Flow and heat transfer due to a stretching cylinder in a porous medium

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1. Heat - Trasmission - Mathematics

<u>**Dedicated to**</u>

My **"Mother"**

Whose prayer has

always been the reason of success and prosperity in my life

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I praise Allah, who is the protector of those who have faith. He brings them out of darkness into light. His greatest blessing for man is knowledge that strengthens faith, provides rational and logical approach towards meaningful human existence.

Through the unbounded grace of the Almighty Allah, I am able to complete my dissertation up to the maximum satisfaction.

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May 12, 2011. Abid Majeed

DECLARATION

I, hereby declare that this dissertation, neither as a whole nor as a part thereof, has been copied out from any source. It is further that I have prepared this dissertation entirely on the basses of my personal effort made under the sincere guidance of my supervisor. No portion of the work, presented in this dissertation, has been submitted in support of any application of any degree or qualification of this or any other university or institute of learning.

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Preface

During the last few decades, the hydromagnetic flow and heat transfer of a viscous fluid along fixed or moving surfaces is one of the basic and important problems in many engineering and technological applications. For example, magnetohydrodynamics (MHD) generators, pumps, meters, bearing and boundary layer control, liquid metals, water mixed with a little acid, hot rolling, wire drawing, glass-fiber and paper production and many others. First Pavlov [1] discussed the effects of transverse magnetic field on the flow past a stretching sheet. Following the pioneering work of Pavlov [1], the flow over a moving surface in the presence of a transverse magnetic field has been studied by many researchers [2-17]. In 1988, Wang [18] has investigated the steady flow of a viscous fluid outside of a stretching hollow cylinder in an ambient fluid at rest. Aldos and Ali [19] discussed the MHD free forced convection from a horizontal cylinder with suction and blowing/injection. Recently, Ganesan and Loganthan [20] presented the effect of the magnetic field on a moving vertical cylinder with constant heat flux. Very recently, Ishak et al. [21] reported the magnetohydrodynamics (MHD) flow and heat transfer analysis due to a stretching cylinder. The dissertation is arranged as follows:

Chapter one contains some basic definitions and equations. Furthermore, the concepts of solution methods also included.

Chapter two is concerned to present the work done by Ishak et al. [21]. The governing partial differential equations are converted into non-linear ordinary differential equations. All the result are reproduced by an implicit finite difference scheme known as Keller box method [22] and perturbation technique.

Chapter three aims to extend the work of [21] into two directions: (i) to analyze the flow in a porous medium and (ii) to include the effects of thermal radiation. The problem is formulated in such a way that the flow equations are transformed into non-linear ordinary differential equations by employing the similarity transformations. The numerical and perturbation solutions of dimensionless velocity and temperature fields are computed. The effects of embedded parameters on the flow and temperature profiles are discuss through graphs. The numerical values of the skin-friction coefficients and the Nusselt number for different parameters are also given.

Contents

Chapter 1

Some basic definitions and equations

The main purpose of this chapter is to present the basic definitions and flow equations. Moreover the basic ideas of solutions techniques, namely, the Keller Box method and Perturbation technique are explained.

1.1 Definitions

1.1.1 Velocity Field

In dealing with fluids in motion, we shall necessarily be concerned with the description of a velocity field. At a given instant the velocity field, $\mathbf{V}$, is a function of the space coordinates (x, y, z) and time t. The velocity at any point in the flow field might vary from one instant to another. Thus the complete representation of velocity is given by

$$\mathbf{V} = \mathbf{V}(x, y, z, t). \tag{1.1}$$

1.1.2 Flow

A material that deforms continuously when different forces act upon it. If the deformation continuously increases without limit,this phenomenon is know as flow.

1.1.3 Fluid

Any liquid or gas that cannot sustain a shearing force when at rest and that undergoes a continuous change in shape when subjected to such as stress.

1.1.4 Viscosity

The internal friction of a fluid, produced by the movement of its molecules against each other. Viscosity causes the fluid to resist flowing.

$$\text{Viscosity} = \frac{\text{Shear stress}}{\text{rate of shear strain}}, \tag{1.2}$$

or

$$\mu = \frac{\tau_{xy}}{\frac{du}{dy}}, \tag{1.3}$$

1.1.5 Density

Density of a fluid is defined as the mass per unit volume. Mathematically, it is denoted by ρ and defined as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}, \tag{1.4}$$

where δv is the total volume element around the point C and δm is the mass of the fluid within δv.

1.1.6 Kinematic Viscosity

It is the ratio of absolute viscosity μ to the density. It is denoted by ν and given as

$$\nu = \frac{\mu}{\rho}. \tag{1.5}$$

1.2 Classification of fluid

1.2.1 Ideal fluid

A fluid which does not depend upon viscosity is called an ideal fluid or perfect fluid. Practically this type of fluid does not exist. However, the fluid with negligible viscosity may be considered as an ideal fluid.

1.2.2 Real fluid

Real fluids are those in which the viscosity of the fluid has significant effects on the fluid motion. In otherworld we can not neglect the viscosity effects on the motion. Real fluids are further divided into two categories.

(i) Newtonian fluid

(ii) Non-Newtonian fluid

1.3 Types of Flow

1.3.1 Laminar Flow

A flow in which the paths of fluid particles are parallel to one another. During laminar flow, all the fluid particles move in distinct and separate layers; there is no mixing between adjacent layers.

1.3.2 Turbulent Flow

The flow of a fluid in which the path of the fluid particles at any point varies rapidly in both magnitude and direction. Turbulent flow is characterized by mixing of adjacent fluid layers.

1.3.3 Uniform Flow

If the flow velocity has the same magnitude and direction at every point in the fluid it is said to be uniform.

1.3.4 Non-Uniform Flow

If at a given instant the velocity of fluid is not the same at every point then the flow will be non-uniform.

1.3.5 Steady Flow

A flow in which properties associated with the motion of the fluid are independent of the time so that the flow pattern remain unchanged with the time, is said to be steady flow.

1.3.6 Unsteady Flow

A flow in which properties associated with the motion of the fluid depend on the time so that the flow pattern varies with time, is said to be unsteady flow.

1.3.7 Compressible Flow

A flow in which the density of the fluid is not constant,is called compressible flow. All the gases are, generally treated as a compressible flow.

1.3.8 Incompressible Flow

A flow in which the density of the fluid remains constant throughout the flow is called incompressible flow. All the liquids are, generally considered as a incompressible flow.

1.4 Fundamentals of heat transfer

1.4.1 Conduction

When a temperature gradient exists in a body, experience has shown that there is an energy transfer from higher temperature region to lower temperature region. We say that energy is transferred by conduction and that the heat transfer rate per unit area is proportional to the normal temperature gradient, given by

$$\frac{q}{A} \propto \frac{\partial \theta}{\partial x},$$

or

$$q = -kA\frac{\partial \theta}{\partial x}, \tag{1.6}$$

where q is the heat transfer rate, A is the area, $\partial\theta/\partial x$ is the temperature gradient in the direction of heat flow and k is the positive constant and is called the thermal conductivity of the material.

1.4.2 Convection

It is well known phenomena that a hot plate of metal will cool faster in front of a fan than when exposed to still air. We can say that the heat is convected away and we call the process convection heat transfer.

1.4.3 Radiation

Radiation heat transfer is concerned with the exchange of thermal radiation energy between two or more bodies. No medium need exist between the two bodies for heat transfer to take place

1.4.4 Specific heat

Specific heat is the amount of heat or thermal energy required to raise the temperature of a unit quantity of a body by one unit. It is denoted by c_p. For example, at a temperature of $15°C$, the heat required to raise the temperature of a water by $1K$ (equivalent to $1°C$) is 4.186 $kJkg^{-1}K^{-1}$.

1.4.5 Fourier's law of heat conduction

The Fourier's law of heat conduction states that the time rate of heat transfer through a material is proportional to the negative at the temperature gradient and to the area at right angles to that gradient through which the heat is flowing.

Mathematically, it is given by

$$\frac{dQ}{dt} = -kA\frac{d\theta}{dx}, \tag{1.7}$$

in which 'Q' is the amount of heat transferred.

1.4.6 Thermal conductivity

Thermal conductivity 'k' is the property of a material that shows its capability to conduct heat. It appears basically in Fourier's Law for heat conduction. Thermal conductivity is measured in watts per Kelvin per metre $\left(WK^{-1}m^{-1}\right)$.

1.4.7 Maxwell's equations

Maxwell's equations are the set of four equations which relate the electric and magnetic field to their sources, charge density and current density. These equations are described as

$$\nabla.\mathbf{E} = \frac{\varrho}{\epsilon_0}, \tag{1.8}$$

$$\nabla.\mathbf{B} = \mathbf{0}, \tag{1.9}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.10}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{1.11}$$

In the above equations ϵ_0 is the permitivity of the free space also called electric constant, μ_0 is the permeability of free space which is also called magnetic constant, ϱ is the total charge density and $\mathbf{J}$ is the total current density. The total magnetic field is $\mathbf{B}$ $(=B_0 + \mathbf{b})$, where $\mathbf{b}$ is induced magnetic field. By *Ohm's law* in generalized form we have

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{V} \times \mathbf{B}\right), \tag{1.12}$$

where σ is the electric conductivity of the fluid. In the present case there is no applied electric field, and the induced magnetic field is neglected due to low magnetic Reynold number. Therefore, the Lorentz force in the direction of the flow in a pipe becomes

$$\left(\mathbf{J} \times \mathbf{B}\right)_z = -\sigma B_0^2 w, \tag{1.13}$$

where B_0 is the applied magnetic field and w is the velocity component normal to the magnetic field and parallel to the flow.

1.5 Dimensionless numbers

1.5.1 Prandtl number

It is the ratio of the product of dynamic viscosity and specific heat to the thermal conductivity, and denoted by the symbol Pr and is given by

$$\mathrm{Pr} = \frac{\mu c_p}{k}. \tag{1.14}$$

1.5.2 Reynolds number

It is the ratio of inertia force to the viscous force. It is denoted by the symbol Re and is given by

$$\mathrm{Re} = \frac{VL}{\nu}, \tag{1.15}$$

where L and V denote the characteristics length and characteristics velocity, respectively.

1.6 Fundamental equations of fluids

1.6.1 Equation of continuity

In any closed system, the mass is always invariant regardless of its changes in shape when external forces are absent or the principle that matter cannot be created or destroyed. In fluid mechanics, this law is named as equation of continuity. In other words the mass of the system remains conserved. Mathematically, it is described as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{1.16}$$

In cylindrical coordinates, this equation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{1.17}$$

1.6.2 Equation of momentum

When some bodies constituting an isolated system act upon one another, the total momentum of the system remains same. In an inertial frame of reference, the general form of equations of fluid motion or the law of conservation of momentum is

$$\rho\frac{D\mathbf{V}}{Dt}=\nabla\cdot\mathbf{T}+\rho\mathbf{f}, \tag{1.18}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{V}$ is the velocity field, D/Dt is the total material derivative and $\mathbf{f}$ is the body force. In cylindrical coordinates the momentum equation in components forms is given by.

r-component:

$$\begin{aligned}\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}+w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial r}+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}(ru)\right)\right]+\\ &\quad \eta\left[\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}\right]+\rho f_r,\end{aligned} \tag{1.19}$$

θ-component:

$$\begin{aligned}\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}+w\frac{\partial v}{\partial z}\right) &= -\frac{1}{r}\frac{\partial p}{\partial r}+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}(rv)\right)\right]+\\ &\quad \eta\left[\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}+\frac{\partial^2 u}{\partial z^2}\right]+\rho f_\theta,\end{aligned} \tag{1.20}$$

z-component:

$$\begin{aligned}\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial r}+\eta\left[\left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r})\right)\right]+\\ &\quad \eta\left[\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right]+\rho f z.\end{aligned} \tag{1.21}$$

1.6.3 Equation of energy

The energy equation is described as

$$\rho c_p\frac{D\theta}{Dt}=\mathbf{T}\cdot\mathbf{L}-\boldsymbol{\nabla}\cdot\mathbf{q}, \tag{1.22}$$

in which

$$\mathbf{L} = \nabla \mathbf{V}. \tag{1.23}$$

In cylindrical coordinates, it is given as

$$\rho c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} + w\frac{\partial T}{\partial z} \right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu\phi, \tag{1.24}$$

where ϕ is the viscous dissipation function.

1.7 Solution methodology

1.7.1 Finite difference method

Finite difference method is an approximate method in the sense that derivative at a point are approximated by difference quotient over a small interval. Assume that a function H and its derivatives are single valued, finite and continuous functions of z, then by Taylor's series we have

$$H(z+h) = H(z) + hH'(z) + \frac{h^2}{2}H''(z) + \frac{h^3}{6}H'''(z) + O(h^4), \tag{1.25}$$

and

$$H(z-h) = H(z) - hH'(z) + \frac{h^2}{2}H''(z) - \frac{h^3}{6}H'''(z) + O(h^4), \tag{1.26}$$

where prime denotes the differentiations with respect to z. From Eqs. (1.25) and (1.26), we may write

$$\left(\frac{dH}{dz} \right)_{z=z} \simeq \frac{H(z+h) - H(z)}{h}, \tag{1.27}$$

$$\left(\frac{dH}{dz} \right)_{z=z} \simeq \frac{H(z) - H(z-h)}{h}, \tag{1.28}$$

with an error of order h, and neglecting the second and higher powers of h. Eqs. (1.27) and (1.28) are called forward difference formula and backward difference formula respectively. Subtract Eq. (1.27) from (1.28) gives

$$\left(\frac{dH}{dz} \right)_{z=z} \simeq \frac{H(z+h) - H(z-h)}{h}, \tag{1.29}$$

with a leading error on the right hand side of order h^2.The equation (1.29) is called a central difference formula. Similarly we can find the approximation for second and third order derivatives.

1.7.2 Keller box method

There are many numerical methods for solving the boundary layer equations in the form of ordinary or partial differential equations in fluid mechanics, but here we used the Keller box method. It is a two point finite-difference scheme, we first express the differential equations as a system first-order equations. The first-order equations are approximated on an arbitrary rectangular net with "centered-difference" derivatives and averages at the midpoints of the net rectangle difference equations. The resulting system of equations which is implicit and nonlinear is linearized by Newton's method and solved by the block-elimination method.

For example, we consider the energy equation for a two dimensional constant density flow in a symmetrical duct with fully developed velocity profile, i.e.,

$$u\frac{\partial T}{\partial x} = \frac{\nu}{P_r}\frac{\partial^2 T}{\partial y^2}, \tag{1.30}$$

The solution of this equation, requires initial and boundary value conditions. We consider the boundary conditions for Eq. (1.30) are given by

$$\begin{aligned} y &= 0, \;\; T = T_w, \\ y &= \delta, \;\; T = T_e. \end{aligned} \tag{1.31}$$

and the initial conditions are written in the form

$$x = x_0, \;\; T = T(y). \tag{1.32}$$

To solve Eq. (1.30) by using Keller box method, we first express it in terms of a system of two first order differential equations by assuming

$$T' = q, \tag{1.33}$$

and Eq. (1.30) as

$$q' = \frac{P_r}{\nu} u \frac{\partial T}{\partial x}. \tag{1.34}$$

Here the primes denote the differentiation with respect to y. The finite difference form of the ordinary differential equation (1.33) is written for the mid point $\left(x_n, y_{\frac{j-1}{2}}\right)$ and the finite difference form of the partial differential equation (1.34) is written for the midpoint $\left(x_{\frac{n-1}{2}}, y_{\frac{j-1}{2}}\right)$. Eqs. (1.33) and (1.34) can be written as

$$\frac{T_j^n - T_{j-1}^n}{h_j} = \frac{q_j^n + q_{j-1}^n}{2} = q_{\frac{j-1}{2}}^n, \tag{1.35}$$

$$\frac{1}{2}\left(\frac{q_j^n - q_{j-1}^n}{h} + \frac{q_j^{n-1} - q_{j-1}^{n-1}}{h_j}\right) = \frac{P_r}{\nu} u_{\frac{j-1}{2}}^{\frac{n-1}{2}} \frac{T_{\frac{j-1}{2}}^n - T_{\frac{j-1}{2}}^{n-1}}{k_n}, \tag{1.36}$$

The above Eqs. (1.35) and (1.36) can written as

$$T_j^n - T_{j-1}^n - \frac{h_j}{2}(q_j^n + q_{j-1}^n) = 0, \tag{1.37}$$

$$(s_1)_j q_j^n + (s_2)_j q_{j-1}^n + (s_3)_j (T_j^n + T_{j-1}^n) = R_{\frac{j-1}{2}}^{n-1}, \tag{1.38}$$

where

$$(s_1)_j = 1, \quad (s_2)_j = -1, \quad (s_3)_j = -\frac{\lambda_j}{2}, \tag{1.39}$$

$$R_{\frac{j-1}{2}}^{n-1} = -\lambda_j T_{\frac{j-1}{2}}^{n-1} + q_{j-1}^{n-1} - q_j^{n-1}, \tag{1.40}$$

$$\lambda_j = \frac{2P_r}{\nu} u_{\frac{j-1}{2}}^{\frac{n-1}{2}} \frac{h_j}{k_n}. \tag{1.41}$$

The superscript on $u_{\frac{j-1}{2}}$ is not necessary but is included for generality. Eqs. (1.37) and (1.38) are imposed for $j = 1, 2, \ldots\ldots, J-1$. At $j = 0$ and J, we have

$$T_0 = T_w, \quad T_j = T_e. \tag{1.42}$$

Since Eqs. (1.37) and (1.38) are linear, with boundary condition (1.42), the system may be written in matrix-vector form as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & & \\ -1 & \frac{-h_1}{2} & 1 & \frac{-h_1}{2} & & \\ (s_3)_j & (s_2)_j & (s_3)_j & (s_1)_j & 0 & 0 \\ 0 & 0 & -1 & \frac{-h_{j+1}}{2} & 1 & \frac{-h_{j+1}}{2} \\ & & (s_3)_j & (s_2)_j & (s_3)_j & (s_1)_j \\ & & 0 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} T_0 \\ q_0 \\ T_j \\ q_j \\ T_J \\ q_J \end{bmatrix} = \begin{bmatrix} (r_1)_0 \\ (r_2)_0 \\ (r_1)_j \\ (r_2)_j \\ (r_1)_j \\ (r_2)_j \end{bmatrix}, \tag{1.43}$$

with

$$\begin{aligned} (r_1)_0 &= T_w, \ \ (r_1)_j = R^{n-1}_{\frac{j-1}{2}}, \ \ 1 \le j \le J, \\ (r_2)_j &= 0, \ \ 0 \le j \le J-1, \ \ (r_2)_j = T_e. \end{aligned} \tag{1.44}$$

The system of equations given by Eq. (1.43) can be rewritten as

$$\bar{\mathbf{A}}\boldsymbol{\delta} = \bar{\mathbf{r}}, \tag{1.45}$$

where

$$\bar{\mathbf{A}} = \begin{pmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C_1 & & \\ & & \ddots & \ddots & \\ & & B_{J-1} & A_{J-1} & C_{J-1} \\ & & & B_J & A_J \end{pmatrix}, \ \bar{\boldsymbol{\delta}} = \begin{pmatrix} \delta_0 \\ \delta_1 \\ \vdots \\ \delta_j \\ \vdots \\ \delta_J \end{pmatrix}, \ \bar{\mathbf{r}} = \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_j \\ \vdots \\ r_J \end{pmatrix}, \tag{1.46}$$

$$\boldsymbol{\delta}_j = \begin{pmatrix} T_j \\ p_j \end{pmatrix}, \ \mathbf{r}_j = \begin{pmatrix} (r_1)_j \\ (r_2)_j \end{pmatrix}, \tag{1.47}$$

and $\mathbf{A}_j, \mathbf{B}_j, \mathbf{C}_j$ are 2×2 matrices defined as follows

$$\begin{aligned}
\mathbf{A}_0 &\equiv \begin{bmatrix} 1 & 0 \\ -1 & -\frac{h_1}{2} \end{bmatrix}, \ \mathbf{A}_j \equiv \begin{bmatrix} (s_3)_j & (s_1)_j \\ -1 & \frac{-h_{j+1}}{2} \end{bmatrix}, \ 1 \le j \le J-1, \\
\mathbf{A}_j &\equiv \begin{bmatrix} (s_3)_J & (s_1)_J \\ 1 & 0 \end{bmatrix}, \ \mathbf{B}_j \equiv \begin{bmatrix} (s_3)_j & (s_1)_j \\ 0 & 0 \end{bmatrix}, \ 1 \le j \le J, \\
\mathbf{C}_j &\equiv \begin{bmatrix} 0 & 0 \\ 1 & \frac{-h_{j+1}}{2} \end{bmatrix}, \ 0 \le j \le J-1.
\end{aligned} \tag{1.48}$$

The solution of Eq. (1.45) by the block-elimination consists of two sweeps. In the forward sweep we compute Γ_j, Δ_j, and w_j from the recursion formulas given by

$$\left.\begin{array}{c}
\mathbf{\Delta}_0 = \mathbf{A}_0, \ \Gamma_j \mathbf{\Delta}_{j-1} = \mathbf{B}_j, \ \mathbf{\Delta}_j = \mathbf{A}_j - \mathbf{\Gamma}_j \mathbf{C}_{j-1}, \ 1 \le j \le J, \\
\mathbf{w}_0 = \mathbf{r}_0, \ \mathbf{w}_j = \mathbf{r}_j - \Gamma_j \mathbf{w}_{j-1}, \ 1 \le j \le J,
\end{array}\right\} \tag{1.49}$$

and Γ_j has the same structure as B_j, i.e.,

$$"\Gamma_j = \begin{bmatrix} (\gamma_{11})_j & (\gamma_{12})_j \\ 0 & 0 \end{bmatrix}, "$$

and although the second row of Δ_j has the same structure as the second row of A_j,

$$\Delta_j = \begin{bmatrix} (\alpha_{11})_j & (\alpha_{12})_j \\ -1 & \frac{-h_{j+1}}{2} \end{bmatrix}.$$

In general we may write

$$\Delta_j = \begin{bmatrix} (\alpha_{11})_j & (\alpha_{12})_j \\ (\alpha_{21})_j & (\alpha_{22})_j \end{bmatrix}.$$

In the backward sweep, δ_j is computed from the following recursion formulas:

$$\mathbf{\Delta}_j \boldsymbol{\delta}_j = \mathbf{w}_j, \ \mathbf{\Delta}_j \boldsymbol{\delta}_j = \mathbf{w}_j - \mathbf{C}_j \boldsymbol{\delta}_{j+1}, \ j = J-1, J-2, ...0. \tag{1.50}$$

Keller's method, often referred to as the box method has several very desirable features that make it appropriate for the solution of all differential equations. The main features of this method are

1. Only slightly more arithmetic to solve as compared to other implicit method.
2. Second-order accuracy with arbitrary or nonuniform x and y spacings.
3. It allows very rapid x variations
4. This method allows easy programming of the solution of large numbers of coupled equations

1.7.3 Perturbation Solution

In order to obtain solutions of equations, by using approximation methods, numerical solution, for example perturbation (asymptotic) method. According to this technique, the solution is given by the first few terms of an expansion. These expansions may be carried out in terms of a parameter (small or large) which appears in the equations.

Let us consider an algebraic equation.

$$u = 1 + \epsilon u^3, \tag{1.51}$$

when $\epsilon = 0$,

$$u = 1.$$

For small $\epsilon (\neq 0)$, we let

$$u = 1 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ..., \tag{1.52}$$

and Eq. (1.51) becomes

$$(\epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ...) = \epsilon(1 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ...)^3 = 0. \tag{1.53}$$

Expanding for small ϵ we rewrite Eq. (1.53) as

$$(\epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + ...) = \epsilon(1 + 3\epsilon u_1 + 3\epsilon^2(u_2 + u_1^2) + ...) = 0. \tag{1.54}$$

Equating the coefficients of like powers of ϵ, we have

$$\epsilon(u_1 - 1) + \epsilon^2(u_2 - 3u_1) + \epsilon^3(u_3 - 3u_2 - 3u_1^2) + ... = 0. \tag{1.55}$$

Since the above equation is an identity in ϵ, each coefficient of ϵ vanishes independently. Thus

$$\left.\begin{aligned} u_1 - 1 &= 0, \\ u_2 - 3u_1 &= 0, \\ u_3 - 3u_2 - 3u_1^2 &= 0, \end{aligned}\right\} \tag{1.56}$$

implies that

$$\left.\begin{aligned} u_1 &= 1, \\ u_2 &= 3u_1 = 3, \\ u_3 &= 3u_2 + 3u_1^2 = 12, \end{aligned}\right\} \tag{1.57}$$

Therefore Eq. (1.52) becomes,

$$u = 1 + \epsilon + 3\epsilon^2 + 12\epsilon^3 + ... \tag{1.58}$$

which is an approximate solution of Eq. (1.50).

Chapter 2

MHD flow and heat transfer due to a stretching cylinder

2.1 Introduction

This chapter deals with the numerical solution of flow and heat transfer due to a stretching cylinder and constant magnetic field $\mathbf{B}$ is applied. The governing partial differential equations are transformed to ordinary differential equations using similarity transformations. The non-linear ordinary differential equations are solved both numerically using Keller Box method and analytically using perturbation technique. The effects of various parameters on the velocity and temperature profiles are discussed through tables and graphs. This chapter is a review of the paper by Ishak et al [21].

2.2 Mathematical Formulation

We consider, the steady flow of an incompressible viscous fluid over a stretching cylinder as shown in Fig. 1, where the z-axis is taken along the axis of the cylinder and the r-axis is in the radial direction . It is assumed that the temperature at the surface of the cylinder is constant 'T_w' and the ambient fluid temperature is T_∞, where $T_w > T_\infty$. The uniform magnetic field B_0 is also appeared in the radial direction and the induced magnetic field is neglected, which is valid when the magnetic Reynolds number is very small. Under these assumptions, the equations for the flow and heat transfer analysis are given as

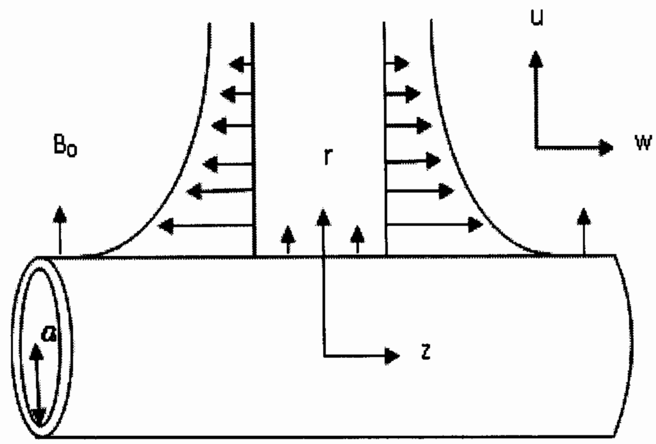

Fig. 2.1: Physical model and coordinate system.

$$\frac{\partial(rw)}{\partial z}+\frac{\partial(ru)}{\partial r}=0, \tag{2.1}$$

$$w\frac{\partial w}{\partial z}+u\frac{\partial w}{\partial r}=\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{\partial w}{r\partial r}\right)-\frac{\sigma B_0^2 w}{\rho}, \tag{2.2}$$

$$w\frac{\partial u}{\partial z}+u\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{r\partial r}-\frac{u}{r^2}\right), \tag{2.3}$$

$$w\frac{\partial T}{\partial z}+u\frac{\partial T}{\partial r}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2}+\frac{\partial T}{r\partial r}\right), \tag{2.4}$$

where u and w are the velocity components in the r- and z- axis direction, ρ is the density of the fluid, σ is the electrical conductivity, ν is the kinematic viscosity, k is the thermal conductivity, c_p is the specific heat and T is the temperature of the fluid.

The boundary conditions of the problem are

$$\left.\begin{array}{c} u(r,z)=0, \quad w(r,z)=w_w, \quad T=T_w \quad \text{at} \quad r=a, \\ w(r,z)\to 0, \quad T\to T_\infty, \quad \text{as} \quad r\to\infty, \end{array}\right\} \tag{2.5}$$

where $w_w = 2cz$ is the stretching velocity of the wall and c is the positive constant having dimension $(\text{time})^{-1}$. We introduce the following similarity transformations,

$$\left.\begin{array}{c}\eta = (\frac{r}{a})^2, \quad u = -ca[f(\eta)/\sqrt{\eta}], \quad w = 2cf'(\eta)z, \\ \theta(\eta)=(T-T_\infty)/(T_w - T_\infty).\end{array}\right\} \qquad 2.6$$

Using the Eq. (2.6), the continuity equation (2.1) is automatically satisfied and Eqs. (2.2) and (2.4) become

$$\eta f''' + f'' - Mf' + \mathrm{Re}(ff'' - f'^2) = 0, \qquad (2.7)$$

$$\eta\theta'' + (1 + \mathrm{Re}\,\mathrm{Pr}\, f)\theta' = 0. \qquad (2.8)$$

with

$$\left.\begin{array}{c}f(1) = 0, \quad f'(1) = 1, \quad f'(\infty) = 0, \\ \theta(1) = 1, \quad \theta(\infty) = 0.\end{array}\right\} \qquad (2.9)$$

Here $\mathrm{Re} = ca^2/2\nu$ is the Reynolds number, $\mathrm{Pr} = \mu c_p/k$ is the Prandtl number and $M = \sigma B_0^2 a^2/4\nu\rho$ is the Hartman number or magnetic parameter. The primes denotes differentiation with respect to dimensionless variable η.

The pressure can be determined from Eq. (2.3) in the form

$$\frac{p - p_\infty}{\rho c\nu} = -\frac{\mathrm{Re}}{\eta} f^2(\eta) - 2f'(\eta). \qquad (2.10)$$

The physical quantities of interest are the skin friction coefficient and the Nusselt number, which are defined as

$$C_f = \frac{\tau_w}{\rho w_w^2/2}, \quad Nu = \frac{aq_w}{k(T_w - T_\infty)}. \qquad (2.11)$$

The shear stress τ_w at the wall and the heat flux q_w at the wall are given by

$$\tau_w = \mu\left(\frac{\partial w}{\partial r}\right)_{r=a}, \quad q_w = -k\left(\frac{\partial T}{\partial r}\right)_{r=a}. \qquad (2.12)$$

Using Eqs. (2.6) and (2.12), Eq. (2.11) becomes

$$C_f(\mathrm{Re}\, z/a) = f''(1), \qquad Nu = -2\theta'(1). \qquad (2.13)$$

2.3 Solution of the problem

2.3.1 Perturbation solution for small M

Following Ariel [23], we can get approximate solutions of Eqs. (2.7) and (2.8) with boundary conditions (2.9), which is valid for $M << 1$. We assume the solution of the form

$$f(\eta) = \sum_{i=0}^{\infty} f_i(\eta) M^i, \tag{2.14}$$

$$\theta(\eta) = \sum_{i=0}^{\infty} \theta_i(\eta) M^i. \tag{2.15}$$

After using Eqs. (2.14) and (2.15) into Eqs. (2.7) and (2.8), we obtain the following set of equations with boundary conditions.

Zeroth-order system:

$$\begin{cases} \eta f_0''' + f_0'' + \mathrm{Re}(f_0 f_0'' - f_0'^2) = 0, \\ \eta\theta_0'' + (1 + \mathrm{Re}\,\mathrm{Pr}\, f_0)\theta_0' = 0, \\ f_0(1) = 0, \quad f_0'(1) = 1, \quad f_0'(\infty) = 0, \\ \theta_0(1) = 1, \qquad \theta_0(\infty) = 0. \end{cases} \tag{2.16}$$

First-order system:

$$\begin{cases} \eta f_1''' + f_1'' + \mathrm{Re}(f_0 f_1'' + f_1 f_0'' - 2f_0' f_1') = 0, \\ \eta\theta_1'' + \mathrm{Re}\,\mathrm{Pr}\, f_1\theta_0' + (1 + \mathrm{Re}\,\mathrm{Pr}\, f_0)\theta_1' = 0, \\ f_1(1) = 0, \quad f_1'(1) = 0, \quad f_1'(\infty) = 0, \\ \theta_1(1) = 0, \qquad \theta_1(\infty) = 0. \end{cases} \tag{2.17}$$

Second-order system:

$$\begin{cases} \eta f_2''' + f_2'' - f_1' + \mathrm{Re}(f_0 f_2'' + f_1 f_1'' - f_2 f_0'' - 2f_0' f_2' - f_1'^2) = 0, \\ \eta\theta_2'' + \mathrm{Re}\,\mathrm{Pr}\, f_1\theta_1' + \mathrm{Re}\,\mathrm{Pr}\, f_2\theta_0' + (1 + \mathrm{Re}\,\mathrm{Pr}\, f_0)\theta_2' = 0, \\ f_2(2) = 0, \quad f_2'(1) = 0, \quad f_2'(\infty) = 0, \\ \theta_2(1) = 0, \qquad \theta_2(\infty) = 0. \end{cases} \tag{2.18}$$

In general, we can write the solution in the form

$$\eta f_i''' + f_i'' - f_{i-1}' + \mathrm{Re}(\sum_{j=0}^{i} f_j f_{i-j}'' - \sum_{j=0}^{i} f_j' f_{i-j}') = 0, \tag{2.19}$$

$$\eta \theta_i'' + \theta_i' + \mathrm{Re}\,\mathrm{Pr} \sum_{j=0}^{i} f_j \theta_{i-j}' = 0, \tag{2.20}$$

for $i \geq 1$, subject to the boundary conditions

$$\begin{aligned} f_i(1) &= 0, \quad f_i'(1) = \delta_{i0}, \quad \theta_i(1) = \delta_{i0}, \\ f_i'(\infty) &= 0, \quad \theta_i(\infty) \to 0. \end{aligned} \tag{2.21}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.22}$$

The above non-linear ordinary differential Eqs. (2.19) and (2.20) with boundary conditions (2.21) are solved numerically using shooting method with Runge-Kutta algorithm. The skin friction coefficient and Nusselt number in Eq. (2.11) are given by

$$\left. \begin{aligned} C_f(\mathrm{Re}\, z/a) &= f_0''(1) + f_1''(1)M + f_2''(1)M^2, \\ Nu &= -2(\theta_0'(1) + \theta_1'(1)\lambda + \theta_2'(1))\lambda^2). \end{aligned} \right\} \tag{2.23}$$

2.3.2 Numerical solution

The non-linear Eqs. (2.7) and (2.8) subject to the boundary conditions (2.9) are solved numerically by using an implicit finite difference method, known as Keller box scheme, which is described in the book by Cebeci and Bradshaw [22].

2.4 Results and discussion

Table 2.1: Values of skin friction coefficient $f''(1)$ for several values of M when $\mathrm{Re} = 10$.

M	Numerical	Small M	Wang [18]
0	-3.3444	-3.3445	-3.34445
0.01	-3.3461	-3.3462	
0.05	-3.3528	-3.3529	
0.1	-3.3612	-3.3613	
0.5	-3.4274	-3.4273	
1	-3.5076		
2	-3.6615		
5	-4.0825		

Table 2.2: Values of the Nusselt number $-\theta'(1)$ for several values of M at $\mathrm{Re} = 10$.

	Pr = 0.7 (air)			Pr = 7 (water)		
M	Numerical	Small M	Wang [18]	Numerical	Small M	Wang [18]
0	1.5687	1.5687	1.568	6.1592	6.1592	6.160
0.01	1.5683	1.5682		6.1588	6.1588	
0.05	1.5665	1.5665		6.1573	6.1573	
0.1	1.5644	1.5643		6.1554	6.1554	
0.5	1.5478	1.5479		6.1402	6.1403	
1	1.5284			6.1219		
2	1.4924			6.0864		
5	1.4012			5.9855		

Table 2.3: Values of skin friction coefficient $f''(1)$ for several values of M and Re.

M	Re = 1			Re = 5		
	Numerical	Small M	Wang [18]	Numerical	Small M	Wang [18]
0	−1.1780	−1.1780	−1.17776	−2.4174	−2.4175	−2.41745
0.01	−1.1839	−1.1839		−2.4199	−2.4199	
0.05	−1.2068	−1.2068		−2.4296	−2.4297	
0.1	−1.2344	−1.2339		−2.4417	−2.4418	
0.5	−1.4269	−1.3928		−2.5352	−2.5338	

Table 2.4: Values of the Nusselt number $-\theta'(1)$ for several values of M and Re when Pr = 7.

	Re = 1			Re = 100		
M	Numerical	Small M	Wang [18]	Numerical	Small M	Wang [18]
0	2.0587	2.0587	2.059	19.1587	19.1587	19.12
0.01	2.0572	2.0573		19.1586	19.1586	
0.05	2.0516	2.0517		19.1581	19.1581	
0.1	2.0449	2.0451		19.1576	19.1576	
0.5	1.9978	2.0082		19.1530	19.1531	

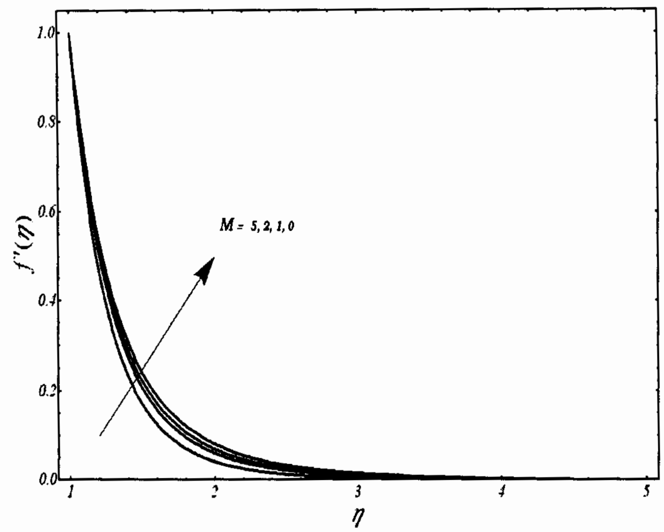

Fig. 2.2: Velocity profile $f'(\eta)$ verses η for various values of M when $\mathrm{Re} = 10$.

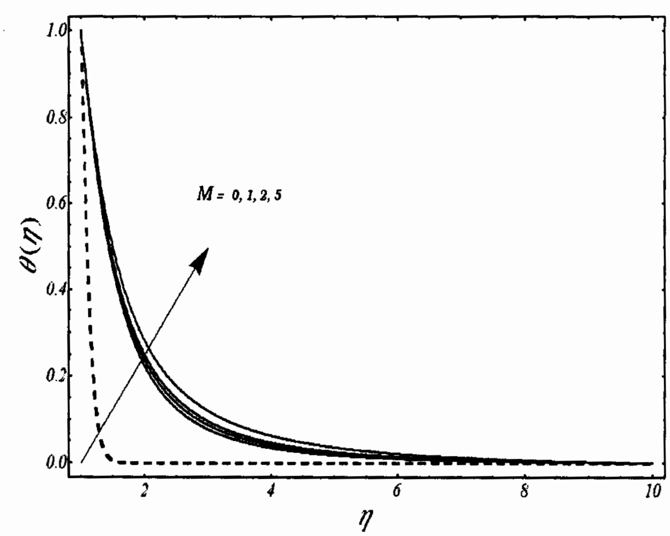

Fig. 2.3: Temperature profile $\theta(\eta)$ verses η for various values of M and Pr: solid line for $\mathrm{Pr} = 0.7$ and dashed line for $\mathrm{Pr} = 7$ when $\mathrm{Re} = 10$.

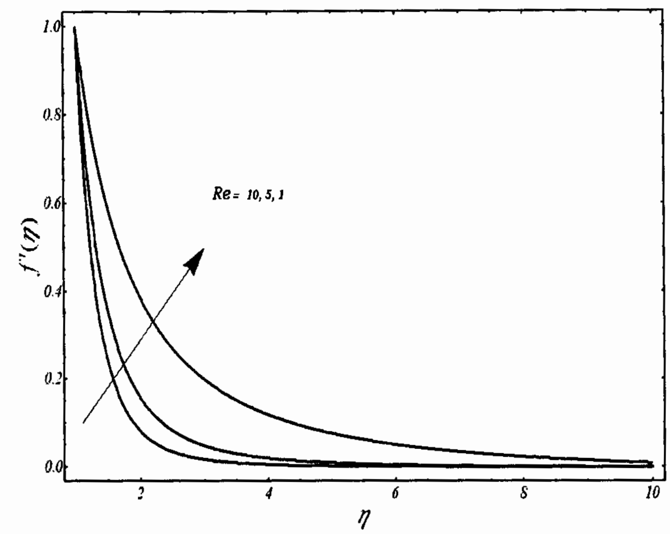

Fig. 2.4: Velocity profile $f'(\eta)$ verses η for various values of Re when $M = 0.1$.

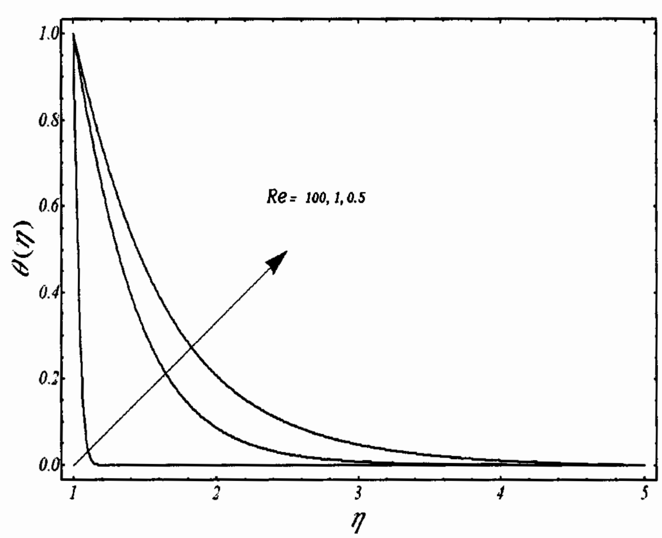

Fig. 2.5: Temperature profile $\theta(\eta)$ verses η for various values of Re when $M = 0.1$ and $\Pr = 7$.

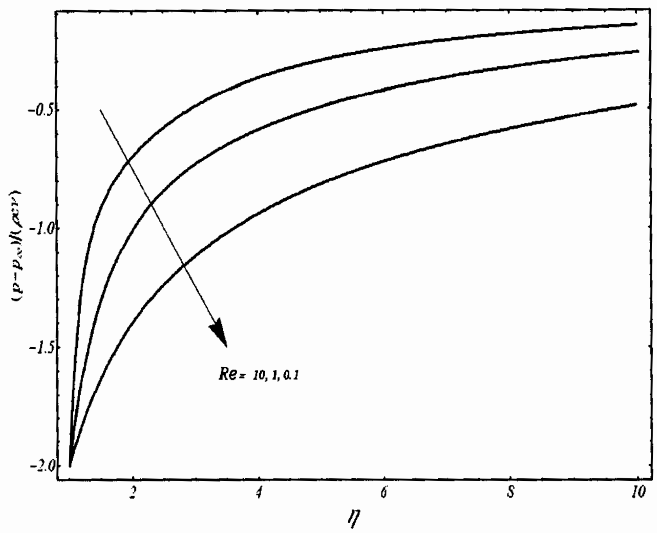

Fig. 2.6: Pressure distribution $(p - p_{\infty})/(\rho c v)$ obtained from Eq. (2.10) for various values of Re.

Figs. $2.2 - 2.6$ are plotted to see the effect of the Reynolds number Re, the Prandtl number Pr and magnetic parameter M on the velocity field $f'(\eta)$ and temperature profile $\theta(\eta)$. Tables $2.1 - 2.4$ show the numerical values of the skin friction coefficient and the Nusselt number. Fig. 2.2 shows the velocity profile $f'(\eta)$ for various values of the magnetic parameter M with $\text{Re} = 10$. The velocity profile show that the rate of transport is significantly reduced with the increase of the magnetic field. This reduction in the velocity curves is caused by the fact that variation of M, leads to produce more resistance to transport phenomena. Fig. 2.3 presents the change in temperature profiles for various values of M, and Pr when $\text{Re} = 10$, both (solid line for $\text{Pr} = 0.7$) and (dashed line for $\text{Pr} = 7$) when $\text{Re} = 10$. The temperature is started to increase as M increases, but it decreases as the distance from the surface increases, and lastly vanishes at some large distance from the surface. For a particular value of M, the thermal boundary layer thickness decreases as Pr increases. Figs. 2.4 and 2.5 demonstrate the velocity and temperature profiles respectively, for various values of Reynolds number Re with $\text{Pr} = 0.7$ and $M = 0.1$. It is noted that the Reynolds number Re indicates the relative significance of the inertia effect compared to the viscous effect. It is also found that both the velocity and temperature profiles decrease as Re increases, which shows similar results as those of Wang [18] for $M = 0$. It is also noted that boundary layer thickness decreases as Re increases. Fig.

2.6 gives the pressure distribution for different values of Reynolds number Re. All curves show that the pressure far away from the surface. Moreover, smaller values of Re result in slower algebraic decay. Table 2.1 show the values of the skin friction coefficient for small values of M when $\mathrm{Re} = 10$. The magnitude of the skin friction coefficients are increased as M increase. For small M, both the numerical and perturbation solutions are in good agreement. Table 2.2 gives the values of the Nusselt number $-\theta'(1)$ for different values of M and Pr when $\mathrm{Re} = 10$. It is noted that the local Nusselt number $-\theta'(1)$ decreases for large values of M and increases as Pr increases. The change in the skin friction $f''(1)$ for small values of M at $\mathrm{Re} = 1$ and 5 is given in table 2.3. The skin friction increases as both M and Re are increased. Table 2.4 presents the values of the Nusselt number $-\theta'(1)$ for small values of M when $\mathrm{Re} = 1100$ and $\mathrm{Pr} = 7$. It is found that the Nusselt number $-\theta'(1)$ increases as Re increases. The comparison of both numerical and perturbation solutions found in excellent agreement with each other and with the results discussed by Wang [18].

Chapter 3

Flow and heat transfer over a stretching cylinder in a porous medium

3.1 Introduction

In this chapter, we investigate the heat transfer analysis for MHD flow of a viscous fluid due to a stretching cylinder in a porous medium. The effects of the thermal radiation is also considered. The governing non-linear partial differential equations are transformed into a system of non-linear ordinary differential equations by employing similarity transformations. The resulting non-linear ordinary differential equations are solved numerically using the implicit finite difference scheme, known as Keller box method and analytically using perturbation technique. The physical influences of the involving parameters on the flow and temperature fields are discussed through graphs and tables. A comparison is made between these two solutions and found in excellent agreement. This chapter is an extension of the work done by Ishak et al [21].

3.2 Formulation of the problem

Consider, steady two-dimensional flow of an incompressible viscous fluid in a porous medium over a stretching cylinder of a radius 'a' in the axial direction, where the z-axis is considered along the axis of the cylinder and the r-axis is taken in the radial direction. The temperature at the surface of the cylinder is constant denoted by 'T_w', and the ambient fluid temperature is T_∞, where $T_w > T_\infty$. A constant magnetic field of strength $\mathbf{B} = (B_0, 0, 0)$ is applied in the radial direction and the induced magnetic field is assumed to be negligible, which is valid when the magnetic Reynolds number is small. Under these assumptions, the equations for the flow and energy are given as [21, 24]:

$$\frac{\partial(rw)}{\partial z} + \frac{\partial(ru)}{\partial r} = 0, \tag{3.1}$$

$$w\frac{\partial w}{\partial z} + u\frac{\partial w}{\partial r} = \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{\partial w}{r\partial r}\right) - \frac{\sigma B_0^2 w}{\rho} - \frac{\nu}{k_1}w, \tag{3.2}$$

$$w\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r\partial r} - \frac{u}{r^2}\right), \tag{3.3}$$

$$w\frac{\partial T}{\partial z} + u\frac{\partial T}{\partial r} = \frac{k}{\rho c_p}\frac{\partial}{r\partial r}\left(r\frac{\partial T}{\partial r}\right) - \frac{1}{\rho c_p}\frac{\partial}{r\partial r}\left(rq_r\right). \tag{3.4}$$

Where u and w are the velocity components in the r- and z- directions respectively, ρ is the density of fluid, ν is the kinematic viscosity, σ is the electrical conductivity, k_1 is the permeability of the porous medium, T is the temperature, c_p is the specific heat, k is the thermal conductivity of the fluid and q_r is the radiative heat flux.

Under the Rosseland approximation for radiation (Brewster [25]), the radiative heat flux q_r is given by

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial r}, \tag{3.5}$$

where σ^* is the Boltzman constant, and k^* is the mean absorption coefficient. Under the assumptions of temperature differences within the flow are sufficiently small, we may express the term T^4 as a linear function of the temperature in a Taylor series about T_∞ and neglecting higher terms, one can obtain

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4. \tag{3.6}$$

With the help of Eqs. (3.5) and (3.6), Eq. (3.4) can be written as

$$w\frac{\partial T}{\partial z} + u\frac{\partial T}{\partial r} = \frac{k}{\rho c_p}\left(1 + \frac{16T_\infty^3\sigma^*}{3k^*k}\right)\frac{\partial}{r\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3.7}$$

subject to the following boundary conditions

$$\left.\begin{aligned} u(r,z)=0, \quad w(r,z)=w_w, \quad T=T_w \quad \text{at } r=a, \\ w(r,z)\to 0, \quad T\to T_\infty, \quad \text{as} \quad r\to\infty. \end{aligned}\right\} \tag{3.8}$$

Where $w_w = 2cz$, is the stretching velocity and c is a constant (> 0), has the dimension $(\text{time})^{-1}$. To simplify the problem, we introduce the following similarity variables

$$\left.\begin{aligned} \eta=\left(\tfrac{r}{a}\right)^2, \quad u=-ca\left(f(\eta)/\sqrt{\eta}\right), \quad w=2cf'(\eta)z, \\ \theta(\eta)=(T-T_\infty)/(T_w-T_\infty). \end{aligned}\right\} \tag{3.9}$$

Using Eq. (3.9), the continuity Eq. (3.1) is automatically satisfied. Eqs. (3.2) and (3.7) become

$$\eta f''' + f'' - \lambda f' + \mathrm{Re}(ff'' - f'^2) = 0, \tag{3.10}$$

$$\eta\theta'' + (1 + k_0\,\mathrm{Re}\,\mathrm{Pr}\,f)\theta' = 0, \tag{3.11}$$

$$\left.\begin{aligned} f(1)=0, \qquad f'(1)=1, \qquad f'(\infty)=0, \\ \theta(1)=1, \qquad \theta(\infty)=0. \end{aligned}\right\} \tag{3.12}$$

Here a prime denotes the differentiation with respect to η, $\mathrm{Pr} = \mu c_p/k$, is the Prandtl number, $\mathrm{Re} = ca^2/2\upsilon$ is the Reynolds number, $Rd = k^*k/4\sigma^*T_\infty^3$ is the radiation parameter and $k_0 = 3Rd/(3Rd+4)$, and $\lambda = \left(\sigma B_0^2 a^2/4\nu\rho + a^2/4k_1\right)$ is a combined parameter due to an applied magnetic field and the permeability of a porous medium. It is noted that, for non-conductivity fluids, $\sigma = 0$, and as a result $\lambda = a^2/4k_1$ corresponds to the classical permeability parameter, and for non porous medium $k_1 \to \infty$, as a result, $\lambda = \sigma B_0^2 a^2/4\nu\rho$ corresponds to the classical Hartman number or the magnetic parameter. It is also worth mentioning here that the classical energy equation (3.11), without thermal radiation influences can be obtained by taking $Rd \to \infty$ (i.e. $k_0 \to 1$).

After finding the values of velocity from Eq. (3.10), one can be determined the pressure from Eq. (3.3) in the following form

$$\frac{p - p_\infty}{\rho c\nu} = -\frac{\mathrm{Re}}{\eta} f^2(\eta) - 2f'(\eta). \tag{3.13}$$

The skin friction coefficient and the Nusselt number are defined as

$$C_f = \frac{\tau_w}{\rho w_w^2/2}, \qquad Nu = \frac{a q_w}{k(T_w - T_\infty)}. \tag{3.14}$$

The shear stress τ_w at the wall and heat flux q_w at the wall are given by

$$\tau_w = \mu \left(\frac{\partial w}{\partial r}\right)_{r=a}, \qquad q_w = -k\left(\left(1 + \frac{16T_\infty^3\sigma^*}{3k^*k}\right)\left(\frac{\partial T}{\partial r}\right)\right)_{r=a}. \tag{3.15}$$

Using Eqs. (3.9) and (3.15), from Eq. (3.14), we get

$$C_f(\mathrm{Re}\, z/a) = f''(1), \qquad Nu = -\frac{2}{k_0}\theta'(1). \tag{3.16}$$

3.3 Solution of the problem

3.3.1 Perturbation solution for small λ

Following Ishak et al. [21] and Ariel [23], we can get approximate solutions of Eqs. (3.10) and (3.11) with boundary conditions (3.12), which is valid for small $\lambda << 1$. We assume the solution of the form

$$f(\eta) = \sum_{i=0}^{\infty} f_i(\eta)\lambda^i, \tag{3.17}$$

$$\theta(\eta) = \sum_{i=0}^{\infty} \theta_i(\eta)\lambda^i. \tag{3.18}$$

Using Eqs. (3.17) and (3.18) into (3.10) and (3.11), we obtain the following set of equations and boundary conditions.

Zeroth-order system:

$$\begin{cases} \eta f_0''' + f_0'' + \mathrm{Re}(f_0 f_0'' - f_0'^2) = 0, \\ \eta\theta_0'' + (1 + k_0 \,\mathrm{Re}\,\mathrm{Pr}\, f_0)\,\theta_0' = 0, \\ f_0(1) = 0, \quad f_0'(1) = 1, \quad f_0'(\infty) = 0, \\ \theta_0(1) = 1, \qquad \theta_0(\infty) = 0. \end{cases} \tag{3.19}$$

First-order system:

$$\begin{cases} \eta f_1''' + f_1'' + \mathrm{Re}(f_0 f_1'' + f_1 f_0'' - 2 f_0' f_1') = 0, \\ \eta \theta_1'' + \mathrm{Re}\,\mathrm{Pr}\, f_1 \theta_0' + (1 + k_0\, \mathrm{Re}\,\mathrm{Pr}\, f_0)\, \theta_1' = 0, \\ f_1(1) = 0, \quad f_1'(1) = 0, \quad f_1'(\infty) = 0, \\ \theta_1(1) = 0, \qquad \theta_1(\infty) = 0. \end{cases} \tag{3.20}$$

Second-order system:

$$\begin{cases} \eta f_2''' + f_2'' - f_1' + \mathrm{Re}(f_0 f_2'' + f_1 f_1'' - f_2 f_0'' - 2 f_0' f_2' - f_1'^2) = 0, \\ \eta \theta_2'' + k_0\, \mathrm{Re}\,\mathrm{Pr}\, f_1 \theta_1' + k_0\, \mathrm{Re}\,\mathrm{Pr}\, f_2 \theta_0' + (1 + k_0\, \mathrm{Re}\,\mathrm{Pr}\, f_0)\, \theta_2' = 0, \\ f_2(2) = 0, \quad f_2'(1) = 0, \quad f_2'(\infty) = 0, \\ \theta_2(1) = 0, \qquad \theta_2(\infty) = 0. \end{cases} \tag{3.21}$$

In general, we can write the solutions of the Eqs. (3.19 – 3.21) in the following form.

$$\eta f_i''' + f_i'' - f_{i-1}' + \mathrm{Re}(\sum_{j=0}^{i} f_j f_{i-j}'' - \sum_{j=0}^{i} f_j' f_{i-j}') = 0, \tag{3.22}$$

$$\eta \theta_i'' + \theta_i' + k_0\, \mathrm{Re}\,\mathrm{Pr} \sum_{j=0}^{i} f_j \theta_{i-j}' = 0. \tag{3.23}$$

for $i \geq 1$,

$$\left. \begin{array}{c} f_i(1) = 0, \quad f_i'(1) = \delta_{i0}, \quad \theta_i(1) = \delta_{i0}, \\ f_i'(\infty) = 0, \quad \theta_i(\infty) \to 0. \end{array} \right\} \tag{3.24}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.25}$$

The above non-linear ordinary differential Eqs. (3.22) and (3.23) with the help of boundary conditions (3.24) are solved numerically for small values of λ using the shooting method with Runge-Kutta algorithm. The skin friction coefficient and the Nusselt number in Eqs. (3.16) are approximately given by

$$\left.\begin{aligned} C_f(\mathrm{Re}\, z/a) &= f_0''(1) + f_1''(1)\lambda + f_2''(1)\lambda^2, \\ Nu &= -2(\theta_0'(1) + \theta_1'(1)\lambda + \theta_2'(1))\lambda^2)/k_0. \end{aligned}\right\} \tag{3.26}$$

3.3.2 Numerical solution

The non-linear ordinary differential Eqs. (3.10) and (3.11) subject to the boundary conditions (3.12) have been solved numerically using an implicit finite difference scheme known as Keller box method, which is described in the book by Cebeci and Bradshaw [22] in detail.

3.4 Result and discussion.

Figs. 3.1 – 3.7 are made in order to see the effects of the involving parameters, for example, λ, Re, Pr and Rd, on the velocity component $f'(\eta)$ and the temperature field $\theta(\eta)$. The values of the skin friction coefficient and Nusselt number for different parameters are given in Table 3.1 – 3.4. The result for $\lambda << 1$, in terms of perturbation solutions are incorporated in these tables.

Table 3.1: Numerical values of skin friction coefficient $f''(1)$ for several values of λ when Re = 10.

λ	Numerical	Small λ
0	−3.3444	−3.3445
0.01	−3.3461	−3.3462
0.05	−3.3528	−3.3529
0.1	−3.3612	−3.3613
0.5	−3.4274	−3.4273
1	−3.5076	
2	−3.6615	
5	−4.0825	

Table 3.2: Numerical values of the Nusselt number Nu for several values of λ and Pr when $Rd = 0.2$ and Re $= 10$.

λ	Pr = 0.7 (air)		Pr = 7 (water)	
	Numerical	Small λ	Numerical	Small λ
0	3.0721	3.0721	14.3323	14.3323
0.01	3.0710	3.0710	14.3289	14.3289
0.05	3.0667	3.0667	14.3156	14.3156
0.1	3.0615	3.0615	14.2991	14.2991
0.5	3.0234	3.0262	14.1709	14.1726
1	2.9821		14.0186	
2	2.9130		13.7380	
5	2.7645		12.9924	

Table 3.3: Numerical values of the skin friction coefficient $f''(1)$ for several values of λ and Re.

λ	Re = 1		Re = 5	
	Numerical	Small λ	Numerical	Small λ
0	−1.1780	−1.1780	−2.4174	−2.4175
0.01	−1.1839	−1.1839	−2.4199	−2.4199
0.05	−1.2068	−1.2068	−2.4296	−2.4297
0.1	−1.2344	−1.2339	−2.4417	−2.4418
0.5	−1.4269	−1.3928	−2.5352	−2.5338

Table 3.4: Numerical values of the Nusselt number Nu for several values of λ, Re and Rd when Pr = 7.

		Re = 1		Re = 100	
λ	Rd	Numerical	Small λ	Numerical	Small λ
0	0.7	3.4180	3.4180	30.1440	30.1441
0.01		3.4131	3.4131	30.1438	30.1437
0.05		3.3942	3.3944	30.1425	30.1420
0.1		3.3718	3.3730	30.1409	30.1400
0.5		3.2187	3.2847	30.1274	30.1266
0.1	0.0	2.0450		19.1576	
	0.7	3.3718		30.1409	
	1	3.0499		27.5730	
	1.5	2.7647		25.2457	
	2	2.6071		23.9388	
	3	2.4368		22.5101	
	5	2.2895		21.2603	

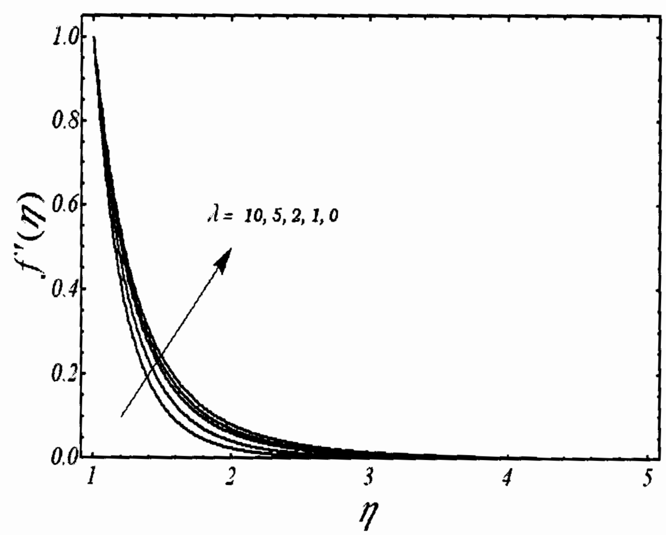

Fig. 3.1: Velocity profile $f'(\eta)$ against η for various values of λ when Re = 10.

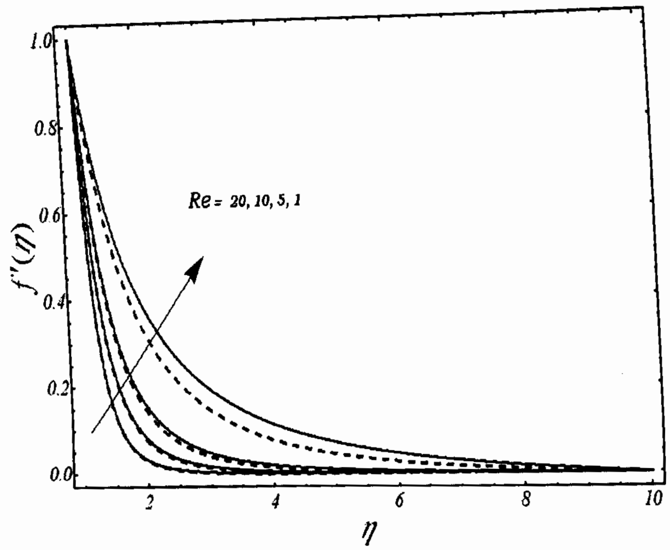

Fig. 3.2: Velocity profile $f'(\eta)$ against η for various values of Re: solid lines for $\lambda = 0$ and dashed lines for $\lambda = 0.1$.

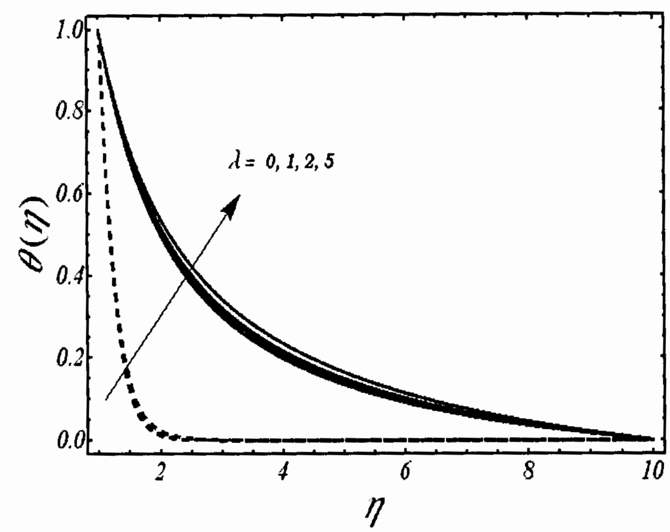

Fig. 3.3: Temperature profile $\theta(\eta)$ against η for various values of λ: solid lines for Pr $= 0.7$ and dashed lines for Pr $= 7$ when $Rd = 0.7$ and Re $= 10$.

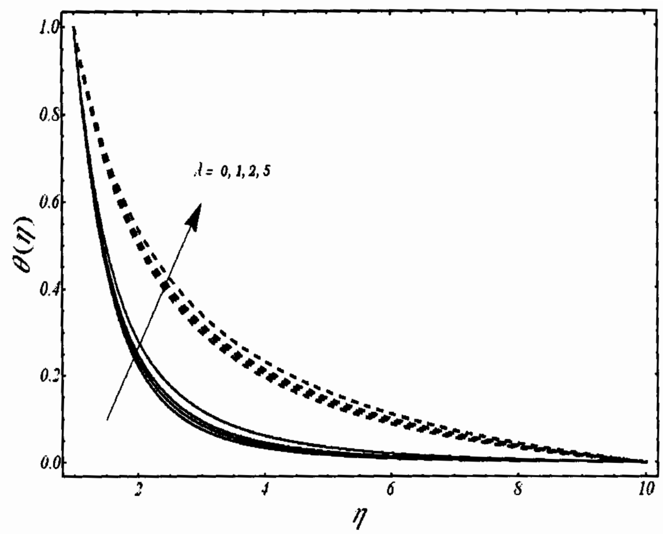

Fig. 3.4: Temperature profile $\theta(\eta)$ against for various values of λ: solid lines for $Rd \to \infty$ (i.e. $k_0 \to 1$) and dashed lines for $Rd = 0.7$ when $\Pr = 0.7$ and $\mathrm{Re} = 10$.

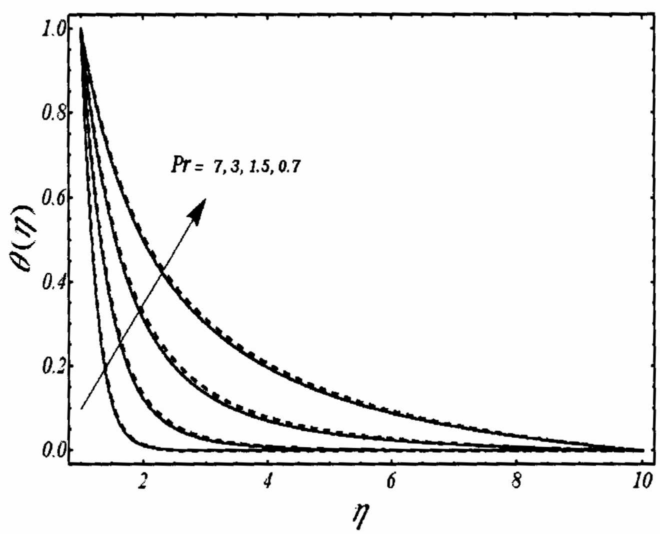

Fig. 3.5: Temperature profile $\theta(\eta)$ against η for several values of Pr: solid lines for $\lambda = 0$ and dashed lines for $\lambda = 1$ when $Rd = 0.7$ and $\mathrm{Re} = 10$.

values of λ and Pr when $Rd = 0.2$ and $\text{Re} = 10$. It can be seen from this table that the values of the Nusselt number decreases as λ increases. It is also found that as we increase the values of Pr, the temperature gradient at the surface of the cylinder increases. The numerical value of the skin friction coefficient $f''(1)$ for various values of Re is given in Table 3.3 when $0 \leq \lambda \leq 0.5$ (small values of λ). For particular values of λ, the skin friction coefficient is increased by increasing the values of Re. Table 3.4 shows the numerical values of the Nusselt number Nu for different values of the λ, Re, Rd when $\text{Pr} = 7$ is fixed. It is observed from this table that the temperature gradient at the wall or the Nusselt number is increased by increasing the values of the Rd and Re. In all these tables, the comparison between the numerical and perturbation solutions is given for small values of λ $(0 \leq \lambda \leq 0.5)$ and found in good agreement.

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