Fixed Point Results for Weak Contractions in Generalized Metric Spaces

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***DOCTOR OF PHILOSOPHY IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD.

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Author's Declaration

I, **Tahair Rasham** Reg. No. **57-FBAS/PHDMA/F15** hereby state that my Ph.D. thesis titled: **Fixed Point Results for Weak Contractions in Generalized Metric Spaces** is my own work and has not been submitted previously by me for taking any degree from this university, **International Islamic University, Sector H-10, Islamabad, Pakistan** or anywhere else in the country/world.

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This is to certify that the research work presented in this thesis, entitled: **Fixed Point Results for Weak Contractions in Generalized Metric Spaces** was conducted by **Mr. Tahair Rasham,** Reg. No. **57-FBAS/PHDMA/F15** under the supervision of **Prof. Dr. M. Arshad Zia** no part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Department of Mathematics & Statistics, FBAS, IIU, Islamabad** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics, Department of Mathematics & Statistics, Faculty of Basic & Applied Science, International Islamic University, Sector H-10, Islamabad, Pakistan.**

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Dedicated to....

In the memory of my beloved father Muhammad Rasham Khan (Late) and my younger brother Mr. Shahid Rasham (Late) for their love and care in my childhood. I also dedicate this thesis for my mother and family who prayed for my success throughout my carrier.

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List of Publications

The list of the research articles, deduced from the work presented in this thesis, published in the international journals of ISI/ non ISI ranking, is given below.

1. T. Rasham, A. Shoaib, M. Arshad, and S. U. Khan, Fixed point results on closed ball for a new rational type contraction mapping in complete dislocated metric spaces, J. Inequal. Spec. Funct., 8(2017), 74-85.
2. T. Rasham, A. Shoaib, N. Hussain, M. Arshad, S. U. Khan, Common fixed point results for new Ciric-type rational multivalued F contraction with an application, J. Fixed Point Theory and Appl., 20(1) (2018), 1-16 pages.
3. T. Rasham, A. Shoaib, B. S. Alamri, M. Arshad, Multivalued fixed point results for new generalized F-Dominted contractive mappings on dislocated metric space with application, Journal of Function Spaces, 2018(2018), Article ID 4808764, 12 pages.
4. T. Rasham, A. Shoaib, B. S. Alamri, M. Arshad, Fixed Point Results for Multivalued Contractive Mappings Endowed With Graphic Structure, Journal of Mathematics, 2018(2018), Article ID 5816364, 8 pages.
5. T. Rasham, A. Shoaib, C. Park, M. Arshad, Fixed Point Results for a Pair of Multi Dominated Mappings on a Smallest Subset in K-Sequentially Dislocated quasi Metric Space with Application, J. Comput. Anal. Appl. 25(5) (2018), pages 975-986.
6. T. Rasham, A. Shoaib, N. Hussain, B. S. Alamri, M. Arshad, Multivalued Fixed Point Results in Dislocated b-Metric Spaces with Application to the System of Nonlinear Integral Equations, Symmetry 2019, 11(1), 40; https://doi.org/10.3390/sym11010040 (registering DOI).
7. A. Shoaib, T. Rasham, N. Hussain, M. Arshad, Fixed point results for a pair of multivalued dominated mappings in dislocated b-metric spaces with applications, Journal of the National Science Foundation of Sri Lanka, 49(2).
8. A. Shoaib, T. Rasham, A. A. Rawashdeh, M. Arshad, DQF-contraction and related fixed point results in DQM spaces with application, J. Math. Anal., 16 pages, (Accepted).

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Preface

fîxȩd þoīnţ theory is an exceptional combination of analysis, topology and geometry. Banach [19] proved a very useful result for ĉonţracţioñ maþþings. Afterward, a huge amount of fîxȩd þoīnţ řesulţs has been published by different authors and they devolped different aspects of the Banach's result. In current literature, various řesulţs have been analysed about that the fîxȩd þoīnţ of maþþings have ĉonţracţioñ over the whole śþace. Let us start with the initial þoint $p_0 \in E$ and define an iterative şequenĉe $\{p_n\}$ of the form $p_{n+1} = Tp_n$ for all $n \geq 0$. We shall assume that $p_n \neq p_{n+1}$ for ēverŷ n. Otherwise, there exists n such that $p_n = p_{n+1}$. Then we will show that $p_n = Tp_n$ and p_n has a fîxȩd þoīnţ of T. To apply ĉonţracţioñ, restriction and further application of the theorems we shall obtain $\lim\limits_{n\to\infty} d(p_n, p_{n+1}) = 0$. Now we are to prove that the şequenĉe $\{p_n\}$ be a Ĉauchŷ şequenĉe in E. Since E is ĉomþleţe mètric śþace. Eaċh Ĉauchŷ şequenĉe $\{p_n\}$ in a ĉomþleţe mètric śþace E converges to a þoint p in E, so $\{p_n\}$ converges to p and hence by using given conditions, we will prove p be a fîxȩd þoīnţ in E of T. Finally, we consider e be another fîxȩd þoīnţ of T we will prove that $p = e$. Hence T has a specific fîxȩd þoīnţ. It is simple to get fîxȩd þoīnţ for such maþþings if they satisfy certain conditions. It has been shown by Hussain et al. [26], the presence of fîxȩd þoīnţ for this ţype of maþþings that fulfill the conditions on a ĉlosed báll. Lateral, Beg et al. [20], proved the sufficient conditions on a ĉlosed báll in an ordered left(right)$-K$ Sequentially ĉomþleţe dislocated quasi mètric śþaces (see also [12, 13, 14, 54, 56, 61, 58, 59, 60]).

Nadler [40], discussed the fîxȩd þoīnţ reults concerned with multivalued maþþings. Several řesulţs on multivalued maþþings have been observed (see [5, 23, 36, 64] . Wardowski [65] introduced new kind of ĉonţracţioñ said $F-$ĉonţracţioñ and showed a new generalized fîxȩd þoīnţ theorem. He observed many previous fîxȩd þoīnţs in a different way. A lot of other řesulţs on $F-$ĉonţracţioñs can be observed in [3, 4, 6, 10, 11, 27, 32, 37, 42, 43, 52, 53]. The theory of setvalued maþs has a faundamental role in many kinds of both pure and applied maths because of its larger number of applications, in real analysis, geometry and complex analysis, algorithims, as well as in functional analysis. Over the years, above theory has raised its importance and hence in the current literature there are varied research articles related with multivalued maþþings. Various authors have discussed different research articles including practical problems and their solutions in multivalued maþþings. Due to the importance of this theory various approaches algorithims and techniques are applied for the developing of this theory. Shoaib et al. [61], discussed the result related to α_*-ψ-Ćirić ţype multifunctions on an intersection of a ȿequenće and ĉlosed báll along with graph.

We have achieved fîxȩd þoīnţ ȓesulţs for new generalized $F-$ĉonţracţioñ on an intersection of a ȿequenće with ĉlosed báll for a more general class of semi α_*-dominated maþþings rather than α_*-admissible maþþings and for a weaker class of strictly increasing maþþings F rather than class of maþþings F used by Wardowski [65]. The notion of multi graph dominated maþþing is also introduced. fîxȩd þoīnţs related to graphic ĉonţracţioñs on a ĉlosed báll for this kind of maþs are developed. Applications are given to investigate the unique common solution of nonlinear Voltera ţype integral equations. Moreover, we investigate our ȓesulţs in a better framework. In 1974, Ćirić [24], introduced quasi ĉonţracţioñ.

This thesis deals with the fîxȩd þoīnţs for weak ĉonţracţioñs in generalized mètric ŝþaces. In this thesis, overview of the fîxȩd þoīnţ theory, fîxȩd þoīnţs for various contractive maþs, fîxȩd þoīnţ ȓesulţs in different mètric ŝþaces, various approaches and methods are discussed. We shall establish new ţypes of fîxȩd þoīnţs for setvalued maþs concerning weak ĉonţracţioñs in generalized mètric ŝþaces. Our findings are depended only for the fact that fîxȩd þoīnţs involving ĉonţracţioñs can be obtained by fîxȩd þoīnţ theory for maþs in different generalized mètric ŝþaces. In our research work, common fîxȩd þoīnţ ȓesulţs locally and globálly contractive maþs in dislocated, dislocated $b-$mètric, and dislocated quasi mètric ŝþaces have been established. New contractive conditions have been introduced. Our ȓesulţs extended some previous theorems to generalized mètric ŝþaces and also restrict that the contractive conditions hold only for sub ŝþace rather than whole ŝþace. Furthermore, we have applied the idea of dominated maþs and weak contractive conditions for the presence of fîxȩd þoīnţs of setvalued contractive maþs in devolpement of generalized mètric ŝþaces. This thesis is based on four chapters. ēverŷ chapter consists of vast introduction having huge findings of material in it.

Chapter 1, is a prospect, of definitions about some generalized mètric ŝþaces for their ĉomþleţeness convergence and Lemmas to determine and recall basic concepts.

Chapter 2, is the study of some fîxȩd þoīnţs for multivalued maþs on generalized rational ţype ĉonţracţioñs. Some fîxȩd þoīnţ ȓesulţs are established in setting of dislocated mètric ŝþace.

In addition to, we have discussed about the fixed points of setvalued F-dominated maps in these spaces.

Chapter 3, discuss the study of some common fixed points for Ćirić type rational multivalued mappings in dislocated b-metric spaces. Furthermore, we introduce the concept of multivalued fixed points for α_*-admissible mapping endowed with graphic structure. Some common fixed point results for a pair of α_*-dominated multivalued mappings on closed ball with applications in dislocated b-metric spaces.

Chapter 4, deals with the some fixed point results in framework of dislocated quasi metric spaces. We delovped fixed point results for α_*-dominated multivalued mappings satisfying generalized $\alpha_* - \Psi$ Ćirić type contraction on dislocated quasi metric spaces. Moreover, we have discussed some fixed points for Ćirić kind rational setvalued F- contractive mappings with applications in these spaces.

I wish to acknowledge that a teacher is a guardian of civilization and it is really true in case of my honourable Supervisor Professor Dr. Muhammad Arshad and Co-Supervisor Dr. Abdullah Shoaib. I pay my humblest gratitude to my teachers who always have been kind to me for the completion of my Ph.D. thesis.

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Tahair Rasham

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Pakistan.

Chapter 1

Introduction and Preliminaries

This chapter aims at developing some clear notions and to explain the nomenclature used in thesis. This chapter discusses some previous well known definitions and ȓesulʈs. Section 1.1, is a vast discussion about the basic material of dislocated mètric ŝþaces. Section 1.2, is related to the basics of dislocated b-mètric ŝþaces. Section 1.3, consists of the concepts of dislocated quasi mètric ŝþaces. Section 1.4, is about some other faundamental basic notions related to it.

1.1 Dislocated Metric Spaces

Definition 1.1.1 [64] Let $\acute{Z} \neq \{\}$ and the maþþing $d : \acute{Z} \times \acute{Z} \to [0, \infty)$ is said a dislocated mètric if (i),(ii) and (iii) stisfiy, for any $q, p, \check{z} \in \acute{Z}$:

(i) If $d_l(q,p) = 0$, then $q = p$;

(ii) $d_l(q,p) = d_l(p,q)$;

(iii) $d_l(q,p) \leq d_l(q,\check{z}) + d_l(\check{z},p)$.

The pair $(\acute{Z}, d_l)$ is said dislocated mètric ŝþace or d_l mètric ŝþace. It is obvious that if $d_l(q,p) = 0$, then from (i), $q = p$. But if $q = p$, $d_l(q,p)$ may not be 0. We use $D.M.S$ instead by dislocated mètric ŝþace.

Definition 1.1.2 [64] Let $(\acute{Z}, d_l)$ is a $D.M.S$.

(i) A şequenće $\{c_{\breve{n}}\}$ in $(\acute{Z}, d_l)$ is said a Ĉauchŷ şequenće if given $\varepsilon > 0$, there must be $\breve{n}_0 \in N$ such that for ēverŷ $\breve{n}, m \geq \breve{n}_0$ we have $d_l(c_m, c_{\breve{n}}) < \varepsilon$ or $\lim\limits_{\breve{n},m\to\infty} d_l(c_{\breve{n}}, c_m) = 0$.

(ii) A şequenće $\{c_{\breve{n}}\}$ dislocated-converges to l if $\lim\limits_{\breve{n}\to\infty} d_l(c_{\breve{n}}, l) = 0$. In this case l is said a

d_l-limit of $\{c_{\check{n}}\}$.

(iii) $(\acute{Z}, d_l)$ is said ĉomþleţe if ėaċh Ĉauchŷ şequenĉe in $\acute{Z}$ converges to a þoint $l \in \acute{Z}$.

Definition 1.1.3 [61] Let $H \neq \{\}$ subset of $D.M.S$ of $\acute{Z}$ and let $\acute{\imath} \in \acute{Z}$. As $v_0 \in H$ is said to be a best aþþroximation in H if

$$d_l(\acute{\imath}, H) = d_l(\acute{\imath}, v_0), \text{ where } d_l(\acute{\imath}, H) = \inf_{y \in H} d_l(\acute{\imath}, y).$$

If ēverŷ $\acute{\imath} \in \acute{Z}$ has at most one best aþþroximation in H, then H is set a þroximinal set. We denote $P(\acute{Z})$ be the set of all ĉlosed þroximinal subsets of $\acute{Z}$.

Definition 1.1.4 [61] The function $H_{d_l} : P(\acute{Z}) \times P(\acute{Z}) \to R^+$, defined by

$$H_{d_l}(N, M) = \max\{\sup_{\check{n} \in N} d_l(\check{n}, M), \sup_{m \in M} d_l(N, m)\}$$

is said dislocated Hausdorff mėtric on $P(\acute{Z})$.

Example 1.1.5 [64] If $\acute{Z} = R^+ \cup \{0\}$, then $d_l(j, k) = j + k$ is a dislocated mėtric d_l on $\acute{Z}$.

Lemma 1.1.6 [46] Let $(\acute{Z}, d_l)$ be a $D.M.S$. Let $(P(\acute{Z}), H_{d_l})$ is a dislocated Hausdorff mėtric śþace on $P(\acute{Z})$. Then for ēverŷ $G, H \in P(\acute{Z})$ and for ėaċh $g \in G$ there must be a $h_g \in H$ satisfies $d_l(g, H) = d_l(g, h_g)$ then $H_{d_l}(G, H) \geq d_l(g, h_g)$.

1.2 Dislocated b-Metric Spaces

Definition 1.2.1 [29] Let $M \neq \{\}$ and let $d_b : M \times M \to [0, \infty)$ is a function, said a dislocated b-mėtric, if for ēverŷ $g, q, \dot{z} \in M$, and $t \geq 1$ the followings hold:

(i) If $d_b(g, q) = 0$, then $g = q$;

(ii) $d_b(g, q) = d_b(q, g)$;

(iii) $d_b(g, q) \leq t[d_b(g, \dot{z}) + d_b(\dot{z}, q)]$.

The pair (M, d_b) is said to be a dislocated b-mėtric śþace. It is obvious that if $d_b(g, q) = 0$, then from (i), $g = q$. But if $g = q$, $d_b(g, q)$ may not be 0. For $g \in M$ and $\varepsilon > 0$, $\overline{B(g, \varepsilon)} = \{q \in M : d_b(g, q) \leq \varepsilon\}$ is a ĉlosed bȧll in (M, d_b). We use $D.B.M.S$ instead dislocated b-mėtric śþace.

Definition 1.2.2 [29] Let (M, d_b) be a $D.B.M.S$.

(i) A şequenće $\{g_n\}$ in (M, d_b) is called Ĉauchŷ şequenće if given $\varepsilon > 0$, there exist $n_0 \in N$ such that for all $n, m \geq n_0$ we have $d_b(g_m, g_n) < \varepsilon$ or $\lim_{n,m \to \infty} d_b(g_n, g_m) = 0$.

(ii) A şequenće $\{g_n\}$ dislocated b-converges (for short d_b -converges) to g if $\lim_{n \to \infty} d_b(g_n, g) = 0$. In this case g is called a d_b-limit of $\{g_n\}$.

(iii) (M, d_b) is said ĉomþleţe if ēverŷ Ĉauchŷ şequenće in M converges to a þoint $g \in M$.

Definition 1.2.3 [49] Let $\hat{H} \neq \{\}$ subset of $D.B.M.S$ of M and let $g \in M$. As $q_0 \in \hat{H}$ is said a best aþþroximation in $\hat{H}$ if

$$d_b(g, \hat{H}) = d_b(g, q_0), \text{ where } d_b(g, \hat{H}) = \inf_{q \in \hat{H}} d_b(g, q).$$

Definition 1.2.4 [51] Let $B, A : M \to P(M)$ be the ĉlosed valued mulifunctions and $\beta : M \times M \to [0, +\infty)$ be a function. We utter that the pair (B, A) is β_*-admissible if for ėaċh $g, q \in M$

$$\beta(g, q) \geq 1 \Rightarrow \beta_*(Bg, Aq) \geq 1, \text{ and } \beta_*(Ag, Bq) \geq 1,$$

where $\beta_*(Ag, Bq) = \inf\{\beta(\bar{a}, b) : \bar{a} \in Ag,\ b \in Bq\}$. When $B = A$, then we obtain the definition of α_*-admissible maþþing given in [9].

Definition 1.2.5 [8] Let (M, d_b) be a $D.B.M.S$, $B : M \to P(M)$ be the setvalued maþþing and $\alpha : M \times M \to [0, +\infty)$. Let $Q \subseteq M$, we utter that the B is semi α_*-admissible on Q, when $\alpha(g, q) \geq 1$ implies $\alpha_*(Bg, Bq) \geq 1$ for all $g, q \in Q$, where $\alpha_*(Bg, Bq) = \inf\{\alpha(\bar{a}, b) : \bar{a} \in Bg,$ $b \in Bq\}$. If $Q = M$, then we utter that the B is α_*-admissible on M.

Definition 1.2.6 [55] The function $H_{d_b} : P(M) \times P(M) \to R^+$, interpreted, by

$$H_{d_b}(\bar{A}, B) = \max\{\sup_{\bar{a} \in \bar{A}} d_b(\bar{a}, B),\ \sup_{b \in B} d_b(\bar{A}, b)\}$$

is said dislocated Hausdorff b-mėtric on $P(M)$.

Example 1.2.7 [29] If $M = \mathbb{R}^+ \cup \{0\}$, then $d_b(g, q) = (g + q)^2$ defines a $D.B.M$ d_b on M.

Lemma 1.2.8 [49] Let (M, d_b) be a $D.B.M.S$. Let $(P(M), H_{d_b})$ is a dislocated Hausdorff b-mėtric śþace on $P(M)$. Then for ēverŷ $\bar{A}, B \in P(M)$ and $\forall\ \bar{a} \in \bar{A}$ there exists $b_{\bar{a}} \in B$ holds $d_b(\bar{a}, B) = d_b(\bar{a}, b_{\bar{a}})$ then $H_{d_b}(\bar{A}, B) \geq d_b(\bar{a}, b_{\bar{a}})$.

1.3 Dislocated Quasi Metric Spaces

Definition 1.3.1 [66] Let $E \neq \{\}$ and $\delta_q : E \times E \to [0, \infty)$ is a function, said a dislocated quasi mètric if (i), (ii) and (iii) hold for ēverŷ $g, s, z \in E$:

(i) If $\delta_q(g, s) = \delta_q(s, g) = 0$, then $g = s$;

(ii) $\delta_q(g, s) \leq \delta_q(g, z) + \delta_q(z, s)$.

The pair (E, δ_q) is said a DQM.

If $\delta_q(g, s) = \delta_q(s, g) = 0$, then from (i), $g = s$. But if $g = s$, $\delta_q(g, s)$ need no be 0. It is noted that if $\delta_q(g, s) = \delta_q(s, g)$ for all $g, s \in E$, then (E, δ_q) becomes a DQM (mètric-like ŝþace) (E, δ_q). For $g \in E$ and $\varepsilon > 0$, $B_{\delta_q}(g, \varepsilon) = \{s \in E : \delta_q(g, s) < \varepsilon \text{ and } \delta_q(s, g) < \varepsilon\}$ and $\overline{B_{\delta_q}(g, \varepsilon)} = \{s \in E : \delta_q(g, s) \leq \varepsilon \text{ and } \delta_q(s, g) \leq \varepsilon\}$ are open and ĉlosed báll in (E, δ_q) respectively. Also $B_{d_l}(g, \varepsilon) = \{s \in E : \delta_q(g, s) \leq \varepsilon\}$ be the ĉlosed báll in (E, d_l). We use DQM for dislocated quasi mètric ŝþace.

Definition 1.3.2 [20] Let (E, δ_q) be a DQM.

(a) A şequenće $\{g_n\}$ in (E, δ_q) is said left K-Cauchy if $\forall\, \varepsilon > 0$, $\exists\, n_0 \in N$ so as $\forall\, n > m \geq n_0$ (for ēverŷ $m > n \geq n_0$), $\delta_q(g_m, g_n) < \varepsilon$.

(b) A şequenće $\{g_n\}$ dislocated quasi-converges to g if $\lim\limits_{n\to\infty} \delta_q(g_n, g) = \lim\limits_{n\to\infty} \delta_q(g, g_n) = 0$ or for any $\varepsilon > 0$, there must be a $n_0 \in N$, so as for ēverŷ $n > n_0$, $\delta_q(g, g_n) < \varepsilon$ and $\delta_q(g_n, g) < \varepsilon$. In above case g is called a δ_q-limit of $\{g_n\}$.

(c) (E, δ_q) is said K-sequentially ĉomþleţe if ėaċh K-Ĉauchŷ şequenće in E converges to a þoint $g \in E$ so as $\delta_q(g, g) = 0$.

Definition 1.3.3 [48] Let (E, δ_q) be a DQM. Let M be a nonempty subŝþace of E and let $g \in E$. An element $s_0 \in M$ is said a best aþþroximation in M if

$$\begin{aligned} \delta_q(g, M) &= \delta_q(g, s_0), \text{ where } \delta_q(g, M) = \inf_{s \in M} \delta_q(g, s) \\ \text{and } \delta_q(M, g) &= \delta_q(s_0, g), \text{ where } \delta_q(M, g) = \inf_{s \in M} \delta_q(s, g). \end{aligned}$$

If ēverŷ $g \in E$ has at minimal one best aþþroximation in M, then M is said a þroximinal set.

It is obvious that if $\delta_q(g, M) = \delta_q(M, g) = 0$, then $g \in M$. But, if $g \in M$, then $\delta_q(g, M)$ or $\delta_q(M, g)$ may not equal to zero. We represent $P(E)$ is the set of all ĉlosed subsets of E.

Definition 1.3.4 [51] Let $(S, T) : E \to P(E)$ and $\beta : E \times E \to [0, +\infty)$ is a function. We where $\beta_\star(Tg, Ss) = \inf\{\beta(a,b) : a \in Tx,\ b \in Ss\}$. When $S = T$ then we are left with single maþþing.

Definition 1.4.2 [65] Let (Z,d) is a mètric and the maþþing $H : Z \to Z$ is A −ĉonţracţioñ if there must be a $\tau > 0$ so as

$$\forall j, k \in Z,\ d(Hj, Hk) > 0 \Rightarrow \tau + A\left(d(Hj, Hk)\right) \leq A\left(d(j,k)\right)$$

with $A : \mathbb{R}_+ \to \mathbb{R}$ real function which satisfies three assumptions:

(F1) A is strictly increasing

(F2) For any şequenće $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers, $\lim_{n\to\infty} \alpha_n = 0$ is equivalen to $\lim_{n\to\infty} A(\alpha_n) = -\infty$;

(F3) There is $k \in (0,1)$ for which $\lim \alpha \to 0^+ \alpha^k A(\alpha) = 0$.

We represent by $\triangle_F$, the set of all functions holding from (F1)-(F3) conditions.

Example 1.4.3 [65] The Family of $\mathcal{F}$ is not empty.

1. $F(g) = \ln(g); g > 0.$
2. $F(g) = g + \ln(g); g > 0.$
3. $F(g) = \frac{-1}{\sqrt{g}}; g > 0.$

Example 1.4.4 [48] Let $E = \mathbb{R}$. Define $\alpha : E \times E \to [0, \infty)$ by

$$\alpha(g,s) = \left\{ \begin{array}{c} 1 \text{ if } g > s \\ \frac{1}{2} \text{ otherwise} \end{array} \right\}.$$

Define the setvalued maþs $S, T : E \to P(E)$ by

$$Sg = \{[g-4, g-3] \text{ if } g \in E\}$$

and,

$$Ts = \{[s-2, s-1] \text{ if } s \in E\}.$$

Suppose $x = 3$ and $y = 2$. As $3 > 2$, then $\alpha(3,2) \geq 1$. Now, $\alpha_\star(S3, T2) = \inf\{\alpha(a,b) : a \in S3, b \in T2\} = \frac{1}{2} \ngeq 1$, this means $\alpha_\star(S3, T2) < 1$, that is, the pair (S,T) is not $\alpha_\star$-admissible. Also, $\alpha_\star(S3, S2) \ngeq 1$ and $\alpha_\star(T3, T2) \ngeq 1$. This implies S and T are not $\alpha_\star$-

admissible individually. As, $\alpha_\star(g, Sg) = \inf\{\alpha(g, b) : b \in Sg\} \geq 1$, for ēverŷ $x \in X$. Hence S is $\alpha_\star$-dominated maþþing. Similarly $\alpha_\star(s, Ts) = \inf\{\alpha(s, b) : b \in Ts\} \geq 1$. Hence it is clear that S and T are not $\alpha_\star$-admissible but $\alpha_\star$-dominated.

Lemma 1.4.5 [60] ēverŷ ĉlosed báll Y in a left (right) K-sequentially ĉomþleţe DQM of E is left (right) K-sequentially ĉomþleţe.

Theorem 1.4.6 [65] Let (E, d) is a mėtric ŝþace and $T : E \to E$ be the F-ĉonţracţioñ. Then $g^* \in E$ is a unique fïxęd þoīnţ of maþþing T and for ėaċh $g \in E$ the şequenĉe $\{T^n g\}_{n \in \mathbb{N}}$ converges to g^*.

Note: We are using C.F.P instead common fïxęd þoīnţ in this thesis.

Chapter 2

Results in Dislocated Metric Spaces

2.1 Introduction

The given theory and results present in this section can be seen in [44, 45, 46].

The fĩxęd þoīnţ theory aims at devolping functional and non linear analysis. Banach [19] proved significant result for ĉonţracţioñ maþþings. Then, a large number of fĩxęd þoīnţ řesulţs were published by different authors and they developed a lot of generalizations of Banach's result.There are many related řesulţs about the fĩxęd þoīnţs of maþþings in which contractive condidions exist on prevail full šþace. It is very simple to show that $A : F \longrightarrow F$ may not be a ĉonţracţioñ but $A : J \longrightarrow F$ be a ĉonţracţioñ, where J is a subset in F. It is convenient to get fĩxęd þoīnţs for such maþþings if they satisfy certain condition. It has been shown by Hussain et al. [26], the presence of fĩxęd þoīnţ for such maþþings that fulfill the certain requirement on a ĉlosed bǻll.

The theory of setvalued maþs has a faundamental role in many ţypes of both pure and applied maths because of its large number of applications, in real analysis and complex analysis, algorithims in the same way in functional analysis. Over the past years, this theory has raised its importance and hence in the current literature there are various research articles related to multivalued maþþings. Nadler [40], underwent the basics of fĩxęd þoīnţs for the setvalued maþþings (see also [17]). Several řesulţs on setvalued maþþings have been observed (see [5, 23, 36, 64]. Wardowski [65] established new family of ĉonţracţioñ maþþings recalled as $F-$ĉonţracţioñ. He generalized many fĩxęd þoīnţ řesulţs in a different aspect. In mètric fĩxęd þoīnţ theory Wardowski, generalize the famous contraction theorem termed as Banach contraction theorem. We generalize $F-$contraction into Ćirić type rational multivalued mappings and showed the applications for nonlinear Voltera type integral equations. We succeded to generalize $F-$contraction by introducing a new Ćirić type rational $F-$contractive multivalued mappings. We further extended it to find fixed point by α_*-dominated multivalued mappings on closed báll. In this chapter we collected these two new ideas by introducing some new rational type multivalued contractive mappings and related fixed point theorem. Many fixed point results for such mappings have been already proved by various authors becomes the corollaries of our results. We show that many other newly fixed points for $F-$contraction in different mètric spaces can be obtained from our results.

From last ten years it can be seen that many authors proved fixed point results endowed with graph. We have applied new approach to proved fixed point results by using graph dominated for an advanced Ćirić type rational $F-$ contractive mappings on closed báll. Secelean [52] asserted fixed points regarding of $F-$contractions by using iterations system. Piri et al. [42] discussed fixed points related to $F-$Suzuki type contractions for self map in the complete mètric space. Acar et al. [4] devolped the idea of $F-$contraction related to multifunctions. Moreover, Acar et al. [3] developed the setvalued $F-$contraction to $\delta-$Distance and to set up fixed points in complete mètric space. Sgroi et al. [53] asserted fixed points for multifunctions $F-$contraction and procured the solution of different functional and integral inclusions, that was a suitable generalization of many setvalued fixed points theorems containing Nadler's result [40]. Many other helpful results related to $F-$contractions can be shown in [6, 11, 27, 37].

In Section 2.2, the concept of multifunctions on a closed set for a new rational type contraction has been introduced. In Section 2.3, we recall the notion of $F-$contraction to have common fixed points for multifunctions on closed subsets justifying an advanced Ćirić kind F-contraction in the frame work of complete dislocated mètric spaces. In Section 2.4, we recalled the idea of $F-$contraction to gain common fixed points for semi α_*-dominated setvalued maps on proximinal sets justifying a rational kind of F-contraction in setting of dislocated mètric spaces.

2.2 Fixed Point Results for a Pair of Rational type Multivalued Contractive Mappings in Dislocated Metric Space

The results given in this section can be shown in [44].

Let (E, d_l) is a $D.M.S$, $y_0 \in E$ and $S, T : E \to P(E)$ are the setvalued mapping on E. Let $y_1 \in Sy_0$ be an element such that $d_l(y_0, Sy_0) = d_l(y_0, y_1)$. Let $y_2 \in Ty_1$ be so as $d_l(y_1, Ty_1) = d_l(y_1, y_2)$. Let $y_3 \in Sy_2$ be such that $d_l(y_2, Sy_2) = d_l(y_2, y_3)$. Proceeding this method, we devolpe sequence y_n in E so as $y_{2n+1} \in Sy_{2n}$ and $y_{2n+2} \in Ty_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_l(y_{2n}, Sy_{2n}) = d_l(y_{2n}, y_{2n+1})$, $d_l(y_{2n+1}, Ty_{2n+1}) = d_l(y_{2n+1}, y_{2n+2})$. We represent this kind of iterative sequence by $\{TS(y_n)\}$. We say that $\{TS(y_n)\}$ is a sequence in E generated by y_0.

Theorem 2.2.1 Let (E, d_l) is a complete $D.M.S$ and y_0 be any arbitrary point in E let the mappings $S, T : E \to P(E)$ satisfy:

$$H_{d_l}(Sy, Tv) \leq \kappa_1 \, d_l(y, v) + \kappa_2 \frac{d_l(y, Sy) . d_l(v, Tv)}{\kappa_4 + d_l(y, Sy)} + \kappa_3 \frac{d_l(y, Sy) . d_l(v, Tv)}{d_l(y, Sy) + d_l(y, v) + d_l(v, Tv)} \tag{2.1}$$

for all $y, v \in \overline{B_{d_l}(y_0, r)} \cap \{TS(y_n)\}$ and $y \neq v$ with $\kappa_1, \kappa_2, \kappa_3, \kappa_4 > 0$ and $\kappa_1 + \kappa_2 + \kappa_3 < 1$,

$$d_l(y_0, Sy_0) \leq (1 - \lambda) r \tag{2.2}$$

where $\lambda = \max\{\frac{\kappa_1 + \kappa_3}{1 - \kappa_2}, \frac{\kappa_1 + \kappa_2}{1 - \kappa_3}\}$. Then $\{TS(y_n)\}$ be the sequence in $\overline{B_{d_l}(y_0, r)}$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(y_n)\} \to h \in \overline{B_{d_l}(y_0, r)}$. Also, if (2.1) holds for h, then h is the C.F.P of both S and T in $\overline{B_{d_l}(y_0, r)}$.

Proof. Let $y_0 \in E$ is an casual point in E define $y_1 \in Sy_0$ and $y_2 \in Ty_1$ then, we have $y_{2n+1} \in Sy_{2n}$ and $y_{2n+2} \in Ty_{2n+1}$, where $n = 0, 1, 2, \ldots$. By Lemma 1.1.6, we have

$$\begin{aligned} d_l(y_1, y_2) &= d_l(y_1, Ty_1) \leq H_{d_l}(Sy_0, Ty_1) \\ &\leq \kappa_1 d_l(y_0, y_1) + \kappa_2 \frac{d_l(y_0, Sy_0) . d_l(y_1, Ty_1)}{\kappa_4 + d_l(y_0, y_1)} \\ &\quad + \kappa_3 \frac{d_l(y_0, Sy_0) . d_l(y_1, Ty_1)}{d_l(y_0, Sy_0) + d_l(y_0, y_1) + d_l(y_1, Ty_1)} \end{aligned}$$

$$
\begin{aligned}
d_l(y_1,y_2) &\leq \kappa_1 d_l(y_0,y_1)+\kappa_2\frac{d_l(y_0,y_1).d_l(y_1,y_2)}{\kappa_4+d_l(y_0,y_1)} \\
&\quad +\kappa_3\frac{d_l(y_0,y_1).d_l(y_1,y_2)}{d_l(y_0,y_1)+d_l(y_0,y_1)+d_l(y_1,y_2)} \\
&\leq \kappa_1 d_l(y_0,y_1)+\kappa_2 d_l(y_1,y_2)+\kappa_3 d_l(y_0,y_1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
d_l(y_1,y_2) &\leq \left(\frac{\kappa_1+\kappa_3}{1-\kappa_2}\right) d_l(y_0,y_1) \\
&\leq \lambda d_l(y_0,y_1)\leq \lambda(1-\lambda)r \quad \text{by using (2.2)} \\
d_l(y_1,y_2) &\leq \lambda(1-\lambda)r.
\end{aligned}
$$

Now,

$$
\begin{aligned}
d_l(y_0,y_2) &\leq d_l(y_0,y_1)+d_l(y_1,y_2) \\
&\leq (1-\lambda)r+\lambda(1-\lambda)r \\
&\leq (1-\lambda^2)r\leq r \\
d_l(y_0,y_2) &\leq r.
\end{aligned}
$$

This implies that $y_2\in\overline{B_{d_l}(y_0,r)}$. Suppose, $y_3,y_4,\cdots,y_j\in\overline{B_{d_l}(y_0,r)}$, for ēverŷ j belongs to N. If $j=2i+1$, where $i=1,2,\cdots,\frac{j-1}{2}$, we get

$$
\begin{aligned}
d_l(y_{2i+1},y_{2i+2}) &= d_l(y_{2i+1},Ty_{2i+1})\leq H_{d_l}(Sy_{2i},Ty_{2i+1}) \\
&\leq \kappa_1 d_l(y_{2i},y_{2i+1})+\kappa_2\frac{d_l(y_{2i},Sy_{2i}).d_l(y_{2i+1},Ty_{2i+1})}{\kappa_4+d_l(y_{2i},Sy_{2i})} \\
&\quad +\kappa_3\frac{d_l(y_{2i},Sy_{2i}).d_l(y_{2i+1},Ty_{2i+1})}{d_l(y_{2i},Sy_{2i})+d_l(y_{2_i},y_{2i+1})+d_l(y_{2i+1},Ty_{2i+1})} \\
&\leq \kappa_1 d_l(y_{2i},y_{2i+1})+\kappa_2\frac{d_l(y_{2i},y_{2i+1}).d_l(y_{2i+1},y_{2i+2})}{\kappa_4+d_l(y_{2i},y_{2i+1})} \\
&\quad +\kappa_3\frac{d_l(y_{2i},y_{2i+1}).d_l(y_{2i+1},y_{2i+2})}{d_l(y_{2i},y_{2i+1})+d_l(y_{2i},y_{2i+1})+d_l(y_{2i+1},y_{2i+2})} \\
&\leq \kappa_1 d_l(y_{2i},y_{2i+1})+\kappa_2 d_l(y_{2i+1},y_{2i+2})+\kappa_3 d_l(y_{2i},y_{2i+1}).
\end{aligned}
$$

Hence

$$\begin{aligned} d_l\left(y_{2i+1}, y_{2i+2}\right) &\leq \left(\frac{\kappa_1+\kappa_3}{1-\kappa_2}\right) d_l\left(y_{2i}, y_{2i+1}\right) \\ &\leq \lambda d_l\left(y_{2i}, y_{2i+1}\right). \end{aligned} \tag{2.3}$$

Similarly, if $j = 2i$, where $i = 1, 2, \cdots, \frac{j-2}{2}$, we have

$$\begin{aligned} d_l\left(y_{2i}, y_{2i+1}\right) &= d_l\left(y_{2i}, Sy_{2i}\right) \leq H_{d_l}\left(Ty_{2i-1}, Sy_{2i}\right) = H_{d_l}\left(Sy_{2i}, Ty_{2i-1}\right) \\ &\leq \kappa_1 d_l\left(y_{2i}, y_{2i-1}\right) + \kappa_2 \frac{d_l\left(y_{2i}, y_{2i+1}\right).d_l\left(y_{2i-1}, y_{2i}\right)}{\kappa_4 + d_l\left(y_{2i}, y_{2i+1}\right)} \\ &\quad + \kappa_3 \frac{d_l\left(y_{2i}, y_{2i+1}\right).d_l\left(y_{2i-1}, y_{2i}\right)}{d_l\left(y_{2i}, y_{2i+1}\right) + d_l\left(y_{2i}, y_{2i-1}\right) + d_l\left(y_{2i-1}, y_{2i}\right)} \\ &\leq \kappa_1 d_l\left(y_{2i-1}, y_{2i}\right) + \kappa_2 d_l\left(y_{2i-1}, y_{2i}\right) + \kappa_3 d_l\left(y_{2i}, y_{2i+1}\right) \\ d_l\left(y_{2i}, y_{2i+1}\right) &\leq \left(\frac{\kappa_1+\kappa_2}{1-\kappa_3}\right) d_l\left(y_{2i}, y_{2i+1}\right) \\ &\leq \lambda d_l\left(y_{2i-1}, y_{2i}\right). \end{aligned} \tag{2.4}$$

Now, (2.3) implies that

$$d_l\left(y_{2i+1}, y_{2i+2}\right) \leq \lambda^{2i+1} d_l\left(y_0, y_1\right). \tag{2.5}$$

Also, (2.4) implies that

$$d_l\left(y_{2i}, y_{2i+1}\right) \leq \lambda^{2i} d_l\left(y_0, y_1\right). \tag{2.6}$$

Now, by combining (2.5) and (2.6), we have

$$d_l\left(y_j, y_{j+1}\right) \leq \lambda^j d_l\left(y_0, y_1\right) \text{ for each } j \in N. \tag{2.7}$$

Now,

$$\begin{aligned} d_l(y_0, y_{j+1}) &\leq d_l(y_0, y_1) + d_l(y_1, y_2) + \cdots + d_l(y_j, y_{j+1}) \\ &\leq d_l(y_0, y_1) + \lambda d_l(y_0, y_1) + \cdots + \lambda^j d_l\left(y_0, y_1\right) \text{ (by (2.7))} \\ &\leq (1 + \lambda + \lambda^2 + \cdots + \lambda^j) d_l\left(y_0, y_1\right) \\ &\leq \frac{1(1-\lambda^j)}{1-\lambda}(1-\lambda) r \leq r. \end{aligned}$$

Thus, $y_{j+1} \in \overline{B_{d_l}(y_0, r)}$. Hence $y_n \in \overline{B_{d_l}(y_0, r)}$ for ėach n belongs to $\mathbb{N} \cup \{0\}$, therefore $\{TS(y_n)\}$ be a şequenće in $\overline{B_{d_l}(y_0, r)}$. Now, we can write inequality (2.7) as

$$d_l\left(y_n, y_{n+1}\right) \leq \lambda^n d_l\left(y_0, y_1\right) \text{ for ėach } n \in N. \tag{2.8}$$

Hence for any $m > n$,

$$\begin{aligned}
d_l\left(y_n, y_m\right) &\leq d_l\left(y_n, y_{n+1}\right) + d_l\left(y_{n+1}, y_{n+2}\right) + \cdots + d_l\left(y_{m-1}, y_m\right), \\
&\leq \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right) d_l\left(y_0, y_1\right), \quad \text{(by using (2.8))} \\
d_l\left(y_n, y_m\right) &\leq \frac{\lambda^n}{1-\lambda} d_l\left(y_0, y_1\right) \longrightarrow 0, \text{ as } m, n \longrightarrow \infty..
\end{aligned}$$

Thus we showed that $\{TS(y_n)\}$ be a Ĉauchŷ şequenće in $(\overline{B_{d_l}(y_0, r)}, d_l)$. As ėach ĉlosed báll in a ĉomþleţe $D.M.S$ is ĉomþleţe, so there must be a $h \in \overline{B_{d_l}(y_0, r)}$ so as $\{TS(y_n)\} \to h$, it follows that

$$\lim_{n \to \infty} d_l(y_n, h) = 0.$$

Now,

$$\begin{aligned}
d_l\left(h, Sh\right) &\leq d_l\left(h, y_{2n+2}\right) + d_l\left(y_{2n+2}, Sh\right) \\
&\leq d_l\left(h, y_{2n+2}\right) + H_{d_l}\left(Ty_{2n+1}, Sh\right), \quad \text{(by Lemma 1.1.6)} \\
d_l\left(h, Sh\right) &\leq d_l\left(h, y_{2n+2}\right) + \kappa_1 d_l\left(h, y_{2n+1}\right) + \kappa_2 \frac{d_l\left(h, Sh\right).d_l\left(y_{2n+1}, Ty_{2n+1}\right)}{\kappa_4 + d_l\left(h, Sh\right)} \\
&\quad + \kappa_3 \frac{d_l\left(h, Sh\right).d_l\left(y_{2n+1}, Ty_{2n+1}\right)}{d_l\left(h, Sh\right) + d_l\left(h, y_{2n+1}\right) + d_l\left(y_{2n+1}, Ty_{2n+1}\right)} \\
&\leq d_l\left(h, y_{2n+2}\right) + \kappa_1 d_l\left(h, y_{2n+1}\right) + \kappa_2 \frac{d_l\left(h, Sh\right).d_l\left(y_{2n+1}, y_{2n+2}\right)}{\kappa_4 + d_l\left(h, Sh\right)} \\
&\quad + \kappa_3 \frac{d_l\left(h, Sh\right).d_l\left(y_{2n+1}, y_{2n+2}\right)}{d_l\left(h, Sh\right) + d_l\left(h, y_{2n+1}\right) + d_l\left(y_{2n+1}, y_{2n+2}\right)}.
\end{aligned}$$

which on making $n \to \infty$, gives rise $d_l\left(h, Sh\right) \leq 0$. Hence $d_l\left(h, Sh\right) = 0$ and so $h \in Sh$. Similarly,

$$\begin{aligned}
d_l\left(h, Th\right) &\leq d_l\left(h, y_{2n+1}\right) + d_l\left(y_{2n+1}, Th\right) \\
&\leq d_l\left(h, y_{2n+1}\right) + H_{d_l}\left(Sy_{2n}, Th\right), \quad \text{(by Lemma 1.6.1)}
\end{aligned}$$

$$
\begin{aligned}
&\le d_l(h, y_{2n+2}) + \kappa_1 d_l(h, y_{2n+1}) + \kappa_2 \frac{d_l(h, Sh) . d_l(y_{2n+1}, Ty_{2n+1})}{\kappa_4 + d_l(h, Sh)} \\
&\quad + \kappa_3 \frac{d_l(h, Sh) . d_l(y_{2n+1}, Ty_{2n+1})}{d_l(h, Sh) + d_l(h, y_{2n+1}) + d_l(y_{2n+1}, Ty_{2n+1})} \\
&\le d_l(h, y_{2n+2}) + \kappa_1 d_l(y_{2n}, h) + \kappa_2 \frac{d_l(y_{2n}, y_{2n+1}) . d_l(h, Th)}{\kappa_4 + d_l(y_{2n}, y_{2n+1})} \\
&\quad + \kappa_3 \frac{d_l(y_{2n}, y_{2n+1}) . d_l(h, Th)}{d_l(y_{2n}, y_{2n+1}) + d_l(y_{2n}, h) + d_l(h, Th)}.
\end{aligned}
$$

Hence $d_l(h, Th) \le 0$ and so $h \in Th$. ■

Example 2.2.2 Let $E = Q^+ \cup \{0\}$ and let $d_l : E \times E \to E$ be the ĉomþleţe $D.M.S$ on E defined by

$$d_l(q, v) = q + v \text{ for ėach } q, v \in E.$$

Define the multivalued maþþing, $S, T : E \times E \to P(E)$ by,

$$
Sq = \begin{cases} [\frac{q}{3}, \frac{2}{3}q] \text{ if } q \in [0,1] \cap E \\ [q, q+1] \text{ if } q \in (1, \infty) \cap E \end{cases}
$$

and,

$$
Tl = \begin{cases} [\frac{l}{4}, \frac{3}{4}l] \text{ if } l \in [0,1] \cap E \\ [l+1, l+3] \text{ if } l \in (1, \infty) \cap E. \end{cases}
$$

Considering, $q_0 = 1, r = 8$, then $\overline{B_{d_l}(q_0, r)} = [0, 7] \cap E$. Now $d_l(q_0, Sq_0) = d_l(1, S1) = d_l(1, \frac{1}{3}) = \frac{4}{3}$. So we obtain a şequenće $\{TS(q_n)\} = \{1, \frac{1}{12}, \frac{1}{144}, \frac{1}{1728}, ...\}$ in E generated by q_0. Let $q, v \in (1, \infty) \cap E$, then by taking $q = 2$, $v = 3$, $\kappa_1 = \frac{1}{3}$, $\kappa_2 = \frac{1}{4}$, and $\kappa_3 = \frac{1}{7}$, $\kappa_4 = 1$, then $H_{d_l}(S2, T3) = 8$. Now,

$$
\begin{aligned}
&= \kappa_1 d_l(2,3) + \kappa_2 \frac{d_l(2, [2, 2+1]).d_l(3, [3+1, 3+3])}{1 + d_l(2, [2, 2+1])} \\
&\quad + \kappa_3 \frac{d_l(2, [2, 2+1]).d_l(3, [3+1, 3+3])}{d_l(2, [2, 2+1]) + d_l(2,3) + d_l(3, [3+1, 3+3])} \\
&= \frac{1}{3} d_l(2,3) + \frac{1}{4} \frac{d_l(2,2).d_l(3,4)}{1 + d_l(2, [2,2])} + \frac{1}{7} \frac{d_l(2,2).d_l(3,4)}{d_l(2,2) + d_l(2,3) + d_l(3,4)} \\
&= \frac{5}{3} + \frac{28}{20} + \frac{28}{112} = 3.31.
\end{aligned}
$$

As $8 > 3.31$, then

$$H_{d_l}(S2,T3) > \frac{1}{3}d_l(2,3) + \frac{1}{4}\frac{d_l\left(2,S2\right).d_l\left(3,T3\right)}{1+d_l\left(2,S2\right)} + \frac{1}{7}\frac{d_l\left(2,S2\right).d_l\left(3,T3\right)}{d_l\left(2,S2\right)+d_l\left(2,3\right)+d_l\left(3,T3\right)}$$

So, the inequality (2.1) is not true for the whole space E. Now for each $q,v \in \overline{B_{d_l}(q_0,r)} \cap \{TS(q_n)\}$, we have

$$\begin{aligned}
H_{d_l}(Sq,Tv) &= [\max\{\sup_{\kappa\in Sq} d_l(\kappa,Tv), \sup_{b\in Tv} d_l(Sq,b)\}] \\
&= \max\{\sup_{\kappa\in Sq} d_l(\kappa,[\frac{v}{4},\frac{3v}{4}]), \sup_{b\in Tv} d_l([\frac{q}{3},\frac{2q}{3}],b)\} \\
&= \max\{d_l(\frac{2q}{3},[\frac{v}{4},\frac{3v}{4}]), d_l([\frac{q}{3},\frac{2q}{3}],\frac{3y}{4})\} \\
&= \max\{d_l(\frac{2q}{3},\frac{v}{4}), d_l(\frac{q}{3},\frac{3v}{4})\} \\
&= \max\{\frac{2q}{3}+\frac{v}{4},\frac{q}{3}+\frac{3v}{4}\}
\end{aligned}$$

$$\begin{aligned}
&= \kappa_1\ d_l(q,v) + \kappa_2\frac{d_l\left(q,Sq\right).d_l\left(v,Tv\right)}{\kappa_4+d_l\left(q,v\right)} + \kappa_3\frac{d_l\left(q,Sq\right).d_l\left(v,Tv\right)}{d_l\left(q,Sq\right)+d_l\left(q,v\right)+d_l\left(v,Tv\right)} \\
&= \kappa_1 d_l(q,v) + \kappa_2\frac{d_l\left(q,\frac{q}{3}\right).d_l\left(v,\frac{v}{4}\right)}{1+d_l\left(q,\frac{q}{3}\right)} + \kappa_3\frac{d_l\left(q,\frac{q}{3}\right).d_l\left(v,\frac{v}{4}\right)}{d_l\left(q,\frac{q}{3}\right)+d_l(q,v)+d_l\left(v,\frac{v}{4}\right)} \\
&= \frac{1}{3}(q+v) + \frac{1}{4}\frac{\frac{4q}{3}.\frac{5v}{4}}{1+q+\frac{q}{3}} + \frac{1}{7}\frac{\frac{4q}{3}.\frac{5v}{4}}{q+\frac{q}{3}+q+v+v+\frac{v}{4}} \\
&\geq \max\{\frac{2q}{3}+\frac{v}{4},\frac{q}{3}+\frac{3v}{4}\} = H_{d_l}(Sq,Tv).
\end{aligned}$$

So, the inequality (2.1) holds on $\overline{B_{d_l}(q_0,r)} \cap \{TS(q_n)\}$. Also,

$$\begin{aligned}
\frac{4}{3} &< (1-\frac{49}{72})\times 8 \\
d_l(q_0,Sq_0) &\leq (1-\lambda)r.
\end{aligned}$$

Hence, all the hypothesis of Theorem 2.2.1 are fulfilled.

Corollary 2.2.3 Let (E,d_l) be a complete $D.M.S$ and q_0 be any arbitrary point in E let the mappings $S, T : E \to P(E)$ satisfy:

$$H_{d_l}(Sq, Tv) \leq \kappa_1 \, d_l(q, v) + \kappa_2 \frac{d_l(q, Sq) . d_l(v, Tv)}{d_l(q, Sq) + d_l(q, v) + d_l(v, Tv)} \tag{2.9}$$

for each $q, v \in \overline{B_{d_l}(q_0, r)} \cap \{TS(q_n)\}$ and $q \neq v$ with $r > 0$,

$$d_l(q_0, Sq_0) \leq (1 - \lambda)r$$

where $\lambda = (\kappa_1 + \kappa_2)$ and, κ_1, κ_2 are positive reals with $\kappa_1 + \kappa_2 < 1$. Then $\{TS(q_n)\}$ is a sequence in $\overline{B_{d_l}(q_0, r)}$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(q_n)\} \to h \in \overline{B_{d_l}(q_0, r)}$. Also, if (2.9) holds for h, then h be the C.F.P of both S and T in $\overline{B_{d_l}(q_0, r)}$.

Theorem 2.2.4 Let (E, d_l) be a complete $D.M.S$ and v_0 be any arbitrary point in E let the mappings $S, T : E \to P(E)$ satisfy:

$$H_{d_l}(S(v), T(f)) \leq a \, d_l(v, f) + b \frac{d_l(v, S(v)) . d_l(f, T(f))}{1 + d_l(v, f)} \tag{2.10}$$

for each $v, f \in \overline{B_{d_l}(v_0, r)} \cap \{TS(v_n)\}$ and $v \neq f$ with $r > 0$,

$$d_l(v_0, Sv_0) \leq (1 - \lambda)r \tag{2.11}$$

where $\lambda = \left(\frac{a}{1-b}\right)$ and a, b are positive reals with $a + b < 1$. Then $\{TS(v_n)\}$ is a sequence in $\overline{B_{d_l}(v_0, r)}$ for each n belongs to $\mathbb{N} \cup \{0\}$, and $\{TS(v_n)\} \to q \in \overline{B_{d_l}(v_0, r)}$. Also, if (2.10) holds for q, then S and T have C.F.P u in $\overline{B_{d_l}(v_0, r)}$.

Proof. Let $v_0 \in E$ and define $v_1 \in S(v_0)$ and $v_2 \in T(v_1)$ then, we have $v_{2n+1} \in S(v_{2n})$ and $v_{2n+2} \in T(v_{2n+1})$, where $n = 0, 1, 2, ...$ By Lemma 1.1.6, we have

$$\begin{aligned}
d_l(v_1, v_2) &= d_l(S(v_0), T(v_1)) \leq H_{d_l}(S(v_0), T(v_1)) \\
&\leq ad_l(v_0, v_1) + b \frac{d_l(v_0, S(v_0)) . d_l(v_1, T(v_1))}{1 + d_l(v_0, v_1)} \\
&\leq ad_l(v_0, v_1) + bd_l(v_1, v_2) \left(\frac{d_l(v_0, v_1)}{1 + d_l(v_0, v_1)} \right) \\
&\leq ad_l(v_0, v_1) + bd_l(v_1, v_2) \\
&\leq \left(\frac{a}{1 - b} \right) d_l(v_0, v_1)
\end{aligned}$$

$$\begin{aligned} &\leq \lambda d_l(v_0, v_1) \\ &\leq \lambda(1-\lambda) \leq r. \text{ by } (2.11) \end{aligned}$$

Where$\left(\frac{a}{1-b}\right) = \lambda$. Now,

$$\begin{aligned} d_l(v_0, v_2) &\leq d_l(v_0, v_1) + d_l(v_1, v_2) \\ &\leq (1-\lambda)r + \lambda(1-\lambda)r \\ &\leq (1-\lambda^2)r \leq r. \end{aligned}$$

This implies that $v_2 \in \overline{B_{d_l}(v_0, r)}$ similarly,

$$\begin{aligned} d_l(v_2, v_3) &= d_l(v_3, v_2) \leq Hd_l(S(v_2), T(v_1)) \text{ by Lemma 1.1.6} \\ &\leq ad_l(v_2, v_1) + b\frac{d_l(v_2, S(v_2)).d_l(v_1, T(v_1))}{1 + d_l(v_2, v_1)} \\ &\leq ad_l(v_2, v_1) + b\frac{d_l(v_2, v_3).d_l(v_1, v_2)}{1 + d_l(v_2, v_1)} \\ &\leq ad_l(v_2, v_1) + bd_l(v_2, v_3)\left(\frac{d_l(v_1, v_2)}{1 + d_l(v_2, v_1)}\right) \\ &\leq ad_l(v_1, v_2) + bd_l(v_2, v_3). \end{aligned}$$

This implies that,

$$\begin{aligned} d_l(v_2, v_3) &\leq \left(\frac{a}{1-b}\right) d_l(v_1, v_2) \\ &\leq \lambda.\lambda d_l(v_0, v_1) \\ &\leq \lambda^2 d_l(v_0, v_1) \\ &\leq \lambda^2(1-\lambda)r \leq r. \end{aligned}$$

Consequently, $v_3, v_4, \cdots, v_j \in \overline{B_{d_l}(v_0, r)}$, for $\bar{e}$ver$\hat{y}$ j belongs to N. If $j = 2i + 1$, where $i = 1, 2, \cdots, \frac{j-1}{2}$ we get

$$d_l(v_{2i+1}, v_{2i+2}) \leq \lambda d_l(v_{2i}, v_{2i+1}) \tag{2.12}$$

Similarly, if $j = 2i+2$, where $i = 0, 1, 2, \cdots, \frac{j-2}{2}$, we have

$$d_l(v_{2i+2}, v_{2i+3}) \leq \lambda d_l(v_{2i+1}, v_{2i+2}). \tag{2.13}$$

Now, (2.12) implies that

$$d_l(v_{2i+1}, v_{2i+2}) \leq \lambda^{2i+1} d_l(v_0, v_1). \tag{2.14}$$

Also, (2.14) implies that

$$d_l(v_{2i+2}, v_{2i+3}) \leq \lambda^{2i+2} d_l(v_0, v_1). \tag{2.15}$$

Now, by combining (2.14) and (2.15), we have

$$d_l(v_j, v_{j+1}) \leq \lambda^j d_l(v_0, v_1) \text{ for each } j \in N. \tag{2.16}$$

Now,

$$\begin{aligned} d_l(v_0, v_{j+1}) &\leq d_l(v_0, v_1) + d_l(v_1, v_2) + \ldots + d_l(v_j, v_{j+1}) \\ &\leq d_l(v_0, v_1) + \lambda d_l(v_0, v_1) + \ldots + \lambda^j d_l(v_0, v_1) \text{ by (2.16)} \\ &\leq (1 + \lambda + \lambda^2 + \ldots + \lambda^j) d_l(v_0, v_1) \\ &\leq \frac{1(1-\lambda^j)}{1-\lambda}(1-\lambda)r \leq r. \end{aligned}$$

Thus, $v_{j+1} \in \overline{B_{d_l}(v_0, r)}$. Hence $v_n \in \overline{B_{d_l}(v_0, r)}$ for each n belongs to $\mathbb{N} \cup \{0\}$, therefore $\{TS(v_n)\}$ be a sequence in $\overline{B_{d_l}(v_0, r)}$. Now, we can write (2.16) as

$$d_l(v_n, v_{n+1}) \leq \lambda^n d_l(v_0, v_1) \text{ for every } n \text{ belongs to } N. \tag{2.17}$$

To show that $\{TS(v_n)\}$ is a Cauchy sequence, we have for any $m > n$,

$$\begin{aligned} d_l(v_n, v_m) &\leq d_l(v_n, v_{n+1}) + d_l(v_{n+1}, v_{n+2}) + \cdots + d_l(v_{m-1}, v_m) \\ &\leq \lambda^n d_l(v_0, v_1) + \lambda^{n+1} d_l(v_0, v_1) + \cdots + \lambda^{m-1} d_l(v_0, v_1) \\ &\leq \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right) d_l(v_0, v_1) \\ &\leq \left(\frac{\lambda^n}{1-\lambda}\right) d_l(v_0, v_1) \longrightarrow 0 \text{ as } m, \ n \longrightarrow \infty. \end{aligned}$$

Thus we proved that $\{TS(v_n)\}$ is a Cauchy in $(\overline{B_{d_l}(v_0,r)}, d_l)$. As ēverŷ ĉlosed báll in a ĉomþleţe $D.M.S$ is ĉomþleţe, so there must be a $u \in \overline{B_{d_l}(v_0,r)}$ such that $\{TS(v_n)\} \to u$, it shows that $u \in Su$, otherwise $d_l(u, Su) = \theta > 0$, that is

$$\lim_{n\to\infty} d_l(v_n, u) = 0. \tag{2.18}$$

Therefore we have,

$$\begin{aligned}
d_l(u,Su) &\leq d_l(u,v_{2n+2}) + d_l(v_{2n+2},Su) \\
&\leq d_l(u,v_{2n+2}) + d_l(T(v_{2n+1}),Su) \\
&\leq d_l(u,v_{2n+2}) + H_{d_l}(Su,T(v_{2n+1})) \text{ by Lemma 1.1.6} \\
&\leq d_l(u,v_{2n+2}) + ad_l(u,v_{2n+1}) + b\frac{d_l(u,Su).d_l(v_{2n+1},T(v_{2n+1}))}{1+d_l(u,v_{2n+1})} \\
&\leq d_l(u,v_{2n+2}) + ad_l(u,v_{2n+1}) + b\frac{d_l(u,Su).d_l(v_{2n+1},v_{2n+2})}{1+d_l(u,v_{2n+1})} \\
&\leq d_l(u,v_{2n+2}) + ad_l(u,v_{2n+1}) + b\frac{\theta.d_l(v_{2n+1},v_{2n+2})}{1+d_l(u,v_{2n+1})}.
\end{aligned}$$

letting $n \to \infty$, and $v_n \longrightarrow u$ by using (2.18) we get,

$$\begin{aligned}
(1-b)\theta &\leq 0 \\
(1-b) &\neq 0 \\
\theta &= d_l(u,Su) \leq 0.
\end{aligned}$$

$d_l(u,Su) < 0$ gives a contracdition so that $u \in Su$. It follows similarly that

$$\begin{aligned}
d_l(u,Tu) &\leq d_l(u,v_{2n+1}) + d_l(v_{2n+1},Tu) \\
&\leq d_l(u,v_{2n+1}) + H_{d_l}(Sv_{2n},Tu) \text{ by Lemma 1.1.6}
\end{aligned}$$

$$\begin{aligned}
&\leq d_l(u,v_{2n+1}) + ad_l(v_{2n},u) + b\frac{d_l(v_{2n},Sv_{2n}).d_l(u,Tu)}{1+d_l(v_{2n},u)} \\
&\leq d_l(u,v_{2n+1}) + ad_l(v_{2n},u) + b\frac{d_l(v_{2n},v_{2n+1}).d_l(u,Tu)}{1+d_l(v_{2n},u)}
\end{aligned}$$

letting $n \to \infty$, and $v_n \longrightarrow u$ by using (2.18) we get,

$$\begin{aligned} d_l(u,Tu) &\leq bd_l(u,Tu) \\ (1-b)d_l(u,Tu) &\leq 0 \\ (1-b) &\neq 0 \\ d_l(u,Tu) &\leq 0. \end{aligned}$$

As $d_l(u,Tu) < 0$, so that $u \in Tu$. Hence u is the C.F.P of both S and T in $\overline{B_{d_l}(v_0,r)}$. Now,

$$\begin{aligned} d_l(u,u) &\leq H_{d_l}(Su,Tu) \text{ by Lemma 1.1.6} \\ &\leq a\ d_l(u,u) + b\frac{d_l(u,Su).d_l(u,Tu)}{1+d_l(u,u)} \\ &\leq a\ d_l(u,u). \end{aligned}$$

This implies that,

$$\begin{aligned} (1-a)\,d_l(u,u) &\leq 0 \\ 1-a &\neq 0 \\ d_l(u,u) &= 0. \end{aligned}$$

This shows that $d_l(u,u) = 0$. ■

Corollary 2.2.5 If $S : E \to E$ is a mapping defined on $D.M.S$ satisfying the condition

$$d_l(Sw,Sl) \leq a\ d_l(w,v) + b\frac{d_l(w,Sw).d_l(l,Sl)}{d+d_l(w,Sw)} + c\frac{d_l(w,Sw).d_l(l,Sl)}{d_l(w,Sw)+d_l(w,l)+d_l(l,Sl)}$$

for all $w, l \in \overline{B_{d_l}(u_0,r)}$ and $u \neq v$ with $r > 0$,

$$d_l(u_0,Su_0) \leq (1-\lambda)r$$

where $\lambda = \max\{\frac{a+c}{1-b}, \frac{a+b}{1-c}\}$ and a, b, c are positive reals with $a+b+c < 1$.Then $\{u_n\}$ is a sequence in $\overline{B_{d_l}(u_0,r)}$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_{d_l}(u_0,r)}$. Then h is the C.F.P of S in $\overline{B_{d_l}(u_0,r)}$.

Corollary 2.2.6 If $S : E \to E$ is a mapping defined on a complete on $D.M.S$, (E, d_l) satisfying the condition :

$$d_l(Su, Sy) \leq a_1\ d_l(u, y) + a_2 \frac{d_l(u, Su) . d_l(y, Sy)}{1 + d_l(u, y)}$$

for all $u, y \in \overline{B_{d_l}(u_0, r)}$ and $u \neq v$ with $r > 0$,

$$d_l(u_0, Su_0) \leq (1 - \lambda)r$$

where $\lambda = \frac{a_1}{1-a_2}$, a_1 and a_2 are positive reals with $a_1 + a_2 < 1$. Then $\{u_n\}$ is a sequence in $\overline{B_{d_l}(u_0, r)}$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $u_n \to h \in \overline{B_{d_l}(u_0, r)}$. Then h is the C.F.P of S in $\overline{B_{d_l}(u_0, r)}$.

2.3 Common Fixed Point Results for new Ciric Type Rational Multivalued F-Contraction with an Application

The given results in this section can be seen in [45].

Let (E, d_l) be a $D.M.S$, $y_0 \in E$ and $S, T : E \to P(E)$ be the setvalued maps on E. Let $y_1 \in Sy_0$ be an element such that $d_l(y_0, Sy_0) = d_l(y_0, y_1)$. Let $y_2 \in Ty_1$ be such that $d_l(y_1, Ty_1) = d_l(y_1, y_2)$. Let $y_3 \in Sy_2$ be such that $d_l(y_2, Sy_2) = d_l(y_2, y_3)$. Proceeding this method, we get a sequence y_n in E such that $y_{2n+1} \in Sy_{2n}$ and $y_{2n+2} \in Ty_{2n+1}$, where $n = 0, 1, 2, \ldots$ Also $d_l(y_{2n}, Sy_{2n}) = d_l(y_{2n}, y_{2n+1})$, $d_l(y_{2n+1}, Ty_{2n+1}) = d_l(y_{2n+1}, y_{2n+2})$. We represent this type of sequence by $\{TS(y_n)\}$. We say that $\{TS(y_n)\}$ is a sequence in E generated by y_0.

We start this section with the definition.

Definition 2.3.1 Let (E, d_l) be a complete $D.M.S$ and $S, T : E \to P(E)$ be two setvalued mappings. The pair (S, T) is said to be a pair of new Ciric type rational $F-$contraction, if for all $w, e \in \{TS(w_n)\}$, we have

$$\tau + F(H_{d_l}(Sw, Te)) \leq F(O_l(w, e)) \tag{2.19}$$

where $F \in \triangle_F$ and $\tau > 0$, and

$$O_l(w,e) = \max\left\{d_l(w,e), \frac{d_l(w,Sw).d_l(e,Te)}{1+d_l(w,e)}, d_l(w,Sw), d_l(e,Te)\right\}. \tag{2.20}$$

Theorem 2.3.2 Let (E, d_l) be a complete $D.M.S$ and (S,T) be a pair of new Ciric type rational multivalued $F-$contraction. Then $\{TS(w_n)\} \to u \in E$. Moreover, if (2.19) also holds for u, then S and T have a C.F.P u in E and $d_l(u,u) = 0$.

Proof. If, $O_l(w,e) = 0$, then clearly $w = e$ is a C.F.P of S and T. Then we have no need to prove and our proof is complete. Let $O_l(e,w)) > 0$ for all $w, e \in \{TS(z_n)\}$ with $w \neq e$. Then from (2.19), and Lemma 1.1.6 we have

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(H_{d_l}(Sw_{2i}, Tw_{2i+1})) \leq F(O_l(w_{2i}, w_{2i+1})) - \tau$$

for each $i \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} O_l(w_{2i}, w_{2i+1}) &= \max\left\{\begin{array}{c} d_l(w_{2i}, w_{2i+1}), \frac{d_l(w_{2i},Sw_{2i}).d_l(w_{2i+1},Tw_{2i+1})}{1+d_l(w_{2i},w_{2i+1})}, \\ d_l(w_{2i}, Sw_{2i}), d_l(w_{2i+1}, Tw_{2i+1}) \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d_l(w_{2i}, w_{2i+1}), \frac{d_l(w_{2i},w_{2i+1}).d_l(w_{2i+1},w_{2i+2})}{1+d_l(w_{2i},w_{2i+1})}, \\ d_l(w_{2i}, w_{2i+1}), d_l(w_{2i+1}, w_{2i+2}) \end{array}\right\} \\ &= \max\{d_l(w_{2i}, w_{2i+1}), d_l(w_{2i+1}, w_{2i+2})\}. \end{aligned}$$

If, $O_l(w_{2i}, w_{2i+1}) = d_l(w_{2i+1}, w_{2i+2})$, then

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_{2i+1}, w_{2i+2})) - \tau,$$

It is not true due to (F1). Therefore,

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_{2i}, w_{2i+1})) - \tau, \tag{2.21}$$

for each i belongs to $\mathbb{N} \cup \{0\}$. Similarly, we have

$$F(d_l(w_{2i}, w_{2i+1})) \leq F(d_l(w_{2i-1}, w_{2i})) - \tau, \tag{2.22}$$

for eȧch i belongs to $\mathbb{N} \cup \{0\}$. By using (2.22) in (2.21), we have

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_{2i-1}, w_{2i})) - 2\tau.$$

Replicating these steps, we have

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_0, w_1)) - (2i+1)\tau. \quad (2.23)$$

Similarly, we have

$$F(d_l(w_{2i}, w_{2i+1})) \leq F(d_l(w_0, w_1)) - 2i\tau, \quad (2.24)$$

We can write (2.23) and (2.24) jointly as

$$F(d_l(w_n, w_{n+1})) \leq F(d_l(w_0, w_1)) - n\tau. \quad (2.25)$$

By using limit $n \to \infty$, eȧch sides of (2.25), we have

$$\lim_{n\to\infty} F(d_l(w_n, w_{n+1})) = -\infty. \quad (2.26)$$

Since $F \in \triangle_F$,

$$\lim_{n\to\infty} d_l(w_n, w_{n+1}) = 0. \quad (2.27)$$

By (2.25), for ēverŷ n belongs to $\mathbb{N}$, we get

$$(d_l(w_n, w_{n+1}))^k((F(d_l(w_n, w_{n+1})) - F(d_l(w_0, w_1))) \leq -(d_l(w_n, w_{n+1}))^k n\tau \leq 0. \quad (2.28)$$

Using the inequalities (2.26), (2.27) and applying $n \to \infty$ in (2.28), we get

$$\lim_{n\to\infty}(n(d_l(w_n, w_{n+1}))^k) = 0. \quad (2.29)$$

Since (2.29) holds, there exist $n_1 \in \mathbb{N}$, such that $n(d_l(w_n, w_{n+1}))^k \leq 1$ for eȧch $n \geq n_1$ or,

$$d_l(w_n, w_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for eȧch } n \geq n_1. \quad (2.30)$$

Using (2.30), we get form $m > n > n_1$,

$$\begin{aligned} d_l(w_n, w_m) &\leq d_l(w_n, w_{n+1}) + d_l(w_{n+1}, w_{n+2}) + \cdots + d_l(w_{m-1}, w_m) \\ &= \sum_{i=n}^{m-1} d_l(w_i, w_{i+1}) \leq \sum_{i=n}^{\infty} d_l(w_i, w_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ entails $\lim_{n,m\to\infty} d_l(w_n, w_m) = 0$. Hence, $\{TS(w_n)\}$ is a Ĉauchŷ şequenĉe in (E, d_l). Since (E, d_l) is a ĉomþleţe $D.M.S$, so there exists $u \in E$ such that $\{TS(w_n)\} \to u$ that is

$$\lim_{n\to\infty} d_l(w_n, u) = 0. \tag{2.31}$$

Now, by Lemma 1.1.6, we have

$$\tau + F(d_l(w_{2n+1}, Tu)) \leq \tau + F(Hd_l(Sw_{2n}, Tu))), \tag{2.32}$$

As (2.19) also must be holds for u, then

$$\tau + F(d_l(w_{2n+1}, Tu)) \leq F(O_l(w_{2n}, u)), \tag{2.33}$$

where,

$$\begin{aligned} O_l(w_{2n}, u) &= \max\left\{ d_l(w_{2n}, u), \frac{d_l(w_{2n}, Sw_{2n}).d_l(u, Tu)}{1 + d_l(w_{2n}, u)}, d_l(w_{2n}, Sw_{2n}), d_l(u, Tu) \right\} \\ &= \max\left\{ d_l(w_{2n}, u), \frac{d_l(w_{2n}, w_{2n+1}).d_l(u, Tu)}{1 + d_l(w_{2n}, u)}, d_l(w_{2n}, w_{2n+1}), d_l(u, Tu) \right\}. \end{aligned}$$

Letting limit $n \to \infty$, and by using (2.31), we have

$$\lim_{n\to\infty} O_l(w_{2n}, u) = d_l(u, Tu). \tag{2.34}$$

Since F is strictly increasing, then (2.33) implies

$$d_l(w_{2n+1}, Tu) < O_l(w_{2n}, u).$$

By using limit $n \to \infty$, and using (2.34), we get

$$d_l(u, Tu) < d_l(u, Tu).$$

It is not true, hence $d_l(u, Tu) = 0$ or $u \in Tu$. Similarly by using (2.31) and Lemma1.1.6 and the inequality

$$\tau + F(d_l(w_{2n+2}, Su)) \leq \tau + F(H_{d_l}(Tw_{2n+1}, Su)),$$

we can setup that $d_l(u, Su) = 0$ or $u \in Su$. Hence S and T have a C.F.P u in E. Now,

$$d_l(u, u) \leq d_l(u, Tu) + d_l(Tu, u) \leq 0.$$

This implies that $d_l(u, u) = 0$. ■

Example 2.3.3 Let $E = \{0\} \cup Q^+$ and $d_l(w, e) = w + e$. Then (E, d_l) is a complete $D.M.S.$ Define $S, T : E \to P(E)$ as follows:

$$S(w) = \left[\frac{1}{3}w, \frac{2}{3}w\right] \text{ and } T(w) = \left[\frac{1}{5}w, \frac{2}{5}w\right] \text{ for all } w \in E.$$

If, $\tau + F(H_{d_l}(Sw, Te)) \leq F(O_l(w, e))$, holds. Define $F : R^+ \to R$ by $F(w) = \ln(w)$ for every $w \in R^+$ and $\tau > 0$. As $w, e \in E$, $\tau = \ln(1.2)$ by taking $w_0 = 7$, we define the sequence $\{TS(w_n)\} = \{7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \cdots\}$ in E generated by $w_0 = 7$. We have

$$\begin{aligned} H_{d_l}(Sw, Te) &= \max\left[\left\{\sup_{a \in Sw} d_l(a, Te), \sup_{g \in Te} d_l(Sw, g)\right\}\right] \\ &= \max\left[\left\{\sup_{a \in Sw} d_l\left(a, \left[\frac{e}{2}, \frac{2e}{5}\right]\right), \sup_{g \in Te} d_l\left(\left[\frac{w}{3}, \frac{2w}{3}\right], g\right)\right\}\right] \\ &= \max\left\{d_l\left(\frac{2w}{3}, \frac{e}{5}\right), d_l\left(\frac{w}{3}, \frac{2e}{5}\right)\right\} \\ &= \max\left\{\frac{2w}{3} + \frac{e}{5}, \frac{w}{3} + \frac{2e}{5}\right\} \end{aligned}$$

where

$$
\begin{aligned}
O_l(w,e) &= \max\left\{\begin{array}{c} d_l(w,e), \frac{d_l(w,\left[\frac{w}{3},\frac{2w}{3}\right]).d_l(e,\left[\frac{e}{5},\frac{2e}{5}\right])}{1+d_l(w,e)}, \\ d_l(w,\left[\frac{w}{3},\frac{2w}{3}\right]), d_l(e,\left[\frac{e}{5},\frac{2e}{5}\right]) \end{array}\right\} \\
&= \max\left\{d_l(w,e), \frac{d_l(w,\frac{w}{3}).d_l(e,\frac{e}{5})}{1+d_l(w,e)}, d_l(w,\frac{w}{3}), d_l(e,\frac{e}{5})\right\} \\
&= \max\left\{w+e, \frac{8we}{15(1+w+e)}, \frac{4w}{3}, \frac{6e}{5}\right\} = w+e.
\end{aligned}
$$

Case (i). If, $\max\left\{\frac{2w}{3}+\frac{e}{5}, \frac{w}{3}+\frac{2e}{5}\right\} = \frac{w}{3}+\frac{2e}{5}$, and $\tau = \ln(1.2)$, then we have

$$
\begin{aligned}
10w + 12e &\leq 25w + 25e \\
\frac{6}{5}(\frac{w}{3}+\frac{2e}{5}) &\leq w+e \\
\ln(1.2) + \ln(\frac{w}{3}+\frac{2e}{5}) &\leq \ln(w+e).
\end{aligned}
$$

which implies that,

$$\tau + F(H_{d_l}(Sw,Te) \leq F(O_l(w,e)).$$

Case (ii). Similarly, if $\max\left\{\frac{2w}{3}+\frac{e}{5}, \frac{w}{3}+\frac{2e}{5}\right\} = \frac{2w}{3}+\frac{e}{5}$, and $\tau = \ln(1.2)$, then we have

$$
\begin{aligned}
20w + 6e &\leq 25w + 25e \\
\frac{6}{5}(\frac{4w}{3}+\frac{2e}{5}) &\leq w+e \\
\ln(1.2) + \ln(\frac{2w}{3}+\frac{e}{5}) &\leq \ln(w+e).
\end{aligned}
$$

Hence,

$$\tau + F(H_{d_l}(Sw,Te) \leq F(O_l(w,e)).$$

Hence all hypothesis of Theorem 2.3.2 are proved so (S,T) have a C.F.P.

Corollary 2.3.4 Let (E,d_l) be a complete $D.M.S$ and $S: E \to P(E)$ is a setvalued mapping such that

$$\tau + F(H_{d_l}(Sw,Se)) \leq F(O_l(w,e)) \tag{2.35}$$

for all $w, e \in \{SS(w_n)\}$, where $F \in \Delta_F$and $\tau > 0$, and

$$O_l(w,e) = \max\left\{d_l(w,e), \frac{d_l\left(w,Sw\right).d_l\left(e,Se\right)}{1+d_l\left(w,e\right)}, d_l(w,Sw), d_l(e,Se)\right\}.$$

Then $\{SS(w_n)\} \to u \in E$. Furthermore, if (2.35) holds for u, then u be the has a fîxẹd þoīnţ of S in E and $d_l(u,u) = 0$.

Remark 2.3.5 By setting the different values of $O_l(l,w)$ in equation (2.20), we can achive different řesulţs as corollaries of Theorem 2.3.2.

$$\begin{aligned}
(1)\ O_l(l,w) &= d_l(l,w)\\
(2)\ O_l(l,w) &= \frac{d_l\left(l,Sl\right).d_l\left(w,Tw\right)}{1+d_l\left(l,w\right)}\\
(3)\ O_l(l,w) &= d_l(l,Sl)\\
(4)\ O_l(l,w) &= d_l(w,Tw)\\
(5)\ O_l(l,w) &= \max\left\{d_l(l,w), \frac{d_l\left(l,Sl\right).d_l\left(w,Tw\right)}{1+d_l\left(l,w\right)}\right\}\\
(6)\ O_l(l,w) &= \max\left\{d_l(l,w), d_l(l,Sl)\right\}\\
(7)\ O_l(l,w) &= \max\left\{d_l(l,w), d_l(w,Tw)\right\}\\
(8)\ O_l(l,w) &= \max\left\{\frac{d_l\left(l,Sl\right).d_l\left(w,Tw\right)}{1+d_l\left(l,w\right)}, d_l(l,Sl)\right\}\\
(9)\ O_l(l,w) &= \max\left\{\frac{d_l\left(l,Sl\right).d_l\left(w,Tw\right)}{1+d_l\left(l,w\right)}, d_l(w,Tw)\right\}\\
(10)\ O_l(l,w) &= \max\left\{d_l(l,Sl), d_l(w,Tw)\right\}\\
(11)\ O_l(l,w) &= \max\left\{d_l(l,w), \frac{d_l\left(l,Sl\right).d_l\left(w,Tw\right)}{1+d_l\left(l,w\right)}, d_l(l,Sl)\right\}\\
(12)\ O_l(l,w) &= \max\left\{d_l(l,w), \frac{d_l\left(l,Sl\right).d_l\left(w,Tw\right)}{1+d_l\left(l,w\right)}, d_l(w,Tw)\right\}\\
(13)\ O_l(l,w) &= \max\left\{d_l(l,w), d_l(l,Sl), d_l(w,Tw)\right\}
\end{aligned}$$

Theorem 2.3.6 Let (E, d_l) be a ĉomþleţe $D.M.S$ and $S, T : E \to P(E)$ be the multivalued maþþings. Assume that if $F \in \Delta_F$and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(H_{d_l}(Sw,Te)) \leq F\left(\delta_1 d_l(w,e) + \delta_2 d_l(w,Sw) + \delta_3 d_l(e,Te) + \delta_4 \frac{d_l^2(w,Sw).d_l(e,Te)}{1+d_l^2(w,e)}\right) \tag{2.36}$$

for all $w, e \in \{TS(w_n)\}$, with $w \neq e$ where $\delta_1, \delta_2, \delta_3, \delta_4 > 0$, $\delta_1+\delta_2+\delta_3+\delta_4 = 1$ and $\delta_3+\delta_4 \neq 1$.

Then $\{TS(w_n)\} \to u \in E$. Furthermore, if (2.36) also holds for u, then u is the C.F.P of S and T.

Proof. As $w_1 \in Sw_0$ and $w_2 \in Tw_1$, by using Lemma 1.1.6

$$\begin{aligned}
\tau + F(d_l(w_1, w_2)) &= \tau + F(d_l(w_1, Tw_1)) \le \tau + F(H_{d_l}(Sw_0, Tw_1)) \\
&\le F\left(\begin{array}{c} \delta_1 d_l(w_0, w_1) + \delta_2 d_l(w_0, w_1) + \delta_3 d_l(w_1, Tw_1) + \\ \delta_4 \frac{d_l^2(w_0, Sw_0).d_l(w_1, Tw_1)}{1+d_l^2(w_0, w_1)} \end{array}\right) \\
&\le F\left(\begin{array}{c} \delta_1 d_l(w_0, w_1) + \delta_2 d_l(w_0, w_1) + \delta_3 d_l(w_1, w_2) + \\ \delta_4 d_l(w_1, Tw_1)\left(\frac{d_l^2(w_0, w_1)}{1+d_l^2(w_0, w_1)}\right) \end{array}\right) \\
&\le F((\delta_1 + \delta_2) d_l(w_0, w_1) + (\delta_3 + \delta_4) d_l(w_1, w_2)).
\end{aligned}$$

Since F is strictly increasing, we have

$$\begin{aligned}
d_l(w_1, w_2) &< (\delta_1 + \delta_2) d_l(w_0, w_1) + (\delta_3 + \delta_4) d_l(w_1, w_2) \\
&< \left(\frac{\delta_1 + \delta_2}{1 - \delta_3 - \delta_4}\right) d_l(w_0, w_1).
\end{aligned}$$

From $\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1$ and $\delta_3 + \delta_4 \neq 1$, we deduce $1 - \delta_3 - \delta_4 > 0$ and so

$$d_l(w_1, w_2) < d_l(w_0, w_1).$$

Consequently

$$F(d_l(w_1, w_2)) \le F(d_l(w_0, w_1)) - \tau.$$

As we have $w_{2i+1} \in Sw_{2i}$ and $w_{2i+2} \in Tw_{2i+1}$ then from (2.36), and Lemma 1.1.6 we have

$$\begin{aligned}
\tau + F(d_l(w_{2i+1}, w_{2i+2})) &= \tau + F(d_l(w_{2i+1}, Tw_{2i+1})) \le \tau + F(H_{d_l}(Sw_{2i}, Tw_{2i+1})) \\
&\le F(\delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, Sw_{2i}) \\
&\quad + \delta_3 d_l(w_{2i+1}, Tw_{2i+1}) + \delta_4 \frac{d_l^2(w_{2i}, Sw_{2i}).d_l(w_{2i+1}, Tw_{2i+1})}{1 + d_l^2(w_{2i}, w_{2i+1})}) \\
&\le F(\delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, w_{2i+1}) + \delta_3 d_l(w_{2i+1}, w_{2i+2}) \\
&\quad + \delta_4 d_l(w_{2i+1}, w_{2i+2}) \frac{d_l^2(w_{2i}, w_{2i+1})}{1 + d_l^2(w_{2i}, w_{2i+1})})
\end{aligned}$$

$$\leq F(\delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, w_{2i+1}) + \delta_3 d_l(w_{2i+1}, w_{2i+2}) + \delta_4 d_l(w_{2i+1}, w_{2i+2})).$$

Since F is strictly increasing, and $\delta_1+\delta_2+\delta_3+\delta_4 = 1$ where $\delta_3+\delta_4 \neq 1$, we deduce $1-\delta_3-\delta_4 > 0$ so we obtain

$$\begin{aligned}
d_l(w_{2i+1}, w_{2i+2}) &< \delta_1 d_l(w_{2i}, w_{2i+1}) + \delta_2 d_l(w_{2i}, w_{2i+1}) + \delta_3 d_l(w_{2i+1}, w_{2i+2}) \\
&\quad + \delta_4 d_l(w_{2i+1}, w_{2i+2})) \\
&< (\delta_1 + \delta_2) d_l(w_{2i}, w_{2i+1}) + (\delta_3 + \delta_4) d_l(w_{2i+1}, w_{2i+2}) \\
d_l(w_{2i+1}, w_{2i+2}) &< \left(\frac{\delta_1 + \delta_2}{1 - \delta_3 - \delta_4}\right) d_l(w_{2i}, w_{2i+1}) = d_l(w_{2i}, w_{2i+1}).
\end{aligned}$$

This implies that,

$$F(d_l(w_{2i+1}, w_{2i+2})) \leq F(d_l(w_{2i}, w_{2i+1})) - \tau$$

Following similar reasons are present in Theorem 2.3.6, we have $\{TS(w_n)\} \to u$ that is

$$\lim_{n\to\infty} d_l(w_n, u) = 0. \tag{2.37}$$

Now, by Lemma 1.6.1, we get

$$\tau + F(d_l(w_{2n+1}, Tu)) \leq \tau + F(Hd_l(Sw_{2n}, Tu)),$$

By using inequality (2.36), we have

$$\begin{aligned}
\tau + F(d_l(w_{2n+1}, Tu)) &\leq F(\delta_1 d_l(w_{2n}, u) + \delta_2 d_l(w_{2n}, Sw_{2n}) + \delta_3 d_l(u, Tu) \\
&\quad + \delta_4 \frac{d_l^2(w_{2n}, Sw_{2n}).d_l(u, Tu)}{1 + d_l^2(w_{2n}, u)}) \\
&\leq F(\delta_1 d_l(w_{2n}, u) + \delta_2 d_l(w_{2n}, w_{2n+1}) + \delta_3 d_l(u, Tu) \\
&\quad + \delta_4 \frac{d_l^2(w_{2n}, w_{2n+1}).d_l(u, Tu)}{1 + d_l^2(w_{2n}, u)}).
\end{aligned}$$

Since F is the strictly increasing maþþings, we have

$$\begin{aligned} d_l(w_{2n+1}, Tu) &< \delta_1 d_l(w_{2n}, u) + \delta_2 d_l(w_{2n}, w_{2n+1}) + \delta_3 d_l(u, Tu) \\ &\quad + \delta_4 \frac{d_l^2(w_{2n}, w_{2n+1}).d_l(u, Tu)}{1 + d_l^2(w_{2n}, u)}. \end{aligned}$$

Letting limit $n \to \infty$, and using the inequality (2.37), we get

$$d_l(u, Tu) < \delta_3 d_l(u, Tu).$$

It is not true, hence $d_l(u, Tu) = 0$ or $u \in Tu$. Similarly by using (2.36), (2.37), Lemma1.1.6 and the inequality

$$\tau + F(d_l(w_{2n+2}, Su)) \leq \tau + F(H_{d_l}(Tw_{2n+1}, Su))$$

we can show that $d_l(u, Su) = 0$ or $u \in Su$. Hence the S and T have a C.F.P u in (E, d_l). Now,

$$d_l(u, u) \leq d_l(u, Tu) + d_l(Tu, u) \leq 0.$$

This implies $d_l(u, u) = 0$. ■

Remark 2.3.7 We can achive all theorems related with partial mètric and mètric ŝþaces as the corollaries of the above theorems, which are not available in the literature.

We are proving řesulţs in this section by using the above definition.

Definition 2.3.8 Let (E, d_l) be a ĉomþleţe $D.M.S.$ The maþþings $S, T : E \to E$ are said to be a pair of new Ciric ţype rational $F-$ĉonţracţioñ, if for ėaċh $w, e \in E$, we have

$$\tau + F(d_l(Sw, Te)) \leq F(O_l(w, e)) \tag{2.38}$$

where $F \in \triangle_F$and $\tau > 0$, and

$$O_l(w, e) = \max\left\{d_l(w, e), \frac{d_l(w, Sw).d_l(e, Te)}{1 + d_l(w, e)}, d_l(w, Sw), d_l(e, Te)\right\}. \tag{2.39}$$

The succeeding theorem is the one of our major řesulţs.

Theorem 2.3.9 Let (E, d_l) be a ĉomþleţe $D.M.S$ and (S, T) be a pair of new Ciric ţype rational $F-$contraction. Then S and T have a C.F.P q in E and $d_l(q,q)=0$.

The proof of is similar as given for Theorem 2.3.2.

In above section, we derive an application of fixed point theorem 2.3.9 in form of Volterra type integral equations.

$$q(t)=\int_0^t L_1(t,n,q(n))dn, \tag{2.40}$$

$$g(t)=\int_0^t L_2(t,n,g(n))dn \tag{2.41}$$

for all $t\in[0,1]$. We find the solution of (2.40) and (2.41). Let $E=\{f: f$ is continuous function from $[0,1]$ to $\mathbb{R}_+\}$, endowed with the complete $D.M.S$. For $q\in E$, define norm as: $\|q\|_\tau=\sup\limits_{t\in[0,1]}\{|q(t)|\,e^{-\tau t}\}$, where $\tau>0$ is taken arbitrary. Then define

$$d_\tau(q,g)=\sup_{t\in[0,1]}\{|q(t)+g(t)|\,e^{-\tau t}\}=\|q+g\|_\tau$$

for each $q,g\in E$, with these settings, (E,d_τ) becomes a complete $D.M.S$.

Theorem 2.3.10 Let the conditions (i) and (ii) are hold:

(i) $L_1,L_2:[0,1]\times[0,1]\times E\to\mathbb{R}$;

(ii) Define

$$\begin{aligned} Sq(t) &= \int_0^t L_1(t,n,q(n))dn,\\ Tg(t) &= \int_0^t L_2(t,n,g(n))dn. \end{aligned}$$

Suppose there exist $\tau>0$, such that

$$|L_1(t,n,u)+L_2(t,n,g)|\le\frac{\tau K(q,g)}{(\tau\sqrt{\|K(q,g)\|_\tau}+1)^2}$$

for each $t, n \in [0,1]$ and $q, g \in E$, where

$$K(q,g) = \max\left\{|q(t)+g(t)|, \frac{|q(t)+Sq(t)|\,|g(t)+Tg(t)|}{1+|q(t)+g(t)|}, |q(t)+Sq(t)|, |g(t)+Tg(t)|\right\},$$

Then integral equations (2.40) and (2.41) has a solution.

By assumption (ii)

$$\begin{aligned}
|Sq(t)+Tg(t)| &= \int_0^t |L_1(t,n,q(n)+L_2(t,n,g(n)))|\,dn,\\
&\le \int_0^t \frac{\tau}{(\tau\sqrt{\|M(q,g)\|_\tau}+1)^2}([M(q,g)]e^{-\tau n})e^{\tau n}dn,\\
&\le \int_0^t \frac{\tau}{(\tau\sqrt{\|M(q,g)\|_\tau}+1)^2}\|M(q,g)\|_\tau e^{\tau n}dn,\\
&\le \frac{\tau\|M(q,g)\|_\tau}{(\tau\sqrt{\|M(q,g)\|_\tau}+1)^2}\int_0^t e^{\tau n}dn,\\
&\le \frac{\|M(q,g)\|_\tau}{(\tau\sqrt{\|M(q,g)\|_\tau}+1)^2}e^{\tau t},
\end{aligned}$$

This implies

$$|Sq(t)+Tg(t)|\,e^{-\tau t} \le \frac{\|K(q,g)\|_\tau}{(\tau\sqrt{\|K(q,g)\|_\tau}+1)^2}.$$

$$\|Sq(t)+Tg(t)\|_\tau \le \frac{\|K(q,g)\|_\tau}{(\tau\sqrt{\|M(q,g)\|_\tau}+1)^2}.$$

$$\frac{\tau\sqrt{\|K(q,g)\|_\tau}+1}{\sqrt{\|K(q,g)\|_\tau}} \le \frac{1}{\sqrt{\|Sq(t)+Tg(t)\|_\tau}}.$$

$$\tau + \frac{1}{\sqrt{\|K(q,g)\|_\tau}} \le \frac{1}{\sqrt{\|Sq(t)+Tg(t)\|_\tau}}.$$

which further implies

$$\tau - \frac{1}{\sqrt{\|Sq(t)+Tg(t)\|_\tau}} \le \frac{-1}{\sqrt{\|K(q,g)\|_\tau}}.$$

So, all the hypothesis of Theorem 2.3.9 are proved for $F(w) = \frac{-1}{\sqrt{w}}$; $w > 0$ and $d_\tau(q,g) = \|q+g\|_\tau$. Hence integral equations (2.40)and (2.41) has a unique solution.

2.4 Multivalued Fixed Point Results for a New Generalized F-Dominated Mappings with Application

The given results in this section can be seen in [46].

Let $(\acute{Z}, d_l)$ be a $D.M.S$, $c_0 \in \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ be the setvalued maps on $\acute{Z}$. Let $c_1 \in \check{S}c_0$ be an element such that $d_l(c_0, \check{S}c_0) = d_l(c_0, c_1)$. Let $c_2 \in \check{T}c_1$ be such that $d_l(c_1, \check{T}c_1) = d_l(c_1, c_2)$. Let $c_3 \in \check{S}c_2$ be such that $d_l(c_2, \check{S}c_2) = d_l(c_2, c_3)$. Proceeding this method, we get a sequence $c_{\check{n}}$ in $\acute{Z}$ so as $c_{2\check{n}+1} \in \check{S}c_{2\check{n}}$ and $c_{2\check{n}+2} \in \check{T}c_{2\check{n}+1}$, where $\check{n} = 0, 1, 2, \ldots$. Also $d_l(c_{2\check{n}}, \check{S}c_{2\check{n}}) = d_l(c_{2\check{n}}, c_{2\check{n}+1})$, $d_l(c_{2\check{n}+1}, \check{T}c_{2\check{n}+1}) = d_l(c_{2\check{n}+1}, c_{2\check{n}+2})$. We represent this type of sequence by $\{\check{T}\check{S}(c_{\check{n}})\}$.

Theorem 2.4.1 Let $(\acute{Z}, d_l)$ be a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e})F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y})F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})}\right) \end{aligned} \tag{2.42}$$

for each $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$ where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \tag{2.43}$$

where $\lambda = \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4}\right)$ and $\eta_3 + \eta_4 \neq 1$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ be the sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$ and $\{\check{T}\check{S}(c_{\check{n}})\} \to \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if the inequality (2.42) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$, then $\bar{u}$ is the C.F.P of $\check{S}$ and $\check{T}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

Proof. Consider a sequence $\{\check{T}\check{S}(c_{\check{n}})\}$. From (2.43), we get

$$d_l(c_0, c_1) \leq d_l(c_0, \check{S}c_0) \leq \check{r}.$$

It means that,

$$c_1 \in \overline{B_{d_l}(c_0, \check{r})}.$$

Let $c_2, \cdots, c_j \in \overline{B_{d_l}(c_0, \check{r})}$ for ēverŷ j belongs to N. If $j = 2\imath + 1$, where $\imath = 1, 2, \ldots, \frac{j-1}{2}$. Since $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ be a semi α_*-dominated maþþings on $\overline{B_{d_l}(c_0, \check{r})}$, so $\alpha_*(c_{2\imath}, \check{S}c_{2\imath}) \geq 1$ and $\alpha_*(c_{2\imath+1}, \check{T}c_{2\imath+1}) \geq 1$. As $\alpha_*(c_{2\imath}, \check{S}c_{2\imath}) \geq 1$, this implies $\inf\{\alpha(c_{2\imath}, b) : b \in \check{S}c_{2\imath}\} \geq 1$. Also $c_{2\imath+1} \in \check{S}c_{2\imath}$, so $\alpha(c_{2\imath}, c_{2\imath+1}) \geq 1$. Now by using Lemma 1.1.6, have,

$$
\begin{aligned}
\tau + F(d_l(c_{2\imath+1}, c_{2\imath+2})) &\leq \tau + F(H_{d_l}(\check{S}c_{2\imath}, \check{T}c_{2\imath+1})) \\
&\leq \max\{\tau + \alpha_*(c_{2\imath}, \check{S}c_{2\imath})F(H_{d_l}(\check{S}c_{2\imath}, \check{T}c_{2\imath+1})), \\
&\qquad \tau + \alpha_*(c_{2\imath+1}, \check{T}c_{2\imath+1})F(H_{d_l}(\check{T}c_{2\imath+1}, \check{S}c_{2\imath}))\} \\
&\leq F[\eta_1 d_l(c_{2\imath}, c_{2\imath+1}) + \eta_2 d_l(c_{2\imath}, \check{S}c_{2\imath}) + \eta_3 d_l(c_{2\imath}, \check{T}c_{2\imath+1}) \\
&\qquad + \eta_4 \frac{d_l^2(c_{2\imath}, \check{S}c_{2\imath}) . d_l(c_{2\imath+1}, \check{T}c_{2\imath+1})}{1 + d_l^2(c_{2\imath}, c_{2\imath+1})}] \\
&\leq F[\eta_1 d_l(c_{2\imath}, c_{2\imath+1}) + \eta_2 d_l(c_{2\imath}, c_{2\imath+1}) \\
&\qquad + \eta_3 d_l(c_{2\imath}, c_{2\imath+1}) + \eta_3 d_l(c_{2\imath+1}, c_{2\imath+2}) \\
&\qquad + \eta_4 \frac{d_l^2(c_{2\imath}, c_{2\imath+1}) . d_l(c_{2\imath+1}, c_{2\imath+2})}{1 + d_l^2(c_{2\imath}, c_{2\imath+1})}] \\
&\leq F((\eta_1 + \eta_2 + \eta_3) d_l(c_{2\imath}, c_{2\imath+1}) - \tau,
\end{aligned}
$$

this implies

$$
\begin{aligned}
F(d_l(c_{2\imath+1}, c_{2\imath+2})) &\leq F((\eta_1 + \eta_2 + \eta_3) d_l(c_{2\imath}, c_{2\imath+1}) \\
&\qquad + (\eta_3 + \eta_4) d_l(c_{2\imath+1}, c_{2\imath+2})),
\end{aligned}
$$

for ėach j belongs to N. As F is strictly increasing, so we obtain

$$
\begin{aligned}
d_l(c_{2\imath+1}, c_{2\imath+2}) &< (\eta_1 + \eta_2 + \eta_3) d_l(c_{2\imath}, c_{2\imath+1}) \\
&\qquad + (\eta_3 + \eta_4) d_l(c_{2\imath+1}, c_{2\imath+2}) \\
(1 - \eta_3 - \eta_4) d_l(c_{2\imath+1}, c_{2\imath+2}) &< (\eta_1 + \eta_2 + \eta_3) d_l(c_{2\imath}, c_{2\imath+1}) \\
d_l(c_{2\imath+1}, c_{2\imath+2}) &< \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4}\right) d_l(c_{2\imath}, c_{2\imath+1}).
\end{aligned}
$$

Here $\lambda = \left(\frac{\eta_1+\eta_2+\eta_3}{1-\eta_3-\eta_4}\right) < 1$. Hence

$$d_l(c_{2i+1}, c_{2i+2}) < \lambda d_l(c_{2i}, c_{2i+1}) < \lambda^2 d_l(c_{2i-1}, c_{2i}) < \cdots < \lambda^j d_l(c_0, c_1). \tag{2.44}$$

Now,

$$\begin{aligned} d_l(c_0, c_{j+1}) &\leq d_l(c_0, c_1) + d_l(c_1, c_2) + \cdots + d_l(c_j, c_{j+1}) \\ &\leq d_l(c_0, c_1)[1 + \lambda + \cdots + \lambda^j] \\ &\leq (1-\lambda)\check{r}\frac{(1-\lambda^{j+1})}{(1-\lambda)} < \check{r}. \end{aligned}$$

Thus $c_{j+1} \in \overline{B_{d_l}(c_0, \check{r})}$. Hence $c_{\check{n}} \in \overline{B_{d_l}(c_0, \check{r})}$, for each $\check{n}$ belongs to N. Proceeding this method, we get

$$\begin{aligned} \tau + F(d_l(c_{\check{n}}, c_{\check{n}+1})) &\leq \tau + F(H_{d_l}(\check{S}c_{\check{n}-1}, \check{T}c_{\check{n}})) \\ &\leq \max\{\tau + \alpha_*(c_{\check{n}-1}, \check{S}c_{\check{n}-1})F(H_{d_l}(\check{S}c_{\check{n}-1}, \check{T}c_{\check{n}})), \\ &\quad \tau + \alpha_*(c_{\check{n}}, \check{T}c_{\check{n}})F(H_{d_l}(\check{T}c_{\check{n}}, \check{S}c_{\check{n}-1}))\} \\ &\leq F[\eta_1 d_l(c_{\check{n}-1}, c_{\check{n}}) + \eta_2 d_l(c_{\check{n}-1}, \check{S}c_{\check{n}-1}) + \eta_3 d_l(c_{\check{n}-1}, \check{T}c_{\check{n}}) \\ &\quad +\eta_4 \frac{d_l^2(c_{\check{n}-1}, \check{S}c_{\check{n}-1}) . d_l(c_{\check{n}}, \check{T}c_{\check{n}})}{1 + d_l^2(c_{\check{n}-1}, c_{\check{n}})}] \\ &\leq F[\eta_1 d_l(c_{\check{n}-1}, c_{\check{n}}) + \eta_2 d_l(c_{\check{n}-1}, c_{\check{n}}) + \eta_3 d_l(c_{\check{n}-1}, c_{\check{n}}) \\ &\quad +\eta_3 d_l(c_{\check{n}}, c_{\check{n}-1}) + \eta_4 \frac{d_l^2(c_{\check{n}-1}, c_{\check{n}}) . d_l(c_{\check{n}}, c_{\check{n}+1})}{1 + d_l^2(c_{\check{n}-1}, c_{\check{n}})}] \\ &\leq F[(\eta_1 + \eta_2 + \eta_3)d_l(c_{\check{n}-1}, c_{\check{n}}) \\ &\quad +(\eta_3 + \eta_4)d_l(c_{\check{n}}, c_{\check{n}+1})] - \tau, \end{aligned}$$

this implies

$$\begin{aligned} F(d_l(c_{\check{n}}, c_{\check{n}+1})) &\leq F[(\eta_1 + \eta_2 + \eta_3)d_l(c_{\check{n}-1}, c_{\check{n}}) \\ &\quad +(\eta_3 + \eta_4)d_l(c_{\check{n}}, c_{\check{n}+1})], \end{aligned}$$

for each $\check{n}$ belongs to N. As F is strictly increasing

$$\begin{aligned} d_l(c_{\check{n}}, c_{\check{n}+1}) &< (\eta_1 + \eta_2 + \eta_3) d_l(c_{\check{n}-1}, c_{\check{n}}) \\ &\quad + (\eta_3 + \eta_4) d_l(c_{\check{n}}, c_{\check{n}+1}) \\ (1 - \eta_3 - \eta_4) d_l(c_{\check{n}}, c_{\check{n}+1}) &< (\eta_1 + \eta_2 + \eta_3) d_l(c_{\check{n}-1}, c_{\check{n}}) \\ d_l(c_{\check{n}}, c_{\check{n}+1}) &< \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3 - \eta_4}\right) d_l(c_{\check{n}-1}, c_{\check{n}}). \end{aligned}$$

Here $\lambda = \left(\frac{\eta_1+\eta_2+\eta_3}{1-\eta_3-\eta_4}\right) < 1$. Hence

$$d_l(c_{\check{n}}, c_{\check{n}+1}) < \lambda d_l(c_{\check{n}-1}, c_{\check{n}}) < d_l(c_{\check{n}-1}, c_{\check{n}}). \tag{2.45}$$

Consequently,

$$\tau + F(d_l(c_{\check{n}}, c_{\check{n}+1})) \leq F(d_l(c_{\check{n}-1}, c_{\check{n}})),$$

which implies,

$$\begin{aligned} F(d_l(c_{\check{n}}, c_{\check{n}+1})) &\leq F(d_l(c_{\check{n}-1}, c_{\check{n}})) - \tau \\ &\vdots \\ &\leq F(d_l(c_0, c_1)) - \check{n}\tau. \end{aligned}$$

Which implies

$$F(d_l(c_{\check{n}}, c_{\check{n}+1})) \leq F(d_l(c_0, c_1)) - \check{n}\tau. \tag{2.46}$$

And so $\lim_{\check{n}\to\infty} F(d_l(c_{\check{n}}, c_{\check{n}+1})) = -\infty$. By (F_2), we find that

$$\lim_{\check{n}\to\infty} F(d_l(c_{\check{n}}, c_{\check{n}+1})) = 0. \tag{2.47}$$

We shall prove that $\{\check{T}\check{S}(c_{\check{n}})\}$ is Cauchy in $(\acute{Z}, d_l)$. So, it suffices to show that $\lim_{\check{n}\to\infty} d_l(c_{\check{n}}, c_m) = 0$. We explain by contradiction. Suppose there must be a $\in > 0$ and şequences $(\check{n}(q))$ and $(m(q))$ of natural no so as

$$m(q) > \check{n}(q) > q,\ d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) \geq \in,\ d_l(c_{\check{n}(q)+1}, c_{m(q)}) < \in \text{ for each } q \in N. \tag{2.48}$$

By triangular inequality, we have

$$\begin{aligned} d_l(c_{\check{n}(q)}, c_{m(q)}) &\leq d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) + d_l(c_{\check{n}(q)+1}, c_{m(q)}) \\ &\leq \in + d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) \\ &\leq \in + d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) \end{aligned} \tag{2.49}$$

From (2.47) there exist q_1 belongs to N such that for ėaċh $q \geq q_1$.

$$d_l(c_{\check{n}(q)}, \check{T}c_{\check{n}(q)}) < \in . \tag{2.50}$$

Combining (2.49) to (2.50) yeilds that

$$d_l(c_{\check{n}(q)}, c_{m(q)}) < 2 \in, \text{ for ėaċh } q \geq q_1. \tag{2.51}$$

As $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ be a semi α_*-dominated maþþings on $\overline{B_{d_l}(c_0, \check{r})}$. So $\alpha_*(c_{\check{n}(q)}, \check{S}c_{\check{n}(q)}) \geq 1$ and $\alpha_*(c_{m(q)}, \check{T}c_{m(q)}) \geq 1$, for ėaċh $m, \check{n}$ belongs to N. Now, by using Lemma 1.1.6 and condition (2.42), we get

$$\begin{aligned} F(\in) &\leq F(H_{d_l}(\check{S}c_{\check{n}(q)}, \check{T}c_{m(q)}) \leq \max\{\tau + \alpha_*(c_{\check{n}(q)}, \check{S}c_{\check{n}(q)})F(H_{d_l}(\check{S}c_{\check{n}(q)}, \check{T}c_{m(q)})) \\ &\quad , \tau + \alpha_*(c_{m(q)}, \check{T}c_{m(q)})F(H_{d_l}(\check{T}c_{m(q)}, \check{S}c_{\check{n}(q)}))\} \\ &\leq F[\eta_1 d_l(c_{\check{n}(q)}, c_{m(q)}) + \eta_2 d_l(c_{\check{n}(q)}, \check{S}c_{\check{n}(q)}) \\ &\quad + \eta_3 d_l(c_{\check{n}(q)}, \check{T}c_{m(q)}) + \eta_4 \frac{d_l^2(c_{\check{n}(q)}, \check{S}c_{\check{n}(q)}).d_l(c_{m(q)}, \check{T}c_{m(q)})}{1 + d_l^2(c_{\check{n}(q)}, c_{m(q)})}] - \tau \\ F(\in) &\leq F[\eta_1 d_l(c_{\check{n}(q)}, c_{m(q)}) + \eta_2 d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) + \eta_3 d_l(c_{\check{n}(q)}, c_{m(q)+1}) \\ &\quad + \eta_4 \frac{d_l^2(c_{\check{n}(q)}, c_{\check{n}(q)+1}).d_l(c_{m(q)}, c_{m(q)+1})}{1 + d_l^2(c_{\check{n}(q)}, c_{m(q)})}] - \tau \\ F(\in) &\leq F[\eta_1 d_l(c_{\check{n}(q)}, c_{m(q)}) + \eta_2 d_l(c_{\check{n}(q)}, c_{\check{n}(q)+1}) + \eta_3 d_l(c_{\check{n}(q)}, c_{m(q)}) \\ &\quad + \eta_3 d_l(c_{m(q)}, c_{m(q)+1}) + \eta_4 \frac{d_l^2(c_{\check{n}(q)}, c_{\check{n}(q)+1}).d_l(c_{m(q)}, c_{m(q)+1})}{1 + d_l^2(c_{\check{n}(q)}, c_{m(q)})}] - \tau \end{aligned}$$

This means that,

$$F(\in) \leq F[2\eta_1 \in + \eta_2 \in + 3\eta_3 \in + \eta_4 \in] - \tau.$$

As, $2\eta_1 + \eta_2 + 3\eta_3 + \eta_4 < 1$, so we get

$$2\eta_1 \in +\eta_2 \in +3\eta_3 \in +\eta_4 \in < \in,$$

we deduce that

$$F(\in) < F(\in),$$

which is not true. Thus $\{\check{T}\check{S}(c_{\check{n}})\}$ be a Ĉauchŷ şequenće in $(\overline{B_{d_l}(c_0, \check{r})}, d_l)$. Since $(\overline{B_{d_l}(c_0, \check{r})}, d_l)$ is a ĉomplete mètric śpace, so there exist $\bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$ such that $\{\check{T}\check{S}(c_{\check{n}})\} \to \bar{u}$ as $\check{n} \to \infty$ then

$$\lim_{\check{n}\to\infty} d_l(c_{\check{n}}, \bar{u}) = 0. \tag{2.52}$$

Since $\alpha_*(\bar{u}, \check{T}\bar{u}) \geq 1$, and $\alpha_*(c_{2\check{n}}, \check{S}c_{2\check{n}}) \geq 1$ by using Lemma 1.1.6, and the inequality (2.42), we have

$$\begin{aligned} F(d_l(c_{2\check{n}+1}, \check{T}\bar{u})) &\leq F(H_{d_l}(\check{S}c_{2\check{n}}, \check{T}\bar{u})) \\ &\leq \max\{\tau + \alpha_*(c_{2\check{n}}, \check{S}c_{2\check{n}})F(H_{d_l}(\check{S}c_{2\check{n}}, \check{T}\bar{u})) \\ &\quad , \tau + \alpha_*(\bar{u}, \check{T}\bar{u})F(H_{d_l}(\check{T}\bar{u}, \check{S}c_{2\check{n}}))\} \end{aligned}$$

$$\begin{aligned} &\leq F[\eta_1 d_l(c_{2\check{n}}, \bar{u}) + \eta_2 d_l(c_{2\check{n}}, \check{S}c_{2\check{n}}) + \eta_3 d_l(c_{2\check{n}}, \check{T}\bar{u}) \\ &\quad + \eta_4 \frac{d_l^2(c_{2\check{n}}, \check{S}c_{2\check{n}}).d_l(\bar{u}, \check{T}\bar{u})}{1 + d_l^2(c_{2\check{n}}, \bar{u})}] - \tau \\ &\leq F[\eta_1 d_l(c_{2\check{n}}, \bar{u}) + \eta_2 d_l(c_{2\check{n}}, \check{S}c_{2\check{n}}) + \eta_3 d_l(c_{2\check{n}}, \bar{u}) \\ &\quad + \eta_3(\bar{u}, \check{T}\bar{u}) + \eta_4 \frac{d_l^2(c_{2\check{n}}, \check{S}c_{2\check{n}}).d_l(\bar{u}, \check{T}\bar{u})}{1 + d_l^2(c_{2\check{n}}, \bar{u})}] - \tau. \end{aligned}$$

By using (2.52) we get

$$F(d_l(\bar{u}, \check{T}\bar{u})) \leq F[\eta_3 d_l(\bar{u}, \check{T}\bar{u})] - \tau.$$

This implies

$$d_l(\bar{u}, \check{T}\bar{u}) < \eta_3 d_l(\bar{u}, \check{T}\bar{u}) < d_l(\bar{u}, \check{T}\bar{u}).$$

Which contradicts to fact, hence $d_l(\bar{u}, \check{T}\bar{u}) = 0$ or $\bar{u} \in \check{T}\bar{u}$. Similarly, by using Lemma 1.1.6, inequality (2.42) and the inequality

$$\begin{aligned} d_l(\bar{u}, \check{S}\bar{u}) &\leq d_l(\bar{u}, c_{2\check{n}+2}) + d_l(c_{2\check{n}+2}, \check{S}\bar{u}) \\ &\leq d_l(\bar{u}, c_{2\check{n}+2}) + d_l(\check{T}_{2\check{n}+1}, \check{S}\bar{u}) \end{aligned}$$

we can prove $d_l(\bar{u}, \check{S}\bar{u}) = 0$. $\bar{u} \in \check{S}\bar{u}$. Hence $\bar{u}$ be the C.F.P of $\check{S}$ and $\check{T}$ in $\overline{B_{d_l}(c_0, \check{r})}$. Now,

$$d_l(\bar{u}, \bar{u}) \leq d_l(\bar{u}, \check{T}\bar{u}) + d_l(\check{T}\bar{u}, \bar{u}) \leq 0.$$

This implies that $d_l(\bar{u}, \bar{u}) = 0$. ■

Example 2.4.2 Let $\acute{Z} = Q^+ \cup \{0\}$ and let $d_l : \acute{Z} \times \acute{Z} \to \acute{Z}$ be the ĉomḃleţe $D.M.S$ on $\acute{Z}$ defined by

$$d_l(i, j) = i + j \text{ for all } i, j \in \acute{Z}.$$

Define, $\check{S}, \check{T} : \acute{Z} \times \acute{Z} \to P(\acute{Z})$ by,

$$\check{S}\acute{z} = \begin{cases} [\frac{\acute{z}}{3}, \frac{2}{3}\acute{z}] \text{ if } \acute{z} \in [0, 7] \cap \acute{Z} \\ [\acute{z}, \acute{z} + 1] \text{ if } \acute{z} \in (7, \infty) \cap \acute{Z} \end{cases}$$

and,

$$\check{T}\acute{z} = \begin{cases} [\frac{\acute{z}}{4}, \frac{3}{4}\acute{z}] \text{ if } \acute{z} \in [0, 7] \cap \acute{Z} \\ [\acute{z} + 1, \acute{z} + 3] \text{ if } \acute{z} \in (7, \infty) \cap \acute{Z}. \end{cases}$$

Taking, $x_0 = 1$, $\check{r} = 8$, $\lambda = \frac{1}{3}$ then $\overline{B_{d_l}(x_0, \check{r})} = [0, 7] \cap \acute{Z}$. Now

$$\begin{aligned} d_l(x_0, \check{S}x_0) &< (1 - \lambda)\check{r} \Rightarrow \frac{4}{3} < (1 - \frac{4}{9})8 \\ 12 &< 40. \end{aligned}$$

So, we obtain a şequenće $\{\check{T}\check{S}(x_{\check{n}})\} = \{1, \frac{1}{12}, \frac{1}{144}, \frac{1}{1728}, ...\}$ in $\acute{Z}$ generated by x_0. Also, $\overline{B_{d_l}(x_0, \check{r})} \cap \{\check{T}\check{S}(x_{\check{n}})\} = \{1, \frac{1}{12}, \frac{1}{144}, ...\}$ and

$$\alpha(c, d) = \begin{cases} 1 \text{ if } c, d \in [0, 1] \\ \frac{3}{2} \quad \text{otherwise.} \end{cases}$$

Now, if $x, y \in \overline{B_{d_l}(x_0, \check{r})} \cap \{\check{T}\check{S}(x_{\check{n}})\}$, then we have the following cases.

Case 1. If

$$\begin{aligned} & \max\{\tau + \alpha_*(x, \check{S}x)F(H_{d_l}(\check{S}x, \check{T}y)), \tau + \alpha_*(y, \check{T}y)F(H_{d_l}(\check{T}y, \check{S}x))\} \\ = \; & \tau + \alpha_*(x, \check{S}x)F(H_{d_l}(\check{S}x, \check{T}y)) \end{aligned}$$

then we consider only

$$\begin{aligned} \alpha_*(x, \check{S}x)H_{d_l}(\check{S}x, \check{T}y) &= 1[\max\{\sup_{\delta \in \check{S}x} d_l(a, \check{T}y), \sup_{b \in \check{T}y} d_l(\check{S}x, b)\}] \\ &= \max\{\sup_{a \in \check{S}x} d_l(a, [\frac{y}{4}, \frac{3y}{4}]), \sup_{b \in \check{T}y} d_l([\frac{x}{3}, \frac{2x}{3}], b)\} \\ &= \max\{d_l(\frac{2x}{3}, [\frac{y}{4}, \frac{3y}{4}]), d_l([\frac{x}{3}, \frac{2x}{3}], \frac{3y}{4})\} \\ &= \max\{d_l(\frac{2x}{3}, \frac{y}{4}), d_l(\frac{x}{3}, \frac{3y}{4})\} \\ &= \max\{\frac{2x}{3} + \frac{y}{4}, \frac{x}{3} + \frac{3y}{4}\} < \frac{1}{5} d_l(x, y) \\ &\quad + \frac{1}{10} d_l(x, [\frac{x}{3}, \frac{2}{3}x]) + \frac{1}{15} d_l(x, [\frac{y}{4}, \frac{3}{4}y]) \\ &\quad + \frac{1}{30} \frac{d_l^2(x, [\frac{x}{3}, \frac{2}{3}x]).d_l(y, [\frac{y}{4}, \frac{3}{4}y])}{1 + d_l^2(x, y)} \\ &= \frac{1}{5}(x + y) + \frac{2x}{15} + \frac{4x + y}{60} + \frac{5x^2y^2}{54\{1 + (x + y)^2\}}. \end{aligned}$$

Thus,

$$H_{d_l}(\check{S}x, \check{T}y) < \eta_1 d_l(x, y) + \eta_2 d_l(x, \check{S}x) + \eta_3 d_l(x, \check{T}y) + \eta_4 \frac{d_l^2(x, \check{S}x).d_l(y, \check{T}y)}{1 + d_l^2(x, y)},$$

which implies that,

$$\tau + \ln(H_{d_l}(\check{S}x, \check{T}y)) \leq \ln \left(\begin{array}{c} \eta_1 d_l(x, y) + \eta_2 d_l(x, \check{S}x) + \eta_3 d_l(x, \check{T}y) \\ + \eta_4 \frac{d_l^2(x, \check{S}x).d_l(y, \check{T}y)}{1 + d_l^2(x, y)} \end{array} \right).$$

That is

$$\tau + F(H_{d_l}(\check{S}x, \check{T}y)) \leq F\left(\begin{array}{c} \eta_1 d_l(x,y) + \eta_2 d_l(x, \check{S}x) + \eta_3 d_l(x, \check{T}y) \\ +\eta_4 \frac{d_l^2(x,\check{S}x).d_l(y,\check{T}y)}{1+d_l^2(x,y)} \end{array}\right).$$

For $\tau = (0, \frac{12}{95}]$, $\eta_1 = \frac{1}{5}$, $\eta_2 = \frac{1}{10}$, $\eta_3 = \frac{1}{15}$, $\eta_4 = \frac{1}{30}$, and $\lambda = \frac{4}{9}$. Thus the mapping $\check{S}$ and $\check{T}$ satisfying all the contractive conditions of Theorem 2.4.1 on closed báll rather than whole space. Now if $x = 8$, $y = 9 \in (7, \infty) \cap \acute{Z}$, then

$$\tau + F(H_{d_l}(\check{S}x, \check{T}y)) > F\left(\begin{array}{c} \eta_1 d_l(x,y) + \eta_2 d_l(x, \check{S}x) + \eta_3 d_l(x, \check{T}y) \\ +\eta_4 \frac{d_l^2(x,\check{S}x).d_l(y,\check{T}y)}{1+d_l^2(x,y)} \end{array}\right).$$

and consequently condition (2.42) not holds on $\acute{Z}$.

Case 2. If $\max\{\tau+\alpha_*(x,\check{S}x)F(H_{d_l}(\check{S}x,\check{T}y)), \tau+\alpha_*(y,\check{T}y)F(H_{d_l}(\check{T}y,\check{S}x))\} = \tau+\alpha_*(y,\check{T}y)F(H_{d_l}(\check{T}y,\check{S}x))$. Then by using the similar arguments of Case1 we can get the same results.

If, we take $\check{S} = \check{T}$ in Theorem 2.4.1, then we are left with the result.

Corollary 2.4.3 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S} : \acute{Z} \to P(\acute{Z})$ be the semi α_*-dominated mapping on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e})F(H_{d_l}(\check{S}\check{e}, \check{S}\hat{y})), \tau + \alpha_*(\hat{y}, \check{S}\hat{y})F(H_{d_l}(\check{S}\hat{y}, \check{S}\check{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{S}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{S}\hat{y})}{1 + d_l^2(x, \hat{y})}\right) \end{aligned} \tag{2.53}$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{S}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$ where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1+\eta_2+\eta_3}{1-\eta_3-\eta_4}\right)$ and $\eta_3+\eta_4 \neq 1$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ be a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$ and $\{\check{S}\check{S}(c_{\check{n}})\} \to \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if the inequality (2.53) holds for $\check{e}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

If, we take $\eta_2 = 0$ in Theorem 2.4.1, then we are left only with the result.

Corollary 2.4.4 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$

exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ be the semi α_*−dominated maþþings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned}
&\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e})F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y})F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e}))\} \\
\leq \; & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})}\right) \qquad (2.54)
\end{aligned}$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$ where $\eta_1, \eta_3, \eta_4 > 0$, $\eta_1 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1 + \eta_3}{1 - \eta_3 - \eta_4}\right)$ and $\eta_3 + \eta_4 \neq 1$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a şequenće in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for ėaċh $\check{n}$ belongs to $N \cup \{0\}$ and $\{\check{T}\check{S}(c_{\check{n}})\} \to \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if the inequality (2.54) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for ėaċh $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

If, we take $\eta_3 = 0$ in Theorem 2.4.1, then we are left only with the result.

Corollary 2.4.5 Let $(\acute{Z}, d_l)$ is a ĉomþleţe $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ be the semi α_*−dominated maþþings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned}
&\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e})F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y})F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e}))\} \\
\leq \; & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})}\right) \qquad (2.55)
\end{aligned}$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$ where $\eta_1, \eta_2, \eta_4 > 0$, $\eta_1 + \eta_2 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1 + \eta_2}{1 - \eta_4}\right)$ and $1 - \eta_4 \neq 0$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a şequenće in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for ėaċh $\check{n}$ belongs to $N \cup \{0\}$ and $\{\check{T}\check{S}(c_{\check{n}})\} \to \bar{u} \in \overline{B_{d_l}(c_0, \check{r})}$. Also if the inequality (2.55) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for ėaċh $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

If, we take $\eta_4 = 0$ in Theorem 2.4.1, then we are left only with the result.

Corollary 2.4.6 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ be the semi α_*−dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + \alpha_*(\breve{e}, \check{S}\breve{e})F(H_{d_l}(\check{S}\breve{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y})F(H_{d_l}(\check{T}\hat{y}, \check{S}\breve{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\breve{e}, \hat{y}) + \eta_2 d_l(\breve{e}, \check{S}\breve{e}) + \eta_3 d_l(\breve{e}, \check{T}\hat{y})\right) \end{aligned} \tag{2.56}$$

for all $\breve{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ with either $\alpha(\breve{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \breve{e}) \geq 1$ where $\eta_1, \eta_2, \eta_3 > 0$, $\eta_1 + \eta_2 + 2\eta_3 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1 + \eta_2 + \eta_3}{1 - \eta_3}\right)$ and $1 - \eta_3 \neq 0$. Then $\{\check{T}\check{S}(c_{\check{n}})\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$ and $\{\check{T}\check{S}(c_{\check{n}})\} \to \bar{u} \in \overline{B_{d_l}(\bar{u}_0, \check{r})}$. Also if the inequality (2.56) holds for $\bar{u}$ and either $\alpha(c_{\check{n}}, \bar{u}) \geq 1$ or $\alpha(\bar{u}, c_{\check{n}}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P $\bar{u}$ in $\overline{B_{d_l}(c_0, \check{r})}$.

Theorem 2.4.7 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$ endowed a graph $\check{G}$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})}$, $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ and let for a sequence $\{\check{T}\check{S}(c_{\check{n}})\}$ in $\acute{Z}$ generated by c_0, with $(c_0, c_1) \in E(\check{G})$. Assume (i) (ii) and (iii)hold:

(i) $\check{S}$ and $\check{T}$ are multi graph dominated for each $\breve{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$;

(ii) there exists some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + \alpha_*(\breve{e}, \check{S}\breve{e})F(H_{d_l}(\check{S}\breve{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y})F(H_{d_l}(\check{T}\hat{y}, \check{S}\breve{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\breve{e}, \hat{y}) + \eta_2 d_l(\breve{e}, \check{S}\breve{e}) + \eta_3 d_l(\breve{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\breve{e}, \check{S}\breve{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\breve{e}, \hat{y})}\right) \end{aligned} \tag{2.57}$$

where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$ such that

$$\tau + F(H_{d_l}(\check{S}\breve{e}, \check{T}\hat{y})) \leq F\left(\begin{array}{c} \eta_1 d_l(\breve{e}, \hat{y}) + \eta_2 d_l(\breve{e}, \check{S}\breve{e}) + \eta_3 d_l(\breve{e}, \check{T}\hat{y}) \\ +\eta_4 \frac{d_l^2(\breve{e}, \check{S}\breve{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\breve{e}, \hat{y})} \end{array}\right) \tag{2.58}$$

for all $\breve{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(c_{\check{n}})\}$ & $(\breve{e}, \hat{y}) \in E(\check{G})$ or $(\hat{y}, \breve{e}) \in E(\check{G})$;

(iii) $\sum_{i=0}^{\check{n}} \lambda^i(d_l(c_0, \check{S}c_0)) \leq \check{r}$ for every $\check{n}$ belongs to $N \cup \{0\}$.

Then, $\{\check{T}\check{S}(c_{\check{n}})\}$ be the sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $(c_{\check{n}}, c_{\check{n}+1}) \in E(\check{G})$ and $\{\check{T}\check{S}(c_{\check{n}})\} \to m^*$. Also, if and the inequality (2.57) holds for m^* and $(c_{\check{n}}, m^*) \in E(\check{G})$ or $(m^*, c_{\check{n}}) \in E(\check{G})$ for each $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P m^* in $\overline{B_{d_l}(c_0, \check{r})}$.

Proof. Define, $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ by

$$\alpha(\check{e}, \hat{y}) = \begin{cases} 1, & \text{if } \check{e} \in \overline{B_{d_l}(c_0, \check{r})},\ (\check{e}, \hat{y}) \in E(\check{G}) \text{ or } (\hat{y}, \check{e}) \in E(\check{G}) \\ 0, & \text{otherwise.} \end{cases}$$

Since $\check{S}$ and $\check{T}$ are semi graph dominated on $\overline{B_{d_l}(c_0, \check{r})}$, then for $\check{e} \in \overline{B_{d_l}(c_0, \check{r})}$, $(\check{e}, \hat{y}) \in E(\check{G})$ for all $\hat{y} \in \check{S}\check{e}$ and $(\check{e}, \hat{y}) \in E(\check{G})$ for all $\hat{y} \in \check{T}\check{e}$. So, $\alpha(\check{e}, \hat{y}) = 1$ for all $\hat{y} \in \check{S}\check{e}$ and $\alpha(\check{e}, \hat{y}) = 1$ for all $\hat{y} \in \check{T}\check{e}$. This implies that $\inf\{\alpha(\check{e}, \hat{y}) : \hat{y} \in \check{S}\check{e}\} = 1$ and $\inf\{\alpha(\check{e}, \hat{y}) : \hat{y} \in \check{T}\check{e}\} = 1$. Hence $\alpha_*(\check{e}, \check{S}\check{e}) = 1$, $\alpha_*(\check{e}, \check{T}\check{e}) = 1$ for all $\check{e} \in \overline{B_{d_l}(c_0, \check{r})}$. So, $\check{S}, \check{T} : \acute{Z} \to P(\acute{Z})$ are the semi α_*-dominated mapping on $\overline{B_{d_l}(c_0, \check{r})}$. Moreover, we can write (2.57) as

$$\begin{aligned} &\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e}) F(H_{d_l}(\check{S}\check{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y}) F(H_{d_l}(\check{T}\hat{y}, \check{S}\check{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})}\right) \end{aligned}$$

for all elements $\check{e}, \hat{y}$ in $\overline{B_{d_l}(c_0, \check{r})} \cap \{\check{T}\check{S}(x_{\check{n}})\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$. Also, (iii) holds. Then, by Theorem 2.4.1, we have $\{\check{T}\check{S}(c_{\check{n}})\}$ is the şequenće in $\overline{B_{d_l}(c_0, \check{r})}$ & $\{\check{T}\check{S}(c_{\check{n}})\} \to m^* \in \overline{B_{d_l}(c_0, \check{r})}$. Now, $c_{\check{n}}, m^* \in \overline{B_{d_l}(c_0, \check{r})}$ and either $(c_{\check{n}}, m^*) \in E(\check{G})$ or $(m^*, c_{\check{n}}) \in E(\check{G})$ implies that either $\alpha(c_{\check{n}}, m^*) \geq 1$ or $\alpha(m^*, c_{\check{n}}) \geq 1$. So, all hypothesis of Theorem 2.4.1 are proved. Hence, by Theorem 2.4.7, $\check{S}$ and $\check{T}$ have a C.F.P m^* in $\overline{B_{d_l}(c_0, \check{r})}$ and $d_l(m^*, m^*) = 0$.

In this section, we discussed new fixȩd þoīnţ řesulţs for one map in ĉomþleţe $D.M.S.$ ■

Theorem 2.4.8 Let $(\acute{Z}, d_l)$ is a ĉomþleţe $D.M.S.$ Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to \acute{Z}$ be the semi α_*-dominated maþþings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e}) F(d_l(\check{S}\check{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y}) F(d_l(\check{T}\hat{y}, \check{S}\check{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})}\right) \qquad (2.59) \end{aligned}$$

for all $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$ where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$,

$\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1-\lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1+\eta_2+\eta_3}{1-\eta_3-\eta_4}\right)$ and $\eta_3 + \eta_4 \neq 1$. Then $\{c_{\check{n}}\}$ is the sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$ and $\{c_{\check{n}}\} \to v \in \overline{B_{d_l}(c_0, \check{r})}$. Also if the inequality (2.59) holds for v and either $\alpha(c_{\check{n}}, v) \geq 1$ or $\alpha(v, c_{\check{n}}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P v in $\overline{B_{d_l}(c_0, \check{r})}$.

Proof. The proof of above Theorem is similar as previous proved Theorem 2.4.1. ■

If, we take $\check{S} = \check{T}$ in Theorem 2.4.8, then we are left only with this result.

Corollary 2.4.9 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S} : \acute{Z} \to \acute{Z}$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e})F(d_l(\check{S}\check{e}, \check{S}\hat{y})), \tau + \alpha_*(\hat{y}, \check{S}\hat{y})F(d_l(\check{S}\hat{y}, \check{S}\check{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_2 d_l(\check{e}, \check{S}\check{e}) + \eta_3 d_l(\check{e}, \check{S}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{S}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})}\right) \end{aligned} \tag{2.60}$$

for each $\check{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \check{e}) \geq 1$ where $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, $\eta_1 + \eta_2 + \eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1-\lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1+\eta_2}{1-\eta_3-\eta_4}\right)$ and $\eta_3 + \eta_4 \neq 1$. Then $\{c_{\check{n}}\}$ is a sequence in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\check{n}}, c_{\check{n}+1}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$ and $\{c_{\check{n}}\} \to v \in \overline{B_{d_l}(c_0, \check{r})}$. Also if the inequality (2.60) holds for v and either $\alpha(c_{\check{n}}, v) \geq 1$ or $\alpha(v, c_{\check{n}}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P v in $\overline{B_{d_l}(c_0, \check{r})}$.

If, we take $\eta_2 = 0$ in Theorem 2.4.8, then we are left only with the result.

Corollary 2.4.10 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0, \infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to \acute{Z}$ are the semi α_*-dominated maps on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + \alpha_*(\check{e}, \check{S}\check{e})F(d_l(\check{S}\check{e}, \check{T}\hat{y})), \tau + \alpha_*(\hat{y}, \check{T}\hat{y})F(d_l(\check{T}\hat{y}, \check{S}\check{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\check{e}, \hat{y}) + \eta_3 d_l(\check{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\check{e}, \check{S}\check{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\check{e}, \hat{y})}\right) \end{aligned} \tag{2.61}$$

for each $\acute{e},\hat{y} \in \overline{B_{d_l}(c_0,\check{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e},\hat{y}) \geq 1$ or $\alpha(\hat{y},\check{e}) \geq 1$ where $\eta_1,\eta_3,\eta_4 > 0$, $\eta_1 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1-\lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1+\eta_3}{1-\eta_3-\eta_4}\right)$ and $\eta_3 + \eta_4 \neq 1$. There is a sequence $\{c_{\check{n}}\}$ in $\overline{B_{d_l}(c_0,\check{r})}$, $\alpha(c_{\check{n}},c_{\check{n}+1}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$ and $\{c_{\check{n}}\} \to v \in \overline{B_{d_l}(c_0,\check{r})}$. Also if the inequality (2.61) holds for v and either $\alpha(c_{\check{n}},v) \geq 1$ or $\alpha(v,c_{\check{n}}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P v in $\overline{B_{d_l}(c_0,\check{r})}$.

If, we take $\eta_3 = 0$ in Theorem 2.4.8, then we are left only with the result.

Corollary 2.4.11 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0,\infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0,\check{r})} \subseteq \acute{Z}$ & $\check{S},\check{T} : \acute{Z} \to \acute{Z}$ are the semi α_*−dominated maps on $\overline{B_{d_l}(c_0,\check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned}
& \max\{\tau + \alpha_*(\check{e},\check{S}\check{e})F(d_l(\check{S}\check{e},\check{T}\hat{y})), \tau + \alpha_*(\hat{y},\check{T}\hat{y})F(d_l(\check{T}\hat{y},\check{S}\check{e}))\} \\
\leq\ & F\left(\eta_1 d_l(\check{e},\hat{y}) + \eta_2 d_l(\check{e},\check{S}\check{e}) + \eta_4 \frac{d_l^2(\check{e},\check{S}\check{e}).d_l(\hat{y},\check{T}\hat{y})}{1+d_l^2(\check{e},\hat{y})}\right)
\end{aligned} \tag{2.62}$$

for each $\check{e},\hat{y} \in \overline{B_{d_l}(c_0,\check{r})} \cap \{c_{\check{n}}\}$ with either $\alpha(\check{e},\hat{y}) \geq 1$ or $\alpha(\hat{y},\check{e}) \geq 1$ where $\eta_1,\eta_2,\eta_4 > 0$, $\eta_1 + \eta_2 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1-\lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1+\eta_2}{1-\eta_4}\right)$ and $1 - \eta_4 \neq 0$. There is a sequence $\{c_{\check{n}}\}$ in $\overline{B_{d_l}(c_0,\check{r})}$, $\alpha(c_{\check{n}},c_{\check{n}+1}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$ and $\{c_{\check{n}}\} \to v \in \overline{B_{d_l}(c_0,\check{r})}$. Also if the inequality (2.62) holds for v and either $\alpha(c_{\check{n}},v) \geq 1$ or $\alpha(v,c_{\check{n}}) \geq 1$ for each $\check{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P v in $\overline{B_{d_l}(c_0,\check{r})}$.

If, we take $\eta_4 = 0$ in Theorem 2.4.8, then we are left only with the result.

Corollary 2.4.12 Let $(\acute{Z}, d_l)$ is a complete $D.M.S$. Suppose a function $\alpha : \acute{Z} \times \acute{Z} \to [0,\infty)$ exists. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0,\check{r})} \subseteq \acute{Z}$ & $\check{S},\check{T} : \acute{Z} \to \acute{Z}$ be the semi α_*−dominated maps on $\overline{B_{d_l}(c_0,\check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned}
& \max\{\tau + \alpha_*(\check{e},\check{S}\check{e})F(d_l(\check{S}\check{e},\check{T}\hat{y})), \tau + \alpha_*(\hat{y},\check{T}\hat{y})F(d_l(\check{T}\hat{y},\check{S}\check{e}))\} \\
\leq\ & F\left(\eta_1 d_l(\check{e},\hat{y}) + \eta_2 d_l(\check{e},\check{S}\check{e}) + \eta_3 d_l(\check{e},\check{T}\hat{y})\right)
\end{aligned} \tag{2.63}$$

for all $\breve{e}, \hat{y} \in \overline{B_{d_l}(\breve{e}_0, \check{r})} \cap \{c_{\breve{n}}\}$ with either $\alpha(\breve{e}, \hat{y}) \geq 1$ or $\alpha(\hat{y}, \breve{e}) \geq 1$ where $\eta_1, \eta_2, \eta_3 > 0$, $\eta_1 + \eta_2 + 2\eta_3 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r},$$

where $\lambda = \left(\frac{\eta_1+\eta_2+\eta_3}{1-\eta_3}\right)$ and $1 - \eta_3 \neq 0$. There is a şequenće $\{c_{\breve{n}}\}$ in $\overline{B_{d_l}(c_0, \check{r})}$, $\alpha(c_{\breve{n}}, c_{\breve{n}+1}) \geq 1$ for ėaċh $\breve{n}$ belongs to $N \cup \{0\}$ and $\{c_{\breve{n}}\} \to v \in \overline{B_{d_l}(c_0, \check{r})}$. Also if the inequality (2.63) holds for v and either $\alpha(c_{\breve{n}}, v) \geq 1$ or $\alpha(v, c_{\breve{n}}) \geq 1$ for ėaċh $\breve{n}$ belongs to $N \cup \{0\}$, then $\check{S}$ and $\check{T}$ have C.F.P v in $\overline{B_{d_l}(c_0, \check{r})}$.

Theorem 2.4.13 Let $(\acute{Z}, d_l)$ be a ĉomþleţe $D.M.S$. Let, $\check{r} > 0$, $c_0 \in \overline{B_{d_l}(c_0, \check{r})} \subseteq \acute{Z}$ & $\check{S}, \check{T} : \acute{Z} \to \acute{Z}$ are the dominated maþs on $\overline{B_{d_l}(c_0, \check{r})}$. Assume that, for some $\tau > 0$,

$$\begin{aligned} &\max\{\tau + F(d_l(\check{S}\breve{e}, \check{T}\hat{y})), \tau + F(d_l(\check{T}\hat{y}, \check{S}\breve{e}))\} \\ \leq\ & F\left(\eta_1 d_l(\breve{e}, \hat{y}) + \eta_2 d_l(\breve{e}, \check{S}\breve{e}) + \eta_3 d_l(\breve{e}, \check{T}\hat{y}) + \eta_4 \frac{d_l^2(\breve{e}, \check{S}\breve{e}).d_l(\hat{y}, \check{T}\hat{y})}{1 + d_l^2(\breve{e}, \hat{y})}\right) \end{aligned} \tag{2.64}$$

for all $\breve{e}, \hat{y} \in \overline{B_{d_l}(c_0, \check{r})} \cap \{c_{\breve{n}}\}$ with $\eta_1, \eta_2, \eta_3, \eta_4 > 0$,where $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$ and

$$d_l(c_0, \check{S}c_0) \leq (1 - \lambda)\check{r}, \tag{2.65}$$

where $\lambda = \left(\frac{\eta_1+\eta_2+\eta_3}{1-\eta_3-\eta_4}\right)$ and $\eta_3 + \eta_4 \neq 1$. There is a şequenće $\{c_{\breve{n}}\}$ in $\overline{B_{d_l}(c_0, \check{r})}$, for ėaċh $\breve{n}$ belongs to $N \cup \{0\}$ and $\{c_{\breve{n}}\} \to v \in \overline{B_{d_l}(c_0, \check{r})}$. Then $\check{S}$ and $\check{T}$ have C.F.P v in $\overline{B_{d_l}(c_0, \check{r})}$.

Proof. The proof of above Theorem is similar as previuos proved Theorem in the previous section. In this section, we discuss the application of fixęd þoīnţ Theorem 2.4.13 in form of Volterra ţype integral equations.

$$\breve{u}(k) = \int_0^k H_1(k, h, \breve{u}(h))dh, \tag{2.66}$$

$$\breve{c}(k) = \int_0^k H_2(k, h, \breve{c}(h))dh \tag{2.67}$$

for ėaċh $k \in [0, 1]$. We find the solution of (2.66) and (2.67). Let $\acute{C} = \{f : f$ is the continuous function from $[0, 1]$ to $\mathbb{R}_+\}$, endowed with the ĉomþleţe $D.M.S$. For $\breve{u}$ belongs to $\acute{C}$, settle norm: $\|\check{u}\|_\tau = \sup_{k\in[0,1]} \{|\check{u}(k)|\, e^{-\tau k}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$d_\tau(\check{u}, \check{c}) = \sup_{k\in[0,1]} \{|\check{u}(k) + \check{c}(k)|\, e^{-\tau k}\} = \|\check{u} + \check{c}\|_\tau$$

for each $\check{u}, \check{c} \in \acute{C}$, with these settings, $(\acute{C}, d_\tau)$ becomes a $D.M.S.$ ∎

Theorem 2.4.14: Let the conditions (i) and (ii) hold:

(i) $H_1, H_2 : [0,1] \times [0,1] \times \acute{C} \to \mathbb{R}_+$;

(ii) Define

$$\begin{aligned} \check{S}\check{u}(k) &= \int_0^k H_1(k, h, \check{u}(h))dh, \\ \check{T}\check{c}(k) &= \int_0^k H_2(k, h, \check{c}(h))dh. \end{aligned}$$

Suppose there exist $\tau > 0$, such that

$$|H_1(k,h,\check{u}) + H_2(k,h,\check{c})| \leq \frac{\tau N(\check{u}, \check{c})}{(\tau\sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2}$$

for each $k, h \in [0,1]$ and $\check{u}, \check{c} \in \acute{C}$, where

$$\begin{aligned} N(\check{u}, \check{c}) &= \eta_1[|\check{u}(k) + \check{c}(k)|] + \eta_2[|\check{u}(k) + \check{S}\check{u}(k)|] + \eta_3[|\check{u}(k) + \check{T}\check{c}(k)|] \\ &+ \eta_4 \frac{[|\check{u}(k) + \check{S}\check{u}(k)|]^2 . [|\check{c}(k) + \check{T}\check{c}(k)|]}{1 + [|\check{u}(k) + \check{c}(k)|]^2}, \end{aligned}$$

where $\eta_1, \eta_2, \eta_3, \eta_4 \geq 0$, and $\eta_1 + \eta_2 + 2\eta_3 + \eta_4 < 1$. Then integral equations (2.66) and (2.67) has a solution.

Proof. By assumption (ii)

$$\begin{aligned} |\check{S}\check{u}(k) + \check{T}\check{c}(k)| &= \int_0^k |H_1(k, h, \check{u}(h) + H_2(k, h, \check{c}(h)))|\, dh, \\ &\leq \int_0^k \frac{\tau}{(\tau\sqrt{\|N(\check{u}, \check{c})\|_\tau} + 1)^2} ([N(\check{u}, \check{c})]e^{-\tau h})e^{\tau h} dh, \end{aligned}$$

$$\leq \int_0^k \frac{\tau}{(\tau\sqrt{\|N(\breve{u},\breve{c})\|_\tau}+1)^2}\|N(\breve{u},\breve{c})\|_\tau e^{\tau h}dh,$$

$$\leq \frac{\tau\|N(\breve{u},\breve{c})\|_\tau}{(\tau\sqrt{\|N(\breve{u},\breve{c})\|_\tau}+1)^2}\int_0^k e^{\tau h}dh,$$

$$\leq \frac{\|N(\breve{u},\breve{c})\|_\tau}{(\tau\sqrt{\|N(\breve{u},\breve{c})\|_\tau}+1)^2}e^{\tau k}.$$

This implies

$$\left|\check{S}\breve{u}(k)+\check{T}\breve{c}(k)\right|e^{-\tau k} \leq \frac{\|N(\breve{u},\breve{c})\|_\tau}{(\tau\sqrt{\|N(\breve{u},\breve{c})\|_\tau}+1)^2}.$$

$$\|\check{S}\breve{u}(k)+\check{T}\breve{c}(k)\|_\tau \leq \frac{\|N(\breve{u},\breve{c})\|_\tau}{(\tau\sqrt{\|N(\breve{u},\breve{c})\|_\tau}+1)^2}.$$

$$\frac{\tau\sqrt{\|N(\breve{u},\breve{c})\|_\tau}+1}{\sqrt{\|N(\breve{u},\breve{c})\|_\tau}} \leq \frac{1}{\sqrt{\|\check{S}\breve{u}(k)+\check{T}\breve{c}(k)\|_\tau}}.$$

$$\tau+\frac{1}{\sqrt{\|N(\breve{u},\breve{c})\|_\tau}} \leq \frac{1}{\sqrt{\|\check{S}\breve{u}(k)+\check{T}\breve{c}(k)\|_\tau}}.$$

which further implies

$$\tau-\frac{1}{\sqrt{\|\check{S}\breve{u}(k)+\check{T}\breve{c}(k)\|_\tau}} \leq \frac{-1}{\sqrt{\|N(\breve{u},\breve{c})\|_\tau}}.$$

So, all hypothesis of Theorem 2.4.13 are proved for $F(\breve{c})=\frac{-1}{\sqrt{\breve{c}}}; \breve{c}>0$ and $d_\tau(\breve{u},\breve{c})=\|\breve{u}+\breve{c}\|_\tau$. Hence (2.66) and (2.67) have unique solution. ■

Chapter 3

Results in Dislocated b-Metric Spaces

3.1 Introduction

Theory present in this section is shown in [47, 55].

fīxęd þoīnţ theory has a fundamental position in functional and mathematical analysis. Aydi et al. [16] proved fīxęd þoīnţ řesulţs for quasi contractive setvalued maþs in b-mètric śþaces. Boriceanu [22] discussed fīxęd þoīnţs for multivalued ĉonţracţioñs on a set with two b-mètrics. Nawab et al. [29] established the new idea of dislocated b-mètric śþace as an extension of b-mètric śþace and proved common fīxęd þoīnţs regarding four maþþing fulfilling the weak ĉonţracţioñ in dislocated b-mètric śþace. Asl et. al [9] gave the idea of α_*-ψ contractive maþþing and got some fīxęd þoīnţ conclusions for these multifunctions (see also [7, 30]). Shoaib et al. [61], discussed the result related to α_*-ψ-Ćirić ţype multifunctions on an intersection of a şequenĉe and ĉlosed báll along with graph. Jachymski, [33], proved the contractive maþþing result on mètric related t graph. The notion of multi graph dominated maþþing is introduced. fīxęd þoīnţs related to graphic ĉonţracţioñs on a ĉlosed set for this kind of maþþings are developed. Moreover, we investigate our řesulţs in a better new framework.

In 1974, Ćirić [24], introduced quasi ĉonţracţioñ. Khan, [38], established some new common fīxęd þoīnţs of generalized rational contractive maþþings in dislocated mètric śþaces with applications. Dislocated mètric śþace (see [25]) is a conception of partial mètric śþace (see [39]).

Another conception of mètric śþace is b-mètric śþace (see [2, 16, 23, 41, 62]).

Nadler [40]. started the study of fîxęd þoīnţs concerning setvalued maþþings (see also [17]). Several ŕesulţs on setvalued maþs have been observed (see [5, 23, 36]). Shoaib [60] introduced the idea of α-dominated map and get common fîxęd þoīnţ theorems (see also [15]). Recently, Alofi et al.[8] devolped the new notion of α-dominated multivalued maþs and showed some fîxęd þoīnţ ŕesulţs on a ĉlosed báll in dislocated quasi b-mètric śþaces. In section 3.2, the concept of new rational ţype multivalued contractive maþþings endowed with graphic structure has been introduced. In section 3.3, we have proved fîxęd þoīnţs for a pair of dominated multivalued maþs in ĉomþleţe dislocated b-mètric śþaces with application has been established. Interesting ŕesulţs in mètric śþace, partial mètric śþace and dislocated mètric śþace can be obtained as corollaries of our theorems, which are still not available in literature.

3.2 Fixed Point Results for Multivalued Contractive Mappings Endowed With Graphic Structure

The results given in this section can be seen in [47].

Let (M, d_b) be a $D.B.M.S$, $g_0 \in W$ and $B : W \to P(W)$ be the multifunctions on W. Then there exist $g_1 \in Bg_0$ such that $d_b(g_0, Bg_0) = d_b(g_0, g_1)$. Let $g_2 \in Bg_1$ be such that $d_b(g_1, Bg_1) = d_b(g_1, g_2)$. Proceeding this method, we get a şequenĉe g_n of þoints in W such that $g_{n+1} \in B_{g_n}$, $d_b(g_n, Bg_n) = d_b(g_n, g_{n+1})$. We represent this ţype of şequenĉe by $\{WB(g_n)\}$. We say that $\{WB(g_n)\}$ be the şequenĉe in W generated by g_0.

Theorem 3.2.1 Let (M, d_b) is a ĉomþleţe $D.B.M.S$, $\acute{r} > 0$, $g_0 \in \overline{B_{d_b}(g_0, \acute{r})}$,and $B : M \to P(M)$ is a semi α_*−admissible setvalued maþs on $\overline{B_{d_b}(g_0, \acute{r})}$ and $\{MB(g_n)\}$ is a şequenĉe in M generated by g_0, $\alpha(g_0, g_1) \geq 1$. Assume that, for some $\psi \in \Psi$ and

$$D_b(g, q) = \max\{d_b(g, q), \frac{d_b(g, Bg).d_b(q, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}$$

where $\bar{a} > 0$. the following hold:

$$\alpha_*(Bg, Bq)H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)) \text{ for ėach } g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{MB(g_n)\} \quad (3.1)$$

$$\sum_{i=0}^{n} B^{i+1}\{\psi^i(d_b(g_0,g_1))\} \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N}\cup\{0\}. \tag{3.2}$$

Then, $\{MB(g_n)\}$ is a sequence in $\overline{B_{d_b}(g_0,\acute{r})}$, $\alpha(g_n,g_{n+1}) \geq 1$ and $\{MB(g_n)\} \to g^* \in \overline{B_{d_b}(g_0,\acute{r})}$. Also if $\alpha(g_n,g^*) \geq 1$ or $\alpha(g^*,g_n) \geq 1$, for each n belongs to $\mathbb{N}\cup\{0\}$ and the inequality (3.1) holds for all $g,q \in \overline{B_{d_b}(g_0,\acute{r})} \cap \{MB(g_n)\} \cup \{g^*\}$. Then B has a C.F.P g^* in $\overline{B_{d_b}(g_0,\acute{r})}$.

Proof. Consider a sequence $\{MB(g_n)\}$ generated by g_0.Then, we have $g_n \in Bg_{n-1}$, and $d_b(g_{n-1},Bg_{n-1}) = d_b(g_{n-1},g_n)$. for each $n \in \mathbb{N}$. By Lemma 1.2.8, we have $d_b(g_n,g_{n+1}) \leq H_{d_b}(Bg_{n-1},Bg_n)$ for each $n \in \mathbb{N}$. If $g_0 = g_1$, then g_0 be a fixed point in $\overline{B_{d_b}(g_0,\acute{r})}$ of B. Let $g_0 \neq g_1$. From (3.2), we have

$$d_b(g_0,g_1) \leq \sum_{i=0}^{n} \psi^i(d_b(g_0,g_1)) \leq \acute{r}.$$

It follows that,

$$g_1 \in \overline{B_{d_b}(g_0,\acute{r})}.$$

If $g_1 = g_2$, then g_1 is a fixed point in $\overline{B_{d_b}(g_0,\acute{r})}$ of B. Let $g_1 \neq g_2$. Since $\alpha(g_0,g_1) \geq 1$ and B is semi α_*-admissible setvalued map on $\overline{B_{d_b}(g_0,\acute{r})}$, so $\alpha_*(Bg_0,Bg_1) \geq 1$. As $\alpha_*(Bg_0,Bg_1) \geq 1$, $g_1 \in Bg_0$ and $g_2 \in Bg_1$, so $\alpha(g_1,g_2) \geq 1$. Let $g_2,\cdots,g_j \in \overline{B_{d_b}(g_0,\acute{r})}$ for each j belongs to $\mathbb{N}$. As $\alpha_*(Bg_1,Bg_2) \geq 1$, we have $\alpha(g_2,g_3) \geq 1$, which further implies $\alpha_*(Bg_2,Bg_3) \geq 1$. Proceeding this process, we have $\alpha_*(Bg_{j-1},Bg_j) \geq 1$. Now, by using Lemma 1.2.8,

$$\begin{aligned}
d_b(g_j,g_{j+1}) &\leq H_{d_b}(Bg_{j-1},Bg_j) \leq \alpha_*(Bg_{j-1},Bg_j)H_{d_b}(Bg_{j-1},Bg_j) \\
&\leq \psi(D_b(g_{j-1},g_j)) \\
&= \psi\left(\max\left\{\begin{array}{c} d_b(g_{j-1},g_j), \frac{d_b(g_{j-1},Bg_{j-1}).d_b(g_j,Bg_j)}{\bar{a}+d_b(g_{j-1},g_j)}, \\ d_b(g_{j-1},Bg_{j-1}), d_b(g_j,Bg_j) \end{array}\right\}\right) \\
&= \psi\left(\max\left\{\begin{array}{c} d_b(g_{j-1},g_j), \frac{d_b(g_{j-1},g_j).d_b(g_j,g_{j+1})}{\bar{a}+d_b(g_{j-1},g_j)}, \\ d_b(g_{j-1},g_j), d_b(g_j,g_{j+1}) \end{array}\right\}\right) \\
&= \psi\left(\max\{d_b(g_{j-1},g_j), d_b(g_j,g_{j+1})\}\right).
\end{aligned}$$

If $\max\{d_b(g_{j-1},g_j),d_b(g_j,g_{j+1})\} = d_b(g_j,g_{j+1})$, then $d_b(g_j,g_{j+1}) \leq \psi(d_b(g_j,g_{j+1}))$. This is contradiction to the fact that $\psi(u) < u$ for each $u > 0$. Hence, we obtain $\max\{d_b(g_{j-1},g_j),d_b(g_j,g_{j+1})\} =$

$d_b(g_{j-1}, g_j)$. Therefore, we have

$$d_b(g_j, g_{j+1}) \leq \psi(d_b(g_{j-1}, g_j)) \leq \cdots \leq \psi^j(d_b(g_0, g_1)). \tag{3.3}$$

Now, by using triangular inequality and by (3.3), we have

$$\begin{aligned} d_b(g_0, g_{j+1}) &\leq td_b(g_0, g_1) + t^2 d_b(g_1, g_2) + \cdots + t^{j+1} d_b(g_j, g_{j+1}) \\ &\leq td_b(g_0, g_1) + t^2 \psi(d_b(g_0, g_1)) + \cdots + t^{j+1} \psi^j(d_b(g_0, g_1)) \\ &\leq \sum_{i=0}^{j} t^{i+1} \left\{ \psi^i(d_b(g_0, g_1)) \right\} \leq \acute{r}. \end{aligned}$$

Thus $g_{j+1} \in \overline{B_{d_b}(g_0, \acute{r})}$. Hence, by induction, $g_n \in \overline{B_{d_b}(g_0, \acute{r})}$. As $\alpha_*(Bg_{j-1}, Bg_j) \geq 1$, $g_j \in Bg_j$, $g_{j+1} \in Bg_j$, then we have $\alpha(g_j, g_{j+1}) \geq 1$. Also B is semi α_*-admissible setvalued maps on $\overline{B_{d_b}(g_0, \acute{r})}$, therefore $\alpha_*(Bg_j, Bg_{j+1}) \geq 1$. This further implies that $\alpha(g_{j+1}, g_{j+2}) \geq 1$. Proceeding this process,we have $\alpha(g_n, g_{n+1}) \geq 1$ for ėach n belongs to $\mathbb{N}$. Now, (3.3) can be expressed as

$$d_b(g_n, g_{n+1}) \leq \psi^n(d_b(g_0, g_1)) \text{ for ėach } n \text{ belongs to } \mathbb{N}. \tag{3.4}$$

Fix $\in > 0$ and let $k_1(\in) \in \mathbb{N}$, such that

$$\sum_{k \geq k_1(\in)} t^k \psi^k(d_b(g_0, g_1)) < \in .$$

Let n, m belong to $\mathbb{N}$ with $m > n > k_1(\in)$. Now,

$$\begin{aligned} d_b(g_n, g_m) &\leq \sum_{k=n}^{m-1} d_b(g_k, g_{k+1}) \\ &\leq \sum_{k=n}^{m-1} t^k \psi^k(d_b(g_0, g_1)), \text{ by (3.4)} \\ d_b(g_n, g_m) &\leq \sum_{k \geq k_1(\in)} t^k \psi^k(d_b(g_0, g_1)) < \in . \end{aligned}$$

Thus, $\{MB(g_n)\}$ is a Cauchy in $(\overline{B_{d_b}(g_0, \acute{r})}, d_b)$. As ėach ĉlosed set in a ĉomþleţe $D.B.M.S$ is complete, so there exist $g^* \in \overline{B_{d_b}(g_0, \acute{r})}$ such that $\{MB(g_n)\} \to g^*$, and

$$\lim_{n\to\infty} d_b(g_n, g^*) = 0. \tag{3.5}$$

Then, we have $\alpha(g_n, g^*) \geq 1$ for every n belongs to $\mathbb{N} \cup \{0\}$. Thus, $\alpha_*(Bg_n, Bg^*) \geq 1$. Now,

$$\begin{aligned}
d_b(g^*, Bg^*) &\leq td_b(g^*, g_{n+1}) + td_b(g_{n+1}, Bg^*) \\
&\leq td_b(g^*, g_{n+1}) + tH_{d_b}(Bg_n, Bg^*) \text{ by Lemma1.2.8} \\
&\leq td_b(g^*, g_{n+1}) + t\{\alpha_*(Bg_n, Bg^*)H_{d_b}(Bg_n, Bg^*)\} \\
&\leq td_b(g^*, g_{n+1}) + t\psi(\max\{d_b(g_n, g^*), \\
&\quad \frac{d_b(g_n, Bg_n).d_b(g^*, Bg^*)}{\bar{a} + d_b(g_n, g^*)}, d_b(g_n, Bg_n), d_b(g^*, Bg^*)\}) \\
&\leq td_b(g^*, g_{n+1}) + t\psi(\max\{d_b(g_n, g^*), \\
&\quad \frac{d_b(g_n, g_{n+1}).d_b(g^*, Bg^*)}{\bar{a} + d_b(g_n, g^*)}, d_b(g_n, g_{n+1}), d_b(g^*, Bg^*)\}).
\end{aligned}$$

Letting $n \to \infty$ and by using inequality (3.5), we obtain $(1 - t)d_b(g^*, Bg^*) \leq 0$. So $(1 - t) \neq 0$, then $d_b(g^*, Bg^*) = 0$. Hence $g^* \in Bg^*$. ■

Corollary 3.2.2 Let $(M, \preceq, d_b)$ is a preordered $D.B.M.S$, $\acute{r} > 0$, $g_0 \in \overline{B_{d_b}(g_0, \acute{r})}$ and $B : M \to P(M)$ be a multifunction on $\overline{B_{d_b}(g_0, \acute{r})}$ and $\{MB(g_n)\}$ is a sequence generated by g_0, with $g_0 \preceq g_1$. Assume that, for some $\psi \in \Psi$ and

$$D_b(g, q) = \max\{d_b(g, q), \frac{d_b(g, Bg).d_b(q, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)) \text{ for all } g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{MB(g_n)\} \text{ with } g \preceq q \tag{3.6}$$

$$\text{and } \sum_{i=0}^{n} t^{i+1}\{\psi^i(d_b(g_0, g_1))\} \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

If $g, q \in \overline{B_{d_b}(g_0, \acute{r})}$, so as $g \preceq q$ implies $Bg \preceq_{\acute{r}} Bq$. Then, $\{MB(g_n)\}$ be the sequence in $\overline{B_{d_b}(g_0, \acute{r})}$, $g_n \preceq g_{n+1}$ and $\{MB(g_n)\} \to g^* \in \overline{B_{d_b}(g_0, \acute{r})}$. Also if $g^* \preceq g_n$ or $g_n \preceq g^*$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.6) holds for all $g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{MB(g_n)\} \cup \{g^*\}$.

Then, g^* is a fixed point of B in $\overline{B_{d_b}(g_0, \acute{r})}$.

Corollary 3.2.3 Let $(M, \preceq, d_b)$ is a preordered complete $D.B.M.S$, $\acute{r} > 0$, $g_0 \in \overline{B_{d_b}(g_0, \acute{r})}$ and $B : M \to P(M)$ be a multifunction on $\overline{B_{d_b}(g_0, \acute{r})}$ and $\{MB(g_n)\}$ is the sequence generated by g_0, with $g_0 \preceq g_1$. Assume that, for some $k \in [0, 1)$ and

$$D_b(g, q) = \max\{d_b(g, q), \frac{d_b(g, Bg).d_b(q, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$H_{d_b}(Bg, Bq) \leq k(D_b(g, q)) \text{ for all } g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{MB(g_n)\} \text{ with } g \preceq q \quad (3.7)$$

$$\text{and } \sum_{i=0}^{n} t^{i+1}\{k^i(d_b(g_0, g_1))\} \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

If $g, q \in \overline{B_{d_b}(g_0, \acute{r})}$, such that $g \preceq q$ implies $Bg \preceq_{\acute{r}} Bq$. Then, $\{MB(g_n)\}$ be a sequence in $\overline{B_{d_b}(g_0, \acute{r})}$, $g_n \preceq g_{n+1}$ and $\{MB(g_n)\} \to g^* \in \overline{B_{d_b}(g_0, \acute{r})}$. Also if $g^* \preceq g_n$ or $g_n \preceq g^*$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.7) holds for all $g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{MB(g_n)\} \cup \{g^*\}$. Then, g^* is a fixed point of B in $\overline{B_{d_b}(g_0, \acute{r})}$.

Corollary 3.2.4 Let $(M, \preceq, d_l)$ is a preordered $D.M$ space, $\acute{r} > 0$, $g_0 \in \overline{B_{d_l}(g_0, \acute{r})}$ and $B : M \to P(M)$ be a multifunction on $\overline{B_{d_b}(g_0, \acute{r})}$ and $\{MB(g_n)\}$ is a sequence generated by g_0, with $g_0 \preceq g_1$. Assume that, for some $\psi \in \Psi$ and

$$D_l(g, q) = \max\{d_l(g, q), \frac{d_l(g, Bg).d_l(q, Bq)}{\bar{a} + d_l(g, q)}, d_l(g, Bg), d_l(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$H_{d_l}(Bg, Bq) \leq \psi(D_l(g, q)) \text{ for all } g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{MB(g_n)\} \text{ with } g \preceq q \quad (3.8)$$

$$\text{and } \sum_{i=0}^{n} \psi^i(d_l(g_0, g_1)) \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

If $g, q \in \overline{B_{d_l}(g_0, \acute{r})}$, such that $g \preceq q$ implies $Bg \preceq_{\acute{r}} Bq$. Then, $\{MB(g_n)\}$ is the sequence in $\overline{B_{d_l}(g_0, \acute{r})}$, $g_n \preceq g_{n+1}$ and $\{MB(g_n)\} \to g^* \in \overline{B_{d_l}(g_0, \acute{r})}$. Also if $g^* \preceq g_n$ or $g_n \preceq g^*$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.8) holds for all $g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{MB(g_n)\} \cup \{g^*\}$.

Then, g^* is a fixed point of B in $\overline{B_{d_l}(g_0, \acute{r})}$.

Corollary 3.2.5 Let $(M, \preceq, d_l)$ is a preordered $D.M$ space, $\acute{r} > 0$, $g_0 \in \overline{B_{d_l}(g_0, \acute{r})}$ and $B : M \to P(M)$ be a multifunction on $\overline{B_{d_b}(g_0, \acute{r})}$ and $\{MB(g_n)\}$ be a sequence generated by g_0, with $g_0 \preceq g_1$. Assume that, for some $k \in [0, 1)$ and

$$D_l(g, q) = \max\{d_l(g, q), \frac{d_l(g, Bg).d_l(q, Bq)}{\bar{a} + d_l(g, q)}, d_l(g, Bg), d_l(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$H_{d_l}(Bg, Bq) \leq k(D_l(g, q)) \text{ for all } g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{MB(g_n)\} \text{ with } g \preceq q \tag{3.9}$$

$$\text{and } \sum_{i=0}^{n} k^i(d_l(g_0, g_1)) \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

If $g, q \in \overline{B_{d_l}(g_0, \acute{r})}$, such that $g \preceq q$ implies $Bg \preceq_{\acute{r}} Bq$. Then, $\{MB(g_n)\}$ is the sequence in $\overline{B_{d_l}(g_0, \acute{r})}$, $g_n \preceq g_{n+1}$ and $\{MB(g_n)\} \to g^* \in \overline{B_{d_l}(g_0, \acute{r})}$. Also if $g^* \preceq g_n$ or $g_n \preceq g^*$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.9) holds for all $g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{MB(g_n)\} \cup \{g^*\}$. Then, g^* is a fixed point of B in $\overline{B_{d_l}(g_0, \acute{r})}$.

Example 3.2.6 Let $M = Q^+ \cup \{0\}$ and let $d_b : M \times M \to M$ be the $D.B.M$ space on M defined by

$$d_b(g, q) = (g + q)^2 \text{ for all } g, q \in M$$

with parameter $t > 1$. Define the multivalued mappings, $B : M \times M \to P(M)$ by,

$$Bg = \begin{cases} [\frac{g}{3}, \frac{2}{3}g] \text{ if } g \in [0, 9] \cap M \\ [g, g + 1] \text{ if } g \in (9, \infty) \cap M, \end{cases}$$

Considering, $g_0 = 1, \acute{r} = 100$, and $\bar{a} = 1$, $b = 2$, then $\overline{B_{d_b}(g_0, \acute{r})} = [0, 9] \cap M$. Now $d_b(g_0, Bg_0) = d_b(1, B1) = d_b(1, \frac{1}{3}) = \frac{16}{9}$. So we obtain a sequence $\{MB(g_n)\} = \{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...\}$ in M generated by g_0. Let $t = 1.2$, $\psi(t) = \frac{4t}{5}$, then $t\psi(t) < t$. Define

$$\alpha(g, q) = \begin{cases} 1 \text{ if } g, q \in [0, 9] \cap M \\ \frac{3}{2} \quad \text{otherwise.} \end{cases}$$

Now,

$$\alpha_*(B10, B11)H_{d_b}(B10, B11) = (\frac{3}{2})(484) > \psi(D_b(g, q)) = \frac{4}{5}(484).$$

So the inequality (3.1) is not true for the whole ṡpace M . Now, for all $g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{MB(g_n)\}$, we have

$$\begin{aligned}
\alpha_*(Bg, Bq)H_{d_b}(Bg, Bq) &= 1\left[\max\left\{\sup_{\bar{a}\in Bg} d_b(\bar{a}, Bq), \sup_{b\in Bq} d_b(Bg, b)\right\}\right] \\
&= \max\left\{\sup_{\bar{a}\in Bg} d_b\left(\bar{a}, \left[\frac{q}{3}, \frac{2q}{3}\right]\right), \sup_{b\in Bq} d_b\left(\left[\frac{g}{3}, \frac{2g}{3}\right], b\right)\right\} \\
&= \max\left\{\sup_{\bar{a}\in Bg} d_b\left(\frac{2g}{3}, \left[\frac{q}{3}, \frac{2q}{3}\right]\right), \sup_{b\in Bq} d_b\left(\left[\frac{g}{3}, \frac{2g}{3}\right], \frac{2q}{3}\right)\right\} \\
&= \max\left\{d_b\left(\frac{2g}{3}, \frac{q}{3}\right), d_b\left(\frac{g}{3}, \frac{2q}{3}\right)\right\} \\
&= \max\left\{\frac{(2g+q)^2}{9}, \frac{(g+2q)^2}{9}\right\} \\
&\leq \psi\left(\max\left\{(g+q)^2, \frac{256g^2q^2}{81\{1+(g+q)^2\}}, \frac{16g^2}{9}, \frac{16q^2}{9}\right\}\right) \\
&= \psi(d_b(g, q)).
\end{aligned}$$

So, the inequality (3.1) holds on $\overline{B_{d_b}(\dot{c}_0, \acute{r})} \cap \{MB(g_n)\}$. As, $t = 1.2 > 1$, then

$$\sum_{i=0}^{n} t^{i+1}\{\psi^i(d_b(g_0, g_1))\} = \frac{16}{9} \times \frac{6}{5}\sum_{i=0}^{n}(\frac{24}{25})^i < 100 = \acute{r}.$$

Hence, all hypothesis of Theorem 3.2.1 are proved. Now, we have $\{MB(g_n)\}$ is a ṣequenće in $\overline{B_{d_b}(g_0, \acute{r})}$, $\alpha(g_n, g_{n+1}) \geq 1$ and $\{MB(g_n)\} \to 0 \in \overline{B_{d_b}(g_0, \acute{r})}$. Furthermore, 0 be a fixẹd þoīnţ of B.

Theorem 3.2.7 Let (M, d_b) is a ĉomþleţe $D.B.M.S$ with a graph $\dot{G}$. Suppose a function $\alpha : M \times M \to [0, \infty)$ exists. Let, $\acute{r} > 0$, $g_0 \in \overline{B_{d_b}(g_0, \acute{r})}$, $B : M \to P(M)$ and let for a ṣequenće $\{MB(g_n)\}$ in M generated by g_0, with $(g_0, g_1) \in Q(G)$. Suppose that (i) and (ii) hold:

(i) B is a graph preserving for all $g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{MB(g_n)\}$;

(ii) there exists $\psi \in \Psi$ and

$$D_b(g,q) = \max\{d_b(g,q), \frac{d_l(g,Bg).d_l(q,Bq)}{\bar{a} + d_l(g,q)}, d_b(g,Bg), d_b(q,Bq)\}$$

where $\bar{a} > 0$ such that

$$H_{d_b}(Bg,Bq) \leq \psi(D_b(g,q)), \tag{3.10}$$

for all $g, q \in \overline{B_{d_b}(g_0,\acute{r})} \cap \{MB(g_n)\}$ and $(g,q) \in Q(G)$;

(iii) $\sum_{i=0}^{n} t^{i+1}\{\psi^i(d_b(g_0,Bg_0))\} \leq \acute{r}$ for èach n belongs to $\mathbb{N} \cup \{0\}$ and $t > 1$.

Then, $\{MB(g_n)\}$ is a şequenće in $\overline{B_{d_b}(g_0,\acute{r})}$, $(g_n, g_{n+1}) \in Q(G)$ and $\{MB(g_n)\} \to g^*$. Also, if and the inequality (3.10) holds for g^* and $(g_n, g^*) \in Q(G)$ or $(g^*, g_n) \in Q(G)$ for ēverŷ n belongs to $\mathbb{N} \cup \{0\}$, then g^* is a fixẹd þoīnţ of B in $\overline{B_{d_b}(g_0,\acute{r})}$.

Proof. Define, $\alpha : M \times M \to [0,\infty)$ by

$$\alpha(g,q) = \begin{cases} 1, & \text{if } g \in \overline{B_{d_b}(g_0,\acute{r})},\ (g,q) \in Q(G) \\ 0, & \text{otherwise.} \end{cases}$$

As $\{MB(g_n)\}$ is a şequenće in M generated by g_0 with $(g_0,g_1) \in Q(G)$, we have $\alpha(g_0,g_1) \geq 1$. Let $\alpha(g,q) \geq 1$, then $(g,q) \in Q(G)$. From (i), we have $(w,p) \in Q(G)$ for all $w \in Bg$ and $p \in Bq$. This implies that $\alpha(w,p) = 1$ for all $w \in Bg$ and $p \in Bq$. This implies that $\inf\{\alpha(w,p) : w \in Bg, p \in Bq\} = 1$. So, $B : M \to P(M)$ is a semi α_*−admissible multifunction on $\overline{B_{d_b}(g_0,\acute{r})}$. Moreover, inequality (3.10) can be written as

$$\alpha_*(Bg,Bq)H_{d_b}(Bg,Bq) \leq \psi(D_b(g,q)),$$

for all elements g, q in $\overline{B_{d_b}(g_0,\acute{r})} \cap \{MB(g_n)\}$ with either $\alpha(g,q) \geq 1$ or $\alpha(q,g) \geq 1$. Also, (iii) holds. Then, by Theorem 3.2.1, we have $\{MB(g_n)\}$ be the şequenće in $\overline{B_{d_b}(g_0,\acute{r})}$ and $\{MB(g_n)\} \to g^* \in \overline{B_{d_b}(g_0,\acute{r})}$. Now, $g_n, g^* \in \overline{B_{d_b}(g_0,\acute{r})}$ and either $(g_n,g^*) \in Q(G)$ or $(g^*,g_n) \in Q(G)$ for èach n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.10) holds for all $g, q \in \overline{B_{d_b}(g_0,\acute{r})} \cap \{MB(g_n)\} \cup \{g^*\}$. Then we have $\alpha(g_n,g^*) \geq 1$ or $\alpha(g^*,g_n) \geq 1$ for èach n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.1) holds for all $g, q \in \overline{B_{d_b}(g_0,\acute{r})} \cap \{MB(g_n)\} \cup \{g^*\}$. So, all hypothesis of Theorem 3.2.1 are proved. Hence, by Theorem 3.2.1, B has a C.F.P g^* in $\overline{B_{d_b}(g_0,\acute{r})}$ and

$d_b(g^*, g^*) = 0.$

In this section we discussed some fixed points for self mapping in complete $D.B.M$ space. Let (M, d_b) be a $D.B.M$ space, $g_0 \in M$ and $B : M \to M$ be a mapping. Let $g_1 = Bg_0$, $g_2 = Bg_1$. Proceeding this metho, we get a sequence g_n of points in M such that $g_{n+1} = B_{g_n}$. We represent this type of sequence by $\{g_n\}$. We say that $\{g_n\}$ is the sequence in M generated by g_0. ■

Theorem 3.2.8 Let (M, d_b) is a complete $D.B.M.S$, $\acute{r} > 0$, $g_0 \in \overline{B_{d_b}(g_0, \acute{r})}$ and $B : M \to M$ is a semi α−admissible function on $\overline{B_{d_b}(g_0, \acute{r})}$ and $\{g_n\}$ is a sequence in M, then $\alpha(g_0, g_1) \geq 1$. Assume that, for some $\psi \in \Psi$ and

$$D_b(g, q) = \max\{d_b(g, q), \frac{d_b(g, Bg).d_b(q, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$\alpha(Bg, Bq)H_{d_b}(Bg, Bq) \leq \psi(D_b(g, q)) \text{ for all } g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{g_n\} \tag{3.11}$$

$$\sum_{i=0}^{n} B^{i+1}\{\psi^i(d_b(g_0, g_1))\} \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

Then, $\{g_n\}$ is a sequence in $\overline{B_{d_b}(g_0, \acute{r})}$, $\alpha(g_n, g_{n+1}) \geq 1$ and $\{g_n\} \to g^* \in \overline{B_{d_b}(g_0, \acute{r})}$. Also if $\alpha(g_n, g^*) \geq 1$ or $\alpha(g^*, g_n) \geq 1$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.11) holds for all $g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{g_n\} \cup \{g^*\}$. Then B has a C.F.P g^* in $\overline{B_{d_b}(g_0, \acute{r})}$.

Proof. The proof of above Theorem is similar as previous proved Theorem 3.2.1. ■

Corollary 3.2.9 Let $(M, \preceq, d_b)$ is a preordered complete $D.B.M.S$, $\acute{r} > 0$, $g_0 \in \overline{B_{d_b}(g_0, \acute{r})}$ and $B : M \to M$ be a self mapping on $\overline{B_{d_b}(g_0, \acute{r})}$ and $\{g_n\}$ is a sequence generated by g_0, with $g_0 \preceq g_1$. Assume that, for some $k \in [0, 1)$ and

$$D_b(g, q) = \max\{d_b(g, q), \frac{d_b(g, Bg).d_b(q, Bq)}{\bar{a} + d_b(g, q)}, d_b(g, Bg), d_b(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$H_{d_b}(Bg, Bq) \leq k(D_b(g, q)) \text{ for all } g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{g_n\} \text{ with } g \preceq q \tag{3.12}$$

$$\text{and} \sum_{i=0}^{n} t^{i+1}\{k^i(d_b(g_0, g_1))\} \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

If $g, q \in \overline{B_{d_b}(g_0, \acute{r})}$, such that $g \preceq q$ implies $Bg \preceq_{\acute{r}} Bq$. Then, $\{g_n\}$ is a sequence in $\overline{B_{d_b}(g_0, \acute{r})}$, $g_n \preceq g_{n+1}$ and $\{g_n\} \to g^* \in \overline{B_{d_b}(g_0, \acute{r})}$. Also if $g^* \preceq g_n$ or $g_n \preceq g^*$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.12) holds for all $g, q \in \overline{B_{d_b}(g_0, \acute{r})} \cap \{g_n\} \cup \{g^*\}$. Then, g^* is a fixed point of B in $\overline{B_{d_b}(g_0, \acute{r})}$.

Corollary 3.2.10 Let $(M, \preceq, d_l)$ is an ordered complete $D.M$ space, $\acute{r} > 0$, $g_0 \in \overline{B_{d_l}(g_0, \acute{r})}$ and $B : M \to M$ be a self mapping on $\overline{B_{d_l}(g_0, \acute{r})}$ and $\{g_n\}$ is a sequence generated by g_0, with $g_0 \preceq g_1$. Assume that, for some $\psi \in \Psi$ and

$$D_l(g, q) = \max\{d_l(g, q), \frac{d_l(g, Bg).d_l(q, Bq)}{\bar{a} + d_l(g, q)}, d_l(g, Bg), d_l(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$H_{d_l}(Bg, Bq) \leq \psi(D_l(g, q)) \text{ for all } g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{g_n\} \text{ with } g \preceq q \tag{3.13}$$

$$\text{and} \sum_{i=0}^{n} \psi^i(d_l(g_0, g_1)) \leq \acute{r} \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

If $g, q \in \overline{B_{d_l}(g_0, \acute{r})}$, such that $g \preceq q$ implies $Bg \preceq_{\acute{r}} Bq$. Then, $\{g_n\}$ be a sequence in $\overline{B_{d_l}(g_0, \acute{r})}$, $g_n \preceq g_{n+1}$ and $\{g_n\} \to g^* \in \overline{B_{d_l}(g_0, \acute{r})}$. Also if $g^* \preceq g_n$ or $g_n \preceq g^*$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.13) holds for all $g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{g_n\} \cup \{g^*\}$. Then, g^* is a fixed point of B in $\overline{B_{d_l}(g_0, \acute{r})}$.

Corollary 3.2.11 Let $(M, \preceq, d_l)$ is an ordered complete $D.M$ space, $\acute{r} > 0$, $g_0 \in \overline{B_{d_l}(g_0, \acute{r})}$ and $B : M \to M$ be a self mapping on $\overline{B_{d_l}(g_0, \acute{r})}$ and $\{g_n\}$ be a sequence in M with initial guess g_0, with $g_0 \preceq g_1$. For some $k \in [0, 1)$ and

$$D_l(g, q) = \max\{d_l(g, q), \frac{d_l(g, Bg).d_l(q, Bq)}{\bar{a} + d_l(g, q)}, d_l(g, Bg), d_l(q, Bq)\}$$

where $\bar{a} > 0$, the following hold:

$$d_l(Bg, Bq) \leq k(D_l(g, q)) \text{ for all } g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{g_n\} \text{ with } g \preceq q \tag{3.14}$$

and $\sum_{i=0}^{j} k^{i}(d_l(g_0, g_1)) \leq \acute{r}$ for each j belongs to $\mathbb{N} \cup \{0\}$.

Then, $\{g_n\}$ be a sequence in $\overline{B_{d_l}(g_0, \acute{r})}$, such that $g_n \preceq g_{n+1}$ and $\{g_n\} \to g^* \in \overline{B_{d_l}(g_0, \acute{r})}$. Also if $g^* \preceq g_n$ or $g_n \preceq g^*$, for each n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (3.14) holds for all $g, q \in \overline{B_{d_l}(g_0, \acute{r})} \cap \{g_n\} \cup \{g^*\}$. Then, g^* be the fixed point of B in $\overline{B_{d_l}(g_0, \acute{r})}$.

3.3 Fixed Point Results for a Pair of Multivalued Dominated Mappings in Dislocated b-Metric Space with Applications

The given results in this section can be seen in [55].

Let (E, d_{lb}) be a $D.B.M.S$, $q_0 \in E$ and $S, T : E \to P(E)$ are the setvalued maps on E. Let $q_1 \in Sq_0$ be an element such that $d_{lb}(q_0, Sq_0) = d_{lb}(q_0, q_1)$. Let $q_2 \in Tq_1$ be such that $d_{lb}(q_1, Tq_1) = d_{lb}(q_1, q_2)$. Let $q_3 \in Sq_2$ be such that $d_{lb}(q_2, Sq_2) = d_{lb}(q_2, q_3)$. Proceeding this method, we get the sequence q_n in E so as $q_{2n+1} \in Sq_{2n}$ and $q_{2n+2} \in Tq_{2n+1}$, where $n = 0, 1, 2, ...$ Also $d_{lb}(q_{2n}, Sq_{2n}) = d_{lb}(q_{2n}, q_{2n+1})$, $d_{lb}(q_{2n+1}, Tq_{2n+1}) = d_{lb}(q_{2n+1}, q_{2n+2})$. We represent this type of sequence by $\{TS(q_n)\}$. We say that $\{TS(q_n)\}$ be the sequence in E generated by q_0. For $q, e \in E$, $a > 0$, we define $D_{lb}(q, e)$ as

$$D_{lb}(q, e) = \max\{d_{lb}(q, e), \frac{d_{lb}(q, Sq) . d_{lb}(e, Te)}{a + d_{lb}(q, e)}, d_{lb}(q, Sq), d_{lb}(e, Te)\}.$$

Theorem 3.3.1 Let (E, d_{lb}) is a complete $D.B.M.S$. Suppose a function $\alpha : E \times E \to [0, \infty)$ exists. Let, $r > 0$, $q_0 \in \overline{B_{d_{lb}}(q_0, r)}$ & $S, T : E \to P(E)$ be two α_*-dominated maps on $\overline{B_{d_{lb}}(q_0, r)}$. Suppose that, for some $\psi_b \in \Psi_b$, the following hold:

$$H_{d_{lb}}(Sq, Te) \leq \psi_b(D_{lb}(q, e)) \tag{3.15}$$

for all $q, e \in \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\}$ with either $\alpha(q, e) \geq 1$ or $\alpha(e, q) \geq 1$. Also

$$\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_{lb}(q_0, Sq_0))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\} \text{ and } b \geq 1. \tag{3.16}$$

Then $\{TS(q_n)\}$ is a sequence in $\overline{B_{d_{lb}}(q_0, r)}$, $\alpha(q_n, q_{n+1}) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$ and

$\{TS(q_n)\} \to q^* \in \overline{B_{d_{lb}}(q_0, r)}$. Also if the inequality (3.15) holds for q^* and either $\alpha(q_n, q^*) \geq 1$ or $\alpha(q^*, q_n) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$, then q^* is the C.F.P of S and T in $\overline{B_{d_{lb}}(q_0, r)}$ and $d_{lb}(q^*, q^*) = 0$.

Proof. Consider a sequence $\{TS(q_n)\}$. From (3.16), we get

$$d_{lb}(q_0, q_1) \leq \sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_{lb}(q_0, Sq_0))\} \leq r.$$

It follows that,

$$q_1 \in \overline{B_{d_{lb}}(q_0, r)}.$$

Let $q_2, \cdots, q_j \in \overline{B_{d_{lb}}(q_0, r)}$ for every j belongs to $\mathbb{N}$. If $j = 2i + 1$, where $i = 1, 2, \ldots, \frac{j-1}{2}$. Since $S, T : E \to P(E)$ be a α_*-dominated mappings on $\overline{B_{d_{lb}}(q_0, r)}$, so $\alpha_*(q_{2i}, Sq_{2i}) \geq 1$ and $\alpha_*(q_{2i+1}, Tq_{2i+1}) \geq 1$. As $\alpha_*(q_{2i}, Sq_{2i}) \geq 1$, this implies $\inf\{\alpha(q_{2i}, b) : b \in Sq_{2i}\} \geq 1$. Also $q_{2i+1} \in Sq_{2i}$, so $\alpha(q_{2i}, q_{2i+1}) \geq 1$. Now by using Lemma 1.2.8, we obtain,

$$\begin{aligned}
d_{lb}(q_{2i+1}, q_{2i+2}) &\leq H_{d_{lb}}(Sq_{2i}, Tq_{2i+1}) \leq \psi_b(D_{lb}(q_{2i}, q_{2i+1})) \\
&\leq \psi_b(\max\{d_{lb}(q_{2i}, q_{2i+1}), \frac{d_{lb}(q_{2i}, q_{2i+1}) . d_{lb}(q_{2i+1}, q_{2i+2})}{a + d_{lb}(q_{2i}, q_{2i+1})}, \\
&\quad d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})\}) \\
&\leq \psi_b(\max\{d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})\}).
\end{aligned}$$

If $\max\{d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})\} = d_{lb}(q_{2i+1}, q_{2i+2})$, then

$$\begin{aligned}
d_{lb}(q_{2i+1}, q_{2i+2}) &\leq \psi_b(d_{lb}(q_{2i+1}, q_{2i+2})) \\
&\leq b\psi_b(d_{lb}(q_{2i+1}, q_{2i+2})).
\end{aligned}$$

Which contradicts that $b\psi_b(t) < t$ for each $t > 0$. So

$$\max\{d_{lb}(q_{2i}, q_{2i+1}), d_{lb}(q_{2i+1}, q_{2i+2})\} = d_{lb}(q_{2i}, q_{2i+1}).$$

Hence, we obtain

$$d_{lb}(q_{2i+1}, q_{2i+2}) \leq \psi_b(d_{lb}(q_{2i}, q_{2i+1})). \tag{3.17}$$

As $\alpha_*(q_{2\acute{i}-1}, Tq_{2\acute{i}-1}) \geq 1$ and $q_{2\acute{i}} \in Tq_{2\acute{i}-1}$, so $\alpha(q_{2\acute{i}-1}, q_{2\acute{i}}) \geq 1$. Now, by using Lemma 1.2.8, we obtain

$$\begin{aligned}
d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1}) &\leq H_{d_{lb}}(Tq_{2\acute{i}-1}, Sq_{2\acute{i}}) \leq \psi_b(D_{lb}(q_{2\acute{i}}, q_{2\acute{i}-1})) \\
&\leq \psi_b(\max\{d_{lb}(q_{2\acute{i}}, q_{2\acute{i}-1}), \frac{d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1}) . d_{lb}(q_{2\acute{i}-1}, q_{2\acute{i}})}{a + d_{lb}(q_{2\acute{i}}, q_{2\acute{i}-1})}, \\
&\quad d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1}), d_{lb}(q_{2\acute{i}-1}, q_{2\acute{i}})\}) \\
&\leq \psi_b(\max\{d_{lb}(q_{2\acute{i}}, q_{2\acute{i}-1}), d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1})\}).
\end{aligned}$$

If $\max\{d_{lb}(q_{2\acute{i}}, q_{2\acute{i}-1}), d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1})\} = d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1})$, then

$$d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1}) \leq \psi_b(d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1})) \leq b\psi_b(d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1})).$$

Which contradicts that $b\psi_b(t) < t$ for each $t > 0$. Hence, we get

$$d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1}) \leq \psi_b(d_{lb}(q_{2\acute{i}-1}, q_{2\acute{i}})). \tag{3.18}$$

As ψ_b is nondecreasing, so

$$\psi_b(d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1})) \leq \psi_b(\psi_b(d_{lb}(q_{2\acute{i}-1}, q_{2\acute{i}}))).$$

By using above inequality in (3.17), we have

$$d_{lb}(q_{2\acute{i}+1}, q_{2\acute{i}+2}) \leq \psi_b^2(d_{lb}(q_{2\acute{i}-1}, q_{2\acute{i}}))).$$

Proceeding in this way, we get

$$d_{lb}(q_{2\acute{i}+1}, q_{2\acute{i}+2}) \leq \psi_b^{2\acute{i}+1}(d_{lb}(q_0, q_1)). \tag{3.19}$$

Now, if $j = 2\acute{i}$, where $\acute{i} = 1, 2, \ldots \frac{j}{2}$. By using (3.18) and similar procedure as above, we have

$$d_{lb}(q_{2\acute{i}}, q_{2\acute{i}+1}) \leq \psi_b^{2\acute{i}}(d_{lb}(q_0, q_1)). \tag{3.20}$$

Now, by combining (3.19) and (3.20)

$$d_{lb}(q_j, q_{j+1}) \leq \psi_b^j(d_{lb}(q_0, q_1)) \text{ for each } j \in N. \tag{3.21}$$

Now, by using triangle inequality and by (3.21), we have

$$\begin{aligned} d_{lb}(q_0, q_{j+1}) &\leq bd_{lb}(q_0, q_1) + b^2 d_{lb}(q_1, q_2) + \ldots + b^{j+1} d_{lb}(q_j, q_{j+1}) \\ &\leq bd_{lb}(q_0, q_1) + b^2 \psi_b(d_{lb}(q_0, q_1)) + \ldots + b^{j+1} \psi_b^j(d_{lb}(q_0, q_1)) \\ &\leq \sum_{i=0}^{j} b^{i+1}\{\psi_b^i(d_{lb}(q_0, q_1))\} \leq r. \end{aligned}$$

Thus q_{j+1} belongs to $\overline{B_{d_{lb}}(q_0, r)}$. Hence q_n belongs to $\overline{B_{d_{lb}}(q_0, r)}$ for every n belongs to N, therefore $\{TS(q_n)\}$ be the sequence in $\overline{B_{d_{lb}}(q_0, r)}$. As S, T are two α_*-dominated maps on $\overline{B_{d_{lb}}(q_0, r)}$, then $\alpha_*(q_{2n}, Sq_{2n}) \geq 1$ and $\alpha_*(q_{2n+1}, Tq_{2n+1}) \geq 1$. This implies $\alpha(q_n, q_{n+1}) \geq 1$. Also inequality (3.21) can be written as

$$d_{lb}(q_n, q_{n+1}) \leq \psi_b^n(d_{lb}(q_0, q_1)), \text{ for each } n \text{ belongs to } N. \tag{3.22}$$

As $\sum_{k=1}^{+\infty} b^k \psi_b^k(t) < +\infty$, then for some $p \in N$, then the series $\sum_{k=1}^{+\infty} b^k \psi_b^k(\psi_b^{p-1}(d_{lb}(q_0, q_1)))$ converges. As $b\psi_b(t) < t$, so

$$b^{n+1}\psi_b^{n+1}(\psi_b^{p-1}(d_{lb}(q_0, q_1))) < b^n \psi_b^n(\psi_b^{p-1}(d_{lb}(q_0, q_1))) \text{ for each } n \in N.$$

Fix $\varepsilon > 0$, then there must be a $p(\varepsilon)$ belongs to N, so as

$$b\psi_b(\psi_b^{p(\varepsilon)-1}(d_{lb}(q_0, q_1))) + b^2\psi_b^2(\psi_b^{p(\varepsilon)-1}(d_{lb}(q_0, q_1))) + \cdots < \varepsilon$$

Let n, m belong to N with $m > n > p(\varepsilon)$, then, we have

$$\begin{aligned}
d_{lb}(q_n, q_m) &\leq bd_{lb}(q_n, q_{n+1}) + b^2 d_{lb}(q_{n+1}, q_{n+2}) + \cdots + b^{m-n} d_{lb}(q_{m-1}, q_m) \\
&\leq b\psi_b^n(d_{lb}(q_0, q_1)) + b^2\psi_b^{n+1}(d_{lb}(q_0, q_1)) + \cdots + b^{m-n}\psi_b^{m-1}(d_{lb}(q_0, q_1)) \\
&= b\psi_b(\psi_b^{n-1}(d_{lb}(q_0, q_1))) + \cdots + b^{m-n}\psi_b^{m-n}(\psi_b^{n-1}(d_{lb}(q_0, q_1))) \\
&< b\psi_b(\psi_b^{p(\varepsilon)-1}(d_{lb}(q_0, q_1))) + b^2\psi_b^2(\psi_b^{p(\varepsilon)-1}(d_{lb}(q_0, q_1))) + \cdots < \varepsilon.
\end{aligned}$$

It is clear that $\{TS(q_n)\}$ is the Cauchy in $(\overline{B_{d_{lb}}(q_0, r)}, d_{lb})$. As èach ĉlosed báll in a ĉomþleţe $D.B.M.S$ is ĉomþleţe, so there must be a $q^* \in \overline{B_{d_{lb}}(q_0, r)}$ so as $\{TS(q_n)\} \to q^*$, that is

$$\lim_{n\to\infty} d_{lb}(q_n, q^*) = 0 \tag{3.23}$$

Now,

$$\begin{aligned}
d_{lb}(q^*, Sq^*) &\leq bd_{lb}(q^*, q_{2n+2}) + bd_{lb}(q_{2n+2}, Sq^*) \\
&\leq bd_{lb}(q^*, q_{2n+2}) + bH_{d_{lb}}(Tq_{2n+1}, Sq^*). \quad \text{(by Lemma 1.2.8)}
\end{aligned}$$

Since $\alpha_*(q^*, Sq^*) \geq 1$ and $\alpha_*(q_{2n+1}, Tq_{2n+1}) \geq 1$ and $\alpha(q_{2n+1}, q^*) \geq 1$, we obtain

$$\begin{aligned}
d_{lb}(q^*, Sq^*) &\leq bd_{lb}(q^*, q_{2n+2}) + b\psi_b(\max\{d_{lb}(q^*, q_{2n+1}), d_{lb}(q^*, Sq^*), \\
&\quad \frac{d_{lb}(q^*, Sq^*).d_{lb}(q_{2n+1}, Tq_{2n+1})}{a + d_{lb}(q^*, q_{2n+1})}, d_{lb}(q_{2n+1}, Tq_{2n+1})\}) \\
&\quad bd_{lb}(q^*, q_{2n+2}) + b\psi_b(\max\{d_{lb}(q^*, q_{2n+1}), d_{lb}(q^*, Sq^*), \\
&\quad \frac{d_{lb}(q^*, Sq^*).d_{lb}(q_{2n+1}, q_{2n+2})}{a + d_{lb}(q^*, q_{2n+1})}, d_{lb}(q_{2n+1}, q_{2n+2})\}).
\end{aligned}$$

Letting $n \to \infty$, and using (3.23), we obtain $d_{lb}(q^*, Sq^*) \leq b\psi_b(d_{lb}(q^*, Sq^*))$. A contradicts to the relaity that $b\psi_b(t) < t$ and hence $d_{lb}(q^*, Sq^*) \leq 0$ or $q^* \in Sq^*$. Similarly, by using the inequality

$$d_{lb}(q^*, Tq^*) \leq bd_{lb}(q^*, q_{2n+1}) + bd_{lb}(q_{2n+1}, Tq^*)$$

and hence $d_{lb}(q^*, Tq^*) \leq 0$ or $q^* \in Tq^*$. Hence q^* is the C.F.P of S and T in $\overline{B_{d_{lb}}(q_0, r)}$. Since $\alpha_*(q^*, Sq^*) \geq 1$ and (S, T) be the pair of α_*-dominated multifunction on $\overline{B_{d_{lb}}(q_0, r)}$, we have

$\alpha_*(q^*, Tq^*) \geq 1$, so $\alpha(q^*, q^*) \geq 1$. Now,

$$\begin{aligned} d_{lb}(q^*, q^*) &\leq d_{lb}(q^*, Tq^*) \leq H_{d_{lb}}(Sq^*, Tq^*) \\ &\leq \psi_b(\max\{d_{lb}(q^*, q^*), \frac{d_{lb}(q^*, Sq^*) . d_{lb}(q^*, Tq^*)}{a + d_{lb}(q^*, q^*)}, \\ &\quad d_{lb}(q^*, Sq^*), d_{lb}(q^*, Tq^*)\}). \end{aligned}$$

This implies that, $d_{lb}(q^*, q^*) = 0$.

We have the following result without ĉlosed báll and $\alpha-$dominated maþþings for one multivalued maþþing. ■

Theorem 3.3.2 Let (E, d_{lb}) is a ĉomþleţe $D.B.M.S$. Suppose $S : E \to P(E)$ is a setvalued map. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$H_{d_{lb}}(Sq, Se) \leq \psi_b(D_{lb}(q, e))$$

for all $q, e \in \{SS(q_n)\}$. Then $\{SS(q_n)\} \to q^* \in E$ and S has a fîxẹd þoīnţ q^* in E and $d_{lb}(q^*, q^*) = 0$.

Theorem 3.3.3 Let $(E, \preceq, d_{lb})$ is an ordered ĉomþleţe $D.B.M.S$. Let, $r > 0$, $q_0 \in \overline{B_{d_{lb}}(q_0, r)}$ and $S, T : E \to P(E)$ are two multi $\preceq$-dominated maþs on $\overline{B_{d_{lb}}(q_0, r)}$. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$H_{d_{lb}}(Sq, Te) \leq \psi_b(D_{lb}(q, e)) \tag{3.24}$$

for all $q, e \in \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\}$ with either $q \preceq e$ or $e \preceq q$. Also

$$\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_{lb}(q_0, q_1))\} \leq r \text{ for ėach } n \text{ belongs to } \mathbb{N} \cup \{0\} \text{ and } b \geq 1. \tag{3.25}$$

Then $\{TS(q_n)\}$ is the şequenće in $\overline{B_{d_{lb}}(q_0, r)}$ and $\{TS(q_n)\} \to q^* \in \overline{B_{d_{lb}}(q_0, r)}$. Also if the inequality (3.24) holds for q^* and either $q_n \preceq q^*$ or $q^* \preceq q_n$ for ėach n belongs to $\mathbb{N} \cup \{0\}$. Then q^* is the C.F.P of S and T in $\overline{B_{d_{lb}}(q_0, r)}$ and $d_{lb}(q^*, q^*) = 0$.

Proof. Let $\alpha : E \times E \to [0, +\infty)$ be a maþþing defined by $\alpha(q, e) = 1$ for all $q \in \overline{B_{d_{lb}}(q_0, r)}$ with either $q \preceq e$ or $e \preceq q$, and $\alpha(q, e) = 0$ for all other elements $q, e \in E$. Since S and T are dominated maþs on $\overline{B_{d_{lb}}(q_0, r)}$, so $q \preceq Sq$ and $q \preceq Tq$ for all $q \in \overline{B_{d_{lb}}(q_0, r)}$. This implies that

$q \preceq b$ for all $b \in Sq$ and $q \preceq c$ for all $c \in Tq$. So, $\alpha(q,b) = 1$ for all $b \in Sq$ and $\alpha(q,c) = 1$ for all $c \in Tq$. This implies that $\inf\{\alpha(q,e) : e \in Sq\} = 1$ and $\inf\{\alpha(q,e) : e \in Tq\} = 1$. Hence $\alpha_*(q, Sq) = 1$, $\alpha_*(q,Tq) = 1$ for all $q \in \overline{B_{d_{lb}}(q_0,r)}$. So, $S,T : E \to P(E)$ are the α_*−dominated mapping on $\overline{B_{d_{lb}}(q_0,r)}$. Moreover, inequality (3.24) can be written as

$$H_{d_{lb}}(Sq,Te) \leq \psi_b(D_{lb}(q,e))$$

for all elements q,e in $\overline{B_{d_{lb}}(q_0,r)} \cap \{TS(q_n)\}$ with either $\alpha(q,e) \geq 1$ or $\alpha(e,q) \geq 1$. Also, inequality (3.25) holds. Then, by Theorem 3.3.1, we have $\{TS(q_n)\}$ be the sequence in $\overline{B_{d_{lb}}(q_0,r)}$ and $\{TS(q_n)\} \to q^* \in \overline{B_{d_{lb}}(q_0,r)}$. Now, $q_n, q^* \in \overline{B_{d_{lb}}(q_0,r)}$ and either $q_n \preceq q^*$ or $q^* \preceq q_n$ implies that either $\alpha(q_n,q^*) \geq 1$ or $\alpha(q^*,q_n) \geq 1$. So, all hypothesis of Theorem 3.3.1 are proved. Hence, by Theorem 3.3.1, q^* is the C.F.P of S and T in $\overline{B_{d_{lb}}(q_0,r)}$ and $d_{lb}(q^*,q^*) = 0$.
We have the following result without closed ball in complete $D.B.M.S$. Also we write the result only for one multivalued mapping. ■

Theorem 3.3.4 Let $(E,\preceq,d_{lb})$ is an ordered complete $D.B.M.S$. Let $S : E \to P(E)$ be two multi $\preceq$-dominated mappings on E. Assume that, for some $\psi_b \in \Psi_b$, the following hold:

$$H_{d_{lb}}(Sq,Se) \leq \psi_b(D_{lb}(q,e)) \tag{3.26}$$

for all $q,e \in \{SS(q_n)\}$ with $q \preceq e$. Then $\{SS(q_n)\} \to q^* \in E$. Also if the inequality (3.26) holds for q^* and either $q_n \preceq q^*$ or $q^* \preceq q_n$ for every n belongs to $\mathbb{N} \cup \{0\}$. Then q^* is the fixed point of S $d_{lb}(q^*,q^*) = 0$.

Example 3.3.5 Let $E = Q^+ \cup \{0\}$ and let $d_{lb} : E \times E \to E$ be the complete $D.B.M.S$ on E defined by

$$d_{lb}(w,k) = (w+k)^2 \text{ for each } w,k \in E$$

with parameter $b = 2$. Define the setvalued maps, $S,T : E \times E \to P(E)$ by,

$$Sq = \begin{cases} [\frac{q}{3}, \frac{2}{3}q] \text{ if } q \in [0,19] \cap E \\ [q, q+1] \text{ if } q \in (19,\infty) \cap E, \end{cases}$$

and

$$Tq = \begin{cases} [\frac{q}{4}, \frac{3}{4}q] \text{ if } q \in [0, 19] \cap E \\ [q+1, q+3] \text{ if } q \in (19, \infty) \cap E. \end{cases}$$

Considering, $q_0 = 1, r = 400$, then $\overline{B_{d_{lb}}(q_0, r)} = [0, 19] \cap E$. Now $d_{lb}(q_0, Sq_0) = d_{lb}(1, S1) = d_{lb}(1, \frac{1}{3}) = \frac{16}{9}$. So we make a şequençe $\{TS(q_n)\} = \{1, \frac{1}{3}, \frac{1}{12}, \frac{1}{36}, ...\}$ in E generated by q_0. Let $\psi_b(t) = \frac{4t}{10}$, then $b\psi_b(t) < t$. Define

$$\alpha(q, e) = \begin{Bmatrix} 1 & \text{if } q > e \\ \frac{1}{2} & \text{otherwise} \end{Bmatrix}.$$

Then $S, T : E \to P(E)$ be the α_*−dominated maþþings on $\overline{B_{d_{lb}}(q_0, r)}$. Now take $20, 21 \in E$ and $a = 1$, then, we have

$$H_{d_{lb}}(S20, T21) = 1936 > \psi_b(D_l(q, e)) = \frac{7396}{10}.$$

So, the inequality (3.15) is not true for the whole şþace E. Now for all $q, e \in \overline{B_{d_q}(q_0, r)} \cap \{TS(q_n)\}$ with either $\alpha(q, e) \geq 1$ or $\alpha(e, q) \geq 1$, we have

$$\begin{aligned} H_{d_{lb}}(Sq, Te) &= \max\{d_{lb}(\frac{2q}{3}, \frac{e}{4}), d_{lb}(\frac{q}{3}, \frac{3e}{4})\} \\ &= \max\left\{\left(\frac{2q}{3} + \frac{e}{4}\right)^2, \left(\frac{q}{3} + \frac{3e}{4}\right)^2\right\} \\ &\leq \psi_b(\max\{(q+e)^2, \frac{25q^2e^2}{9(1+(q+e)^2)}, \left(\frac{4q}{3}\right)^2, \left(\frac{5e}{4}\right)^2\}). \end{aligned}$$

So, the inequality (3.15) holds on $\overline{B_{d_q}(q_0, r)} \cap \{TS(q_n)\}$. Also, for èaćh n belongs to $\mathbb{N} \cup \{0\}$, we have

$$\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_{lb}(q_0, q_1))\} = \frac{16}{9} \times 2\sum_{i=0}^{n}(\frac{4}{5})^i < 400 = r.$$

Now, we have $\{TS(q_n)\}$ be the şequençe in $\overline{B_{d_l}(q_0, r)}$, $\alpha(q_n, q_{n+1}) \geq 1$ and $\{TS(q_n)\} \to 0 \in \overline{B_{d_l}(q_0, r)}$. Also, $\alpha(q_n, 0) \geq 1$ or $\alpha(0, q_n) \geq 1$ for ēverŷ n belongs to $\mathbb{N} \cup \{0\}$. Hence, all hypothesis of Theorem 3.3.1 are proved.

Theorem 3.3.6 Let (E, d_{lb}) is a ĉomþleţe $D.B.M.S$ endowed a graph G. Let, $r > 0$,

$q_0 \in \overline{B_{d_{lb}}(q_0, r)}$, $S, T : E \to P(E)$ and $\{TS(q_n)\}$ be a şequenće in E generated by q_0. Suppose (i), (ii) and (iii) hold:

(i) S and T are multi graph dominated on $\overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\}$;

(ii) there exists $\psi_b \in \Psi_b$, so as

$$H_{d_{lb}}(Sq, Te) \leq \psi_b(D_{lb}(q, e)), \tag{3.27}$$

for all $q, e \in \overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\}$ and $(q,e) \in W(G)$ or $(e,q) \in W(G)$;

(iii) $\sum_{i=0}^{n} b^{i+1}\{\psi_b^i(d_{lb}(q_0, Sq_0))\} \leq r$ for ėach n belongs to $\mathbb{N} \cup \{0\}$ and $b \geq 1$.

Then, $\{TS(q_n)\}$ is a şequenće in $\overline{B_{d_{lb}}(q_0, r)}$, $(q_n, q_{n+1}) \in W(G)$ and $\{TS(q_n)\} \to q^*$. Also, if the inequality (3.27) holds for q^* and $(q_n, q^*) \in W(G)$ or $(q^*, q_n) \in W(G)$ for ėach n belongs to $\mathbb{N} \cup \{0\}$, then q^* is the C.F.P of both S and T in $\overline{B_{d_{lb}}(q_0, r)}$ and $d_{lb}(q^*, q^*) = 0$.

Proof. Define, $\alpha : E \times E \to [0, \infty)$ by

$$\alpha(q, e) = \begin{cases} 1, & \text{if } q \in \overline{B_{d_{lb}}(q_0, r)},\ (q, e) \in W(G) \text{ or } (e,q) \in W(G) \\ 0, & \text{otherwise.} \end{cases}$$

Given S and T are graph dominated on $\overline{B_{d_{lb}}(q_0, r)}$, then for $q \in \overline{B_{d_{lb}}(q_0, r)}$, $(q, e) \in W(G)$ for all $e \in Sq$ and $(q, e) \in W(G)$ for all $e \in Tq$. So, $\alpha(q, e) = 1$ for all $e \in Sq$ and $\alpha(q, e) = 1$ for all $e \in Tq$. This implies that $\inf\{\alpha(q, e) : e \in Sq\} = 1$ and $\inf\{\alpha(q, e) : e \in Tq\} = 1$. Hence $\alpha_*(q, Sq) = 1$, $\alpha_*(q, Tq) = 1$ for all $q \in \overline{B_{d_{lb}}(q_0, r)}$. So, $S, T : E \to P(E)$ are the α_*-dominated maþþing on $\overline{B_{d_{lb}}(q_0, r)}$. Moreover, inequality (3.27) can be written as

$$H_{d_{lb}}(Sq, Te) \leq \psi_b(D_{lb}(q, e)),$$

for all elements q, e in $\overline{B_{d_{lb}}(q_0, r)} \cap \{TS(q_n)\}$ with either $\alpha(q, e) \geq 1$ or $\alpha(e, q) \geq 1$. Also, (iii) holds. Then, by Theorem 3.3.1, we have $\{TS(q_n)\}$ is the şequenće in $\overline{B_{d_{lb}}(q_0, r)}$ and $\{TS(q_n)\} \to q^* \in \overline{B_{d_{lb}}(q_0, r)}$. Now, $q_n, q^* \in \overline{B_{d_{lb}}(q_0, r)}$ and either $(q_n, q^*) \in W(G)$ or $(q^*, q_n) \in W(G)$ implies that either $\alpha(q_n, q^*) \geq 1$ or $\alpha(q^*, q_n) \geq 1$. So, all hypothesis of Theorem 3.3.1 are proved. Hence, by Theorem 3.3.1, q^* be the C.F.P of S and T in $\overline{B_{d_{lb}}(q_0, r)}$ and $d_{lb}(q^*, q^*) = 0$.

We have the following result without ĉlosed báll in ĉomþleţe $D.B.M.S$ for multi graph dominated mapping. Also we write the result only for one multivalued mapping and for $D_{lb}(q,e) = d_{lb}(q,e)$. ■

Theorem 3.3.7 Let (E, d_{lb}) is a complete $D.B.M.S$ endowed a graph G. Let, $r > 0$, $q_0 \in \overline{B_{d_{lb}}(q_0, r)}$, $S : E \to P(E)$ and $\{SS(q_n)\}$ be the sequence in E generated by q_0. Assume that (i) and (ii) hold:

(i) S is a multi graph dominated on$\{SS(q_n)\}$;

(ii) there exists $\psi_b \in \Psi_b$ so as

$$H_{d_{lb}}(Sq, Se) \leq \psi_b(d_{lb}(q,e)), \tag{3.28}$$

for all $q, e \in \{TS(q_n)\}$ and $(q,e) \in W(G)$ or $(e,q) \in W(G)$;

Then, $(q_n, q_{n+1}) \in W(G)$ and $\{SS(q_n)\} \to q^*$. Also, if the inequality (3.28) holds for q^* and $(q_n, q^*) \in W(G)$ or $(q^*, q_n) \in W(G)$ for each n belongs to $\mathbb{N} \cup \{0\}$, then q^* is the C.F.P of both S and T in E and $d_{lb}(q^*, q^*) = 0$.

3.4 Fixed Point Results for Multivalued Dominated Mappings in Dislocated b-Metric Spaces with Application

Results given in this section can be seen in [49].

Let (Z, d_l) be a $D.B.M.S$, $g_0 \in Z$ and $S, T : Z \to P(Z)$ be the setvalued maps on Z. Let $g_1 \in Sg_0$ be an element such that $d_l(g_0, Sg_0) = d_l(g_0, g_1)$. Let $g_2 \in Tg_1$ be such that $d_l(g_1, Tg_1) = d_l(g_1, g_2)$. Let $g_3 \in Sg_2$ be such that $d_l(g_2, Sg_2) = d_l(g_2, g_3)$. Proceeding this method, we get sequence g_n in Z such that $g_{2n+1} \in Sg_{2n}$ and $g_{2n+2} \in Tg_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_l(g_{2n}, Sg_{2n}) = d_l(g_{2n}, g_{2n+1})$, $d_l(g_{2n+1}, Tg_{2n+1}) = d_l(g_{2n+1}, g_{2n+2})$. We represent this type of sequence by $\{TS(g_n)\}$. We say that $\{TS(g_n)\}$ be a sequence in Z generated by g_0.

Theorem 3.4.1 Let (Z, d_l) is a complete $D.B.M.S$ with coefficient $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0, r)}$. Assume (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$ and a strictly increasing maþþing F such that

$$\tau + F(H_{d_l}(Se,Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e,y) + \eta_2 d_l(e,Se) \\ +\eta_3 d_l(e,Ty) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{array}\right), \tag{3.29}$$

whenever $e, y \in \overline{B_{d_l}(g_0,r)} \cap \{TS(g_n)\}$, $\alpha(e,y) \geq 1$ and $H_{d_l}(Se,Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3-\eta_4}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r. \tag{3.30}$$

Then $\{TS(g_n)\}$ is the şequenće in $\overline{B_{d_l}(g_0,r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for ėach n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0,r)}$. Also, if the inequality (3.29) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for ėach n belongs to $\mathbb{N} \cup \{0\}$,then u is the C.F.P of both S and T in $\overline{B_{d_l}(g_0,r)}$.

Proof. Consider a şequenće $\{TS(g_n)\}$. From (3.30), we get

$$d_l(g_0, g_1) = d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r < r.$$

It implies that,

$$g_1 \in \overline{B_{d_l}(g_0,r)}.$$

Let $g_2, \cdots, g_j \in \overline{B_{d_l}(g_0,r)}$ for ēverŷ j belongs to $\mathbb{N}$. If j is odd, then $j = 2\imath + 1$ for some $\imath \in \mathbb{N}$. Since $S, T : Z \to P(Z)$ be a semi α_*-dominated maþþings on $\overline{B_{d_l}(g_0,r)}$, so $\alpha_*(g_{2\imath}, Sg_{2\imath}) \geq 1$ and $\alpha_*(g_{2\imath+1}, Tg_{2\imath+1}) \geq 1$. As $\alpha_*(g_{2\imath}, Sg_{2\imath}) \geq 1$, this implies $\inf\{\alpha(g_{2\imath}, b) : b \in Sg_{2\imath}\} \geq 1$. Also $g_{2\imath+1} \in Sg_{2\imath}$, so $\alpha(g_{2\imath}, g_{2\imath+1}) \geq 1$. Now, by using Lemma 1.2.8, we get

$$\tau + F(d_l(g_{2\imath+1}, g_{2\imath+2})) \leq \tau + F(H_{d_l}(Sg_{2\imath}, Tg_{2\imath+1}))$$

Now, by using (3.29), we get

$$\begin{aligned} \tau + F(d_l(g_{2\imath+1}, g_{2\imath+2})) \leq\ & F[\eta_1 d_l(g_{2\imath}, g_{2\imath+1}) + \eta_2 d_l(g_{2\imath}, Sg_{2\imath}) + \eta_3 d_l(g_{2\imath}, Tg_{2\imath+1}) \\ & +\eta_4 \frac{d_l^2(g_{2\imath}, Sg_{2\imath}).d_l(g_{2\imath+1}, Tg_{2\imath+1})}{1 + d_l^2(g_{2\imath}, g_{2\imath+1})}] \end{aligned}$$

$$
\begin{aligned}
&\leq F[\eta_1 d_l(g_{2i}, g_{2i+1}) + \eta_2 d_l(g_{2i}, g_{2i+1}) + \eta_3 d_l(g_{2i}, g_{2i+2}) \\
&\quad + \eta_4 \frac{d_l^2(g_{2i}, g_{2i+1}) . d_l(g_{2i+1}, g_{2i+2})}{1 + d_l^2(g_{2i}, g_{2i+1})}] \\
&\leq F[\eta_1 d_l(g_{2i}, g_{2i+1}) + \eta_2 d_l(g_{2i}, g_{2i+1}) + b\eta_3 d_l(g_{2i}, g_{2i+1}) \\
&\quad + b\eta_3 d_l(g_{2i+1}, g_{2i+2}) + \eta_4 \frac{d_l^2(g_{2i}, g_{2i+1}) . d_l(g_{2i+1}, g_{2i+2})}{1 + d_l^2(g_{2i}, g_{2i+1})}]
\end{aligned}
$$

$$
\leq F((\eta_1 + \eta_2 + b\eta_3) d_l(g_{2i}, g_{2i+1}) + (b\eta_3 + \eta_4) d_l(g_{2i+1}, g_{2i+2})).
$$

This implies

$$
\begin{aligned}
F(d_l(g_{2i+1}, g_{2i+2})) &< F((\eta_1 + \eta_2 + b\eta_3) d_l(g_{2i}, g_{2i+1}) \\
&\quad + (b\eta_3 + \eta_4) d_l(g_{2i+1}, g_{2i+2})).
\end{aligned}
$$

As F is the strictly increasing maþþings. So,

$$
\begin{aligned}
d_l(g_{2i+1}, g_{2i+2}) &< (\eta_1 + \eta_2 + b\eta_3) d_l(g_{2i}, g_{2i+1}) \\
&\quad + (b\eta_3 + \eta_4) d_l(g_{2i+1}, g_{2i+2}).
\end{aligned}
$$

Which implies

$$
\begin{aligned}
(1 - b\eta_3 - \eta_4) d_l(g_{2i+1}, g_{2i+2}) &< (\eta_1 + \eta_2 + b\eta_3) d_l(g_{2i}, g_{2i+1}) \\
d_l(g_{2i+1}, g_{2i+2}) &< \left(\frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3 - \eta_4} \right) d_l(g_{2i}, g_{2i+1}).
\end{aligned}
$$

As $\lambda = \frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3 - \eta_4} < 1$. Hence

$$
d_l(g_{2i+1}, g_{2i+2}) < \lambda d_l(g_{2i}, g_{2i+1}) < \lambda^2 d_l(g_{2i-1}, g_{2i}) < \cdots < \lambda^{2i+1} d_l(g_0, g_1).
$$

Similarly, if j is even, we have

$$
d_l(g_{2i+2}, g_{2i+3}) < \lambda^{2i+2} d_l(g_0, g_1).
$$

Now, we have

$$
d_l(g_j, g_{j+1}) < \lambda^j d_l(g_0, g_1) \text{ for each } j \text{ belongs to } \mathbb{N}. \tag{3.31}
$$

Now,

$$
\begin{aligned}
d_l(x_0, g_{j+1}) &\leq bd_l(g_0, g_1) + b^2 d_l(g_1, g_2) + \cdots + b^{j+1} d_l(g_j, g_{j+1}) \\
&\leq bd_l(g_0, g_1) + b^2\lambda(d_l(g_0, g_1)) + \cdots \\
&\quad + b^{j+1}\lambda^{j+1}(d_l(g_0, g_1)), \qquad \text{(by (3.31))} \\
d_l(g_0, g_{j+1}) &\leq \frac{b(1-(b\lambda)^{j+1})}{1-b\lambda}\lambda(1-b\lambda)r < r,
\end{aligned}
$$

which means g_{j+1} belongs to $\overline{B_{d_l}(g_0, r)}$. Hence, by induction $g_n \in \overline{B_{d_l}(g_0, r)}$ for every n belongs to N. Also $\alpha(g_n, g_{n+1}) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$. Now,

$$
d_l(g_n, g_{n+1}) < \lambda^n d_l(g_0, g_1) \text{ for each } n \in \mathbb{N}. \tag{3.32}
$$

Now, for non negative integers m, n $(n > m)$, we get

$$
\begin{aligned}
d_l(g_m, g_n) &\leq b(d_l(g_m, g_{m+1})) + b^2(d_l(g_{m+1}, g_{m+2})) + \cdots \\
&\quad + b^{n-m}(d_l(g_{n-1}, g_n)),
\end{aligned}
$$

$$
\begin{aligned}
&< b\lambda^m d_l(g_0, g_1) + b^2\lambda^{m+1} d_l(g_0, g_1) + \cdots \\
&\quad + b^{n-m}\lambda^{n-1} d_l(g_0, g_1), \qquad \text{(by (3.32))} \\
&< b\lambda^m(1 + b\lambda + \cdots)d_l(g_0, g_1)
\end{aligned}
$$

As $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, $b \geq 1$ and $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$, so $|b\lambda| < 1$. Then, we have

$$
d_l(g_m, g_n) < \frac{b\lambda^m}{1-b\lambda} d_l(g_0, g_1) \to 0 \text{ as } m \to \infty.
$$

Hence $\{TS(g_n)\}$ is Cauchy in $\overline{B_{d_l}(g_0, r)}$. Since $(\overline{B_{d_l}(g_0, r)}, d_l)$ is a complete metric space, so there exist $u \in \overline{B_{d_l}(g_0, r)}$ so as $\{TS(g_n)\} \to u$ as $n \to \infty$, then

$$
\lim_{n\to\infty} d_l(g_n, u) = 0. \tag{3.33}
$$

By assumption, $\alpha(g_n, u) \geq 1$. Suppose that $d_l(u, Tg) > 0$, then there exist positive integer k so as $d_l(g_n, Tu) > 0$ for each $n \geq k$. For $n \geq k$, we get

$$\begin{aligned} d_l(u, Tu) &\leq d_l(u, g_{2n+1}) + d_l(g_{2n+1}, Tu) \\ &\leq d_l(u, g_{2n+1}) + H_{d_l}(Sg_{2n}, Tu) \end{aligned}$$

$$\begin{aligned} &< d_l(u, g_{2n+1}) + \eta_1 d_l(g_{2n}, u) + \eta_2 d_l(g_{2n}, Sg_{2n}) \\ &\quad + \eta_3 d_l(g_{2n}, Tu) + \eta_4 \frac{d_l^2(g_{2n}, Sg_{2n}).d_l(u, Tu)}{1 + d_l^2(g_{2n}, u)} \\ &< d_l(u, g_{2n+1}) + \eta_1 d_l(g_{2n}, u) + \eta_2 d_l(g_{2n}, g_{2n+1}) \\ &\quad + \eta_3 d_l(g_{2n}, Tu) + \eta_4 \frac{d_l^2(g_{2n}, g_{2n+1}).d_l(u, Tu)}{1 + d_l^2(g_{2n}, u)}. \end{aligned}$$

Letting $n \to \infty$, and by using (3.33) we get

$$d_l(u, Tu) < \eta_3 d_l(u, Tu) < d_l(u, Tu),$$

which contradicts. So our supposition is wrong. Hence $d_l(u, Tu) = 0$ or $u \in Tu$. Similarly, by using Lemma 1.2.8, inequality (3.29),

$$\begin{aligned} d_l(u, Su) &\leq d_l(u, g_{2n+2}) + d_l(g_{2n+2}, Su) \\ &\leq d_l(u, g_{2n+2}) + H_{d_l}(Tg_{2n+1}, Su) \\ &< d_l(u, g_{2n+2}) + \eta_1 d_l(u, g_{2n+1}) \\ &\quad + \eta_2 d_l(u, Su) + \eta_3 d_l(u, Tg_{2n+1}) \\ &\quad + \eta_4 \frac{d_l^2(u, Su).d_l(g_{2n+1}, Tg_{2n+1})}{1 + d_l^2(u, g_{2n+1})} \end{aligned}$$

$$\begin{aligned} &< d_l(u, g_{2n+2}) + \eta_1 d_l(u, g_{2n+1}) + \eta_2 d_l(u, Su) \\ &\quad + \eta_3 d_l(u, g_{2n+2}) + \eta_4 \frac{d_l^2(u, Su).d_l(g_{2n+1}, g_{2n+2})}{1 + d_l^2(u, g_{2n+1})}. \end{aligned}$$

Letting $n \to \infty$, and by using (3.33) we get

$$d_l(u, Su) < \eta_2 d_l(u, Su) < d_l(u, Su),$$

which contradicts. So our supposition is wrong. Hence $d_l(u, Su) = 0$ or $u \in Su$. Hence the S and T have a C.F.P u in $\overline{B_{d_l}(g_0, r)}$. Now,

$$d_l(u, u) \leq bd_l(u, Tu) + bd_l(Tu, u) \leq 0.$$

This implies $d_l(u, u) = 0$. ■

Example 3.4.2 Let $Z = Q^+ \cup \{0\}$ and let $d_l : Z \times Z \to Z$ be the complete $D.B.M.S$ defined by

$$d_l(v, p) = (v + p)^2 \text{ for all } v, p \in Z.$$

with $b = 2$. Define the multivalued mapping, $S, T : Z \times Z \to P(Z)$ by,

$$Sg = \begin{cases} [\frac{g}{3}, \frac{2}{3}g] \text{ if } g \in [0, 14] \cap Z \\ [g, g+1] \text{ if } g \in (14, \infty) \cap Z \end{cases}$$

and,

$$Tp = \begin{cases} [\frac{z}{4}, \frac{3}{4}z] \text{ if } p \in [0, 14] \cap Z \\ [p+1, p+3] \text{ if } p \in (14, \infty) \cap Z. \end{cases}$$

Suppose that, $g_0 = 1$, $r = 225$, then $\overline{B_{d_l}(g_0, r)} = [0, 14] \cap Z$ and $\{TS(g_n)\} = \{1, \frac{1}{3}, \frac{1}{12}, ...\}$. Take $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$, $\eta_3 = \frac{1}{60}$, $\eta_4 = \frac{1}{30}$, then $b\eta_1 + b\eta_2 + (1 + b)b\eta_3 + \eta_4 < 1$ and $\lambda = \frac{11}{56}$. Now

$$d_l(g_0, Sg_0) = \frac{16}{9} < \frac{11}{56}(1 - \frac{22}{56})225 = \lambda(1 - b\lambda)r$$

Consider the mapping $\alpha : Z \times Z \to [0, \infty)$ by

$$\alpha(g, p) = \begin{Bmatrix} 1 \text{ if } g > p \\ \frac{1}{2} \text{ otherwise} \end{Bmatrix}.$$

Now, if $g, p \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$ with $\alpha(g, p) \geq 1$, we have

$$\begin{aligned}
H_{d_l}(Sg,Tp) &= \max\{\sup_{a\in Sg} d_l(a,Tp), \sup_{b\in Tp} d_l(Sg,b)\} \\
&= \max\{\sup_{a\in Sg} d_l(a,[\frac{p}{4},\frac{3p}{4}]), \sup_{b\in Tp} d_l([\frac{g}{3},\frac{2g}{3}],b)\} \\
&= \max\{d_l(\frac{2g}{3},[\frac{p}{4},\frac{3p}{4}]), d_l([\frac{g}{3},\frac{2g}{3}],\frac{3p}{4})\} \\
&= \max\{d_l(\frac{2g}{3},\frac{p}{4}), d_l(\frac{g}{3},\frac{3p}{4})\}
\end{aligned}$$

$$\begin{aligned}
&= \max\left\{\left(\frac{2g}{3}+\frac{p}{4}\right)^2, \left(\frac{g}{3}+\frac{3p}{4}\right)^2\right\} \geq \\
&< \frac{1}{10}(g+p)^2 + \frac{4g^2}{45} + \frac{(4g+p)^2}{960} + \frac{40g^4p^2}{243\{1+(g+p)^4\}} \\
&= \frac{1}{10}d_l(g,p) + \frac{1}{20}d_l(g,[\frac{g}{3},\frac{2}{3}g]) + \frac{1}{60}d_l(g,[\frac{p}{4},\frac{3}{4}p]) \\
&\quad + \frac{1}{30}\frac{d_l^2(g,[\frac{g}{3},\frac{2}{3}g]).d_l(p,[\frac{p}{4},\frac{3}{4}p])}{1+d_l^2(g,p)}.
\end{aligned}$$

Thus,

$$H_{d_l}(Sg,Tp) < \eta_1 d_l(g,p) + \eta_2 d_l(g,Sg) + \eta_3 d_l(g,Tp) + \eta_4 \frac{d_l^2(g,Sg).d_l(p,Tp)}{1+d_l^2(g,p)},$$

which implies that, for any $\tau \in (0, \frac{12}{95}]$ and for a strictly increasing mapping $F(s) = \ln s$, we have

$$\tau + F(H_{d_l}(Sg,Tp)) \leq F\left(\begin{array}{c} \eta_1 d_l(g,p) + \eta_2 d_l(g,Sg) + \eta_3 d_l(g,Tp) \\ +\eta_4 \frac{d_l^2(g,Sg).d_l(p,Tp)}{1+d_l^2(g,p)} \end{array}\right).$$

Note that, for $16, 15 \in X$, then $\alpha(16,15) \geq 1$. But, we have

$$\tau + F(H_{d_l}(S16,T15)) > F\left(\begin{array}{c} \eta_1 d_l(16,15) + \eta_2 d_l(16,S16) + \eta_3 d_l(16,T15) \\ +\eta_4 \frac{d_l^2(16,S16).(15,T15)}{1+d_l^2(16,15)} \end{array}\right).$$

So condition (3.29) does not hold on Z. Thus maps S and T are satisfying all requirements of Theorem 3.4.1 only for $g, p \in \overline{B_{d_l}(g_0,r)} \cap \{TS(g_n)\}$ with $\alpha(g,p) \geq 1$. Hence S and T have a C.F.P.

If, we take $S = T$ in Theorem 3.4.1, then we are left only with the result.

Corollary 3.4.3 Let (Z, d_l) is a complete $D.B.M.S$ with coefficient $w \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S : Z \to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0, r)}$. Suppose (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $w\eta_1 + w\eta_2 + (1+w)w\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Se, Sy)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Sy) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Sy)}{1+d_l^2(x,y)} \end{array}\right), \tag{3.34}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{SS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Sy) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+w\eta_3}{1-w\eta_3-\eta_4}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1 - w\lambda)r.$$

Then $\{SS(g_n)\}$ be the sequence in $\overline{B_{d_l}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $\{SS(g_n)\} \to u \in \overline{B_{d_l}(g_0, r)}$. Also, if the inequality (3.34) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for every n belongs to $\mathbb{N} \cup \{0\}$, then u be the fixed point of S in $\overline{B_{d_l}(g_0, r)}$.

If, we take $\eta_2 = 0$ in Theorem 3.4.1, then we are left only with the result.

Corollary 3.4.4 Let (Z, d_l) is a complete $D.B.M.S$ with coefficient $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0, r)}$. Suppose (i) and (ii) hold::

(i) There exist $\tau, \eta_1, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + (1+b)b\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Se, Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_3 d_l(e, Ty) \\ +\eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{array}\right), \tag{3.35}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1+b\eta_3}{1-b\eta_3-\eta_4}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{TS(g_n)\}$ be the sequence in $\overline{B_{d_l}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0, r)}$. Also, if the inequality (3.35) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for every n belongs to $\mathbb{N} \cup \{0\}$, then u is the C.F.P of both S and T in $\overline{B_{d_l}(g_0, r)}$.

If, we take $\eta_3 = 0$ in Theorem 3.4.1, then we are left only with the result.

Corollary 3.4.5 Let (Z, d_l) is a complete $D.B.M.S$ with coefficient $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0, r)}$. Suppose (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Se, Ty)) \leq F\begin{pmatrix} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_4 \frac{d_l^2(e, Se).d_l(y, Ty)}{1 + d_l^2(e, y)} \end{pmatrix}, \tag{3.36}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_4}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{TS(g_n)\}$ be the sequence in $\overline{B_{d_l}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0, r)}$. Also, if the inequality (3.36) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for every n belongs to $\mathbb{N} \cup \{0\}$, then then u is the C.F.P of both S and T in $\overline{B_{d_l}(g_0, r)}$.

If, we take $\eta_4 = 0$ in Theorem 3.4.1, then we are left only with the result.

Corollary 3.4.6 Let (Z, d_l) is a complete $D.B.M.S$ with coefficient $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to P(Z)$ are the semi α_*-dominated maps on $\overline{B_{d_l}(g_0, r)}$. Suppose (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_3 > 0$ satisfying $b\eta_1 + b\eta_2 + (1 + b)b\eta_3 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_3 d_l(e, Ty)\right), \tag{3.37}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1-b\lambda)r.$$

Then $\{TS(g_n)\}$ be the şequenĉe in $\overline{B_{d_l}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for ėaċh n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0, r)}$. Also, if the inequality (3.37) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for ēverŷ n belongs to $\mathbb{N} \cup \{0\}$, then u is the C.F.P of both S and T in $\overline{B_{d_l}(g_0, r)}$.

Now we presents an application of Theorem 3.4.1 in graph theory. Jachymski [33] proved the result concerning for contractive maþþings with a graph. Hussain et al. [31] introduced the fîxẹd þoīnţs theorem for graphic ĉonţracţioñ and gave an application. Furtheremore, avoiding sets condition is ĉlosed related to fîxẹd þoīnţ and is applied to the study of multi-agent systems (see [46]).

Definition 3.4.7 Let $Z \neq \{\}$ and $Q = (V(Q), W(Q))$ be a graph such that $V(Q) = Z$, $A \subseteq Z$. $S : Z \to P(Z)$ be the multi graph dominated on A if $(p, q) \in W(Q)$, for all $q \in Sp$ and $q \in A$.

Theorem 3.4.8 Let (Z, d_l) is a ĉomþleţe $D.B.M.S$ endowed a graph Q with coefficient $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)}$ and $S, T : Z \to P(Z)$. Assume (i), (ii) and (iii) satisfy:

(i) S and T are multi graph dominated on $\overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$.

(ii) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$ and a strictly increasing maþþing F such that

$$\tau + F(H_{d_l}(Sp, Tq)) \leq F\left(\begin{array}{c} \eta_1 d_l(p, q) + \eta_2 d_l(p, Sp) \\ +\eta_3 d_l(p, Tq) + \eta_4 \frac{d_l^2(p, Sp).d_l(q, Tq)}{1+d_l^2(p, q)} \end{array}\right), \tag{3.38}$$

whenever $p, q \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $(p,q) \in W(Q)$ and $H_{d_l}(Sp, Tq) > 0$.

(iii) $d_l(g_0, Sg_0) \leq \lambda(1-b\lambda)r$, where $\lambda = \frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3-\eta_4}$.

Then. $\{TS(g_n)\}$ be the şequenĉe in $\overline{B_{d_l}(g_0, r)}$. $\{TS(g_n)\} \to m^*$ and $(g_n, g_{n+1}) \in W(Q)$, where $g_n, g_{n+1} \in \{TS(g_n)\}$. Also, if the inequality (3.38) holds for $p, q \in \{m^*\}$ and $(g_n, m^*) \in W(Q)$ or $(m^*, g_n) \in W(Q)$ for ēverŷ n belongs to $\mathbb{N} \cup \{0\}$, then m^* is the C.F.P of both S and T in $\overline{B_{d_l}(g_0, r)}$.

Proof. Define, $\alpha : Z \times Z \to [0, \infty)$ by

$$\alpha(w,e) = \begin{cases} 1, & \text{if } w \in \overline{B_{d_l}(g_0,r)},\ (w,e) \in W(Q) \\ 0, & \text{otherwise.} \end{cases}$$

Given S and T are semi graph dominated on $\overline{B_{d_l}(g_0,r)}$, then for $p \in \overline{B_{d_l}(g_0,r)}$, $(p,q) \in W(Q)$ for all $q \in Sp$ and $(p,q) \in W(Q)$ for all $q \in Tp$. So, $\alpha(p,q) = 1$ for all $q \in Sp$ and $\alpha(p,q) = 1$ for all $q \in Tp$. This implies that $\inf\{\alpha(p,q) : q \in Sp\} = 1$ and $\inf\{\alpha(p,q) : q \in Tp\} = 1$. Hence $\alpha_*(p, Sp) = 1$, $\alpha_*(p, Tp) = 1$ for all $p \in \overline{B_{d_l}(g_0,r)}$. So, $S, T : Z \to P(Z)$ are the semi α_*-dominated mapping on $\overline{B_{d_l}(g_0,r)}$. Moreover, inequality (2.38) can be written as

$$\tau + F(H_{d_l}(Sp,Tq)) \leq F\left(\begin{array}{c} \eta_1 d_l(p,q) + \eta_2 d_l(p,Sp) \\ +\eta_3 d_l(p,Tq) + \eta_4 \frac{d_l^2(p,Sp).d_l(q,Tq)}{1+d_l^2(p,q)} \end{array} \right)$$

whenever $p, q \in \overline{B_{d_l}(g_0,r)} \cap \{TS(g_n)\}$, $\alpha(p,q) \geq 1$ and $H_{d_l}(Sp,Tq) > 0$. Also, (iii) holds. Then, by Theorem 3.4.1, we have $\{TS(g_n)\}$ be the sequence in $\overline{B_{d_l}(g_0,r)}$ and $\{TS(g_n)\} \to m^* \in \overline{B_{d_l}(g_0,r)}$. Now, $g_n, m^* \in \overline{B_{d_l}(g_0,r)}$ and either $(g_n, m^*) \in W(Q)$ or $(m^*, g_n) \in W(Q)$ implies that either $\alpha(g_n, m^*) \geq 1$ or $\alpha(m^*, g_n) \geq 1$. So, all hypothesis of Theorem 3.4.1 are proved. Hence, by Theorem 3.4.1, S and T have a C.F.P m^* in $\overline{B_{d_l}(g_0,r)}$ and $d_l(m^*, m^*) = 0$. ■

In this section, we have discussed some new fixed point results for single valued mapping in complete $D.B.M.S$. Let (Z, d_l) be a $D.B.M.S$, $c_0 \in Z$ and $S, T : Z \to Z$ be the mappings. Let $c_1 = Sc_0$, $c_2 = Tc_1$, $c_3 = Sc_2$. Proceeding this method, we make a sequence c_n of points in Z such that $c_{2n+1} = Sc_{2n}$ and $c_{2n+2} = Tc_{2n+1}$, where $n = 0, 1, 2, \ldots$. We represent this type of sequence by $\{TS(c_n)\}$. Then $\{TS(c_n)\}$ be the sequence in Z generated by c_0.

Theorem 3.4.7 Let (Z, d_l) is a complete $D.B.M.S$. Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0,r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated maps on $\overline{B_{d_l}(c_0,r)}$. Assume (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + (1+t)t\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(d_l(Se,Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e,y) + \eta_2 d_l(e,Se) \\ +\eta_3 d_l(e,Ty) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{array} \right), \tag{3.39}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+t\eta_3}{1-t\eta_3-\eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{TS(c_n)\}$ be the iterative şequenĉe in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for ėaĉh n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also if the inequality (3.39) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for ėaĉh n belongs to $\mathbb{N} \cup \{0\}$, then u is the C.F.P of both S and Tin $\overline{B_{d_l}(c_0, r)}$.

Proof. The proof of above Theorem is similar as previous proved Theorem 3.4.1. ■

If, we take $S = T$ in Theorem 3.4.7, then we are left only with the result.

Corollary 3.4.8 Let (Z, d_l) be a ĉomþleţe *D.B.M.S.* Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S : Z \to Z$ be the semi α-dominated maþs on $\overline{B_{d_l}(c_0, r)}$. Suppose (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + (1 + t)t\eta_3 + \eta_4 < 1$ and a strictly increasing maþþing F such that

$$\tau + F(d_l(Se, Sy)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Sy) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Sy)}{1+d_l^2(e,y)} \end{array}\right), \tag{3.40}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{SS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Sy) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+t\eta_3}{1-t\eta_3-\eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{SS(c_n)\}$ be a şequenĉe in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for ėaĉh n belongs to $\mathbb{N} \cup \{0\}$ and $\{SS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also if the inequality (3.40) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for ēverŷ n belongs to $\mathbb{N} \cup \{0\}$, then u is the fîxȩd þoīnţ of S in $\overline{B_{d_l}(c_0, r)}$.

If, we take $\eta_2 = 0$ in Theorem 3.4.7, then we are left only with the result.

Corollary 3.4.9 Let (Z, d_l) is a ĉomþleţe *D.B.M.S.* Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated maþs on $\overline{B_{d_l}(c_0, r)}$. Suppose (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_3, \eta_4 > 0$ satisfying $t\eta_1 + (1 + t)t\eta_3 + \eta_4 < 1$ and a strictly increasing maþþing F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_3 d_l(e, Ty) + \eta_4 \frac{d_l^2(e, Se).d_l(y, Ty)}{1 + d_l^2(e, y)}\right), \tag{3.41}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + t\eta_3}{1 - t\eta_3 - \eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{TS(c_n)\}$ be the şequenĉe in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for ėach n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also if the inequality (3.41) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for ėach n belongs to $\mathbb{N} \cup \{0\}$, then u is the C.F.P of both S and T in $\overline{B_{d_l}(c_0, r)}$.

If, we take $\eta_3 = 0$ in Theorem 3.4.7, then we are left with the result.

Corollary 3.4.10 Let (Z, d_l) is a ĉomþleţe $D.B.M.S.$ Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated maþs on $\overline{B_{d_l}(c_0, r)}$. Suppose (i) and (ii) are hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + \eta_4 < 1$ and a strictly increasing maþþing F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_4 \frac{d_l^2(e, Se).d_l(y, Ty)}{1 + d_l^2(e, y)}\right), \tag{3.42}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{TS(c_n)\}$ be the şequenĉe in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for ėach n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also if the inequality (3.42) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for ėach n belongs to $\mathbb{N} \cup \{0\}$, then u is the C.F.P of both S and T in $\overline{B_{d_l}(c_0, r)}$.

If, we take $\eta_4 = 0$ in Theorem 3.4.7, then we are left only with the result.

Corollary 3.4.11 Let (Z, d_l) be a ĉomþleţe $D.B.M.S.$ Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated maþs on $\overline{B_{d_l}(c_0, r)}$. Assume that (i) and (ii) hold:

(i) There exist $\tau, \eta_1, \eta_2, \eta_3 > 0$ satisfying $t\eta_1 + t\eta_2 + (1+t)t\eta_3 < 1$ and a strictly increasing mapping F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_3 d_l(e, Ty)\right), \tag{3.43}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2 + t\eta_3}{1 - t\eta_3}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{TS(c_n)\}$ be a sequence in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also if, (3.43) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$, then S and T have C.F.P u in $\overline{B_{d_l}(c_0, r)}$.

Theorem 3.4.12 Let (Z, d_l) be a complete $D.B.M.S.$ Let $c_0 \in Z$ and $S, T : Z \to Z$. Assume that, There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that the following satisfy:

$$\tau + F(d_l(Se, Ty)) \leq F\begin{pmatrix} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Ty) + \eta_4 \frac{d_l^2(e, Se).d_l(y, Ty)}{1 + d_l^2(e, y)} \end{pmatrix}, \tag{3.44}$$

whenever $e, y \in \{TS(c_n)\}$ and $d(Se, Ty) > 0$. Then $\{TS(c_n)\} \to g \in Z$. Also if the inequality (3.44) holds for g, then g is the unique C.F.P of both S and T in Z.

Proof. The proof of above Theorem is similar as previous proved Theorem 3.4.1. We have to prove the uniqueness only. Let p be another C.F.P of S and T. Suppose $d_l(Sg, Tp) > 0$. Then, we have

$$\tau + F(d_l(Sg, Tp)) \leq F\left(\eta_1 d_l(g, p) + \eta_2 d_l(g, Sg) + \eta_3 d_l(g, Tp) + \eta_4 \frac{d_l^2(g, Sg).d_l(p, Tp)}{1 + d_l^2(g, p)}\right)$$

This implies that

$$d_l(g, p) < \eta_1 d_l(g, p) + \eta_3 d_l(g, p) < d_l(g, p),$$

which is a contradiction. So $d_l(Sg, Tp) = 0$. Hence $g = p$.

Now, we derive the application of fixed point Theorem 3.4.12 in form of Volterra type integral equations.

$$g(k) = \int_0^k H_1(k, h, g(h))dh, \tag{3.45}$$

$$p(k) = \int_0^k H_2(k, h, p(h))dh \tag{3.46}$$

for èach $k \in [0, 1]$. We find the solution of (2.45) and (2.46). Let $\dot{G} = \{f : f$ is a continuous function from $[0, 1]$ to $\mathbb{R}_+\}$, endowed with the $D.B.M.S.$ For $g \in \dot{G}$, define norm as: $\|g\|_\tau = \sup_{k\in[0,1]} \{|g(k)|\, e^{-\tau k}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$d_\tau(g, p) = \left[\sup_{k\in[0,1]} \{|g(k) + p(k)|\, e^{-\tau k}\} \right]^2 = \|g + p\|_\tau^2$$

for èach $g, p \in \dot{G}$, with these settings, $(\dot{G}, d_\tau)$ becomes a ĉomplete $D.B.M.S.$ with constant $b = 2$. ■

Theorem 3.4.13 Assume (i), (ii) and (iii) are satisfied:

(i) $H_1, H_2 : [0, 1] \times [0, 1] \times \dot{G} \to \mathbb{R}$;

(ii) Define

$$\begin{aligned} Sg(k) &= \int_0^k H_1(k, h, g(h))dh, \\ Tp(k) &= \int_0^k H_2(k, h, p(h))dh. \end{aligned}$$

Suppose there exist $\tau > 0$, such that

$$|H_1(k, h, g) + H_2(k, h, p)| \leq \frac{\tau H(g, p)}{\tau \|H(g, p)\|_\tau + 1}$$

for èach $k, h \in [0, 1]$ and $g, p \in \dot{G}$, where

$$\begin{aligned} H(g(h), p(h)) &= \eta_1[|g(h) + p(h)|]^2 + \eta_2[|g(h) + Sg(h)|]^2 + \eta_3[|g(h) + Tp(h)|]^2 \\ &\quad + \eta_4 \frac{[|g(h) + Sg(h)|]^4.[|p(h) + Tp(h)|]^2}{1 + [|g(h) + p(h)|]^4}, \end{aligned}$$

where $\eta_1, \eta_2, \eta_3, \eta_4 \geq 0$, and $2\eta_1 + 2\eta_2 + 6\eta_3 + \eta_4 < 1$. Then integral equations (2.45) and (2.46) has a solution.

Proof. By assumption (ii)

$$
\begin{aligned}
|Sg(k) + Tp(k)| &= \int_0^k |H_1(k,h,g(h) + H_2(k,h,p(h)))|\, dh, \\
&\leq \int_0^k \frac{\tau}{\tau\|H(g,p)\|_\tau + 1}([H(g,p)]e^{-\tau h})e^{\tau h} dh \\
&\leq \int_0^k \frac{\tau}{\tau\|H(g,p)\|_\tau + 1}\|H(g,p)\|_\tau e^{\tau h} dh
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\tau\|H(g,p)\|_\tau}{\tau\|M(g,p)\|_\tau + 1}\int_0^k e^{\tau h} dh, \\
&\leq \frac{\|H(g,p)\|_\tau}{\tau\|H(g,p)\|_\tau + 1} e^{\tau k}.
\end{aligned}
$$

This implies

$$|Sg(k) + Tp(k)|\, e^{-\tau k} \leq \frac{\|H(g,p)\|_\tau}{\tau\|H(g,p)\|_\tau + 1}.$$

$$\|Sg(k) + Tp(k)\|_\tau \leq \frac{\|H(g,p)\|_\tau}{\tau\|H(g,p)\|_\tau + 1}.$$

$$\frac{\tau\|H(g,p)\|_\tau + 1}{\|H(g,p)\|_\tau} \leq \frac{1}{\|Sg(k) + Tp(k)\|_\tau}.$$

$$\tau + \frac{1}{\|H(g,p)\|_\tau} \leq \frac{1}{\|Sg(k) + Tp(k)\|_\tau}.$$

which further implies

$$\tau - \frac{1}{\|Sg(k) + Tp(k)\|_\tau} \leq \frac{-1}{\|H(g,p)\|_\tau}.$$

So, all the hypothesis of Theorem 3.4.12 are proved for $F(p) = \frac{-1}{\sqrt{p}}; p > 0$ and $d_\tau(g,p) = \|g+p\|_\tau^2$, $b = 2$. Hence integral equations (3.45) and (3.46) has a unique common solution. ■

Example 3.4.14 Consider the integral equations

$$g(k) = \frac{1}{3}\int_0^k g(h)dh, \quad p(k) = \frac{1}{4}\int_0^k p(h))dh, \text{ where } k \in [0,1].$$

Define $H_1, H_2 : [0,1] \times [0,1] \times \dot{G} \to \mathbb{R}$ by $H_1 = \frac{1}{3}g(h)$, $H_2 = \frac{1}{4}p(h)$. Now,

$$Sg(k) = \frac{1}{3}\int_0^k g(h)dh, \quad Tp(k) = \frac{1}{4}\int_0^k p(h))dh$$

Take $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$, $\eta_3 = \frac{1}{60}$, $\eta_4 = \frac{1}{30}$, $\tau = \frac{12}{95}$, then $2\eta_1 + 2\eta_2 + 6\eta_3 + \eta_4 < 1$. Moreover, requirements of Theorem 3.4.13 are proved and $g(k) = p(k) = 0$ for each k, is a unique common solution to the shown integral equations.

Chapter 4

Results in Dislocated Quasi Metric Space

4.1 Introduction

The theory present in this section is published in [48] and accepted for publication in [57].

Recall that a maþþing $B : W \longrightarrow P(W)$ has a fîxęd þoīnţ $y \in W$, if $y \in By$. There are many generalizations of mètric ŝþace and various researchers make different kind of mètric ŝþaces. Dislocated quasi mètric ŝþace is one of the most important and famous generalizations of mètric ŝþace and it has a faundamental value in mètric fîxęd þoīnţ theory. It is very easy to say that the work on dislocated quasi mètric ŝþace is more better than other mètric versions. Many authors obtained fîxęd þoīnţ theorems in ĉomþleţe DQM (see [15, 20, 50, 61, 59, 66, 67]) which is a more general setting of partial mètric ŝþace, mètric-like ŝþace, quasi-partial mètric ŝþace (see [54, 35]), and mètric ŝþace. fîxęd þoīnţ ŕesulţs are a tool to estimate the particular solution of functional, differential and integral equations. It is simple to prove that $Q : F \longrightarrow F$ is not a ĉonţracţioñ but $Q : L \longrightarrow F$ is a ĉonţracţioñ, where L is a subset in F. It is possible for one to get fîxęd þoīnţ for such maþþings if they satisfiy certain condition. It has been shown by Beg et al. [20], the presence of fîxęd þoīnţ for such maþþings that fulfill the certain conditions on a ĉlosed set rather then whole ŝþace. Some common fîxęd þoīnţ ŕesulţs for a pair of α_*-dominated multivalued maþs on ĉlosed báll with graph in dislocated quasi ŝþaces have been proved. We delovped fixed points for α_*-dominated setvalued maps satisfying generalized $\alpha_* - \Psi$ Ćirić type contraction on DQM.

The theory of setvalued maps has a faundamental role in many types of both pure and applied maths because of its large number of applications, in geometry, real analysis and complex analysis, algorithims as well as in functional analysis. Over the past years, above theory has raised its importance and hence in current literature there are several research articles related to multivalued mappings. Various authors have discussed different research articles including practical problems and their solutions in multivalued mappings. Due to the importance of this theory various approaches algorithims and techniques are applied for the developing of multivalued fixed point theory.

Wardowski [65] devolped $F-$contraction principle to investigate fixed points in the setting of complete mètric space. This result has a faundamental postion in the field of fixed point. Afterwards, several authors generalized many fixed point results in a fruitful way by introducing $F-$contraction (see [3, 4, 6, 11, 27, 37, 34, 42, 52]). These results bring about the modern fixed point theory foundation which is mostly related to contractive type mappings. Rasham et al. [45] obtained fixed points for the pair of setvalued $F-$contractive maps, and showed an application for integral equations which extended some multivalued fixed point theorems in current literature. In Section 4.2 we proved some common fixed points of multivalued maps satisfying a new generalized $\alpha_* - \Psi$ Ćirić type contraction in the context of DQM. Also we apply graphic contraciton to get unique fixed point in these spaces. Example is presented on setvalued mappings and it is observed that the contraction which does not prevail on full space but it is holds only on subspace. In Section 4.3 we have achieved common fixed points for the pair of setvalued proximinal maps satisfying a new Ćirić kind rational $F-$contraction in complete dislocated quasi mètric spaces. An example has been derived in which we have discussed different cases for $F-$contractive mappings to show the variety of our theorem. An application is derived on non linear Voltera type integral equations to find unique solutions.

4.2 Fixed Point Results for a Pair of Multi Dominated Mappings on a Smallest Subset with Graph

Results given in this section can be seen in [48].

Let (E, d_q) be a DQM, $b_0 \in E$ and $S, T : E \to P(E)$ be the setvalued maps on E. Let $b_1 \in Sb_0$ be an element such that $d_q(b_0, Sb_0) = d_q(b_0, b_1)$. Let $b_2 \in Tb_1$ be such that $d_q(b_1, Tb_1) = d_q(b_1, b_2)$. Let $b_3 \in Sb_2$ be such that $d_q(b_2, Sb_2) = d_q(b_2, b_3)$. Proceeding this method, we gain a sequence b_n in E so as $b_{2n+1} \in Sb_{2n}$ and $b_{2n+2} \in Tb_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_q(b_{2n}, Sb_{2n}) = d_q(b_{2n}, b_{2n+1})$, $d_q(b_{2n+1}, Tb_{2n+1}) = d_q(b_{2n+1}, b_{2n+2})$. We represent this type of sequence by $\{TS(b_n)\}$.

Theorem 4.2.1 Let (E, d_q) be a left (right) K-sequentially complete DQM space. Assume a function $\alpha : E \times E \to [0, \infty)$ exists. Let, $r > 0$, $b_0 \in \overline{B_{d_q}(b_0, r)}$ and $S, T : E \to P(E)$ be a semi α_*-dominated maps on $\overline{B_{d_q}(b_0, r)}$. Suppose that, for some $\psi \in \Psi$ and $D_q(b, g) = \max\{d_q(b, g), d_q(b, Sb), d_q(g, Tg)\}$, the following hold:

$$\max\{\alpha_*(b, Sb)H_{dq}(Sb, Tg), \alpha_*(g, Tg)H_{dq}(Tg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\} \tag{4.1}$$

for all $b, g \in \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\}$ with either $\alpha(b, g) \geq 1$ or $\alpha(g, b) \geq 1$ whenever $b \in Sg$. Also

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0)), \psi^i(d_q(b_0, b_1))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}. \tag{4.2}$$

Then $\{TS(b_n)\}$ is the sequence in $\overline{B_{d_q}(b_0, r)}$ and $\{TS(b_n)\} \to b^* \in \overline{B_{d_q}(b_0, r)}$. Also, if the inequality (4.1) holds for b^* and either $\alpha(b_n, b^*) \geq 1$ or $\alpha(b^*, b_n) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$. Then b^* is the C.F.P of both S and T in $\overline{B_{d_q}(b_0, r)}$ and $d_q(b^*, b^*) = 0$.

Proof. Consider a sequence $\{TS(b_n)\}$ generated by b_0. Then, we have $b_{2n+1} \in Sb_{2n}$ and $b_{2n+2} \in Tb_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_q(b_{2n}, Sb_{2n}) = d_q(b_{2n}, b_{2n+1})$, $d_q(b_{2n+1}, Tb_{2n+1}) = d_q(b_{2n+1}, b_{2n+2})$. By Lemma 1.3.8, we have

$$d_q(b_{2n}, b_{2n+1}) \leq H_{d_q}(Tb_{2n-1}, Sb_{2n}) \tag{4.3}$$

$$d_q(b_{2n+1}, b_{2n+2}) \leq H_{d_q}(Sb_{2n}, Tb_{2n+1}) \tag{4.4}$$

for each $n = 1, 2, \ldots$. From (4.2), we have

$$\max\{d_q(b_1, b_0), d_q(b_0, b_1)\} \leq \sum_{i=0}^{j} \max\{\psi^i(d_q(b_1, b_0), \psi^i(d_q(b_0, b_1))\} \leq r.$$

It follows that, $d_q(b_1, b_0) \leq r$ and $d_q(b_0, b_1) \leq r$. Hence, we have

$$b_1 \in \overline{B_{d_q}(b_0, r)}.$$

Let $b_2, \cdots, b_j \in \overline{B_{d_q}(b_0, r)}$ for every j belongs to $\mathbb{N}$. If $j = 2i + 1$, where $i = 1, 2, \ldots, \frac{j-1}{2}$. Since $S, T : E \to P(E)$ be a semi α_*-dominated maps on $\overline{B_{d_q}(b_0, r)}$, so $\alpha_*(b_{2i}, Sb_{2i}) \geq 1$ and $\alpha_*(b_{2i+1}, Tb_{2i+1}) \geq 1$. As $\alpha_*(b_{2i}, Sb_{2i}) \geq 1$, this implies $\inf\{\alpha(b_{2i}, b) : b \in Sb_{2i}\} \geq 1$. Also $b_{2i+1} \in Sb_{2i}$, so $\alpha(b_{2i}, b_{2i+1}) \geq 1$. Now by using (4.3), we obtain

$$\begin{aligned}
d_q(b_{2i+1}, b_{2i+2}) &\leq H_{d_q}(Sb_{2i}, Tb_{2i+1}) \leq \max\{\alpha_*(b_{2i}, Sb_{2i})H_{d_q}(Sb_{2i}, Tb_{2i+1}), \\
&\quad \alpha_*(b_{2i+1}, Tb_{2i+1})H_{d_q}(Tb_{2i+1}, Sb_{2i})\} \\
&\leq \min\{\psi(D_q(b_{2i}, b_{2i+1})), \psi(D_q(b_{2i+1}, b_{2i}))\} \leq \psi(D_q(b_{2i}, b_{2i+1})) \\
&\leq \psi(\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i}, Sb_{2i}), d_q(b_{2i+1}, Tb_{2i+1})\}) \\
&\leq \psi(\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, b_{2i+2})\}) \\
&\leq \psi(\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, b_{2i+2})\}).
\end{aligned}$$

If $\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, b_{2i+2})\} = d_q(b_{2i+1}, b_{2i+2})$, then $d_q(b_{2i+1}, b_{2i+2}) \leq \psi(d_q(b_{2i+1}, b_{2i+2}))$. Which contradicts the reality $\psi(t) < t$ for each $t > 0$. So $\max\{d_q(b_{2i}, b_{2i+1}), d_q(b_{2i+1}, b_{2i+2})\} = d_q(b_{2i}, b_{2i+1})$. Hence,

$$d_q(b_{2i+1}, b_{2i+2}) \leq \psi(d_q(b_{2i}, b_{2i+1})) \tag{4.5}$$

As $\alpha_*(b_{2i-1}, Tb_{2i-1}) \geq 1$ and $b_{2i} \in Tb_{2i-1}$, so $\alpha(b_{2i-1}, b_{2i}) \geq 1$. Now, by using (4.4), we have

$$\begin{aligned}
d_q(b_{2i}, b_{2i+1}) &\leq H_{d_q}(Tb_{2i-1}, Sb_{2i}) \\
&\leq \max\{\alpha_*(b_{2i}, Sb_{2i})H_{d_q}(Sb_{2i}, Tb_{2i-1}), \\
&\quad \alpha_*(b_{2i-1}, Tb_{2i-1})H_{d_q}(Tb_{2i-1}, Sb_{2i})\}
\end{aligned}$$

$$\begin{aligned}
&\leq \min\{\psi(D_q(b_{2i}, b_{2i-1})), \psi(D_q(b_{2i-1}, b_{2i}))\} \leq \psi(D_q(b_{2i}, b_{2i-1})) \\
&\leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, Sb_{2i}), d_q(b_{2i-1}, Tb_{2i-1})\}) \\
&\leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, b_{2i+1}), d_q(b_{2i-1}, b_{2i})\}) \\
&\leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, b_{2i+1}), d_q(b_{2i-1}, b_{2i})\}).
\end{aligned}$$

$$\begin{aligned}
&\text{If } \max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i}, b_{2i+1}), d_q(b_{2i-1}, b_{2i})\} \\
= \ & d_q(b_{2i}, b_{2i+1}), \text{ then } d_q(b_{2i}, b_{2i+1}) \leq \psi(d_q(b_{2i}, b_{2i+1})).
\end{aligned}$$

This is contradicts to the reality $\psi(t) < t$ for each $t > 0$. Hence, we have

$$d_q(b_{2i}, b_{2i+1}) \leq \psi(\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i-1}, b_{2i})\}).$$

If $\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i-1}, b_{2i})\} = d_q(b_{2i-1}, b_{2i})$, then

$$d_q(b_{2i}, b_{2i+1})) \leq \psi(d_q(b_{2i-1}, b_{2i})).$$

As ψ is nondecreasing function, so

$$\psi(d_q(b_{2i}, b_{2i+1})) \leq \psi^2(d_q(b_{2i-1}, b_{2i})).$$

By using the above inequality in (4.5), we obtain

$$d_q(b_{2i+1}, b_{2i+2}) \leq \psi^2(d_q(b_{2i-1}, b_{2i})). \tag{4.6}$$

If $\max\{d_q(b_{2i}, b_{2i-1}), d_q(b_{2i-1}, b_{2i})\} = d_q(b_{2i}, b_{2i-1})$, then

$$d_q(b_{2i+1}, b_{2i+2}) \leq \psi^2(d_q(b_{2i}, b_{2i-1})) \tag{4.7}$$

Now, by combining (4.6) and (4.7), we obtain

$$d_q(b_{2i+1}, b_{2i+2}) \leq \max\{\psi^2(d_q(b_{2i}, b_{2i-1})), \psi^2(d_q(b_{2i-1}, b_{2i}))\}$$

Proceeding this way, we get

$$d_q(b_{2i+1}, b_{2i+2}) \leq \max\{\psi^{2i+1}(d_q(b_1, b_0)), \psi^{2i+1}(d_q(b_0, b_1))\} \tag{4.8}$$

Now, if $j = 2i$, where $i = 1, 2, \dots \frac{j}{2}$. Then, similarly, we have

$$d_q(b_{2i}, b_{2i+1}) \leq \max\{\psi^{2i}(d_q(b_1, b_0)), \psi^{2i}(d_q(b_0, b_1))\} \tag{4.9}$$

Now, by combining (4.8) and (4.9), we obtain

$$d_q(b_j, b_{j+1}) \leq \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\} \text{ for some } j \in \mathbb{N}. \tag{4.10}$$

Now, by Lemma 1.3.8 and inequality (4.1), we have

$$\begin{aligned} d_q(b_{2i+2}, b_{2i+1}) &\leq H_{d_q}(Tb_{2i+1}, Sb_{2i}) \\ &\leq \max\{\alpha_*(b_{2i}, Sb_{2i})H_{d_q}(Sb_{2i}, Tb_{2i+1}), \\ &\quad \alpha_*(b_{2i+1}, Tb_{2i+1})H_{d_q}(Tb_{2i+1}, Sb_{2i})\} \\ &\leq \min\{\psi(D_q(b_{2i}, b_{2i+1})), \psi(D_q(b_{2i+1}, b_{2i}))\} \end{aligned}$$

In similar way, we used to solve inequality (4.10), we get

$$d_q(b_{j+1}, b_j) \leq \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\} \text{ for } \bar{e}\text{ver}\hat{y} \ j \text{ belongs to } \mathbb{N}. \tag{4.11}$$

Now,

$$\begin{aligned} d_q(b_0, b_{j+1}) &\leq d_q(b_0, b_1) + \dots + d_q(b_j, b_{j+1}) \\ &\leq d_q(b_0, b_1) + \dots + \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\} \\ &\leq \sum_{i=0}^{j} \max\{\psi^i(d_q(b_1, b_0), \psi^i(d_q(b_0, b_1))\} \leq r. \end{aligned} \tag{4.12}$$

Also,

$$\begin{aligned} d_q(b_{j+1}, b_0) &\leq d_q(b_{j+1}, b_j) + ... + d_q(b_1, b_0) \\ &\leq \max\{\psi^j(d_q(b_1, b_0)), \psi^j(d_q(b_0, b_1))\} + ... + d_q(b_1, b_0) \\ &\leq \sum_{i=0}^{j} \max\{\psi^i(d_q(b_1, b_0), \psi^i(d_q(b_0, b_1))\} \leq r. \end{aligned} \tag{4.13}$$

By (4.12) and (4.13), we have $b_{j+1} \in \overline{B_{dq}(b_0, r)}$. Hence by mathematical induction $b_n \in \overline{B_{dq}(b_0, r)}$ for ēverŷ n belongs to $\mathbb{N}$. Therefore, $\{TS(b_n)\}$ be the şequenĉe in $\overline{B_{d_q}(b_0, r)}$. As $S, T : E \to P(E)$ be a semi α_*−dominated maƀs on $\overline{B_{d_q}(b_0, r)}$, so $\alpha_*(b_n, Sb_n) \geq 1$ and $\alpha_*(b_n, Tb_n) \geq 1$, for ėaĉh $n \in \mathbb{N}$. Now we can write (4.8) and (4.9) in result as

$$d_q(b_n, b_{n+1}) \leq \max\{\psi^n(d_q(b_1, b_0)), \psi^n(d_q(b_0, b_1))\}, \text{ for ėaĉh } n \text{ belongs to } \mathbb{N}. \tag{4.14}$$

$$d_q(b_{n+1}, b_n) \leq \max\{\psi^n(d_q(b_1, b_0)), \psi^n(d_q(b_0, b_1))\}, \text{ for ėaĉh } n \text{ belongs to } \mathbb{N}. \tag{4.15}$$

Fix $\varepsilon > 0$ and let $k_1(\varepsilon)$ belongs to $\mathbb{N}$ so as $\sum_{k \geq k_1(\varepsilon)} \max\{\psi^k(d_q(b_1, b_0)), \psi^k(d_q(b_0, b_1))\} < \varepsilon$. Let n, m belongs to $\mathbb{N}$ with $m > n > k_1(\varepsilon)$, then, we obtain,

$$\begin{aligned} d_q(b_n, b_m) &\leq \sum_{k=n}^{m-1} d_q(b_k, b_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \max\{\psi^k(d_q(b_1, b_0)), \psi^k(d_q(b_0, b_1))\}, \text{ by (4.14)} \\ d_q(b_n, b_m) &\leq \sum_{k \geq k_1(\varepsilon)} \max\{\psi^n(d_q(b_1, b_0)), \psi^n(d_q(b_0, b_1))\} < \varepsilon. \end{aligned}$$

Thus we have showed that $\{TS(b_n)\}$ be a left K-Cauchy in $(\overline{B_{d_q}(b_0, r)}, d_q)$. Similarly, by using (4.15) we have

$$d_q(b_m, b_n) \leq \sum_{k=n}^{m-1} d_q(b_{k+1}, b_k) < \varepsilon$$

Hence, $\{TS(b_n)\}$ is a right K-Cauchy in $(\overline{B_{d_q}(b_0, r)}, d_q)$. As ėaĉh ĉlosed bǻll in left(right) K-sequentially ĉomƀleţe DQM is left(right) K-sequentially ĉomƀleţe, so there must be a $b^* \in$

$\overline{B_{d_q}(b_0, r)}$ so as $\{TS(b_n)\} \to b^*$, that is

$$\lim_{n\to\infty} d_q(b_n, b^*) = \lim_{n\to\infty} d_q(b^*, b_n) = 0 \tag{4.16}$$

Now,

$$\begin{aligned} d_q(b^*, Tb^*) &\leq d_q(b^*, b_{2n+1}) + d_q(b_{2n+1}, Tb^*) \\ &\leq d_q(b^*, b_{2n+1}) + H_{d_q}(Sb_{2n}, Tb^*), \text{ by Lemma 1.3.8} \end{aligned} \tag{4.17}$$

Since $\alpha_*(b^*, Tb^*) \geq 1$, $\alpha_*(b_{2n}, Sb_{2n}) \geq 1$ and $\alpha(b_{2n}, b^*) \geq 1$, we obtain

$$\begin{aligned} H_{d_q}(Sb_{2n}, Tb^*) &\leq \max\{\alpha_*(b_{2n}, Sb_{2n})H_{d_q}(Sb_{2n}, Tb^*), \alpha_*(b^*, Tb^*)H_{d_q}(Tb^*, Sb_{2n})\} \\ &\leq \min\{\psi(D_q(b_{2n}, b^*)), \psi(D_q(b^*, b_{2n}))\} \\ &\leq \psi(\max\{d_q(b_{2n}, b^*), d_q(b_{2n}, b_{2n+1}), d_q(b^*, Tb^*)\}) \\ &\leq \psi(\max\{d_q(b_{2n}, b^*), d_q(b_{2n}, b^*) + d_q(b^*, b_{2n+1}), d_q(b^*, Tb^*)\}). \end{aligned} \tag{4.18}$$

By using inequality (4.18) in inequality (4.17), we have

$$d_q(b^*, Tb^*) \leq d_q(b^*, b_{2n+1}) + \psi(\max\{d_q(b_{2n}, b^*), d_q(b_{2n}, b^*) + d_q(b^*, b_{2n+1}), d_q(b^*, Tb^*)\}).$$

Letting $n \to \infty$, and by using the inequality (4.16), we obtain $d_q(b^*, Tb^*) \leq \psi(d_q(b^*, Tb^*))$ and hence $d_q(b^*, Tb^*) = 0$. Now,

$$\begin{aligned} d_q(Tb^*, b^*) &\leq d_q(Tb^*, b_{2n+1}) + d_q(b_{2n+1}, b^*) \\ &\leq H_{d_q}(Tb^*, Sb_{2n}) + d_q(b_{2n+1}, b^*), \text{ by Lemma 1.3.8} \end{aligned}$$

By using similar arguments, we obtain $d_q(Tb^*, b^*) = 0$ or $b^* \in Tb^*$. Similarly, by using Lemma 1.3.8 inequality (4.16) and the inequality

$$d_q(b^*, Sb^*) \leq d_q(b^*, b_{2n+2}) + d_q(b_{2n+2}, Sb^*),$$

we can show that $d_q(b^*, Sb^*) = 0$. $b^* \in Sb^*$. Similarly, $d_q(Sb^*, b^*) = 0$. Hence b^* is the C.F.P of both the maps S and T in $\overline{B_{d_q}(b_0, r)}$. Now,

$$d_q(b^*, b^*) \leq d_q(b^*, Tb^*) + d_q(Tb^*, b^*) \leq 0$$

This means that, $d_q(b^*, b^*) = 0$. ∎

Corollary 4.2.2 Let (E, d_q) is a left (right) K-Sequentially complete DQM. Suppose a function $\alpha : E \times E \to [0, \infty)$ exists. Let, $r > 0$, $b_0 \in \overline{B_{d_q}(b_0, r)}$ and $S : E \to P(E)$ are two semi α_*−dominated maps on $\overline{B_{d_q}(b_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_q(b, g) = \max\{d_q(b, g), d_q(b, Sb), d_q(g, Sg)\}$, the following hold:

$$\max\{\alpha_*(b, Sb)H_{dq}(Sb, Sg),\ \alpha_*(g, Sg)H_{dq}(Sg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\} \quad (4.19)$$

for all $b, g \in \overline{B_{d_q}(b_0, r)} \cap \{S(b_n)\}$, with either $\alpha(b, g) \geq 1$ or $\alpha(g, b) \geq 1$. Also

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0), \psi^i(d_q(b_0, b_1))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

Then $\{S(b_n)\}$ be the sequence in $\overline{B_{d_q}(b_0, r)}$ and $\{S(b_n)\} \to b^* \in \overline{B_{d_q}(b_0, r)}$. Also, if the inequality (4.19) holds for b^* and either $\alpha(b_n, b^*) \geq 1$ or $\alpha(b^*, b_n) \geq 1$ for every n belongs to $\mathbb{N} \cup \{0\}$. Then b^* is the fixed point of S in $\overline{B_{d_q}(b_0, r)}$ and $d_q(b^*, b^*) = 0$.

Corollary 4.2.3 Let (E, d_l) is a complete DQM. Suppose a function $\alpha : E \times E \to [0, \infty)$ exists. Let, $r > 0$, $b_0 \in \overline{B_{d_l}(b_0, r)}$ and $S, T : E \to P(E)$ are semi α_*−dominated maps on $\overline{B_{d_l}(b_0, r)}$. Assume that, for some $\psi \in \Psi$ and $D_l(b, g) = \max\{d_l(b, g), d_l(b, Sb), d_l(g, Tg)\}$, the following hold:

$$\max\{\alpha_*(b, Sb)H_{d_l}(Sb, Tg),\ \alpha_*(g, Tg)H_{d_l}(Sb, Tg)\} \leq \psi(D_l(b, g)) \quad (4.20)$$

for all $b, g \in \overline{B_{d_l}(b_0, r)} \cap \{TS(b_n)\}$ with either $\alpha(b, g) \geq 1$ or $\alpha(g, b) \geq 1$. Also

$$\sum_{i=0}^{n} \psi^i(d_l(b_0, b_1)) \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

Then $\{TS(b_n)\}$ is a sequence in $\overline{B_{d_l}(b_0, r)}$ and $\{TS(b_n)\} \to b^* \in \overline{B_{d_l}(b_0, r)}$. Also, if the inequality (4.20) holds for b^* and either $\alpha(b_n, b^*) \geq 1$ or $\alpha(b^*, b_n) \geq 1$ for each n belongs to $\mathbb{N} \cup \{0\}$. Then, b^* is the C.F.P of both the maps S and T in $\overline{B_{d_q}(b_0, r)}$ and $d_q(b^*, b^*) = 0$.

Corollary 4.2.4 Let (E, d_l) is a complete DQM. Suppose a function $\alpha : E \times E \to [0, \infty)$ exists. Let, $r > 0$, $b_0 \in \overline{B_{d_l}(b_0, r)}$ and $S : E \to P(E)$ is semi α_*-dominated map on $\overline{B_{d_l}(b_0, r)}$. Suppose that, for some $\psi \in \Psi$ and $D_l(b, g) = \max\{d_l(b, g), d_l(b, Sb), d_l(g, Sg)\}$, the following hold:

$$\max\{\alpha_*(b, Sb)H_{d_l}(Sb, Sg),\ \alpha_*(g, Sg)H_{d_l}(Sb, Sg)\} \leq \psi(D_l(b, g)) \tag{4.21}$$

for all $b, g \in \overline{B_{d_l}(b_0, r)} \cap \{S(b_n)\}$ with either $\alpha(b, g) \geq 1$ or $\alpha(g, b) \geq 1$. Also

$$\sum_{i=0}^{n} \psi^i(d_l(b_0, b_1)) \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}.$$

Then $\{S(b_n)\}$ is the sequence in $\overline{B_{d_l}(b_0, r)}$ and $\{S(b_n)\} \to b^* \in \overline{B_{d_l}(b_0, r)}$. Also, if the inequality (4.21) holds for b^* and either $\alpha(b_n, b^*) \geq 1$ or $\alpha(b^*, b_n) \geq 1$ for every n belongs to $\mathbb{N} \cup \{0\}$. Then S has a fixed point b^* in $\overline{B_{d_l}(b_0, r)}$ and $d_q(b^*, b^*) = 0$.

Corollary 4.2.5 Let $(E, \preceq, d_q)$ is a left (right) K-sequentially ordered complete DQM. Let, $r > 0$, $b_0 \in \overline{B_{d_q}(b_0, r)}$ and $S, T : E \to P(E)$ are semi dominated maps on $\overline{B_{d_q}(b_0, r)}$. Suppose that, for some $\psi \in \Psi$ and $D_q(b, g) = \max\{d_q(b, g), d_q(b, Sb), d_q(g, Tg)\}$, the following hold:

$$\max\{H_{dq}(Sb, Tg), H_{dq}(Tg, Sb)\} \leq \min\{\psi(D_q(b, g)), \psi(D_q(g, b))\}, \tag{4.22}$$

for all $b, g \in \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\}$ with either $b \preceq g$ or $g \preceq b$. Also

$$\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1, b_0), \psi^i(d_q(b_0, b_1))\} \leq r \text{ for each } n \text{ belongs to } \mathbb{N} \cup \{0\}. \tag{4.23}$$

Then $\{TS(b_n)\}$ be the sequence in $\overline{B_{d_q}(b_0, r)}$ and $\{TS(b_n)\} \to b^* \in \overline{B_{d_q}(b_0, r)}$. Also if the inequality (4.22) holds for b^* and either $b_n \preceq b^*$ or $b^* \preceq b_n$ for each n belongs to $\mathbb{N} \cup \{0\}$. Then b^* is the C.F.P of both the maps S and T in $\overline{B_{d_q}(b_0, r)}$ and $d_q(b^*, b^*) = 0$.

Proof. Let $\alpha : E\times E \to [0,+\infty)$ be a maþþing defined by $\alpha(b,g) = 1$ for ēverŷ $b \in \overline{B_{d_q}(b_0,r)}$ with either $b \preceq g$ or $g \preceq b$, and $\alpha(b,g) = 0$ for all other elements $b, g \in E$. Given S and T be the semi dominated maþs on $\overline{B_{d_q}(b_0,r)}$, so $b \preceq Sb$ and $b \preceq Tb\ \forall\ b \in \overline{B_{d_q}(b_0,r)}$. This means that $b \preceq b$ for ēverŷ $b \in Sb$ and $b \preceq c$ for èaċh $c \in Tb$. So, $\alpha(b,b) = 1$ for all $b \in Sb$ and $\alpha(b,c) = 1\ \forall\ c \in Tb$. This implies $\inf\{\alpha(b,g) : g \in Sb\} = 1$ and $\inf\{\alpha(b,g) : g \in Tb\} = 1$. Hence $\alpha_*(b,Sb) = 1$, $\alpha_*(b,Tb) = 1$ for all $b \in \overline{B_{d_q}(b_0,r)}$. So, $S,T : E \to P(E)$ are the semi α_*-dominated map on $\overline{B_{d_q}(b_0,r)}$. Moreover, inequality (4.22) can be written as

$$\max\{\alpha_*(b,Sb)H_{dq}(Sb,Tg),\ \alpha_*(g,Tg)H_{dq}(Tg,Sb)\} \le \min\{\psi(D_q(b,g)), \psi(D_q(g,b))\},$$

for all elements b, g in $\overline{B_{d_q}(b_0,r)}\cap\{TS(b_n)\}$ with either $\alpha(b,g) \ge 1$ or $\alpha(g,b) \ge 1$. Also, inequality (4.23) holds. Then, by Theorem 4.2.1, we have $\{TS(b_n)\}$ be the şequenĉe in $\overline{B_{d_q}(b_0,r)}$ and $\{TS(b_n)\} \to b^* \in \overline{B_{d_q}(b_0,r)}$. Now, $b_n, b^* \in \overline{B_{d_q}(b_0,r)}$ and either $b_n \preceq b^*$ or $b^* \preceq b_n$ implies that either $\alpha(b_n,b^*) \ge 1$ or $\alpha(b^*,b_n) \ge 1$. So, all hypothesis of Theorem 4.2.1 are proved. Hence, by Theorem 4.2.1, b^* is C.F.P of both S and T in $\overline{B_{d_q}(b_0,r)}$ and $d_q(b^*,b^*) = 0$. ∎

Example 4.2.6 Let $E = Q^+ \cup \{0\}$ and $d_q : E \times E \to E$ is a DQM on E clafried by

$$d_q(w,e) = w + e\ \forall\ w, e \in E.$$

Define, $S,T : E \times E \to P(E)$ by,

$$Sw = \begin{cases} [\frac{w}{3}, \frac{2}{3}w] & \text{if } w \in [0,1] \cap E \\ [w, w+1] & \text{if } w \in (1,\infty) \cap E \end{cases}$$

and,

$$Tw = \begin{cases} [\frac{w}{4}, \frac{3}{4}w] & \text{if } w \in [0,1] \cap E \\ [w+1, w+3] & \text{if } w \in (1,\infty) \cap E. \end{cases}$$

Considering, $b_0 = 1$, $r = 8$, then $\overline{B_{d_q}(b_0,r)} = [0,7] \cap E$. Now $d_q(b_0, Sb_0) = d_q(1, S1) = d_q(1, \frac{1}{3}) = \frac{4}{3}$. So we obtain a şequenĉe $\{TS(b_n)\} = \{1, \frac{1}{12}, \frac{1}{144}, \frac{1}{1728}, \dots\}$ in E generated by

b_0. Also, $\overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\} = \{1, \frac{1}{12}, \frac{1}{144}, ...\}$. Let $\psi(t) = \frac{4t}{5}$ and

$$\alpha(b, g) = \begin{cases} 1 \text{ if } b, g \in [0, 1] \\ \frac{3}{2} \quad \text{otherwise.} \end{cases}$$

Now, if $b, g \notin \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\}$, then the availabe cases are given.

Case 1. If $\max\{\alpha_*(b, Sb)H_{d_q}(Sb, Tg), \alpha_*(g, Tg)H_{d_q}(Tg, Sb)\} = \alpha_*(b, Sb)H_{d_q}(Sb, Tg)$ then for $b = 2$,and $g = 3$, we have

$$\alpha_*(2, S2)Hd_q(S2, T3) = \frac{3}{2}(8) > \psi(D_q(b, g)) = \frac{28}{5}.$$

Case 2. If $\max\{\alpha_*(b, Sb)H_{d_q}(Sb, Tg), \alpha_*(g, Tg)H_{dq}(Tg, Sb)\} = \alpha_*(g, Tg)H_{dq}(Tg, Sb)$ then for $b = 2$,and $g = 3$, we have

$$\alpha_*(3, T3)Hd_q(T3, S2) = \frac{3}{2}(8) > \psi(D_q(g, b)) = \frac{28}{5}.$$

So, the inequality (4.1) is not true for the whole space E.

Now, for all $b, g \in \overline{B_{d_q}(b_0, r)} \cap \{TS(b_n)\}$, we have

Case 3. If $\max\{\alpha_*(b, Sb)H_{d_q}(Sb, Tg), \alpha_*(g, Tg)H_{d_q}(Tg, Sb)\} = \alpha_*(b, Sb)H_{d_q}(Sb, Tg)$, then, we have

$$\begin{aligned} \alpha_*(b, Sb)H_{d_q}(Sb, Tg) &= 1[\max\{\sup_{a \in Sb} d_q(a, Tg), \sup_{g \in Tg} d_q(Sb, g)\}] \\ &= \max\{\sup_{a \in Sb} d_q(a, [\frac{g}{4}, \frac{3g}{4}]), \sup_{g \in Tg} d_q([\frac{b}{3}, \frac{2b}{3}], g)\} \\ &= \max\{d_q(\frac{2b}{3}, [\frac{g}{4}, \frac{3g}{4}]), d_q([\frac{b}{3}, \frac{2b}{3}], \frac{3g}{4})\} \\ &= \max\{d_q(\frac{2b}{3}, \frac{g}{4}), d_q(\frac{b}{3}, \frac{3g}{4})\} \\ &= \max\{\frac{2b}{3} + \frac{g}{4}, \frac{b}{3} + \frac{3g}{4}\} \\ &\leq \psi(\max\{b + g, \frac{4b}{3}, \frac{5g}{4}\}) = \psi(D_q(b, g)). \end{aligned}$$

Case 4. If $\max\{\alpha_*(b, Sb)H_{d_q}(Sb, Tg), \alpha_*(g, Tg)H_{d_q}(Tg, Sb)\} = \alpha_*(g, Tg)H_{d_q}(Tg, Sb)$ then, we have

$$
\begin{aligned}
\alpha_*(g,Tg)H_{d_q}(Tg,Sb) &= 1[\max\{\sup_{b\in Tg} d_q(Sb,b), \sup_{a\in Sb} d_q(a,Tg)\}] \\
&= \max\{\sup_{b\in Tg} d_q([\frac{b}{3},\frac{2b}{3}],b), \sup_{a\in Sb} d_q(a,[\frac{g}{4},\frac{3g}{4}])\} \\
&= \max\{d_q([\frac{b}{3},\frac{2b}{3}],\frac{3g}{4}), d_q(\frac{2b}{3},[\frac{g}{4},\frac{3g}{4}])\} \\
&= \max\{d_q(\frac{b}{3},\frac{3g}{4}), d_q(\frac{2b}{3},\frac{g}{4})\} \\
&= \max\{\frac{b}{3}+\frac{3g}{4}, \frac{2b}{3}+\frac{g}{4}\} \\
&\leq \psi(\max\{g+b,\frac{5g}{4},\frac{4b}{3}\}) = \psi(D_q(g,b))
\end{aligned}
$$

So, the inequality (4.1) holds on $\overline{B_{d_q}(b_0,r)} \cap \{TS(b_n)\}$. Also,

$$
\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1,b_0),\psi^i(d_q(b_0,b_1))\} = \frac{4}{3}\sum_{i=0}^{n}(\frac{4}{5})^i < 8 = r.
$$

Hence, all the hypothesis of Theoorem 4.2.1 are proved. Now, we have $\{TS(b_n)\}$ be the şequenće in $\overline{B_{d_q}(b_0,r)}$ and $\{TS(b_n)\} \to 0 \in \overline{B_{d_q}(b_0,r)}$. Also, $\alpha(b_n,0) \geq 1$ or $\alpha(0,b_n) \geq 1$ for ėach n belongs to $\mathbb{N} \cup \{0\}$.

Definition 4.2.7 Let $E \neq \Phi$ and $G = (V(G), W(G))$ is a graph so as $V(G) = E$, and $S : E \to CB(E)$ is called to be semi graph dominated on $A \subseteq E$, if for ėach $b \in A$, then $(b,g) \in W(G)$, for all $g \in Sb$. If $A = E$, then we say that S is graph dominated on E.

Theorem 4.2.8 Let (E,d_q) is a ĉomþleţe DQM endowed with graph G. Let, $r > 0$, $b_0 \in \overline{B_{d_q}(b_0,r)}$, $S,T : E \to P(E)$ and $\{TS(b_n)\}$ be the şequenće in E generated by b_0. Suppose that (i), (ii) and (iii) hold:

(i) S and T are semi graph dominated on $\overline{B_{d_q}(b_0,r)}$;

(ii) there exists $\psi \in \Psi$ and $D_q(b,g) = \max\{d_q(b,g), d_q(b,Sb), d_q(g,Tg)\}$, such that

$$
\max\{H_{d_q}(Sb,Tg), H_{d_q}(Tg,Sb)\} \leq \min\{\psi(D_q(b,g)), \psi(D_q(g,b))\} \tag{4.24}
$$

for all $b,g \in \overline{B_{d_q}(b_0,r)} \cap \{TS(b_n)\}$ with $(b,g) \in W(G)$ or $(g,b) \in W(G)$;

(iii) $\sum_{i=0}^{n} \max\{\psi^i(d_q(b_1,b_0),\psi^i(d_q(b_0,b_1))\} \leq r$ for ėach n belongs to $\mathbb{N} \cup \{0\}$.

Then, $\{TS(b_n)\}$ is the şequenće in $\overline{B_{d_q}(b_0,r)}$ and $\{TS(b_n)\} \to b^*$. Also, if $(b_n, b^*) \in W(G)$ or $(b^*, b_n) \in W(G)$ for ėaċh n belongs to $\mathbb{N} \cup \{0\}$ and the inequality (4.24) holds for b^*.Then b^* is the C.F.P of both S and T in $\overline{B_{d_q}(b_0,r)}$.

Proof. Define, $\alpha : E \times E \to [0, \infty)$ by

$$\alpha(b,g) = \begin{cases} 1, & \text{if } b \in \overline{B_{d_q}(b_0,r)},\ (b,g) \in W(G) \text{ or } (g,b) \in W(G) \\ 0, & \text{otherwise.} \end{cases}$$

Given S and T are semi graph dominated on $\overline{B_{d_q}(b_0,r)}$, then for $b \in \overline{B_{d_q}(b_0,r)}$, $(b,g) \in W(G)$ for ēverŷ $g \in Sb$ and $(b,g) \in W(G)$ for ēverŷ $g \in Tb$. So, $\alpha(b,g) = 1$ for all $g \in Sb$ and $\alpha(b,g) = 1$ for all $g \in Tb$. This implies that $\inf\{\alpha(b,g) : g \in Sb\} = 1$ and $\inf\{\alpha(b,g) : g \in Tb\} = 1$. Hence $\alpha_*(b, Sb) = 1$, $\alpha_*(b,Tb) = 1$ for all $b \in \overline{B_{d_q}(b_0,r)}$. So, $S,T : E \to P(E)$ are the semi α_*-dominated maþþing on $\overline{B_{d_q}(b_0,r)}$. Moreover, inequality (4.24) can be written as

$$\max\{\alpha_*(b,Sb)H_{dq}(Sb,Tg),\ \alpha_*(g,Tg)H_{dq}(Tg,Sb)\} \leq \min\{\psi(D_q(b,g)), \psi(D_q(g,b))\},$$

for all elements b, g in $\overline{B_{d_q}(b_0,r)} \cap \{TS(b_n)\}$ with either $\alpha(b,g) \geq 1$ or $\alpha(g,b) \geq 1$. Also, (iii) holds. Then, by Theorem 4.2.1, we have $\{TS(b_n)\}$ be the şequenće in $\overline{B_{d_q}(b_0,r)}$ and $\{TS(b_n)\} \to b^* \in \overline{B_{d_q}(b_0,r)}$. Now, $b_n, b^* \in \overline{B_{d_q}(b_0,r)}$ and either $(b_n, b^*) \in W(G)$ or $(b^*, b_n) \in W(G)$ implies that either $\alpha(b_n, b^*) \geq 1$ or $\alpha(b^*, b_n) \geq 1$. So, all hypothesis of Theorem 4.2.1 are proved. Hence, by Theorem 4.2.1, u is the C.F.P of S and T in $\overline{B_{d_q}(b_0,r)}$ and $d_q(b^*, b^*) = 0$. ■

4.3 DQF-contraction and Related Fixed Point Results in DQM Spaces with Application

Results given in this section can be seen in [48].

Let (Y, d_q) be a DQM śþace, $y_0 \in Y$ and $S,T : Y \to P(Y)$ be setvalued maþs on Y. Let $y_1 \in Sy_0$ be an element such that $d_q(y_0, Sy_0) = d_q(y_0, y_1)$, let $y_2 \in Ty_1$ be such that $d_q(y_1, Ty_1) = d_q(y_1, y_2)$, let $y_3 \in Sy_2$ be such that $d_q(y_2, Sy_2) = d_q(y_2, y_3)$ and so on. Then, we get şequenće y_n in Y so as $y_{2n+1} \in Sy_{2n}$ and $y_{2n+2} \in Ty_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_q(y_{2n}, Sy_{2n}) = d_q(y_{2n}, y_{2n+1})$, $d_q(y_{2n+1}, Ty_{2n+1}) = d_q(y_{2n+1}, y_{2n+2})$. We denote this ţype of iterative şequenće by $\{TS(y_n)\}$. We say that $\{TS(y_n)\}$ is a şequenće in Y generated by y_0. If $T = S$, then we say that $\{S(y_n)\}$ is a şequenće in Y generated by y_0.

Definition 4.3.1 Let (Y, d_q) be a ĉomþleţe DQM š þace and $S, T : Y \to P(Y)$ be two setvalued maþs. The pair (S, T) is called $DQF-$ĉonţracţioñ, if there must be a $F \in \mathcal{F}$ and $\tau, a > 0$ such that for ēverŷ two consecutive þoints l, w belonging to the range of an iterative şequenće $\{TS(y_n)\}$ with $\max\{D_q(l,w), D_q(w,l)\} > 0$, we have

$$\tau + \max\{F(H_{d_q}(Sl, Tw)), F(H_{d_q}(Tw, Sl))\} \leq \min\{F(D_q(l,w)), F(D_q(w,l))\} \tag{4.25}$$

where,

$$D_q(l,w)) = \max\left\{d_q(l,w), \frac{d_q(l,Sl).d_q(w,Tw)}{a+\max\{d_q(l,w), d_q(w,l)\}}, d_q(l,Sl), d_q(w,Tw)\right\}, \tag{4.26}$$

Theorem 4.3.2 Let (Y, d_q) be a ĉomþleţe DQM šþace and (S,T) be a $DQF-$ĉonţracţioñ. Then $\{TS(y_n)\} \to u \in Y$. Also, if u satisfies (4.25), then u is the C.F.P of S and T in Y and $d_q(u,u) = 0$.

Proof. Let $\{TS(y_n)\}$ be the iterative şequenće in Y generated by a þoint $y_0 \in Y$. Let y_{2n}, y_{2n+1} be elements of this şequenće. Clearly, if $\max\{D_q(y_{2n}, y_{2n+1}), D_q(y_{2n+1}, y_{2n})\} = 0$, then $D_q(y_{2n}, y_{2n+1}) = 0$ and $D_q(y_{2n+1}, y_{2n}) = 0$. If $D_q(y_{2n}, y_{2n+1}) = 0$, then

$$\max\left\{\begin{array}{c} d_q(y_{2n}, y_{2n+1}), \frac{d_q(y_{2n}, y_{2n+1}).d_q(xy_{2n+1}, y_{2n+2})}{a+\max\{d_q(y_{2n}, y_{2n+1}), d_q(y_{2n+1}, y_{2n})\}}, \\ d_q(y_{2n}, y_{2n+1}), d_q(y_{2n+1}, y_{2n+2}) \end{array}\right\} = 0,$$

So $d_q(y_{2n}, y_{2n+1}) = d_q(y_{2n+1}, y_{2n+2}) = 0$. Also $D_q(y_{2n+1}, y_{2n}) = 0$ implies $d_q(y_{2n+1}, y_{2n}) = d_q(y_{2n+2}, y_{2n+1}) = 0$. Hence $y_{2n} = y_{2n+1} = y_{2n+2}$ is a C.F.P of (S,T) the argument is satisfied. In order to find C.F.P of both S and T, when $\min\{(D_q(e,c)), (D_q(c,e))\} > 0$ for ēverŷ $e, c \in Y$ with $e \neq c$, we make a şequenće $\{TS(y_n)\}$ generated by y_0. Then, we have $y_{2n+1} \in Sy_{2n}$ and $y_{2n+2} \in Ty_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $d_q(y_{2n}, Sy_{2n}) = d_q(y_{2n}, y_{2n+1})$, $d_q(y_{2n+1}, Ty_{2n+1}) = d_q(y_{2n+1}, y_{2n+2})$. By Lemma 1.3.8 we gain

$$d_q(y_{2n}, y_{2n+1}) \leq H_{d_q}(Ty_{2n-1}, Sy_{2n}),\ d_q(y_{2n+1}, y_{2n}) \leq H_{d_q}(Sy_{2n}, Ty_{2n-1}), \tag{4.27}$$

and

$$d_q(y_{2n+1}, y_{2n+2}) \leq H_{d_q}(Sy_{2n}, Ty_{2n+1}),\ d_q(y_{2n+2}, y_{2n+1}) \leq H_{d_q}(Ty_{2n+1}, Sy_{2n}). \tag{4.28}$$

Then from (4.28), we get

$$\begin{aligned} F(d_q(y_{2p+1}, y_{2p+2})) &\leq F(H_{d_q}(Sy_{2p}, Ty_{2p+1})) \\ &\leq \max\{F(H_{d_q}(Sy_{2p}, Ty_{2p+1})), F(H_{d_q}(Ty_{2p}, Sy_{2p+1}))\} \\ &\leq \min\{F(D_q(y_{2p}, y_{2p+1})), F(D_q(y_{2p+1}, y_{2p}))\} - \tau \\ &\leq F(D_q(y_{2p}, y_{2p+1}) - \tau, \end{aligned}$$

for each $p \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned} D_q(y_{2p}, y_{2p+1}) &= \max\left\{ \begin{array}{c} d_q(y_{2p}, y_{2p+1}), \frac{d_q(y_{2p}, Sy_{2p}).d_q(y_{2p+1}, Ty_{2p+1})}{a+\max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p})\}}, \\ d_q(y_{2p}, Sy_{2p}), d_q(y_{2p+1}, Ty_{2p+1}) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d_q(y_{2p}, y_{2p+1}), \frac{d_q(y_{2p}, y_{2p+1}).d_q(y_{2p+1}, y_{2p+2})}{a+\max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p})\}}, \\ d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2}) \end{array} \right\} \\ &\leq \max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2})\}. \end{aligned}$$

If, $\max\{d_q(y_{2p}, y_{2p+1}), d_q(y_{2p+1}, y_{2p+2})\} = d_q(y_{2p+1}, y_{2p+2})$, then

$$F(d_q(y_{2p+1}, y_{2p+2})) \leq F(d_q(y_{2p+1}, y_{2p+2})) - \tau,$$

which is not true due to F_1. Therefore,

$$F(d_q(y_{2p+1}, y_{2p+2})) \leq F(d_q(y_{2p}, y_{2p+1})) - \tau, \tag{4.29}$$

and this implies

$$F(d_q(y_{2p+1}, y_{2p+2})) \leq \max\{F(d_q(y_{2p}, y_{2p+1})), F(d_q(y_{2p+1}, y_{2p}))\} - \tau \tag{4.30}$$

Again using (4.27), we have

$$
\begin{aligned}
F(d_q(y_{2p}, y_{2p+1})) &\leq F(H_{d_q}(Ty_{2p-1}, Sy_{2p})) \\
&\leq \max\{F(H_{d_q}(Sy_{2p-1}, Ty_{2p})), F(H_{d_q}(Ty_{2p-1}, Sy_{2p}))\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \min\{F(D_q(y_{2p-1}, y_{2p})), F(D_q(y_{2p}, y_{2p-1}))\} - \tau \\
&\leq F(D_q(y_{2p}, y_{2p-1})) - \tau.
\end{aligned}
$$

Now,

$$
\begin{aligned}
D_q(y_{2p}, y_{2p-1}) &= \max\left\{\begin{array}{c} d_q(y_{2p}, y_{2p-1}), \frac{d_q(y_{2p}, Sy_{2p}).d_q(y_{2p-1}, Ty_{2p-1})}{a+\max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p})\}}, \\ d_q(y_{2p}, Sy_{2p}), d_q(y_{2p-1}, Ty_{2p-1}) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d_q(y_{2p}, y_{2p-1}), \frac{d_q(y_{2p}, y_{2p+1}).d_q(y_{2p-1}, y_{2p})}{a+\max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p})\}}, \\ d_q(y_{2p}, y_{2p+1}), d_q(y_{2p-1}, y_{2p}) \end{array}\right\} \\
&\leq \max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p}), d_q(y_{2p}, y_{2p+1})\}.
\end{aligned}
$$

If $\max\{d_q(y_{2p}, y_{2p-1}), d_q(y_{2p-1}, y_{2p}), d_q(y_{2p}, y_{2p+1})\} = d_q(y_{2p}, y_{2p+1})$, then we obtain

$$
F(d_q(y_{2p}, y_{2p+1})) \leq F(d_q(y_{2p}, y_{2p+1})) - \tau,
$$

which is not true due to F_1. Therefore,

$$
F(d_q(y_{2p}, y_{2p+1})) \leq F\{\max(d_q(y_{2p-1}, y_{2p}), d_q(y_{2p}, y_{2p-1}))\} - \tau. \tag{4.31}
$$

By using (4.31) in (4.29), we get

$$
F(d_q(y_{2p+1}, y_{2p+2})) \leq \max\{F(d_q(y_{2p-1}, y_{2p})), F(d_q(y_{2p}, y_{2p-1}))\} - 2\tau. \tag{4.32}
$$

Now, from (4.28) and (F1), we have

$$
\begin{aligned}
&\le \max\{F(d_q(y_{2p-1}, y_{2p}), Fd_q(y_{2p}, y_{2p-1})\} \\
&\le \max\{F(H_{d_q}(Sy_{2p-2}, Ty_{2p-1})), F(H_{d_q}(Ty_{2p-1}, Sy_{2p-2}))\}
\end{aligned}
$$

$$
\begin{aligned}
&\le \min\{F(D_q(y_{2p-2}, y_{2p-1})), F(D_q(y_{2p-1}, y_{2p-2}))\} - \tau \\
&\le F(D_q(y_{2p-2}, y_{2p-1})) - \tau.
\end{aligned}
$$

Now,

$$
\begin{aligned}
D_q(y_{2p-2}, y_{2p-1}) &= \max\left\{\begin{array}{c} d_q(y_{2p-2}, y_{2p-1}), \frac{d_q(y_{2p-2}, Sy_{2p-2}).d_q(y_{2p-1}, Ty_{2p-1})}{a+\max\{d_q(y_{2p-2}, y_{2p-1}), d_q(y_{2p-1}, y_{2p-2})\}}, \\ d_q(y_{2p-2}, Sy_{2p-2}), d_q(y_{2p-1}, Ty_{2p-1}) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d_q(y_{2p-2}, y_{2p-1}), \frac{d_q(y_{2p-2}, y_{2p-1}).d_q(y_{2p-1}, y_{2p})}{a+\max\{d_q(y_{2p-2}, y_{2p-1}), d_q(y_{2p-1}, y_{2p-2})\}}, \\ d_q(y_{2p-2}, y_{2p-1}), d_q(y_{2p-1}, y_{2p}) \end{array}\right\} \\
&\le \max\{d_q(y_{2p-2}, y_{2p-1}), d_q(y_{2p-1}, y_{2p})\}.
\end{aligned}
$$

Now, $\max\{d_q(y_{2p-2}, y_{2p-1}), d_q(y_{2p-1}, y_{2p})\} = d_q(y_{2p-1}, y_{2p})$ gives a contradiction. So, we have

$$\max\{F(d_q(y_{2p-1}, y_{2p}), F(d_q(y_{2p}, y_{2p-1}))\} \le F(d_q(y_{2p-2}, y_{2p-1})) - \tau.$$

Using the above inequality in (4.32), we get

$$F(d_q(y_{2p+1}, y_{2p+2})) \le \max\{F(d_q(y_{2p-2}, y_{2p-1})), F(d_q(y_{2p-1}, y_{2p-2}))\} - 3\tau.$$

Observing (4.30), (4.32), the above inequality and proceeding in this way, we have

$$F(d_q(y_{2p+1}, y_{2p+2})) \le \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\} - (2p+1)\tau, \tag{4.33}$$

for each $p \in \mathbb{N} \cup \{0\}$. Similarly, we have

$$F(d_q(y_{2p}, y_{2p+1})) \le \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\} - (2p)\tau \tag{4.34}$$

Combining (4.33) and (4.34), we get

$$F(d_q(y_n, y_{n+1})) \leq \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\} - n\tau. \tag{4.35}$$

Similarly, we get

$$F(d_q(y_{n+1}, y_n)) \leq \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\} - n\tau. \tag{4.36}$$

On taking limit $n \to \infty$, both sides of (4.35) and (4.36) , we have

$$\lim_{n\to\infty} F(d_q(y_n, y_{n+1})) = \lim_{n\to\infty} F(d_q(y_{n+1}, y_n)) = -\infty. \tag{4.37}$$

Since $F \in \mathcal{F}$,

$$\lim_{n\to\infty} d_q(y_n, y_{n+1}) = \lim_{n\to\infty} d_q(y_{n+1}, y_n) = 0. \tag{4.38}$$

By (4.35), for each $n \in \mathbb{N}$, we obtain

$$0 \leq (d_q(y_n, y_{n+1}))^k((F(d_q(y_n, y_{n+1})) - \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\},$$

which implies,

$$0 \leq -(d_q(y_n, y_{n+1}))^k n\tau \leq 0. \tag{4.39}$$

By using (4.37), (4.38) and taking limit $n \to \infty$ in inequality (4.39), we get

$$\lim_{n\to\infty}(n(d_q(y_n, y_{n+1}))^k) = 0. \tag{4.40}$$

Same result can be obtain by using (4.36),

$$\begin{aligned} 0 &\leq (d_q(y_{n+1}, y_n))^k((F(d_q(y_{n+1}, y_n)) - \max\{F(d_q(y_0, y_1)), F(d_q(y_1, y_0))\} \\ &\leq -(d_q(y_{n+1}, y_n))^k n\tau \leq 0. \end{aligned}$$

By using (4.37), (4.38) and letting $n \to \infty$, we have

$$\lim_{n\to\infty}(n(d_q(y_{n+1}, y_n))^k) = 0. \tag{4.41}$$

As (4.40) satisfies, there is $n_1 \in \mathbb{N}$, and $n(d_q(y_n, y_{n+1}))^k \leq 1$, for each $n \geq n_1$ or,

$$d_q(y_n, y_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for each } n \geq n_1. \tag{4.42}$$

Similarly, by using (4.41), there exists $n_2 \in \mathbb{N}$, so as $n(d_q(y_{n+1}, y_n))^k \leq 1$, for each $n \geq n_2$, we have

$$d_q(y_{n+1}, y_n) \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for each } n \geq n_2. \tag{4.43}$$

Using (4.42), we get form $m > n > n_1$,

$$\begin{aligned} d_q(y_n, y_m) &\leq d_q(y_n, y_{n+1}) + d_q(y_{n+1}, y_{n+2}) + \cdots + d_q(y_{m-1}, y_m) \\ &= \sum_{p=n}^{m-1} d_q(y_p, y_{p+1}) \leq \sum_{p=n}^{\infty} d_q(y_p, y_{p+1}) \leq \sum_{p=n}^{\infty} \frac{1}{p^{\frac{1}{k}}}. \end{aligned}$$

The convergence of this series $\sum_{p=n}^{\infty} \frac{1}{p^{\frac{1}{k}}}$ demands $\lim_{n,m\to\infty} d_q(y_n, y_m) = 0$. Now, by treating the inequality (4.43) we get, $\lim_{m,n\to\infty} d_q(y_m, y_n) = 0$. Hence, $\{TS(y_n)\}$ is a Cauchy in (Y, d_q). Since (Y, d_q) is a complete DQM space, so there must be a $u \in y$ so as $\{TS(y_n)\} \to u$ that is

$$\lim_{n\to\infty} d_q(y_n, u) = \lim_{n\to\infty} d_q(u, y_n) = 0. \tag{4.44}$$

Now, by Lemma 1.3.8 we have

$$\begin{aligned} \tau + F(d_q(y_{2n+1}, Tu) &\leq \tau + F(H_{d_q}(Sy_{2n}, Tu) \\ &\leq \tau + \max\{F(H_{d_q}(Sy_{2n}, Tu), F(H_{d_q}(Ty_{2n}, Su)\}. \end{aligned}$$

Using inequality (4.25),

$$\tau + F(d_q(y_{2n+1}, Tu) \leq \min\{F(D_q(y_{2n}, u), F(D_q(u, y_{2n})\}$$

$$\tau + F(d_q(y_{2n+1}, Tu) \leq F(D_q(y_{2n}, u)), \tag{4.45}$$

where,

$$\begin{aligned} D_q(y_{2n}, u) &= \max \left\{ \begin{array}{c} d_q(y_{2n}, u), \frac{d_q(y_{2n}, Sy_{2n}).d_q(u, Tu)}{a+\max\{d_q(y_{2n}, u), d_q(u, y_{2n})\}}, \\ d_q(y_{2n}, Sy_{2n}), d_q(u, Tu) \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} d_q(y_{2n}, u), \frac{d_q(y_{2n}, y_{2n+1}).d_q(u, Tu)}{a+\max\{d_q(y_{2n}, u), d_q(u, y_{2n})\}}, \\ d_q(y_{2n}, y_{2n+1}), d_q(u, Tu) \end{array} \right\}. \end{aligned}$$

Applying limit $n \to \infty$, on inequality (4.44), we get

$$D_q(y_{2n}, u) = d_q(u, Tu). \tag{4.46}$$

Using (4.46) in (4.45), we get

$$\tau + F(d_q(y_{2n+1}, Tu) \leq F(d_q(u, Tu)).$$

Since F, is the continuous strictly increasing real function, we get

$$d_q(y_{2n+1}, Tu) < d_q(u, Tu).$$

Applying limit $n \to \infty$, we get

$$d_q(u, Tu) < d_q(u, Tu).$$

It is not true , so $d_q(u, Tu) = 0$. Now, by Lemma 1.3.8

$$\tau + d_q(Tu, y_{2n+1})) \leq \tau + F(H_{d_q}(Tu, Sy_{2n})),$$

by using the similar reasons as above, we get $d_q(Tu, u) = 0$. Hence $u \in Tu$. Similarly by using (4.44), Lemma 1.3.8, and the inequality

$$\tau + d_q(Su, y_{2n+2}) \leq \tau + F(H_{d_q}(Su, Ty_{2n+1}))$$

we can show that $d_q(Su, u) = 0$. Similarly, $d_q(u, Su) = 0$. Hence, the pair (S, T) have a C.F.P

u in (Y, d_q). Now,

$$d_q(u,u) \leq d_q(u,Tu) + d_q(Tu,u) \leq 0.$$

This implies that $d_q(u,u) = 0$. Hence the proof is completed. ■

Example 4.3.3 Let $Y = \{0\} \cup \mathbb{Q}^+$ and $d_q(x,y) = x + 2y$. Then (Y, d_q) be a DQM space. Define $S, T : Y \to P(Y)$ by:

$$S(y) = \left\{ \begin{array}{lr} \left[\frac{1}{3}y, \frac{2}{3}y\right] \cap \mathbb{Q}^+, \text{ for all } y \in \{0, 7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \cdots\}, & \\ {[y, y+5]} \cap \mathbb{Q}^+, & \text{otherwise.} \end{array} \right\}$$

$$T(y) = \left\{ \begin{array}{lr} \left[\frac{1}{5}y, \frac{2}{5}y\right] \cap \mathbb{Q}^+, \text{ for all } y \in \{0, 7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \cdots\}, & \\ {[y+2, y+6]} \cap \mathbb{Q}^+, & \text{otherwise.} \end{array} \right\}$$

Case 1: If,

$$\begin{aligned} & \tau + \max\{F(H_{d_q}(Se, Tc)), F(H_{d_q}(Te, Sc))\} \\ = \ & \tau + F(H_{d_q}(Se, Tc)) \leq \min\{F(D_q(e,c)), F(D_q(c,e))\} \end{aligned}$$

holds. Define $F : R^+ \to R$ real function by $F(u) = \ln(u)$ for every $u \in R^+$ and $\tau > 0$. As $x, y \in Y$, $\tau = \ln(1.2)$ and by taking $y_0 = 7$, we define the sequence $\{TS(y_n)\} = \{7, \frac{7}{3}, \frac{7}{15}, \frac{7}{45}, \cdots\}$ in Y generated by $y_0 = 7$. Now, if $x, y \in \{TS(y_n)\}$, we have

$$\begin{aligned} H_{d_q}(Sx, Ty) &= \max\left[\left\{\sup_{a \in Sx} d_q(a, Ty), \sup_{b \in Ty} d_q(Sx, b)\right\}\right] \\ &= \max\left[\left\{\sup_{a \in Sx} d_q\left(a, \left[\frac{y}{2}, \frac{2y}{5}\right]\right), \sup_{b \in Ty} d_q\left(\left[\frac{x}{3}, \frac{2x}{3}\right], b\right)\right\}\right] \\ &= \max\left\{d_q\left(\frac{2x}{3}, \frac{y}{5}\right), d_q\left(\frac{x}{3}, \frac{2y}{5}\right)\right\} \\ &= \max\left\{\frac{2x}{3} + \frac{2y}{5}, \frac{x}{3} + \frac{4y}{5}\right\}. \end{aligned}$$

Also

$$\begin{aligned}
D_q(x,y) &= \max\left\{\begin{array}{c} d_q(x,y), \frac{d_q(x,\left[\frac{x}{3},\frac{2x}{3}\right]).d_q(y,\left[\frac{y}{5},\frac{2y}{5}\right])}{1+\max\{d_q(x,y),d_q(y,x)\}}, \\ d_q(x,\left[\frac{x}{3},\frac{2x}{3}\right]), d_q(y,\left[\frac{y}{5},\frac{2y}{5}\right]) \end{array}\right\} \\
&= \max\left\{d_q(x,y), \frac{d_q(x,\frac{x}{3}).d_q(y,\frac{y}{5})}{1+\max\{d_q(x,y),d_q(y,x)\}}, d_q(x,\frac{x}{3}), d_q(y,\frac{y}{5})\right\} \\
&= \max\left\{x+2y, \frac{7xy}{3(1+x+2y)}, \frac{5x}{3}, \frac{7y}{5}\right\} = x+2y.
\end{aligned}$$

Case (i). If $\max\left\{\frac{2x}{3}+\frac{2y}{5}, \frac{x}{3}+\frac{4y}{5}\right\} = \frac{x}{3}+\frac{4y}{5}$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned}
10x+24y &\leq 25x+50y \\
\frac{6}{5}(\frac{x}{3}+\frac{4y}{5}) &\leq x+2y \\
\ln(1.2)+\ln(\frac{x}{3}+\frac{4y}{5}) &\leq \ln(x+2y).
\end{aligned}$$

Which shows that,

$$\tau + F(H_{d_q}(Sx,Ty) \leq F(D_q(x,y)).$$

Case (ii). Similarly, if $\max\left\{\frac{2x}{3}+\frac{2y}{5}, \frac{x}{3}+\frac{4y}{5}\right\} = \frac{2x}{3}+\frac{2y}{5}$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned}
20x+12y &\leq 25x+50y \\
\frac{6}{5}(\frac{2x}{3}+\frac{2y}{5}) &\leq x+2y \\
\ln(1.2)+\ln(\frac{2x}{3}+\frac{2y}{5}) &\leq \ln(x+2y).
\end{aligned}$$

Hence,

$$\tau + F(H_{d_q}(Sx,Ty) \leq F(D_q(x,y)).$$

Case 2: If

$$\begin{aligned}
&\max\{\tau + F(H_{d_q}(Sx,Ty)), \tau + F(H_{d_q}(Tx,Sy))\} \\
= \ &\tau + F(H_{d_q}(Tx,Sy)) \leq \min\{F(D_q(x,y)), F(D_q(y,x))\}
\end{aligned}$$

holds.

$$
\begin{aligned}
H_{d_q}(Tx,Sy) &= \max\left[\left\{\sup_{b\in Tx} d_q(Sy,b), \sup_{a\in Sy} d_q(a,Tx)\right\}\right]\\
&= \max\left[\left\{\sup_{b\in Tx} d_q\left(\left[\frac{y}{3},\frac{2y}{3}\right],b\right), \sup_{a\in Sy} d_q\left(a,\left[\frac{x}{5},\frac{2x}{5}\right]\right)\right\}\right]\\
&= \max\left\{d_q\left(\frac{2y}{3},\frac{x}{5}\right), d_q\left(\frac{y}{3},\frac{2x}{5}\right)\right\}\\
&= \max\left\{\frac{2y}{3}+\frac{2x}{5}, \frac{y}{3}+\frac{4x}{5}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
D_q(y,x) &= \max\left\{\begin{array}{c} d_q(y,x), \frac{d_q(x,[\frac{x}{3},\frac{2x}{3}]).d_q(y,[\frac{y}{5},\frac{2y}{5}])}{1+\max\{d_q(x,y),d_q(y,x)\}},\\ d_q(x,[\frac{x}{3},\frac{2x}{3}]), d_q(y,\left[\frac{y}{5},\frac{2y}{5}\right]) \end{array}\right\}\\
&= \max\left\{\begin{array}{c} d_q(y,x), \frac{d_q(x,\frac{x}{3}).d_q(y,\frac{y}{5})}{1+\max\{d_q(x,y),d_q(y,x)\}},\\ d_q(x,\frac{x}{3}), d_q(y,\frac{y}{5}) \end{array}\right\}\\
&= \max\left\{y+2x, \frac{7xy}{3(1+y+2x)}, \frac{5x}{3}, \frac{7y}{5}\right\} = y+2x.
\end{aligned}
$$

Case (i). If, $\max\left\{\frac{2y}{3}+\frac{2x}{5}, \frac{y}{3}+\frac{4x}{5}\right\} = \frac{y}{3}+\frac{4x}{5}$, and $\tau = \ln(1.2)$, then we have

$$
\begin{aligned}
10y+24x &\leq 25y+50x\\
\frac{6}{5}\left(\frac{y}{3}+\frac{4x}{5}\right) &\leq y+2x\\
\ln(1.2)+\ln\left(\frac{y}{3}+\frac{4x}{5}\right) &\leq \ln(y+2x),\\
\text{so, } \tau + F(H_{d_q}(Tx,Sy) &\leq F(D_q(y,x)).
\end{aligned}
$$

Case (ii). Similarly, if $\max\left\{\frac{2y}{3}+\frac{2x}{5}, \frac{y}{3}+\frac{4x}{5}\right\} = \frac{2y}{3}+\frac{2x}{5}$, and $\tau = \ln(1.2)$, then we have

$$
\begin{aligned}
20y+24x &\leq 25x+50y\\
\frac{6}{5}\left(\frac{2y}{3}+\frac{2x}{5}\right) &\leq y+2x\\
\ln(1.2)+\ln\left(\frac{2y}{3}+\frac{2x}{5}\right) &\leq \ln(y+2x).\\
\text{Hence, } \tau + F(H_{d_q}(Tx,Sy) &\leq F(D_q(y,x)).
\end{aligned}
$$

Now, if $x, y \notin \{TS(y_n)\}$, then the contraction does not hold. Hence all hypothesis of Theorem 4.3.2 are proved so S and T have a C.F.P.

If we take $S = T$ in Theorem 4.3.2, then we are left with the theorem.

Theorem 4.3.4 Let (Y, d_q) be a complete DQM space and $S : Y \to P(Y)$ be the setvalued map such that

$$\tau + F(H_{d_q}(Sl, Sp)) \leq F(D_q(l, p)), \tag{4.47}$$

for each $l, p \in \{S(y_n)\}$, with $D_q(l, p) > 0$, $F \in \mathcal{F}$, $\tau, a > 0$, and

$$D_q(l,p) = \max\left\{d_q(l,p), \frac{d_q(l, Sl) . d_q(p, Sp)}{a + d_q(l,p)}, d_q(l, Sl), d_q(p, Sp)\right\}.$$

Then $\{S(y_n)\} \to u \in Y$. Moreover, if (4.47) holds for u, then S has a fixed point u in Y and $d_q(u, u) = 0$.

Definition 4.3.5 Let $S, T : Y \to Y$ be two maps and $x_0 \in Y$. Let $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$ and so on. Proceeding this method, we get the sequence x_n in X so as

$$x_{2p+1} = Sx_{2p} \text{ and } x_{2p+2} = Tx_{2p+1}, \text{ (where } p = 0, 1, 2, \dots).$$

We say that $\{TS(x_n)\}$ be the sequence in Y generated by x_0.

Definition 4.3.6 Let (Y, d_q) be a DQM space and $S, T : Y \to Y$ be two maps. The pair (S, T) is said a $FDQ-$contraction, if for all $e, g \in \{TS(e_n)\}$, we get

$$\tau + \max\{F(d_q(Se, Tg)), F(d_q(Te, Sg))\} \leq \min\{F(D_q(e, g)), F(D_q(g, e))\} \tag{4.48}$$

where $F \in \mathcal{F}$ and $\tau > 0$, and

$$D_q(e, g)) = \max\left\{d_q(e, g), \frac{d_q(e, Se) . d_q(g, Tg)}{1 + \max\{d_q(e, g), d_q(g, e)\}}, d_q(e, Se), d_q(g, Tg)\right\}. \tag{4.49}$$

Then we deduce the following main result.

Theorem 4.3.7 Let (Y, d_q) be a complete DQM space and (S, T) be a $FDQ-$contraction. Then $\{TS(x_n)\} \to u \in X$. Also, if u satisfies (4.48), then u is the C.F.P of S and T in X and $d_q(u. u) = 0$.

Now, we have shown an application, of Theorem 4.3.4 to find unique solution of systems of non linear Volterra type integral inclusions. Let,

$$w(t) = \int_0^t L_1(t, s, w(s))ds + f(t), \tag{4.50}$$

$$c(t) = \int_0^t L_2(t, s, c(s))ds + g(t) \tag{4.51}$$

for all $t \in [0, 1]$. We find the solution of (4.50) and (4.51). Let $E = \{f : f[$ is continuous function from $[0, 1]$ to $\mathbb{R}_+\}$, endowed with the complete DQM. For $w \in E$, identify the norm as: $\|w\|_\tau = \sup_{t\in[0,1]} \{w(t)e^{-\tau t}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$d_\tau(w, c) = \sup_{t\in[0,1]} \{(w(t) + 2c(t))e^{-\tau t}\} = \|w + 2c\|_\tau$$

for all $w, c \in E$, with these settings, (E, d_τ) becomes a DQM space.

Theorem 4.3.8 Assume (i), (ii) and (iii) are satisfied:

(i) $L_1, L_2 : [0, 1] \times [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $f, g : [0, 1] \to \mathbb{R}_+$ are real and continuous;

(ii) Define

$$\begin{aligned} Sw(t) &= \int_0^t L_1(t, s, w(s)ds + f(t), \\ Tc(t) &= \int_0^t L_2(t, h, c(s))ds + g(t). \end{aligned}$$

Suppose there exist $\tau > 1$, such that

$$\max\{L_1(t, s, w) + 2L_2(t, s, c), L_2(t, s, c) + 2L_1(t, s, w)\} \le \tau e^{-\tau} \min\{P(w, c), P(c, w)\} \tag{4.52}$$

for each t, s belong to $[0, 1]$ and w, c belong to $C([0,1], \mathbb{R})$, where

$$P(w,c) = \max\left\{ \begin{array}{c} w(t) + 2c(t), \frac{(w(t)+2Sw(t))(c(t)+2Tc(t))}{a+\max\{w(t)+2c(t), c(t)+2w(t)\}}, \\ w(t) + 2Sw(t), c(t) + 2Tc(t) \end{array} \right\}$$

Then integral equations (4.50) and (4.51) has a solution.

Proof. By assumption (ii)

$$\begin{aligned}
\max\{Sw + 2Tv, Tw + 2Sc\} &= \max\{\int_0^t (K_1(t,s,w) + 2K_2(t,s,c))ds, \\
&\quad \int_0^t (K_2(t,s,c) + 2K_1(t,s,w))ds\} \\
&\leq \int_0^t \tau e^{-\tau} \min\{M(w,c), M(c,w)\} ds \\
&\leq \int_0^t \tau e^{-\tau} [\min\{P(w,c), P(c,w)\} e^{-\tau s}] e^{\tau s} ds \\
&\leq \int_0^t \tau e^{-\tau} \| \min\{P(w,c), P(c,w)\} \|_\tau e^{\tau s} ds \\
&\leq \tau e^{-\tau} \| \min\{P(w,c), P(c,w)\} \|_\tau \int_0^t e^{\tau s} ds \\
&\leq \tau e^{-\tau} \| \min\{P(w,c), P(c,w)\} \|_\tau \frac{1}{\tau} e^{\tau t} \\
&\leq e^{-\tau} \| \min\{P(w,c), P(c,w)\} \|_\tau e^{\tau t}.
\end{aligned}$$

This implies

$$\max\{Sw + 2Tc, Tw + 2Sc\} e^{-\tau t} \leq e^{-\tau} \| \min\{P(w,c), P(c,w)\} \|_\tau.$$

That is $\| \max\{Su + 2Tv, Tu + 2Sv\} \|_\tau \leq e^{-\tau} \| \min\{P(w,c), P(c,w)\} \|_\tau$, which further implies

$$\tau + \ln \| \max\{Sw + 2Tc, Tw + 2Sc\} \|_\tau \leq \ln \| \min\{P(w,c), P(c,w)\} \|_\tau.$$

So, all hypothesis of Theorem 4.3.7 are proved. Hence, (4.50)and (4.51) have a common solution.

Remark 3.4.10 By setting different values of $P(w, c)$ in equation (4.52), we can obtain different weak contractive inequalities and řesulţs as corollaries of Theorem 4.3.8. ■

Bibliography

[1] M. Abbas, B. Ali and S. Romaguera, Fixed and Periodic Points of Generalized Contractions in Metric Spaces, Fixed Point Theory Appl., 2013, 2013:243.

[2] M. Abbas, I. Z. Chema, A. Razani, Existence of common fixed point for b-metric rational type contraction, Filomat 30:6 (2016), 1413-1429.

[3] Ö. Acar and I. Altun, A Fixed Point Theorem for Multivalued Mappings with δ-Distance, Abstr. Appl. Anal., 2014, Article ID 497092, 5 pages.

[4] Ö. Acar, G. Durmaz and G. Minak, Generalized multivalued $F-$contractions on complete metric spaces , Bulletin of the Iranian Mathematical Society, 40(2014), 1469-1478.

[5] A. Ahmed, N. Abdou, Common Fixed Point Results for multivalued Mappings with some examples, J. Nonlinear. Sci. Appl., 9(2016), 787-798.

[6] J. Ahmad, A. Al-Rawashdeh and A. Azam, Some New Fixed Point Theorems for Generalized Contractions in Complete Metric Spaces, Fixed Point Theory and Appl., 2015, 2015:80.

[7] M. U. Ali, T. Kamran, E. Karapınar, Further discussion on modified multivalued $\alpha^* - \psi$-contractive type mapping, Filomat, 29(8), 2015, 1893–1900.

[8] A. S. M. Alofi, A. E. Al-Mazrooei, B. T. Leyew, M. Abbas, Common fixed points of α-dominated multivalued mappings on closed balls in a dislocated quasi b-metric space, J. Nonlinear Sci. Appl., 10(7), 2017, 3456–3476.

[9] J. H. Asl, S. Rezapour and N. Shahzad, On fixed points of $\alpha - \psi$ contractive multifunctions, Fixed Point Theory Appl., 2012 2012:212.

[10] E. Ameer, M. Arshad, Two new generalization for F-contraction on closed ball and fixed point theorem with application, J. Mathematical Exten. 11(2017), 1-24.

[11] M. Arshad, S. U. Khan, J. Ahmad, Fixed point results for F-contractions involving some new rational expressions, JP Journal of Fixed Point Theory and Appl., 11(1), 2016, 79-97 pages.

[12] M. Arshad, A. Shoaib, and P. Vetro, Common Fixed Points of a Pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Dislocated Metric Spaces, Journal of Function Spaces, 2013 (2013), Article ID 63818.

[13] M. Arshad , A. Shoaib, M. Abbas and A. Azam, Fixed Points of a pair of Kannan Type Mappings on a Closed Ball in Ordered Partial Metric Spaces, Miskolc Math. Notes, 14(3), 2013, 769–784.

[14] M. Arshad, A. Azam, M. Abbas and A. Shoaib, Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity U.P.B. Sci. Bull., Series A, 76(2), 2014.

[15] M. Arshad, Z. Kadelburg, S. Radenović, A. Shoaib, S. Shukla, Fixed Points of α-Dominated Mappings on Dislocated Quasi Metric Spaces, Filomat, 31(11) (2017), 3041–3056.

[16] H . Aydi, M. F. Bota, E. Karapinar and S. Mitrović, fixed point theorem for set-valued quasi-contractions in b-metric spaces Fixed Point Theory Appl., 2012:88, 8 pages.

[17] A. Azam, M. Arshad, Fixed points of a sequence of locally contractive multi-valued maps, Comp. Math. Appl., 57 (2009), 96-100.

[18] A. Azam and M. Arshad, Fixed points of a sequence of locally contractive multi-valued maps, Comp. Math. Appl., 57 (2009), 96-100.

[19] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux equations itegrales, Fund. Math., 3 (1922) 133–181.

[20] I. Beg, M. Arshad , A. Shoaib,Fixed Point on a Closed Ball in ordered dislocated Metric Space, Fixed Point Theory, 16(2), 2015.

[32] A. Hussain, M. Arshad, M. Nazim, Connection of Ciric type $F-$contraction involving fixed point on closed ball, Ghazi. Uni. Journal of Sci., 30(1)(2017), 283-291.

[33] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc., 1 (136) (2008) 1359-1373.

[34] E. Karapınar, H. Piri, and H. H. Alsulami, Fixed points of modified F-contractive mappings in complete metric-like spaces, Journal of Function Spaces, Article ID 270971, 9 pages.

[35] E. Karapınar, İ.M. Erhan and A. Öztürk, Fixed point theorems on quasi-partial metric spaces, Math. Comp. Model., (2012), doi:10.1016/j.mcm.2012.06.036.

[36] M. S. Khan, Common fixed Point Theorems for Multivalued Mappings, Pacific J. Maths Vol (95), 12 pages.

[37] S. U. Khan, M. Arshad, A. Hussain, M. Nazam,Two new types of fixed point theorems for F-contraction, J. Adv. Stud. In. Topology, 7:4 (2016), 251-260.

[38] S.U. Khan, M. Arshad ,T. Rasham, A. Shoaib, Some New Common Fixed Points of Generalized Rational Contractive Mappings in Dislocated Metric Spaces with Applications, Honam Math. J., 39(2)(2017), 161-174 pages.

[39] S. G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., 728 (1994), 183-197.

[40] S. B J. Nadler, Multi-valued contraction mappings. Pacific J. Math. 30(1969), 475-478.

[41] V. Ozturk, D. Turkoglu, Fixed points for generalized contractions in b-metric spaces, J. Nonlinear Convex Anal., 16(10)(2015), 2059–2066.

[42] H. Piri and P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl., 2014, 2014:210.

[43] H. Piri, S. Rahrovi, H. Morasi, P. Kumam, Fixed point theorem for F-Khan-contractions on complete metric spaces and application to the integral equations, J. Nonlinear Sci. Appl., 10(2017), 4564-4869.

[44] T. Rasham, A. Shoaib, M. Arshad, and S. U. Khan, Fixed point results on closed ball for a new rational type contraction mapping in complete dislocated metric spaces, J. Inequal. Spec. Funct., 8(2017), 74-85.

[45] T. Rasham, A. Shoaib, N. Hussain, M. Arshad, S.U. Khan, Common fixed point results for new Ciric-type rational multivalued F contraction with an application, J. Fixed Point Theory. Appl., 20(1)(2018), 1-16 pages.

[46] T. Rasham, A. Shoaib, B. S. Alamri, M. Arshad, Multivalued fixed point results for new generalized F-Dominted contractive mappings on dislocated metric space with application, Journal of Function Spaces, 2018(2018), Article ID 4808764, 12 pages.

[47] T. Rasham, A. Shoaib, B. S. Alamri, M. Arshad, Fixed Point Results for Multivalued Contractive Mappings Endowed With Graphic Structure, Journal of Mathematics, 2018 (2018) Article ID 5816364, 8 pages.

[48] T. Rasham, A. Shoaib, C. Park, M. Arshad, Fixed Point Results for a Pair of Multi Dominated Mappings on a Smallest Subset in K-Sequentially Dislocated quasi Metric Space with Application, J. Comput. Anal. Appl., 25(5) (2018) pages 975-986.

[49] T. Rasham, A. Shoaib, N. Hussain, B. S. Alamri, M. Arshad, Multivalued Fixed Point Results in Dislocated $b-$Metric Spaces with Application to the System of Nonlinear Integral Equations, Symmetry 2019, 11(1), 40; https://doi.org/10.3390/sym11010040 (registering DOI)

[50] M. Sarwar, Mujeeb Ur Rahman and G. Ali, Some fixed point results in dislocated quasi metric (dq-metric) spaces, J. Inequal. Appl., 2014, 2014:278.

[51] T. Senapati, L. K. Dey, Common Fixed Point Theorems for Multivalued β_*-ψ-Contractive Mappings, Thai Journal of Maths, 15(3) (2017) 747–759.

[52] N. A. Secelean, Iterated function systems consisting of F-contractions. Fixed Point Theory Appl. 2013, Article ID 277 (2013). doi:10.1186/1687-1812-2013-277.

[53] M. Sgroi and C. Vetro, Multi-valued $F-$contractions and the solution of certain functional and integral equations, Filomat, 27(7), (2013), 1259-1268.

[54] A. Shoaib, M. Arshad and J. Ahmad, Fixed point results of locally cotractive mappings in ordered quasi-partial metric spaces, The Scientific World Journal, 2013 (2013), Article ID 194897, 8 pages.

[55] A. Shoaib, T. Rasham, N. Hussain, M. Arshad, Fixed point results for a pair of multivalued dominated mappings in dislocated $b-$metric spaces with applications, Journal of the National Science Foundation of Sri Lanka, 47(2), pp.235–243. DOI: http://doi.org/10.4038/jnsfsr.v47i2.9165.

[56] A. Shoaib, M. Arshad and M. A. Kutbi, Common fixed points of a pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Partial Metric Spaces, J. Comput. Anal. Appl., 17(2014), 255-264.

[57] A. Shoaib, T. Rasham, A. A. Rawashdeh, M. Arshad, DQF-contraction and related fixed point results in DQM spaces with application, J. Math. Anal., 16 pages, (accepted).

[58] A. Shoaib, M. Arshad, T. Rasham M. Abbas, Unique fixed point results on closed ball for dislocated quasi G-metric spaces, Transactions of A. Razmadze Mathematical Institute, 2017, http://dx.doi.org/10.1016/j.trmi.2017.01.002

[59] A. Shoaib, Fixed Point Results for α_*-ψ-multivalued Mappings, Bull. Math. Anal. Appl., 8(4), 2016, 43-55.

[60] A. Shoaib, $\alpha - \eta$ Dominated Mappings and Related Common Fixed Point Results in Closed Ball, J. Concr. Appl. Math., 13(1-2), 2015, 152-170.

[61] A. Shoaib, A. Hussain, M. Arshad, and A. Azam, Fixed point results for α_*-ψ-Ciric type multivalued mappings on an intersection of a closed ball and a sequence with graph, J. Math. Anal., 7(3), 2016, 41-50.

[62] W. Sintunavarat, Nonlinear integral equations with new admissibility types in b-metric spaces, J. Fixed Point Theory Appl., 18(2016), 397–416.

[63] J. Tiammee, S. Suantai, Coincidence point theorems for graph-preserving multi-valued mappings, Fixed Point Theory Appl., 2014, 2014:70.

[64] B. R. Wadkar, R. Bhardawaj, B. K.Singh, A common fixed point Theorem in Dislocared metric space, Int. J. Engg Research and Devolpment, 10(2014), 14-17.

[65] D. Wardowski, Fixed point theory of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl., 2012, Article ID 94 (2012).

[66] F. M. Zeyada, G. H. Hassan, and M. A. Ahmed, A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces, Arabian J. Science and Engineering, 31(1),(2006), 111–114.

[67] L. Zhu, C. Zhu, C. Chen and Z. Stojanović, Multidimensional fixed points for generalized Ψ-quasi-contractions in quasi-metric-like spaces, J. Inequal. Appl., 2014, 2014:27.