)(\beta(d(p, x_k))d(p, x_k)\end{aligned}$$

which implies that

$$d(Sp, x_{k+1}) \leq \beta(d(p, x_k))d(p, x_k)$$

Letting $k \to \infty$ we obtain $d(p, Sp) = 0$. Thus $p = Sp$. Let q be another fixed point of S

$$\begin{aligned}\eta(p, Sp)\eta(q, Sq)\, d(Sp, Sq) &\leq \alpha(p, Sp)\alpha(q, Sq)\, d(Sp, Sq) \\ &\leq \eta(p, Sp)\eta(q, Sq)\, \beta(d(p, q))d(p, q)\end{aligned}$$

which implies that

$$d(Sp, Sq) \leq \beta(d(p, q))d(p, q)$$

By definition of β, $\beta(d(p, q)) = 1$, implies $d(p, q) = 0$. Then $p = q$. Hence S has a unique fixed (Requires proof) point ■

If $\eta(x, y) = 1$ in Theorem 2.2.5, get the following corollary

2.2.6 Corollary [65]

Let (X, d) be a complete metric space and S an α-admissible mapping. Assume that there exists a function $\beta \in \Omega$ such that

$$\alpha(x, Sx)\alpha(y, Sy)d(Sx, Sy) \leq \beta\,(d(x, y))\, d(x, y)$$

for all $x, y \in X$. Also suppose that either

(i) S is continuous, or

(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then

$$\alpha(p, Sp) \geq 1$$

If there exists $x_0, x_1 \in X$ such that $\alpha(x_0, x_1) \geq 1$, then S has a fixed point

2.2.7 Remark

Our results are more general than those in [65, 113, 114] and improve several results existing in the literature

2.3 Modified α-Geraghty Ćirić type theorems

Results given in this section have been published in [22].

In this section, we improve the notion of Geraghty contraction type mappings and establish some common fixed point theorems for a pair of α-admissible mappings under the improved notion of α-Geraghty contractive type condition in a complete metric space

Cho, Bae and Karapinar [46] established new fixed point theorems for α-Geraghty contraction type mappings in a complete metric spaces. We have the following extension/generalization of these results

Let (X, d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Two mappings $S, T : X \to X$ is called a pair of generalized α-Geraghty contraction type mappings if there exists $\beta \in \Omega$ such that for all $x, y \in X$

$$\alpha(x, y)d(Sx, Ty) \leq \beta(M(x, y))M(x, y) \tag{2.17}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(y, Sx) + d(x, Ty)}{2}\right\}$$

If $S = T$ then T is called generalized α-Geraghty contraction type mapping if there exists $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x, y)d(Sx, Ty) \leq \beta(N(x, y))N(x, y)$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} M(x_{2i}, x_{2i+1}) &= \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i}, Sx_{2i}), d(x_{2i+1}, Tx_{2i+1}), \frac{d(x_{2i}, Tx_{2i+1}) + d(x_{2i+1}, Sx_{2i})}{2}\right\} \\ &= \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+2})}{2}\right\} \\ &\leq \max\left\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}), \frac{d(x_{2i}, x_{2i+1}) + d(x_{2i+1}, x_{2i+2})}{2}\right\} \\ &= \max\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\} \end{aligned}$$

Thus

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta(M(x_{2i}, x_{2i+1}))M(x_{2i}, x_{2i+1}) \\ &\leq \beta(d(x_{2i}, x_{2i+1}))d((x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}) \end{aligned} \tag{2.18}$$

This implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N} \cup \{0\}$$

So the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. Now we prove that $d(x_n, x_{n+1}) \to 0$. It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. Therefore there exists some nonnegative number r such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. From (2.18), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) \leq 1$$

By taking the limit $n \to \infty$, we have

$$1 \leq \beta(d(x_n, x_{n+1})) \leq 1,$$

that is

$$\lim_{n\to\infty} \beta(d(x_n, x_{n+1})) = 1$$

By definition of β, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{2.19}$$

Now we show that sequence $\{x_n\}$ is Cauchy. Suppose on the contrary that $\{x_n\}$ is not a

Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that for all positive integers k, we have $m_k > n_k > k$,

$$d(x_{m_k}, x_{n_k}) \geq \epsilon$$

and

$$d(x_{m_k}, x_{n_{k-1}}) < \epsilon.$$

By the triangle inequality we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{n_k}) \\ &< \epsilon + d(x_{n_{k-1}}, x_{n_k}) \end{aligned} \qquad (2.20)$$

for all $k \in \mathbb{N}$. In the view of (2.20), (2.19), we have

$$\lim_{k \to +\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \qquad (2.21)$$

Again, using the triangle inequality, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_{k+1}}) + d(x_{m_{k+1}}, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k})$$

and

$$d(x_{m_{k+1}}, x_{n_{k+1}}) \leq d(x_{m_{k+1}}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_{k+1}}).$$

Taking the limit as $k \to +\infty$, and using (2.19) and (2.21), we obtain

$$\lim_{k \to +\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon.$$

By Lemma 1.3.15 $\alpha(x_{n_k}, x_{m_{k+1}}) \geq 1$. Thus

$$\begin{aligned} d(x_{n_{k+1}}, x_{m_{k+2}}) &= d(Sx_{n_k}, Tx_{m_{k+1}}) \leq \alpha(x_{n_k}, x_{m_{k+1}}) d(Sx_{n_k}, Tx_{m_{k+1}}) \\ &\leq \beta(M(x_{n_k}, x_{m_{k+1}})) M(x_{n_k}, x_{m_{k+1}}) \end{aligned}$$

Finally, we conclude that

$$\frac{d(x_{n_{k+1}}, x_{m_{k+2}})}{M(x_{n_k}, x_{m_{k+1}})} \leq \beta(M(x_{n_k}, x_{m_{k+1}}))$$

Keeping (2.19) in mind and letting $k \to +\infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \beta(d(x_{n_k}, x_{m_{k+1}})) = 1$$

So $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = 0 < \epsilon$, which is a contradiction. Using similar technique for other cases, it can be easily seen that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a $p \in X$ such that $x_n \to p$ implies that $x_{2i+1} \to p$ and $x_{2i+2} \to p$. As S and T are continuous, we get $Tx_{2i+1} \to Tp$ and $Sx_{2i+2} \to Sp$. Thus $p = Sp$. Similarly, $p = Tp$, so we have $Sp = Tp = p$. Then (S, T) have a common fixed point. ■

In the following Theorem 2.3.2, we have removed the continuity assumption.

2.3.2 Theorem

Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ a function and $S, T : X \to X$ two mappings. Suppose that the following hold:

(i) (S, T) is a pair of generalized α-Geraghty contraction type mappings;

(ii) (S, T) is triangular α-admissible;

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k.

Then (S, T) have a common fixed point.

Proof. Using an argument similar to that of Theorem 2.3.1. Define a sequence $x_{2i+1} = Sx_{2i}$ and $x_{2i+2} = Tx_{2i+1}$, where $i = 0, 1, 2, \ldots$ converges to $p \in X$. By hypotheses of (iv) there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, p) \geq 1$ for all k. Now by using (2.17) for all k we have

$$\begin{aligned} d(x_{2n_k+1}, Tp) &= d(Sx_{2n_k}, Tp) \leq \alpha(x_{2n_k}, p)d(Sx_{2n_k}, Tp) \\ &\leq \beta(M(x_{2n_k}, p))\, M(x_{2n_k}, p) \end{aligned}$$

On the other hand, we obtain

$$M(x_{2n_k}, p) = \max\left\{d(x_{2n_k}, p), d(x_{2n_k}, Sx_{2n_k}), d(p, Tp), \frac{d(x_{2n_k}, Tp) + d(p, Sx_{2n_k})}{2}\right\}$$

Letting $k \to \infty$ we have

$$\lim_{k\to\infty} M(x_{2n_k}, p) = d(p, Tp) \tag{2.22}$$

Suppose that $d(p, Tp) > 0$. From (2.22), for k large enough, we have $M(x_{2n_k}, p) > 0$, which implies that

$$\beta(M(x_{2n_k}, p)) < M(x_{2n_k}, p)$$

Then

$$d(x_{2n_k}, Tp) < M(x_{2n_k}, p) \tag{2.23}$$

Letting $k \to \infty$ inequality (2.23), we obtain that $d(p, Tp) < d(p, Tp)$, which is a contradiction. Thus, we find that $d(p, Tp) = 0$, implies that $p = Sp$. Thus $p = Tp = Sp$. ■

If $N(x, y) = \max\left\{d(x, y), d(x, Sx), d(y, Sy), \frac{d(y, Sx)+d(x, Sy)}{2}\right\}$ and $S = T$ in Theorem 2.3.1 and Theorem 2.3.2, we have the following corollaries.

2.3.3 Corollary

Let (X, d) be a complete metric space and S an α-admissible mapping such that the following hold:

(i) S is a generalized α-Geraghty contraction type mapping;

(ii) S is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iv) S is continuous.

Then S has a unique fixed point $p \in X$ and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

2.3.4 Corollary

Let (X, d) be a complete metric space and S an α-admissible mapping such that the following hold:

(i) S is a generalized α-Geraghty contraction type mapping,

(ii) S is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$ then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k

Then S has a unique fixed point $p \in X$ and S is a Picard operator that is $\{S^n x_0\}$ converges to p

If $N(x, y) = \max\{d(x, y), d(x, Sx), d(y, Sy)\}$ and $S = T$ in Theorem 2 3 1 and Theorem 2 3 2 we obtain the following corollaries

2.3.5 Corollary [46]

Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ a function and $S : X \to X$ a mapping Assume that the following hold

(i) S is a generalized α-Geraghty contraction type mapping

(ii) S is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$

(iv) S is continuous

Then S has a unique fixed point $p \in X$, and S is a Picard operator that is $\{S^n x_0\}$ converges to p

2.3.6 Corollary [46]

Let (X, d) be a complete metric space $\alpha : X \times X \to \mathbb{R}$ a function and $S : X \to X$ a mapping Assume that the following hold

(i) S is a generalized α-Geraghty contraction type mapping

(ii) S is triangular α-admissible

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$ then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k

Then S has a unique fixed point $p \in X$, and S is a Picard operator that is $\{S^n x_0\}$ converges to p

2.3.11 Example

Let $X = \{i, j, k\}$ with metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \frac{5}{7} \text{ if } x, y \in X - \{j\} \\ 1 \text{ if } x, y \in X - \{k\} \\ \frac{4}{7} \text{ if } x, y \in X - \{i\} \end{cases}$$

and

$$\alpha(x, y) = \left\{ \begin{array}{ll} 1 & \text{if } x, y \in X \\ 0 & \text{otherwise} \end{array} \right\}$$

Define a mapping $T : X \to X$ as follows

$$T(x) = \begin{cases} i \text{ if } x \neq j \\ k \text{ if } x = j, \end{cases}$$

and $\beta : [0, +\infty) \to [0, 1)$. Then

$$\alpha(x, y)d(Tx, Ty) \nleq \beta(M(x, y))M(x, y)$$

Indeed, let $x = j$ and $y = k$ then

$$\begin{aligned} M(j, k) &= \max\left\{d(j, k), d(j, T(j)), d(k, T(k)), \frac{d(j, T(k)) + d(k, T(j))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, \frac{4}{7}, \frac{5}{7}, \frac{1}{2}\right\} = \frac{5}{7} \end{aligned}$$

Theorem 2.1[16] cannot be used to obtain a fixed point of T, since

$$\alpha(j, k)d(T(j), T(k)) \nleq \beta(M(j, k))M(j, k)$$

Now we prove that Theorem 2.3.1 can be applied to common fixed point of S and T. Let $S : X \to X$ be a mapping such that $Sx = i$ for each $x \in X$.

where

$$\begin{aligned} M(j,k) &= \max\left\{d(j,k),\ d(j,S(j)),\ d(k,T(k)),\ \frac{d(j,T(k))+d(k,S(j))}{2}\right\} \\ &= \max\left\{\frac{4}{7}, 1, \frac{5}{7}, \frac{12}{14}\right\} = 1, \end{aligned}$$

and

$$d(Sj, Tk) = d(i,i) = 0$$

$$\alpha(x,y)d(Sx, Ty) \leq \beta(M(x,y))(M(x,y)).$$

Hence all of the hypothesis of the Theorem 2.3.1 are satisfied, so S and T have a common fixed point.

2.4 Fixed point results for rational α-Geraghty contraction

Results given in this section will appear in [21].

In this section an effort has been made to improve the notion of α-Geraghty contraction type mappings and establish some common fixed point theorems for a pair of α-admissible mappings under the improved approach of a generalized rational α-Geraghty contractive type condition in a complete metric space.

Let (X,d) be a metric space, $\alpha : X \times X \to \mathbb{R}$ a function and $S, T : X \to X$. S and T are called a pair of a generalized rational α-Geraghty contraction type mapping if there exists a $\beta \in \Omega$ such that, for all $x, y \in X$

$$\alpha(x,y)d(Sx, Ty) \leq \beta(R(x,y))R(x,y) \tag{2.25}$$

where

$$R(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Sx)d(y,Ty)}{1+d(Sx,Ty)}\right\}$$

If $S = T$ then T is called a generalized rational α-Geraghty contraction type mapping if there exists a $\beta \in \Omega$ such that, for all $x, y \in X$,

$$\alpha(x,y)d(Tx,Ty) \leq \beta\left(Q(x,y)\right)Q(x,y),$$

where

$$Q(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}$$

2.4.1 Theorem

Let (X,d) be a complete metric space $\alpha : X \times X \to \mathbb{R}$ a function and $S,T : X \to X$ two mappings such that the following hold

(i) (S,T) is pair of generalized rational α-Geraghty contraction type mapping

(ii) (S,T) is triangular α-admissible

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$

(iv) S and T are continuous

Then (S,T) have a unique common fixed point

Proof. Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0,1,2,$$

By assumption $\alpha(x_0, x_1) \geq 1$ and the pair (S,T) is α-admissible by Lemma 1.3.15. We have

$$\alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \cup \{0\}$$

Then

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &= d(Sx_{2i}, Tx_{2i+1}) \leq \alpha(x_{2i}, x_{2i+1})d(Sx_{2i}, Tx_{2i+1}) \\ &\leq \beta\left(R(x_{2i}, x_{2i+1})\right)R(x_{2i}, x_{2i+1}) \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned}
R(x_{2i}, x_{2i+1}) &= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, Sx_{2i})d(x_{2i+1}, Tx_{2i+1})}{1+d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, Sx_{2i})d(x_{2i+1}, Tx_{2i+1})}{1+d(Sx_{2i}, Tx_{2i+1})}\right\} \\
&= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, x_{2i+1})d(x_{2i+1}, x_{2i+2})}{1+d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, x_{2i+1})d(x_{2i+1}, x_{2i+2})}{1+d(x_{2i+1}, x_{2i+2})}\right\} \\
&\leq \max\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\}
\end{aligned}$$

Thus

$$\begin{aligned}
d(x_{2i+1}, x_{2i+2}) &\leq \beta\left(R(x_{2i}, x_{2i+1})\right) R(x_{2i}, x_{2i+1}) \\
&\leq \beta\left(d(x_{2i}, x_{2i+1})\right) d((x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}) \qquad (2.26)
\end{aligned}$$

which implies

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N} \cup \{0\}$$

So the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. We shall now prove that $d(x_n, x_{n+1}) \to 0$. It is clear that $\{d(x_n, x_{n-1})\}$ is a decreasing sequence. Therefore, there exists some nonnegative number r such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. From (2.26), we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(d(x_n, x_{n+1})) \leq 1$$

Taking the limit as $n \to \infty$, we have

$$1 \leq \beta(d(x_n, x_{n+1})) \leq 1,$$

that is

$$\lim_{n\to\infty} \beta(d(x_n, x_{n+1})) = 1$$

From defintion of β, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \qquad (2.27)$$

Now we shall show that sequence $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$

On the other hand, we obtain

$$R(x_{2n_k}, p) = \max\left\{d(x_{2n_k}, p), \frac{d(x_{2n_k}, Sx_{2n_k})\, d(p, Tp)}{1 + d(x_{2n_k}, p)}, \frac{d(x_{2n_k}, Sx_{2n_k})\, d(p, Tp)}{1 + d(Sx_{2n_k}, Tp)}\right\}$$

Letting $k \to \infty$ we have

$$\lim_{k\to\infty} R(x_{2n_k}, p) = \max\{d(p, Sp), d(p, Tp)\} \tag{2.30}$$

Case I $\lim_{k\to\infty} R(x_{2n_k}, p) = d(p, Tp)$. Suppose that $d(p, Tp) > 0$. From (2.30), for large enough k, we have $R(x_{2n_k}, p) > 0$, which implies that

$$\beta(R(x_{2n_k}, p)) < R(x_{2n_k}, p)$$

Then we have

$$d(x_{2n_k}, Tp) < R(x_{2n_k}, p) \tag{2.31}$$

Letting $k \to \infty$ in inequality (2.31), we obtain that $d(p, Tp) < d(p, Tp)$, which is a contradiction. Thus $d(p, Tp) = 0$, which implies that $p = Tp$.

Case II $\lim_{k\to\infty} M(x_{2n_k}, p) = d(p, Sp)$. Similarly $p = Sp$. Thus $p = Tp = Sp$. ■

If $Q(x, y) = \max\{d(x, y), d(x, Sx)d(y, Sy)/1 + d(x, y), d(x, Sx)d(y, Sy)/1 + d(Sx, Sy)\}$ and $S = T$ in Theorem 2.4.1 and Theorem 2.4.2, we have the following corollaries.

2.4.3 Corollary

Let (X, d) be a complete metric space and S an α-admissible mapping such that the following hold:

(i) S is a generalized rational α-Geraghty contraction type mapping,

(ii) S is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,

(iv) S is continuous.

Then S has a unique fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p.

2.4.4 Corollary

Let (X,d) be a complete metric space and S an $\alpha-$admissible mapping such that the following hold

(i) S is a generalized rational α-Geraghty contraction type mapping

(ii) S is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$,

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to p \in X$ as $n \to +\infty$ then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq 1$ for all k

Then S has a unique fixed point $p \in X$, and S is a Picard operator that is, $\{S^n x_0\}$ converges to p

Let (X, d) be a metric space, and let $\alpha, \eta : X \times X \to \mathbb{R}$ be functions The maps $S, T : X \to X$ are called the pair of a generalized rational α-Geraghty contraction type mappings if there exists a $\beta \in \Omega$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow d(Sx, Ty) \leq \beta(R(x, y))R(x, y) \tag{2.32}$$

where

$$R(x, y) = \max\left\{d(x, y), \frac{d(x,Sx)d(y, Ty)}{1+d(x, y)}, \frac{d(x, Sx)d(y, Ty)}{1+d(Sx, Ty)}\right\}$$

2.4.5 Theorem

Let (X,d) be a complete metric space and (S,T) are α-admissible mappings with respect to η such that the following hold

(i) (S,T) is a generalized rational α-Geraghty contraction type mapping

(ii) (S,T) is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$,

(iv) S and T are continuous,

Then (S,T) have a unique common fixed point

Proof Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$ Continuing this process, we construct a sequence $\{x_n\}$ of points in X such that

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1} \text{ where } i = 0, 1, 2, \ldots$$

By assumption $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$, and the pair (S, T) is α-admissible with respect to η. Thus $\alpha(Sx_0, Tx_1) \geq \eta(Sx_0, Tx_1)$, from which we deduce that $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ which also implies that $\alpha(Tx_1, Sx_2) \geq \eta(Tx_1, Sx_2)$. Continuing in this way we obtain $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &= d(Sx_{2i}, Tx_{2i+1}) \leq \alpha(x_{2i}, x_{2i+1})d(Sx_{2i}, Tx_{2i+1}) \\ &\leq \beta\left(R(x_{2i}, x_{2i+1})\right) R(x_{2i}, x_{2i+1}) \end{aligned}$$

for all $i \in \mathbb{N} \cup \{0\}$. Now

$$\begin{aligned} R(x_{2i}, x_{2i+1}) &= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, Sx_{2i})d(x_{2i+1}, Tx_{2i+1})}{1 + d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, Sx_{2i})d(x_{2i+1}, Tx_{2i+1})}{1 + d(Sx_{2i}, Tx_{2i+1})}\right\} \\ &= \max\left\{d(x_{2i}, x_{2i+1}), \frac{d(x_{2i}, x_{2i+1})d(x_{2i+1}, x_{2i+2})}{1 + d(x_{2i}, x_{2i+1})}, \frac{d(x_{2i}, Sx_{2i})d(x_{2i+1}, Tx_{2i+1})}{1 + d(x_{2i+1}, x_{2i+2})}\right\} \\ &\leq \max\{d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2})\} \end{aligned}$$

From the definition of β, the case $R(x_{2i}, x_{2i+1}) = d(x_{2i+1}, x_{2i+2})$ is impossible.

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta\left(R(x_{2i}, x_{2i+1})\right) R(x_{2i}, x_{2i+1}) \\ &\leq \beta\left(d(x_{2i+1}, x_{2i+2})\right) d(x_{2i+1}, x_{2i+2}) < d(x_{2i+1}, x_{2i+2}) \end{aligned}$$

which is a contradiction. Thus

$$\begin{aligned} d(x_{2i+1}, x_{2i+2}) &\leq \beta\left(R(x_{2i}, x_{2i+1})\right) R(x_{2i}, x_{2i+1}) \\ &\leq \beta\left(d(x_{2i}, x_{2i+1})\right) d((x_{2i}, x_{2i+1}) < d(x_{2i}, x_{2i+1}) \end{aligned}$$

This implies that

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}$$

Using an argument similar to that of Theorem 2.4.1, p is unique common fixed point of S and T. ∎

2.4.6 Theorem

Let (X, d) be a complete metric space and (S, T) α-admissible mappings with respect to η such that the following hold

(i) (S, T) is a generalized rational α-Geraghty contraction type mapping

(ii) (S, T) is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$,

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$ then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all k

Then S and T have a common fixed point

Proof. The proof is similar to that of Theorem 2.4.2 ∎

If $Q(x, y) = \max\{d(x, y), d(x, Sx)d(y, Sy)/1 + d(x, y), d(x, Sx)d(y, Sy)/1 + d(Sx, Sy)\}$ and $S = T$ in Theorem 2.4.5 and Theorem 2.4.6 we get the following corollaries

2.4.7 Corollary

Let (X, d) be a complete metric space and S an α-admissible mapping with respect to η such that the following hold

(i) S is a generalized rational α-Geraghty contraction type mapping,

(ii) S is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$

(iv) S is continuous

Then S has a unique fixed point $p \in X$ and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p

2.4.8 Corollary

Let (X, d) be a complete metric space and S an α-admissible mapping with respect to η such that the following hold

(i) S is a generalized rational α-Geraghty contraction type mapping

(ii) S is triangular α-admissible,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to p \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, p) \geq \eta(x_{n_k}, p)$ for all k

Then S has a unique fixed point $p \in X$, and S is a Picard operator, that is, $\{S^n x_0\}$ converges to p

2.4.9 Example

Let $X = \{1, 2, 3\}$ with metric

$$\begin{aligned} d(1,3) &= d(3,1) = \frac{5}{7}\ d(1,1) = d(2,2) = d(3,3) = 0 \\ d(1,2) &= d(2,1) = 1,\ d(2,3) = d(3,2) = \frac{4}{7} \\ \alpha(x,y) &= \left\{ \begin{array}{ll} 1 & \text{if } x, y \in X, \\ 0 & \text{otherwise} \end{array} \right\} \end{aligned}$$

Define the mappings $S, T : X \to X$ as follows

$$\begin{aligned} Sx &= 1 \text{ for each } x \in X \\ T(1) &= T(3) = 1,\ T(2) = 3, \end{aligned}$$

and $\beta : [0, +\infty) \to [0, 1]$, and

$$\alpha(x, y) d(Tx, Ty) \leq \beta(M(x, y)) M(x, y)$$

Let $x = 2$ and $y = 3$ Then condition of Theorem 2.1 [46] is not satisfied

$$d(T(2), T(3)) = d(3, 1) = \frac{5}{7},$$

$$\begin{aligned} M(x,y) &= \max\{d(2,3), d(2,T(2)), d(3,T(3))\} \\ &= \max\left\{\frac{4}{7},\frac{4}{7},\frac{5}{7}\right\} = \frac{5}{7} \end{aligned}$$

and

$$\alpha(2,3)d(T(2),T(3)) \nleq \beta(M(x,y))M(x,y)$$

If

$$\begin{aligned} Q(x,y) &= \max\left\{d(2,3), \frac{d(2,T(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,T(2))d(3,T(3))}{1+d(T2,T3)}\right\} \\ &= \max\left\{\frac{4}{7},\frac{20}{77},\frac{20}{84}\right\} = \frac{4}{7}. \end{aligned}$$

Then the contractive condition does not hold

$$\alpha(2,3)d(T(2),T(3)) \nleq \beta(Q(x,y))Q(x,y)$$

We now prove that Theorem 2.4.1 can be applied to S and T. Let $x,y \in X$. Clearly (S,T) $\alpha-$admissible mapping such that $\alpha(x,y) \geq 1$. Let $x,y \in X$ and so that $Sx, Ty \in X$ and $\alpha(Sx,Ty) = 1$. Hence (S,T) α-admissible. We show that condition (2.25) of Theorem 2.4.1 is satisfied. If $x,y \in X$ then $\alpha(x,y) = 1$. Thus

$$\alpha(x,y)d(Sx,Ty) \leq \beta(R(x,y))\,(R(x,y)),$$

where

$$\begin{aligned} R(x,y) &= \max\left\{d(2,3), \frac{d(2,S(2))d(3,T(3))}{1+d(2,3)}, \frac{d(2,S(2))d(3,T(3))}{1+d(S2,T3)}\right\} \\ &= \max\left\{\frac{4}{7},\frac{20}{77},\frac{20}{19}\right\} = \frac{4}{7}, \end{aligned}$$

and

$$d(S2,T3) = d(1,1) = 0$$

$$\alpha(x,y)d(Sx,Ty) \leq \beta(R(x,y))\,(R(x,y))$$

Hence all of the hypotheses of Theorem 2.4.1 are satisfied, and S, T have a common fixed point

2.4.10 Remark

For more detail, applications and examples, see [46] and the references therein. Our results are more general than those in [46, 68, 113] and improve several results existing in the literature.

Conclusion. This chapter contains some fixed and common fixed point theorems for single and a pair of α-admissible mappings, under the more general notion of an α-Geraghty contractive type condition. The presented theorems extend, generalize and improve many new and classical results in fixed point theory, in particular, the very famous Banach contraction principle. The present version of these results make significant and useful contribution in the existing literature.

Chapter 3

Single-valued and Multivalued Theorems for F-Contractions in Metric Spaces

3.1 Introduction

In metric fixed point theory, the contractive conditions on underlying functions play an important role to finding solutions of fixed point problems. The Banach contraction principle [34] is a fundamental result in metric fixed point theory. Due to its importance and simplicity several authors have generalized/extended it in different directions. In 2012, Wardowski [123] presented a new type of contraction called an F-contraction and established a Banach fixed point theorem for F-contraction. His findings were followed by Secelean [115], Piri and Kumam [107], Cosentino and Vetro [49], Acar, Durmaz and Minak [11], Acar and Altun [10], Minak, Halvaci and Altun [97] and many others have continued these investigation on F-contraction and obtained fixed point theorems.

Sgroi et al [117] established fixed point theorems for multivalued F-contractions and obtained the solution of certain functional and integral equations, which was a proper generalization of some multivalued fixed point theorems including Nadler's theorem. Recently, Ahmad et al. [12, 58] using the concept of an F-contraction obtained some fixed point and common fixed point results in the context of complete metric spaces. Recently Kutbi et al. [93] extended the concept of an F-contraction to obtain some fixed point results in a complete metric space.

In Section 3.2 we extend the concept of an F-contraction and introduce the notion of a Ćirić type α-η-GF-contraction for a single valued mapping and obtain some new fixed point theorems in a complete metric space.

In Section 3.3 we improve the notion of a Ćirić type α-η-GF-contraction for multivalued mappings and obtain some new fixed point theorems.

In Section 3.4 we introduce a generalization of an F-contraction and establish fixed point theorems for multivalued mappings under α_*-τ-F-contraction on a metric space.

In Section 3.5 we extend an α_*-τ-F-contraction to an α_*-η-τF-contraction and obtain some new Wardowski type fixed point theorems in the setting of a complete metric space.

3.2 Fixed point results for Ćirić type α-η-GF-contractions

Results given in this section have been published in [93]

In this section we define a new contraction, called a Ćirić type α-η-GF-contraction, and obtain some new fixed point theorems for such contractions in the setting of complete metric spaces. We define a Ćirić type α-η-GF-contraction as follows.

3.2.1 Definition

Let (X, d) be a metric space and T a self-mapping on X. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ are two functions. We say that T is Ćirić type α-η-GF-contraction if for all $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y)) \tag{3.1}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

$G \in \Delta_G$ and $F \in I$

Now we state our result

3.2.2 Theorem

Let (X, d) be a complete metric space. Suppose T is a Ćirić type α-η-GF-contraction satisfying the following assertions

(i) T is an α-admissible mapping with respect to η,

(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,

(iii) T is $\alpha - \eta$-continuous.

Then T has a fixed point in X. Moreover T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let x_0 in X be such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$. Continuing this process $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now, since T is an α-admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this process, we have

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N} \tag{3.2}$$

If there exists an $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, there is nothing to prove. So we assume that $x_n \neq x_{n+1}$ with

$$d(Tx_{n-1}, Tx_n) = d(x_n, Tx_n) > 0 \ \forall n \in \mathbb{N}$$

Since T is a Ćirić type α-η-GF-contraction, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} & G(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})) \\ +F(d(Tx_{n-1}, Tx_n)) \leq & \ F(M(x_{n-1}, x_n)) \end{aligned}$$

which implies that

$$\begin{aligned} & G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) \\ +F(d(Tx_{n-1}, Tx_n)) \leq & \ F(M(x_{n-1}, x_n)) \end{aligned} \tag{3.3}$$

Now by the definition of G, $d(x_{n-1}, x_n) d(x_n, x_{n+1}) d(x_{n-1}, x_{n+1}) 0 = 0$ there exists a $\tau > 0$

such that

$$G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) = \tau$$

Therefore

$$F\left(d(x_n, x_{n+1})\right) = F\left(d(Tx_{n-1}, Tx_n)\right) \leq F\left(M(x_{n-1}, x_n)\right) - \tau \tag{3.4}$$

Now

$$\begin{aligned}
M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\right\} \\
&= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\
&\leq \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\right\} \\
&= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}
\end{aligned}$$

so we have

$$F\left(d(x_n, x_{n+1})\right) = F\left(d(Tx_{n-1}, Tx_n)\right) \leq F\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\right) - \tau$$

In this case $M(x_{n-1}, x_n) = \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} = d(x_n, x_{n+1})$, which is impossible because

$$F\left(d(x_n, x_{n+1})\right) = F\left(d(Tx_{n-1}, Tx_n)\right) \leq F\left(d(x_n, x_{n+1})\right) - \tau < F\left(d(x_n, x_{n+1})\right)$$

which is a contradiction. So

$$M(x_{n-1}, x_n) = \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} = d(x_{n-1}, x_n)$$

Thus from (3.4), we have

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_{n-1}, x_n)\right) - \tau$$

Continuing this process we get

$$\begin{aligned} F(d(x_n, x_{n+1})) &\le F(d(x_{n-1}, x_n)) - \tau \\ &= F(d(Tx_{n-2}, Tx_{n-1})) - \tau \\ &\le F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &= F(d(Tx_{n-3}, Tx_{n-2})) - 2\tau \\ &\le F(d(x_{n-3}, x_{n-2})) - 3\tau \\ &\vdots \\ &\le F(d(x_0, x_1)) - n\tau \end{aligned}$$

This implies that

$$F(d(x_n, x_{n+1})) \le F(d(x_0, x_1)) - n\tau \tag{3.5}$$

From (3.5), we obtain $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Since $F \in \mathcal{F}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{3.6}$$

From $(F3)$, there exists a $k \in (0,1)$ such that

$$\lim_{n\to\infty} \left((d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1})) \right) = 0 \tag{3.7}$$

From (3.5), for all $n \in \mathbb{N}$, we obtain

$$(d(x_n, x_{n+1}))^k (F(d(x_n, x_{n+1})) - F(d(x_0, x_1))) \le -(d(x_n, x_{n+1}))^k n\tau \le 0 \tag{3.8}$$

By using (3.6), (3.7) and letting $n \to \infty$ in (3.8), we have

$$\lim_{n\to\infty} \left(n (d(x_n, x_{n+1}))^k \right) = 0 \tag{3.9}$$

We observe that, from (3.9) there exists an $n_1 \in \mathbb{N}$, such that $n(d(x_n, x_{n+1}))^k \le 1$ for all

$n \geq n_1$. Thus

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1 \tag{3.10}$$

Now for $m, n \in \mathbb{N}$ such that $m > n \geq n_1$, using the triangle inequality, and from (3.10), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \qquad (3.11) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} i^{-1/k} \end{aligned}$$

The series $\sum_{i=n}^{\infty} i^{-1/k}$ is convergent. By taking the limit as $n \to \infty$ in (3.11), we have $\lim_{n,m \to \infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space there exists an $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. T is α-η-continuous and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. Thus $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$, that is $x^* = Tx^*$. Hence x^* is a fixed point of T. To prove uniqueness, let $x \neq y$ be any two fixed points of T. Then from (3.1) we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y))$$

which implies that

$$\tau + F(d(x, y)) \leq F(d(x, y))$$

which is a contradiction. Hence $x = y$ and therefore T has a unique fixed point. ■

3.2.3 Theorem

Let (X, d) be a complete metric space. Suppose that T is a self mapping satisfying the following assertions

(i) T is an α-admissible mapping with respect to η,

(ii) T is Ciric type α-η-GF-contraction

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x^*$ as $n \to \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

hold for all $n \in \mathbb{N}$

Then T has a fixed point in X. Moreover T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$

Proof. The proof is similar to that of Theorem 3 2 2. We can

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \to x^* \text{ as } n \to \infty$$

$$\alpha(Tx_n, x^*) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x^*) \geq \eta(T^2x_n, T^3x_n)$$

for all $n \in \mathbb{N}$. This implies that

$$\alpha(x_{n+1}, x^*) \geq \eta(x_{n+1}, x_{n+2}) \text{ or } \alpha(x_{n+2}, x^*) \geq \eta(x_{n+2}, x_{n+3}), \text{ for all } n \in \mathbb{N}$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x^*)$$

From (3 1) we have

$$\begin{aligned}
& G(d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), d(x_{n_k}, Tx^*), d(x^*, Tx_{n_k})) + F(d(Tx_{n_k}, Tx^*)) \\
\leq\ & F(M(x_{n_k}, x^*)) \\
=\ & F\left(\max\left\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2}\right\}\right) \\
=\ & F\left(\max\left\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1})}{2}\right\}\right)
\end{aligned}$$

Using the continuity of F and the fact that

$$\lim_{k\to\infty} d(x_{n_k}, x^*) = 0 = \lim_{k\to\infty} d(x_{n_k+1}, x^*)$$

we obtain

$$\tau + F\left(d(x^*, Tx^*)\right) \leq F\left(d(x^*, Tx^*)\right),$$

which is a contradiction. Therefore $d(x^*, Tx^*) = 0$ implies that follows along lines similar to those of the argument in Theorem 3.2.2. ∎

In the following we extend the Wardowski type fixed point theorem.

3.2.4 Theorem

Let T be a continuous self-mapping of a complete metric space X. If, for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F\left(d(Tx, Ty)\right) \leq F\left(M(x, y)\right)$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

$G \in \Delta_G$ and $F \in \digamma$. Then T has a fixed point in X.

Proof. Let us define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = d(x, y) \text{ and } \eta(x, y) = d(x, y) \text{ for all } x, y \in X.$$

Now, $d(x, y) \leq d(x, y)$ for all $x, y \in X$, so $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 3.2.2 hold true. Since T is continuous, T is α-η-continuous. Since $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have $d(x, Tx) \leq d(x, y)$. Then

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F\left(d(Tx, Ty)\right) \leq F\left(M(x, y)\right)$$

That is, T is a Ćirić type α-η-GF-contraction mapping. Hence, all of the conditions of Theorem 3.2.2 are satisfied and T has a fixed point. Let T be a continuous self-mapping on a complete metric space X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have

$$\tau + F\left(d(Tx, Ty)\right) \leq F\left(M(x, y)\right),$$

where $\tau > 0$ and $F \in \Gamma$, then T has a fixed point in X ∎

3.2.5 Corollary

Let T be a continuous selfmapping on a complete metric space X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have

$$\tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right),$$

where $\tau > 0$, and $F \in \Gamma$, then T has a fixed point in X

3.2.6 Corollary [67]

Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a self-mapping satisfying the following assertions

(i) T is an α-admissible mapping with respect to η,

(ii) T is an α-η-GF-contraction,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$

(iv) T is $\alpha - \eta$-continuous

Then T has a fixed point in X. Moreover T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$

3.2.7 Corollary [67]

Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a self-mapping satisfying the following assertions

(i) T is an α-admissible mapping with respect to η,

(ii) T is an α-η-GF-contraction,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,

(iv)if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n-1})$ with $x_n \to x^*$ as $n \to \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in Fix(T)$

3.2.8 Example

Consider the sequence,

$S_1 = 1 \times 3$

$S_2 = 1 \times 3 + 2 \times 5$

$S_3 = 1 \times 3 + 2 \times 5 + 3 \times 7$

$S_n = 1 \times 3 + 2 \times 5 + 3 \times 7 \quad + n \times (2n+1) = n(n+1)(4n+5)/6$

Let $X=\{S_n \; n \in \mathbb{N}\}$ and $d(x,y) = |x-y|$. Then $(X\; d)$ is a complete metric space. If $F(\alpha) = \alpha + \ln \alpha$, $\alpha > 0$ and $G(t_1, t_2\; t_3, t_4) = \tau$ where $\tau = 1$. Define the mapping $T \; X \to X$ by, $T(S_1) = S_1$ and $T(S_n) = S_{n-1}, n \geq 1$, and $\alpha(x\; y) = 1$ if $x \in X$ $\eta(x\; Tx) = 1/2$ for all $x \in X$ we have

$$\lim_{n\to\infty} \frac{d(T(S_n), T(S_1))}{d(S_n, S_1)} = \lim_{n\to\infty} \frac{S_{n-1}-3}{S_n - 3} = \frac{(n-1)n(4n+1)-18}{n(n+1)(4n+5)-18} = 1$$

So we conclude the following two cases

Case 1

For every $m \in \mathbb{N}$ $m > 2, n = 1$ or $n = 1$ and $m > 1$. Then $\alpha(S_m\; S_n) \geq \eta(S_m\; T(S_m))$ and we have

$$\begin{aligned}
\frac{d(T(S_m), T(S_1))}{M(S_m, S_1)} e^{d(T(S_m),T(S_1))-M(S_m\, S_1)} &= \frac{S_{m-1}-3}{S_m-3} e^{S_{m-1}-S_m} \\
&= \frac{(m-1)m(4m+1)-18}{m(m+1)(4m+5)-18} e^{-\frac{m(m+1)(4m+5)}{6}} \\
&< e^{-1}
\end{aligned}$$

Case 2

For $m > n > 1$ $\alpha(S_m, S_n) \geq \eta(S_m, T(S_m))$, and

$$\frac{d(T(S_m),T(S_n))}{M(S_m,S_n)}e^{d(T(S_m),T(S_n))-M(S_m,S_n)}$$
$$= \frac{S_{m-1}-S_{n-1}}{S_m-S_n}e^{S_n-S_{n-1}+S_{m-1}-S_m}$$
$$= \frac{(m-1)m(4m+1)-(n-1)n(4n+1)}{m(m+1)(4m+5)-n(n+1)(4n+5)}e^{\frac{n(n+1)(4n+5)}{6}-\frac{n(m+1)(4m+5)}{6}} \leq e^{-1}$$

Thus all of the conditions of Theorem 3 2 2 and Theorem 3 2 3 are satisfied so T has a fixed point in X

Let $(X,d,\preceq)$ be a partially ordered metric space Let T $X \to X$ such that for x $y \in X$ if $x \preceq y$ implies $Tx \preceq Ty$ then the mapping T is said to be non-decreasing We derive following important result in partially ordered metric spaces

3.2.9 Theorem

Let $(X,d,\preceq)$ be a complete partially ordered metric space Assume that the following assertions are true

(i) T is nondecreasing and ordered GF-contraction

(ii) there exists an $x_0 \in X$ such that $x_0 \preceq Tx_0$

(iii) either for a given $x \in X$ and sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ we have $Tx_n \to Tx$,

or if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$ then either

$$Tx_n \preceq x \text{ or } T^2x_n \preceq x$$

holds for all $n \in \mathbb{N}$

Then T has a fixed point in X

Define $\Gamma = \{\phi$ $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ ϕ is a Lebesgue integral mapping which is summable nonnegative and satisfies $\int_0^\epsilon \phi(t)dt > 0$, for each $\epsilon > 0\}$

We can easily deduce following result involving integral type inequalities

3.2.10 Theorem

Let T be a continuous selfmapping on a complete metric space X. If, for $x, y \in X$ and

$$\int_0^{d(x,Tx)} \phi(t)dt \leq \int_0^{d(x,y)} \phi(t)dt \text{ and } \int_0^{d(Tx,Ty)} \phi(t)dt > 0,$$

we have

$$G\left(\int_0^{d(x,Tx)} \phi(t)dt, \int_0^{d(y,Ty)} \phi(t)dt, \int_0^{d(x,Ty)} \phi(t)dt, \int_0^{d(y,Tx)} \phi(t)dt\right) + F\left(\int_0^{d(Tx,Ty)} \phi(t)dt\right)$$
$$\leq F\left(\int_0^{M(x,y)} \phi(t)dt\right),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}$$

$\phi \in \Gamma, G \in \Delta_G$ and $F \in F$. Then T has a fixed point in X.

3.3 Fixed point results for multivalued Ćirić type α-η-GF-contractions

Results given in this section will be published in [23].

In this section we introduce multivalued Ćirić type α-η-GF-contraction and establish some new fixed point results in a complete metric space. We extend the concept of an F-contraction to multivalued mappings as follows.

3.3.1 Definition

Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ are two functions. We say that T is a multivalued Ćirić type α-η-GF-contraction if for $x, y \in X$ with $\eta_*(x, Tx) \leq \alpha_*(x, y)$ and $Tx \neq Ty$ we have

$$2G\left(D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)\right) + F\left(H(Tx, Ty)\right) \leq F\left(M(x, y)\right) \qquad (3.12)$$

where

$$M(x,y) = \max\left\{d(x,y),\ D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\right\}$$

$G \in \Delta_G$ and $F \in F$

3.3.2 Theorem

Let (X,d) be a complete metric space. Suppose that $T: X \longrightarrow CB(X)$ satisfying the following assertions

(i) T is an α_*-admissible mapping with respect to η,

(ii) T is a multivalued Ćirić type α-η-GF-contraction,

(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq \eta_*(x_0, Tx_0)$,

(iv) T is an $\alpha-\eta$-continuous multivalued mapping.

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be such that $\alpha_*(x_0, Tx_0) \geq \eta_*(x_0, Tx_0)$. Since T is an α_*-admissible mapping with respect to η, there exists an $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha_*(x_0, Tx_0) \geq \eta_*(x_0, Tx_0) = \eta(x_0, x_1) \tag{3.13}$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T. So we assume that $x_0 \neq x_1$, then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$\begin{aligned} F(hH(Tx_0, Tx_1)) &< F(H(Tx_0, Tx_1)) + \\ &\quad G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)) \end{aligned}$$

Now, from $D(x_1, Tx_1) < hH(Tx_0, Tx_1)$, we deduce that there exists an $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq hH(Tx_0, Tx_1)$. Consequently, we obtain

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(hH(Tx_0, Tx_1)) \\ &< F(H(Tx_0, Tx_1)) + G(D(x_0, Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)) \end{aligned}$$

which implies that

$$
\begin{aligned}
& 2G\left(D(x_0,Tx_0), D(x_1,Tx_1), D(x_0,Tx_1), D(x_1,Tx_0)\right) + F(D(x_1, x_2)) \\
\leq\; & 2G\left(D(x_0,Tx_0), D(x_1,Tx_1), D(x_0,Tx_1), D(x_1,Tx_0)\right) + F\left(H(Tx_0,Tx_1)\right) + \\
& G\left(D(x_0,Tx_0), D(x_1,Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)\right) \\
\leq\; & F\left(M(x_0, x_1)\right) + G\left(D(x_0,Tx_0), D(x_1, Tx_1), D(x_0, Tx_1), D(x_1, Tx_0)\right)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& G(D(x_0,Tx_0), D(x_1,Tx_1), D(x_0,Tx_1), 0) + F\left(M(Tx_0, Tx_1)\right) \\
\leq\; & F\left(\max d(x_0,x_1), D(x_0,Tx_0), D(x_1,Tx_1), \frac{D(x_0,Tx_1) + D(x_1,Tx_0)}{2}\right)
\end{aligned}
\tag{3.14}
$$

Now since $d(x_0, x_1)\, d(x_1, x_2)\, d(x_0, x_2)\, 0 = 0$, so from (G) there exists a $\tau > 0$ such that

$$G(D(x_0,Tx_0), D(x_1,Tx_1), D(x_0,Tx_1), 0) = \tau$$

Therefore from (3.14), we deduce that

$$
\begin{aligned}
& \tau + F\left(d(x_1,Tx_1)\right) \\
\leq\; & F\left(\max\left\{d(x_0, x_1), D(x_0,Tx_0), D(x_1,Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2}\right\}\right) \\
=\; & F\left(\max\left\{d(x_0, x_1), D(x_0,Tx_0), D(x_1,Tx_1), \frac{D(x_0, Tx_1) + D(x_1, Tx_0)}{2}\right\}\right) - \tau \\
=\; & F\left(\max\left\{d(x_0, x_1), D(x_0, Tx_0), D(x_1,Tx_1), \frac{D(x_0, x_2) + D(x_1, x_1)}{2}\right\}\right) - \tau \\
\leq\; & F\left(\max\left\{D(x_0, Tx_0), D(x_1, Tx_1)\right\}\right) - \tau
\end{aligned}
\tag{3.15}
$$

If $\max\left\{D(x_0,Tx_0), D(x_1,Tx_1)\right\} = D(x_1,Tx_1)$, then (3.15) becomes

$$F\left(D(x_1, Tx_1)\right) \leq F\left(D(x_1, Tx_1)\right) - \tau$$

which is not true Thus $\max\{D(x_0, Tx_0), D(x_1, Tx_1)\} = D(x_0, Tx_0)$ Consequently

$$F(D(x_1, Tx_1)) \leq F(D(x_0, Tx_0)) - \tau$$

By continuing this process and we obtain a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$,

$$\eta(x_{n-1}, x_n) = \eta_*(x_{n-1}, Tx_{n-1}) \leq \alpha_*(x_{n-1}, Tx_{n-1}) = \alpha(x_{n-1}, x_n) \tag{3.16}$$

and

$$\begin{aligned}
& \tau + F(d(x_n, x_{n+1})) \leq \\
& F\left(\max\left\{d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \frac{D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})}{2}\right\}\right) \\
= \ & F\left(\max\left\{d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \frac{D(x_{n-1}, Tx_n)}{2}\right\}\right) - \tau \\
\leq \ & F(\max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\}) - \tau
\end{aligned}$$

If $\max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\} = D(x_n, Tx_n)$ then

$$F(D(x_n, Tx_n)) \leq F(D(x_n, Tx_n)) - \tau$$

Thus $\max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\} = D(x_{n-1}, Tx_{n-1})$ we obtain

$$F(d(x_n, x_{n+1})) \leq F(D(x_{n-1}, Tx_{n-1})) - \tau \tag{3.17}$$

for all $n \in \mathbb{N} \cup \{0\}$ By (3.17) we have

$$\begin{aligned}
F(d(x_n, x_{n+1})) &\leq F(D(x_{n-1}, Tx_{n-1})) - \tau \\
&\leq F(D(x_{n-2}, Tx_{n-2})) - 2\tau \\
& \\
&\leq F(D(x_0, Tx_0)) - n\tau
\end{aligned} \tag{3.18}$$

for all $n \in \mathbb{N}$. Since $F \in \digamma$, by taking limit as $n \longrightarrow \infty$ in (3.18), we deduce that

$$\lim_{n \longrightarrow \infty} F(d(x_n, x_{n+1})) = -\infty \iff \lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = 0 \tag{3.19}$$

From (F_3), there exists a k, $0 < k < 1$ such that

$$\lim_{n \longrightarrow \infty} [d(x_n, x_{n+1})]^k F(d(x_n, x_{n+1})) = 0 \tag{3.20}$$

By (3.17) we have

$$\begin{aligned} & d(x_n, x_{n+1})^k F(d(x_n, x_{n+1})) - d(x_n, x_{n+1})^k F(d(x_0, x_1)) \qquad (3.21) \\ \leq\ & d(x_n, x_{n+1})^k [F(d(x_0, x_1) - n\tau)] - d(x_n, x_{n+1})^k F(d(x_0, x_1)) \\ =\ & -n\tau [d(x_n, x_{n+1})]^k \leq 0 \end{aligned}$$

Letting $n \longrightarrow \infty$ in (3.21) and applying (3.19) and (3.20), we have

$$\lim_{n \longrightarrow \infty} n [d(x_n, x_{n+1})]^k = 0 \tag{3.22}$$

From (3.22), there exists an $n_1 \in \mathbb{N}$, such that $n(d(x_n, x_{n+1}))^k \leq 1$ for all $n \geq n_1$, which implies that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1 \tag{3.23}$$

Now $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then, by the triangle inequality and from (3.23), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \end{aligned}$$

The series $\sum_{i=n}^{\infty} i^{-1/k}$ is convergent. This implies that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists an $x^* \in X$ such that $\lim_{n \longrightarrow \infty} d(x_n, x^*) = 0$. By (3.16) and the α-η-continuity of the multi-valued mapping T, we get

$$\lim_{n \longrightarrow \infty} H(Tx_n, Tx^*) = 0$$

Now we obtain

$$D(x^*, Tx^*) = \lim_{n \longrightarrow \infty} D(x_{n+1}, Tx^*) \leq \lim_{n \longrightarrow \infty} H(Tx_n, Tx^*) = 0$$

Therefore, $x^* \in Tx^*$ and hence T has a fixed point. ■

3.3.3 Theorem

Let (X, d) be a complete metric space. Suppose $T : X \longrightarrow CB(X)$ satisfies the following assertions:

(i) T is an α_*-admissible mapping with respect to η,

(ii) T is a multivalued Ćirić type α-η-GF-contraction,

(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq \eta_*(x_0, Tx_0)$,

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \rightarrow x$ as $n \rightarrow \infty$ then either

$$\alpha_*(Tx_n, x) \geq \eta_*(Tx_n, Tx_{n+1}) \text{ or } \alpha_*(Tx_{n+1}, x) \geq \eta_*(Tx_{n+1}, Tx_{n+2})$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof The proof is similar to that of Theorem 3.3.2, we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \rightarrow x \text{ as } n \rightarrow \infty$$

$$\alpha_*(Tx_n, x) \geq \eta_*(Tx_n, Tx_{n+1}) \text{ or } \alpha_*(Tx_{n+1}, x) \geq \eta_*(Tx_{n+1}, Tx_{n+2})$$

holds for all $n \in \mathbb{N}$, we have

$$\alpha(x_{n+1}, x) \geq \eta(x_{n+1}, x_{n+2}) \text{ or } \alpha(x_{n+2}, x) \geq \eta(x_{n+2}, x_{n+3})$$

Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_{n_k+1} \in Tx_{n_k}$ such that

$$\eta_*(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha_*(x_{n_k}, x)$$

and so, from (3.12), we deduce that

$$\begin{aligned} & 2G\left(D(x_{n_k}, Tx_{n_k}), D(x, Tx), D(x_{n_k}, Tx), D(x, Tx_{n_k})\right) + F\left(H(Tx_{n_k}, Tx)\right) \\ \leq \; & F\left(\max\left\{d\left(x_{n_k}, x\right), D\left(x_{n_k}, Tx_{n_k}\right), D\left(x, Tx\right), \frac{D\left(x_{n_k}, Tx_{n_k}\right) + D\left(x, Tx_{n_k}\right)}{2}\right\}\right) \end{aligned}$$

that is

$$\begin{aligned} & 2G\left(D(x_{n_k}, Tx_{n_k}), D(x, Tx), D(x_{n_k}, Tx), D(x, Tx_{n_k})\right) + F\left(D(x_{n_k+1}, Tx)\right) \\ \leq \; & 2G\left(D(x_{n_k}, Tx_{n_k}), D(x, Tx), D(x_{n_k}, Tx), D(x, Tx_{n_k})\right) + F\left(H(Tx_{n_k}, Tx)\right) \\ \leq \; & F\left(\max\left\{d\left(x_{n_k}, x\right), D\left(x_{n_k}, Tx_{n_k}\right), D\left(x, Tx\right), \frac{D\left(x_{n_k}, Tx_{n_k}\right) + D\left(x, Tx_{n_k}\right)}{2}\right\}\right) \end{aligned}$$

Using the continuity of F and the fact that

$$\lim_{k \to \infty} d(x_{n_k}, x) = 0 = \lim_{k \to \infty} d(x_{n_k+1}, x)$$

we obtain

$$\tau + F\left(D(x, Tx)\right) \leq F\left(D(x, Tx)\right)$$

that is $D(x, Tx) = 0$. Since Tx is closed, we get that $x \in Tx$ and hence x is a fixed point of T. ■

3.3.4 Theorem

Let $T : X \longrightarrow CB(X)$ be a continuous multivalued mapping in a complete metric space X. If for $x, y \in X$ with $D(x, Tx) \leq d(x,y)$ and $Tx \neq Ty$, we have

$$\begin{aligned} & 2G(D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)) + F(H(Tx, Ty)) \\ \leq \; & F(M(x,y)), \end{aligned}$$

where

$$M(x,y) = \max\left\{ d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2} \right\},$$

and $G \in \Delta_G$, $F \in F$, and T has a fixed point in X.

Proof. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = d(x,y) \text{ and } \eta(x,y) = d(x,y) \text{ for all } x, y \in X.$$

Now, $d(x,y) \leq d(x,y)$ for all $x, y \in X$, so $\alpha(x,y) \geq \eta(x,y)$ for all $x, y \in X$, that is, conditions (i) and (iii) of Theorem 3.3.2 are satisfied. T is continuous and T is an α-η-continuous multivalued mapping. Let $\eta_*(x, Tx) \leq \alpha_*(x,y)$ and $Tx \neq Ty$. Then $D(x, Tx) \leq d(x,y)$ with $Tx \neq Ty$, and

$$\begin{aligned} & 2G(D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)) + F(H(Tx, Ty)) \\ \leq \; & F(M(x,y)), \end{aligned}$$

that is, T is a multivalued Ćirić type α-η-GF-contraction mapping. Hence all of the conditions Theorem 3.3.2 are satisfied and T has a fixed point. ■

3.3.5 Example

Let $X = [0,1]$, $T : X \to CB(X)$ be defined by $Tx = [0, x/2]$ and d be the usual metric on X. Define $\alpha, \eta : X \times X \longrightarrow [0, \infty)$, $G : R^{+^4} \longrightarrow R^+$ and $F : R^+ \longrightarrow R$ by

$$\alpha(x,y) = \begin{cases} e^{xy} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \eta(x,y) = \begin{cases} e^{x-y} & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

$G(t_1, t_2, t_3, t_4) = \tau$ where $\tau = \ln(\sqrt{4})$ and $F(t) = \ln(t) + t$ for all $t > 0$. Then for all $x, y \in X$, $Tx \neq Ty$, we obtain

$$
\begin{aligned}
& 2G(D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)) + F(H(Tx, Ty)) \\
= \ & 2\tau + F(H(Tx, Ty)) \\
= \ & \ln(4) + \ln(H(Tx, Ty)) + H(Tx, Ty) \\
= \ & \ln(4) + \ln(\frac{1}{2}|y-x|) + \frac{1}{2}|y-x| \\
\leq \ & \ln(4) + \ln(\frac{1}{4}) + \ln(\frac{1}{2}|y-x|) + \frac{1}{2}|y-x| \\
= \ & F(d(x,y)) \\
\leq \ & F(M(x,y))
\end{aligned}
$$

Therefore T is a multivalued Ćirić type α-η-GF-contraction. Thus, all of the above conditions of Theorem 3.3.2 are satisfied, and 0 is a fixed point of T.

3.3.6 Theorem

Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ satisfying the following assertions:

(i) T is an α_*-admissible mapping

(ii) T is a multivalued Ćirić type α-GF-contraction

(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$

(iv) T is continuous

Then T has a fixed point in X.

3.3.7 Corollary

Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a Ćirić type α-η-GF-contraction satisfying the following assertions:

(i) T is an α-admissible mapping with respect to η,

(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,

(iii) T is $\alpha - \eta$-continuous.

Then T has a fixed point in X. Moreover T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

3.3.8 Corollary

Let (X, d) be a complete metric space. Suppose that T is a selfmap satisfying the following assertions:

(i) T is an α-admissible mapping with respect to η;

(ii) T is Ćirić type α-η-GF-contraction;

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x^*$ as $n \to \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X. Moreover T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

3.3.9 Corollary

Let T be a continuous selfmapping of a complete metric space X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y))$$

where $\tau > 0$, and $F \in \mathcal{F}$, then T has a fixed point in X.

3.3.10 Corollary

Let T be a continuous selfmapping of a complete metric space X. If, for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

where $\tau > 0$, and $F \in \mathcal{F}$, then T has a fixed point in X

3.3.11 Corollary [67]

Let (X, d) be a complete metric space. Suppose that $T : X \to X$ is a self-mapping satisfying the following assertions

(i) T is an α-admissible mapping with respect to η;

(ii) T is an α-η-GF-contraction,

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,

(iv) T is $\alpha - \eta$-continuous

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$

3.4 Generalization of fixed point results for F-contraction

The definition and results given in this section have been published in [63]

In this section we define a contraction, called an α_*-τ-F-contraction for a multivalued mapping and obtain some new fixed point theorems for such contractions in the setting of complete metric spaces. We define a multivalued α_*-τ-F-contraction as follows

3.4.1 Definition

Let (X, d) be a metric space and $T : X \longrightarrow CB(X)$ an α_*-admissible multivalued mapping. Also suppose that $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function. We say that T is a multivalued α_*-τ-F-contraction if for $x, y \in X$, and $H(Tx, Ty) > 0$ we have

$$2\tau(M(x, y)) + \alpha_*(Tx, Ty)F(H(Tx, Ty)) \leq F(M(x, y)) \tag{3.24}$$

where

$$M(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty)\}$$

$$\alpha_*(A, B) = \inf\{\alpha(x, y) : x \in A, y \in B\},$$

and $F \in \mathcal{F}$

Our main result is the following

3.4.2 Theorem

Let (X, d) be a complete metric space. Suppose that $T: X \longrightarrow CB(X)$ satisfies the following assertions

(i) T is an α_*-admissible multivalued mapping,

(ii) T is multivalued α_*-τ-F-contraction

(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$,

(iv) $\forall_{t\geq 0} \liminf_{s \to t^+} \tau(s) > 0$

(v) T is continuous

Then T has a fixed point in X

Proof. Let $x_0 \in X$, such that $\alpha_*(x_0, Tx_0) \geq 1$. Since T is an α_*-admissible mapping there exists an $x_1 \in Tx_0$ such that

$$\alpha_*(x_0, Tx_0) \geq 1$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T. So we assume that $x_0 \neq x_1$. Then $Tx_0 \neq Tx_1$. Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau(\max\{d(x_0, x_1)D(x_0, Tx_0), D(x_1, Tx_1)\})$$

From $D(x_1, Tx_1) < hH(Tx_0, Tx_1)$ we deduce that there exists an $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq hH(Tx_0, Tx_1)$. Consequently we obtain

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(hH(Tx_0, Tx_1)) \\ &< F(H(Tx_0, Tx_1)) + \tau(\max\{d(x_0, x_1)D(x_0, Tx_0), D(x_1, Tx_1)\}) \end{aligned}$$

there exists a $c > 0$ and $n \in \mathbb{N}$ such that $\tau(d(x_n, x_{n+1})) > c$ for all $n > n_0$. We then obtain

$$
\begin{aligned}
F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau(d(x_{n-1}, x_n)) \\
&\leq F(d(x_{n-2}, x_{n-1})) - \tau(d(x_{n-2}, x_{n-1})) - \tau(d(x_{n-1}, x_n)) \\
&\vdots \\
&\leq F(d(x_0, x_1)) - \tau(d(x_0, x_1)) - \cdots - \tau(d(x_{n-1}, x_n)) \\
&= F(d(x_0, x_1)) - (\tau(d(x_0, x_1)) + \cdots + \tau(d(x_{n_0-1}, x_{n_0}))) \\
&\quad - (\tau(d(x_{n_0}, x_{n_0+1})) + \cdots + \tau(d(x_{n-1}, x_n))) \\
&\leq F(d(x_0, x_1)) - (n - n_0)\, c
\end{aligned} \tag{3.25}
$$

Since $F \in \mathcal{F}$, by taking limit as $n \longrightarrow \infty$ in (3.25), we deduce that

$$\lim_{n \longrightarrow \infty} F(d(x_n, x_{n+1})) = -\infty \iff \lim_{n \longrightarrow \infty} d(x_n, x_{n+1}) = 0 \tag{3.26}$$

From (F_3), there exists a k, $0 < k < 1$ such that

$$\lim_{n \longrightarrow \infty} [d(x_n, x_{n+1})]^k F(d(x_n, x_{n+1})) = 0 \tag{3.27}$$

By (3.25) we have

$$
\begin{aligned}
& d(x_n, x_{n+1})^k F(d(x_n, x_{n+1})) - d(x_n, x_{n+1})^k F(d(x_0, x_1)) \\
\leq\; & d(x_n, x_{n-1})^k [F(d(x_0, x_1) - (n - n_0)c)] - d(x_n, x_{n+1})^k F(d(x_0, x_1)) \\
=\; & -(n - n_0)\, c\, [d(x_n, x_{n+1})]^k \leq 0
\end{aligned} \tag{3.28}
$$

Letting $n \longrightarrow \infty$ in (3.28) and applying (3.26) and (3.27), we have

$$\lim_{n \longrightarrow \infty} n\, [d(x_n, x_{n-1})]^k = 0 \tag{3.29}$$

We observe that from (3.29), there exists an $n_1 \in \mathbb{N}$, such that $n\,(d(x_n, x_{n+1}))^k \leq 1$ for all $n \geq n_1$, which implies that

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1 \tag{3.30}$$

Now $m, n \in \mathbb{N}$ are such that $m > n \geq n_1$. Then by the triangle inequality, and from (3.30), we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \quad (3.31)$$
$$= \sum_{i=n}^{m-1} d(x_i, x_{i+1})$$
$$\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1})$$
$$\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$$

The series $\sum_{i=n}^{\infty} i^{-1/k}$ is convergent. By taking the limit as $n \to \infty$ in (3.31), we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists an $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. By (v), T is continuous. Thus

$$\lim_{n\to\infty} H(Tx_n, Tx^*) = 0$$

Now we obtain

$$D(x^*, Tx^*) = \lim_{n\to\infty} D(x_{n+1}, Tx^*) \leq \lim_{n\to\infty} H(Tx_n, Tx^*) = 0$$

Therefore $x^* \in Tx^*$, and hence T has a fixed point. ■

3.4.3 Theorem

Let (X, d) be a complete metric space and $T : X \longrightarrow CB(X)$ satisfies the following assertions:

(i) T is multivalued α_*-admissible mapping;

(ii) T is multivalued α_*-τ-F-contraction;

(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \geq 1$;

(iv) $\forall_{t\geq 0} \liminf_{s\to t^+} \tau(s) > 0$;

(v) if $\{x_n\}$ is a sequence in X such that $\alpha_*(x_n, x_{n+1}) \geq 1$ with $x_n \to x^*$ as $n \to \infty$, then $\alpha_*(x_n, x^*) \geq 1$ holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof. The proof is similar to that of Theorem 3.4.2. By (v), $\alpha_*(x_{n+1}, x^*) \geq 1$ for all $n \in \mathbb{N}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha_*(x_{n_k+1}, x^*) \geq 1$$

From (3.24) we have

$$2\tau(M(x_{n_k}, x^*)) + \alpha(Tx_{n_k}, Tx^*)F(H(Tx_{n_k}, Tx^*)) \leq F(M(x_{n_k}, x^*))$$

which implies that

$$\begin{aligned} & 2\tau(\max\{d(x_{n_k}, x^*), D(x_{n_k}, Tx_{n_k}), D(x^*, Tx^*)\}) + \alpha(Tx_{n_k}, Tx^*)F(H(Tx_{n_k}, Tx^*)) \\ \leq \; & F(\max\{d(x_{n_k}, x^*), D(x_{n_k}, Tx_{n_k}), D(x^*, Tx^*)\}) \end{aligned}$$

Using the continuity of F and the fact that

$$\lim_{k\to\infty} d(x_{n_k}, x^*) = 0 = \lim_{k\to\infty} d(x_{n_k+1}, x^*),$$

we obtain

$$2\tau(D(x^*, Tx^*)) + F(D(x^*, Tx^*)) \leq F(D(x^*, Tx^*)),$$

which is a contradiction. Therefore $x^* \in Tx^*$ implies that x^* is a fixed point of T. ■

3.5 Modified fixed point results for F-contraction

The definition and results given in this section have been published in [63].

In this section we extend an α_*-τ-F-contraction to an α_*-η-τF-contraction and obtain some new Wardowski type fixed point theorems in the setting of a complete metric space. We define an α_*-η-τF-contraction as follows

3.5.1 Definition

Let (X,d) be a metric space and $T : X \longrightarrow CB(X)$ an α_*-admissible multivalued mapping with respect to η_*. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty), \tau : \mathbb{R}_+ \to \mathbb{R}_+$ are three functions. We say that T is a multivalued α_*-η-τF-contraction if for all $x, y \in X$, with $\eta_*(x, Tx) \le \alpha_*(x, y)$ and $H(Tx, Ty) > 0$, we have

$$2\tau(M(x,y)) + F(H(Tx,Ty)) \le F(M(x,y)),$$

where

$$M(x,y) = \max\{d(x,y), D(x,Tx), D(y,Ty)\}$$

$$\alpha_*(A,B) = \inf\{\alpha(x,y) : x \in A, y \in B\}, \eta_*(A,B) = \sup\{\eta(x,y) : x \in A, y \in B\},$$

and $F \in \Gamma$.

Now we state our result.

3.5.2 Theorem

Let (X,d) be a complete metric space and $T : X \longrightarrow CB(X)$ satisfies the following assertions:

(i) T is multivalued α_*-admissible mapping with respect to η,

(ii) T is multivalued α_*-η-τF-contraction,

(iii) there exists an $x_0 \in X$ such that $\alpha_*(x_0, Tx_0) \ge 1$,

(iv) $\forall_{t \ge 0} \liminf_{s \to t^+} \tau(s) > 0$,

(v) T is an $\alpha - \eta$-continuous multivalued mapping.

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ for which $\alpha_*(x_0, Tx_0) \ge \eta_*(x_0, Tx_0)$. Since T is an α_*-admissible mapping with respect to η, then there exists an $x_1 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha_*(x_0, Tx_0) \ge \eta_*(x_0, Tx_0) = \eta(x_0, x_1).$$

If $x_1 \in Tx_1$, then x_1 is a fixed point of T. So, we shall that $x_0 \ne x_1$. Then $Tx_0 \ne Tx_1$. Since

F is continuous from the right there exists a real number $h > 1$ such that

$$F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau(\max\{d(x_0, x_1) D(x_0, Tx_0), D(x_1, Tx_1)\})$$

From $D(x_1, Tx_1) < hH(Tx_0, Tx_1)$, it follows that there exists an $x_2 \in Tx_1$ such that $d(x_1, x_2) \leq hH(Tx_0, Tx_1)$. Consequently we obtain

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq F(hH(Tx_0, Tx_1)) \\ &< F(H(Tx_0, Tx_1)) + \tau(\max\{d(x_0, x_1) D(x_0, Tx_0), D(x_1, Tx_1)\}) \end{aligned}$$

which implies that

$$\begin{aligned} & 2\tau(\max\{d(x_0, x_1) D(x_0, Tx_0), D(x_1, Tx_1)\}) + F(d(x_1, x_2)) \\ \leq\ & 2\tau(\max\{d(x_0, x_1) D(x_0, Tx_0), D(x_1, Tx_1)\}) + F(H(Tx_0, Tx_1)) + \\ & \tau(\max\{D(x_0, Tx_0), D(x_1, Tx_1)\}) \\ \leq\ & F(\max\{D(x_0, Tx_0), D(x_1, Tx_1)\}) + \tau(\max\{D(x_0, Tx_0), D(x_1, Tx_1)\}) \end{aligned}$$

In this case, $\max\{D(x_0, Tx_0), D(x_1, Tx_1)\} = D(x_1, Tx_1)$ is impossible because

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq \alpha_*(Tx_0, Tx_1) F(H(Tx_0, Tx_1)) \\ &\leq F(D(x_1, Tx_1)) - \tau(D(x_1, Tx_1)) \\ &< F(D(x_1, Tx_1)) \end{aligned}$$

Which is a contradiction. Thus

$$\begin{aligned} F(D(x_1, Tx_1)) &\leq \alpha_*(Tx_0, Tx_1) F(H(Tx_0, Tx_1)) \\ &\leq F(D(x_0, Tx_0)) - \tau(D(x_0, Tx_0)) \end{aligned}$$

Continuing this process, we obtain a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$,

$$\eta(x_{n-1}, x_n) = \eta_*(x_{n-1}, Tx_{n-1}) \leq \alpha_*(x_{n-1}, Tx_{n-1}) = \alpha(x_{n-1}, x_n)$$

The rest of the proof is similar to that of Theorem 3.4.2 ■

3.5.3 Corollary [67]

Let (X,d) be a complete metric space. Suppose that $T: X \to X$ is a self-mapping satisfying the following assertions

(i) T is an α-admissible mapping with respect to η,

(ii) If, for $x, y \in X$ with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

where $\tau > 0$ and $F \in F$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,

(iv) T is an $\alpha - \eta$-continuous

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in Fix(T)$

3.5.4 Example

Let $X = [0, 1]$, and $T: X \to CB(X)$ be defined by $Tx = [0, x/3]$ and d be the usual metric on X. Define $\alpha, \eta: X \times X \longrightarrow [0, \infty)$, $\tau: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $F: \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$\alpha(x, y) = \begin{cases} e^{x+y} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\eta(x, y) = \begin{cases} e^{x-y} & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases}$$

$$\tau(t) = \begin{cases} -\ln t & \text{for } t \in (0, 1) \\ \ln t & \text{for } t \in [1, \infty) \end{cases}$$

and $F(t) = \ln(t) + t$ for all $t > 0$ Then, for all $x, y \in X$ $Tx \neq Ty$ we obtain

$$\begin{aligned} & \tau(M(x,y)) + F(d(Tx,Ty)) \\ = \;& \ln(t) + \ln(d(Tx,Ty)) + d(Tx,Ty) \\ \leq \;& \ln(t) + \ln(\frac{1}{3}|y-x|) + \frac{1}{3}|y-x| \\ \leq \;& \ln(t) + \ln(\frac{1}{t}) + \ln(\frac{1}{3}|y-x|) + \frac{1}{3}|y-x| \\ = \;& F(d(x,y)) \\ \leq \;& F(M(x,y)) \end{aligned}$$

Therefore T is an α_*-η-τF-contraction, and all of the conditions of Theorem 3 3 2 and Theorem 3 4 2 are satisfied

Conclusion: The main aim of our chapter is to present new concepts of a Ciric type F-contraction for single-valued and multivalued mapping, different from the F-contractions given in [67 107, 123] The existence of fixed point results for single-valued and multivalued mapping of F-contraction in a complete metric space are established In each section we introduce new concepts of F-contraction and establish some results The new concepts lead to further investigations and applications It will be also interesting to apply these concepts to different metric spaces

Chapter 4

Dynamic Process for Generalized (f, L)-Almost F-Contraction

The aim of this chapter is to introduce the notion of a dynamic process for generalized (f, L)-almost F-contraction mappings, and to obtain coincidence and common fixed point results for generalized dynamic processes. It is worth mentioning that our results do not rely on the commonly used range inclusion condition. We provide some examples to support our results. As an application of our results, we obtain the existence and uniqueness of solutions of dynamic programming and integral equations. Our results provide an extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Abbas et al. ([4]) extended the concept of an F-contraction mapping and obtained common fixed point results. They employed their results to obtain fixed points of a generalized nonexpansive mappings on star shaped subsets of normed linear spaces. Recently, Abbas et al. [2] introduced the concept of a multivalued f-almost F-contraction which generalizes the class of multivalued almost contraction mappings, and obtained coincidence point results. Minak [97] proved some fixed point results for a Ciric type generalized F- contractions on complete metric spaces. Very recently, Budhia et al. [15] introduced two new concepts of an α-type almost-F-contraction and an α-type F Suzuki contraction and proved some fixed point theorems for such mappings in a complete metric space.

4.1 Introduction

Theory, Examples and the definition given in this section have been published in [60]

In this section we introduce a concept of a generalized dynamic process $D(f, T, x_0)$. Let x_0 be an arbitrary but fixed element in X. The set

$$D(f, T, x_0) = \left\{ (fx_n)_{n \in \mathbb{N} \cup \{0\}} : fx_n \in Tx_{n-1} \text{ for all } n \in \mathbb{N} \right\}$$

is called a generalized dynamic process of f and T starting at x_0. Note that $D(f, T, x_0)$ reduces to the dynamic process of T starting at x_0 if $f = I_X$ (the identity map on X) [90]. The generalized dynamic process $D(f, T, x_0)$ will simply be written as (fx_n). The sequence $\{x_n\}$ for which (fx_n) is a generalized dynamic process is called an f-iterative sequence of T starting at x_0.

Note that, if the hybrid pair $\{f, T\}$ is said to satisfy the range inclusion condition then for any $x_0 \in X$, construction of an f-iterative sequence of T, starting at x_0, is immediate and hence $D(f, T, x_0)$ is nonemtpy.

There are many situations where $D(f, T, x_0)$ is nonempty, even when the range inclusion condition does not hold. The following are the examples of such cases.

4.1.1 Example

Let $X = [0, \infty)$. Define $f : X \to X$ and $T : X \to CL(X)$ by $f(x) = 2x$, $Tx = [1 + x, \infty)$ respectively. Note that one can construct several f-iterative sequences of T starting at some point $x_0 \in X$.

$$x_n = \frac{3}{2}(1 + x_{n-1})$$

is an f-iterative sequence of T starting at 0.

4.1.2 Example

Let $X = [0, \infty)$. Define $f: X \to X$ and $T: X \to CL(X)$ by $f(x) = x^2$, $Tx = [2 + x, \infty)$ respectively. The sequence $\{x_n\}$, where

$$x_n = \sqrt{x_{n-1} + 2}$$

is an f-iterative sequence of T starting at the point 0.

4.1.3 Example

Let $X = \mathbb{R}$. Define $f: X \to X$ and $T: X \to CL(X)$ by $f(x) = x - 1/2$ and

$$Tx = \begin{cases} [\frac{1}{4}, \frac{x}{2}] \text{ when } x > 0 \\ \{0\} \text{ otherwise} \end{cases},$$

respectively. Define a sequence $\{x_n\}$ by $x_n = x_{n-1} + 1$. If $x_0 = 1$, then

$$\begin{aligned} f(x_1) &= \frac{1}{2} \in Tx_0 = [\frac{1}{4}, \frac{1}{2}] \\ f(x_2) &= 1 \in Tx_1 = [\frac{1}{4}, 1] \\ f(x_3) &= \frac{3}{2} \in Tx_2 = [\frac{1}{4}, \frac{3}{2}] \text{ and so on} \end{aligned}$$

Here

$$D(f, T, 1) = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$$

is a generalized dynamic process of f and T starting at $x_0 = 1$.

4.1.4 Definition

Let $f: X \to X$ and x_0 be an arbitrary point in X. A multivalued mapping $T: X \to CL(X)$ is called a generalized multivalued (f, L)-almost F-contraction with respect to a generalized dynamic process $D(f, T, x_0)$, if there exist $F, \tau: \mathbb{R}_+ \to \mathbb{R}_+$, non decreasing, and $L \geq 0$ such that

$$\forall_{n\in\mathbb{N}}\ d(fx_n, fx_{n+1}) > 0 \Rightarrow \tau(M(x_{n-1}, x_n)) + F(d(fx_n, fx_{n+1})) \leq F(M(x_{n-1}, x_n) + LN(x_{n-1}, x_n))$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\}, \\ N(x_{n-1}, x_n) &= \min\{d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), d(fx_{n-1}, Tx_n), d(fx_n, Tx_{n-1})\} \end{aligned}$$

and $\liminf_{s\to t^+} \tau(s) > 0$ for all $t \geq 0$

4.1.5 Remark

Take $F(x) = \ln x$ in Definition 4.1.1 to obtain

$$\tau(M(x_{n-1}, x_n)) + \ln(d(fx_n, fx_{n+1})) \leq \ln(M(x_{n-1}, x_n) + LN(x_{n-1}, x_n)),$$

that is

$$\begin{aligned} d(fx_n, fx_{n+1}) &\leq e^{-\tau(M(x_{n-1}, x_n))}M(x_{n-1}, x_n) + e^{-\tau(M(x_{n-1}, x_n))}LN(x, y) \\ &= \theta_2 M(x_{n-1}, x_n) + L_2 N(x_{n-1}, x_n) \end{aligned}$$

where $\theta_2 = e^{-\tau(M(x_{n-1}, x_n))} \in (0, 1)$ and $L_2 = e^{-\tau(M(x_{n-1}, x_n))}L \geq 0$. Thus we obtain a generalized multivalued $(f, L)-$almost $F-$contraction with respect to a dynamic process.

4.1.6 Example

Consider Example 4.1.3. For the arbitrary points $x = 0$ and $y = 2$, we have

$$\begin{aligned} M(0, 2) &= \max\{d(f0, f2), d(f0, T0), d(f2, T2), \frac{d(f0, T2) + d(f2, T0)}{2}\} \\ &= \max\left\{d(-\frac{1}{2}, \frac{1}{2}), d(-\frac{1}{2}, 0), d(\frac{1}{2}, [\frac{1}{4}, 1]), \frac{d(-\frac{1}{2}, [\frac{1}{4}, 1]) + d(\frac{1}{2}, 0)}{2}\right\} \\ &= \max\{1, \frac{1}{2}, 0, \frac{5}{8}\} = 1 \end{aligned}$$

and

$$\begin{aligned} N(0,2) &= \min\{d(f0,T0), d(f2,T2)\ d(f0,T2), d(f2,T0)\} \\ &= \min\left\{\frac{1}{2}\ 0\ \frac{3}{1}, \frac{1}{2}\right\} = 0 \end{aligned}$$

Take $F(x) = \ln x, \tau > 0$, and $L \geq 0$ to get

$$\begin{aligned} 2\tau + F(H(T0\ T2)) &\nleq F(M(0,2) + LN(0\ 2)) \\ 2\tau + \ln\frac{1}{4} &\nleq \ln(1) \end{aligned}$$

Hence T is not a generalized multivalued (f, L)–almost F–contraction On the other hand the contractive condition is satisfied for every point in the set $D(f\ T\ 1)$ For example, take $\frac{1}{2}$ and 1 in the set $D(f, T\ 1)$, we obtain

$$\begin{aligned} M(\frac{1}{2},1) &= \max\{d(f\frac{1}{2},f1), d(f\frac{1}{2},T\frac{1}{2}), d(f1,T1), \frac{d(f\frac{1}{2},T1)+d(f1,T\frac{1}{2})}{2}\} \\ &= \max\left\{d(-\frac{1}{4}\ 0), d(-\frac{1}{4},\frac{1}{4},), d(0,[\frac{1}{4},\frac{1}{2}]), \frac{d(-\frac{1}{4}\ [\frac{1}{4},\frac{1}{2}])+d(0\ \frac{1}{1})}{2}\right\} \\ &= \max\{\frac{1}{4},\frac{1}{2}\ \frac{1}{4}\ \frac{3}{8}\} = \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} N(\frac{1}{2}\ 1) &= \min\{d(f\frac{1}{2}\ T\frac{1}{2})\ d(f1\ T1), d(f\frac{1}{2},T1)\ d(f1,T\frac{1}{2})\} \\ &= \min\left\{\frac{1}{2}\ \frac{1}{4}\ \frac{1}{2},\frac{1}{4}\right\} = \frac{1}{4} \end{aligned}$$

We have

$$d(f(x_2), f(x_3)) = d(1, \frac{3}{2}) = \frac{1}{2} > 0$$

$$F(x) = \ln x \text{ and } \tau(t) = \begin{cases} -\ln(t+\frac{1}{2}) \text{ for } t \in (0,1) \\ \ln 3 \text{ for } t \in [1, \infty) \end{cases} \text{ and } L = 1,$$

$$\begin{aligned} \tau\left(M(\frac{1}{2}, 1)\right) + F(\frac{1}{2}) &\leq F(M(\frac{1}{2}, 1) + LN(\frac{1}{2}, 1)) \\ \tau\left(\frac{1}{2}\right) + F(\frac{1}{2}) &\leq F(\frac{1}{2} + 1 \cdot \frac{1}{4}) \\ -\ln 1 + \ln\frac{1}{2} &\leq \ln\frac{3}{4} \end{aligned}$$

Hence T is a generalized multivalued (f, L)-almost F-contraction with respect to the generalized dynamic process $D(f, T, 1)$

4.1.7 Example

Let $X = [0,1]$ and d be the usual metric on X. Define $f : X \to X$ and $T : X \to CL(X)$ by

$$f(x) = \begin{cases} 0 \text{ if } x \in [0, \frac{3}{4}) \\ 1 \text{ otherwise} \end{cases} \text{ and } Tx = \begin{cases} [0, \frac{x}{2}] \text{ if } x \in (0, 1] \\ [1, 2] \text{ if } x = 0 \end{cases}$$

Then, for any two points $x = 0$ and $y = 1$, we have

$$\begin{aligned} M(0,1) &= \max\{d(f0, f1), d(f0, T0), d(f1, T1), \frac{d(f0, T1) + d(f1, T0)}{2}\} \\ &= \max\left\{d(0, 1), d(0, [1, 2]), d(1, \left[0, \frac{1}{2}\right]), \frac{d(0, [0, \frac{1}{2}]) + d(1, [1, 2])}{2}\right\} \\ &= \max\{1, 1, \frac{1}{2}, 0\} = 1 \end{aligned}$$

and

$$\begin{aligned} N(0,1) &= \min\{d(f0, T0), d(f1, T1), d(f0, T1), d(f1, T0)\} \\ &= \min\left\{d(0, [1, 2]), d(1, \left[0, \frac{1}{2}\right]), d(0, \left[0, \frac{1}{2}\right]), d(1, [1, 2])\right\} \\ &= \min\{1, \frac{1}{2}, 0, 0\} = 0 \end{aligned}$$

Consequently the contractive condition is not satisfied,

$$2\tau + F(H(T0, T1)) \nleq F(M(0,1) + LN(0,1))$$

$F(x) = \ln x$ and $\tau > 0$ and $L \geq 0$ Then

$$2\tau + \ln \frac{1}{2} \nleq \ln 1,$$

and hence T is not a generalized multivalued (f, L)-almost F-contraction

4.2 Fixed point results for generalized (f, L)-almost F-contraction

Results given in this section have been published in [60]

In this section we assume that the mapping F is right continuous In the sequel we will consider only the dynamic processes (fx_n) satisfying the following condition

$$(D) \text{ For any } n \text{ in } \mathbf{N} \quad d(fx_n, fx_{n+1}) > 0 \Rightarrow d(fx_{n-1}\ fx_n) > 0$$

If a dynamic processes (fx_n) does not satisfy property (D) then there exists an $n_0 \in \mathbb{N}$ such that $d(fx_{n_0}, fx_{n_0+1}) > 0$ and $d(fx_{n_0-1}, fx_{n_0}) = 0$, which implies that $fx_{n_0-1} = fx_{n_0} \in Tx_{n_0-1}$, that is, the set of coincidence points of a hybrid pair (f, T) is nonempty Under suitable conditions on the hybrid pair (f, T), one obtain the existence of a common fixed point of (f, T)

4.2.1 Theorem

Let x_0 be an arbitrary point in X and $T \quad X \to CL(X)$ a generalized multivalued $(f\ L)$- almost F- contraction with respect to a dynamic process $D(f, T, x_0)$ Then $C(f, T) \neq \emptyset$ provide that $f(X)$ is complete and F is continuous or T is a closed multivalued mapping Moreover $F(f\ T)$ $\neq \emptyset$ if one of the following conditions holds

(a) for some $x \in C(f\ T)$ $\quad f$ is T- weakly commuting at x $\quad f^2x = fx$

(b) $f(C(f, T))$ is a singleton subset of $C(f\ T)$

Proof Let x_0 be a given point in X Since T is a generalized multivalued (f, L)- almost

F- contraction with respect to dynamic process $D(f, T, x_0)$ we have

$$\tau(M(x_{n-1}, x_n)) + F(d(fx_n, fx_{n+1})) \leq F(M(x_{n-1}, x_n) + LN(x_{n-1}, x_n))$$

which implies that

$$\begin{aligned}
F(d(fx_n, fx_{n+1})) &\leq F(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\} \\
&\quad + L\min\{d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), d(fx_{n-1}, Tx_n), d(fx_n, Tx_{n-1})\}) \\
&\quad - \tau(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2} \\
&\leq F(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\} \\
&\quad + L\min\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)\}) \\
&\quad - \tau(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2} \\
&\leq F(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\}) \\
&\quad - \tau(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\}) \\
&= F(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}) - \tau(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})
\end{aligned}$$

Then we have a sequence $\{x_n\}$ in X such that $fx_{n+1} \in Tx_n \subseteq T(X)$ and it satisfies

$$F(d(fx_n, fx_{n+1})) \leq F(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}) - \tau(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})$$

for all $n \in \mathbb{N}$. As F is strictly increasing

$$d(fx_n, fx_{n+1}) < \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}$$

If

$$\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\} = d(fx_n, fx_{n+1})$$

for some n, then

$$d(fx_n, fx_{n+1}) < d(fx_n, fx_{n+1})$$

give a contradiction and hence we have

$$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$$

Consequently,

$$\tau(d(fx_{n-1}, fx_n)) + F(d(fx_n, fx_{n+1})) \leq F(d(fx_{n-1}, fx_n))$$

for all $n \in \mathbb{N}$ By the assumption on τ, there exists a $b > 0$ and an $n \in \mathbb{N}$ such that $\tau(d(x_n, x_{n+1})) > b$ for all $n > n_0$ Thus we obtain that

$$\begin{aligned}
F(d(fx_n, fx_{n+1})) &\leq F(d(fx_{n-1}, fx_n)) - \tau(d(fx_{n-1}, fx_n)) \\
&\leq F(d(fx_{n-2}, fx_{n-1})) - \tau(d(fx_{n-2}, fx_{n-1})) - \tau(d(fx_{n-1}, fx_n)) \\
&\quad \vdots \\
&\leq F(d(fx_0, fx_1)) - \tau(d(fx_0, fx_1)) - \cdots - \tau(d(fx_{n-1}, fx_n)) \\
&= F(d(fx_0, fx_1)) - (\tau(d(fx_0, fx_1)) + \cdots + \tau(d(fx_{n_0-1}, fx_{n_0}))) \\
&\quad - (\tau(d(fx_{n_0}, fx_{n_0+1})) + \cdots + \tau(d(fx_{n-1}, fx_n))) \\
&\leq F(d(fx_0, fx_1)) - (n - n_0)b
\end{aligned}$$

On taking the limit as $n \to \infty$, we have $\lim_{n\to\infty} F(d(fx_n, fx_{n+1})) = -\infty$ By (F2) $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$ By (F3) there exists an $r \in (0,1)$ such that

$$\lim_{n\to\infty} \{d(fx_n, fx_{n+1})\}^r F(d(fx_n, fx_{n+1})) = -\infty$$

Hence it follows that

$$\begin{aligned}
&\{d(fx_n, fx_{n+1})\}^r F(d(fx_n, fx_{n+1})) - \{d(fx_n, fx_{n+1})\}^r F(d(fx_0, fx_1)) \\
\leq\ & d(fx_n, fx_{n+1})^r [F(d(fx_0, fx_1) - (n - n_0)b)] - d(x_n, x_{n+1})^r F(d(fx_0, fx_1)) \\
=\ & -(n - n_0)b[d(fx_n, fx_{n+1})]^r \leq 0
\end{aligned}$$

On taking limit as n tends to ∞ we obtain that $\lim_{n\to\infty} n\{d(fx_n, fx_{n+1})\}^r = 0$, that is

$$\lim_{n\to\infty} n^{1/r} d(fx_n, fx_{n+1}) = 0$$

This implies that $\sum_{n=1}^{\infty} d(fx_n, fx_{n+1})$ is convergent and hence that the sequence $\{fx_n\}$ is a Cauchy sequence in $f(X)$ There is a $p \in f(X)$ such that $\lim_{n\to\infty} fx_n = p$ Suppose that there exists a u^* in X such that $fu^* = p$ We claim that $fu^* \in Tu^*$ If not then $d(fu^*, Tu^*) > 0$ as Tu^* is closed Since F is strictly increasing we deduce from Definition 1 4 11 that

$$H(Tx_n, Tu^*) < M(x_n, u^*) + LN(x_n, u^*)$$

for all $n \in \mathbb{N}$ Therefore

$$d(fx_{n+1}, Tu^*) \leq H(Tx_n, Tu^*) < M(x_n, u^*) + LN(x_n, u^*)$$

From condition (F1) we have

$$\tau(M(x_n, u^*)) + F(d(fx_{n+1}, Tu^*)) \leq F(M(x_n, u^*) + LN(x_n, u^*)),$$

for all $n \in \mathbb{N}$ Next suppose that F is continuous Since

$$\lim_{n\to\infty} d(fx_n, Tu^*) = d(fu^*, Tu^*)$$

we deduce that

$$\lim_{n\to\infty} M(x_n, u^*) = d(fu^*, Tu^*)$$

Moreover

$$\lim_{n\to\infty} N(x_n, u^*) = 0$$

so, by the continuity of F,

$$\tau(d(fu^*, Tu^*)) + F(d(fu^*, Tu^*)) \leq F(d(fu^*, Tu^*))$$

which is a contradiction We conclude that $d(fu^*, Tu^*) = 0$, and thus $fu^* \in Tu^*$

Now suppose that (a) hold, that is for $x \in C(f,T)$, f is T−weakly commuting at x We then get $f^2x \in Tfx$ By the given hypothesis, $fx = f^2x$ and hence $fx = f^2x \in Tfx$ Consequently $fx \in F(f,T)$ (b) The conditions $f(C(f,T)) = \{x\}$ (say) and $x \in C(f, T)$ imply that $x = fx \in Tx$ Thus $F(f,T) \neq \emptyset$ ∎

4.2.2 Example

Let $X = [1, \infty)$ be the usual metric space Define $f : X \to X$, $\tau : \mathbb{R}_+ \longrightarrow \mathbb{R}$, $T : X \to CL(X)$ by $fx = x^2$ and $Tx = [x+2, \infty)$ for all $x \in X$ and $\tau(t) = \begin{cases} -\ln t \text{ for } t \in (0, 1) \\ \ln 3 \text{ for } t \in [1, \infty) \end{cases}$ and $F(t) = \ln(t)$ for all $t > 0$ Note that $f(X)$ is complete It is easy to check that for all $x, y \in X$ with $Tx \neq Ty$ (equivalently with $x \neq y$), one has

$$\tau(M(x, y)) + F(H(Tx, Ty)) \leq F(M(x, y) + LN(x, y))$$

Now apply Theorem 4 2 1

4.3 Applications

Applications given in this section have been published in [60]

In this section, we discuss applications of Theorem 4 2 1 We have obtained the existence and uniqueness of a common solution of a system of functional equations in dynamical programing and the existence and uniqueness of common solution of system of integral equations

(1) Application to functional equations in dynamic programming.

Decision space and a state space are two basic components of dynamic programming problems The state space is a set of states including initial states action states and transitional states So a state space is set of parameters representing different states A decision space is the set of possible actions that can be taken to solve the problem These general settings allow us to formulate many problems in mathematical optimization and computer programming In particular the problem of dynamic programming related to multistage process reduces to the problem of solving functional equations

$$p(x) = \sup_{y\in D}\{g(x,y) + G_1(x,y,p(\xi(x,y)))\}, \text{ for } x \in W, \quad (4.1)$$

$$q(x) = \sup_{y\in D}\{g'(x,y) + G_2(x,y,q(\xi(x,y)))\}, \text{ for } x \in W, \quad (4.2)$$

where U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$ and

$$\xi \quad W \times D \longrightarrow W,$$

$$g, g' \quad W \times D \longrightarrow \mathbb{R}$$

$$G_1, G_2 \quad W \times D \times \mathbb{R} \longrightarrow \mathbb{R}$$

For more details on dynamic programming we refer to [35, 36, 37, 38, 105]. Suppose that W and D are the state and decision spaces respectively. We aim to give the existence and uniqueness of common and bounded solutions of functional equations given in (4.1) and (4.2). Let $B(W)$ denote the set of all bounded real valued functions on W. For an arbitrary $h \in B(W)$ define $\|h\| = \sup_{x\in W} |h(x)|$. Then $(B(W), \|\ \|)$ is a Banach space endowed with the metric d defined as

$$d(h,k) = \sup_{x\in W} |hx - kx| \quad (4.3)$$

Suppose that the following conditions hold

$(C1)$ G_1, G_2, g, and g' are bounded

$(C2)$ for $x \in W$, $h \in B(W)$ and $b > 0$, define

$$Kh(x) = \sup_{y\in D}\{g(x,y) + G_1(x,y,h(\xi(x,y)))\}, \quad (4.4)$$

$$Jh(x) = \sup_{y\in D}\{g'(x,y) + G_2(x,y,h(\xi(x,y)))\} \quad (4.5)$$

Moreover, assume that $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ and $L \geq 0$ are such that for every $(x,y) \in W \times D$, $h,k \in B(W)$ and $t \in W$,

$$|G_1(x,y,h(t)) - G_1(x,y,k(t))| \leq e^{-\tau(t)}[M(h(t),k(t)) + LN(h(t),k(t))] \quad (4.6)$$

where

$$\begin{aligned}
M((h(t), k(t))) &= \max\{d(Jh(t), Jk(t)), d(Jk(t), Kk(t)), d(Jh(t), Kh(t)), \\
&\quad \frac{d(Jh(t), Kk(t)) + d(Jk(t), Kh(t))}{2}\}, \\
N((h(t), k(t))) &= \min\{d(h(t), Kh(t)), d(k(t), Kk(t)), d(h(t), Kk(t)), d(k(t), Kh(t))\}
\end{aligned}$$

$(C3)$ for any $h \in B(W)$ there exists a $k \in B(W')$ such that, for $x \in W$

$$Kh(x) = Jk(x)$$

$(C4)$ There exists an $h \in B(W)$ such that

$$Kh(x) = Jh(x) \text{ implies that } JKh(x) = KJh(x)$$

4.3.1 Theorem

Assume that conditions $(C1) - (C4)$ are satisfied. If $J(B(W))$ is a closed convex subspace of $B(W)$, then the functional equations (4.1) and (4.2) have a unique common and bounded solution.

Proof. Note that $(B(W), d)$ is a complete metric space. By $(C1)$, J, K are selfmaps of $B(W')$. The condition $(C3)$ implies that $K(B(W)) \subseteq J(B(W))$. It follows from $(C4)$ that J and K commute at their coincidence points. Let λ be an arbitrary positive number and $h_1, h_2 \in B(W')$. Choose $x \in W$ and $y_1, y_2 \in D$ such that

$$Kh_j < g(x, y_j) + G_1(x, y_j, h_j(x_j)) + \lambda, \tag{4.7}$$

where $x_j = \xi(x, y_j)$, $j = 1, 2$. Further, from (4.4) and (4.5), we have

$$\begin{aligned}
Kh_1 &\geq g(x, y_2) + G_1(x, y_2, h_1(x_2)), & (4.8) \\
Kh_2 &\geq g(x, y_1) + G_1(x, y_1, h_2(x_1)) & (4.9)
\end{aligned}$$

Then (4 7) and (4 9) together with (4 6), imply

$$
\begin{aligned}
Kh_1(x) - Kh_2(x) &< G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2)) + \lambda \\
&\le |G_1(x, y_1, h_1(x_1)) - G_1(x\ y_1\ h_2(x_2))| + \lambda \\
&\le e^{-\tau(t)}(M((h(t), k(t)) + LN(h(t), k(t))) + \lambda \qquad (4\ 10)
\end{aligned}
$$

Then (4 7) and (4 8), together with (1 6), imply

$$
\begin{aligned}
Kh_2(x) - Kh_1(x) &\le G_1(x, y_1, h_2(x_2)) - G_1(x\ y_1\ h_1(x_1)) \\
&\le |G_1(x\ y_1\ h_1(x_1)) - G_1(x\ y_1\ h_2(x_2))| \\
&\le e^{-\tau(t)}(M((h(t), k(t)) + LN(h(t)\ k(t))) \qquad (4\ 11)
\end{aligned}
$$

From (4 10) and (4 11) we have

$$
|Kh_1(x) - Kh_2(x)| \le e^{-\tau(t)}(M((h(t)\ k(t)) + LN(h(t)\ k(t))) \qquad (4\ 12)
$$

Inequality (4 12) implies

$$
d(Kh_1(x) - Kh_2(x)) \le e^{-\tau(t)}[(M((h(t), k(t)) + LN(h(t)\ k(t)))]
$$

$$
\tau(t) + \ln[d(Kh_1(x) - Kh_2(x))] \le \ln[(M((h(t)\ k(t)) + LN(h(t)\ k(t)))]
$$

Therefore, by Theorem 4 2 1, the pair (K, J) has a common fixed point h^* that is $h^*(x)$ is the unique, bounded and common solution of (4 1) and (4 2) ■

(2) Application to systems of integral equations.

We now discuss an application of a fixed point theorem which we have proved in the previous section in solving the system of Volterra type integral equations Such a system is given by the following equations

$$u(t) = \int_0^t K_1(t, s, u(s))ds + g(t) \tag{4.13}$$

$$u'(t) = \int_0^t K_2(t, s, u(s))ds + f(t) \tag{4.14}$$

for $t \in [0, a]$, where $a > 0$ We first find a solution of the systems (4.13) and (4.14) Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$ For $u \in C([0, a], \mathbb{R})$, define supremum norm as $\|u\|_\tau = \sup_{t\in[0,a]} \{u(t)e^{-\tau(t)t}\}$ where $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is taken as a function Let $C([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t\in[0,a]} \| |u(t) - v(t)| e^{-\tau(t)t} \|_\tau \tag{4.15}$$

for all $u, v \in C([0, a], \mathbb{R})$ With these, setting $C([0, a], \mathbb{R}, \| \ \|_\tau)$ becomes a Banach space

Now we prove the following theorem to ensure the existence of a solution of the system of integral equations For more details on such applications we refer the reader to [29, 103]

4.3.2 Theorem

Assume the following conditions are satisfied

(i) $K_1, K_2 : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ and $f, g : [0, a] \to \mathbb{R}$ are continuous

(ii) define

$$Tu(t) = \int_0^t K_1(t, s, u(s))ds + g(t),$$

$$Su(t) = \int_0^t K_2(t, s, u(s))ds + f(t)$$

Suppose there exist $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ and $L \geq 0$ such that

$$|K_1(t, s, u) - K_1(t, s, v)| \leq \tau(t)e^{-\tau(t)t}[M(u, v) + L N(u, v)$$

for all $t, s \in [0, a]$ and $u, v \in C([0, a], \mathbb{R})$, where

$$\begin{aligned} M(u,v) &= \max\left\{|Su(t) - Sv(t)|, |Sv(t) - Tv(t)|, |Su(t) - Tu(t)|, \frac{|Su(t) - Tv(t)| + |Sv(t) - Tu(t)|}{2}\right\} \\ N(u,v) &= \min\{|u(t) - Tu(t)|, |v(t) - Tv(t)|, |u(t) - Tv(t)|, |v(t) - Tu(t)|\} \end{aligned}$$

(iii) there exists $u \in C([0, a], \mathbb{R})$ such that $Tu(t) = Su(t)$ implies $TSu(t) = STu(t)$. Then the system of integral equations given in (4.13) and (4.14) has a solution.

Proof. By assumption (iii)

$$\begin{aligned} |Tu(t) - Tv(t)| &= \int_0^t |K_1(t, s, u(s)) - K_1(t, s, v(s)))| \, ds \\ &\leq \int_0^t \tau(t) e^{-\tau(t)} ([M(u,v) + LN(u,v)] e^{-\tau(t)s}) e^{\tau(t)s} ds \\ &\leq \int_0^t \tau(t) e^{-\tau(t)} \|M(u,v) + LN(u,v)\|_\tau e^{\tau(t)s} ds \\ &\leq \tau(t) e^{-\tau(t)} \|M(u,v) + LN(u,v)\|_\tau \int_0^t e^{\tau(t)s} ds \\ &\leq \tau(t) e^{-\tau(t)} \|M(u,v) + LN(u,v)\|_\tau \frac{1}{\tau(t)} e^{\tau(t)t} \\ &\leq e^{-\tau(t)} \|M(u,v) + LN(u,v)\|_\tau e^{\tau(t)t} \end{aligned}$$

This implies that

$$|Tu(t) - Tv(t)| \, e^{-\tau(t)t} \leq e^{-\tau(t)} \|M(u,v) + LN(u,v)\|_\tau,$$

that is

$$\|Tu(t) - Tv(t)\|_\tau \leq e^{-\tau(t)} \|M(u,v) + LN(u,v)\|_\tau$$

which further implies that

$$\tau(t) + \ln \|Tu(t) - Tv(t)\|_\tau \leq \ln \|M(u,v) + LN(u,v)\|_\tau$$

So all of the conditions of Theorem 4 2 1 are satisfied Hence the system of integral equations given in (4 13) and (4 14) has a unique common solution ■

Conclusion The main aim of our chapter is to present new concepts of generalized dynamic process for generalized (f L)-almost F-contraction, different from the F-contractions given in [67 107 123] The existence of coincidence and common fixed for generalized dynamic process in a complete metric space are established Our results provide extention as well as substantial generalizations and improvements of seiveral well known results in the existing and comparable literature The new concepts lead to further investigations and applications It will be also interesting to apply these concepts to different metric spaces

Chapter 5

Fixed Point Theorems for Local F-Contraction on a Closed Ball in Metric Spaces

5.1 Introduction

This chapter is a continuation of the investigation of F-contractions. We introduce a new approach of Ćirić type F-contractions on a closed ball, and establish fixed point theorems for an F-contraction on a closed ball in the framework of a complete metric space. There are many situations in which such mappings are not contractive on the whole space, but they are contractive on its subsets.

In 1971, Ćirić [47] introduced generalized contractions and proved some fixed point theorems. Since then many generalizations have been given in the literature see [48] and many others. Shoaib et al. [119] presented significant results concerning the existence of fixed points of dominated self-mappings satisfying some contractive conditions on a closed ball in a 0-complete quasi-partial metric space. Other results on a closed ball can be seen in [27, 28, 32, 33].

For $x \in X$ and $\varepsilon > 0$, $\overline{B(x,\varepsilon)} = \{y \in X : d(x,y) \leq \varepsilon\}$ is a closed ball in (X, d). The following result, regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball, is given in [91, Theorem 5.1.1]. The result is very useful in the sense that it requires the contraction of the mapping only on a closed ball instead of on the whole space

5.1.1 Theorem [91]

Let (X, d) be a complete metric space, $T : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose that there exists a $k \in [0, 1)$ with

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, Tx_0) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Tx^*$

5.2 Fixed point theorems for Ćirić type F-contraction on a closed ball

The result given in this section has been published in [64]

In this section we introduce a fixed point theorem for a modified F-contraction on a closed ball in a complete metric spaces

Now we state our Theorem

5.2.1 Theorem

Let T be a continuous self-map in a complete metric space (X, d) and x_0 an arbitrary point in X. Assume that $\tau > 0$ and $F \in \mathcal{F}$ for all $x, y \in \overline{B(x_0, r)} \subseteq X$ with $d(Tx, Ty) > 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y)) \tag{5.1}$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}$$

Moreover

$$\sum_{j=0}^{N} d(x_0, Tx_0) \leq r, \text{ for all } j \in N \text{ and } r > 0 \tag{5.2}$$

Then there exists a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$

Proof. Choose a point x_1 in X such that $x_1 = Tx_0$. Continuing in this way we have $x_{n+1} = Tx_n$, for all $n \geq 0$ and this implies that (x_n) is a nonincreasing sequence. First we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$ by using mathematical induction. From (5.2) we have

$$d(x_0, x_1) = d(x_0, Tx_0) \leq r \tag{5.3}$$

Thus, $x_1 \in \overline{B(x_0, r)}$. Suppose that $x_2, \ldots, x_j \in \overline{B(x_0, r)}$ for some $j \in N$. From (5.1) we obtain

$$\begin{aligned}
F(d(x_j, x_{j+1})) &= F(d(Tx_{j-1}, Tx_j)) \leq F(M(x_{j-1}, x_j)) - \tau \\
&= F\left(\max\left\{d(x_{j-1}, x_j), d(x_{j-1}, x_j), d(x_j, x_{j+1}), \frac{d(x_{j-1}, x_{j-1}) + d(x_j, x_j)}{2}\right\}\right) - \tau \\
&= F\left(\max\left\{d(x_{j-1}, x_j), d(x_{j-1}, x_j), d(x_j, x_{j+1}), \frac{d(x_{j-1}, x_{j+1})}{2}\right\}\right) - \tau \\
&\leq F\left(\max\left\{d(x_{j-1}, x_j), d(x_j, x_{j+1}), \frac{d(x_{j-1}, x_j) + d(x_j, x_{j+1})}{2}\right\}\right) - \tau \\
&= F\left(\max\left\{d(x_{j-1}, x_j), d(x_j, x_{j+1})\right\}\right) - \tau
\end{aligned}$$

So we have

$$F(d(x_j, x_{j+1})) = F\left(d(Tx_{j-1}, Tx_j)\right) \leq F\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\right) -$$

In this case, $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ is impossible, because

$$F(d(x_j, x_{j+1})) \leq F(d(x_j, x_{j+1})) - \tau,$$

which implies that $\tau \leq 0$, a contradiction. So

$$\max\{d(x_{j-1}, x_j), d(x_j, x_{j+1})\} = d(x_{j-1}, x_j)$$

As F is strictly increasing, we have

$$d(x_j, x_{j+1}) < d(x_{j-1}, x_j) \tag{5.4}$$

Now

$$\begin{aligned} d(x_0, x_{j+1}) &\leq d(x_0, x_1) + \cdots + d(x_j, x_{j+1}) \\ &\leq \sum_{j=0}^{\mathbf{N}} d(x_0, x_1) \leq r \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Continuing this process, we get

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\ &\leq F(d(Tx_{n-2}, Tx_{n-1})) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\leq F(d(Tx_{n-3}, Tx_{n-2})) - 2\tau \\ &\leq F(d(x_{n-3}, x_{n-2})) - 3\tau \\ &\;\;\vdots \\ &\leq F(d(x_0, x_1)) - n\tau \end{aligned}$$

This implies that

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \tag{5.5}$$

From (5.5), we obtain $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Since $F \in \mathcal{F}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{5.6}$$

From $(F3)$, there exists a $\kappa \in (0, 1)$ such that

$$\lim_{n\to\infty} \left((d(x_n, x_{n+1}))^{\kappa} F(d(x_n, x_{n+1}))\right) = 0 \tag{5.7}$$

From (5.5), for all $n \in \mathbf{N}$, we obtain

$$(d(x_n, x_{n+1}))^{\kappa} \left(F(d(x_n, x_{n+1})) - F(d(x_0, x_1))\right) \leq -(d(x_n, x_{n+1}))^{\kappa} n\tau \leq 0 \tag{5.8}$$

By using (5.6), (5.7) and letting $n \to \infty$ in (5.8), we have

$$\lim_{n\to\infty} \left(n \left(d(x_n, x_{n+1})\right)^{\kappa}\right) = 0 \tag{5.9}$$

We observe that, from (5.9), there exists an $n_1 \in \mathbb{N}$ such that $n\left(d(x_n, x_{n+1})\right)^{\kappa} \leq 1$ for all $n \geq n_1$, so that we get

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1 \tag{5.10}$$

Now, let $m, n \in \mathbb{N}$ be such that $m > n \geq n_1$. Then by the triangle inequality and from (5.10), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}} \end{aligned} \tag{5.11}$$

The series $\sum_{i=n}^{\infty} i^{-1/k}$ is convergent. By taking the limit as $n \to \infty$, in (5.11), we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space there exists an $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$. Since T is a continuous, $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$, that is, $x^* = Tx^*$. Hence x^* is a fixed point of T. To prove uniqueness, let $x, y \in \overline{B_p(x_0, r)}$ and $x \neq y$ be any two fixed point of T, then from (5.1), we have

$$\tau + F\left(d(Tx, Ty)\right) \leq F\left(M(x, y)\right),$$

from which we obtain

$$\tau + F\left(d(x, y)\right) \leq F\left(d(x, y)\right)$$

which is a contradiction. Hence $x = y$. Therefore T has a unique fixed point in $\overline{B(x_0, r)}$. ■

5.2.2 Example

Let $X = \mathbb{R}^+$ and $d(x, y) = |x - y|$ Then (X, d) is a complete metric space Define the mapping $T : X \to X$ by,

$$T(x) = \begin{cases} \frac{x}{4} \text{ if } x \in [0, 1] \\ x - \frac{1}{2} \text{ if } x \in (1, \infty) \end{cases}$$

$x_0 = 1, r = 2, \overline{B(x_0, r)} = [0, 1]$ If $F(\alpha) = \ln \alpha$ $\alpha > 0$ and $\tau > 0$, then

$$d(1, T1) = \left|1 - \frac{1}{4}\right| = \frac{3}{4} < r$$

If $x, y \in \overline{B(x_0, r)}$, then

$$\begin{aligned} \frac{1}{4}|x - y| &< |x - y| \\ \frac{x}{4} - \frac{y}{4} &< |x - y| \\ d(Tx, Ty) &< d(x, y) \leq M(x, y) \end{aligned}$$

This implies that

$$\tau + F(d(Tx, Ty)) = \tau + \ln d(Tx, Ty) \leq \ln M(x, y) = F(M(x, y))$$

If $x, y \in (1, \infty)$, then

$$\begin{aligned} \left|x - \frac{1}{2} - y + \frac{1}{2}\right| &= |x - y| \\ \tau + |Tx - Ty| &> |x - y| \\ \tau + F(d(Tx, Ty)) &> F(d(x, y)) \end{aligned}$$

then the contractive condition does not hold on X

5.3 Fixed point theorems for Ćirić type GF-contraction on a closed ball

The result given in this section has been published in [64]

In this section we define a new contraction called an α-η-GF-contraction, on a closed ball and obtain a fixed point theorem for such a contraction, in the setting of complete metric spaces. We define a Ćirić type α-η-GF-contraction on a closed ball as follows.

5.3.1 Definition

Let T be a self mapping in a metric space (X, d) and let x_0 be an arbitrary point in X. Also suppose that $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty), \eta : X \times X \to \mathbb{R}^+$ are two functions. We say that T is called a Ćirić type α-η-GF-contraction on a closed ball if for all $x, y \in \overline{B(x_0, r)} \subseteq X$, with $\eta(x, Tx) \le \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have

$$G(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) + F(d(Tx,Ty)) \le F(M(x,y)) \tag{5.12}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\},$$

and

$$\sum_{j=0}^{N} d(x_0, Tx_0) \le r, \text{ for all } j \in N \text{ and } r > 0 \tag{5.13}$$

where $G \in \Delta_G$ and $F \in \mathcal{F}$.

5.3.2 Theorem

Let (X, d) be a complete metric space. Let $T : X \to X$ be a Ćirić type α-η-GF-contraction mapping on a closed ball satisfying the following assertions

(i) T is an α-admissible mapping with respect to η,

(ii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge \eta(x_0, Tx_0)$,

(iii) T is $\alpha - \eta$-continuous.

Then there exists a point x^* in $\overline{B(x_0, r)}$ such that $Tx^* = x^*$.

Proof. Let x_0 in X be such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$ we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$. Continuing this way, $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now, since T is an α-admissible mapping with respect to η, then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this this manner we have

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N} \tag{5.14}$$

If there exists an $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, there is nothing to prove. So we assume that $x_n \neq x_{n+1}$ with

$$d(Tx_{n-1}, Tx_n) = d(x_n, Tx_n) > 0, \text{ for all } n \in \mathbb{N}$$

First we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$. Since T is a Ćirić type α-η-GF-contraction on a closed ball, we have

$$d(x_0, x_1) = d(x_0, Tx_0) \leq r \tag{5.15}$$

Thus, $x_1 \in \overline{B(x_0, r)}$. Suppose that $x_2, \dots, x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$, such that

$$\begin{aligned} & G(d(x_{j-1}, Tx_{j-1}), d(x_j, Tx_j), d(x_{j-1}, Tx_j), d(x_j, Tx_{j-1})) \\ + F(d(Tx_{j-1}, Tx_j)) \leq \; & F(M(x_{j-1}, x_j)) \end{aligned}$$

Then

$$\begin{aligned} & G(d(x_{j-1}, x_j), d(x_j, x_{j+1}), d(x_{j-1}, x_{j+1}), 0) \\ + F(d(Tx_{j-1}, Tx_j)) \leq \; & F(M(x_{j-1}, x_j)) \end{aligned} \tag{5.16}$$

Using the definition of G, $d(x_{j-1}, x_j)\, d(x_j, x_{j+1})\, d(x_{j-1}, x_{j+1})\, 0 = 0$, there exists a $\tau > 0$ such that

$$G(d(x_{j-1}, x_j), d(x_j, x_{j+1}), d(x_{j-1}, x_{j+1}), 0) = \tau$$

Therefore

$$F(d(x_j, x_{j+1})) = F(d(Tx_{j-1}, Tx_j)) \leq F(M(x_{j-1}, x_j)) - \tau \tag{5.17}$$

The rest of the proof is similar to that of Theorem 5.2.1. Since X is a complete metric space there exists an $x^* \in \overline{B(x_0, r)}$ such that $x_n \to x^*$ as $n \to \infty$. Since T is an α-η-continuous and $\eta(x_{n-1}, x_n) \le \alpha(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$, $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$, that is $x^* = Tx^*$. Hence x^* is a fixed point of T. ■

5.3.3 Example

Let $X = \mathbb{R}^+$ and d be the usual metric on X. Define $T : X \to X$, $\alpha : X \times X \to \{-\infty\} \cup (0, +\infty)$, $\eta : X \times X \to \mathbb{R}^+$, $G : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$Tx = \begin{cases} \sqrt{x} & \text{if } x \in [0, 1], \\ 2x & \text{if } x \in (1, \infty), \end{cases} \quad \alpha(x, y) = \begin{cases} e^{x+y} & \text{if } x \in [0, 1] \\ \frac{1}{3} & \text{otherwise} \end{cases}$$

$$\eta(x, y) = 1/2 \text{ for all } x, y \in X,\ G(t_1, t_2, t_3, t_4) = \tau > 0 \text{ and } F(t) = \ln t \text{ with } t > 0$$

$x_0 = \frac{1}{2}, r = 1, \overline{B(x_0, r)} = [0, 1]$, then

$$d(\frac{1}{2}, T\frac{1}{2}) = \left|\frac{1}{2} - \frac{1}{\sqrt{2}}\right| = 0.20710 < r$$

If $x, y \in \overline{B(x_0, r)}$, then $\alpha(x, y) = e^{x+y} \ge \frac{1}{2} = \eta(x, y)$. On the other hand $Tx \in [0, 1]$ for all $x \in [0, 1]$. Thus $\alpha(Tx, Ty) \ge \eta(x, Tx)$ with $d(Tx, Ty) = |\sqrt{x} - \sqrt{y}| > 0$ and, clearly $\alpha(0, T0) \ge \eta(0, T0)$. Hence we have

$$d(Tx, Ty) = \left|\frac{\sqrt{x} - \sqrt{y} \times \sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right| = \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| < |x - y| \le M(x, y)$$

Consequently,

$$\tau + F(d(Tx, Ty)) = \tau + \ln d(Tx, Ty) \le \ln M(x, y) = F(M(x, y))$$

If $x \notin \overline{B(x_0, r)}$ or $y \notin \overline{B(x_0, r)}$ then $\alpha(x, y) = 1/3 \ngeq 1/2 = \eta(x, y)$, either

$$\begin{aligned} 2|x - y| &> |x - y| \\ |2x - 2y| &> |x - y| \\ |Tx - Ty| &> |x - y| \end{aligned}$$

or

$$\tau + F\left(d(Tx, Ty)\right) \geq F\left(d(x, y)\right)$$

Then the contractive condition does not hold on X

Conclusion: The main aim of our chapter is to present new concepts of a Ciric type F-contraction on a closed ball, different from the F-contractions given in [67 107 123] The existence of fixed point results for such a type of F-contraction on a closed ball in a complete metric space are established The results of such a study are very useful, in the sense that they require the F-contraction mapping is defined only on a closed ball instead of on the whole space The new concepts lead to further investigations and applications It will be also interesting to apply these concepts to different metric spaces

Bibliography

[1] M Abbas, Coincidence points of multivalued f−almost nonexpansive mappings Fixed Point Theory, 13 (1) (2012), 3-10

[2] M Abbas, B Ali and S Romaguera, Coincidence points of generalized multivalued (f, L)−almost F−contraction with applications, J Nonlinear Sci Appl 8 (2015), 919–934

[3] M Abbas B Ali and S Romaguera, Generalized contraction and invariant approximation results on nonconvex subsets of normed spaces, Abstr Appl Anal (2014) Article 391952, 5 pages

[4] M Abbas, B Ali and S Romaguera Fixed and periodic points of generalized contractions in metric spaces, Fixed Point Theory Appl (2013) 2013 243

[5] M Abbas and G Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, J Math Anal Appl 341(2008), 416-420

[6] M Abbas and A R Khan, Common fixed points of generalized contractive hybrid pairs in symmetric spaces, Fixed Point Theory Appl 2009 (2009), 11 pages Article ID 869407

[7] M Abbas and B E Rhoades, Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type Fixed Point Theory Appl (2007), 9 pages, Article ID 54101

[8] M Abbas and B E Rhoades, Common fixed points theorems for occasionally weakly compatible mappings satisfying a generalized contractive condition, Math Communications, 13 (2008), 295-301

[9] T. Abdeljawad, Meir-Keeler α-contractive fixed and common fixed point theorems, Fixed Point Theory Appl. (2013), doi 10.1186/1687-1812-2013-19

[10] Ö. Acar and I. Altun, A fixed point theorem for multivalued mappings with δ-distance. Abstr. Appl. Anal. (2014), Article ID 497092, 5 pages

[11] O. Acar, G. Durmaz and G. Minak, Generalized multivalued F-contractions on complete metric spaces, Bull. Iran. Math. Soc., 40 (2014), 1469-1478

[12] J. Ahmad, A. Al-Rawashdeh and A. Azam, Some new fixed point theorems for generalized contractions in complete metric spaces. Fixed Point Theory Appl. (2015), 2015:80

[13] J. Ahmad, N. Hussain, A. R. Khan and A. Azam, Fixed point results for generalized multi-valued contractions, J. Nonlinear Sci. Appl. 8 (6) (2015), 909-918

[14] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, J. Math. Anal. Appl. 322 (2006) 796-802

[15] M. U. Ali, T. Kamran and Q. Kiran, Fixed point theorem for (α, ψ, ϕ)-contractive mappings on spaces with two metrics, J. Adv. Math. Stud., 7 (2) (2014), 8-11

[16] M. U. Ali, T. Kamran and E. Karapinar, Fixed point of α-ψ-contractive type mappings in uniform spaces, Fixed Point Theory Appl. (2014), 2014:150

[17] M. A. Al-Thagafi and N. Shahzad, Coincidence points, generalized I- nonexpansive multimaps and applications, Nonlinear Anal. 67 (2007), 2180-2188

[18] I. Altun, G. Minak and H. Dag, Multivalued F-contraction on complete metric spaces, J. Nonlinear Convex Anal. accepted

[19] I. Altun, G. Minak and M. Olgun, Fixed points of multivalued nonlinear F-contractions on complete metric spaces, Nonlinear Analysis: Modelling and Control, 21 (2) (2016), 201-210

[20] M. Arshad, E. Ameer and A. Hussain, Hardy-Rogers-type fixed point theorems for α-GF-contractions, Archivum Mathematicum, 51 (2015), 129–141

[21] M Arshad and A Hussain, Fixed Point results for generalized rational α-Geraghty contraction Miskolc Mathematical Notes, accepted

[22] M Arshad A Hussain and A Azam, Fixed Point of α-Geraghty Contraction with Applications U P B Sci Bull , Series A, 78 (2) (2016), 67-78

[23] M Arshad A Hussain and M A Kutbi, , Multivalued Ćirić type α-η-GF-Contractions J Comp Anal Appl , accepted

[24] M Arshad , Fahimuddin A Shoaib and A Hussain, Fixed point results for α-ψ-locally graphic contraction in dislocated quasi metric spaces, Math Sci (2014) 1-7

[25] M Arshad Some fixed point results for α^*-ψ-contractive multi-valued mapping in partial metric spaces, J Adv Res Pure Math , 6 (4) (2014) 47-57

[26] M Arshad, A Azam and P Vetro, Some common fixed point results in cone metric spaces, Fixed Point Theory Appl (2009) Article ID 493965, 11 pages

[27] M Arshad , A Shoaib and I Beg, Fixed point of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space, Fixed Point Theory Appl (2013), 2013 115, 15 pp

[28] M Arshad, A Shoaib, and P Vetro, Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered dislocated metric spaces, Journal of Functions Spaces and Appl 2013 (2013), article ID 638181, 9 pages

[29] A Augustynowicz, Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating argument of the derivative of unknown function Serdica Mathematical Journal, 23 (1997) 203-210

[30] H Aydi E Karapinar and B Samet Fixed points for generalized (α-ψ) contractions on generalized metric spaces J Inequal Appl 2014 (2014) 16 pages

[31] H Aydi, M Jellali, E Karapinar Common fixed points for generalized α-implicit contractions in partial metric spaces, Consequences and Application, RACSAM 2014 (2014) 1-18

[32] A. Azam, S. Hussain and M. Arshad, Common fixed points of Chatterjea type fuzzy mappings on closed balls, Neural Computing & Applications (2012) 21 (Suppl 1) S313-S317

[33] A. Azam, M. Waseem and M. Rashid, Fixed point theorems for fuzzy contractive mappings in quasi-pseudo-metric spaces, Fixed Point Theory Appl. (2013) 2013:27

[34] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux equations intégrales, Fund. Math. 3 (1922), 133-181

[35] R. Baskaran, P.V. Subrahmanyam, A note on the solution of a class of functional equations, Appl. Anal. 22 (3-4), 235-241

[36] R. Bellman, Methods of Nonlinear Analysis, Vol. II, Mathematics in Science and Engineering, vol. 61, Academic Press, New York (1973)

[37] R. Bellman and E.S. Lee, Functional equations in dynamic programming, Aequ. Math. 17 (1978), 1-18

[38] P.C. Bhakta and S. Mitra, Some existence theorems for functional equations arising in dynamic programming, J. Math. Anal. Appl. 98 (1984), 348-362

[39] V. Berinde, Approximating fixed points of weak contraction using the Picard iteration, Nonlinear Anal. Forum., 9 (2004), 43-53

[40] M. Berinde and V. Berinde, On general class of multivalued weakly Picard mappings, J. Math. Anal. Appl., 326 (2007), 772-782

[41] V. Berinde and M. Păcurar, Fixed points and continuity of almost contractions, Fixed Point Theory, 9 (1) (2008), 23-34

[42] V. Berinde, Some remarks on a fixed point theorem for Ciric-type almost contractions, Carpathian J. Math. 25 (2) (2009), 157-162

[43] V. Berinde and M. Păcurar, A note on the paper " Remarks on fixed point theorems of Berinde", Nonlinear Anal. Forum. 14 (2009), 119-124

[44] A Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int J Math Math Sci , 29 (9) (2002), 531-536

[45] L B Budhia, P Kumam, J Martínez-Moreno and D Gopal Extensions of almost-F and F-Suzuki contractions with graph and some applications to fractional calculus Fixed Point Theory Appl (2016), 2016 2

[46] S Cho, J Bae and E Karapinar Fixed point theorems for α−Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl (2013) 2013 329

[47] L B Ciríc, Generalized contractions and fixed point theorems Publ Inst Math 12 (26) (1971), 19–26

[48] L B Ćirić, A generalization of Banach's contraction principle Proc Am Math Soc 45 (1974), 267-273

[49] M Cosentino and P Vetro, Fixed point results for F-contractive mappings of Hardy-Rogers-type, Filomat 28 (4) (2014), 715-722

[50] M Edelstein, On fixed and periodic points under contractive mappings J Lond Math Soc , 37 (1962), 74-79

[51] B Fisher, Set-valued mappings on metric spaces, Fundamenta Mathematicae 112 (2) (1981), 141–145

[52] U C Gairola and A S Rawat, A fixed point theorem for integral type inequality, Int J Math Anal , 2 (15) (2008), 709-712

[53] M Geraghty, On contractive mappings Proc Amer Math Soc , 40 (1973) 604-608

[54] F Gu and H Ye, Common fixed point theorems of altman integral type mappings in G-metric spaces, Abstr Appl Anal , (2012) 13 pages Article ID 630457

[55] V Gupta and N Mani, A common fixed point theorem for two weakly compatible mappings satisfying a new contractive condition of integral type, Mathematical Theory and Modeling, 1 (1) (2011), 1-6

[56] R H Haghi Sh Rezapour and N shahzad Some fixed point generalizations are not real generalizations Nonlinear Anal , 74 (2011), 1799-1803

[57] J Hasanzade Asl, S Rezapour and N Shahzad, On fixed points of α-ψ contractive multifunctions Fixed Point Theory Appl , 2012 Article ID 212 (2012)

[58] N Hussain J Ahmad and A Azam, On Suzuki-Wardowski type fixed point theorems, J Nonlinear Sci Appl , 8 (2015), 1095–1111

[59] N Hussain J Ahmad and A Azam Generalized fixed point theorems for multi-valued α-ψ-contractive mappings, J Inequal Appl (2014), 2014 31

[60] N Hussain, M Arshad, M Abbas and A Hussain Generalized dynamic process for generalized (f, L)-almost F-contraction with Applications, J Nonlinear Sci Appl 9 (4) (2016) 1702-1715

[61] N Hussain, J Ahmad, L Ciric and A Azam, Coincidence point theorems for generalized contractions with application to integral equations Fixed Point Theory Appl (2015) 2015 78

[62] N Hussain, M Arshad, A Shoaib and Fahimuddin, Common fixed point results for $\alpha - \psi$-contractions on a metric space endowed with graph, J Inequal Appl (2014) 2014 136

[63] A Hussain, M Arshad and S U Khan, τ-Generalization of fixed point results for F-contractions, Bangmod Int J Math & Comp Sci , 1 (1) (2015), 136-146

[64] A Hussain M Arshad and M Nazam Connection of Ciric type F-contraction involving fixed point on closed ball, Gazi Uni J Sciences, in press

[65] N Hussain E Karapinar P Salimi and F Akbar α-Admissible mappings and related fixed point theorems, J Inequal Appl 114 (2013), 1-11

[66] N Hussain and M Abbas, Common fixed point results for two new classes of hybrid pairs in symmetric spaces, Appl Math Comp , 218 (2011), 542-547

[67] N Hussain and P Salimi suzuki-wardowski type fixed point theorems for α-GF-contractions, Taiwanese J Math , 20 (20) (2014) 1-20

[68] N Hussain, P Salimi and A Latif Fixed point results for single and set-valued $\alpha-\eta-\psi$-contractive mappings, Fixed Point Theory Appl (2013) 2013 212

[69] N Hussain and Y J Cho, Weak contractions, common fixed points and invariant approximations J Inequal Appl 2009 (2009), 10 pages

[70] N Hussain E Karapinar P Salimi and P Vetro, Fixed point results for G^m-Meir-Keeler contractive and G-(α,ψ)-Meir-Keeler contractive mappings Fixed Point Theory Appl (2013), 2013 34

[71] N Hussain, S Al-Mezel and P Salimi, Fixed points for α-ψ-graphic contractions with application to integral equations Abstr Appl Anal (2013) article 575869

[72] N Hussain and G Jungck, Common fixed point and invariant approximation results for noncommuting generalized (f,g)-nonexpansive maps J Math Anal Appl 321 (2006) 851-861

[73] M Jleli, E Karapinar and B Samet Fixed point results for α-ψ_λ-contractions on gauge spaces and applications Abstr Appl Anal, (2013), article ID 730825

[74] M Jleli, H K Nashine, B Samet and C Vetro On multivalued weakly Picard operators in partial Hausdorff metric spaces, Fixed Point Theory Appl (2015), 2015 52

[75] G Jungck, Commuting mappings and fixed points Amer Math Monthly 83 (1976) 261-263

[76] G Jungck, Compatible mappings and common fixed points, Int J Math Math Sci 9 (1986) 771-779

[77] G Jungck, Compatible mappings and common fixed points (2), Int J Math Math Sci 9 (1986) 285-288

[78] G Jungck, Common fixed points for commuting and compatible maps on compacta Proc Amer Math Soc 103 (1988) 977-983

[79] G Jungck and B E Rhoades Fixed points for set valued functions without continuity Indian J Pure Appl Math, 29 (3) (1998) 227-238

[80] Z. Kadelburg, P. Kumam, S. Radenović and W. Sintunavarat. Common coupled fixed point theorems for Geraghty-type contraction mappings using monotone property. Fixed Point Theory Appl. (2015), 2015:27.

[81] T. Kamran, Multivalued f−weakly Picard mappings, Nonlinear Anal. 67 (2007), 2289-2296.

[82] R. Kannan, Some results on fixed points. Bull. Calcutta Math. Soc. 60 (1968), 71-76.

[83] E. Karapınar, A Discussion on "alpha-psi-Geraghty contraction type mappings. Filomat 28 (4) (2014), 761-766.

[84] E. Karapınar, alpha-psi-Geraghty contraction type mappings and some related fixed point results, Filomat. 28 (1) (2014), 37-48.

[85] E. Karapınar, H.H. Alsulami and M. Noorwali. Some extensions for Geragthy type contractive mappings, J. Inequal. Appl. (2015), 2015:303.

[86] E. Karapınar, M.A. Kutbi, H. Piri and D. O'Regan. Fixed points of conditionally F-contractions in complete metric-like spaces. Fixed Point Theory Appl. (2015), 2015:126.

[87] E. Karapınar, P. Kumam and P. Salimi, On $\alpha-\psi$-Meir-Keeler contractive mappings, Fixed Point Theory Appl. (2013), article ID 94 (2013).

[88] E. Karapınar and B. Samet, Generalized $(\alpha-\psi)$ contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. (2012), article id 793486.

[89] J.L. Kelley, General Topology and D. Van Nostrand Co. Ins. Priceton New Jersey, 1959.

[90] D. Klim and D. Wardowski, Fixed points of dynamic processes of set-valued F-contractions and application to functional equations. Fixed Point Theory Appl. (2015), 2015:22.

[91] E. Kryeyszig, Introductory Functional Analysis with Applications. John Wiley & Sons New York. (Wiley Classics Library Edition) 1989.

[92] M.A. Kutbi, M. Arshad and A. Hussain. On modified α-η-contractive mappings. Abstr. Appl. Anal. (2014), 7 pages.

[93] M A Kutbi, M Arshad and A Hussain Fixed point results for Ćirić type α-η-GF-contractions, J Comput Anal Appl 21 (1) (2016) 466-481

[94] M A Kutbi and W Sintunavarat On new fixed point results for (α, ψ, ξ)-contractive multi-valued mappings on α-complete metric spaces their consequences Fixed Point Theory Appl (2015), 2015 2

[95] Y Li, F Gu Common fixed point theorem of Altman integral type mappings, J Nonlinear Sci Appl, 2 (4) (2009) 214–218

[96] M A Miandaragh, M Postolache and Sh Rezapour Some approximate fixed point results for generalized (α, ψ)-contractive mappings U P B Sci Bull, Ser A 75 (2) (2013), 3-10

[97] G Minak A Halvaci and I Altun, Ćirić type generalized F-contractions on complete metric spaces and fixed point results, Filomat, 28 (6) (2014) 1143-1151

[98] B Mohammadi and Sh Rezapour, On modified α-ϕ-contractions J Adv Math Stud 6 (2) (2013), 162-166

[99] B Mohammadi Sh Rezapour and N Shahzad Some results on fixed points of α-ψ-Ćirić generalized multifunctions, Fixed Point Theory Appl (2013) article ID 24 10 pp 2013

[100] S Moradi and M Omid, A fixed point theorem for integral type inequality depending on another function, Research Journal of Applied Sciences Engineering and Technology, 2 (3) (2010), 239-2442

[101] S B Nadler Jr, Multi-valued contraction mappings Pacific J Math 30 (1969) 475-488

[102] D B Ojha, M K Mishra and U Katoch, A common fixed point theorem satisfying integral type for occasionally weakly compatible maps R J Appl Sci Eng Tech 2(3) (2010), 239-244

[103] D ÓRegan and A Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, J Math Anal Appl 341 (2008) 1241 1252

[104] H K Pathak, R Tiwari and M S Khan, A common fixed point theorem satisfying integral type implicit relations, Appl Math E-notes, 7 (2007), 222-228

[105] H K Pathak, Y J Cho, S M Kang and B S Lee, Fixed point theorems for compatible mappings of type P and applications to dynamic programming Matematiche 50 (1995), 15-33

[106] SB Nadler, Multivalued contraction mappings Pacific J Math, 30 (1969) 475-488

[107] H Piri and P Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl (2014), 2014 210

[108] N Redjel, A Dehicia, E Karapinar and I M Erhan, Fixed point theorems for (α, ψ)-Meir-Keeler-Khan mappings J Nonlinear Sci Appl, 8 (2015) 955-964

[109] Sh Rezapour and M E Samei, Some fixed point results for α-ψ-contractive type mappings on intuitionistic fuzzy metric spaces J Adv Math Stud, 7 (1) (2014), 176-181

[110] B E Rhoades, A comparison of various definitions of contractive mappings Trans Amer Math Soc 226 (1977) 257-290

[111] B E Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type Int J Math Math Sci, 63 (2003), 4007-4013

[112] V L Rosa and P Vetro Common fixed points for α-ψ-ϕ-contractions in generalized metric spaces, Nonlinear Anal Model Control 19 (1) (2014) 43–51

[113] P Salimi A Latif and N Hussain Modified $\alpha-\psi$-contractive mappings with applications Fixed Point Theory Appl (2013) 2013 151

[114] B Samet C Vetro and P Vetro Fixed point theorems for $\alpha-\psi$-contractive type mappings Nonlinear Anal 75 (2012) 2154–2165

[115] NA Secelean, Iterated function systems consisting of F-contractions, Fixed Point Theory Appl (2013), article ID 277 (2013)

[116] S Sessa, On a weak commutativity condition of mappings in fixed points considerations Publ Inst Math 32 (1982) 149-153

[117] M Sgroi and C Vetro Multi-valued F-contractions and the solution of certain functional and integral equations, Filomat 27 (7) (2013) 1259 1268

[118] N Shahzad and E Karapinar, A Francisco and R L de-Hierro On some fixed point theorems under (α,ψ,φ)-contractivity conditions in metric spaces endowed with transitive binary relations, Fixed Point Theory Appl (2015), 2015 121

[119] A Shoaib, M Arshad and J Ahmad Fixed point results of locally cotractive mappings in ordered quasi-partial metric spaces The Scientific World Journal 2013 (2013) article ID 194897, 8 pages

[120] P K Shrivastava, N P S Bawa, and S K Nigam, Fixed point theorems for hybrid contractions, Varahmihir J Mathematical Sciences, 2 (2) (2002) 275-281

[121] T Sistani and M Kazemipour, Fixed point theorems for α-ψ-contractions on metric spaces with a graph J Adv Math Stud 7 (1) (2014) 65-79

[122] P Vijayaraju, B E Rhoades, and R Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Int J Math Math Sci 15 (2005), 2359-2364

[123] D Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl (2012), article ID 94

[124] D Wardowski and N V Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstratio Mathematics (1) (2014), 146-155

[125] X Zhang, Common fixed point theorems for some new generalized contractive type mappings, J Math Anal Appl, 333 (2007), 780-786

[93] M A Kutbi, M Arshad and A Hussain, Fixed point results for Ćirić type α-η-GF-contractions, J Comput Anal Appl, 21 (1) (2016) 466-481

[94] M A Kutbi and W Sintunavarat On new fixed point results for (α, ψ, ξ)-contractive multi-valued mappings on α-complete metric spaces their consequences Fixed Point Theory Appl (2015) 2015 2

[95] Y Li, F Gu, Common fixed point theorem of Altman integral type mappings, J Nonlinear Sci Appl, 2 (4) (2009), 214–218

[96] M A Miandaragh M Postolache and Sh Rezapour, Some approximate fixed point results for generalized (α, ψ)-contractive mappings U P B Sci Bull Ser A 75 (2) (2013) 3-10

[97] G Minak, A Halvacı and I Altun, Ćirić type generalized F-contractions on complete metric spaces and fixed point results Filomat, 28 (6) (2014) 1143-1151

[98] B Mohammadi and Sh Rezapour, On modified α-φ-contractions J Adv Math Stud 6 (2) (2013), 162-166

[99] B Mohammadi Sh Rezapour and N Shahzad Some results on fixed points of α-ψ-Ćirić generalized multifunctions Fixed Point Theory Appl (2013) article ID 24 10 pp 2013

[100] S Moradi and M Omid, A fixed point theorem for integral type inequality depending on another function, Research Journal of Applied Sciences Engineering and Technology 2 (3) (2010), 239-2442

[101] S B Nadler Jr, Multi-valued contraction mappings Pacific J Math 30 (1969) 475-488

[102] D B Ojha, M K Mishra and U Katoch, A common fixed point theorem satisfying integral type for occasionally weakly compatible maps, R J Appl Sci Eng Tech 2(3) (2010), 239-244

[103] D O'Regan and A Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, J Math Anal Appl 341 (2008) 1241–1252

[104] H K Pathak R Tiwari and M S Khan A common fixed point theorem satisfying integral type implicit relations, Appl Math E-notes, 7 (2007), 222-228

[105] H K Pathak Y J Cho, S M Kang and B S Lee, Fixed point theorems for compatible mappings of type P and applications to dynamic programming Matematiche 50 (1995) 15-33

[106] SB Nadler Multivalued contraction mappings Pacific J Math 30 (1969) 475-488

[107] H Piri and P Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl (2014), 2014 210

[108] N Redjel, A Dehicia, E Karapinar and I M Erhan Fixed point theorems for (α, ψ)-Meir-Keeler-Khan mappings, J Nonlinear Sci Appl, 8 (2015) 955–964

[109] Sh Rezapour and M E Samei, Some fixed point results for α-ψ-contractive type mappings on intuitionistic fuzzy metric spaces J Adv Math Stud 7 (1) (2014) 176-181

[110] B E Rhoades, A comparison of various definitions of contractive mappings Trans Amer Math Soc 226 (1977) 257-290

[111] B E Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type Int J Math Math Sci, 63 (2003), 4007-4013

[112] V L Rosa and P Vetro Common fixed points for α-ψ-ϕ-contractions in generalized metric spaces, Nonlinear Anal Model Control, 19 (1) (2014), 43–54

[113] P Salimi A Latif and N Hussain Modified $\alpha - \psi$-contractive mappings with applications Fixed Point Theory Appl (2013) 2013 151

[114] B Samet, C Vetro and P Vetro, Fixed point theorems for $\alpha - \psi$-contractive type mappings Nonlinear Anal 75 (2012) 2154–2165

[115] NA Secelean, Iterated function systems consisting of F-contractions Fixed Point Theory Appl (2013), article ID 277 (2013)

[116] S Sessa, On a weak commutativity condition of mappings in fixed points considerations Publ Inst Math, 32 (1982), 149-153

[117] M Sgroi and C Vetro, Multi-valued F-contractions and the solution of certain functional and integral equations, Filomat 27 (7) (2013) 1259-1268

[118] N Shahzad and E Karapinar A Francisco and R L de-Hierro On some fixed point theorems under $(\alpha\ \psi\ \varphi)$-contractivity conditions in metric spaces endowed with transitive binary relations, Fixed Point Theory Appl (2015), 2015 124

[119] A Shoaib, M Arshad and J Ahmad Fixed point results of locally cotractive mappings in ordered quasi-partial metric spaces The Scientific World Journal, 2013 (2013) article ID 194897, 8 pages

[120] P K Shrivastava N P S Bawa and S K Nigam, Fixed point theorems for hybrid contractions, Varahmihir J Mathematical Sciences 2 (2) (2002) 275-281

[121] T Sistani and M Kazemipour Fixed point theorems for α-ψ-contractions on metric spaces with a graph J Adv Math Stud , 7 (1) (2014) 65-79

[122] P Vijayaraju, B E Rhoades, and R Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, Int J Math Math Sci 15 (2005), 2359-2364

[123] D Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl (2012), article ID 94

[124] D Wardowski and N V Dung, Fixed points of $F-$weak contractions on complete metric spaces, Demonstratio Mathematics (1) (2014), 146-155

[125] X Zhang, Common fixed point theorems for some new generalized contractive type mappings, J Math Anal Appl , 333 (2007) 780-786