Spaces of Continuous Functions with Analysis in Fixed Point Theory

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY ISLAMABAD, PAKISTAN.

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DEDICATED TO....

My mother

And

Sisters

Who

prayed for my success

throughout my carrier.

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ABSTRACT

The aim and intent of this dissertation titled "SPACES OF CONTINUOUS FUNCTIONS WITH ANALYSIS IN FIXED POINT THEORY", embodies a brief account of general function paces along with the investigation of some fixed point and common fixed point results under the supervision of Professor and Dean of sciences Dr. Muhammad Arshad Zia, International Islamic University Islamabad.

The main aim of this work is to study, generalize, extend and obtain fixed point theorems in the setting of complete, compact, pseudo-compact and b-metric spaces.

The work presented in this thesis has been divided into five chapters. Chapter first is introductory. In this chapter, we present basic definitions and known results without proof. In chapters, 2, 3, 4 and 5, we present generalization of fixed point theorems proved by several authors in the literature of fixed point theory.

Contents

Chapter 1

Introduction

1.1 Historical Perspective

The purpose of the work presented in this thesis, includes a brief account of functional analysis, function spaces and fixed point theory is necessary. In this section, a little bit of the history of the above mentioned concepts is being presented.

Functional analysis was born at the end of nineteenth and beginning of twentieth century. The development of analysis, with its wide range of applications was one of the major mathematical achievements of the twentieth century. It is that branch of analysis that link together and generalizes linear algebra and topology. It is now a broad field encompassing all branches of modern mathematics. In fact, it would be difficult to give a precise and concise definition of what it is today. The term "functional analysis" was for the first time coined by J. Hadamard at the turn of nineteenth century, who is known to have introduced the radius of convergence of power series. At the beginning of twentieth century the theory of this branch of mathematics began to be developed among others by V. Volterra, I. Fredholm, Ascoli, C. Arzela, S. Pincherle, Hilbert, the Bourbaki group, I. M. Gelfand and other representatives of the French and Italian mathematical schools. For a more extensive and penetrating details on the history and development of analysis we refer to see [16]. It play an important role in mathematics and its applications in various fields ranging from differential equations, numerical analysis, fixed point theory to biological, physical and social sciences.

The theory of function spaces has a long and rich history of which we will present few main results and definitions without proof. In analysis we apply two standard methods for investigating behaviors of various functions namely classical and modern. In the former we study individual functions for understanding their behavior in terms of graphs, integrations or derivatives etc. In the later we consider sets or collections of functions and regard each function as point in the set for investigating geometric and algebraic properties of the set as a whole. A set X together with a particular norm that assigns a nonnegative real number $\|g\|_X$ to every function in the collections of functions under consideration is called a normed functional space. In order to fully understand and study functional space various abstract and systematic techniques have been developed as the theory of functional analysis with a focus on Banach spaces. Thus, the study of function spaces has great importance in all branches of mathematical analysis. The development of functional analysis and its applications in the last 20th century has made possible the importance of research in the theory of functional space. The abstract topological, geometric, order structure and the interpolation of operators are properties that are still in demand of deeper and extensive research activities. Function spaces play a vital role in all branches of both pure and applied mathematics, operator theory, ordinary and partial differential equations, physics, engineering. It has in fact, important applications in almost every area of mathematics. Function spaces continues to be a very fruitful and rich area of research for mathematicians ever since its introduction by Maurice Fréchet [37].

The origin of fixed point theory can be go back to 1890, when Picard, used the method of approximation in the solutions of differential and other functional equations. However, in the beginning of the twentieth century fixed point theory began flourish an important part of modern analysis. The credit of all this goes to the pioneering work of the Polish mathematician Stefan Banach [10], who publish his remarkable work in 1922, which provide a contractive method for finding fixed points of mappings. Banach used the idea of shrinking map for obtaining the outstanding contraction mapping theorem. The celebrated Banach contraction principles has been generalized and studied by several authors in various directions in all fields of natural sciences for single valued and multiple valued mappings under different contractive conditions in terms of complete metric spaces.

Fixed point theory is an important branch of analysis having wide range of applications. Numerous problems in physics, chemistry, biology and economics lead to various nonlinear differential and integral equations. In order to find a solution to such equations there is a need to find fixed point by reducing them to functional equations. In functional analysis there are branches of fixed point theory depends on metric space and the type of contraction. Kannan [48] used it with mappings not necessarily continuous. Thus various generalizations concerning fixed point, common fixed point and coincidence points have been under taken for mappings satisfying different contraction conditions in different settings by several authors viz. Bailey [8], Ciric [26], Edelstein [33], Kirk [49], Pant [60], Rhoades [65] and Singh [76]. Thus, fixed point theory has been extensively generalized in terms of metric, topological, order and Banach spaces and enriched it in different approaches see for example Mukheimer [54], Nashine [58], Rao [63], Sintunavarat [70] and Shukla [73]. This advancement in fixed point theory has greatly diversified the applications of various fixed point results in different areas such as the existence theory of differential and integral equations, dynamic programming, chaos theory, discrete mathematics, system analysis, optimization and game theory, perturbation and other diverse disciplines of mathematical sciences.

The thesis consists of following chapters:
In Chapter 1 we give some basic definitions, prepositions, lemmas, examples and other necessary background materials from topological , metric, normed, Banach spaces and used throughout the thesis.

Chapter 2 is divided into two sections. In the first section we have proved and generalized the theorems of Bailey, Edelstein and Fisher in compact metric spaces. The second section deals with the generalizations of fixed point results in pseudo compact tichnov spaces by Fisher and Pathak.

Chapter 3 deals briefly with the concept of common fixed point of weakly commuting and compatible mappings introduced by Jungck in 1976. The most important result of this chapter is the generalization of fixed point theorems of Sahu and Sharma for four weakly compatible mappings.

The main thrust of Chapter 4 is to study generalizations of fixed point theorems in b-metric spaces introduced by Bakhtin in 1989. The main concern of this Chapter is the generalization of fixed point results by Fisher, Pachpatte, Sahu and Sharma.

In 5th and last Chapter, our work is devoted to the extension and generalizations of common fixed point theorems in compact and hausdorff spaces for a single pair of weakly commuting maps. This chapter include the generalization of some fixed point theorems of Fisher, Jungck, Mukherjee, Pachpatte and Sahu and Sharma.

1.2 Basic Concepts

We begin this section, by providing some basic concepts and definitions necessary for the upcoming chapters from norm, metric and topological spaces that are needed in the forthcoming chapters. For further study of related materials see Keyszig [51], Rudin [67], Simmon [74] and others.

Definition 1.2.1. A function $\rho : \Im \times \Im \to R^+$ is called metric on a non-empty set $\Im$ if for all $\xi_1, \xi_2, \xi_3 \in \Im$ it satisfying the following conditions:

(i) $\rho(\xi_1, \xi_2) \geq 0$.

(ii) $\rho(\xi_1, \xi_2) = 0 \Leftrightarrow \xi_1 = \xi_2$.

(iii) $\rho(\xi_1, \xi_2) = \rho(\xi_2, \xi_1)$.

(iv) $\rho(\xi_1, \xi_3) \leq \rho(\xi_1, \xi_2) + \rho(\xi_2, \xi_3)$.

then $\Im$ together with ρ is called metric space.

Examples 1.2.2. (*i*) Every non-empty set $\Im$ can be given a metric and hence can be converted into a metric space and is called discrete metric space. (*ii*) If $\Im$ is any set and $F(\Im) = \{f : \Im \to R : f(\Im) \text{ is bounded}\}$, then the set $F(\Im)$ together with the metric $\rho_\infty : F(\Im) \times F(\Im) \to R^+$ is defined as $\rho_\infty(g_1, g_2) = \sup_{\zeta \in X}(\rho(g_1(\zeta) - g_2(\zeta)))$ and is called uniform metric. (*iii*) For $\Im = R$ with the metric $\rho(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|$, is called usual metric on $\Im$.

Definition 1.2.3. If $\Im$ together with ρ is a metric space. Then, the sequence $\{\xi_n\}$ in $\Im$ is a function $f : N \to \Im$ which is denoted by $\xi_n = f(n)$ for all $n \in N$.

Definition 1.2.4. $\{\xi_n\}$ in $\Im$ is called *convergent* if there exists $\xi \in \Im$ such that $\lim_{n \to \infty} \xi_n = \xi$ for $n > N$.

Definition 1.2.5. A subset Ω in $(\rho, \Im)$ is called,

(*i*) Closed if the limit of any convergent sequence $\{\xi_n\}$ in $\Omega \in K$.

(*ii*) Compact if $\{\xi_n\}$ in Ω has a subsequence $\{\xi_{n_k}\}$ which converges to ξ.

(*iii*) *Relatively compact* if the closure $\overline{\Omega} \subset X$ is a compact subset.

(*iv*) *Bounded* if, for some $\zeta \in \Omega$ and radius $r > 0$, we have $\Omega \subset B_r(\zeta)$.

Definition 1.2.6. $\{\xi_n\}$ in $(\Im, \rho)$ is called *Cauchy* if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\rho(\xi_n, \xi_j) \to 0$ as $n, j \to \infty$.

Definition 1.2.7. The line in R^n through two points ζ_1 and ζ_2 is the set of points ξ satisfying $\xi = \zeta_1 + \tau(\zeta_2 - \zeta_1)$, where τ is any real parameter. In set notation we can write $[\zeta_1, \zeta_2] = \{\xi : \xi = (1 - \tau)\zeta_1 + \tau\zeta_2, \ -\infty < \tau < \infty\}$.

Definition 1.2.8. A subset Ω of R^n is called *convex* if given any two points $\zeta_1, \zeta_2 \in \Omega$, the line segment $[\zeta_1, \zeta_2]$ joining these points is contained in Ω where $[\zeta_1, \zeta_2] = \{\zeta_1 : \zeta_1 = \lambda\zeta_1 + \gamma\zeta_2, \ \lambda \geq 0, \ \gamma \geq 0, \ \lambda + \gamma = 1\}$.

Lemma 1.2.9. If a set Δ is convex then the *closure* $\overline{\Delta}$ and *interior* Δ^0 are convex.

Example 1.2.10. (*i*) The *empty* set Φ, R^n and the *singleton* set $\{\zeta\}$ are convex.

(*ii*) Any *subspace* of a vector space is *convex*.

(*iii*) The two sets $U_1 = g \in V : |g(\zeta)| \leq 1$and $U_2 = \{f \in V : |f(\zeta)| < 1\}$ are *convex subspaces* of

$$V = \{f : [\zeta_1, \zeta_2] \to R : |f(\zeta)| \leq \alpha, \alpha \geq 0\}.$$

(*iv*) The two *unit balls* $\{\zeta \in \Im : |\zeta| \leq 1\}$ and $\{\zeta \in \Im : |\zeta| < 1\}$
in a normed linear space are convex.

(*v*) The upper *half-plane* $\{(\zeta_1, \zeta_2) \in R^2 : \zeta_2 > 0\}$ is convex.

(*vi*) The *open* or *closed* $\|.\|$-balls around any point ζ_0 that is $\{\zeta \in \Im : \|\zeta - \zeta_0\| < r\}$ and $\{\zeta \in \Im : \|\zeta - \zeta_0\| \leq r\}$ are convex in a norm linear space $\Im$.

Definition 1.2.11. A *cover* of a set Δ in a metric space $(\Im, \rho)$ is a *collection* F of sub-sets of such that $\Delta \subseteq \cup F$. If all the members of F are open sets, then F is an *open cover* of Δ. A *subcover* of F is a subcollection of that is also a cover of Δ. In other words, a collection of sets χ is a subcover of Δ, if and only if $\chi \subseteq F$ and $F \subseteq \cup G$. If a subcover has a finite number of member, then it is called a finite subcover.

Definition 1.2.12. A set $\Omega \subseteq (\Im, \rho)$ is *compact* if every open cover of Ω has a finite subcover. A set $\Delta \subset \Im$ is *precompact* if $\overline{\Delta}$ is compact.

Proposition 1.2.13. *Compact* sets in a metric $(\Im, \rho)$ are *closed*.

Definition 1.2.14. A *normed space* is a linear space L together with the norm function $\|.\| : L \to R^+$ defined on L such that the following axioms are satisfied:

(*i*) For all $\zeta \in L$, $\|\zeta\| \geq 0$.

(*ii*) For all $\zeta \in L$, $\|\zeta\| = 0$ iff $\zeta = 0$.

(*iii*) For $\zeta \in L$ and $\eta \in F$ we have $\|\eta\zeta\| = |\eta| \, \|\zeta\|$ (where $F = R$ or C).

(iv) For all $\zeta_1, \zeta_2 \in L$ we have $\|\zeta_1 + \zeta_2\| \leq \|\zeta_1\| + \|\zeta_2\|$.

Example 1.2.15. (i) The set of *real* and *complex* numbers $\mathbb{R}$ and $\mathbb{C}$ are one *dimensional* normed linear spaces with the absolute value norm $\|.\| = |.|$.

(ii) The set $\mathbb{R}^2$ may be with the norms$\|\xi\|_1$, $\|\zeta\|_2$ or $\|\zeta\|_\infty$.

(iii) For any compact set $\Omega \subset R$, the norm on the space $C(\Omega)$ is given by $\|f\| = \sup_{\zeta \in K} |f(\zeta)|$.

Proposition 1.2.16. If $(\Im, \|.\|)$ is a *normed vector space*, then $\rho : \Im \times \Im \to R^+$ given by $\rho(\zeta_1, \zeta_2) = \|\zeta_1 - \zeta_2\|$ is a metric space on $\Im$.

Definition 1.2.17. (i) The sequence $\{\xi_n\}$ in a normed space $\Im$ is called *convergent* if for some $\xi \in \Im$, $\lim_{n\to\infty} \|\xi_n - \xi\| = 0$.

(ii) The sequence $\{\xi_n\}$ in a normed space $\Im$ is said to be *Cauchy*

if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ and for $j, n > N$ we obtain $\|\xi_n - \xi_j\| \to 0$ as $j, m \to \infty$.

Definition 1.2.18. If $\Im$ together with ρ is a metric space, then *open, closed balls and sphere* with center Δ_0 and radius $r > 0$, are given by

$$\begin{aligned}
\delta(\Delta_0, r) &= \{\Delta \in \Im : \rho(\Delta, \Delta_0) < r\}, \\
\bar{\delta}(\Delta_0, r) &= \{\Delta \in \Im : \rho(\Delta, \Delta_0) \leq r\}, \\
\hat{\delta}(\Delta_0, r) &= \{\Delta \in \Im : \rho(\Delta, \Delta_0) = r\}.
\end{aligned}$$

Definition 1.2.19. If $\Im$ together with a metric ρ is a metric space, then

(i) A set $\Omega \subset \Im$ is called *open*, if for all $\zeta \in \Omega$ and $r > 0$, there exists open ball $\delta_r(x)$ such that $\delta_r(x) \subset \Omega$.

(ii) A set $\Omega \subset \Im$ is a *neighborhood* of $\zeta \in \Im$, if the open ball $\delta_r(\zeta) \subset \Omega$ for some $r > 0$.

(iii) The set $\Omega \subset \Im$, is *closed* if $\Omega^c = \{\zeta \in \Im : \zeta \notin \Omega\}$ is open,

(iv) A set $\Omega \subset \Im$ is *bounded*, if for $\zeta \in \Im$ and $0 \leq R < \infty$, with $\rho(\zeta_1, \zeta_2) \leq R$ for all $\zeta_2 \in \Omega$.

(v) Let $\Omega \subset \Im$ be non-empty, then the *diameter* of Ω is defined by $diam(S) = \sup_{\zeta_1, \zeta_2 \in S} \rho(\zeta_1, \zeta_2)$.

Definition 1.2.20. If $(\Im, \rho)$ is a metric space

and $\mathcal{F} \subset \Im$ is a subset of $\Im$ then

(i) A point $\zeta \in \mathcal{F}$ is an *interior point* of $\mathcal{F}$ if the ball $\delta_r(\zeta) \subset \mathcal{F}$ for some $r > 0$.

(ii) A point $\zeta \in \mathcal{F}$ is an *isolated point* of if $\delta_r(\zeta) \cap \mathcal{F} = \{\zeta\}$ for some $r > 0$.

(iii) A point $\zeta \in \Im$ is *boundary point* of $\mathcal{F}$ if for every $r > 0$,

the ball $\delta_r(\zeta)$ contains points in $\mathcal{F}$ and points not in $\mathcal{F}$.

(iv) A point $\zeta \in \Im$ is an *accumulation point* of $\mathcal{F}$ if for every $r > 0$,

the ball $\delta_r(\zeta)$ contains a point $\xi \in \mathcal{F}$ such that $\xi \neq \zeta$.

Definition 1.2.21. If $(\Im_1, \rho_1)$ and $(\Im_2, \rho_2)$ are metric spaces and $\mathcal{L} : \Im_1 \to \Im_2$ is a mapping, then $\mathcal{L}$ is *continuous* at a point $\mu_0 \in \Im_1$if for every $\in > 0$, there is a $\eta > 0$ such that $\rho(\mu_1, \mu_2) < \eta$ $\rho(\mathcal{L}(\mu_1), \mathcal{L}(\mu_2)) < \in$.

Definition 1.2.22. The map $\mathcal{L}$ is said to be *uniformly continuous* if for every $\in > 0$, there is a $\delta > 0$ such that $\rho(\mu_1, \mu_2) < \delta$ implies $\rho(\mathcal{L}(\mu_1), \mathcal{L}(\mu_2)) < \in$.

Definition 1.2.23. A subset $\mathcal{F} \subseteq (\Im, \rho)$ is called *totally bounded* if for every $\in > 0$, there is a finite number of balls $\delta(\xi_1, \in), \delta(\xi_2, \in), \ldots, (\xi_n, \in)$ of small radius $\in$ that covers $\mathcal{F}$ such that $\mathcal{F} \subseteq \delta(\xi_1, \in) \cup \delta(\xi_2, \in) \cup \delta(\xi_3, \in) \ldots \cup (\xi_n, \in)$.

Theorem 1.2.24. A subset of $\Omega \subseteq (\Im, \rho)$ is *compact* iff it is closed and *totally bounded.*

Definition 1.2.25. A topological space $\top$ is called *locally compact* if for each $\zeta \in \tau$, there is an open set u and a closed set Δ such that $\zeta \in u \subset \bar{\Delta}$, with $\bar{\Delta}$ compact.

Definition 1.2.26. Let $\Im$ is a metric space together with a metric ρ and $\in > 0$ is given, then a subset δ of $\Im$ is called *an* $\in -net$ if δ is finite and $\Im = \cup_{\zeta \subset \delta} \delta_{\in}(\zeta)$.

In the next section, we discuss some spaces of continuous functions over compact subsets of the real line $\mathbb{R}$ or complex plane $\mathbb{C}$ which forms a part of this dissertation with some basic concepts, definitions and related theorems widely used in fixed point theory and modern and classical analysis.

1.3 Spaces of Continuous Functions

The formal study of function spaces began with the work of Arzela [4] and Ascoli [5]. Their papers mark not only the beginning of the theory of function spaces but of general topology. The first space of functions to be investigated extensively was that of the space of continuous functions $C([\zeta_1, \zeta_2])$, on compact domain $[\zeta_1, \zeta_2]$, in the real line. The space of continuous functions is a known example of function spaces with each function continuous on the closed interval $[\zeta_1, \zeta_2]$. It is natural to understand that the most commonly used norm associated with this space is the supremum norm $\|g\|_\infty = \sup\{g(\zeta) : \zeta \in [\zeta_1, \zeta_2]\}$. The associated sup norm in the space of continuous functions is also a norm of uniform convergence for functional sequences in compact metric and topological spaces. The important aspect of uniform convergence is that it preserves the notion of uniform continuity. It is a well-known fact that the concept of compactness is central to both pure and applied mathematics. The idea of compactness for finite dimensional spaces are attributed to Heine-Boral, who succeeded in providing essential tools for the characterizations of compact subsets in n-dimensional Euclidean spaces on compact domains. Continuous functions and compact sets go well together. The Arzela-Ascoli theorem gives us necessary and sufficient criterion for the characterization of compact subsets of function spaces using the concept of equicontinuity. One can also use the Arzela-Ascoli theorem in proving Picard-Lindelof theorem for the existence of ordinary differential equations and also in the proof of Riemann mapping theorem in complex analysis. The Stone Weierstrass theorem is another important theorem for polynomial approximations of functions and has the advantage over Taylor expansion in that it does not require differentiability with the condition that the domain is a compact one. The space of continuous functions is also a good example of Banach algebra which is closed under multiplication.

It is impossible to list all function spaces with complete characterization in this section. Here we give examples of a limited number of function spaces and conclude the section with some related definitions and theorems for spaces of continuous functions on compact domains. The proofs of most of the results are being omitted which can be seen in the standard text books, for example see [1, 5, 45, 53].

Examples 1.3.1. (i) On a topological space the space $C(\tau)$ the set of all *continuous real-valued or complex* valued functions.

(ii) The space $C_c(\Omega)$ is a space of continuous functions with *compact support.*

(iii) $C_0(\Im)$ continuous functions which *vanish at infinity.*

(iv) On a metric space we have the notion of *Holder space* $C^{0,\sigma}$ with the Holder norm defined as $\sup_{\zeta,\xi\in\Im}\frac{|g(\zeta)-f(\xi)|}{d(\zeta,\xi)^{\sigma}}, \sigma \in (0,1)$.

(v) Space of *Lipscitz functions* $C^{0,1}$for $0 < \sigma < 1$.

(vi) On a differential we have the space $C^{\infty}(\Im)$ of continuous functions that *vanishes at infinity.*

(vii) $C_c^{\infty}(\Im)$ space of *smooth functions* with compact support.

$(viii)$ Space of continuous functions $C^j(\Im)$ of j-time *differentiable functions.*

(ix) *Schwartz function* spaces $S(\Im)$ of real or complex functions is a subspace of $C^{\infty}(\Im)$.

Definition 1.3.2. Suppose $\Omega \subseteq R$ and for each $n \in N$ there is a function $g_n : \Omega \to R$. The collection of functions $\{g_n : n \in N\}$ is a *sequence of functions* defined on Ω.

Definition 1.3.3. Let $\{g_n(\zeta)\}$ be a sequence of functions defined on a domain $\Omega = [\zeta_1, \zeta_2] \subseteq R$, then

(*i*) We say that $\{g_n(\zeta)\}$ converges *pointwise* to $g(\zeta)$ on Ω if for each $\zeta \in G$, $g_n(\zeta) \to g(\zeta)$ as $n \to \infty$, that is $|g_n(\zeta) - g(\zeta)| \to 0$ as $n \to \infty$ for all $\zeta \in \Omega$.

(*ii*) We say $\{g_n(\zeta)\}$ converges *uniformly* to $g(\zeta)$ on Ω if $\sup |g_n(\zeta) - g(\zeta)| \to 0$ as $n \to \infty$.

(*iii*) A sequence of functions $g_n(\zeta) : \Omega \to R$ is *uniformly Cauchy* on Ω if for every $\in > 0$, $\exists\ N \in \mathbb{N}$ and $j, n > N$ implies $|g_j(\zeta) - g_n(\zeta)| < \in$ for all $\zeta \in \Omega$.

Remark 1.3.4. Every *uniformly convergent* sequence is *pointwise convergent* and the uniform limit function is same as the pointwise limit. But the converse is not true.

Theorem 1.3.5 (Dini's Theorem). If $\{g_n\}$ is a sequence of real-valued continuous functions converging pointwise to a continuous limit function g on a compact set $\overline{\Delta}$ and if $g_n(\zeta) \geq g_{n+1}(\zeta)$ for each $\zeta \in \overline{\Delta}$ and every $n = 1, 2, 3, \ldots$, then $g_n \to g$ uniformly on $\overline{\Delta}$.

Definition 1.3.6. A series of functions $\sum_{n=1}^{\infty} g_n(\zeta)$ *converges uniformly* to a function $g(\zeta)$ on $\Omega = [\zeta_1, \zeta_2]$ if the sequence of its partial sum $\{S_n\}$ given by $S_n(\zeta) = \sum_{j=1}^{n} g_j(\zeta)$ converges uniformly on $[\zeta_1, \zeta_2]$.

Definition 1.3.7. A series of functions $\sum_{n=1}^{\infty} g_n(\zeta)$ converges uniformly to g on $\Omega = [\zeta_1, \zeta_2]$ if for $\in > 0$ and all $\zeta \in \Omega$, there is an integer N independent of ζ and dependent on $\in$ such that $|g_1(\zeta) + g_2(\zeta) + \ldots + g_n(\zeta) - g(\zeta)| < \in$ for $n \geq N$.

Theorem 1.3.8 (Cauchy's Criterion and Uniform Convergence). The sequence of functions $\{g_n(\zeta)\}$ *converges uniformly* to $g(\zeta)$ on $[\zeta_1, \zeta_2]$ if and only if for every $\in > 0$ and for every $\zeta \in [\zeta_1, \zeta_2]$ there exists an N so that $|g_{n+p}(\zeta) - g_n(\zeta)| < \in$ and $n \geq N$, $p \geq 1$.

Weierstrass M-Test 1.3.9. Suppose $\{g_n(\zeta)\}$ is a sequence of functions $g_n : \Omega \to R$ with $g : \Omega \to R$ and that $\lim_{n\to\infty} g_n(\zeta) = g(\zeta)$ for some $\zeta \in \Omega$.

Suppose that $K_n = \sup_{\zeta\in\Omega} |g_n(\zeta) - g(\zeta)|$ exists for all n, then $\{g_n(\zeta)\}$ converges uniformly to g on Ω if and only if $K_n = 0$.

Theorem 1.3.10 For a real-valued, continuous function g defined on a closed interval $[\zeta_1, \zeta_2]$, there exists a sequence of real polynomials $\{P_n(\zeta)\}$ such that $\lim_{n\to\infty} P_n(\zeta) = g(\zeta)$ *uniformly* on $[\zeta_1, \zeta_2]$.

Definition 1.3.11. A family F of functions is said to be *compact* if the limits of all the converging sequences of functions of F are functions belonging to F. The family F is called *conditionally* compact on a subset $\Omega \subseteq R$ if every sequence $\{g_n\} \in F$ contains a subsequence $\{g_{n_k}\}$ which is uniformly convergent on every compact subset of Ω.

Definition 1.3.12. If $\Delta \subseteq R$ is compact domain such that $F \subset \overline{\Delta}$:

(*i*) The family F is *normal family* in Δ if and only if each sequence of members of F has a subsequence which converges uniformly on each compact subset of $\overline{\Delta}$.

(*ii*) The family of functions $F \subset \overline{\Delta}$ is said to be *uniformly bounded* if there exist an $\mathrm{F} \in X$ such that $\{g_n\}$ is bounded by F for each $n = 0, 1, 2...$

(*iii*) The family of functions F is *pointwise bounded* if for each $\zeta_0 \in \Omega$, the set $\{g(\zeta_0) : g \in K\}$ is bounded.

(*iv*) The family of functions $F \subset C(\Delta)$ is called *equicontinuous*. If for every $\in > 0$ there is a $\delta > 0$ with $|g(\zeta_1) - g(\zeta_2)| < \in$ and $\zeta_1, \zeta_2 \in X$ satisfying $|\zeta_1 - \zeta_2| < \delta$, $g \in F$.

Theorem 1.3.13 (Arzela-Ascoli Theorem) [74]. Let $\{g_n\}_{n=1}^{\infty}$ be a uniformly equicontinuous family of uniformly bounded functions on $[\zeta_1, \zeta_2]$. Then every *subsequence* $\{g_{n_k}\}$ of $\{g_n\}$ *converges uniformly* to g on $[\zeta_1, \zeta_2]$.

Theorem 1.3.14 (Peano) [74]. Let $g(\zeta, \xi)$ be continuous in a domain Ω of the plane in $R^2$and let (ζ_0, ξ_0) belong to the interior of Ω. Then there is a small $k > 0$ and a function $g(\zeta)$ continuously differentiable on $|\zeta - \zeta_0| < k$ such that $(\zeta, g(\zeta))$ remains in Ω for $|\zeta - \zeta_0| < k$ and $g(\zeta)$ is one solution of the problem with given inetial comdition $\frac{d\xi}{d\zeta} = g(\zeta, \xi), g(\zeta_0) = \xi_0$.

Definition 1.3.15. Let $F > 0$ be a constant and g be a function define in a domain Ω of the R^2-plane. A *Lipchitz condition* is the inequality $|g(\zeta, \xi_1) - g(\zeta, \xi_2)| \leq F|\xi_1 - \xi_2|$, assumed to hold for all (ζ, ξ_1) and(ζ, ξ_2) in Ω.

Theorem 1.3.16 (Picard Theorem). Let the initial value problem be given by the equations $\xi' = g(\zeta, \xi), \xi(\zeta_0) = \xi_0$. Suppose $g(\zeta, \xi)$ and $\frac{\partial g}{\partial \xi}(\zeta, \xi)$ are continuous in some open rectangular region $\varpi = \{(\zeta, \xi) : \zeta_1 < \zeta < \zeta_2, \xi_1 < \xi < \xi_2\}$ containing (ζ_0, ξ_0). Then, the initial value problem possesses unique solution in some closed interval $J = [\zeta_0 + k, \xi_0 + k]$, where $k > 0$. Moreover, the *Piterations* defined by $\xi_{n+1}(\zeta) = \xi_0 + \int_{\zeta_0}^{\zeta} g(t, \xi_n(t))\, dt$, produces a sequence of functions $\{\xi_n(\zeta)\}$ that converge to this solution uniformly on J.

A lot of literature is available on the extension, generalization and improvement of Banach contraction mapping. In the next section, we give some preliminary concepts and generalizations of Banach fixed point theorem by some authors.

1.4 Fixed Point Theorems with Some Basic Concepts

Definition 1.4.1. Let $\Im$ be non-empty and $g : \Im \to \Im$ is a mapping of such that $g(\zeta) = \zeta$, then ζ is called *fixed point* of g in $\Im$.

Theorem 1.4.2. If a mapping $g(\zeta)$ is *defined and continuous* on a closed interval $[\zeta_1, \zeta_2]$ and $g(\zeta) \in [\zeta_1, \zeta_2]$ for all $\zeta \in [\zeta_1, \zeta_2]$. Then the point $v \in [\zeta_1, \zeta_2]$ is such that $\Im(v) = v$.

Definition 1.4.3. The mapping $g : \Im \to \Im$ is called a *contraction* if $\rho(g(\zeta), g(\xi)) \leq \sigma\rho(\zeta, \xi)$,for some $0 \leq \sigma < 1$ and all $\zeta, \xi \in \Im$.

Lemma 1.4.4. A *continuous* mapping $g : \Im \to \Im$ is always *contraction* mapping.

Theorem 1.4.5 [10]. If $(\Im, \rho)$ is *complete* and $g : \Im \to \Im$ is mapping of $\Im$ then g has *unique fixed* point in $\Im$.

Proposition 1.4.6. If $J = [\zeta_1, \zeta_2] \subseteq R$ is a closed and bounded interval in R with $g : [\zeta_1, \zeta_2] \to [\zeta_1, \zeta_2]$, a *continuously differentiable* function with $|g'(\zeta)| < 1$, for all $\zeta \in J$, then g is a *contraction.*

Definition 1.4.7. The map $g : \Im \to \Im$ is called

(*i*) *Lipschitz if* $\exists\ \sigma > 0$ such that for $\zeta, \xi \in \Im$ implies $\rho(g(\zeta), g(\xi)) \leq \sigma\rho(\zeta, \xi)$,

(*ii*) *Banach contraction,* if there exists $\sigma \in [0, 1)$ such that for all $\zeta, \xi \in \Im$, we have $\rho(g(\zeta), g(\xi)) \leq \sigma\rho(\zeta, \xi)$,

(*iii*) *Non-expensive,* if for all $\zeta, \xi \in \Im$, $\rho(g(\zeta), g(\xi)) \leq \rho(\zeta, \xi)$,

(*iv*) *Contractive,* if for all $\zeta, \xi \in \Im$ with $\zeta \neq \xi$, $\rho(g(\zeta), g(\xi)) < \rho(\zeta, \xi)$.

Definition 1.4.8 [23]. If $(\Im, \rho)$ is a complete. Then the map $\mathbf{F} : \Im \to \Im$ is called a *Chatterjea mapping* or *Chatterjea contraction* if for all $\zeta_1, \zeta_2 \in \Im$ and $\sigma \in [0, \frac{1}{2}) \Rightarrow \rho(\mathbf{F}(\zeta_1), \mathbf{F}(\zeta_2)) \leq \sigma(\rho(\zeta_1, \mathbf{F}(\zeta_2)) + \rho(\zeta_1, \mathbf{F}(\zeta_2)))$.

Definition 1.4.9 [26]. If $\mathbf{F} : \Im \to \Im$ is a mappig. Then $\mathbf{F}$ is called a *Ciric-contraction* if for $\xi_1, \xi_2 \in \Im$ and $0 \leq \sigma < 1$ satisfies

$$\rho(\mathbf{F}(\xi_1), \mathbf{F}(\xi_2)) \leq \sigma \max\{\rho(\xi_1, \xi_2), \rho(\xi_1, \mathbf{F}(\xi_1)), \rho(\xi_2, \mathbf{F}(\xi_2)), \rho(\xi_1, \mathbf{F}(\xi_2)), \rho(\xi_1, \mathbf{F}(\xi_2))\}.$$

Definition 1.4.10 [46]. If $g_1, g_2 : \Im \to \Im$ are mappings. Then, g_1 and g_2 are said to be (*i*) *Commuting* if $g_1(g_2(\zeta)) = g_2(g_1(\zeta))$ for $\zeta \in X$,

(*ii*) *Weakly commuting,* if $\rho(g_1 g_2 \zeta, g_2 g_1 \zeta) \leq \rho(g_1(\zeta), g_2(\zeta))$,

(*iii*) *Compatible,* if $\lim_{n\to\infty} \rho(g_1 g_2 \zeta_n, g_2 g_1 \zeta_n) = 0$ for $\{\zeta_n\}$ in $\Im$ such that $\lim_{n\to\infty} g_1 \zeta_n = \lim_{n\to\infty} g_2 \zeta_n = v$, and $v \in \Im$,

(*iv*) If they commute at coincidence point such that $g_1 v = g_2 v$, then they are called weakly compatible.

Definition 1.4.11 [9]. If $\Im$ is non-empty and $\sigma \geq 1$. A function $\rho : \Im \times \Im \to [0, \infty)$ is called *b-metric* space iff for $\eta_1, \eta_2, \eta_3 \in \Im$ the following conditions are holds:

(*i*) $\rho(\eta_1, \eta_2) = 0 \Leftrightarrow \eta_1 = \eta_2$,

(*ii*) $\rho(\eta_1, \eta_2) = \rho(\eta_2, \eta_1)$,

(*iii*) $\rho(\eta_1, \eta_2) \leq \sigma(\rho(\eta_1, \eta_2) + \rho(\eta_2, \eta_3))$, then the pair $(\Im, \rho)$ is called b-metric space.

Definition 1.4.12 [46]. Two mappings $\mathbf{F}_1$, $\mathbf{F}_2 : \Im \to \Im$ satisfying the conditions $\rho(\mathbf{F}_1(\xi_1), \mathbf{F}_1(\xi_2)) \leq \sigma\rho(\mathbf{F}_2(\xi_1), \mathbf{F}_2(\xi_2))$ for $\xi_1, \xi_2 \in \Im$ and $\sigma \in [0, 1)$ is called *Jungck contraction.*

Theorem 1.4.13 [33]. The map $\mathbf{F} : \Im \to \Im$ is called contractive if $\rho(\mathbf{F}(\zeta_1), \mathbf{F}(\zeta_2)) < \rho(\zeta_1, \zeta_2)$, ζ_1, $\zeta_2 \in \Im$, $\zeta_1 \neq \zeta_2$, then $\mathbf{F}$ has a unique fixed point in $\Im$ if the iterative sequence $\{\mathbf{F}^n(\zeta)\}$ converges to the unique fixed point of $\Im$.

Theorem 1.4.14 [8]. If $\mathbf{F}$: $\Im \to \Im$ is contractive and for every $\alpha_1, \alpha_2 \in \Im$, $\alpha_1 \neq \alpha_2$ and $\delta = \delta(\alpha_1, \alpha_2)$ satisfy $\rho\left(g^{\delta(\alpha_1,\alpha_2)}, g^{\delta(\alpha_1,\alpha_2)}\right) < \rho(\alpha_1, \alpha_2)$ then $\mathbf{F}$ has fixed point in $\Im$.

Theorem 1.4.15 [34] If $\mathbf{F} : \Im \to \Im$ is continuous and satisfies

$$\rho(\mathbf{F}\delta_1, \mathbf{F}\hbar_2) < \frac{1}{2}(\rho(\delta_1, \mathbf{F}\delta_1) + \rho(\delta_2, \mathbf{F}\delta_2))$$

for some $\delta_1, \delta_2 \in \Im$, $\delta_1 \neq \delta_2$,then $\mathbf{F}$ has a fixed point in $\Im$.

Theorem 1.4.16 [59]. If $\mathbf{F} : \Im \to \Im$ satisfies the inequality

$$(\rho(\mathbf{F}\delta_1, \mathbf{F}\delta_2))^2 \leq \alpha_1 (\rho(\delta_1, \mathbf{F}\delta_1)\, \rho(\delta_2, \mathbf{F}\delta_2) + \rho(\delta_1, \mathbf{F}\delta_2)\, \rho(\delta_2, \mathbf{F}\delta_1)) + \alpha_2 (\rho(\delta_1, \mathbf{F}\delta_1)\, \rho(\delta_2, \mathbf{F}\delta_1) + \rho(\delta_1, \mathbf{F}\delta_2)\, \rho(\delta_2, \mathbf{F}\delta_2)),$$

for δ_1, δ_2 in $\Im$ where α_1, $\alpha_2 \geq 0$ and $\alpha_1 + 2\alpha_2 < 1$, then g possesses a fixed point.

Chapter 2

Generalization of Fixed Point Results in Compact and Pseudo-Compact-Tichnov Spaces

2.1 Generalized Fixed Point Results in Compact Metric Spaces

In the second chapter, we have generalized some fixed point theorems in the settings of compact and pseudo-compact tichnovo spaces. The present chapter is divided into two sections namely, fixed point results for contractive mappings in (i) compact and (ii) pseudo-compact tichnov spaces.

In section first of the 2nd chapter, we have proved several fixed point results on compact metric spaces using the conditions of contractive mapping. The obtained results are generalizations of some theorems by Bailey [8], Edelstein [33] and Fisher [34].

Before starting the main results first, we are giving some fundamental results.

Theorem 2.1.1 [33]. If $\pounds : \Im \to \Im$ satisfies

$$\rho(\pounds\mu_1, \pounds\mu_2) < \rho(\mu_1, \mu_2), \tag{2.1}$$

for $\mu_1, \mu_2 \in \Im$, $\mu_1 \neq \mu_2$, then $\pounds$ has a unique fixed point in $\Im$.

Theorem 2.1.2 [8]. If $(\Im, \rho)$ compact with the metic ρ and $\pounds$ is a continuous self-map of $\Im$ such that for $\xi_1, \xi_2 \in \Im$, $\xi_1 \neq \xi_2$ satisfy the condition

$$\rho\left(\pounds^{\delta(\xi_1,\xi_2)}, \pounds^{\delta(\xi_1,\xi_2)}\right) < \rho(\xi_1, \xi_2), \tag{2.2}$$

where $\delta = \delta(\xi_1, \xi_2)$ is a positive integer, then $\pounds$ has a unique fixed point in $\Im$.

Theorem 2.1.3 [34]. If $\pounds : \Im \to \Im$ satisfy the condition

$$\rho(\pounds\mu_1, \pounds\mu_2) < \frac{1}{2}\left(\rho(\mu_1, \pounds\mu_1) + \rho(\mu_2, \pounds\mu_2)\right), \tag{2.3}$$

for some $\mu_1, \mu_2 \in \Im$, $\mu_1 \neq \mu_2$, then $\pounds$ has a unique fixed point in $\Im$.

Theorem 2.1.4 [34]. If $\pounds : \Im \to \Im$ is a map with $\Im$ compact and satisfying the condition given by

$$\rho(\pounds\mu_1, \pounds\mu_2) < \frac{1}{2}\left(\rho(\mu_1, \pounds\mu_2) + \rho(\mu_2, \pounds\mu_1)\right), \tag{2.4}$$

for some $\mu_1, \mu_2 \in \Im$, $\mu_1 \neq \mu_2$,then $\pounds$ has fixed point in $\Im$.

2.2 Main Results

Theorem 2.2.1. If $\pounds : \Im \to \Im$ is continuous with the metric ρ such that $(\Im, \rho)$ is compact metric space and $\pounds$ satisfying the condition:

$$\begin{aligned}\rho(\pounds(\alpha_1), \pounds(\alpha_2)) &< \eta_1\left(\frac{\rho(\alpha_1, \pounds(\alpha_1))\,\rho(\alpha_2, \pounds(\alpha_2))}{\rho(\alpha_1, \alpha_2)}\right) + \eta_2\rho(\alpha_2, \pounds(\alpha_1)) + \rho(\alpha_2, \pounds(\alpha_2)) + \\ &\quad \eta_3(\rho(\alpha_1, \pounds(\alpha_1)) + \rho(\alpha_2, \pounds(\alpha_2))) + \eta_4\rho(\alpha_1, \pounds(\alpha_2)) + \rho(\alpha_2, \pounds(\alpha_1)) \\ &\quad +\eta_5\rho(\alpha_1, \alpha_2),\end{aligned}$$

for some $\alpha_1, \alpha_2 \in \Im$, $\alpha_1 \neq \alpha_2$, where $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$ are nonnegative real numbers such that $\eta_1 + \eta_2 + 2(\eta_3 + \eta_4) + \eta_5 < 1$, then $\pounds$ has unique fixed point in $\Im$.

Proof First we define a function λ on $\Im$, such that $\lambda(\alpha_1) = \rho(\alpha_1, \pounds\alpha_1)$ for all $\alpha_1 \in \Im$. Since ρ and $\pounds$ are continuous on $\Im$, therefore, λ is also continuous on $\Im$. Since, $\Im$ is compact there a point $\varphi \in \Im$ such that $\lambda(\phi) = \inf\{\lambda(\alpha_1) : \alpha_1 \in \Im\}$. If $\lambda(\phi) \neq 0$, it follows that $\pounds\phi \neq \phi$ and

$$\begin{aligned}\lambda(\pounds\varphi) &= \rho(\pounds(\phi), \pounds(\pounds(\varphi))) < \eta_1\left(\frac{\rho(\phi, \pounds(\phi))\,\rho(\pounds(\varphi), \pounds(\pounds(\phi)))}{\rho(\phi, \pounds(\phi))}\right) + \\ &\quad +\eta_2(\rho(\pounds(\phi), \pounds(\phi)) + \rho(\pounds(\phi), \pounds(\pounds(\phi)))), \\ &\quad +\eta_3(\rho(\varphi, \pounds(\varphi)) + \rho(\pounds(\varphi), \pounds(\pounds(\varphi)))) \\ &\quad +\eta_4(\rho(\pounds(\phi), \pounds(\pounds(\phi))) + \rho(\pounds(\phi), \pounds(\phi))) + \eta_5\rho(\phi, \pounds(\phi)), \\ &\Rightarrow \rho(\pounds(\phi), \pounds(\pounds(\varphi))) < (\eta_1 + \eta_2 + \eta_3 + \eta_4)\,\rho(\pounds(\phi), \pounds(\pounds(\phi))) \\ &\quad (\eta_3 + \eta_4 + \eta_5)\,\rho(\phi, \pounds(\phi)), \\ &\Rightarrow \rho(\pounds(\phi), \pounds(\pounds(\phi))) < \frac{(\eta_3 + \eta_4 + \eta_5)}{(1 - (\eta_1 + \eta_2 + \eta_3 + \eta_4))}\rho(\phi, \pounds(\varphi)), \\ &\Rightarrow \rho(\pounds(\phi), \pounds(\pounds(\varphi))) < \rho(\varphi, \pounds(\phi)),\end{aligned}$$

because $\eta_1 + \eta_2 + 2(\eta_3 + \eta_4) + \eta_5 < 1$. Hence, $\lambda(\pounds(\phi)) < \lambda(\varphi)$ and this contradicts the definition of ϕ and condition (2.1), therefore, $\phi = \pounds(\phi)$ and ϕ is fixed point of $\pounds$.

Next, to prove uniqueness, suppose $\phi_1 \neq \phi_2$ is another fixed point of $\pounds$, then we have

$$\begin{aligned}
\rho(\varphi_1, \varphi_2) &= \rho(\pounds(\phi_1), \pounds(\phi_2)) < \eta_1 \left(\frac{\rho(\phi_1, \pounds(\phi_2)) \rho(\phi_2, \pounds(\phi_2))}{\rho(\varphi_1, \varphi_2)} \right), \\
&+\eta_2 (\rho(\phi_2, \pounds(\phi_1)) + \rho(\phi_2, \pounds(\phi_2))) + \eta_3 (\rho(\phi_1, \pounds(\phi_2)) + \rho(\phi_2, \pounds(\varphi_1))) \\
&+\eta_4 (\rho(\phi_1, \pounds(\phi_1)) + \rho(\phi_2, \pounds(\phi_2))) + \eta_5 \rho(\phi_1, \phi_2), \\
&\Rightarrow \rho(\phi_1, \varphi_2) < \eta_1 \left(\frac{\rho(\phi_1, \pounds(\phi_1)) \rho(\phi_2, \pounds(\phi_2))}{\rho(\varphi_1, \varphi_2)} \right), \\
&\Rightarrow \rho(\phi_1, \phi_2) < \eta_2 \rho(\phi_1, \phi_2) + \eta_5 \rho(\phi_1, \phi_2), \\
&\Rightarrow (1 - (\eta_2 + \eta_5)) \ \rho(\phi_1, \phi_3) < 0,
\end{aligned}$$

which is a contradiction, because $\eta_1 + \eta_2 + 2(\eta_3 + \eta_4) + \eta_5 < 1$. So that ϕ is fixed point of $\pounds$ in $\Im$.

Remark 2.2.2. Put $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 0$, $\eta_5 = 1$, in Theorem 2.2.1, we get result of Edelstein [33].

Corollary 2.2.3. If $\Im$ is compact and $\pounds : \Im \to \Im$ satisfies

$$\begin{aligned}
\rho(\pounds(\alpha_1), \pounds(\alpha_2)) &< \eta_1 \left(\frac{\rho(\alpha_1, \pounds(\alpha_1)) \rho(\alpha_2, \pounds(\alpha_2))}{\rho(\alpha_1, \alpha_2)} \right) + \eta_2 (\rho(\alpha_2, \pounds(\alpha_1)) + \rho(\alpha_2, \pounds(\alpha_2))) \\
&+ \eta_4 (\rho(\alpha_1, \pounds(\alpha_2)) + \rho(\alpha_2, \pounds(\alpha_1))) \ + \eta_5 \rho(\alpha_1, \alpha_2),
\end{aligned}$$

for $\alpha_1, \alpha_2 \in \Im$, $\alpha_1 \neq \alpha_2$, where $\eta_1, \eta_2, \eta_4, \eta_5$ are nonnegative real numbers such that $\eta_1 + \eta_2 + 2\eta_4 + \eta_5 < 1$, then $\pounds$ has a unique fixed point.

Corollary 2.2.4. If $\Im$ is compact and $\pounds : \Im \to \Im$ satisfies

$$\begin{aligned}
\rho(\pounds(\alpha_1), \pounds(\alpha_2)) &< \eta_1 \left(\frac{\rho(\alpha_1, \pounds(\alpha_1)) \rho(\alpha_2, \pounds(\alpha_2))}{\rho(\alpha_1, \alpha_2)} \right) + \eta_2 (\rho(\alpha_2, \pounds(\alpha_1)) + \rho(\alpha_2, \pounds(\alpha_2))) \\
&+\eta_3 (\rho(\alpha_1, \pounds(\alpha_1)) + \rho(\alpha_2, \pounds(\alpha_2))) + \eta_5 \rho(\alpha_1, \alpha_2),
\end{aligned}$$

for all $\alpha_1, \alpha_2 \in \Im, \alpha_1 \neq \alpha_2$, where $\eta_1, \eta_2, \eta_3, \eta_5$ are nonnegative real numbers such that $\eta_1 + \eta_2 + 2\eta_3 + \eta_5 < 1$, then $\pounds$ has a unique fixed point.

Corollary 2.2.5. If $\Im$ is compact and $\pounds$ self-map of $\Im$ satisfying

$$\begin{aligned}\rho(\pounds(\alpha_1), \pounds(\alpha_2)) &< \eta_1 \left(\frac{\rho(\alpha_1, \pounds(\alpha_1)) \rho(\alpha_2, \pounds(\alpha_2))}{\rho(\alpha_1, \alpha_2)} \right) + \eta_2 (\rho(\alpha_2, \pounds(\alpha_1)) + \rho(\alpha_1, \pounds(\alpha_2))) \\ &+ \eta_5 \rho(\alpha_1, \alpha_2),\end{aligned}$$

for $\alpha_1, \alpha_2 \in \Im$, $\alpha_1 \neq \alpha_2$, where η_1, η_2, η_5 are nonnegative real numbers such that $\eta_1 + \eta_2 + \eta_5 < 1$, then $\pounds$ admit a fixed point.

Corollary 2.2.6. If $\Im$ is compact and $\pounds$ is self-map of $\Im$ satisfying

$$\rho(\pounds(\alpha_1), \pounds(\alpha_2)) < \eta_1 \left(\frac{\rho(\alpha_1, \pounds(\alpha_1)) \rho(\alpha_2, \pounds(\alpha_2))}{\rho(\alpha_1, \alpha_2)} \right) + \eta_5 \rho(\alpha_1, \alpha_2),$$

for $\alpha_1, \alpha_2 \in \Im, \alpha_1 \neq \alpha_2$, where η_1, η_5 are nonnegative real numbers such that $\eta_1 + \eta_5 < 1$, then $\pounds$ has a unique fixed point.

Example 2.2.7. If $\Im = \{0, 1, 3\}$is non-empty and with the usual metric $\rho : \Im \times \Im \to R^+$ and $\pounds$ on $\Im$ is given by $\pounds 0 = \pounds 3 = 1, \pounds 1 = 1$. Then it is easy to see that Example 2.2.8 satisfy Theorems 2.2.1 with 1 as the unique fixed point of $\pounds$ in $\Im$.

Theorem 2.2.8. If $\pounds$ is a continuous mapping as in Theorem 2.2.1 with $0 \leq \eta_1 + \eta_2 + 2(\eta_3 + \eta_4) + \eta_5 < 1$, and satisfy the inequality given below

$$\begin{aligned} \rho(\pounds(\mu_1), \pounds(\mu_2)) < \; & \eta_1 \left(\frac{\rho(\mu_1, \pounds(\mu_1))\, \rho(\mu_2, \pounds(\mu_2))}{\rho(\mu_1, \mu_2)} \right) + \\ & \eta_2 \left(\frac{\rho(\mu_1, \pounds(\mu_1)) + \rho(\mu_2, \pounds(\mu_1))}{1 + \rho(\mu_1, \pounds(\mu_1))\, \rho(\mu_2, \pounds(\mu_1))} \right) \\ & + \eta_3 \left(\frac{\rho(\mu_1, \mu_2) + \rho(\mu_1, \pounds(\mu_1)) + \rho(\pounds(\mu_1), \pounds(\mu_2))}{1 + \rho(\mu_1, \mu_2)\, \rho(\mu_1, \pounds(\mu_1))\, \rho(\pounds(\mu_1), \pounds(\mu_2))} \right) \\ & + \eta_4 \left(\frac{\rho(\mu_1, \mu_2) + \rho(\mu_1, \pounds(\mu_1))\, \rho(\mu_2, \pounds(\mu_1)) + \rho(\pounds(\mu_1), \pounds(\mu_2))}{1 + \rho(\mu_1, \mu_2)\, \rho(\mu_1, \pounds(\mu_1))\, \rho(\mu_2, \pounds(\mu_1))\, \rho(\pounds(\mu_1), \pounds(\mu_2))} \right) + \\ & \eta_5 \rho(\mu_1, \mu_2), \end{aligned}$$

then for every $\phi \in \Im$, $\{\pounds^n \phi\}$ possesses a fixed point of $\pounds$.

Proof By Theorem 2.2.1, $\pounds$ has a unique fixed point (say) ϕ_0 in $\Im$. Now for each $n = 0, 1, 2, ...,$ define $\rho_n = \rho(\pounds^n \phi, \phi_0)$ for every $\varphi \in \Im$, $\phi \neq \phi_0$. We consider the following two cases:

Case 1. If $\rho_n = 0$ for some n, then $\pounds^j \phi = \phi_0$ for some $j \geq n$ and hence the sequence $\{\pounds^n \phi\}$ converges to ϕ_0.

Case 2. If $\rho_n \neq 0$ for each n, then $\rho_{n+1} = \rho(\pounds^{n+1}\phi, \phi_0) = \rho(\pounds^{n+1}\phi, \pounds^{n+1}\phi_0)$ and we use

$$
\begin{aligned}
\rho_{n+1} &< \eta_1 \left(\frac{\rho\left(\mathcal{L}^n(\phi_0), \mathcal{L}^{n+1}(\phi_0)\right)\rho\left(\mathcal{L}^n(\phi), \mathcal{L}^{n+1}(\phi)\right)}{\mu_1} \right) + \\
&\quad \eta_2 \left(\frac{\rho\left(\mathcal{L}^n(\phi_0), \mathcal{L}^{n+1}(\phi_0)\right) + \rho\left(\mathcal{L}^n(\varphi), \mathcal{L}^{n+1}(\phi_0)\right)}{1+\mu_2} \right) + \\
&\quad \eta_3 \left(\frac{\begin{array}{c} \rho\left(\mathcal{L}^n(\phi_0), \mathcal{L}^n(\phi)\right) + \rho\left(\mathcal{L}^n(\varphi_0), \mathcal{L}^{n+1}(\phi_0)\right) + \\ \rho\left(\mathcal{L}^{n+1}(\phi_0), \mathcal{L}^{n+1}(\phi)\right) \end{array}}{1+\mu_3} \right) + \\
&\quad \eta_4 \left(\frac{\begin{array}{c} \rho\left(\mathcal{L}^n(\phi_0), \mathcal{L}^n(\phi)\right) + \\ \rho\left(\mathcal{L}^n(\varphi_0), \mathcal{L}^{n+1}(\phi_0)\right)\rho\left(\mathcal{L}^n(\phi), \mathcal{L}^{n+1}(\phi_0)\right) + \\ \rho\left(\mathcal{L}^{n+1}(\phi_0), \mathcal{L}^{n+1}(\phi)\right) \end{array}}{1+\mu_4} \right) + \\
&\quad +\eta_5 \rho\left(\mathcal{L}^n(\phi_0), \mathcal{L}^n(\phi)\right), \\
&\leq \eta_1(0) + \eta_2\rho_n + \eta_3\left(\rho_n + \rho_{n+1}\right) + \eta_4\left(\rho_n + \rho_{n+1}\right) + \eta_5\rho_n, \\
\rho_{n+1} &< (\eta_3+\eta_4)\rho_{n+1} + (\eta_2+\eta_3+\eta_4+\eta_5)\rho_n, \\
(1-(\eta_3+\eta_4))\rho_{n+1} &< (\eta_2+\eta_3+\eta_4+\eta_5)\rho_n, \qquad (2.6) \\
\rho_{n+1} &< \frac{(\eta_2+\eta_3+\eta_4+\eta_5)}{(1-(\eta_3+\eta_4))}\rho_n, \\
\rho_{n+1} &< \rho_n,
\end{aligned}
$$

where

$$0 \leq \mu_1 + \mu_2 + 2(\mu_3+\mu_4) + \mu_5 < 1,$$

and $\mu_1, \mu_2, \mu_3, \mu_4$ are given as under

$$\begin{aligned}
\mu_1 &= \rho\left(\pounds^n(\phi_0), \pounds^n(\phi)\right), \\
\mu_2 &= 1+\rho\left(\pounds^n(\varphi_0), \pounds^{n+1}(\varphi_0)\right)\rho\left(\pounds^n(\varphi), \pounds^{n+1}(\varphi_0)\right), \\
\mu_3 &= 1+\rho\left(\pounds^n(\phi_0), \pounds^n(\phi)\right)\rho\left(\pounds^n(\phi_0), \pounds^{n+1}(\phi_0)\right)\rho\left(\pounds^{n+1}(\phi_0), \pounds^{n+1}(\phi)\right), \\
\mu_4 &= 1+\rho\left(\pounds^n(\varphi_0), \pounds^n(\phi)\right)\rho\left(\pounds^n(\phi_0), \pounds^{n+1}(\phi_0)\right)\rho\left(\pounds^n(\phi), \pounds^{n+1}(\phi_0)\right), \\
& \quad \rho\left(\pounds^{n+1}(\phi_0), \pounds^{n+1}(\phi)\right).
\end{aligned}$$

Hence, $\{\rho_n\}$ is a non- increasing and hence converges to a real $c \geq 0$, which is the greatest lower bound of the sequence $\{\rho_n\}$. By the compactness of $\Im$, the sequence $\{\pounds^n(\varphi)\}$ has a convergent subsequence $\{\pounds^{n_k}(\phi)\}$ which converges to $\psi \in \Im$ (say). Since $\pounds$ is continuous, $\pounds^{n_{k+1}}(\varphi) = \pounds\left(\pounds^{n_k}(\varphi)\right) \to \pounds\psi$ as $k \to \infty$. By the continuity of the metric ρ, letting $k \to \infty$, then $\rho_{n_k} = \rho\left(\pounds^{n_k}(\phi), \phi_0\right) \to \rho(\pounds\psi, \phi_0) = l$, where the sequence $\left\{\rho_{n_k}\right\}$ is a subsequence of $\{\rho_n\}$. Since the sequence $\left\{\rho_{n_{k+1}}\right\}$ is a subsequence of $\{\rho_n\}$, so

$$l = \rho(\psi, \phi_0) = \rho(\pounds\psi, \phi_0). \tag{2.7}$$

Now, we claim $l = 0$. Suppose $l \neq 0$. Then $\psi \neq \phi_0$. By (2.6), we get $\rho(\pounds\psi, \phi_0) = \rho(\pounds\psi, \pounds\phi_0) < \rho(\psi, \phi_0)$, which contradicts (2.7). Hence, $\psi = \phi_0$ which means $l = \rho(\psi, \phi_0) = 0$. This shows $\{\rho_n\} \to 0$ as $n \to \infty$ and this complete the proof of the theorem.

Theorem 2.2.9. If $\pounds$ is maps of a compact metric space $\Im$ into itself such that for some $n \geq 1$, $\pounds^n$ is continuous and satisfy the condition:

$$\begin{aligned}
\rho(\pounds^n(\gamma_1), \pounds^n(\gamma_2)) &< \mu_1 \left(\frac{\rho(\gamma_1, \pounds^n(\gamma_1))\, \rho(\gamma_2, \pounds^n(\gamma_2))}{\rho(\gamma_1, \gamma_2)} \right) \\
&\quad + \mu_2 (\rho(\gamma_2, \pounds^n(\gamma_1)) + \rho(\gamma_2, \pounds^n(\gamma_2))) \\
&\quad + \mu_3 (\rho(\gamma_1, \pounds^n(\gamma_1)) + \rho(\gamma_2, \pounds^n(\gamma_2))) \\
&\quad + \mu_4 (\rho(\gamma_1, \pounds^n(\gamma_2)) + \rho(\gamma_2, \pounds^n(\gamma_1))), \\
&\quad + \mu_5 \rho(\gamma_1, \gamma_2),
\end{aligned}$$

for $\gamma_1, \gamma_2 \in \Im$, $\gamma_1 \neq \gamma_2$ and $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ are nonnegative real numbers such that $\mu_1 + \mu_2 + 2(\mu_3 + \mu_4) + \mu_5 < 1$. Then $\pounds$ has a fixed point in $\Im$.

Proof Define $\eta : \Im \to R^+$ by $\lambda(\gamma_1) = \rho(\gamma_1, \pounds^n(\gamma_1))$ for every $\gamma_1 \in \Im$. Suppose $\gamma_1 \neq \pounds\gamma_1$, then $\eta(\pounds^n(\gamma_1)) = \rho(\pounds^n(\gamma_1), \pounds^n(\pounds^n(\gamma_1)))$ and

$$\begin{aligned}
\eta(\pounds^n(\gamma_1)) &< \mu_1 \left(\frac{\rho(\gamma_1, \pounds^n(\gamma_1))\, \rho(\pounds^n(\gamma_1), \pounds^n(\pounds^n(\gamma_1)))}{\rho(\gamma_1, \pounds^n(\gamma_1))} \right) \\
&\quad + \mu_2 (\rho(\pounds^n(\gamma_1), \pounds^n(\gamma_1)) + \rho(\pounds^n(\gamma_1), \pounds^n(\pounds^n(\gamma_1)))) + \\
&\quad \mu_3 (\rho(\gamma_1, \pounds^n(\gamma_1)) + \rho(\pounds^n(\gamma_1), \pounds^n(\pounds^n(\gamma_1)))) + \\
&\quad \mu_4 (\rho(\gamma_1, \pounds^n(\pounds^n(\gamma_1))) + \rho(\pounds^n(\gamma_1), \pounds^n(\gamma_1))) + \mu_5 \rho(\gamma_1, \pounds^n(\gamma_1)), \\
&\Rightarrow \eta(\pounds^n(\gamma_1)) < (\mu_1 + \mu_2 + \mu_3 + \mu_4)\, \rho(\pounds^n(\gamma_1), \pounds^n(\pounds^n(\gamma_1))) \\
&\quad + (\mu_3 + \mu_4 + \mu_5)\, \rho(\gamma_1, \pounds^n(\gamma_1)), \\
&\Rightarrow \eta(\pounds^n(\gamma_1)) < \frac{(\mu_3 + \mu_4 + \mu_5)}{(1 - (\mu_1 + \mu_2 + \mu_3 + \mu_4))} \rho(\gamma_1, \pounds^n(\gamma_1)), \\
&\Rightarrow \eta(\pounds^n(\gamma_1)) < \rho(\gamma_1, \pounds^n(\gamma_1)),
\end{aligned}$$

which is contradiction, because $\mu_1 + \mu_2 + 2(\mu_3 + \mu_4) + \mu_5 < 1$ and hence

$$\eta(\pounds^n(\gamma_1)) < \eta(\gamma_1), \ \gamma_1 \neq \pounds^n(\gamma_1). \tag{2.8}$$

Since $\mathcal{L}^n$ and η are continuous on the compact metric space $\Im$, hence it attains its minimum on $\Im$ at the point (say) v_0. Suppose, $F(v_0) = \rho(v_0, \mathcal{L}^n(v_0)) > 0$. Then by (2.8) we obtain $\eta(\mathcal{L}^n(v_0)) < \eta(v_0)$, which contradicts minimality of the value of η at v_0. Hence, our supposition $\eta(v_0) > 0$ is false. Therefore, $\eta(v_0) = \rho(v_0, \mathcal{L}^n(v_0)) = 0$ so that v_0 is a fixed point of $\mathcal{L}^n$.

Suppose, if possible $v \neq v_0$ is another fixed point of $\mathcal{L}^n$ then $\rho(v_0, v) = \rho(\mathcal{L}^n(v_0), \mathcal{L}^n(v))$ amd

$$\begin{aligned}
\rho(v_0, v) &< \mu_1\left(\frac{\rho(v_0, \mathcal{L}^n(v_0))\,\rho(v, \mathcal{L}^n(v))}{\rho(v_0, v)}\right) \\
&\quad +\mu_2(\rho(v, \mathcal{L}^n(v_0)) + \rho(v, \mathcal{L}^n(v))) + \mu_3(\rho(v_0, \mathcal{L}^n(v_0)) + \rho(v, \mathcal{L}^n(v_0))) \\
&\quad +\mu_4(\rho(v_0, \mathcal{L}^n(v)) + \rho(v, \mathcal{L}^n(v_0))) + \mu_5\rho(v_0, v), \\
\rho(v_0, v) &< \mu_2\rho(v_0, v) + \mu_3\rho(v_0, v) + 2\mu_4\rho(v_0, v) + \mu_5\rho(v_0, v),
\end{aligned}$$

which is a contradiction, because $\mu_1 + \mu_2 + 2(\mu_3 + \mu_4) + \mu_5 < 1$.

Now, let v_0 is a fixed point of $\mathcal{L}^n$ and since $\mathcal{L}^n(\mathcal{L}(v_0)) = \mathcal{L}(\mathcal{L}^n(v_0))$, then $\mathcal{L}^n(\mathcal{L}(v_0)) = \mathcal{L}(\mathcal{L}^n(v_0)) = \mathcal{L}(v_0) = v_0$ which shows that any fixed point of $\mathcal{L}$ is also a fixed point of $\mathcal{L}^n$.

Remark 2.2.10. Putting $n = 1$ in Theorem 2.2.10, we obtain Theorem 2.2.1.

Example 2.2.11. Let $\Im = \{1, 3, 5, 7\}$ and define $\rho : \Im \times \Im \to R^+$ by $\rho(\gamma_1, \gamma_2) = |\gamma_1 - \gamma_2|$ for all $\gamma_1, \gamma_2 \in \Im$. If $\mathcal{L} : \Im \to \Im$ is a mapping given by

$$\mathcal{L}1 = \mathcal{L}5 = \mathcal{L}7 = 3, \mathcal{L}3 = 3.$$

Then, $\mathcal{L}$ satisfy both Theorem 2.2.9 and Theorem 2.2.10 and 3 a unique fixed point of $\mathcal{L}$ in $\Im$.

Theorem 2.2.12. If $\Im$ is compact together with ρ and $\pounds$ is a continuous self-map of $\Im$ into itself satisfying

$$\rho(\pounds\gamma_1, \pounds\gamma_2) < \max\left\{\begin{array}{c} \rho(\gamma_1, \gamma_2), \rho(\gamma_1, \pounds(\gamma_1)), \rho(\gamma_2, \pounds(\gamma_2)), \\ \frac{1}{2}(\rho(\gamma_1, \pounds(\gamma_1)) + \rho(\gamma_2, \pounds(\gamma_2))), \\ \frac{1}{2}(\rho(\gamma_1, \pounds(\gamma_2)) + \rho(\gamma_2, \pounds(\gamma_1))) \end{array}\right\},$$

for all $\gamma_1, \gamma_2 \in \Im$, $\gamma_1 \neq \gamma_2$. Then $\pounds$ possesses a unique fixed point in $\Im$.

Proof Define a real-valued function η on $\Im$ by $\eta(\gamma_1) = \rho(\gamma_1, \pounds(\gamma_1))$ for $\gamma \in \Im$. Since ρ and $\pounds$ are continuous on $\Im$, it follows that η is also continuous on $\Im$. As $\Im$ is compact, for $\phi_1 \in \Im$, we have

$$\eta(\phi_1) = \inf\{\eta(\gamma_1) : \gamma_1 \in \Im\}. \tag{2.9}$$

Assume that $\pounds(\phi_1) \neq \phi_2$, then $\mu(\pounds(\phi_1)) = \rho(\pounds(\phi_1), \pounds(\pounds(\phi_1)))$ and we have

$$\begin{aligned} \mu(\pounds(\phi_1)) &< \max\left\{\begin{array}{c} \rho(\phi_1, \pounds(\phi_1)), \rho(\phi_1, \pounds(\phi_1)), \rho(\pounds(\phi_1), \pounds(\pounds(\phi_1))), \\ \frac{1}{2}[\rho(\phi_1, \pounds(\phi_1)) + \rho(\pounds(\phi_1), \pounds(\pounds(\phi_1)))], \frac{1}{2}\rho(\phi_1, \pounds(\pounds(\phi_1))) \end{array}\right\}, \\ &< \max\left\{\rho(\phi_1, \pounds(\phi_1)), \frac{1}{2}(\rho(\phi_1, \pounds(\phi_1)) + \rho(\pounds(\phi_1), \pounds(\pounds(\phi_1))))\right\}, \\ &= \max\left\{\mu(\phi_1), \frac{1}{2}(\mu(\phi_1) + \mu(\pounds(\phi_1)))\right\}, \end{aligned}$$

either $\mu(\pounds(\phi_1)) < \mu(\phi_1)$ or $\mu(\pounds(\phi_1)) < \frac{1}{2}(\mu(\phi_1) + \mu(\pounds(\phi_1)))$, which gives us a contradiction in both the cases. Hence our assumption was false and we must have, $\pounds(\phi_1) = \phi_1$. Thus, ϕ_1 is a fixed point of $\pounds$. For uniqueness, suppose $\pounds$ has a second fixed point ϕ_2 distinct from ϕ_1. Then, we have $\rho(\phi_1, \phi_2) = \rho(\pounds(\phi_1), \pounds(\phi_2))$ and

$$\rho(\phi_1, \phi_2) < \max\left\{\begin{array}{c} \rho(\phi_1, \phi_2), \rho(\phi_1, \pounds(\phi_1)), \rho(\phi_2, \pounds(\phi_2)), \\ \frac{1}{2}(\rho(\phi_1, \pounds(\phi_1)) + \rho(\phi_2, \pounds(\phi_2))), \\ \frac{1}{2}(\rho(\phi_1, \pounds(\phi_2)) + \rho(\phi_2, \pounds(\phi_1))) \end{array}\right\} = \rho(\phi_1, \phi_2),$$

which gives us a contradiction. Hence, the fixed point is unique.

Example 2.2.13. If $\Im = \{5, 6, 7\}$ with $\rho : \Im \times \Im \to R^+$ is defined by $\rho(\gamma_1, \gamma_2) = 0$, $\rho(\gamma_1, \gamma_2) = \rho(\gamma_2, \gamma_1)$ for $\gamma_1, \gamma_2 \in \Im$. Clearly, ρ is a compact metric space on $\Im$ such that

$$\rho(5, 6) = \rho(6, 7) = 1, \rho(5, 7) = 2.$$

If $\pounds : \Im \to \Im$ is mapping given by $\pounds 5 = \pounds 7 = 6$, $\pounds 6 = 6$. Thus, the map $\pounds$ satisfy Theorem 2.2.13 and 6 is the fixed point of $\Im$.

2.3 Fixed Point Theorems in Pseudo-Compact Tichnov Spaces

In the present section, we give yet another generalization of contractive mapping theorems for fixed points in pseudo-compact tichnov spaces. In the present section, we prove fixed point results for a pair of weakly commutative and continuous self-mappings in terms tichonov spaces satisfying contractive conditions.

The following fixed-point theorems were proved in [34] and [61].

Theorem 2.3.1 [34]. If $\pounds : \Im \to \Im$ satisfy

$$(\rho(\pounds\xi_1, \pounds\xi_2))^2 \leq \alpha_1 (\rho(\xi_1, \pounds\xi_1) \rho(\xi_2, \pounds\xi_2)) + \alpha_2 (\rho(\xi_1, \pounds\xi_2) \rho(\xi_2, \pounds\xi_1)),$$

for ξ_1, ξ_2 in $\Im$ where $0 \leq \alpha_1 < 1$ and $0 \leq \alpha_2$ then $\pounds$ has a fixed point.

Theorem 2.3.2 [61]. If $\Im$ is Pseudo-Compact Tichonov space and $\rho : \Im \times \Im \to R^+$satisfy the conditions:

$$(i) \begin{cases} \rho(\zeta_1, \zeta_1) = 0 \text{ for all } \zeta_1 \in \Im \text{ and} \\ \rho(\zeta_1, \zeta_3) \leq \rho(\zeta_1, \zeta_2) + \rho(\zeta_3, \zeta_2) \text{ for all } \zeta_1, \zeta_2, \zeta_3 \in \Im. \end{cases}$$

If $\pounds_1$and $\pounds_2$ are two continuous mappings of $\Im$ satisfying (ii) $\pounds_1\pounds_2 = \pounds_2\pounds_1$, and (iii)

$$\begin{aligned}\rho(\pounds_1\pounds_2\zeta_1, \pounds_1\zeta_2) &< \alpha_1\rho(\pounds_2\zeta_1, \zeta_2) + \alpha_2\rho(\pounds_1\pounds_2\zeta_1, \pounds_2\zeta_1) + \alpha_3\rho((\pounds_1\pounds_2\zeta_1, \zeta_2)) + \\ &\quad \alpha_4\rho(\pounds_2\zeta_1, \pounds_1\zeta_2) + \alpha_5\rho(\pounds_1\zeta_2, \zeta_2) + \\ &\quad \alpha_6\left(\frac{\rho(\pounds_1\pounds_2\zeta_1, \pounds_2\zeta_1)\,\rho(\pounds_1\zeta_2, \zeta_2)}{\rho(\pounds_2\zeta_1, \zeta_2)}\right) + \\ &\quad \alpha_7\left(\frac{\rho(\pounds_2\zeta_1, \pounds_1\zeta_2)\,\rho(\pounds_1\pounds_2\zeta_1, \zeta_2)}{\rho(\pounds_2\zeta_1, \zeta_2)}\right),\end{aligned}$$

for distinct ζ_1, $\zeta_2 \in \Im$ with $\pounds_2\zeta_1 \neq \zeta_2$, where $\alpha_3 \geq 0$, $\alpha_2 + \alpha_3 + \alpha_6 < 1$, $\alpha_1 + \alpha_2 + 2\alpha_2 + \alpha_5 + \alpha_6 \leq 1$. Then $\pounds_1$ and $\pounds_2$ have common fixed point in $\Im$ which is unique if $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_7 \leq 1$.

Definition 2.3.3 [56]. A topological space $\mathcal{L}$ is called pseudo-compact if every real valued continuous function on $\mathcal{L}$ is bounded. It may be noted that every compact space is pseudo-compact, but converse is not necessarily true. However, in case of the notion of a metric compact and pseudo-compact coincide. By Tichonov space, we mean a completely regular Hausdorff space.

Definition 2.3.4 [56]. The pair of maps $\pounds_1$ and $\pounds_2$ on $(\Im, \rho)$ are said to be weakly commutative if $\rho(\pounds_1\pounds_2\zeta, \pounds_2\pounds_1\zeta) \leq \rho(\pounds_2\zeta, \pounds_1\zeta)$ for some ζ in $\Im$.

Theorem 2.3.5. If $\Im$ is a pseudo-compact tichnov space and ρ on $\Im \times \Im$ satisfies the conditions

(i) $\rho(\zeta_1, \zeta_1) = 0$ for all $\zeta_1 \in \Im$,

(ii) $\rho(\zeta_1, \zeta_2) \leq \rho(\zeta_1, \zeta_3) + \rho(\zeta_2, \zeta_3)$ for all ζ_1, ζ_2, $\zeta_3 \in \Im$. If $\pounds_1$ and $\pounds_2$ are continuous self-maps of $\Im$ satisfy the following condition:

$$\pounds_1\pounds_2 = \pounds_2\pounds_1 \tag{2.10}$$

and

$$
\begin{aligned}
(\rho(\pounds_1\pounds_2\zeta_1,\pounds_1\zeta_2))^2 &< \alpha_1(\rho(\pounds_2\zeta_1,\zeta_2)\rho(\pounds_2\zeta_1,\pounds_1\pounds_2\zeta_1)) \\
&+\alpha_2\left(\frac{(\rho(\pounds_2\zeta_1,\pounds_1\pounds_2\zeta_1))^2\rho(\zeta_2,\pounds_1\zeta_2)}{\rho(\pounds_2\zeta_1,\zeta_2)}\right) \\
&+\alpha_3(\rho(\pounds_1\pounds_2\zeta_1,\zeta_2)\rho(\pounds_1\pounds_2\zeta_1,\pounds_1\zeta_2)) \\
&+\alpha_4\left(\frac{(\rho(\pounds_2\zeta_1,\zeta_2))\rho(\pounds_2\zeta_1,\pounds_1\zeta_2)}{1+\rho(\zeta_2,\pounds_1\zeta_2)}\right) \\
&+\alpha_5\left(\frac{(1+\rho(\pounds_2\zeta_1,\zeta_2))(\rho(\pounds_2\zeta_1,\pounds_1\pounds_2\zeta_1))\rho(\pounds_2\zeta_1,\zeta_2)}{1+\rho(\zeta_2,\pounds_1\zeta_2)}\right) \\
&+\alpha_6\left(\frac{\rho(\zeta_2,\pounds_1\zeta_2)\rho(\pounds_1\pounds_2\zeta_1,\pounds_2\zeta_1)}{\rho(\pounds_2\zeta_1,\zeta_2)}\right) \\
&+\alpha_7\left(\frac{\rho(\zeta_2,\pounds_1\pounds_2\zeta_1)\rho(\pounds_2\zeta_1,\pounds_1\zeta_2)}{\rho(\pounds_2\zeta_1,\zeta_2)}\right)^2 \qquad (2.11)
\end{aligned}
$$

for distinct ζ_1, $\zeta_2 \in \Im$ with $\pounds_2\zeta_1 \neq \zeta_2$ and $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6<1$, then $\pounds_1$ and $\pounds_2$ have a common fixed point in $\Im$ which is unique whenever $\alpha_3+\alpha_4+\alpha_7<1$.

Proof Define $\eta : \Im \to R^+$on $\Im$ by $\eta(\zeta_1) = \rho(\pounds_1\pounds_2\zeta_1, \pounds_2\zeta_1)$ for $\zeta_1 \in \Im$. Thus, there exists a point $\gamma_1 \in \Im$ such that $\eta(\gamma_1) = \inf\{\eta(\gamma_1) : \gamma_1 \in \Im\}$. We now affirm that γ_1 is a fixed point for $\pounds_1$. If not, let us suppose that $\pounds_1\gamma_1 \neq \gamma_1$, then using (2.11), we can write

$(\eta(\pounds_1\gamma_1))^2 = (\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_2\pounds_1\gamma_1))^2$ and, therefore,

$$
\begin{aligned}
(\eta(\pounds_1\gamma_1))^2 &< \alpha_1(\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1)\rho(\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\pounds_1\gamma_1)) \\
&+\alpha_2\left(\frac{(\rho(\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\pounds_1\gamma_1))^2\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1)}{\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1)}\right) \\
&+\alpha_3(\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1)\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\gamma_1)) \\
&+\alpha_4\left(\frac{(\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1))\rho(\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\gamma_1)}{(1+\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1))}\right) + \\
&+\alpha_5\left(\frac{(1+\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1))(\rho(\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\pounds_1\gamma_1))\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1)}{(1+\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1))}\right) \\
&+\alpha_6\left(\frac{\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1)\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_2\pounds_1\gamma_1)}{\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1)}\right)^2 \\
&+\alpha_7\left(\frac{\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\pounds_1\gamma_1)\rho(\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\gamma_1)}{\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1)}\right)^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\gamma_1)) &< \alpha_1(\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1) + \alpha_2\rho(\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\pounds_1\gamma_1)) + \\
&\alpha_3\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1) + \alpha_5\rho(\pounds_2\pounds_1\gamma_1, \pounds_2\gamma_1) + \\
&\alpha_6\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_2\pounds_1\gamma_1),
\end{aligned}
$$

or

$$
\begin{aligned}
(1-(\alpha_2+\alpha_3+\alpha_6))\eta(\pounds_1\gamma_1) &< (\alpha_1+\alpha_3+\alpha_5)\eta(\gamma_1), \\
\eta(\pounds_1\gamma_1) &< \frac{(\alpha_1+\alpha_3+\alpha_5)}{(1-(\alpha_2+\alpha_3+\alpha_6))}\eta(\gamma_1), \\
\eta(\pounds_1\gamma_1) &< \eta(\gamma_1),
\end{aligned}
$$

which is contradiction, because $\rho(\pounds_1\pounds_2\pounds_1\gamma_1, \pounds_1\pounds_2\gamma_1) \geq 0$. Hence, $\gamma_1 \in \Im$ is a fixed point for $\pounds_1$, that is $\pounds_1(\gamma_1) = \gamma_1$. Using (2.10) ,we have

$$
\pounds_1\pounds_2(\gamma_1) = \pounds_2\pounds_1(\gamma_1) = \pounds_2(\gamma_1). \tag{2.12}
$$

Now, we shall prove that $\pounds_2(\gamma_1) = \gamma_1$. If possible, let $\pounds_2(\gamma_1) \neq \gamma_1$, then, we use the equality, $(\rho(\pounds_2\gamma_1, \gamma_1))^2 = (\rho(\pounds_1\pounds_2\gamma_1, \pounds_1\gamma_1))^2$ utilizing (2.11) and (2.12), we can write

$$\begin{aligned}
(\rho(\pounds_2\gamma_1, \gamma_1))^2 &< \alpha_1(\rho(\pounds_2\gamma_1, \gamma_1)\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1)) \\
&\quad +\alpha_2\left(\frac{(\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1))^2\rho(\gamma_1, \pounds_1\gamma_1)}{\rho(\pounds_2\gamma_1, \gamma_1)}\right) \\
&\quad +\alpha_3(\rho(\pounds_1\pounds_2\gamma_1, \gamma_1)\rho(\pounds_1\pounds_2\gamma_1, \pounds_1\gamma_1)) \\
&\quad +\alpha_4\left(\frac{(\rho(\pounds_2\gamma_1, \gamma_1))\rho(\pounds_2\gamma_1, \pounds_1\gamma_1)}{1+\rho(\gamma_1, \pounds_1\gamma_1)}\right) \\
&\quad +\alpha_5\left(\frac{(1+\rho(\pounds_2\gamma_1, \gamma_1))(\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1))^2}{1+\rho(\gamma_1, \pounds_1\gamma_1)}\right) \\
&\quad +\alpha_6\left(\frac{\rho(\gamma_1, \pounds_1\gamma_1)\rho(\pounds_1\pounds_2\gamma_1, \pounds_2\gamma_1)}{\rho(\pounds_2\gamma_1, \gamma_1)}\right)^2 \\
&\quad +\alpha_7\left(\frac{\rho(\gamma_1, \pounds_1\pounds_2\gamma_1)\rho(\pounds_2\gamma_1, \pounds_1\gamma_1)}{\rho(\pounds_2\gamma_1, \gamma_1)}\right)^2 \\
&< (\alpha_3+\alpha_4+\alpha_7)(\rho(\pounds_2\gamma_1, \gamma_1))^2, \\
&< (\rho(\pounds_2\gamma_1, \gamma_1))^2.
\end{aligned}$$

Which is a contradiction because $\alpha_3+\alpha_4+\alpha_7 < 1$. Hence, $\gamma_1 \in \Im$ is a fixed point of $\pounds_2$ i.e. $\pounds_2\gamma_1 = \gamma_1$. For uniqueness of γ_1, let γ_2 is another fixed point such that $\gamma_1 = \pounds_1\gamma_1 = \pounds_2\gamma_1$ and $\gamma_2 = \pounds_1\gamma_2 = \pounds_2\gamma_2$ $(\gamma_2 \neq \gamma_1)$. Then, using $(\rho(\gamma_1, \gamma_2))^2 = (\rho(\pounds_1\pounds_2\gamma_1, \pounds_1\gamma_2))^2$ and condition (2.11), we have

$$\begin{aligned}
(\rho(\gamma_1, \gamma_2))^2 &< \alpha_1(\rho(\pounds_2\gamma_1, \gamma_2)\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1)) \\
&\quad +\alpha_2\left(\frac{(\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1))^2 d(\gamma_2, \pounds_1\gamma_2)}{\rho(\pounds_2\gamma_1, \gamma_2)}\right) \\
&\quad +\alpha_3(\rho(\pounds_1\pounds_2\gamma_1, \gamma_2)\rho(\pounds_1\pounds_2\gamma_1, \pounds_1\gamma_2)) \\
&\quad +\alpha_4\left(\frac{(\rho(\pounds_2\gamma_1, \gamma_2))\rho(\pounds_2\gamma_1, \pounds_1\gamma_2)}{1+\rho(\gamma_2, \pounds_1\gamma_2)}\right) + \\
&\quad \alpha_5\left(\frac{(1+\rho(\pounds_2\gamma_1, \gamma_2))(\rho(\pounds_2\gamma_1, \pounds_1\pounds_2\gamma_1))\rho(\pounds_2\gamma_1, \gamma_2)}{1+\rho(\gamma_2, \gamma_2)}\right) + \\
&\quad \alpha_7\left(\frac{\rho(\gamma_2, \pounds_1\pounds_2\gamma_1)\rho(\pounds_2\gamma_1, \pounds_1\gamma_2)}{\rho(\pounds_2\gamma_1, \gamma_2)}\right)^2,
\end{aligned}$$

giving us

$$(\rho(\gamma_1,\gamma_2))^2 < (\alpha_3+\alpha_4+\alpha_7)(\rho(\gamma_1,\gamma_2))^2,$$

and this leads us to a contradiction, because $\alpha_3+\alpha_4+\alpha_7<1$, which proves that $\gamma_1 \in \Im$ is unique.

Corollary 2.3.6. If $\Im$ is tichnov space and ρ on $\Im\times\Im$ satisfies: (i) $\rho(\zeta_1,\zeta_1)=0$ for $\zeta_1\in\Im$. (ii) $\rho(\zeta_1,\zeta_3)\le\rho(\zeta_1,\zeta_3)+\rho(\zeta_3,\zeta_2)$ for some $\zeta_1,\zeta_2,\zeta_3\in\Im$, and $\pounds_1:\Im\to\Im$ is a continuous map satisfying the inequality

$$\begin{aligned}
\left(\rho\left(\pounds_1^2\zeta_1,\pounds_1\zeta_2\right)\right)^2 &< \alpha_1\left(\rho(\pounds_1\zeta_1,\zeta_2)\,\rho\left(\pounds_1\zeta_1,\pounds_1^2\zeta_1\right)\right)+\alpha_2\left(\frac{\left(\rho\left(\pounds_1\zeta_1,\pounds_1^2\zeta_1\right)\right)^2\rho(\zeta_2,\pounds_1\zeta_2)}{\rho(\pounds_1\zeta_1,\zeta_2)}\right)\\
&+\alpha_3\left(\rho\left(\pounds_1^2\zeta_1,\zeta_2\right)\rho\left(\pounds_1^2\zeta_1,\pounds_1\zeta_2\right)\right)+\alpha_4\left(\frac{(\rho(\zeta_1,\zeta_2))\,\rho(\pounds_1\zeta_1,\pounds_1\zeta_2)}{1+\rho(\zeta_2,\pounds_1\zeta_2)}\right)\\
&+\alpha_5\left(\frac{(1+\rho(\pounds_1\zeta_1,\zeta_2))\left(\rho\left(\pounds_1\zeta_1,\pounds_1^2\zeta_1\right)\right)\rho(\pounds_1\zeta_1,\zeta_2)}{1+\rho(\zeta_2,\pounds_1\zeta_2)}\right)+\\
&\alpha_6\left(\frac{\rho(\zeta_2,\pounds_1\zeta_2)\,\rho\left(\pounds_1^2\zeta_1,\pounds_1\zeta_1\right)}{\rho(\pounds_1\zeta_1,\zeta_2)}\right)^2+\alpha_7\left(\frac{\rho\left(\zeta_2,\pounds_1^2\zeta_1\right)\rho(\pounds_1\zeta_1,\pounds_1\zeta_2)}{\rho(\pounds_1\zeta_1,\zeta_2)}\right)^2.
\end{aligned}$$

for some distinct ζ_1, $\zeta_2\in\Im$ with $\pounds_1\zeta_1\ne\zeta_2$ and $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6<1$, then $\pounds_1$ has fixed point in $\Im$.

Proof On taking $\pounds_2=\pounds_1$ Theorem 2.5.3 guaranttees that $\pounds_1$ has a fixed point in $\Im$.

Theorem 2.3.7. If $\Im$ is pseudo-compact and ρ on $\Im\times\Im$ satisfying (i) $\rho(\mathcal{L}_1,\mathcal{L}_1)=0$ for $\mathcal{L}_1\in\Im$, and (ii) $\rho(\mathcal{L}_1,\mathcal{L}_2)\le\rho(\mathcal{L}_1,\mathcal{L}_3)+\rho(\mathcal{L}_2,\mathcal{L}_3)$ for some $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3\in\Im$. If $\pounds_1$and $\pounds_2$ are continuous self maps of $\Im$ satisfying

$$\pounds_1\pounds_2=\pounds_2\pounds_1, \tag{2.13}$$

and

$$
\begin{aligned}
&(\rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}_2))^2 < \alpha_1(\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2)\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}\mathcal{L}_1)) \\
&+\alpha_2\left(\frac{(\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1))^2\rho(\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2)}{\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2)}\right) \\
&+\alpha_3(\rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2)\rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}_2)) + \alpha_4\left(\frac{(\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2))\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}_2)}{1+\rho(\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2)}\right) + \\
&\alpha_5\left(\frac{(1+\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2))(\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1))\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2)}{1+\rho(\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2)}\right) \\
&+\alpha_6\left(\frac{\rho(\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2)\rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1, \mathcal{L}\mathcal{L}_1)}{\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2)}\right)^2 + \alpha_7\left(\frac{\rho(\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1)\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}_2)}{\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2)}\right)^2 \\
&+\alpha_8\max\left\{\begin{array}{c}\rho(\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2)\rho(\mathcal{L}_1\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1), \rho(\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1)\rho(\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1), \\ \rho(\mathcal{L}_1\mathcal{L}_2, \mathcal{L}_2\mathcal{L}_1)\rho(\mathcal{L}_2, \mathcal{L}_2\mathcal{L}_1), (\rho(\mathcal{L}_1\mathcal{L}_2, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1))^2\end{array}\right\},
\end{aligned}
\tag{2.14}
$$

for $\mathcal{L}_1, \mathcal{L}_2 \in \Im$, $\mathcal{L}_2\mathcal{L}_1 \neq \mathcal{L}_2$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ are nonnegative real numbers such that $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_5 + \alpha_6 + 2\alpha_8 < 1$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ possess a fixed point in $\Im$, which is unique if $\alpha_3 + \alpha_4 + \alpha_7 + \alpha_8 < 1$.

Proof Define $\eta : \Im \to R^+$on $\Im$ by $\eta(\mathcal{L}_1) = \rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1, \mathcal{L}_2\mathcal{L}_1)$ for some $\mathcal{L}_1 \in \Im$. Thus, there exists a point $\ell_1 \in \Im$ such that $\eta(\ell_1) = \inf\{\eta(\mathcal{L}_1) : \mathcal{L}_1 \in \Im\}$. We now affirm that ℓ_1 is a fixed point for $\mathcal{L}_1$. If not, let us suppose that $\mathcal{L}_1\ell_1 \neq \ell_1$, then using (2.14) we can write $\eta((\mathcal{L}_1\ell_1))^2 = (\rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1))^2$ and

$$
\begin{aligned}
\eta((\mathcal{L}_1\ell_1))^2 \; < \; & \alpha_1(\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1)\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1)) + \\
& \alpha_2\left(\frac{\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1)^2\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)}{\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1)}\right) \\
& +\alpha_3(\rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1)\rho(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)) \\
& +\alpha_4\left(\frac{\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1)\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)}{1+\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)}\right) \\
& +\alpha_5\left(\frac{(1+\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1))(\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1))\rho(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1)}{1+\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)}\right)
\end{aligned}
$$

$$+\alpha_6\left(\frac{\rho(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1)\,\rho(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_2\pounds_1\ell_1)}{\rho(\pounds_2\pounds_1\ell_1,\pounds_1\ell_1)}\right)^2$$
$$+\alpha_7\left(\frac{\rho(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1)\,\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\ell_1)}{\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)}\right)^2$$
$$+\alpha_8\max\left\{\begin{array}{c}\rho(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1)\,\rho(\pounds_1\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1),\\ \rho(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1)\,\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1),\\ \rho(\pounds_1\pounds_2\ell_1,\pounds_2\pounds_1\ell_1)\,\rho(\pounds_2\ell_1,\pounds_2\pounds_1\ell_1),(\rho(\pounds_1\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1))^2\end{array}\right\}.$$

Case (i)- If $\max\left\{\begin{array}{c}\rho(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1)\,\rho(\pounds_1\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1),\\ \rho(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1)\,\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1),\\ \rho(\pounds_1\pounds_2\ell_1,\pounds_2\pounds_1\ell_1)\,\rho(\pounds_2\ell_1,\pounds_2\pounds_1\ell_1),(\rho(\pounds_1\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1))^2\end{array}\right\}$

$$=(\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1))^2.$$

Then, we have

$$\begin{aligned}\eta((\pounds_1\ell_1))^2 &< \alpha_1(\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)\,\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1))\\ &+\alpha_2\left(\frac{\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1)^2\,\rho(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1)}{\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)}\right)\\ &+\alpha_3(\rho(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)\,\rho(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\ell_1))\\ &+\alpha_4\left(\frac{\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)\,\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\ell_1)}{1+\rho(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1)}\right)+\\ &\alpha_5\left(\frac{(1+\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1))\,(\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1))\,\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)}{1+\rho(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1)}\right)\\ &+\alpha_6\left(\frac{\rho(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1)\,\rho(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_2\pounds_1\ell_1)}{\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)}\right)^2\\ &+\alpha_7\left(\frac{\rho(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1)\,\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\ell_1)}{\rho(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1)}\right)^2\\ &+\alpha_8(\rho(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1)),\end{aligned}$$

or

$$\begin{aligned}
\eta\left(\left(\pounds_1\ell_1\right)\right) &< \alpha_1\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right) + \alpha_2\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)+\alpha_3\rho\left(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)+ \\
&\quad \alpha_5\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)+\alpha_6\rho\left(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_2\pounds_1\ell_1\right)+\alpha_8\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right), \\
\eta\left(\pounds_1\ell_1\right) &< \frac{\left(\alpha_1+\alpha_3+\alpha_5\right)}{\left(1-\left(\alpha_2+\alpha_3+\alpha_6+\alpha_8\right)\right)}\eta_1\left(\ell_1\right), \\
\eta\left(\pounds_1\ell_1\right) &< \eta_1\left(\ell_1\right).
\end{aligned}$$

where $\eta_1=\frac{(\alpha_1+\alpha_3+\alpha_5)}{(1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8))}$, which is a contradiction because $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6+\alpha_8<1$.

Case (ii)- If $\max\left\{\begin{array}{c}\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1\right)\rho\left(\pounds_1\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right), \\ \rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right), \\ \rho\left(\pounds_1\pounds_2\ell_1,\pounds_2\pounds_1\ell_1\right)\rho\left(\pounds_2\ell_1,\pounds_2\pounds_1\ell_1\right),\left(\rho\left(\pounds_1\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\right)^2\end{array}\right\}$

$$=\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right).$$

Then, we can write

$$\begin{aligned}
\eta\left(\left(\pounds_1\ell_1\right)\right)^2 &< \alpha_1\left(\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\right) \\
&\quad +\alpha_2\left(\frac{\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)^2\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1\right)}{\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)}\right) \\
&\quad +\alpha_3\left(\rho\left(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)\rho\left(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\ell_1\right)\right) \\
&\quad +\alpha_4\left(\frac{\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\ell_1\right)}{1+\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1\right)}\right) \\
&\quad +\alpha_5\left(\frac{\left(1+\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)\right)\left(\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\right)\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)}{1+\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1\right)}\right) \\
&\quad +\alpha_6\left(\frac{\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\ell_1\right)\rho\left(\pounds_1\pounds_2\pounds_1\ell_1,\pounds_2\pounds_1\ell_1\right)}{\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)}\right)^2 \\
&\quad +\alpha_7\left(\frac{\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\ell_1\right)}{\rho\left(\pounds_2\pounds_1\ell_1,\pounds_2\ell_1\right)}\right)^2 \\
&\quad +\alpha_8\left(\rho\left(\pounds_2\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\rho\left(\pounds_2\pounds_1\ell_1,\pounds_1\pounds_2\pounds_1\ell_1\right)\right),
\end{aligned}$$

This gives

$$
\begin{aligned}
(\eta(\pounds_1\ell_1)) &< \alpha_1\rho(\pounds_2\pounds_1\ell_1, \pounds_2\ell_1) + \alpha_2\rho(\pounds_2\pounds_1\ell_1, \pounds_1\pounds_2\pounds_1\ell_1) \\
&\quad +\alpha_3\rho(\pounds_1\pounds_2\pounds_1\ell_1, \pounds_2\ell_1) + \alpha_5\rho(\pounds_2\pounds_1\ell_1, \pounds_2\ell_1) \\
&\quad +\alpha_6\rho(\pounds_1\pounds_2\pounds_1\ell_1, \pounds_2\pounds_1\ell_1) + \alpha_8\rho(\pounds_2\ell_1, \pounds_1\pounds_2\pounds_1\ell_1),
\end{aligned}
$$

or

$$
\begin{aligned}
(1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8))(\eta(\pounds_1\ell_1)) &< (\alpha_1+\alpha_3+\alpha_5+\alpha_8)\eta_2(\ell_1), \\
(\eta(\pounds_1\ell_1)) &< \eta_2(\ell_1),
\end{aligned}
$$

where $\eta_2 = \frac{(\alpha_1+\alpha_3+\alpha_5+\alpha_8)}{(1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8))}$ is a contradiction because $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6+2\alpha_8 < 1$.

Case-(iii) If $\max\left\{\begin{array}{c} \rho(\pounds_2\ell_1, \pounds_1\pounds_2\ell_1)\rho(\pounds_1\pounds_2\ell_1, \pounds_1\pounds_2\pounds_1\ell_1), \\ \rho(\pounds_2\ell_1, \pounds_1\pounds_2\pounds_1\ell_1)\rho(\pounds_2\pounds_1\ell_1, \pounds_1\pounds_2\pounds_1\ell_1), \\ \rho(\pounds_1\pounds_2\ell_1, \pounds_2\pounds_1\ell_1)\rho(\pounds_2\ell_1, \pounds_2\pounds_1\ell_1), (\rho(\pounds_1\pounds_2\ell_1, \pounds_1\pounds_2\pounds_1\ell_1))^2 \end{array}\right\}$

$$
= \rho(\pounds_2\ell_1, \pounds_1\pounds_2\ell_1)\rho(\pounds_1\pounds_2\ell_1, \pounds_1\pounds_2\pounds_1\ell_1).
$$

We can write, again

$$\begin{aligned}
\eta\left((\mathcal{L}_1\ell_1)\right)^2 &< \alpha_1\left(\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1\right)\right) \\
&+\alpha_2\left(\frac{\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1\right)^2\rho\left(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)}{\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)}\right) \\
&+\alpha_3\left(\rho\left(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)\rho\left(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)\right) \\
&+\alpha_4\left(\frac{\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)}{1+\rho\left(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)}\right) \\
&+\alpha_5\left(\frac{\left(1+\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)\right)\left(\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1\right)\right)\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)}{1+\rho\left(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)}\right) \\
&+\alpha_6\left(\frac{\rho\left(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)\rho\left(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\mathcal{L}_1\ell_1\right)}{\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)}\right)^2 \\
&+\alpha_7\left(\frac{\rho\left(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1\right)\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)}{\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)}\right)^2 \\
&+\alpha_8\left(\rho\left(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right)\rho\left(\mathcal{L}_1\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1\right)\right),
\end{aligned}$$

or

$$\begin{aligned}
\eta\left(\mathcal{L}_1\ell_1\right) &< \alpha_1\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)+\alpha_2\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1\right) \\
&+\alpha_3\rho\left(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)+\alpha_5\rho\left(\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1\right)+ \\
&\alpha_6\rho\left(\mathcal{L}_1\mathcal{L}_2\mathcal{L}_1\ell_1, \mathcal{L}_2\mathcal{L}_1\ell_1\right)+\alpha_8\rho\left(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1\right), \\
\eta\left(\mathcal{L}_1\ell_1\right) &< \frac{(\alpha_1+\alpha_3+\alpha_5+\alpha_8)}{(1-(\alpha_2+\alpha_3+\alpha_6))}\eta_3\left(\ell_1\right), \\
\eta\left(\mathcal{L}_1\ell_1\right) &< \eta_3\left(\ell_1\right),
\end{aligned}$$

where, $\eta_3=\frac{(\alpha_1+\alpha_3+\alpha_5+\alpha_8)}{(1-(\alpha_2+\alpha_3+\alpha_6))}$ and is contradiction because $\alpha_1+\alpha_2+2\alpha_3+\alpha_5+\alpha_6+\alpha_8<1$

Now, if $\eta=\max\left\{\frac{(\alpha_1+\alpha_3+\alpha_5)}{[1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8)]}, \frac{(\alpha_1+\alpha_3+\alpha_5+\alpha_8)}{[1-(\alpha_2+\alpha_3+\alpha_6+\alpha_8)]}, \frac{(\alpha_1+\alpha_3+\alpha_5+\alpha_8)}{[1-(\alpha_2+\alpha_3+\alpha_6)]}\right\}=\max\left\{\eta_1,\eta_2,\eta_3\right\}$. Then, $\ell_1\in\Im$ is a fixed point of $\mathcal{L}_1$ and so $\mathcal{L}_1\left(\ell_1\right)=\ell_1$. Using (2.13), we have

$$\mathcal{L}_1\mathcal{L}_2\left(\ell_1\right)=\mathcal{L}_2\mathcal{L}_1\left(\ell_1\right)=\mathcal{L}_2\left(\ell_1\right). \tag{2.15}$$

Now we shall prove that $\mathcal{L}_2\left(\ell_1\right)=\ell_1$. If possible, let $\mathcal{L}_2\left(\ell_1\right)\neq\ell_1$, then by using (2.13) and (2.14) we have $(\rho(\mathcal{L}_2\ell_1, \ell_1))^2 = (\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_1))^2$ and

$$\begin{aligned}(\rho(\mathcal{L}_2\ell_1, \ell_1))^2 &< \alpha_1(\rho(\mathcal{L}_2\ell_1, \ell_1)\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)) \\ &+\alpha_2\left(\frac{(\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1))^2\rho(\ell_1, \mathcal{L}_1\ell_1)}{\rho(\mathcal{L}_2\ell_1, \ell_1)}\right) \\ &+\alpha_3(\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \ell_1)\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_1)) \\ &+\alpha_4\left(\frac{(\rho(\mathcal{L}_2\ell_1, \ell_1))\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_1)}{1+\rho(\ell_1, \mathcal{L}_1\ell_1)}\right) \\ &+\alpha_5\left(\frac{(1+\rho(\mathcal{L}_2\ell_1, \ell_1))(\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1))\rho(\mathcal{L}_2\ell_1, \ell_1)}{1+\rho(\ell_1, \mathcal{L}_1\ell_1)}\right) \\ &+\alpha_6\left(\frac{\rho(\ell_1, \mathcal{L}_1\ell_1)\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \mathcal{L}_2\ell_1)}{\rho(\mathcal{L}_2\ell_1, \ell_1)}\right)^2 \\ &+\alpha_7\left(\frac{\rho(\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_1)}{\rho(\mathcal{L}_2\ell_1, \ell_1)}\right)^2 \\ &+\alpha_8\left\{\begin{array}{c}\rho(\ell_1, \mathcal{L}_1\ell_1)\rho(\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1), \rho(\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1), \\ \rho(\mathcal{L}_1\ell_1, \mathcal{L}_2\ell_1)\rho(\ell_1, \mathcal{L}_2\ell_1), (\rho(\mathcal{L}_1\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1))^2\end{array}\right\}, \\ &< (\alpha_3+\alpha_4+\alpha_7+\alpha_8)(\rho(\ell_1, \mathcal{L}_2\ell_1))^2,\end{aligned}$$

which is contradiction, because $\alpha_3+\alpha_4+\alpha_7+\alpha_8 < 1$. Hence $\ell_1 \in \Im$ is a fixed point of $\mathcal{L}_2$, that is $\mathcal{L}_2(\ell_1) = \ell_1$.

Let, if possible $\ell_1 \neq \ell_2$ are two fixed pioints of $\mathcal{L}_1$ and $\mathcal{L}_2$ that is $\ell_1 = \mathcal{L}_1(\ell_1) = \mathcal{L}_2(\ell_1)$ and $\ell_2 = \mathcal{L}_1(\ell_2) = \mathcal{L}_2(\ell_2)$. Using (2.14) and $(\rho(\ell_1, \ell_2))^2 = (\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_2))^2$, we obtain

$$\begin{aligned}(\rho(\ell_1, \ell_2))^2 &< \alpha_1(\rho(\mathcal{L}_2\ell_1, \ell_2)\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1)) + \alpha_2\left(\frac{(\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1))^2\rho(\ell_2, \mathcal{L}_1\ell_2)}{\rho(\mathcal{L}_2\ell_1, \ell_2)}\right) \\ &+\alpha_3(\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \ell_2)\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_2)) + \alpha_4\left(\frac{(\rho(\mathcal{L}_2\ell_1, \ell_2))\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_2)}{1+\rho(\ell_2, \mathcal{L}_1\ell_2)}\right) \\ &+\alpha_5\left(\frac{(1+\rho(\mathcal{L}_2\ell_1, \ell_2))(\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1))\rho(\mathcal{L}_2\ell_1, \ell_2)}{1+\rho(\ell_2, \mathcal{L}_1\ell_2)}\right) \\ &+\alpha_6\left(\frac{\rho(\ell_2, \mathcal{L}_1\ell_2)\rho(\mathcal{L}_1\mathcal{L}_2\ell_1, \mathcal{L}_2\ell_1)}{\rho(\mathcal{L}_2\ell_1, \ell_2)}\right)^2 + \alpha_7\left(\frac{\rho(\ell_2, \mathcal{L}_1\mathcal{L}_3\ell_1)\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\ell_1)}{\rho(\mathcal{L}_2\ell_1, \ell_2)}\right)^2 \\ &+\alpha_8\max\left\{\begin{array}{c}\rho(\ell_2, \mathcal{L}_1\ell_2)\rho(\mathcal{L}_1\ell_2, \mathcal{L}_1\mathcal{L}_2\ell_1), \rho(\ell_2, \mathcal{L}_1\mathcal{L}_2\ell_1)\rho(\mathcal{L}_2\ell_1, \mathcal{L}_1\mathcal{L}_2\ell_1), \\ \rho(\mathcal{L}_1\ell_2, \mathcal{L}_2\ell_1)\rho(\ell_2, \mathcal{L}_2\ell_1), \{\rho(\mathcal{L}_1\ell_2, \mathcal{L}_1\mathcal{L}_2\ell_1)\}^2\end{array}\right\}.\end{aligned}$$

Or

$$(\rho(\ell_1, \ell_2))^2 < (\alpha_3 + \alpha_4 + \alpha_7 + \alpha_8)(\rho(\ell_1, \ell_2))^2.$$

This is a contradiction, because $\alpha_3 + \alpha_4 + \alpha_7 + \alpha_8 < 1$. Hence $\ell_1 \in \Im$ is unique fixed point of $\pounds_1$ and $\pounds_2$.

Remark 2.3.8. Putting $\Im_8 = 0$ Theorem 2.5.5 yields Theorem 2.5.3.

Example 2.3.9. If $\Im = \{1, 3, 5, 7\}$ and let $\top$ is the discrete topology on $\Im$ and define $\pounds_1, \pounds_2 : \Im \to \Im$ by $\pounds_1 1 = 1$, $\pounds_1 3 = 5$, $\pounds_1 5 = 7$, $\pounds_1 7 = 1$,$\pounds_2 1 = 1$, $\pounds_2 3 = 7$, $\pounds_2 5 = 3$, $\pounds_2 7 = 5$ and ρ on $\Im \times \Im$ is given by $\rho(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|$, for all $\zeta_1 \neq \zeta_2 \in \Im$. It is evident that $\Im$ is a pseudo-compact tichnov space with $g_1$and g_2 are continuous on $\Im$ satisfying Theorem 2.5.3 and Theorem 2.5.5 with 1 is the unique common fixed point on putting $\zeta_1 = 3$, $\zeta_2 = 5$ for $\alpha_1 = 0$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{12}$, $\alpha_4 = \frac{1}{2}$, $\alpha_5 = 0$, $\alpha_6 = \alpha_7 = \alpha_8 = \frac{1}{2}$.

Chapter 3

Common Fixed Point Theorems for Four Weakly Compatible Maps in Complete Metric Spaces

3.1 Introduction

In the third chapter, we prove several fixed-point results using the idea of weakly compatible maps. We generalize the corresponding results proved in [35].

Theorem 3.1.1 [35]. If $\mathcal{L} : \Im \to \Im$ is a mapping on $\Im$ with metric ρ satisfies

$$\rho(\mathcal{L}\xi_1, \mathcal{L}\xi_2) \leq \max \left\{ \begin{array}{c} 2\alpha_1 \rho(\xi_1, \xi_2), \alpha_2 (\rho(\xi_1, \mathcal{L}\xi_1) + \rho(\xi_2, \mathcal{L}\xi_2)), \\ \alpha_3 (\rho(\xi_1, \mathcal{L}\xi_2) + \rho(\xi_2, \mathcal{L}\xi_1)) \end{array} \right\},$$

for some $\xi_1, \xi_2 \in \Im$ and $0 \leq \alpha_1, \alpha_2, \alpha_3 < \frac{1}{2}$ then the map $\mathcal{L}$ admit a fixed point in $\Im$.

Theorem 3.1.2 [35]. If $\mathcal{L} : \Im \to \Im$ satisfies

$$(\rho(\mathcal{L}\xi_1, \mathcal{L}\xi_2))^2 \le \max \left\{ \begin{array}{c} 2\alpha_1\rho(\xi_1,\xi_2)(\rho(\xi_1,\mathcal{L}\xi_1)+\rho(\xi_2,\mathcal{L}\xi_2)), \\ 2\alpha_2\rho(\xi_1,\xi_2)(\rho(\xi_1,\mathcal{L}\xi_2)+\rho(\xi_2,\mathcal{L}\xi_1)), \\ \alpha_3(\rho(\xi_1,\mathcal{L}\xi_1)+\rho(\xi_2,\mathcal{L}\xi_2))(\rho(\xi_1,\mathcal{L}\xi_2)+\rho(\xi_2,\mathcal{L}\xi_1)) \end{array} \right\},$$

for some $\xi_1, \xi_2 \in \Im$ and $0 \le \alpha_1, \alpha_2, \alpha_3 < \frac{1}{2}$,the map $\mathcal{L}$ has fixed point in $\Im$.

Definition 3.1.3. A point $\xi \in \Im$ is called point of coincidence of the mappings $\mathcal{L}_1$ and $\mathcal{L}_2$ if $\mathcal{L}_1\xi = \mathcal{L}_2\xi$.

Definition 3.1.4 [46]. Two self-maps $\mathcal{L}_1, \mathcal{L}_2 : \Im \to \Im$ on $\Im$ are said to be commuting maps if $\mathcal{L}_1(\mathcal{L}_2(\mathbf{\mathcal{L}})) = \mathcal{L}_2(\mathcal{L}_1(\mathbf{\mathcal{L}}))$ for some $\mathbf{\mathcal{L}} \in \Im$.

Definition 3.1.5 [71]. The maps $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\Im$ are weakly commuting if

$$\rho(\mathcal{L}_1\mathcal{L}_2\mathbf{\mathcal{L}}, \mathcal{L}_2\mathcal{L}_1\mathbf{\mathcal{L}}) \le \rho(\mathcal{L}_1\mathbf{\mathcal{L}}, \mathcal{L}_2\mathbf{\mathcal{L}})\, for \mathbf{\mathcal{L}} \in \Im.$$

Definition 3.1.6 [46]. Maps $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\Im$ are called compatible if

$$\lim_{n\to\infty} \rho(\mathcal{L}_1\mathcal{L}_2\mathbf{\mathcal{L}}_n, \mathcal{L}_2\mathcal{L}_1\mathbf{\mathcal{L}}_n) = 0,$$

where $\mathbf{\mathcal{L}}_n \in \Im$ such that $\lim_{n\to\infty} \mathcal{L}_1\mathbf{\mathcal{L}}_n = \lim_{n\to\infty} \mathcal{L}_2\mathbf{\mathcal{L}}_n = z \in \Im$.

Lemma 3.1.7 [46]. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are compatible mappings on $\Im$. Suppose that $\lim_{n\to\infty} \mathcal{L}_1\mathbf{\mathcal{L}}_n = \lim_{n\to\infty} \mathcal{L}_2\mathbf{\mathcal{L}}_n = z$, for some $z \in \Im$ then $\lim_{n\to\infty} \mathcal{L}_2\mathcal{L}_1\mathbf{\mathcal{L}}_n = \mathcal{L}_1 z$, if $\mathcal{L}_1$ is continuous.

Example 3.1.8. The mapping $\rho : \Im \times \Im \to R^+$ on $\Im$ given by $\rho(\zeta_1, \zeta_2) = |\zeta_1 - \zeta_2|$ for some $\zeta_1, \zeta_2 \in \Im$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\Im$ are given by $\mathcal{L}_1(\zeta) = \frac{1}{10}$, $\mathcal{L}_2(\zeta) = \frac{10\zeta+1}{20}$. Then, $\mathcal{L}_1$

and $\pounds_2$ commute with each other such that $\pounds_1(\pounds_2(\zeta)) = \pounds_2(\pounds_1(\zeta)) = \frac{1}{10}$, with $\frac{1}{10}$ as the unique common fixed point of $\pounds_1$ and $\pounds_2$ and so are weakly compatible on $\Im$.

3.2 Main results

We begin with a simple but useful lemma that will be used in the sequel.

Lemma 3.2.1. If $\pounds_1, \pounds_2, \pounds_3, \pounds_4 : \Im \to \Im$ are mappings of $\Im$ with metric ρ and satisfy the conditions

(*i*) $\pounds_1(\Im) \subseteq \pounds_4(\Im)$ and $\pounds_2(\Im) \subseteq \pounds_3(\Im)$,

(*ii*)

$$\rho(\pounds_1(\mathcal{L}_1), \pounds_2(\mathcal{L}_2)) \leq \max \left\{ \begin{array}{c} 2\lambda_1 \rho(\pounds_3(\mathcal{L}_1), \pounds_4(\mathcal{L}_2)), \lambda_2 \left(\begin{array}{c} \rho(\pounds_1(\mathcal{L}_1), \pounds_4(\mathcal{L}_2)) \\ +\rho(\pounds_2(\mathcal{L}_2), \pounds_3(\mathcal{L}_1)) \end{array} \right) \\ , \lambda_3 (\rho(\pounds_1(\mathcal{L}_1), \pounds_3(\mathcal{L}_1)) + \rho(\pounds_2(\mathcal{L}_2), \pounds_4(\mathcal{L}_2))) \end{array} \right\},$$

for $\mathcal{L}_1, \mathcal{L}_2 \in \Im$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$, are nonnegative real numbers such that $0 \leq \lambda_1, \lambda_2, \lambda_3 < \frac{1}{2}$, then every sequence $\{\mathcal{L}_n\}$converges in $\Im$.

Proof For $\mathcal{L}_0 \in \Im$, choose $\mathcal{L}_1 \in \Im$ such that $\pounds_4\mathcal{L}_1 = \pounds_1\mathcal{L}_0$.and for $\mathcal{L}_1$ there exists $\mathcal{L}_2 \in \Im$ such that $\pounds_3\mathcal{L}_2 = \pounds_2\mathcal{L}_1$. In this way, we construct sequences $\{\mathcal{L}_n\}$ and $\{\xi_n\}$ in $\Im$ given by

$$\begin{cases} \xi_{2n} = \pounds_1\mathcal{L}_{2n} = \pounds_4\mathcal{L}_{2n+1}, \\ \xi_{2n+1} = \pounds_2\mathcal{L}_{2n+1} = \pounds_3\mathcal{L}_{2n+2}. \end{cases}$$

Suppose, there exists $\eta \in 0 \leq \mathcal{L} < 1$ such that $\rho\left(\xi_n, \xi_{n+1}\right) \leq \eta\rho\left(\xi_{n-1}, \xi_n\right)$ for $n \geq 1$. For the convergence of $\{\xi_n\}$ is in $\Im$, we use (i) and (ii) and obtain

$$\rho\left(\xi_{2n}, \xi_{2n+1}\right) = \rho\left(\mathcal{L}_4\mathcal{L}_{2n+1}, \mathcal{L}_3\mathcal{L}_{2n+2}\right) =$$

$$\begin{aligned} \rho\left(\mathcal{L}_1\mathcal{L}_{2n}, \mathcal{L}_2\mathcal{L}_{2n+1}\right) &\leq \max\left\{\begin{array}{c} 2\lambda_1\rho\left(\mathcal{L}_3\left(\mathcal{L}_{2n}\right), \mathcal{L}_4\left(\mathcal{L}_{2n+1}\right)\right), \\ \lambda_2\left(\rho\left(\mathcal{L}_1\left(\mathcal{L}_{2n}\right), \mathcal{L}_4\left(\mathcal{L}_{2n+1}\right)\right) + \rho\left(\mathcal{L}_2\left(\mathcal{L}_{2n+1}\right), \mathcal{L}_3\left(\mathcal{L}_{2n}\right)\right)\right), \\ \lambda_3\left(\rho\left(\mathcal{L}_1\left(\mathcal{L}_{2n}\right), \mathcal{L}_3\left(\mathcal{L}_{2n}\right)\right) + \rho\left(\mathcal{L}_2\left(\mathcal{L}_{2n+1}\right), \mathcal{L}_4\left(\mathcal{L}_{2n+1}\right)\right)\right) \end{array}\right\}, \\ &\leq \max\left\{\begin{array}{c} 2\lambda_1\rho\left(\xi_{2n-1}, \xi_{2n}\right), \\ \lambda_2\left(\rho\left(\xi_{2n}, \xi_{2n}\right) + \rho\left(\xi_{2n+1}, \xi_{2n-1}\right)\right), \\ \lambda_3\left(\rho\left(\xi_{2n}, \xi_{2n-1}\right) + \rho\left(\xi_{2n+1}, \xi_{2n}\right)\right) \end{array}\right\}; \\ &\leq \max\left\{2\eta\rho\left(\xi_{2n-1}, \xi_{2n}\right), \eta\left(\rho\left(\xi_{2n-1}, \xi_{2n}\right) + \rho\left(\xi_{2n}, \xi_{2n+1}\right)\right)\right\}, \end{aligned}$$

where $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\} < \frac{1}{2}$. Hence, either

$$\eta\left(\xi_{2n}, \xi_{2n+1}\right) \leq 2\lambda\rho\left(\xi_{2n-1}, \xi_{2n}\right),$$

or

$$\rho\left(\xi_{2n}, \xi_{2n+1}\right) \leq \frac{\lambda}{(1-\lambda)}\rho\left(\xi_{2n-1}, \xi_{2n}\right).$$

In either case, we have

$$\rho\left(\xi_{2n}, \xi_{2n+1}\right) \leq \eta\rho\left(\xi_{2n-1}, \xi_{2n}\right).$$

Similarly,

$$\rho\left(\xi_{2n-1}, \xi_{2n}\right) \leq \eta^2\rho\left(\xi_{2n-2}, \xi_{2n-1}\right).$$

where $\eta = \max\left\{2\lambda, \frac{\lambda}{(1-\lambda)}\right\} < 1$, and since $0 \leq \lambda < \frac{1}{2}$ we have $0 \leq \eta < 1$. Therefore, for some $n \in \mathbb{N}$, we can write

$$\rho\left(\xi_{n+1}, \xi_{n+2}\right) \leq \eta\rho\left(\xi_n, \xi_{n+1}\right) \leq \ldots \leq \eta^{n+1}\rho\left(\xi_0, \xi_1\right).$$

Now, if $j, n \in \mathbb{N}$ then for $j > n$, we have

$$\begin{aligned}
\rho(\xi_n, \xi_j) &\leq \rho(\xi_n, \xi_{n-1}) + \rho(\xi_{n-1}, \xi_{n-2}) + \ldots + \rho(\xi_{j-1}, \xi_j), \\
&\leq \eta^n \rho(\xi_0, \xi_1) + \eta^{n+1} \rho(\xi_0, \xi_1) + \ldots + \eta^{m-1} \rho(\xi_0, \xi_1), \\
&= \frac{\eta^n}{(1-\eta)} d(\xi_0, \xi_1) \to 0, \text{ as } n, \ j \to \infty.
\end{aligned}$$

Thus,

$$\rho(\xi_n, \xi_j) \to 0 \text{ as } n \to \infty.$$

The following result extends and generalizes Theorem 3.1.2 of Fisher in [35] for four compatible mappings.

Theorem 3.2.2. If $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ self-maps of $\Im$ and satisfy

(i) $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$, $\mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im)$, and

(ii) $\mathcal{L}_4(\Im)$ or $\mathcal{L}_3(\Im)$ is a complete subspace of $\Im$,

(iii) the pairs $\{\mathcal{L}_1, \mathcal{L}_3\}$ and $\{\mathcal{L}_2, \mathcal{L}_4\}$ are weakly compatible,

(iv) $\mathcal{L}_1$and $\mathcal{L}_2$ satisfy the inequality

$$\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_2(\mu_2)) \leq \max \left\{ \begin{array}{c} 2\lambda_1 \rho(\mathcal{L}_3(\mu_1), \mathcal{L}_4(\mu_2)), \lambda_2 \left(\begin{array}{c} \rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) \\ +\rho(\mathcal{L}_2(\mu_2), \mathcal{L}_3(\mu_1)) \end{array} \right), \\ \lambda_3 (\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_3(\mu_1)) + \rho(\mathcal{L}_2(\mu_2), \mathcal{L}_4(\mu_2))) \end{array} \right\}, \tag{3.1}$$

for $\mu_1, \mu_2 \in \Im$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$, are nonnegative real numbers such that

$$0 \leq \lambda_1, \lambda_2, \lambda_3 < \frac{1}{2}, \tag{3.2}$$

then $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ have common fixed point in $\Im$.

Proof In view of condition (i), we define $\{\mu_n\}$ in $\Im$ as $\mu_{2n} = \mathcal{L}_1\mu_{2n} = \mathcal{L}_4\mu_{2n+1}$ and $\mu_{2n+1} = \mathcal{L}_2\mu_{2n+1} = \mathcal{L}_3\mu_{2n+2}$, $n = 0, 1, 2, 3, ..$ First, we use condition (iv) to show that

$\{\mu_n\}$ is a Cauchy sequence in $\Im$. On substituting $\mu_1 = \mu_{2n}$ and $\mu_2 = \mu_{2n+1}$ in inequality (3.1) gives us

$$\begin{aligned}
\rho\left(\mu_{2n}, \mu_{2n+1}\right) &= \rho\left(\mathcal{L}_1\mu_{2n}, \mathcal{L}_2\mu_{2n+1}\right), \\
\rho\left(\mu_{2n}, \mu_{2n+1}\right) &\leq \max\left\{\begin{array}{c} 2\lambda_1\rho\left(\mathcal{L}_3\left(\mu_{2n}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right), \\ \lambda_2\left(\rho\left(\mathcal{L}_1\left(\mu_{2n}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right) + \rho\left(\mathcal{L}_2\left(\mu_{2n+1}\right), \mathcal{L}_3\left(\mu_{2n}\right)\right)\right), \\ \lambda_3\left(\rho\left(\mathcal{L}_1\left(\mu_{2n}\right), \mathcal{L}_3\left(\mu_{2n}\right)\right) + \rho\left(\mathcal{L}_2\left(\mu_{2n+1}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right)\right) \end{array}\right\},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\rho\left(\mu_{2n}, \mu_{2n+1}\right) &\leq \max\left\{\begin{array}{c} 2\lambda_1\rho\left(\mu_{2n-1}, \mu_{2n}\right), \lambda_2\left(\rho\left(\mu_{2n}, \mu_{2n}\right) + \rho\left(\mu_{2n+1}, \mu_{2n-1}\right)\right), \\ \lambda_3\left(\rho\left(\mu_{2n}, \mu_{2n-1}\right) + \rho\left(\mu_{2n+1}, \mu_{2n}\right)\right) \end{array}\right\}, \\
&\leq \max\left\{2\lambda\rho\left(\mu_{2n-1}, \mu_{2n}\right), \lambda\left(\rho\left(\mu_{2n-1}, \mu_{2n}\right) + \rho\left(\mu_{2n}, \mu_{2n+1}\right)\right)\right\},
\end{aligned}$$

where $\lambda = \max\{\lambda_1, \lambda_2, \lambda_3\} < \frac{1}{2}$. Hence, either

$$\rho\left(\mu_{2n}, \mu_{2n+1}\right) \leq 2\lambda\rho\left(\mu_{2n-1}, \mu_{2n}\right),$$

or

$$\rho\left(\mu_{2n}, \mu_{2n+1}\right) \leq \frac{\lambda}{(1-\lambda)}\rho\left(\mu_{2n-1}, \mu_{2n}\right).$$

In either case, we get

$$\rho\left(\mu_{2n}, \mu_{2n+1}\right) \leq \eta\rho\left(\mu_{2n-1}, \mu_{2n}\right). \tag{3.3}$$

Similarly, we can write

$$\rho\left(\mu_{2n-1}, \mu_{2n}\right) \leq \eta^2\rho\left(\mu_{2n-2}, \mu_{2n-1}\right), \tag{3.4}$$

and the above inequalities is possible only if $\rho(\gamma, \mathcal{L}_2 v_1) = 0 \Rightarrow \gamma = \mathcal{L}_2 v_1 = \mathcal{L}_4 v_1$. In other words, v_1 is a coincidence point of $\mathcal{L}_2$ and $\mathcal{L}_4$. Since, $\mathcal{L}_2$ and $\mathcal{L}_4$ are weakly compatible, they commute at coincident point. Therefore, $\mathcal{L}_2\mathcal{L}_4(v_1) = \mathcal{L}_4\mathcal{L}_2(v_1)$ and so $\mathcal{L}_2\gamma = \mathcal{L}_4\gamma$. If $\gamma \neq \mathcal{L}_2\gamma$, then by using (3.1) we get

$$\rho(\mathcal{L}_1\mu_{2n}, \mathcal{L}_2\gamma) \leq \left\{ \begin{array}{c} 2\lambda\rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_4\gamma), \lambda(\rho(\mathcal{L}_1\mu_{2n}, \mathcal{L}_4\gamma) + \rho(\mathcal{L}_2\gamma, \mathcal{L}_3\mu_{2n})), \\ \lambda(\rho(\mathcal{L}_1\mu_{2n}, \mathcal{L}_3\mu_{2n}) + \rho(\mathcal{L}_2\gamma, \mathcal{L}_4\gamma)) \end{array} \right\},$$

as $n \to \infty$, we have

$$\begin{aligned} (\gamma, \mathcal{L}_2\gamma) &\leq \max\left\{ \begin{array}{c} 2\lambda\rho(\gamma, \mathcal{L}_4\gamma), \lambda(\rho(\gamma, \mathcal{L}_4\gamma) + \rho(\gamma, \mathcal{L}_2\gamma)), \\ \lambda(\rho(\gamma, \gamma) + \rho(\mathcal{L}_2\gamma, \mathcal{L}_4\gamma)) \end{array} \right\}, \\ &= \{2\lambda\rho(\gamma, \mathcal{L}_2\gamma), 2\lambda\rho(\gamma, \mathcal{L}_2\gamma), 0\}, \\ &= 2\lambda\rho(\gamma, \mathcal{L}_2\gamma), \end{aligned}$$

and this implies that $(1 - 2\lambda)\rho(\gamma, \mathcal{L}_2\gamma) \leq 0$, which is possible only if $\rho(\gamma, \mathcal{L}_2\gamma) = 0 \Rightarrow \gamma = \mathcal{L}_2\gamma$. Since, $\mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im)$ there exists $u_1 \in \Im$ such that $\mathcal{L}_3 u_1 = \gamma$. If $\mathcal{L}_1 u_1 \neq \gamma$, by (3.1), we have

$$\rho(\mathcal{L}_1 u_1, \mathcal{L}_2\gamma) \leq \max\left\{ \begin{array}{c} 2\lambda\rho(\mathcal{L}_3 u_1, \mathcal{L}_4\gamma), \lambda(\rho(\mathcal{L}_1 u_1, \mathcal{L}_4\gamma) + \rho(\mathcal{L}_2\gamma, \mathcal{L}_3 u_1)), \\ \lambda(\rho(\mathcal{L}_1 u_1, \mathcal{L}_3 u_1) + \rho(\mathcal{L}_2\gamma, \mathcal{L}_4\gamma)) \end{array} \right\},$$

and this gives us

$$\begin{aligned} \rho(\mathcal{L}_1 u_1, \gamma) &\leq \max\left\{ \begin{array}{c} 2\lambda\rho(\mathcal{L}_3 u_1, \gamma), \lambda(\rho(\mathcal{L}_1 u_1, \gamma) + \rho(\gamma, \mathcal{L}_3 u_1)), \\ \lambda(\rho(\mathcal{L}_1 u_1, \mathcal{L}_3 u_1) + \rho(\gamma, \gamma)) \end{array} \right\}, \\ &= \lambda\rho(\mathcal{L}_1 u_1, \gamma), \\ &\Rightarrow (1 - \lambda)\rho(\mathcal{L}_1 u_1, \gamma) \leq 0. \end{aligned}$$

and the inequality is possible only if $\rho(\mathcal{L}_1 u_1, \gamma) = 0 \Rightarrow \mathcal{L}_1 u_1 = \gamma$ and hence, $\mathcal{L}_1 u_1 = \mathcal{L}_3 u_1 = \gamma$. Since, $\mathcal{L}_1$ and $\mathcal{L}_3$ are weakly compatible, $\mathcal{L}_1\mathcal{L}_3 u_1 = \mathcal{L}_3\mathcal{L}_1 u_1$ so, $\mathcal{L}_1\gamma = \mathcal{L}_3\gamma$.

Hence, $\rho(\mathcal{L}_1\gamma,\gamma)=\rho(\mathcal{L}_1\gamma,\mathcal{L}_2\gamma)$ and

$$\rho(\mathcal{L}_1\gamma,\gamma)\le\max\left\{\begin{array}{c}2\lambda\rho(\mathcal{L}_3\gamma,\gamma),\lambda(\rho(\mathcal{L}_1\gamma,\gamma)+\rho(\gamma,\mathcal{L}_3\gamma)),\\ \lambda(\rho(\mathcal{L}_1\gamma,\mathcal{L}_3\gamma)+\rho(\gamma,\gamma))\end{array}\right\},$$

If $\mathcal{L}_1\gamma\neq\gamma$, again by (3.1) we have

$$\begin{aligned}&\max\{2\lambda\rho(\mathcal{L}_1\gamma,\gamma),0\},\\ =\;&2\lambda\rho(\mathcal{L}_1\gamma,\gamma),\end{aligned}$$

This implies that, $(1-2\lambda)\lambda(\mathcal{L}_1u_1,\gamma)\le 0$, which is possible only if $\rho(\mathcal{L}_1u_1,\gamma)=0$. Hence, $\gamma=\mathcal{L}_1u_1$. Thus, $\mathcal{L}_1\gamma=\mathcal{L}_2\gamma=\mathcal{L}_3\gamma=\mathcal{L}_4\gamma=\gamma$ and so γ is a common fixed point of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$.

For uniqueness, suppose $\gamma^*\in\Im$ is another fixed point of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$and $\mathcal{L}_4$ such that $\mathcal{L}_1(\gamma^*)=\mathcal{L}_2(\gamma^*)=\mathcal{L}_3(\gamma^*)=\mathcal{L}_4(\gamma^*)=\gamma^*$. Using (3.1), we have

$$\begin{aligned}\rho(\gamma,\gamma^*)&\le\max\left\{\begin{array}{c}2\lambda\rho(\mathcal{L}_3(\gamma),\mathcal{L}_4(\gamma^*)),\\ \lambda(\rho(\mathcal{L}_1(\gamma),\mathcal{L}_4(\gamma^*))+\rho(\mathcal{L}_2(\gamma^*),\mathcal{L}_3(\gamma))),\\ \lambda(\rho(\mathcal{L}_1(\gamma),\mathcal{L}_3(\gamma))+\rho(\mathcal{L}_2(\gamma^*),\mathcal{L}_4(\gamma^*)))\end{array}\right\},\\ &\le\max\left\{\begin{array}{c}2\lambda\rho(\gamma,\gamma^*),\lambda(\rho(\gamma,\gamma^*)+\rho(\gamma^*,\gamma)),\\ \lambda(\rho(\gamma,\gamma)+\rho(\gamma^*,\gamma^*))\end{array}\right\},\\ &=2\lambda\rho(\gamma,\gamma^*),\end{aligned}$$

which is possible only if $\rho(\gamma,\gamma^*)=0$ and since $\lambda<\frac{1}{2}$, it follows that $\gamma=\gamma^*$, which giving uniqueness of the common fixed point γ of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ in $\Im$.

Example 3.2.3. If $\Im=[0,1]$ with $\rho(\mu_1,\mu_2)=|\mu_1-\mu_2|$ and $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3$ and $\mathcal{L}_4$ on $\Im$ are given by

$$\mathcal{L}_1(\mu)=\mathcal{L}_2(\mu)=\left\{\tfrac{1}{3}\right\}\text{ and}$$

$$g_3(\mu) = \begin{cases} \frac{1}{3}, & 0 \le \mu < 1 \\ 1, & \mu = 1 \end{cases}, \quad g_4(\mu) = \begin{cases} \frac{1}{3}, & 0 \le \mu < \frac{1}{3} \\ 1, & \frac{1}{3} < \mu \le 1. \end{cases}$$

Then $\mathcal{L}_1(\Im) = \mathcal{L}_2(\Im) = \{\frac{1}{3}\}$ and $\mathcal{L}_3(\Im) = \mathcal{L}_4(\Im) = \{\frac{1}{3}, 1\}$. We see that $\mathcal{L}_1(\Im) = \mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im) = \mathcal{L}_4(\Im)$ with $\mathcal{L}_3(\Im)$ and $\mathcal{L}_4(\Im)$ is complete subspace of $\Im$.

Also, we have $\mathcal{L}_1(\mathcal{L}_3(\mu)) = \mathcal{L}_3(\mathcal{L}_1(\mu))$. Similarly, $\mathcal{L}_2(\mathcal{L}_4(\mu)) = \mathcal{L}_4(\mathcal{L}_2(\mu))$. So, the pairs $\{\mathcal{L}_1, \mathcal{L}_3\}$and $\{\mathcal{L}_2, \mathcal{L}_4\}$ commute at coincidence point and are compatible. Hence, these mappings satisfy Theorem 3.2.2 with $\frac{1}{3}$ is the common fixed point of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$and $\mathcal{L}_4$ which is unique.

Corollary 3.2.4. If $(\Im, \rho)$ is complete and $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_4 : \Im \to \Im$ are mappings satisfying the following conditions:

(i) $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$, $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_3(\Im)$ and

(ii) $\mathcal{L}_3(\Im)$ or $\mathcal{L}_4(\Im)$ is subspace of $\Im$ which is complete

(iii) the pairs $\{\mathcal{L}_1, \mathcal{L}_3\}$ and $\{\mathcal{L}_1, \mathcal{L}_4\}$ are weakly compatible and satisfy the inequality

$$\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_1(\mu_2)) \le \max \left\{ \begin{array}{c} 2\lambda_1 \rho(\mathcal{L}_3(\mu_1), \mathcal{L}_4(\mu_2)), \\ \lambda_2(\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) + \rho(\mathcal{L}_1(\mu_2), \mathcal{L}_3(\mu_1))), \\ \lambda_3(\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_3(\mu_1)) + \rho(\mathcal{L}_1(\mu_2), \mathcal{L}_4(\mu_2))) \end{array} \right\},$$

for $\mu_1, \mu_2 \in \Im$ and $\lambda_1, \lambda_2, \lambda_3 \ge 0$, are nonnegative real numbers such that $0 \le \lambda_1, \lambda_2, \lambda_3 < \frac{1}{2}$, then $\mathcal{L}_1$, $\mathcal{L}_3$and $\mathcal{L}_4$ posses a common fixed point in $\Im$.

Proof For the proof see Theorem 3.2.2 by putting $\mathcal{L}_2 = \mathcal{L}_1$.

Corollary 3.2.5. If $(\Im, \rho)$ be a complete metric space and let $\mathcal{L}_1, \mathcal{L}_4 : \Im \to \Im$ are commuting maps such that

(i) $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$,

(ii) $\mathcal{L}_4(\Im)$ is a complete subspace of $\Im$.

(iii) the pair$\{\mathcal{L}_1, \mathcal{L}_4\}$ is weakly compatible,

(iv) $\pounds_1$and $\pounds_4$ satisfy the inequality

$$\rho(\pounds_1(\mu_1), \pounds_1(\mu_2)) \leq \max \left\{ \begin{array}{c} 2\lambda_1 \rho(\pounds_4(\mu_1), \pounds_4(\mu_2)), \\ \lambda_2(\rho(\pounds_1(\mu_1), \pounds_4(\mu_2)) + \rho(\pounds_1(\mu_2), \pounds_4(\mu_1))), \\ \lambda_3(\rho(\pounds_1(\mu_1), \pounds_4(\mu_1)) + \rho((\mu_2), \pounds_4(\mu_2))) \end{array} \right\},$$

for $\mu_1, \mu_2 \in \Im$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$, are nonnegative reals such that $0 \leq \lambda_1, \lambda_2, \lambda_3 < \frac{1}{2}$, then $\pounds_1$and $\pounds_4$ possesses a common fixed point in $\Im$.

Proof For the proof see Theorem 3.2.2 by putting $\pounds_2 = \pounds_1$and $\pounds_3 = \pounds_4$.

Corollary 3.2.6. If $\pounds_4 : \Im \to \Im$ satisfies

$$\rho(\pounds_4\mu_1, \pounds_4\mu_2) \leq \max \left\{ \begin{array}{c} 2\lambda_1\rho(\mu_1, \mu_2), \lambda_2(\rho(\mu_1, \pounds_4\mu_1) + \rho(\mu_2, \pounds_4\mu_2)), \\ \lambda_3(\rho(\mu_1, \pounds_4\mu_2) + \rho(\mu_2, \pounds_4\mu_1)) \end{array} \right\},$$

for some $\mu_1, \mu_2 \in \Im$ and $0 \leq \lambda_1, \lambda_2, \lambda_3 < \frac{1}{2}$, then $\pounds_4$ has a unique fixed point.

Proof For the proof see Theorem 3.2.2 on putting $\pounds_1 = \pounds_2 = \pounds_4$ and $\pounds_3 = I_\pounds$ (Identity mapping).

Our next theorem is an extension of Theorem 3.1.3 in [35].

Theorem 3.2.7. If $\pounds_1$, $\pounds_2$, $\pounds_3$ and $\pounds_4 : \Im \to \Im$ are mappings and $\Im$ is complete such that

(i) $\pounds_1(\Im) \subseteq \pounds_4(\Im)$, $\pounds_2(\Im) \subseteq \pounds_3(\Im)$,

(ii) $\pounds_3$and $\pounds_4$ are continuous and (*iii*) the pairs $\{\mathcal{L}_1, \mathcal{L}_3\}$ and $\{\mathcal{L}_2, \mathcal{L}_4\}$ are compatible on $\Im$ and satisfy the inequality

$$(\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_2(\mu_2)))^2 \leq \max\left\{\begin{array}{c} 2\alpha_1\rho(\mathcal{L}_3(\mu_1), \mathcal{L}_4(\mu_2))\left(\begin{array}{c}\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_3(\mu_1)) \\ +\rho(\mathcal{L}_2(\mu_2), \mathcal{L}_4(\mu_2))\end{array}\right), \\ 2\alpha_2(\rho\mathcal{L}_3(\mu_1), \mathcal{L}_4(\mu_2))\left(\begin{array}{c}\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) \\ +\rho(\mathcal{L}_2(\mu_2), \mathcal{L}_3(\mu_1))\end{array}\right), \\ \alpha_3(\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_3(\mu_1)) + \rho(\mathcal{L}_2(\mu_2), \mathcal{L}_4(\mu_2))) \times \\ (\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) + \rho(\mathcal{L}_2(\mu_2), \mathcal{L}_3(\mu_1))) \end{array}\right\}, \tag{3.6}$$

for $\mu_1, \mu_2 \in \Im$ and $0 \leq \alpha_1, \alpha_2, \alpha_3 < \frac{1}{4}$, then $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ possesses fixed point in $\Im$.

Proof If $\mu_0 \in \Im$ is any point and $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$,choose $\mu_1 \in \Im$ such that $\mathcal{L}_1(\mu_0) = \mathcal{L}_4(\mu_1)$. Since, $\mu_1 \in \mathcal{L}_3(\Im)$, we can choose $\mu_2 \in \Im$ such that $\mathcal{L}_2\mu_1 = \mathcal{L}_3\mu_2$ and μ_{n+1}, μ_{2n+2} so that we can define the Picard sequence $\{\mu_n\}$ in $\Im$, given by $\mu_{2n} = \mathcal{L}_1\mu_{2n} = \mathcal{L}_4\mu_{2n+1}$and $\mu_{2n+1} = \mathcal{L}_2\mu_{2n+1} = \mathcal{L}_3\mu_{2n+2}$ for some $n \geq 0$ and use $(\rho(\mu_{2n+1}, \mu_{2n+2}))^2 = (\rho(\mathcal{L}_1(\mu_{2n}), \mathcal{L}_2(\mu_{2n+1})))^2$ in (3.6) ,we have

$$\begin{aligned}(\rho(\mu_{2n+1}, \mu_{2n+2}))^2 &\leq \max\left\{\begin{array}{c} 2\alpha_1\rho(\mathcal{L}_3(\mu_{2n}), \mathcal{L}_4(\mu_{2n+1})) \times \\ (\rho(\mathcal{L}_1(\mu_{2n}), \mathcal{L}_3(\mu_{2n})) + \rho(\mathcal{L}_2(\mu_{2n+1}), \mathcal{L}_4(\mu_{2n+1}))), \\ 2\alpha_2\rho(\mathcal{L}_3(\mu_{2n}), \mathcal{L}_4(\mu_{2n+1})) \times \\ (\rho(\mathcal{L}_1(\mu_{2n}), \mathcal{L}_4(\mu_{2n+1})) + \rho(\mathcal{L}_2(\mu_{2n+1}), \mathcal{L}_3(\mu_{2n}))), \\ \alpha_3(\rho(\mathcal{L}_1(\mu_{2n}), \mathcal{L}_3(\mu_{2n})) + \rho(\mathcal{L}_2(\mu_{2n+1}), \mathcal{L}_4(\mu_{2n+1}))) \times \\ (\rho(\mathcal{L}_1(\mu_{2n}), \mathcal{L}_4(\mu_{2n+1})) + \rho(\mathcal{L}_2(\mu_{2n+1}), \mathcal{L}_3(\mu_{2n}))) \end{array}\right\}, \\ &= \max\left\{\begin{array}{c} 2\alpha_1\rho(\mu_{2n-1}, \mu_{2n})(\rho(\mu_{2n}, \mu_{2n-1}) + \rho(\mu_{2n+1}, \mu_{2n})), \\ 2\alpha_2\rho(\mu_{2n-1}, \mu_{2n})(\rho(\mu_{2n}, \mu_{2n}) + \rho(\mu_{2n+1}, \mu_{2n-1})), \\ \alpha_3(\rho(\mu_{2n}, \mu_{2n-1}) + \rho(\mu_{2n+1}, \mu_{2n})) \times \\ (\rho(\mu_{2n}, \mu_{2n}) + \rho(\mu_{2n+1}, \mu_{2n-1})) \end{array}\right\}, \\ &\leq \max\left\{\begin{array}{c} 2\alpha d(\mu_{2n-1}, \mu_{2n})(\rho(\mu_{2n}, \mu_{2n-1}) + \rho(\mu_{2n+1}, \mu_{2n})), \\ \alpha(\rho(\mu_{2n+1}, \mu_{2n}) + \rho(\mu_{2n}, \mu_{2n-1}))^2 \end{array}\right\}.\end{aligned}$$

where $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$.

Now, $\Im$ is complete, there exist $\gamma \in \Im$ with $\lim_{n\to\infty} \gamma_1\mu_{2n} = \lim_{n\to\infty} \mathcal{L}_4\mu_{2n+1} = \lim_{n\to\infty} \mathcal{L}_2\mu_{2n+1} = \lim_{n\to\infty} \mathcal{L}_3\mu_{2n+2} = \gamma$.

For a common fixed point γ of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$and $\mathcal{L}_4$. By the continuity of $\mathcal{L}_3$, we have, $\lim_{n\to\infty} \mathcal{L}_3^2\mu_{2n+2} = \mathcal{L}_3\gamma$ and $\lim_{n\to\infty} \mathcal{L}_1\mu_{2n} = \gamma$. Since, the pair $\{\mathcal{L}_1, \mathcal{L}_3\}$ is compatible on $\Im$, so $\lim_{n\to\infty} (\mathcal{L}_1\mathcal{L}_3\mu_{2n}, \mathcal{L}_3\mathcal{L}_1\mu_{2n}) = 0$. By Lemma 3.1.10, we have $\lim_{n\to\infty} \mathcal{L}_1\mathcal{L}_3\mu_{2n} = \mathcal{L}_3\gamma$. If $\mu_1 = \mathcal{L}_3\mu_{2n}$ and $\mu_2 = \mu_{2n+1}$and using $(\rho(\mathcal{L}_3\gamma, \gamma))^2 = (\rho(\mathcal{L}_1(\mathcal{L}_3\mu_{2n}), \mathcal{L}_2(\mu_{2n+1})))^2$ in (3.6), we obtain

$$(\rho(\mathcal{L}_3\gamma,\gamma))^2 \le \max\left\{\begin{array}{c} 2\alpha_1\rho\left(\mathcal{L}_3\left(\mathcal{L}_3\mu_{2n}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right) \times \\ \left(\rho\left(\mathcal{L}_1\left(\mathcal{L}_3\mu_{2n}\right), \mathcal{L}_3\left(\mathcal{L}_3\mu_{2n}\right)\right) + \rho\left(\mathcal{L}_2\left(\mu_{2n+1}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right)\right) \\ , 2\alpha_2\rho\left(\mathcal{L}_3\left(\mathcal{L}_3\mu_{2n}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right) \times \\ \left(\rho\left(\mathcal{L}_1\left(\mathcal{L}_3\mu_{2n}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right) + \rho\left(\mathcal{L}_2\left(\mu_{2n+1}\right), \mathcal{L}_3\left(\mathcal{L}_3\mu_{2n}\right)\right)\right) \\ , \alpha_3\left(\rho\left(\mathcal{L}_1\left(\mathcal{L}_3\mu_{2n}\right), \mathcal{L}_3\left(\mathcal{L}_3\mu_{2n}\right)\right) + \rho\left(\mathcal{L}_2\left(\mu_{2n+1}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right)\right) \times \\ \left(\rho\left(\mathcal{L}_1\left(\mathcal{L}_3\mu_{2n}\right), \mathcal{L}_4\left(\mu_{2n+1}\right)\right) + \rho\left(\mathcal{L}_2\left(\mu_{2n+1}\right), \mathcal{L}_3\left(\mu_{2n}\right)\right)\right) \end{array}\right\},$$

taking limit as $n \to \infty$, yields

$$\begin{aligned} (\rho(\mathcal{L}_3\gamma,\gamma))^2 &\le \max\left\{\begin{array}{c} 2\alpha\rho(\mathcal{L}_3\gamma,\gamma)(\rho(\mathcal{L}_3\gamma,\mathcal{L}_3\gamma)+\rho(\gamma,\gamma)), \\ 2\alpha\rho(\mathcal{L}_3\gamma,\gamma)(\rho(\mathcal{L}_3\gamma,\gamma)+\rho(\gamma,\mathcal{L}_3\gamma)), \\ \alpha(\rho(\mathcal{L}_3\gamma,\mathcal{L}_3\gamma)+\rho(\gamma,\gamma)) \times \\ (\rho(\mathcal{L}_3\gamma,\gamma)+\rho(\gamma,\mathcal{L}_3\gamma)) \end{array}\right\}, \\ &\le \max\left\{0, 4\alpha(\rho(\gamma,\mathcal{L}_3\gamma))^2, 0\right\}, \\ (\rho(\mathcal{L}_3\gamma,\gamma))^2 &\le 4\alpha(\rho(\gamma,\gamma))^2. \end{aligned}$$

where $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\} < \frac{1}{4}$ and the above inequality is possible only if $(\rho(\mathcal{L}_3\gamma,\gamma))^2 = 0 \Rightarrow \rho(\mathcal{L}_3\gamma,\gamma) = 0 \Rightarrow \mathcal{L}_3\gamma = \gamma$ since $0 \le \alpha < \frac{1}{4}$. Next, we will show that $\mathcal{L}_3\gamma = \mathcal{L}_4\gamma = \gamma$. Using continuity of $\mathcal{L}_4$, we have $\lim_{n\to\infty} \mathcal{L}_3(\mathcal{L}_4\mu_{2n+1}) = \mathcal{L}_4\gamma$ and $\lim_{n\to\infty} \mathcal{L}_4\mathcal{L}_2\mu_{2n+1} = \mathcal{L}_4\gamma$. Since, $\mathcal{L}_2$ and $\mathcal{L}_4$ are compatible, $\lim_{n\to\infty} \rho(\mathcal{L}_2\mathcal{L}_4\mu_{2n}, \mathcal{L}_4\mathcal{L}_2\mu_{2n}) = 0$. By Lemma

3.1.8, we have $\lim_{n\to\infty} \pounds_2\pounds_4\mu_{2n} = \pounds_4\gamma$. Putting $\mu_1 = \mu_{2n}$ and $\mu_2 = \pounds_4\mu_{2n+1}$ and using $(\rho(\gamma, \pounds_4\gamma))^2 = (\rho(\pounds_1(\mu_{2n}), \pounds_2(\pounds_4\mu_{2n+1})))^2$ in (3.6), we obtain

$$(\rho(\gamma, \pounds_4\gamma))^2 \leq \max\left\{\begin{array}{c} 2\alpha_1\rho(\pounds_3(\mu_{2n}), \pounds_4(\pounds_4\mu_{2n+1})) \times \\ (\rho(\pounds_1(\mu_{2n}), \pounds_3(\mu_{2n})) + \rho(\pounds_2(\pounds_4\mu_{2n+1}), \pounds_4(\pounds_4\mu_{2n+1}))), \\ 2\alpha_2\rho(\pounds_3(\mu_{2n}), \pounds_4(\pounds_4\mu_{2n+1})) \times \\ (\rho(\pounds_1(\mu_{2n}), \pounds_4(\pounds_4\mu_{2n+1})) + \rho(\pounds_2(\pounds_4\mu_{2n+1}), \pounds_3(\mu_{2n}))), \\ \alpha_3(\rho(\pounds_1(\mu_{2n}), \pounds_3(\mu_{2n})) + \rho(\pounds_3(\pounds_4\mu_{2n+1}), \pounds_4(\pounds_4\mu_{2n+1}))) \times \\ (\rho(\pounds_1(\mu_{2n}), \pounds_4(\pounds_4\mu_{2n+1})) + \rho(\pounds_2(\pounds_4\mu_{2n+1}), \pounds_3(\mu_{2n}))) \end{array}\right\},$$

taking limit as $n \to \infty$, we obtain

$$\begin{aligned} (\rho(\gamma, \pounds_4\gamma))^2 &\leq \max\left\{\begin{array}{c} 2\alpha\rho(\gamma, \pounds_4\gamma)(\rho(\gamma,\gamma) + \rho(\pounds_4\gamma, \pounds_4\gamma)), \\ 2\alpha\rho(\gamma, \pounds_4\gamma)(\rho(\gamma, \pounds_4\gamma) + \rho(\pounds_4\gamma, \gamma)), \\ \alpha(\rho(\gamma,\gamma) + \rho(\pounds_4\gamma, \pounds_4\gamma)) \times (\rho(\gamma, \pounds_4\gamma) + \rho(\pounds_4\gamma, \gamma)) \end{array}\right\}, \\ &\leq \max\left\{0, 4\alpha(\rho(\gamma, \pounds_4\gamma))^2, 0\right\}, \\ (\rho(\gamma, \pounds_4\gamma))^2 &\leq 4\alpha(\rho(\gamma, \pounds_4\gamma))^2, \end{aligned}$$

which is contradiction, because $0 \leq \alpha < \frac{1}{4} \Rightarrow (\rho(\pounds_4\gamma, \gamma))^2 \leq 0$ and the inequality is possible only if $\pounds_4\gamma = \gamma$. Hence, $\pounds_4\gamma = \pounds_3\gamma = \gamma$. Again, utilizing condition (3.6) and using $\rho(\pounds_1\gamma, \gamma) = \rho(\pounds_1\gamma, \pounds_2(\mu_{2n+1}))$, we obtain

$$(\rho(\pounds_1\gamma, \gamma))^2 \leq \max\left\{\begin{array}{c} 2\alpha_1\rho(\pounds_3\gamma, \pounds_4(\mu_{2n+1})) \times \\ (\rho(\pounds_1\gamma, \pounds_3\gamma) + \rho(\pounds_2(\mu_{2n+1}), \pounds_4(\mu_{2n+1}))), \\ 2\alpha_2\rho(\pounds_3\gamma, \pounds_4(\mu_{2n+1})) \times \\ (\rho(\pounds_1\gamma, \pounds_4(\mu_{2n+1})) + \rho(\pounds_2(\mu_{2n+1}), \pounds_3\gamma)), \\ \alpha_3(\rho(\pounds_1\gamma, \pounds_3\gamma) + \rho(\pounds_2(\mu_{2n+1}), \pounds_4(\mu_{2n+1}))) \times \\ (\rho(\pounds_1(\gamma), \pounds_4(\mu_{2n+1})) + \rho(\pounds_2(\mu_{2n+1}), \pounds_3\gamma)) \end{array}\right\},$$

taking limit as $n \to \infty$, and use $\pounds_3\gamma = \pounds_4\gamma = \gamma$, we have

$$\begin{aligned}
(\rho(\pounds_1\gamma,\gamma))^2 &\leq \max\left\{\begin{array}{c} 2\alpha\rho(\pounds_3\gamma,\gamma)(\rho(\pounds_1\gamma,\pounds_3\gamma)+\rho(\gamma,\gamma)), \\ 2\alpha\rho(\pounds_3\gamma,\gamma)(\rho(\pounds_1\gamma,\gamma)+\rho(\gamma,\pounds_3\gamma)), \\ \alpha(\rho(\pounds_1\gamma,\pounds_3\gamma)+\rho(\gamma,\gamma))\times \\ (\rho(\pounds_1\gamma,\gamma)+\rho(\gamma,\pounds_3\gamma)) \end{array}\right\}, \\
&= \max\left\{0,0,\alpha(\rho(\pounds_1\gamma,\gamma))^2\right\}, \\
&= \alpha(\rho(\gamma,\pounds_1\gamma))^2, \\
(\rho(\pounds_1\gamma,\gamma))^2 &\leq \alpha(\rho(\gamma,\pounds_1\gamma))^2,
\end{aligned}$$

which implies that $\rho(\pounds_1\gamma,\gamma) = 0 \Rightarrow \pounds_1\gamma = \gamma$, since $0 \leq \alpha < \frac{1}{4}$. Finally, using condition (3.6) and the fact that $\pounds_3\gamma = \pounds_4\gamma = \pounds_1\gamma = \gamma$, we have $(\rho(\gamma,\pounds_2\gamma))^2 = (\rho(\pounds_1\gamma,\pounds_2\gamma))^2$

$$\begin{aligned}
(\rho(\gamma,\pounds_2\gamma))^2 &\leq \max\left\{\begin{array}{c} 2\alpha\rho(\pounds_3\gamma,\pounds_4\gamma)(\rho(\pounds_1\gamma,\pounds_3\gamma)+\rho(\pounds_2\gamma,\pounds_4\gamma)), \\ 2\alpha\rho(\pounds_3\gamma,\pounds_4\gamma)(\rho(\pounds_1\gamma,\pounds_4\gamma)+\rho(\pounds_2\gamma,\pounds_3\gamma)), \\ \alpha(\rho(\pounds_1\gamma,\pounds_3\gamma)+\rho(\pounds_2\gamma,\pounds_4\gamma))\times \\ (\rho(\pounds_1\gamma,\pounds_4\gamma)+\rho(\pounds_2\gamma,\pounds_3\gamma)) \end{array}\right\}, \\
(\rho(\gamma,\pounds_2\gamma))^2 &= (\rho(\pounds_1\gamma,\pounds_2\gamma))^2, \\
&= \max\left\{0,0,\alpha(\rho(\gamma,\pounds_2\gamma))^2\right\}, \\
&= \alpha(\rho(\gamma,\pounds_2\gamma))^2, \\
&\Rightarrow (\rho(\gamma,\pounds_2\gamma))^2 \leq \alpha(\rho(\gamma,\pounds_2\gamma))^2,
\end{aligned}$$

and the above inequality is possible only, if $\rho(\gamma,\pounds_2\gamma) = 0$ implies $\pounds_2\gamma = \gamma$. Hence, $\pounds_1\gamma = \pounds_2\gamma = \pounds_3\gamma = \pounds_4\gamma = \gamma$.

For uniqueness, let $\gamma \neq \gamma_1$ is another common fixed point of $\pounds_1$, $\pounds_2$, $\pounds_3$and $\pounds_4$. We prove that $\gamma = \gamma_1$.

Putting $\mu_1 = \gamma$ and $\mu_2 = \gamma_1$ in (3.6), we obtain

$$\begin{aligned}
(\rho(\gamma,\gamma_1))^2 &\leq \max\left\{\begin{array}{c} 2\alpha\rho(\mathcal{L}_3\gamma,\mathcal{L}_4\gamma_1)(\rho(\mathcal{L}_1\gamma,\mathcal{L}_3\gamma)+\rho(\mathcal{L}_2\gamma_1,\mathcal{L}_4\gamma_1)), \\ 2\alpha\rho(\mathcal{L}_3\gamma,\mathcal{L}_4\gamma_1)(\rho(\mathcal{L}_1\gamma,\mathcal{L}_4\gamma_1)+\rho(\mathcal{L}_2\gamma_1,\mathcal{L}_3\gamma)), \\ \alpha(\rho(\mathcal{L}_1\gamma,\mathcal{L}_3\gamma)+\rho(\mathcal{L}_2\gamma_1,\mathcal{L}_4\gamma_1))\times \\ (\rho(\mathcal{L}_1\gamma,\mathcal{L}_4\gamma_1)+\rho(\mathcal{L}_1\gamma_1,\mathcal{L}_3\gamma)) \end{array}\right\}, \\
&= \max\left\{\begin{array}{c} 2\alpha\rho(\gamma,\gamma_1)(\rho(\gamma,\gamma)+\rho(\gamma_1,\gamma_1)), 2\alpha\rho(\gamma,\gamma_1)(\rho(\gamma,\gamma_1)+\rho(\gamma_1,\gamma)), \\ \alpha(\rho(\gamma,\gamma)+\rho(\gamma_1,\gamma_1))\times(\rho(\gamma,\gamma_1)+\rho(\gamma_1,\gamma)) \end{array}\right\}, \\
&= \max\left\{0, 4\alpha(\rho(\gamma,\gamma_1))^2, 0\right\}, \\
&= 4\alpha(\rho(\gamma,\gamma_1))^2, \\
&\Rightarrow (\rho(\gamma,\gamma_1))^2 \leq 4\alpha(\rho(\gamma,\gamma_1))^2,
\end{aligned}$$

or

$$(1-4\alpha)(\rho(\gamma,\gamma_1))^2 \leq 0,$$

and is possible only if $(\rho(\gamma,\gamma_1))^2 = 0 \Rightarrow \rho(\gamma,\gamma_1) = 0$ or $\gamma = \gamma_1$ giving us uniqueness of the common fixed point γ of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$and $\mathcal{L}_4$.

Example 3.2.8. If $\Im = [0,1]$ with the metric $\rho(\mu_1,\mu_2) = |\mu_1 - \mu_2|$ and define self-maps $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$and $\mathcal{L}_4$ on $\Im$ by

$$\mathcal{L}_1(\mu) = \begin{cases} \frac{\mu}{10}, & 0 \leq \mu \leq 1 \\ 0, & \mu = 1, \end{cases}, \ \mathcal{L}_2(\mu) = \begin{cases} \frac{\mu}{4}, & 0 \leq \mu \leq \frac{1}{2} \\ 0 & , \frac{1}{2} < \mu \leq 1 \end{cases},$$

$$\mathcal{L}_3(\mu) = \mu \text{ and } \mathcal{L}_4(\mu) = \frac{\mu}{2}.$$

Clearly, $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$ and $\mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im)$ for some $\mu_1, \mu_2 \in \Im$. Furthermore, the pairs $\{\mathcal{L}_1, \mathcal{L}_3\}$ and $\{\mathcal{L}_2, \mathcal{L}_4\}$ are weakly compatible. Therefore, $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$and $\mathcal{L}_4$ satisfy all conditions of Theorem 3.2.7 with $\mu = 0$ is the unique common fixed point in $\Im$.

Corollary 3.2.9. If $\mathcal{L}_1$, $\mathcal{L}_3$and $\mathcal{L}_4 : \Im \to \Im$ are mappings of $\Im$ which is complete such that

(i) $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$, $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_3(\Im)$,

(ii) $\mathcal{L}_3$and $\mathcal{L}_4$ are continuous and

(iii) the pairs $\{\mathcal{L}_1, \mathcal{L}_3\}$ and $\{\mathcal{L}_1, \mathcal{L}_4\}$ are compatible on $\Im$ and satisfy the inequality

$$(\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_1(\mu_2)))^2 \leq \max \left\{ \begin{array}{c} 2\alpha_1 \rho(\mathcal{L}_3(\mu_1), \mathcal{L}_4(\mu_2)) \left(\begin{array}{c} \rho(\mathcal{L}_1(\mu_1), \mathcal{L}_3(\mu_1)) \\ +\rho(\mathcal{L}_1(\mu_2), \mathcal{L}_4(\mu_2)) \end{array} \right), \\ 2\alpha_2 \rho(\mathcal{L}_3(\mu_1), \mathcal{L}_4(\mu_2)) \left(\begin{array}{c} \rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) \\ +\rho(\mathcal{L}_1(\mu_2), \mathcal{L}_3(\mu_1)) \end{array} \right), \\ \alpha_3 (\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_3(\mu_1)) + \rho(\mathcal{L}_1(\mu_2), \mathcal{L}_4(\mu_2))) \times \\ (\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) + \rho(\mathcal{L}_1(\mu_2), \mathcal{L}_3(\mu_1))) \end{array} \right\},$$

for $\mu_1, \mu_2 \in \Im$ and $0 \leq \alpha_1, \alpha_2, \alpha_3 < \frac{1}{4}$, then $\mathcal{L}_1$, $\mathcal{L}_3$ and $\mathcal{L}_4$ possesses a fixed point in $\Im$.

Proof For the proof put $\mathcal{L}_2 = \mathcal{L}_1$in Theorem 3.2.7.

Corollary 3.2.10. If $(\Im, \rho)$ is complete and $\mathcal{L}_1$, $\mathcal{L}_4 : \Im \to \Im$ are mappings of $\Im$ such that

(i) $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$, $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_4(\Im)$,

(ii) $\mathcal{L}_1$or $\mathcal{L}_4$ is continuous and

(iii) the pair $\{\mathcal{L}_1, \mathcal{L}_4\}$ is compatible on $\Im$ and satisfy the inequality

$$(\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_1(\mu_2)))^2 \leq \max \left\{ \begin{array}{c} 2\alpha_1 \rho(\mathcal{L}_4(\mu), \mathcal{L}_4(\mu_2)) \left(\begin{array}{c} \rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_1)) \\ +\rho(\mathcal{L}_1(\mu_2), \mathcal{L}_4(\mu_2)) \end{array} \right), \\ 2\alpha_2 \rho(\mathcal{L}_4(\mu_1), \mathcal{L}_4(\mu_2)) \left(\begin{array}{c} \rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) \\ +\rho(\mathcal{L}_1(\mu_2), \mathcal{L}_4(\mu_1)) \end{array} \right), \\ \alpha_3 (\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_1)) + \rho(\mathcal{L}_1(\mu_2), \mathcal{L}_4(\mu_2))) \times \\ (\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_4(\mu_2)) + \rho(\mathcal{L}_1(\mu_2), \mathcal{L}_4(\mu_1))) \end{array} \right\},$$

for $\mu_1, \mu_2 \in \Im$ and $0 \leq \alpha_1, \alpha_2, \alpha_3 < \frac{1}{4}$, then $\mathcal{L}_1$and $\mathcal{L}_4$ have a unique common fixed point in $\Im$.

Proof For the proof put $\mathcal{L}_2 = \mathcal{L}_1$, $\mathcal{L}_3 = \mathcal{L}_4$ in Theorem 3.2.7 and then follow the result.

Corollary 3.2.11. If $\mathcal{L}_4 : \Im \to \Im$ is a mapping of $\Im$ and satisfy the inequality

$$(\rho(\mathcal{L}_4(\mu_1), \mathcal{L}_4(\mu_2)))^2 \leq \max \left\{ \begin{array}{c} 2\alpha_1((\mu_1), \mathcal{L}_4(\mu_2)) \begin{pmatrix} \rho(\mathcal{L}_4(\mu_1), (\mu_1)) \\ +\rho(\mathcal{L}_4(\mu_2), \mathcal{L}_4(\mu_2)) \end{pmatrix}, \\ 2\alpha_2\rho((\mu_1), \mathcal{L}_4(\mu_2)) \begin{pmatrix} \rho(\mathcal{L}_4(\mu_1), \mathcal{L}_4(\mu_2)) \\ +\rho(\mathcal{L}_4(\mu_2), (\mu_1)) \end{pmatrix}, \\ \alpha_3(\rho(\mathcal{L}_4(\mu_1), (\mu_1)) + \rho(\mathcal{L}_4(\mu_2), \mathcal{L}_4(\mu_2))) \times \\ (\rho(\mathcal{L}_4(\mu_1), \mathcal{L}_4(\mu_2)) + \rho(\mathcal{L}_4(\mu_2), (\mu_1))) \end{array} \right\},$$

for $\mu_1, \mu_2 \in \Im$ and $0 \leq \alpha_1, \alpha_2, \alpha_3 < \frac{1}{4}$, then $\mathcal{L}_4$ possesses a fixed point.

Proof The proof follows from Theorem 3.2.7. by taking $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{L}_4$ and $\mathcal{L}_3 = I_{\mathcal{L}}$.

Chapter 4

b-Metric Generalization of Some Fixed Point Theorems

4.1 Introduction

In this chapter, we continue with the idea of compatible maps and try to prove and generalize several fixed-point results in the setting of b-metric spaces for three mappings. We have proved common fixed-point theorems in complete b-metric spaces for three weakly compatible self mapping. The obtained results are generalizations of b-metric variant of fixed point theorems of Fisher, Pachpatte and Sahu and Sharma.

Following are some of the fixed-point theorems proved in [34], [59], [68].

Theorem 4.1.1 [34]. If $\pounds : \Im \to \Im$ is a mapping of $\Im$ and satisfy $(\rho(\pounds\mu_1, \pounds\mu_2))^2 \leq \alpha_1 (\rho(\mu_1, \pounds\mu_1)\rho(\mu_2, \pounds\mu_2)) + \alpha_2(\rho(\mu_1, \pounds\mu_2)\rho(\mu_2, g\mu_1))$, for some $\mu_1, \mu_2 \in \Im$ and $0 \leq \alpha_1 < 1$, $0 \leq \alpha_2$, then $\pounds$ admits of a fixed point.

Theorem 4.1.2 [59]. If $\pounds : \Im \to \Im$ is a mapping of $\Im$ and satisfy

$$(\rho(\pounds\mu_1, \pounds\mu_2))^2 \leq \alpha_1 (\rho(\mu_1, \pounds\mu_1)\rho(\mu_2, \pounds\mu_2) + \rho(\mu_1, \pounds\mu_2)\rho(\mu_2, \pounds\mu_1)) + \alpha_2 (\rho(\mu_1, \pounds\mu_1)\rho(\mu_2, \pounds\mu_1) + \rho(\mu_1, \pounds\mu_2)\rho(\mu_2, \pounds\mu_2)),$$

for μ_1, μ_2 in $\Im$ where $\alpha_1,\ \alpha_2,\ \alpha_3 \geq 0$, and $\alpha_1 + 2\alpha_2 < 1$, then the map $\pounds$ possesses a fixed point.

Theorem 4.1.3 [68]. If the map $\pounds : \Im \to \Im$ satisfy the inequality

$$\begin{aligned}(\rho(\pounds\mu_1, \pounds\mu_2))^2 \leq\ & \alpha_1 (\rho(\mu_1, \pounds\mu_1)\rho(\mu_2, \pounds\mu_2) + \rho(\mu_1, \pounds\mu_2)\rho(\mu_2, \pounds\mu_1)) + \\ & \alpha_2 (\rho(\mu_1, \pounds\mu_1)\rho(\mu_2, \pounds\mu_1) + \rho(\mu_1, \pounds\mu_2)\rho(\mu_2, \pounds\mu_2)) + \\ & \alpha_3 \left((\rho(\mu_2, \pounds\mu_1))^2 + (\rho(\mu_2, \pounds\mu_2))^2\right),\end{aligned}$$

for $\mu_1,\ \mu_2$ in $\Im$ where $\alpha_1, \alpha_2, \alpha_3 \geq 0$, and $\alpha_1 + 2\alpha_2 + \alpha_3 < 1$, then $\pounds$ gives us a fixed point.

Definition 4.1.4 [29]. (i) $\rho(\zeta_1, \zeta_2) = 0$ if and only if $\zeta_1 = \zeta_2$,

(ii) $\rho(\zeta_1, \zeta_2) = \rho(\zeta_2, \zeta_1)$ for all $\zeta_1, \zeta_2 \in \Im$,

(iii) $\rho(\zeta_1, \zeta_2) \leq k(\rho(\zeta_1, \zeta_3) + \rho(\zeta_3, \zeta_2))$ for all $\zeta_1,\ \zeta_2,\ \zeta_3 \in \Im$.

Then ρ is metric on $\Im$ and $\Im$ together with ρ is called b-metric space.

The following example illustrates the above remarks.

Example 4.1.5 [19]. If $\Im$ is a finite set $\Im = \{0, 1, 2\}$. Define $\rho : \Im \times \Im \to R^+$ by $\rho(0,0) = \rho(1,1) = \rho(2,2) = 0, \rho(1,2) = \rho(2,1) = \rho(1,0) = \rho(0,1) = 1, \rho(0,2) = \rho(2,0) = c \geq 2$ for $k = \frac{c}{2}$. Example 4.1.5 is a b-metric space but is not a metric space for $c > 2$.

Proposition 4.1.6. If the mappings $\pounds_1,\ \pounds_2,\ \pounds_3 : \Im \to \Im$ defined on $\Im$ and the pairs $\{\pounds_1, \pounds_3\}$ and $\{\pounds_2, \pounds_3\}$ are weakly compatible and admits of a point of coincidence then, $\pounds_1,\ \pounds_2,\ \pounds_3$ possesses a common fixed point.

Definition 4.1.7. If $\{\eta_n\}$ converges in a b-metric space $(\Im, \rho)$, it is called complete.

4.2 Common fixed point results

Lemma 4.2.1. If $(\Im, \rho)$ is a b-metric space which is complete for the coefficient $k \geq 1$, and $\pounds_1,\ \pounds_2,\ \pounds_3 : \Im \to \Im$ satisfying the following conditions:

(i) $\pounds_1(\Im) \cup \pounds_2(\Im) \subseteq \pounds_3(\Im)$,

(ii)

$$\begin{aligned}(\rho(\pounds_1\mu_1, \pounds_2\mu_2))^2 \leq\ & \alpha_1(\rho(\pounds_3\mu_2, \pounds_1\mu_1)\rho(\pounds_3\mu_2, \pounds_2\mu_2) + \rho(\pounds_3\mu_1, \pounds_2\mu_2)\rho(\pounds_3\mu_2, \pounds_1\mu_1)) + \\ & \alpha_2(\rho(\pounds_3\mu_2, \pounds_2\mu_1)\rho(\pounds_3\mu_2, \pounds_1\mu_1) + \rho(\pounds_3\mu_1, \pounds_2\mu_2)\rho(\pounds_3\mu_2, \pounds_2\mu_2)) + \\ & \alpha_3\left((\rho(\pounds_3\mu_2, \pounds_1\mu_1))^2 + (\rho(\pounds_3\mu_2, \pounds_2\mu_2))^2\right), \qquad (4.1)\end{aligned}$$

for $\mu_1, \mu_2 \in \Im$; $\alpha_1, \alpha_2, \alpha_3 \geq 0$, such that

$$k\alpha_1 + \left(k^2 + k\right)\alpha_2 + \alpha_3 < 1, \qquad (4.2)$$

then every sequence $\{\eta_n\}$ that converges in $\Im$ is called Cauchy in $\Im$.

Proof For $\mu_0 \in \Im$ consider $\mu_1 \in \Im$ such that $\pounds_3\mu_1 = \pounds_1\mu_0$ and for μ_1 choose $\mu_2 \in \Im$ such that $\pounds_3\mu_2 = \pounds_2\mu_1$, continuing in this way get the sequences $\{\mu_n\}$ and $\{\eta_n\}$ in $\Im$ given by $\eta_{2n} = \pounds_3\mu_{2n+1} = \pounds_1\mu_{2n}$ and $\eta_{2n+1} = \pounds_3\mu_{2n+1} = \pounds_2\mu_{2n+1}$ for $n \geq 0$. Suppose that for $\kappa \in [0, \frac{1}{k})$ we have $\rho\left(\eta_n, \eta_{n+1}\right) \leq \kappa\rho\left(\eta_{n-1}, \eta_n\right)$ for $n \geq 1$. We show that $\{\eta_n\}$ is a Cauchy sequence in $\Im$. Using (4.1) with

$$\left(\rho\left(\eta_{2n}, \eta_{2n+1}\right)\right)^2 = \left(\rho\left(\pounds_3\mu_{2n+1}, \pounds_3\mu_{2n+2}\right)\right)^2 = \left(\rho\left(\pounds_1\mu_{2n}, \pounds_2\mu_{2n+1}\right)\right)^2,$$

we have

$$
\begin{aligned}
\left(\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)^2 &\leq \alpha_1\left(\begin{array}{c}\rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_1\mu_{2n}\right)\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_2\mu_{2n+1}\right)+\\ \rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_2\mu_{2n+1}\right)\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_1\mu_{2n}\right)\end{array}\right)+\\
&\quad \alpha_2\left(\begin{array}{c}\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_2\mu_{2n}\right)\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_1\mu_{2n}\right)+\\ \rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_2\mu_{2n+1}\right)\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_2\mu_{2n+1}\right)\end{array}\right)+\\
&\quad \alpha_3\left(\left(\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_3\mu_{2n+1}\right)\right)^2+\left(\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_3\mu_{2n+2}\right)\right)\right)^2,\\
&\leq \alpha_1\left(\begin{array}{c}\rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_3\mu_{2n+1}\right)\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_3\mu_{2n+2}\right)+\\ \rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_3\mu_{2n+2}\right)\left(\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_3\mu_{2n+1}\right)\right)\end{array}\right)+\\
&\quad \alpha_2\left(\begin{array}{c}\rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_3\mu_{2n+!}\right)\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_3\mu_{2n+1}\right)+\\ \rho\left(\mathcal{L}_3\mu_{2n}\mathcal{L}_3\mu_{2n+2}\right)\rho\left(\mathcal{L}_3\mu_{2n+1},\mathcal{L}_3\mu_{2n+2}\right)\end{array}\right)+\\
&\quad \alpha_3\left(\left(\rho\left(\eta_{2n},\eta_{2n}\right)\right)^2+\left(\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)\right)^2,\\
&\leq \alpha_1\left(\begin{array}{c}\rho\left(\eta_{2n-1},\eta_{2n}\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)+\\ \rho\left(\eta_{2n-1},\eta_{2n+1}\right)\rho\left(\eta_{2n},\eta_{2n}\right)\end{array}\right)+\\
&\quad \alpha_2\left(\begin{array}{c}\rho\left(\eta_{2n-1},\eta_{2n}\right)\rho\left(\eta_{2n},\eta_{2n}\right)+\\ \rho\left(\eta_{2n-1},\eta_{2n+1}\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)\end{array}\right)+\\
&\quad \alpha_3\left(\left(\rho\left(\eta_{2n},\eta_{2n}\right)\right)^2+\left(\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)\right)^2.
\end{aligned}
$$

equivalently

$$\begin{aligned}
\rho\left(\eta_{2n},\eta_{2n+1}\right) &\leq \alpha_1\rho\left(\eta_{2n-1},\eta_{2n}\right)+\alpha_2\rho\left(\eta_{2n-1},\eta_{2n+1}\right)+\alpha_3\rho\left(\eta_{2n},\eta_{2n+1}\right)\\
&\leq \alpha_1\rho\left(\eta_{2n-1},\eta_{2n}\right)+k\alpha_2\left(\rho\left(\eta_{2n-1},\eta_{2n}\right)+\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)+\\
&\quad \alpha_3\rho\left(\eta_{2n},\eta_{2n+1}\right),\\
&\leq \left(\alpha_1+k\alpha_2\right)\rho\left(\eta_{2n-1},\eta_{2n}\right)+\left(k\alpha_2+\alpha_3\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)\\
\left(1-\left(k\alpha_2+\alpha_3\right)\right)\rho\left(\eta_{2n},\eta_{2n+1}\right) &\leq \left(\alpha_1+k\alpha_2\right)\rho\left(\eta_{2n-1},\eta_{2n}\right),\\
\rho\left(\eta_{2n},\eta_{2n+1}\right) &\leq \frac{\left(\alpha_1+k\alpha_2\right)}{\left(1-ks\alpha_2+\alpha_3\right)}\rho\left(\eta_{2n-1},\eta_{2n}\right),\\
&\leq \phi\rho\left(\eta_{2n},\eta_{2n-1}\right),\\
\rho\left(\eta_{2n},\eta_{2n+1}\right) &\leq \phi\rho\left(\eta_{2n},\eta_{2n-1}\right),
\end{aligned}$$

where $\phi=\frac{(\alpha_1+k\alpha_2)}{(1-ks\alpha_2+\alpha_3)}$. Therefore, for all $k\in\mathbb{N}$, we can write

$$\rho\left(\eta_{n+1},\eta_{n+2}\right)\leq\phi\rho\left(\eta_n,\eta_{n+1}\right)\leq\ldots\leq\phi^{n+1}\rho\left(\eta_0,\eta_1\right).$$

Now, for any j, $n\in\mathbb{N}$, $j>n$, we have

$$\begin{aligned}
\rho\left(\eta_k,\eta_j\right) &\leq k\rho\left(\eta_n,\eta_{n+1}\right)+k\rho\left(\eta_{n+1},\eta_j\right),\\
&\leq k\rho\left(\eta_n,\eta_{n+1}\right)+k^2\rho\left(\eta_{n+1},\eta_{n+2}\right)+k^3\rho\left(\eta_{n+2},\eta_j\right),\\
&\leq k\rho\left(\eta_n,\eta_{n+1}\right)+k^2\rho\left(\eta_{n+1},\eta_{n+2}\right)+k^3\rho\left(\eta_{n+2},\eta_{n+3}\right)+\ldots+\\
&\quad k^{j-n-1}\rho\left(\eta_{j-2},\eta_{j-1}\right)+k^{j-n-1}\rho\left(\eta_{j-1},\eta_j\right),\\
&\leq \left(k\phi^n+k^2\phi^{n+1}+k^3\phi^{n+2}+\ldots+k^{j-n}\phi^{j-1}\right)\rho\left(\eta_1,\eta_0\right),\\
&\leq \frac{k\phi^n}{(1-k\phi)}\rho\left(\eta_1,\eta_0\right)\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Thus, $\rho\left(\eta_n,\eta_j\right)\to 0$ as $n\to\infty$. Hence $\{\eta_n\}$ is a converges in b-metric space $(\Im,\rho)$.

Our first theorem is b-metric variant of Theorem 4.1.3 in [68].

Theorem 4.2.2. If $(\Im, \rho)$ is a b-metric space which is complete with the coefficient $k \geq 1$. Suppose, that the mappings $\pounds_1,\ \pounds_2,\ \pounds_3 : \Im \to \Im$ satisfy the condition:

$$\begin{aligned}(\rho(\pounds_1\mu_1, \pounds_2\mu_2))^2 \leq\ & \alpha_1 (\rho(\pounds_3\mu_1, \pounds_1\mu_1)\rho(\pounds_3\mu_2, \pounds_2\mu_2) + \rho(\pounds_3\mu_1, \pounds_2\mu_2)\rho(\pounds_3\mu_2, \pounds_1\mu_1)) + \\ & \alpha_2 (\rho(\pounds_3\mu_2, \pounds_2\mu_1)\rho(\pounds_3\mu_2, \pounds_1\mu_1) + \rho(\pounds_3\mu_1, \pounds_2\mu_2)\rho(\pounds_3\mu_2, \pounds_2\mu_2)) + \\ & \alpha_3 \left((\rho(\pounds_3\mu_2, \pounds_1\mu_1))^2 + (\rho(\pounds_3\mu_2, \pounds_2\mu_2))^2\right), \end{aligned} \tag{4.3}$$

where $\alpha_1,\ \alpha_2,\ \alpha_3 \geq 0$, are nonnegative reals with

$$k\alpha_1 + \left(k^2 + k\right)\alpha_2 + \alpha_3 < 1. \tag{4.4}$$

If $\pounds_1(\Im) \cup \pounds_2(\Im) \subseteq \pounds_3(\Im)$ and $\pounds_3(\Im)$ is a complete subspace of $\Im$. Then the maps $\pounds_1$, $\pounds_2$ and $\pounds_3$ have point of coincidence δ in $\Im$. Moreover, if $\{\pounds_1,\ \pounds_3\}$ and $\{\pounds_2,\ \pounds_3\}$ are weakly compatible pairs. Then $\pounds_1$, $\pounds_2$ and $\pounds_3$ possesses a unique common fixed point in $\Im$.

Proof If for some $\mu_0 \in \Im$ we have the sequence $\{\eta_n\}$ in $\Im$ such that $\eta_{2n} = \pounds_3\mu_{2n+1} = \pounds_1\mu_{2n}$ and $\eta_{2n+1} = \pounds_3\mu_{2n+2} = \pounds_2\mu_{2n+1}$ for $n = 0, 1, 2, \ldots$ Then, we show that $\{\eta_n\}$ is a

Cauchy sequence and by (4.1) $\left(\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)^2=\left(\rho\left(\pounds_1\mu_{2n},\pounds_2\mu_{2n+1}\right)\right)^2$ and

$$\begin{aligned}
\left(\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)^2 &\leq \alpha_1\begin{pmatrix}\rho\left(\pounds_3\mu_{2n},\pounds_1\mu_{2n}\right)\rho\left(\pounds_3\mu_{2n+1},\pounds_2\mu_{2n+1}\right)+\\ \rho\left(\pounds_3\mu_{2n},\pounds_2\mu_{2n+1}\right)\rho\left(\pounds_3\mu_{2n+1},\pounds_1\mu_{2n}\right)\end{pmatrix}+\\
&\quad \alpha_2\begin{pmatrix}\rho\left(\pounds_3\mu_{2n+1},\pounds_2\mu_{2n}\right)\rho\left(\pounds_3\mu_{2n+1},\pounds_1\mu_{2n}\right)+\\ \rho\left(\pounds_3\mu_{2n},\pounds_2\mu_{2n+1}\right)\rho\left(\pounds_3\mu_{2n+1},\pounds_2\mu_{2n+1}\right)\end{pmatrix}+\\
&\quad \alpha_3\left(\left(\rho\left(\pounds_3\mu_{2n+1},\pounds_1\mu_{2n}\right)\right)^2+\left(\rho\left(\pounds_3\mu_{2n+1},\pounds_2\mu_{2n+1}\right)\right)^2\right),\\
&\leq \alpha_1\begin{pmatrix}\rho\left(\eta_{2n-1},\eta_{2n}\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)+\\ \rho\left(\eta_{2n-1},\eta_{2n}\right)\rho\left(\eta_{2n},\eta_{2n}\right)\end{pmatrix}+\\
&\quad \alpha_2\begin{pmatrix}\rho\left(\eta_{2n},\eta_{2n}\right)\rho\left(\eta_{2n},\eta_{2n}\right)+\\ \rho\left(\eta_{2n-1},\eta_{2n+1}\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)\end{pmatrix}+\\
&\quad \alpha_3\left(\left(\rho\left(\eta_{2n},\eta_{2n}\right)\right)^2+\left(\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)^2\right),
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_1\left(\rho\left(\eta_{2n-1},\eta_{2n}\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)+\\
&\quad \alpha_2\left(\rho\left(\eta_{2n-1},\eta_{2n+1}\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)+\alpha_3\left(\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)^2,\\
&\leq \alpha_1\rho\left(\eta_{2n-1},\eta_{2n}\right)+\alpha_2\rho\left(\eta_{2n-1},\eta_{2n+1}\right)+\alpha_3\rho\left(\eta_{2n},\eta_{2n+1}\right),\\
&\leq \alpha_1\rho\left(\eta_{2n-1},\eta_{2n}\right)+k\alpha_2\left(\rho\left(\eta_{2n-1},\eta_{2n}\right)+\rho\left(\eta_{2n},\eta_{2n+1}\right)\right)\\
&\quad +\alpha_3\rho\left(\eta_{2n},\eta_{2n+1}\right),\\
&\leq \left(\alpha_1+k\alpha_2\right)\rho\left(\eta_{2n},\eta_{2n+1}\right)+\left(k\alpha_2+\alpha_3\right)\rho\left(\eta_{2n-1},\eta_{2n}\right),\\
\rho\left(\eta_{2n},\eta_{2n+1}\right) &\leq \frac{\left(\alpha_1+k\alpha_3\right)}{\left(1-\left(k\alpha_2+\alpha_3\right)\right)}\rho\left(\eta_{2n-1},\eta_{2n}\right),\\
&\leq \delta\rho\left(\eta_{2n-1},\eta_{2n}\right),\\
\rho\left(\eta_{2n},\eta_{2n+1}\right) &\leq \delta\rho\left(\eta_{2n-1},\eta_{2n}\right).
\end{aligned}$$

Similarly, we can show that

$$\rho\left(\eta_{2n+1}, \eta_{2n+2}\right) \leq \delta^2 \rho\left(\eta_{2n}, \eta_{2n+1}\right),$$

where $\delta = \frac{(\alpha_1 + k\alpha_2)}{(1-(k\alpha_2+\alpha_3))} < \frac{1}{k}$. Therefore, for all $j,\, n \in \mathbb{N}$ with $j > n$, we get

$$\begin{aligned}
\left(\eta_{n+1}, \eta_{n+2}\right) &\leq \delta\rho\left(\eta_n, \eta_{n+1}\right), \ldots \leq \delta^{n+1}\rho\left(\eta_0, \eta_1\right). \\
\rho\left(\eta_k, \eta_j\right) &\leq k\rho\left(\eta_n, \eta_{n+1}\right) + k\rho\left(\eta_{n+1}, \eta_j\right), \\
&\leq k\rho\left(\eta_n, \eta_{n+1}\right) + k^2\rho\left(\eta_{n+1}, \eta_{n+2}\right) + k^3\rho\left(\eta_{n+2}, \eta_j\right), \\
&\leq k\rho\left(\eta_n, \eta_{n+1}\right) + k^2\rho\left(\eta_{n+1}, \eta_{n+2}\right) + \ldots + \\
&\quad k^3\rho\left(\eta_{n+2}, \eta_{n+3}\right) + k^4\rho\left(\eta_{n+3}, \eta_j\right), \\
&\leq k\rho\left(\eta_n, \eta_{n+1}\right) + k^2\rho\left(\eta_{n+1}, \eta_{n+2}\right) + \ldots + \\
&\quad k^3\rho\left(\eta_{n+2}, \eta_{n+3}\right) + \ldots + k^{j-n-2}\rho\left(\eta_{j-3}, \eta_{j-2}\right) + \\
&\quad k^{j-n-1}\rho\left(\eta_{j-2}, \eta_{j-1}\right) + k^{j-n}\rho\left(\eta_{j-1}, \eta_j\right), \\
\rho\left(\eta_n, \eta_j\right) &\leq \frac{k\delta^n}{(1-k\delta)}\rho\left(\eta_0, \eta_1\right).
\end{aligned}$$

Thus, as $n \to \infty$, $\rho\left(\eta_n, \eta_j\right) \to 0$. From Lemma 4.2.1 it follows that $\{\eta_n\}$ converges to some $\eta \in \Im$. Therefore,

$$\lim_{n\to\infty} \mathcal{L}_1\mu_{2n} = \lim_{n\to\infty} \mathcal{L}_2\mu_{2n+1} = \lim_{n\to\infty} \mathcal{L}_3\mu_{2n+1} = \lim_{n\to\infty} \mathcal{L}_3\mu_{2n+2} = \gamma.$$

Since, $\mathcal{L}_1(\Im) \cup \mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im)$ implies either $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_3(\Im)$ or $\mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im)$.

Case (i) Let $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_3(\Im)$. Since $\mathcal{L}_3(\Im)$ is a complete subspace of $\mathcal{L}_1(\Im) \cup \mathcal{L}_2(\Im)$ and $\mathcal{L}_1(\Im) \subseteq \mathcal{L}_3(\Im)$ implies $\mathcal{L}_3(\Im)$ is closed. Hence, there exist $\gamma_1, \gamma \in \Im$ such that

$\mathcal{L}_3\gamma_1 = \gamma$. If $\mathcal{L}_2\gamma_1 \neq \gamma$, then by using (4.3) we get

$$\begin{aligned}
(\rho(\mathcal{L}_1\mu_{2n}, \mathcal{L}_2\gamma_1))^2 &\leq \alpha_1 (\rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_1\mu_{2n}) \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\gamma_1) + \rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_2\gamma_1) \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_1\mu_{2n})) \\
&\quad \alpha_2 (\rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\mu_{2n}) \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_1\mu_{2n}) + \rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_2\gamma_1) \rho(\mathcal{L}_3\gamma_1, g_2\gamma_1)) + \\
&\quad \alpha_3 \left((\rho(\mathcal{L}_3\gamma_1, \mathcal{L}_1\mu_{2n}))^2 + (\rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\gamma_1))^2\right),
\end{aligned}$$

taking limit as $n \to \infty$, yields us

$$\begin{aligned}
(\rho(\mathcal{L}_1\mu_{2n}, \mathcal{L}_2\gamma_1))^2 &\leq \alpha_1 (\rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_1\mu_{2n}) \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\gamma_1) + \rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_2\gamma_1) \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_1\mu_{2n})) \\
&\quad \alpha_2 (\rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\mu_{2n}) \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_1\mu_{2n}) + \rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_2\gamma_1) \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\gamma_1)) + \\
&\quad \alpha_3 \left((\rho(\mathcal{L}_3\gamma_1, \mathcal{L}_1\mu_{2n}))^2 + (\rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\gamma_1))^2\right), \\
(\rho(\gamma, \mathcal{L}_2\gamma_1))^2 &\leq \alpha_1 (\rho(\gamma, \gamma) \rho(\gamma, \mathcal{L}_2\gamma_1) + \rho(\gamma, \mathcal{L}_2\gamma_1) \rho(\gamma, \gamma)) \\
&\quad \alpha_2 (\rho(\gamma, \gamma) \rho(\gamma, \gamma) + \rho(\gamma, \mathcal{L}_2\gamma_1) \rho(\gamma, \mathcal{L}_2\gamma_1)) + \\
&\quad \alpha_3 \left((\rho(\gamma, \gamma))^2 + (\rho(\gamma, \mathcal{L}_2\gamma_1))^2\right), \\
&\leq (\alpha_2 + \alpha_3) (\rho(\gamma, \mathcal{L}_2\gamma_1))^2, \\
&\Rightarrow (1 - (\alpha_2 + \alpha_3)) (\rho(\gamma, \mathcal{L}_2\gamma_1))^2 \leq 0,
\end{aligned}$$

and the above inequality is possible only if $(\rho(\gamma, \mathcal{L}_2\gamma_1))^2 = 0$ implies $\gamma = \mathcal{L}_2\gamma_1$. It follows that $\mathcal{L}_3\gamma_1 = \gamma = \mathcal{L}_2\gamma_1$. Since $\mathcal{L}_2$ and $\mathcal{L}_3$ are weakly compatible, hence $\mathcal{L}_3\mathcal{L}_2\gamma_1 = \mathcal{L}_2\mathcal{L}_3\gamma_1$ and so $\mathcal{L}_3\gamma = \mathcal{L}_2\gamma$ (4.5)

If $\gamma \neq \mathcal{L}_2\gamma$, then by (4.3) we have

$$\begin{aligned}
(\rho(\mathcal{L}_1\mu_{2n}, \mathcal{L}_2\gamma))^2 &\leq \alpha_1 \begin{pmatrix} \rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_1\mu_{2n}) \rho(\mathcal{L}_3\gamma, \mathcal{L}_2\gamma) + \\ \rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_2\gamma) \rho(\mathcal{L}_3\gamma, \mathcal{L}_1\mu_{2n}) \end{pmatrix} + \\
&\quad \alpha_2 \begin{pmatrix} \rho(\mathcal{L}_3\gamma, \mathcal{L}_2\mu_{2n}) \rho(\mathcal{L}_3\gamma, \mathcal{L}_1\mu_{2n}) + \\ \rho(\mathcal{L}_3\mu_{2n}, \mathcal{L}_2\gamma) \rho(\mathcal{L}_3\gamma, \mathcal{L}_2\gamma) \end{pmatrix} + \\
&\quad \alpha_3 \left((\rho(\mathcal{L}_3\gamma, \mathcal{L}_1\mu_{2n}))^2 + (\rho(\mathcal{L}_3\gamma, \mathcal{L}_2\gamma))^2\right),
\end{aligned}$$

As $n \to \infty$, we have, $\rho(\gamma, \mathcal{L}_2\gamma) = 0 \Longrightarrow \gamma = \mathcal{L}_2\gamma$ and using (4.5), we get

$$\mathcal{L}_3\gamma = \mathcal{L}_2\gamma = \gamma. \tag{4.6}$$

Case (ii) If $\mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im)$, again there exists points $\gamma_1, \gamma \in \Im$ such that $\gamma = \mathcal{L}_3\gamma_1$ if $\gamma \neq \mathcal{L}_1\gamma_1$ using (4.3) we get

$$\begin{aligned}(\rho(\mathcal{L}_1\gamma_1, \mathcal{L}_2\gamma))^2 \leq\ & \alpha_1(\rho(\mathcal{L}_3\gamma_1, \mathcal{L}_1\gamma_1)\rho(\mathcal{L}_3\gamma, \mathcal{L}_2\gamma) + \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\gamma)\rho(\mathcal{L}_3\gamma, \mathcal{L}_1\gamma_1)) + \\ & \alpha_2(\rho(\mathcal{L}_3\gamma, \mathcal{L}_2\gamma_1)\rho(\mathcal{L}_3\gamma, \mathcal{L}_1\gamma_1) + \rho(\mathcal{L}_3\gamma_1, \mathcal{L}_2\gamma)\rho(\mathcal{L}_3\gamma, \mathcal{L}_2\gamma)) + \\ & \alpha_3\left((\rho(\mathcal{L}_3\gamma, \mathcal{L}_1\gamma_1))^2 + (\rho(\mathcal{L}_3\gamma, \mathcal{L}_2\gamma))^2\right),\end{aligned}$$

It follows that

$$\begin{aligned}(\rho(\mathcal{L}_1\gamma_1, \gamma))^2 \leq\ & \alpha_1(\rho(\gamma, \mathcal{L}_1\gamma_1)\rho(\gamma, \gamma) + \rho(\gamma, \gamma)\rho(\gamma, \mathcal{L}_1\gamma_1)) + \\ & \alpha_2(\rho(\gamma, \gamma)\rho(\gamma, \mathcal{L}_1\gamma_1) + \rho(\gamma, \gamma)\rho(\gamma, \gamma)) + \\ & \alpha_3\left((\rho(\gamma, \mathcal{L}_1\gamma_1))^2 + (\rho(\gamma, \gamma))^2\right), \\ \Rightarrow\ & (1-\alpha_3)(\rho(\gamma, \mathcal{L}_1\gamma_1))^2 \leq 0,\end{aligned}$$

which is possible only, if $\rho(\gamma, \mathcal{L}_1\gamma) = 0 \Rightarrow \gamma = \mathcal{L}_1\gamma_1 \Longrightarrow \mathcal{L}_1\gamma_1 = \mathcal{L}_3\gamma_1 = \gamma$.

Since $\mathcal{L}_1$ and $\mathcal{L}_3$ are compatible hence, $\mathcal{L}_1\mathcal{L}_3\gamma_1 = \mathcal{L}_3\mathcal{L}_1\gamma_1$ and $\mathcal{L}_1\gamma = \mathcal{L}_3\gamma$. By (4.6), we have $\gamma = \mathcal{L}_1\gamma = \mathcal{L}_2\gamma = \mathcal{L}_3\gamma$.

Thus, γ is the common fixed point of self mappings $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.This completes the proof of the theorem.

For uniqueness, let $\gamma_1 \neq \gamma_2$ common fixed points. Then, using (4.3), we have

$$\begin{aligned}(\rho(\gamma_1, \gamma_2))^2 \leq\ & \alpha_1(\rho(g_3\gamma_1, g_1\gamma_1)\rho(g_3\gamma_2, g_2\gamma_2) + \rho(g_3\gamma_1, g_2\gamma_2)\rho(g_3\gamma_2, g_1\gamma_1)) + \\ & \alpha_2(\rho(g_3\gamma_2, g_2\gamma_1)\rho(g_3\gamma_2, g_1\gamma_1) + \rho(g_3\gamma_1, g_2\gamma_2)\rho(g_3\gamma_2, g_2\gamma_2)) + \\ & \alpha_3\left((\rho(g_3\gamma_2, g_1\gamma_1))^2 + (\rho(g_3\gamma_2, g_2\gamma_2))^2\right).\end{aligned}$$

and is,

$$\begin{aligned}
&\leq \alpha_1 (\rho(\gamma_1, \gamma_1) \rho(\gamma_2, \gamma_2) + \rho(\gamma_1, \gamma_2) \rho(\gamma_1, \gamma_2)) + \\
&\quad \alpha_2 (\rho(\gamma_2, \gamma_1) \rho(\gamma_2, \gamma_1) + \rho(\gamma_1, \gamma_2) \rho(\gamma_2, \gamma_2)) + \\
&\quad \alpha_3 \left((\rho(\gamma_1, \gamma_2))^2 + (\rho(\gamma_2, \gamma_2))^2\right), \\
&\leq (\alpha_1 + \alpha_2 + \alpha_3)(\rho(\gamma_1, \gamma_2))^2, \\
&\Rightarrow (1 - (\alpha_1 + \alpha_2 + \alpha_3))(\rho(\gamma_1, \gamma_2))^2 \leq 0,
\end{aligned}$$

which is possible only if $\rho(\gamma_1, \gamma_2) = 0 \Rightarrow \gamma_1 = \gamma_2$, gives us uniqueness of γ.

Remark.4.2.3. If we put $\alpha_3 = 0$, then we get Theorem 4.1.2 in [59].

Now, we present the modified form of Theorem 4.2.2 in terms of b-metric spaces.

Theorem 4.2.4. If $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 : \Im \to \Im$ are mappings of a complete b-metric space with the coefficient $k \geq 1$ and satisfy

$$\begin{aligned}
&(\rho(\mathcal{L}_1\mu_1, \mathcal{L}_2\mu_2))^2 \leq \alpha_1 (\rho(\mathcal{L}_3\mu_1, \mathcal{L}_1\mu_1) \rho(\mathcal{L}_3\mu_2, \mathcal{L}_2\mu_2)) + \\
&\alpha_2 (\rho(\mathcal{L}_3\mu_1, \mathcal{L}_2\mu_2) \rho(\mathcal{L}_3\mu_2, \mathcal{L}_2\mu_2)(1 + \rho(\mathcal{L}_1\mu_1, \mathcal{L}_3\mu_2))) + \\
&\quad \alpha_3 (\rho(\mathcal{L}_3\mu_1, \mathcal{L}_3\mu_2) \rho(\mathcal{L}_3\mu_2, \mathcal{L}_2\mu_2)(1 + \rho(\mathcal{L}_1\mu_1, \mathcal{L}_3\mu_2))) + \\
&\quad \alpha_4 (\rho(\mathcal{L}_3\mu_1, \mathcal{L}_2\mu_2) \rho(\mathcal{L}_3\mu_2, \mathcal{L}_2\mu_2)) + \\
&\quad \alpha_5 \left((\rho(\mathcal{L}_3\mu_2, \mathcal{L}_1\mu_1))^2 + (\rho(\mathcal{L}_3\mu_2, \mathcal{L}_2\mu_2))^2\right),
\end{aligned} \tag{4.7}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \geq 0$, are nonnegative reals with

$$k\alpha_1 + \left(k^2 + k\right)\alpha_2 + k\alpha_3 + \left(k^2 + k\right)\alpha_4 + \alpha_5 < 1. \tag{4.8}$$

If $\mathcal{L}_1(\Im) \cup \mathcal{L}_2(\Im) \subseteq \mathcal{L}_3(\Im)$ and $\mathcal{L}_3(\Im)$ is a complete subspace of $\Im$. Then the maps $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$possesses point of coincidence point δ in $\Im$. Moreover, if $\{\mathcal{L}_1, \mathcal{L}_3\}$ and $\{\mathcal{L}_2, \mathcal{L}_3\}$ are weakly compatible. Then $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ admits of a common fixed point.

Proof If $\mu_0 \in \Im$ is arbitrary. Then, we define a sequence $\{\eta_n\}$ in $\Im$ as follows $\eta_{2n} = \pounds_3\mu_{2n+1} = \pounds_1\mu_{2n}$ and $\eta_{2n+1} = \pounds_3\mu_{2n+2} = \pounds_2\mu_{2n+1}$ for $n \geq 0$. If $(\rho(\eta_{2n}, \eta_{2n+1}))^2 = (\rho(\pounds_1\mu_{2n}, \pounds_2\mu_{2n+1}))^2$, then by (4.7) we can write

$$\begin{aligned}
(\rho(\eta_{2n}, \eta_{2n+1}))^2 \leq\ & \alpha_1 (\rho(\pounds_3\mu_{2n}, \pounds_1\mu_{2n}) \rho(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1})) + \\
& \alpha_2 \begin{pmatrix} \rho(\pounds_3\mu_{2n}, \pounds_2\mu_{2n+1}) \rho(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}) \\ (1 + \rho(\pounds_1\mu_{2n}, \pounds_3\mu_{2n+1})) \end{pmatrix} + \\
& \alpha_3 \begin{pmatrix} \rho(\pounds_3\mu_{2n}, \pounds_3\mu_{2n+1}) \rho(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}) \\ (1 + \rho(\pounds_1\mu_{2n}, \pounds_3\mu_{2n+1})) \end{pmatrix} + \\
& \alpha_4 (\rho(\pounds_3\mu_{2n}, \pounds_2\mu_{2n+1}) \rho(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1})) + \\
& \alpha_5 \left((\rho((\pounds_3\mu_{2n+1}, \pounds_1\mu_{2n})))^2 + (\rho(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}))^2 \right), \\
\leq\ & \alpha_1 (\rho(\eta_{2n-1}, \eta_{2n}) \rho(\eta_{2n}, \eta_{2n+1})) + \\
& \alpha_2 (\rho(\eta_{2n-1}, \eta_{2n+1}) \rho(\eta_{2n}, \eta_{2n+1}) (1 + \rho(\eta_{2n}, \eta_{2n}))) + \\
& \alpha_3 (\rho(\eta_{2n-1}, \eta_{2n}) \rho(\eta_{2n}, \eta_{2n+1}) (1 + \rho(\eta_{2n}, \eta_{2n}))) + \\
& \alpha_4 (\rho(\eta_{2n-1}, \eta_{2n+1}) \rho(\eta_{2n+1}, \eta_{2n})) + \\
& \alpha_5 \left((\rho(\eta_{2n}, \eta_{2n}))^2 + (\rho(\eta_{2n}, \eta_{2n+1}))^2 \right),
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
\rho(\eta_{2n}, \eta_{2n+1}) \leq\ & \alpha_1\rho(\eta_{2n-1}, \eta_{2n}) + \alpha_2\rho(\eta_{2n-1}, \eta_{2n+1}) + \alpha_3\rho(\eta_{2n-1}, \eta_{2n}) + \\
& \alpha_4\rho(\eta_{2n-1}, \eta_{2n+1}) + \alpha_5\rho(\eta_{2n}, \eta_{2n+1}), \\
\leq\ & \alpha_1\rho(\eta_{2n-1}, \eta_{2n}) + k\alpha_2 (\rho(\eta_{2n-1}, \eta_{2n}) + \rho(\eta_{2n}, \eta_{2n+1})) + \\
& \alpha_3\rho(\eta_{2n-1}, \eta_{2n}) + k\alpha_4 (\rho(\eta_{2n-1}, \eta_{2n}) + \rho(\eta_{2n}, \eta_{2n+1})) + \\
& \alpha_5\rho(\eta_{2n}, \eta_{2n+1}),
\end{aligned}$$

and the above inequality implies

$$\begin{aligned}
\left(1-\left(k\alpha_2+k\alpha_4+\alpha_5\right)\right)\rho\left(\eta_{2n},\eta_{2n+1}\right) &\leq \left(\alpha_1+k\alpha_2+\alpha_3+k\alpha_4\right)\rho\left(\eta_{2n},\eta_{2n-1}\right), \\
\rho\left(\eta_{2n+1},\eta_{2n}\right) &\leq \frac{\left(\alpha_1+k\alpha_2+\alpha_3+k\alpha_4\right)}{\left(1-\left(k\alpha_2+k\alpha_4+\alpha_5\right)\right)}\rho\left(\eta_{2n},\eta_{2n-1}\right), \\
\rho\left(\eta_{2n+1},\eta_{2n}\right) &\leq \Phi\rho\left(\eta_{2n},\eta_{2n-1}\right),
\end{aligned}$$

where $\Phi=\frac{(\alpha_1+k\alpha_2+\alpha_3+k\alpha_4)}{[1-(k\alpha_2+k\alpha_4+\alpha_5)]}<\frac{1}{k}$, and for $n\in N$, we get

$$\rho\left(\eta_{n+1},\eta_{n+2}\right)\leq\Phi\rho\left(\eta_n,\eta_{n+1}\right),\ldots\leq\Phi^{n+1}\rho\left(\eta_0,\eta_1\right).$$

Now, for any $j>n$, we have

$$\begin{aligned}
\rho\left(\eta_n,\eta_j\right) &\leq \rho\left(\eta_n,\eta_{n+1}\right)+\rho\left(\eta_{n+1},\eta_{n+2}\right)+\ldots+\rho\left(\eta_{j-1},\eta_j\right) \\
&\leq \left(\Phi^n+\Phi^{n+1}+\ldots+\Phi^{j-1}\right)\rho\left(\eta_1,\eta_0\right), \\
&\leq \frac{\Phi^n}{(1-\Phi)}\rho\left(\eta_1,\eta_0\right)
\end{aligned}$$

Therefore, from Lemma 4.2.1, we have $\rho\left(\eta_n,\eta_j\right)\leq\frac{k\Phi^n}{(1-k\Phi)}\rho\left(\eta_1,\eta_0\right)\to 0$ as $j,\ n\to\infty$ where $k\Phi<1$. It follows that $\{\eta_n\}$ converges to some $\gamma\in\Im$ due to completeness of $\Im$ and therefore,

$$\lim_{n\to\infty}\eta_n=\lim_{n\to\infty}\mathcal{L}_1\mu_{2n}=\lim_{n\to\infty}\mathcal{L}_2\mu_{2n+1}=\lim_{n\to\infty}\mathcal{L}_3\mu_{2n+1}=\lim_{n\to\infty}\mathcal{L}_3\mu_{2n+2}=\gamma.$$

Since $\mathcal{L}_3(\Im)$ is a complete subspace of $\Im$, there exists $\delta,\ \gamma\in\Im$ such that $\mathcal{L}_3\delta=\gamma$. If $\mathcal{L}_2\delta\neq\gamma$ then using (4.7), we get

$$\begin{aligned}
\left(\rho\left(\mathcal{L}_1\mu_{2n},\mathcal{L}_2\delta\right)\right)^2 \leq\ & \alpha_1\left(\rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_1\mu_{2n}\right)\rho\left(\mathcal{L}_3\delta,\mathcal{L}_2\delta\right)\right)+\alpha_2\left(\begin{array}{c}\rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_2\delta\right)\rho\left(\mathcal{L}_3\delta,\mathcal{L}_2\delta\right)\\ \left(1+\rho\left(\mathcal{L}_1\mu_{2n},\mathcal{L}_3\delta\right)\right)\end{array}\right)+ \\
& \alpha_3\left(\begin{array}{c}\rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_3\delta\right)\rho\left(\mathcal{L}_3\delta,\mathcal{L}_2\delta\right)\\ \left(1+\rho\left(\mathcal{L}_1\mu_{2n},\mathcal{L}_3\delta\right)\right)\end{array}\right)+\alpha_4\left(\rho\left(\mathcal{L}_3\mu_{2n},\mathcal{L}_2\delta\right)\rho\left(\mathcal{L}_3\delta,\mathcal{L}_2\delta\right)\right) \\
& +\alpha_5\left(\left(\rho\left(\mathcal{L}_3\delta,\mathcal{L}_1\mu_{2n}\right)\right)^2+\left(\rho\left(\mathcal{L}_3\delta,g_2\delta\right)\right)^2\right),
\end{aligned}$$

on taking limit as $n \to \infty$, we have

$$\begin{aligned}
(\rho(\gamma, \pounds_2\delta))^2 &\leq \alpha_1 (\rho(\gamma,\gamma)\,\rho(\gamma,\pounds_2\delta)) + \alpha_2 (\rho(\gamma,\pounds_2\delta)\,\rho(\gamma,\pounds_2\delta)(1+\rho(\gamma,\gamma))) + \\
&\quad \alpha_3 (\rho(\gamma,\gamma)\,\rho(\gamma,\pounds_2\delta)(1+\rho(\gamma,\gamma))) + \alpha_4 (\rho(\gamma,\pounds_2\delta)\,\rho(\gamma,\pounds_2\delta)) + \\
&\quad \alpha_5 \left((\rho(\gamma,\gamma))^2 + (\rho(\gamma,\pounds_2\delta))^2\right), \\
&\leq (\alpha_2+\alpha_4+\alpha_5)(\rho(\gamma,\pounds_2\delta))^2, \\
&\Rightarrow (1-(\alpha_2+\alpha_4+\alpha_5))(\rho(\gamma,\pounds_2\delta))^2 \leq 0,
\end{aligned}$$

which is possible only, if $(\rho(\gamma, \pounds_2\delta))^2 = 0 \Rightarrow \gamma = \pounds_2\delta$. It follows that $\pounds_3\delta = \gamma = \pounds_2\delta$. Since the pair of mappings $\{g_2, g_3\}$ is weakly compatible, we have $\pounds_2\pounds_3\delta = \pounds_3\pounds_2\delta$, hence $\pounds_2\gamma = \pounds_3\gamma$. If $\gamma \neq \pounds_2\gamma$, by (4.7), we get

$$\begin{aligned}
(\rho(\pounds_1\mu_{2n}, \pounds_2\gamma))^2 &\leq \alpha_1 (\rho(\pounds_3\mu_{2n},\pounds_1\mu_{2n})\,\rho(\pounds_3\gamma,\pounds_2\gamma)) + \\
&\quad \alpha_2 (\rho(\pounds_3\mu_{2n},\pounds_2\gamma)\,\rho(\pounds_3\gamma,\pounds_2\gamma)(1+\rho(\pounds_1\mu_{2n},\pounds_3\gamma))) + \\
&\quad \alpha_3 (\rho(\pounds_3\mu_{2n},\pounds_3\gamma)\,\rho(\pounds_3\gamma,\pounds_2\gamma)(1+\rho(\gamma,\gamma))) + \\
&\quad \alpha_4 (\rho(\pounds_3\mu_{2n},\pounds_2\gamma)\,\rho(\pounds_3\gamma,\pounds_2\gamma)) + \\
&\quad \alpha_5 \left((\rho(\pounds_3\gamma,\pounds_1\mu_{2n}))^2 + (\rho(\pounds_3\gamma,\pounds_2\gamma))^2\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
(\rho(\gamma, \pounds_2\gamma))^2 &\leq \alpha_1 (\rho(\gamma,\gamma)\,\rho(\pounds_2\gamma,\pounds_2\gamma)) + \alpha_2 (\rho(\gamma,\pounds_2\gamma)\,\rho(\gamma,\pounds_2\gamma)(1+\rho(\gamma,\gamma))) + \\
&\quad \alpha_3 (\rho(\gamma,\gamma)\,\rho(\gamma,\pounds_2\gamma)(1+\rho(\gamma,\gamma))) + \\
&\quad \alpha_4 (\rho(\gamma,\pounds_2\gamma)\,\rho(\gamma,\pounds_2\gamma)) + \\
&\quad \alpha_5 \left((\rho(\gamma,\gamma))^2 + (\rho(\gamma,\pounds_2\gamma))^2\right), \\
&\leq (\alpha_2+\alpha_4+\alpha_5)(\rho(\gamma,\pounds_2\gamma))^2, \\
&\Rightarrow (1-(\alpha_2+\alpha_4+\alpha_5))(\rho(\gamma,\pounds_2\gamma))^2 \leq 0.
\end{aligned}$$

And the above inequality is possible only if $(\rho(\gamma, \pounds_2\gamma))^2 = 0 \Longrightarrow \gamma = \pounds_2\gamma$. Using $\pounds_3\gamma = \pounds_2\gamma$, we get

$$\gamma = \pounds_2\gamma = \pounds_3\gamma. \tag{4.9}$$

Again, if $\gamma \neq \pounds_1\delta$, by (4.7) we have

$$\begin{aligned}
\left(\rho\left(\pounds_1\delta, \pounds_2\mu_{2n+1}\right)\right)^2 &= (\rho(\gamma, \pounds_1\delta))^2 \leq \alpha_1\left(\rho\left(\pounds_3\delta, \pounds_1\delta\right)\rho\left(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}\right)\right) + \\
&\alpha_2\left(\rho\left(\pounds_3\delta, \pounds_2\mu_{2n+1}\right)\rho\left(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}\right)\left(1 + \rho\left(\pounds_1\delta, \pounds_3\mu_{2n+1}\right)\right)\right) + \\
&\alpha_3\left(\rho\left(\pounds_3\delta, \pounds_3\mu_{2n+1}\right)\rho\left(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}\right)\left(1 + \rho\left(\pounds_1\delta, \pounds_3\mu_{2n+1}\right)\right)\right) + \\
&\alpha_4\left(\rho\left(\pounds_3\delta, \pounds_2\mu_{2n+1}\right)\rho\left(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}\right)\right) + \\
&\alpha_5\left(\left(\rho\left(\pounds_3\mu_{2n+1}, \pounds_1\delta\right)\right)^2 + \left(\rho\left(\pounds_3\mu_{2n+1}, \pounds_2\mu_{2n+1}\right)\right)^2\right),
\end{aligned}$$

taking limit as $n \to \infty$, we get

$$\begin{aligned}
(\rho(\gamma, \pounds_1\delta))^2 &\leq \alpha_1(\rho(\gamma, \pounds_1\delta)\rho(\gamma,\gamma)) + \alpha_2(\rho(\gamma,\gamma)\rho(\gamma,\gamma)(1 + \rho(\pounds_1\delta,\gamma))) + \\
&\alpha_3(\rho(\gamma,\gamma)\rho(\gamma,\gamma)(1 + \rho(\pounds_1\delta,\gamma))) + \alpha_4(\rho(\gamma,\gamma)\rho(\gamma,\gamma)) + \\
&\alpha_5\left((\rho(\gamma, \pounds_1\delta))^2 + (\rho(\gamma,\gamma))^2\right),
\end{aligned}$$

and this implies

$$\begin{aligned}
&\Rightarrow \quad (\rho(\gamma, \pounds_1\delta))^2 \leq \alpha_5(\rho(\gamma, \pounds_1\delta))^2, \\
&\Rightarrow \quad (1 - \alpha_5)(\rho(\gamma, \pounds_1\delta))^2 \leq 0,
\end{aligned}$$

which is possible only if $(\rho(\pounds_1\delta, \gamma))^2 = 0 \Longrightarrow \gamma = \pounds_1\delta$. Since $\pounds_1$ and $\pounds_3$ are weakly compatible, $\pounds_1\pounds_3\delta = \pounds_3\pounds_1\delta$

$$\Longrightarrow \pounds_1\gamma = \pounds_2\gamma. \tag{4.10}$$

By (4.9) and (4.10), we have

$$\pounds_1\gamma = \pounds_2\gamma = \pounds_3\gamma = \gamma.$$

Thus, γ is the unique common fixed point of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$.

To see uniqueness, let γ_1 and γ_2 be two distinct common fixed points of $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ such that $\gamma_1 \neq \gamma_2$. Then by using (4.7) we get

$$\begin{aligned}
(\rho(\gamma_1,\gamma_2))^2 &\leq \alpha_1(\rho(\mathcal{L}_3\gamma_1,\mathcal{L}_1\gamma_1)\rho(\mathcal{L}_3\gamma_2,\mathcal{L}_2\gamma_2))+\\
&\quad \alpha_2(\rho(\mathcal{L}_3\gamma_1,\mathcal{L}_2\gamma_2)\rho(\mathcal{L}_3\gamma_2,\mathcal{L}_2\gamma_2)(1+\rho(\mathcal{L}_1\gamma_1,\mathcal{L}_3\gamma_2)))+\\
&\quad \alpha_3(\rho(\mathcal{L}_3\gamma_1,\mathcal{L}_3\gamma_2)\rho(\mathcal{L}_3\gamma_2,\mathcal{L}_2\gamma_2)(1+\rho(\mathcal{L}_1\gamma_1,\mathcal{L}_3\gamma_2)))+\\
&\quad \alpha_4(\rho(\mathcal{L}_3\gamma_1,\mathcal{L}_2\gamma_2)\rho(\mathcal{L}_3\gamma_2,\mathcal{L}_2\gamma_2))+\\
&\quad \alpha_5\left((\rho(\mathcal{L}_3\gamma_2,\mathcal{L}_1\gamma_1))^2+(\rho(\mathcal{L}_3\gamma_2,\mathcal{L}_2\gamma_2))^2\right),\\
&\leq \alpha_1(\rho(\gamma_1,\gamma_1)\rho(\gamma_2,\gamma_2))+\alpha_2(\rho(\gamma_1,\gamma_2)\rho(\gamma_2,\gamma_2)(1+\rho(\gamma_1,\gamma_2)))+\\
&\quad \alpha_3(\rho(\gamma_1,\gamma_2)\rho(\gamma_2,\gamma_2)(1+\rho(\gamma_1,\gamma_2)))+\alpha_4(\rho(\gamma_1,\gamma_2)\rho(\gamma_2,\gamma_2))+\\
&\quad \alpha_5\left((\rho(\gamma_2,\gamma_1))^2+(\rho(\gamma_2,\gamma_2))^2\right),\\
&\leq \alpha_5(\rho(\gamma_1,\gamma_2))^2,\\
&\Rightarrow (1-\alpha_5)(\rho(\gamma_1,\gamma_2))^2 \leq 0,
\end{aligned}$$

and the inequality is possible only if $(\rho(\gamma_1,\gamma_2))^2 = 0$, which implies that $\gamma_1 = \gamma_2$ and the common fixed point is unique.

Theorem 4.2.4 yields the following Corollaries.

Corollary 4.2.5. If $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3 : \Im \to \Im$ are any three mappings of b-metric $\Im$ which is complete and satisfy

$$(\rho(\mathcal{L}_1\mu_1,\mathcal{L}_2\mu_2))^2 \leq \alpha_1\begin{pmatrix} \rho(\mathcal{L}_3\mu_1,\mathcal{L}_1\mu_1)\rho(\mathcal{L}_3\mu_2,\mathcal{L}_2\mu_2)+ \\ \rho(\mathcal{L}_3\mu_1,\mathcal{L}_2\mu_2)\rho(\mathcal{L}_3\mu_2,\mathcal{L}_2\mu_2)(1+\rho(\mathcal{L}_1\mu_1,\mathcal{L}_3\mu_2)) \end{pmatrix}+$$
$$\alpha_3(\rho(\mathcal{L}_3\mu_1,\mathcal{L}_3\mu_2)\rho(\mathcal{L}_3\mu_2,\mathcal{L}_2\mu_2)(1+\rho(\mathcal{L}_1\mu_1,\mathcal{L}_3\mu_2))+\rho(\mathcal{L}_3\mu_1,\mathcal{L}_2\mu_2)\rho(\mathcal{L}_3\mu_2,\mathcal{L}_2\mu_2)),$$

for μ_1, $\mu_2 \in \Im$, and $\alpha_1, \alpha_3 \geq 0$, such that $\alpha_1 + (k^2+1)\alpha_3 < 1$ then $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ possesses a fixed point in $\Im$.

Proof Putting $\alpha_5 = 0$ and $\alpha_2 = \alpha_1$, $\alpha_4 = \alpha_3$ in Theorem 4.2.4 gives us the result.

Corollary 4.2.6. If $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 : \Im \to \Im$ are three mappings satisfying the inequality

$$\begin{aligned}(\rho(\mathcal{L}_1\mu_1, \mathcal{L}_2\mu_2))^2 &\le \alpha_1(\rho(\mathcal{L}_3\mu_1, \mathcal{L}_1\mu_1)\rho(\mathcal{L}_3\mu_2, \mathcal{L}_2\mu_2)) \\ &\quad +\alpha_2(\rho(\mathcal{L}_3\mu_1, \mathcal{L}_2\mu_2)\rho(\mathcal{L}_3\mu_2, \mathcal{L}_2\mu_2)(1+\rho(\mathcal{L}_1\mu_1, \mathcal{L}_3\mu_2))),\end{aligned}$$

for $\mu_1, \mu_2 \in \Im$ such that $\alpha_1 + (k^2+k)\alpha_2 < 1$. Then $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$admit a common fixed point in $\Im$.

Proof Putting $\alpha_3 = \alpha_4 = \alpha_5 = 0$ in Theorem 4.2.4 and the result follows.

Remark 4.2.7. Corollary 4.2.4, is the result of Pachpatte [59].

Remark 4.2.8. Corollary 4.2.5, is the result of Fisher [34].

Example 4.2.9. Let $\Im = \{3, 4, 5\}$, and $\rho : \Im \times \Im \to R^+$ is defined as follows

$$\begin{aligned}\rho(3,4) &= \rho(4,3) = 1,\ \rho(5,4) = \rho(4,5) = \frac{20}{25},\ \rho(3,5) = \rho(5,3) = \frac{1}{25}, \\ \rho(3,3) &= \rho(4,4) = \rho(5,5) = 0.\end{aligned}$$

So, we see that $(\Im, \rho)$ is a b-metric space with parameter $k = \frac{4}{5}$. Define the mappings $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3 : \Im \to \Im$ by

$$\begin{aligned}\mathcal{L}_{13} &= \mathcal{L}_{15} = 4, \mathcal{L}_{14} = 5, \\ \mathcal{L}_{23} &= \mathcal{L}_{24} = \mathcal{L}_{25} = 5, \\ \mathcal{L}_{33} &= 4, \mathcal{L}_{34} = 3, \mathcal{L}_{35} = 5.\end{aligned}$$

For $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \frac{1}{2}$, the maps $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$ satisfy all the conditions of Theorem 4.2.2 and Theorem 4.2.4 with 5 as the only common fixed point in $\Im$.

Chapter 5

Generalization of Common Fixed Point Theorems for Two Mappings

5.1 Introduction

We want to establish fixed point results in complete, compact and Hausdorff spaces for a pair of commuting maps.

The obtained results are generalizations of some fixed-point theorems of Fisher [34], Jungck [46], Mukherjee [55], Pachpatte [59] and Sahu and Sharma [68].

The following fixed point theorems were proved in [34], [46], [55], [59] and [68].

Theorem 5.1.1[34]. If $\pounds$ is map of the complete metric space $\Im$ into itself satisfying the inequality

$$(\rho(\pounds\mu_1, \pounds\mu_2))^2 \leq \eta_1 (\rho(\mu_1, \pounds\mu_1)\rho(\mu_2, \pounds\mu_2)) + \eta_2 (\rho(\mu_1, \pounds\mu_2)\rho(\mu_2, \pounds\mu_1)),$$

for $\mu_1, \mu_2 \in \Im$, $0 \leq \eta_1 < 1$ and $0 \leq \eta_2$, then $\pounds$ possesses a fixed point in $\Im$.

Theorem 5.1.2 [46]. If $\pounds$ is continuous mapping of a metric space $(\Im, \rho)$ into itself. Then $\pounds$ admit a fixed point in $\Im$ if and only if there exists $\eta \in (0, 1)$ and a mapping

$\mathcal{L}_1 : \Im \to \Im$ which commutes with $\mathcal{L}$ and satisfies $\mathcal{L}_1(\Im) \subset \mathcal{L}(\Im)$ and

$$(\rho(\mathcal{L}_1(\mu_1), \mathcal{L}_1(\mu_2))) \leq \eta\rho(\mathcal{L}(\mu_1), \mathcal{L}(\mu_2)),$$

for $\mu_1, \mu_2 \in \Im$, then $\mathcal{L}$ and $\mathcal{L}_1$ have fixed point in $\Im$.

Theorem 5.1.3 [55]. If $\mathcal{L}$, $\mathcal{L}_1 : \Im \to \Im$ are commuting and continuous such that $\mathcal{L}_1(\Im) \subset \mathcal{L}(\Im)$. Also, $\mathcal{L}$ satisfiy the condition:

$$\begin{aligned}(\rho(\mathcal{L}\mu_1, \mathcal{L}\mu_2)) \leq \eta_1\rho(\mathcal{L}_1\mu_1, \mathcal{L}\mu_1) + \eta_2\rho(\mathcal{L}_1\mu_2, \mathcal{L}\mu_2) +\\ \eta_3\rho(\mathcal{L}_1\mu_1, \mathcal{L}\mu_2) + \eta_4\rho(\mathcal{L}_1\mu_2, \mathcal{L}\mu_1) +\\ \eta_5\rho(\mathcal{L}\mu_1, \mathcal{L}\mu_2),\end{aligned}$$

with $\eta_i \geq 0$, for all i and $\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + \eta_5 < 1$, then $\mathcal{L}$ and $\mathcal{L}_1$possesses a common fixed point in $\Im$.

Theorem 5.1.4 [59]. If $\mathcal{L}$ is a map of the complete metric space $\Im$ into itself and satisfy the condition

$$\begin{aligned}(\rho(\mathcal{L}\mu_1, \mathcal{L}\mu_2))^2 \leq \eta_1(\rho(\mu_1, \mathcal{L}\mu_1)\rho(\mu_2, \mathcal{L}\mu_2) + \rho(\mu_1, \mathcal{L}\mu_2)\rho(\mu_2, \mathcal{L}\mu_1)) +\\ \eta_2(\rho(\mu_1, \mathcal{L}\mu_1)\rho(\mu_2, \mathcal{L}\mu_1) + \rho(\mu_1, \mathcal{L}\mu_2)\rho(\mu_2, \mathcal{L}\mu_2)),\end{aligned}$$

for $\mu_1, \mu_2 \in \Im$, and $\eta_1, \eta_2 \geq 0$, such that $\eta_1 + 2\eta_2 < 1$, then $\mathcal{L}$ has a unique fixed point.

Theorem 5.1.5 [68]. If $\mathcal{L} : \Im \to \Im$ satisfy

$$\begin{aligned}(\rho(\mathcal{L}\mu_1, \mathcal{L}\mu_2))^2 \leq \eta_1(\rho(\mu_1, \mathcal{L}\mu_1)\rho(\mu_2, \mathcal{L}\mu_2) + \rho(\mu_1, \mathcal{L}\mu_2)\rho(\mu_2, \mathcal{L}\mu_1)) +\\ \eta_2(\rho(\mu_1, \mathcal{L}\mu_1)\rho(\mu_2, \mathcal{L}\mu_1) + \rho(\mu_1, \mathcal{L}\mu_2)\rho(\mu_2, \mathcal{L}\mu_2)) +\\ \eta_3((\rho(\mu_2, \mathcal{L}\mu_1))^2 + (\rho(\mu_2, \mathcal{L}\mu_2))^2),\end{aligned}$$

for some $\mu_1, \mu_2 \in \Im$ and $\eta_1, \eta_2, \eta_3 \geq 0$ such that $\eta_1 + 2\eta_2 + \eta_3 < 1$, then $\mathcal{L}$ possesses a fixed point in $\Im$.

Definition 5.1.6. The map $\rho : \Im \times \Im \to R^+$is called sequentially convergent if for every sequence $\{\mu_n\}$ of $\Im$ if $\{\pounds\mu_n\}$ is convergent then $\{\mu_n\}$ has a convergent subsequence.

5.2 Main results

In this section, we state and prove our main result on the common fixed points of commuting maps in the frame work of complete metric spaces. We use jungck's idea of common fixed point for commuting maps with one of the functions need to be continuous. We start with the following result.

Theorem 5.2.1. If $\pounds$, $\pounds_1 : \Im \to \Im$ are commuting self-maps of $\Im$ with $\pounds$ continuous such that $\pounds_1(\Im) \subset \pounds(\Im)$ and $\pounds_1$ satisfy the condition

$$\begin{aligned}(\rho(\pounds_1(\mu_1), \pounds_1(\mu_2)))^2 \leq\ & \eta_1 (\rho(\pounds(\mu_1), \pounds_1(\mu_1)) \rho(\pounds(\mu_2), \pounds_1(\mu_2))) \\ & + \eta_2 (\rho(\pounds(\mu_2), \pounds_1(\mu_2))^2 (1 + \rho(\pounds(\mu_2), \pounds_1(\mu_1)))) \\ & + \eta_3 \left(\begin{array}{c} \rho(\pounds(\mu_2), \pounds_1(\mu_2)) \rho(\pounds_1(\mu_1), \pounds_1(\mu_2)) \\ (1 + \rho(\pounds(\mu_2), \pounds_1(\mu_1))) \end{array} \right) \\ & + \eta_4 (\rho(\pounds(\mu_1), \pounds_1(\mu_2)) \rho(\pounds(\mu_2), \pounds_1(\mu_2))) \\ & + \eta_5 \left(\frac{(1 + \rho(\pounds(\mu_2), \pounds(\mu_1))) \rho(\pounds(\mu_2), \pounds_1(\mu_2))}{(1 + \rho(\pounds(\mu_1), \pounds_1(\mu_1)))} \right)^2 \\ & + \eta_6 \left(\frac{\rho(\pounds(\mu_2), \pounds_1(\mu_1)) \rho(\pounds(\mu_2), \pounds_1((\mu_2)))}{(1 + \rho(\pounds(\mu_1), \pounds_1(\mu_1)))} \right)^2, \end{aligned} \tag{5.1}$$

for some $\mu_1, \mu_2 \in \Im$, where $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6 \geq 0$, such that

$$\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + \eta_5 < 1, \tag{5.2}$$

then $\pounds$ and $\pounds_1$ admits a common fixed point in $\Im$.

Proof If $\mu_0 \in \Im$ is arbitrary. Then for $\mu_1 \in \Im$ we have $\pounds_1(\mu_0) = \pounds(\mu_1)$. We construct sequence $\{\mu_n\}$ in $\Im$ such that $\pounds_1(\mu_n) = \pounds(\mu_{n+1})$. Since, $\pounds_1(\Im) \subset \pounds(\Im)$. Using (5.1), we get

$$\begin{aligned}
\left(\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)\right)^2 &\leq \eta_1\left(\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+1}\right)\right)\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)\right)+\\
&\eta_2\left(\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)^2\left(1+\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+1}\right)\right)\right)\right)\\
&+\eta_3\left(\begin{array}{c}\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)\\ \left(1+\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+1}\right)\right)\right)\end{array}\right)\\
&+\eta_4\left(\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+2}\right)\right)\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)\right)\\
&+\eta_5\left(\frac{\left(1+\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_n\right)\right)\right)\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)}{\left(1+\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+1}\right)\right)\right)}\right)^2\\
&\eta_6\left(\frac{\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+1}\right)\right)\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\left(\mu_{n+2}\right)\right)\right)}{\left(1+\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+1}\right)\right)\right)}\right)^2\\
&\leq \eta_1\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+1}\right)\right)+\eta_2\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)+\\
&+\eta_3\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)+\eta_4\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+2}\right)\right)\\
&+\eta_5\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\left(\mu_{n+2}\right)\right)\right)
\end{aligned}$$

or

$$\begin{aligned}
\left(1-\left(\eta_2+\eta_3+\eta_4\right)\right)\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right) &\leq \left(\eta_1+\eta_4\right)\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+1}\right)\right),\\
\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right) &\leq \frac{\left(\eta_1+\eta_4\right)}{\left(1-\left(\eta_2+\eta_4+\eta_5\right)\right)}\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+1}\right)\right),\\
\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right) &\leq \eta\rho\left(\pounds\left(\mu_n\right),\pounds\left(\mu_{n+1}\right)\right),
\end{aligned}$$

where $\eta=\frac{(\eta_1+\eta_4)}{[1-(\eta_2+\eta_4+\eta_5)]}<1$, on continuous repetition of the above process for $n\in N$, we get

$$\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)\leq\eta^n\rho\left(\pounds\left(\mu_1\right),\pounds\left(\mu_0\right)\right). \tag{5.3}$$

On taking limit as $n\to\infty$, we get $\lim_{n\to\infty}\rho\left(\pounds\left(\mu_{n+1}\right),\pounds\left(\mu_{n+2}\right)\right)=0$. Since $\Im$ is complete, there exists $\omega\in\Im$ such that $\lim_{n\to\infty}\pounds_1\left(\mu_n\right)=\lim_{n\to\infty}\pounds\left(\mu_{n+1}\right)=\omega$. Since $\pounds$ is continuous and $\pounds$, $\pounds_1$ commute, we have, $\pounds\omega=\pounds\left(\lim_{n\to\infty}\pounds\mu_n\right)=\lim_{n\to\infty}\pounds^2\mu_n$.

Also $\pounds\omega = \pounds\left(\lim_{n\to\infty}\pounds_1\mu_n\right) = \lim_{n\to\infty}\pounds\pounds_1\mu_n = \lim_{n\to\infty}\pounds_1\pounds\mu_n$ and by (5.1) we get

$$\begin{aligned}
\left(\rho\left(\pounds\left(\omega\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\right)^2 \leq\ & \eta_1\left(\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\right)+ \\
& \eta_2\left(\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)^2\left(1+\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\right)\right) \\
& +\eta_3\begin{pmatrix}\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\rho\left(\pounds_1\left(\pounds\left(\mu_n\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\right) \\ \left(1+\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\right)\end{pmatrix} \\
& \eta_4\left(\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\right)+ \\
& \eta_5\left(\frac{\left(1+\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds\left(\pounds\left(\mu_n\right)\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)}{\left(1+\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\right)}\right)^2 \\
& +\eta_6\left(\frac{\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\left(\pounds_1\left(\omega\right)\right)\right)\right)}{\left(1+\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\pounds\left(\left(\mu_n\right)\right)\right)\right)}\right)^2
\end{aligned}$$

Now, since $\pounds$ is continuous, we have, $\pounds_1\left(\pounds\left(\mu_n\right)\right) = \pounds\left(\pounds_1\left(\mu_n\right)\right) = \pounds\left(\pounds\left(\mu_{n+1}\right)\right) = \pounds\left(\omega\right)$ from which we have

$$\begin{aligned}
\left(\rho\left(\pounds\left(\omega\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\right)^2 \leq\ & \eta_1\left(\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\right)+ \\
& \eta_2\left(\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)^2\left(1+\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right)\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\right)\right) \\
& +\eta_3\begin{pmatrix}\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\rho\left(\pounds_1\left(\pounds\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right) \\ \left(1+\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\right)\end{pmatrix} \\
& \eta_4\left(\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)\right)+ \\
& +\eta_5\left(\frac{\left(1+\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds\left(\pounds\left(\mu_n\right)\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds_1\left(\omega\right)\right)\right)}{\left(1+\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\right)}\right)^2 \\
& +\eta_6\left(\frac{\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\pounds\left(\mu_n\right)\right)\right)\rho\left(\pounds\left(\pounds_1\left(\omega\right)\right),\pounds_1\left(\left(\pounds_1\left(\omega\right)\right)\right)\right)}{\left(1+\rho\left(\pounds\left(\pounds\left(\mu_n\right)\right),\pounds_1\left(\pounds\left(\left(\mu_n\right)\right)\right)\right)\right)}\right)^2,
\end{aligned}$$

on taking limit as $n \to \infty$ and use continuity, we obtain

$$
\begin{aligned}
(\rho((\pounds(\omega)), \pounds_1(\pounds_1(\omega))))^2 \leq\ & \eta_1(\rho(\pounds(\omega), (\pounds(\omega)))\rho(\pounds(\pounds_1(\omega)), \pounds_1(\pounds_1(\omega)))) + \\
& \eta_2(\rho(\pounds(\pounds_1(\omega)), \pounds_1(\pounds_1(\omega)))^2(1+\rho(\pounds(\pounds_1(\omega)), \pounds(\omega)))) \\
& +\eta_3(\rho(\pounds(\pounds_1(\omega)), \pounds_1(\pounds_1(\omega)))^2(1+\rho(\pounds(\pounds_1(\omega)), \pounds(\omega)))) \\
& +\eta_4(\rho((\pounds(\omega)), \pounds_1(\pounds_1(\omega)))\rho(\pounds(\pounds_1(\omega)), \pounds_1(\pounds_1(\omega)))) \\
& +\eta_5\left(\frac{(1+\rho(\pounds(\pounds_1(\omega)), (\pounds(\omega))))\rho(\pounds(\pounds_1(\omega)), \pounds_1(\pounds_1(\omega)))}{(1+\rho((\pounds(\omega)), (\pounds(\omega))))}\right)^2 \\
& +\eta_6\left(\frac{\rho(\pounds(\pounds_1(\omega)), (\pounds(\omega)))\rho(\pounds(\pounds_1(\omega)), \pounds_1((\pounds_1(\omega))))}{(1+\rho((\pounds(\omega)), \pounds((\omega))))}\right)^2.
\end{aligned}
$$

Since, $\pounds$ and $\pounds_1$commute, we have

$$
\begin{aligned}
(\rho((\pounds(\omega)), \pounds_1(\pounds_1(\omega))))^2 \leq\ & \eta_1(\rho(\pounds(\omega), (\pounds(\omega)))\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega)))) + \\
& \eta_2(\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega)))^2(1+\rho(\pounds(\omega), \pounds(\omega)))) + \\
& \eta_3(\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega)))(1+\rho(\pounds(\omega), \pounds(\omega)))) + \\
& \eta_4(\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega)))\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega)))) + \\
& \eta_5\left(\frac{(1+\rho(\pounds(\omega), (\pounds(\omega))))\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega)))}{(1+\rho(\pounds(\omega), \pounds(\omega)))}\right)^2 \\
& +\eta_6\left(\frac{\rho(\pounds(\omega), (\pounds(\omega)))\rho(\pounds(\pounds_1(\omega)), \pounds_1((\pounds_1(\omega))))}{(1+\rho(\pounds(\omega), \pounds(\omega)))}\right)^2,
\end{aligned}
$$

which implies,

$$
\begin{aligned}
\rho((\pounds(\omega)), \pounds_1(\pounds_1(\omega))) &\leq \eta_2\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega))) + \eta_3\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega))) \\
& \quad \eta_4\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega))) + \eta_5\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega))), \\
&\leq (\eta_2+\eta_3+\eta_4+\eta_5)\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega))), \\
&\Rightarrow (1-(\eta_2+\eta_3+\eta_4+\eta_5))\rho(\pounds(\omega), \pounds_1(\pounds_1(\omega))) \leq 0, \\
&\Rightarrow \rho(\pounds(\omega), \pounds_1(\pounds_1(\omega))) = 0, \\
&\Rightarrow \pounds(\omega) = \pounds_1(\pounds_1(\omega)).
\end{aligned}
$$

Next, we prove that $\pounds_1(\omega)$ is a common fixed point of $\pounds$ and $\pounds_1$. So, by (5.1), we have

$$\begin{aligned}(\rho(\pounds_1(\pounds_1(\omega)),\pounds_1(\omega)))^2 \leq\ & \eta_1(\rho(\pounds(\pounds_1(\omega)),\pounds_1(\pounds_1(\omega)))\rho(\pounds(\omega),\pounds_1(\omega)))+\\ & \eta_2\left(\rho(\pounds(\omega),\pounds_1(\omega))^2(1+\rho(\pounds(\omega),\pounds_1(\pounds_1(\omega))))\right)+\\ & \eta_3\left(\begin{array}{c}\rho(\pounds(\omega),\pounds_1(\omega))\rho(\pounds_1(\pounds_1(\omega)),\pounds_1(\omega))\\ (1+\rho(\pounds(\omega),\pounds_1(\pounds_1(\omega))))\end{array}\right)+\\ & \eta_4(\rho(\pounds(\pounds_1(\omega)),\pounds_1(\omega))\rho(\pounds(\omega),\pounds_1(\omega)))+\\ & \eta_5\left(\frac{(1+\rho(\pounds(\omega),\pounds(\pounds_1(\omega))))\rho(\pounds(\omega),\pounds_1(\omega))}{(1+\rho(\pounds(\pounds_1(\omega)),\pounds_1(\pounds_1(\omega))))}\right)^2+\\ & +\eta_6\left(\frac{\rho(\pounds(\omega),\pounds_1(\pounds_1(\omega)))\rho(\pounds(\omega),\pounds_1((\omega)))}{(1+\rho(\pounds(\pounds_1(\omega)),\pounds_1(\pounds_1(\omega))))}\right)^2,\end{aligned}$$

or

$$\begin{aligned}(\rho(\pounds(\omega),\pounds_1(\omega)))^2 \leq\ & \eta_1(\rho(\pounds(\omega),\pounds((\omega)))\rho(\pounds(\omega),\pounds_1(\omega)))+\\ & \eta_2\left(\rho(\pounds(\omega),\pounds_1(\omega))^2(1+\rho(\pounds(\omega),\pounds(\omega)))\right)+\\ & \eta_3(\rho(\pounds(\omega),\pounds_1(\omega))\rho(\pounds(\omega),\pounds_1(\omega))(1+\rho(\pounds(\omega),\pounds(\omega))))\\ & +\eta_4(\rho(\pounds(\omega),\pounds_1(\omega))\rho(\pounds(\omega),\pounds_1(\omega)))+\\ & \eta_5\left(\frac{(1+\rho(\pounds(\omega),\pounds(\omega)))\rho(\pounds(\omega),\pounds_1(\omega))}{(1+\rho(\pounds(\omega),\pounds(\omega)))}\right)^2\\ & +\eta_6\left(\frac{\rho(\pounds(\omega),\pounds(\omega))\rho(\pounds(\omega),\pounds_1((\omega)))}{(1+\rho(\pounds(\omega),\pounds(\omega)))}\right)^2,\end{aligned}$$

$$\begin{aligned}\Rightarrow \rho(\pounds(\omega),\pounds_1(\omega)) \leq\ & \eta_2\rho(\pounds(\omega),\pounds_1(\omega))+\eta_3\rho(\pounds(\omega),\pounds_1(\omega))+\\ & \eta_4\rho(\pounds(\omega),\pounds_1(\omega))+\eta_5\rho(\pounds(\omega),\pounds_1(\omega)),\end{aligned}$$

implies

$$(1-(\eta_2+\eta_3+\eta_4+\eta_5))\rho(\pounds(\omega),\pounds_1(\omega)) \leq 0,$$

and the inequality is possible only if $\rho(\pounds(\omega),\pounds_1(\omega))=0 \Rightarrow \pounds(\omega)=\pounds_1(\omega)$. Hence, we use $\pounds(\pounds_1(\omega))=\pounds_1(\pounds(\omega))=\pounds_1(\pounds_1(\omega))=\pounds(\omega)=\pounds_1(\omega)$.

To see uniqueness, suppose $\omega_1=\pounds(\omega_1)=\pounds_1(\omega_1)$ and $\pounds_2=\pounds(\omega_2)=\pounds_2(\omega_2)$. By

(5.1) we get

$$
\begin{aligned}
(\rho(\omega_1.\omega_2))^2 \leq\ & \eta_1 (\rho(\pounds(\omega_1), \pounds_1(\omega_1)) \rho(\pounds(\omega_2), \pounds_1(\omega_2))) + \\
& \eta_2 \left(\rho(\pounds(\omega_2), \pounds_1(\omega_2))^2 (1+\rho(\pounds(\omega_2), \pounds_1(\omega_1)))\right) + \\
& \eta_3 (\rho(\pounds(\omega_2), \pounds_1(\omega_2)) \rho(\pounds_1(\omega_1), \pounds_1(\omega_2)) (1+\rho(\pounds(\omega_2), \pounds_1(\omega_1)))) + \\
& \eta_4 (\rho(\pounds(\omega_1), \pounds_1(\omega_2)) \rho(\pounds(\omega_2), \pounds_1(\omega_2))) + \\
& \eta_5 \left(\frac{(1+\rho(\pounds(\omega_2), \pounds(\omega_1))) \rho(\pounds(\omega_2), \pounds_1(\omega_2))}{(1+\rho(\pounds(\omega_1), \pounds_1(\omega_1)))}\right)^2 + \\
& \eta_6 \left(\frac{\rho(\pounds(\omega_2), \pounds_1(\omega_1)) \rho(\pounds(\omega_2), \pounds_1(\omega_2))}{(1+\rho(\pounds(\omega_1), \pounds_1(\omega_1)))}\right)^2, \\
\leq\ & \eta_1 (\rho(\omega_1, \omega_1) \rho(\omega_2, \omega_2)) + \eta_2 \left(\rho(\omega_2, \omega_2)^2 (1+\rho(\omega_2, \omega_1))\right) \\
& \eta_3 (\rho(\omega_2, \omega_2) \rho(\omega_1, \omega_2) (1+\rho(\omega_2, \omega_1))) + \eta_4 (\rho(\omega_1, \omega_2) \rho(\omega_2, \omega_2)) + \\
& \eta_5 \left(\frac{(1+\rho(\omega_2, \omega_1)) \rho(\omega_2, \omega_2)}{(1+\rho(\omega_1, \omega_1))}\right)^2 + \eta_6 \left(\frac{\rho(\omega_2, \omega_1) \rho(\omega_2, \omega_2)}{(1+\rho(\omega_1, \omega_1))}\right)^2, \\
\Rightarrow\ & (\rho(\omega_1.\omega_2))^2 \leq 0,
\end{aligned}
$$

which is possible only if $\rho(\omega_1, \omega_2) = 0 \Rightarrow \omega_1 = \omega_2$, which shows the uniqueness of fixed point of mappings $\pounds$ and $\pounds_1$.

Corollary 5.2.2. If $\pounds_1 : \Im \to \Im$ satisfy the condition:

$$
\begin{aligned}
(\rho(\pounds_1(\mu_1), \pounds_1(\mu_2)))^2 \leq\ & \eta_1 (\rho(\mu_1, \pounds_1(\mu_1)) \rho(\mu_2, \pounds_1(\mu_2))) + \\
& \eta_2 \left(\rho(\mu_2, \pounds_1(\mu_2))^2 (1+\rho(\mu_2, \pounds_1(\mu_1)))\right) + \\
& \eta_3 (\rho(\mu_2, \pounds_1(\mu_2)) \rho(\pounds_1(\mu_1), \pounds_1(\mu_2)) (1+\rho(\pounds_2, \pounds_1(\mu_1)))) + \\
& \eta_4 (\rho(\mu_1, \pounds_1(\mu_2)) \rho(\mu_2, \pounds_1(\mu_2))) + \\
& \eta_5 \left(\frac{(1+\rho(\mu_2, \mu_1)) \rho(\mu_2, \pounds_1(\mu_2))}{(1+\rho(\mu_1, \pounds_1(\mu_1)))}\right)^2 + \\
& \eta_6 \left(\frac{\rho(\mu_2, \pounds_1(\mu_1)) \rho(\mu_2, \pounds_1((\mu_2)))}{(1+\rho(\mu_1, \pounds_1(\mu_1)))}\right)^2
\end{aligned}
$$

for some $\mu_1, \mu_2 \in \Im$, where $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6 \geq 0$, such that $\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + \eta_5 < 1$, then $\mathcal{L}_1$ has a unique fixed point in $\Im$.

Proof Put $\mathcal{L} = I_{\mathcal{L}}$ (Identity mapping) in Theorem 5.2.1 and follow the result.

Remark 5.2.3. If we put $\eta_6 = 0$, then we get Theorem 5.1.3 in [55].

Remark 5.2.4. If we put $\eta_2 = \eta_3 = \eta_4 = \eta_5 = \eta_6 = 0$, then we get Theorem 5.1.2 in [46].

Remark 5.2.5. If $\eta_1 = \eta_2 = \lambda_1$, $\eta_3 = \eta_4 = \lambda_2$ and $\eta_5 = \eta_6 = \lambda_3$, we get Theorem 5.1.5 in [68].

Example 5.2.6. If $\Im = \left[0, \frac{1}{3}\right]$ is any set and $\rho : \Im \times \Im \to R^+$ is a metric on $\Im$ given by $\rho(\mu_1, \mu_2) = |\mu_1 - \mu_2|$ for $\mu_1, \mu_2 \in \Im$. If $\mathcal{L}$ and $\mathcal{L}_1$ on $\Im$ are given by $\mathcal{L}(\mu) = \mu^2$, $\mathcal{L}_1(\mu) = \mu^3$. Then $\mathcal{L}$ and $\mathcal{L}_1$ commute with each other such that $\mathcal{L}(\mathcal{L}_1(\mu)) = \mathcal{L}_1(\mathcal{L}(\mu)) = \mu^6$ and $\mathcal{L}_1(\Im) = \left[0, \frac{1}{27}\right] \subseteq \left[0, \frac{1}{18}\right] = \mathcal{L}(\Im)$ with 0 as the only common fixed point of $\mathcal{L}$ and $\mathcal{L}_1$.

Our next theorem, is a generalization of Theorem 5.2.1 in the context of a compact metric space. In concluding the section, we observe that Theorem 5.2.1 remains valid even if $\Im$ happens to be a compact metric space.

Theorem 5.2.7. If $\mathcal{L}, \mathcal{L}_1 : \Im \times \Im \to R^+$ are two mappings of a compact metric space with $\mathcal{L}$ is continuous and $\mathcal{L}, \mathcal{L}_1$ commute with each other such that $\mathcal{L}_1(\Im) \subset \mathcal{L}(\Im)$ and

$\mathcal{L}_1$ satisfy the following condition:

$$\begin{aligned}
\left(\rho\left(\mathcal{L}_1\left(\mu_1\right),\mathcal{L}_1\left(\mu_2\right)\right)\right)^2 \leq\ & \eta_1\left(\rho\left(\mathcal{L}\left(\mu_1\right),\mathcal{L}_1\left(\mu_1\right)\right)\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_2\right)\right)\right)+ \\
& \eta_2\left(\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_2\right)\right)^2\left(1+\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_1\right)\right)\right)\right)+ \\
& \eta_3\left(\begin{array}{c}\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_2\right)\right)\rho\left(\mathcal{L}_1\left(\mu_1\right),\mathcal{L}_1\left(\mu_2\right)\right)\times \\ \left(1+\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_1\right)\right)\right)\end{array}\right)+ \\
& \eta_4\left(\rho\left(\mathcal{L}\left(\mu_1\right),\mathcal{L}_1\left(\mu_2\right)\right)\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_2\right)\right)\right)+ \\
& \eta_5\left(\frac{\left(1+\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}\left(\mu_1\right)\right)\right)\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_2\right)\right)}{\left(1+\rho\left(\mathcal{L}\left(\mu_1\right),\mathcal{L}_1\left(\mu_1\right)\right)\right)}\right)^2+ \\
& \eta_6\left(\frac{\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\mu_1\right)\right)\rho\left(\mathcal{L}\left(\mu_2\right),\mathcal{L}_1\left(\left(\mu_2\right)\right)\right)}{\left(1+\rho\left(\mathcal{L}\left(\mu_1\right),\mathcal{L}_1\left(\mu_1\right)\right)\right)}\right)^2,
\end{aligned}$$

for μ_1, $\mu_2 \in \Im$, $\mu_1 \neq \mu_2$, $\eta_i \geq 0$ and

$$\eta_1+\eta_2+\eta_3+2\eta_4+\eta_5=1, \tag{5.4}$$

then $\mathcal{L}$ and $\mathcal{L}_1$ have a unique common fixed point in $\Im$.

Proof If $\mu_0 \in \Im$ is arbitrary and $\mathcal{L}_1(\Im) \subset \mathcal{L}(\Im)$, there exists $\mu_1 \in \Im$ such that $\mathcal{L}_1\mu_0 = \mathcal{L}\mu_1$. We construct a sequence $\{\mu_n\}$ in $\Im$ satisfying $\mathcal{L}_1\mu_n = \mathcal{L}\mu_{n+1}$ using (5.4) and $\left(\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right)\right)^2 = \left(\rho\left(\pounds_1\left(\mu_n\right), \pounds_1\left(\mu_{n+1}\right)\right)\right)^2$ we get

$$
\begin{aligned}
\left(\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right)\right)^2 \leq\ & \eta_1\left(\rho\left(\pounds\left(\mu_n\right), \pounds_1\left(\mu_n\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_{n+1}\right)\right)\right)+ \\
& \eta_2\left(\rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_{n+1}\right)\right)^2\left(1+\rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_n\right)\right)\right)\right) \\
& +\eta_3\left(\begin{array}{c}\rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_{n+1}\right)\right) \rho\left(\pounds_1\left(\mu_n\right), \pounds\left(\mu_{n+1}\right)\right) \\ \left(1+\rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_n\right)\right)\right)\end{array}\right) \\
& +\eta_4\left(\rho\left(\pounds\left(\mu_n\right), \pounds_1\left(\mu_{n+1}\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_{n+1}\right)\right)\right) \\
& +\eta_5\left(\frac{\left(1+\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_n\right)\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_{n+1}\right)\right)}{\left(1+\rho\left(\pounds\left(\mu_n\right), \pounds_1\left(\mu_n\right)\right)\right)}\right)^2 \\
& \eta_6\left(\frac{\rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\mu_n\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds_1\left(\left(\mu_{n+1}\right)\right)\right)}{\left(1+\rho\left(\pounds\left(\mu_n\right), \pounds_1\left(\mu_n\right)\right)\right)}\right)^2 \\
\leq\ & \eta_1\left(\rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n+1}\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right)\right)+ \\
& \eta_2\left(\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right)^2\left(1+\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+1}\right)\right)\right)\right)+ \\
& \eta_3\left(\begin{array}{c}\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right) \\ \left(1+\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+1}\right)\right)\right)\end{array}\right)+ \\
& \eta_4\left(\rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n+2}\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right)\right)+ \\
& \eta_5\left(\begin{array}{c}\left(\frac{\left(1+\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_n\right)\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right)}{\left(1+\rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n+1}\right)\right)\right)}\right)^2+ \\ \left(\frac{\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+1}\right)\right) \rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\left(\mu_{n+2}\right)\right)\right)}{\left(1+\rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n-1}\right)\right)\right)}\right)^2\end{array}\right),
\end{aligned}
$$

which gives us $\rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right) \leq \rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n+1}\right)\right)$ for $n \geq 0$. Hence, $\rho_{n+1} \leq \rho_n$, where $\rho_n = \rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n+1}\right)\right)$ and $\rho_{n+1} = \rho\left(\pounds\left(\mu_{n+1}\right), \pounds\left(\mu_{n+2}\right)\right)$, $\eta_1+\eta_2+\eta_3+2\eta_4+\eta_5 = 1$. Hence, $\rho_n = \left\{\rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n+1}\right)\right)\right\}$ is non-increasing and so converges to a limit $\rho \geq 0$ such that $\lim_{n\to\infty} \rho_n = \lim_{n\to\infty}\left\{\rho\left(\pounds\left(\mu_n\right), \pounds\left(\mu_{n+1}\right)\right)\right\} = \rho$. Since $\Im$ is compact, using sequential compactness of $\Im$, there exists a subsequence $\left\{\mu_{n_k}\right\}$ of $\left\{\mu_n\right\}$ such that for any $\theta \in \Im$ and $k \to \infty$, we get $\lim_{k\to\infty} \mu_{n_k} = \theta$. Using continuity of $\pounds_1$ and ρ, to obtain $\rho_{n_k} = \rho\left(\mu_{n_k}, \mu_{n_{k+1}}\right) = \rho\left(\mu_{n_k}, \pounds_1\mu_{n_k}\right) \to \rho\left(\theta, \pounds_1\theta\right)$ as $k \to \infty$. Since

$\rho_{n_k} \to \rho$, we have $\rho = \rho(\theta, \pounds_1\theta)$. Similarly,

$$\rho_{n_{k+1}} = \rho\left(\mu_{n_{k+1}}, \mu_{n_{k+2}}\right) = \rho\left(\pounds_1\mu_{n_k}, \pounds_1\pounds_1\mu_{n_k}\right) \to \rho(\pounds_1\theta, \pounds_1\pounds_1\theta) = \rho \qquad (5.6)$$

as $k \to \infty$. Since, the sequence $\left\{\rho_{n_{k+1}}\right\}$ is a subsequence of the sequence $\{\rho_n\}$, we get $\rho = \rho(\theta, \pounds_1\theta) = \rho(\pounds_1\theta, \pounds_1\pounds_1\theta)$. Next, we claim $\rho = 0$. Suppose $\rho \neq 0$, then $\theta \neq \pounds_1\theta$ and by (5.6) we obtain

$$\rho = \lim_{k\to\infty} \rho_{n_{k+1}} = \lim_{k\to\infty} \rho\left(\pounds_1\left(\mu_{n_k}\right), \pounds_1\left(\pounds_1\left(\mu_{n_k}\right)\right)\right) = \rho\left(\pounds_1(\theta), \pounds_1(\pounds_1(\theta))\right) < \rho(\theta, \pounds_1(\theta))$$

which is contradiction, hence $\rho = 0$. Hence,

$$\lim_{n\to\infty} \rho\left(\pounds\mu_n, \pounds\mu_{n+1}\right) = 0. \qquad (5.7)$$

Since, $\pounds_1\mu_n = \pounds\mu_{n+1}$ for each $n = 0, 1, 2, \ldots$.From (5.7), we get

$$\inf\{d(\pounds(\mu), \pounds_1(\mu)) : \mu \in \Im\} = 0. \qquad (5.8)$$

Since the mapping $\rho : \Im \to R^+$ defined by $\mu \to \rho(\pounds(\mu), \pounds_1(\mu))$ is continuous, so for $\theta_1 \in \Im$ we get

$$\rho(\pounds(\theta_1), \pounds_1(\theta_1)) = \inf\{\rho(\pounds(\mu), \pounds_1(\mu)) : \mu \in \Im\}.$$

By (5.8), $\rho(\pounds(\theta_1), \pounds_1(\theta_1)) = 0$ and so $\pounds(\theta_1) = \pounds_1(\theta_1) = \theta$. Now, since $\pounds$ and $\pounds_1$ commute, we have $\pounds(\theta) = \pounds(\pounds_1(\theta_1)) = \pounds_1(\pounds(\theta_1)) = \pounds_1(\theta)$. Thus,

$$\pounds(\theta) = \pounds_1(\theta) = \theta, \qquad (5.9)$$

and so θ is a common fixed point of $\pounds$ and $\pounds_1$.

To prove, θ is unique. Suppose, on the contrary that there exists another point $\omega \in \Im$

such that $\pounds(\omega) = \pounds_1(\omega) = \omega$ with $\pounds(\omega) \neq \pounds(\theta)$. Using condition (5.4) we get

$$\begin{aligned}
(\rho(\theta,\omega))^2 &\leq \eta_1 (\rho(\pounds(\theta), \pounds_1(\theta)) \rho(\pounds(\omega), \pounds_1(\omega))) + \\
&\quad \eta_2 \left(\rho(\pounds(\omega), \pounds_1(\omega))^2 (1 + \rho(\pounds(\omega), \pounds_1(\theta)))\right) + \\
&\quad \eta_3 (\rho(\pounds(\omega), \pounds_1(\omega)) \rho(\pounds_1(\omega), \pounds_1(\omega)) (1 + \rho(\pounds(\omega), \pounds_1(\omega)))) + \\
&\quad \eta_4 (\rho(\pounds(\theta), \pounds_1(\omega)) \rho(\pounds(\omega), \pounds_1(\omega))) + \\
&\quad \eta_5 \left(\frac{(1 + \rho(\pounds(\omega), \pounds(\theta))) \rho(\pounds(\omega), \pounds_1(\omega))}{(1 + \rho(\pounds(\theta), \pounds_1(\theta)))}\right)^2 + \\
&\quad \eta_6 \left(\frac{\rho(\pounds(\omega), \pounds_1(\theta)) \rho(\pounds(\omega), \pounds_1((\omega)))}{(1 + \rho(\pounds(\theta), \pounds_1(\theta)))}\right)^2, \\
&\leq \eta_1 (\rho(\theta,\theta) \rho(\omega,\omega)) + \eta_2 \left(\rho(\omega,\omega)^2 (1 + \rho(\omega,\theta))\right) + \\
&\quad \eta_3 (\rho(\omega,\omega) \rho(\theta,\omega) (1 + \rho(\omega,\theta))) + \eta_4 (\rho(\theta,\omega) \rho(\omega,\omega)) + \\
&\quad \eta_5 \left(\frac{(1 + \rho(\omega,\theta)) \rho(\omega,\omega)}{(1 + \rho(\theta,\theta))}\right)^2 + \eta_6 \left(\frac{\rho(\omega,\upsilon) \rho(\omega,\omega)}{(1 + \rho(\theta,\theta))}\right),
\end{aligned}$$

this gives us

$$\begin{aligned}
(\rho(\theta,\omega))^2 &\leq 0, \\
(\rho(\theta,\omega))^2 &= 0, \\
\rho(\theta,\omega) &= 0,
\end{aligned}$$

which implies that $\theta = \omega$ and this gives us uniqueness of θ.

Corollary 5.2.8. If $\pounds$, $\pounds_1 : \Im \to \Im$ are self maps and $\Im$ with ρ is compact and satisfy

$$(\rho(\pounds_1(\mu_1), \pounds_1(\mu_2)))^2 \leq \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$$

where

$$
\begin{aligned}
X_1 &= \rho\left(\mathcal{L}\left(\mu_1\right), \mathcal{L}_1\left(\mu_1\right)\right) \rho\left(\mathcal{L}\left(\mu_2\right), \mathcal{L}_1\left(\mu_2\right)\right)+\rho\left(\mathcal{L}\left(\mu_2\right), \mathcal{L}_1\left(\mu_2\right)\right)^2 \\
&\quad \left(1+\rho\left(\mathcal{L}\left(\mu_2\right), \mathcal{L}_1\left(\mu_1\right)\right)\right) \\
X_2 &= \rho\left(\mathcal{L}\left(\mu_2\right), \mathcal{L}_1\left(\mu_2\right)\right) \rho\left(\mathcal{L}_1\left(\mu_1\right), \mathcal{L}_1\left(\mu_2\right)\right)\left(1+\rho\left(\mathcal{L}\left(\mu_2\right), \mathcal{L}_1\left(\mu_1\right)\right)\right) \\
&\quad +\rho\left(\mathcal{L}\left(\mu_1\right), \mathcal{L}_1\left(\mu_2\right)\right) \rho\left(\mathcal{L}\left(\mu_2\right), \mathcal{L}_1\left(\mu_2\right)\right) \\
X_3 &= \left(\frac{(1+\rho(\mathcal{L}(\mu_2),\mathcal{L}(\mu_1)))\rho(\mathcal{L}(\mu_2),\mathcal{L}_1(\mu_2))}{(1+\rho(\mathcal{L}(\mu_1),\mathcal{L}_1(\mu_1)))}\right)^2+\left(\frac{\rho(\mathcal{L}(\mu_2),\mathcal{L}_1(\mu_1))\rho(\mathcal{L}(\mu_2),\mathcal{L}_1((\mu_2)))}{(1+\rho(\mathcal{L}(\mu_1),\mathcal{L}_1(\mu_1)))}\right)^2,
\end{aligned}
$$

for μ_1, $\mu_2 \in \Im$, where λ_1, λ_2, $\lambda_3 \geq 0$ with $\lambda_1 + 2\lambda_2 + \lambda_3 = 1$, then $\mathcal{L}$ and $\mathcal{L}_1$have unique common fixed point in $\Im$.

Proof Putting $\eta_1 = \eta_2 = \lambda_1$, $\eta_3 = \eta_4 = \lambda_2$ and $\eta_5 = \lambda_3$ in Theorem 5.2.4 we get the required result.

Corollary 5.2.9. If $\mathcal{L}_1 : \Im \to \Im$ satisfy

$$
\begin{aligned}
\left(\rho\left(\mathcal{L}_1\left(\mu_1\right), \mathcal{L}_1\left(\mu_2\right)\right)\right)^2 \leq\; & \eta_1\left(\rho\left(\mu_1, \mathcal{L}_1\left(\mu_1\right)\right) \rho\left(\mathcal{L}_2, \mathcal{L}_1\left(\mu_2\right)\right)\right)+ \\
& \eta_2\left(\rho\left(\mu_2, \mathcal{L}_1\left(\mu_2\right)\right)^2\left(1+\rho\left(\mu_2, \mathcal{L}_1\left(\mu_1\right)\right)\right)\right)+ \\
& \eta_3\left(\rho\left(\mu_2, \mathcal{L}_1\left(\mu_2\right)\right) \rho\left(\mathcal{L}_1\left(\mu_1\right), \mathcal{L}_1\left(\mu_2\right)\right)\left(1+\rho\left(\mu_2, \mathcal{L}_1\left(\mu_1\right)\right)\right)\right)+ \\
& \eta_4\left(\rho\left(\mu_1, \mathcal{L}_1\left(\mu_2\right)\right) \rho\left(\mu_2, \mathcal{L}_1\left(\mu_2\right)\right)\right) \\
& \eta_4\left(\rho\left(\mu_1, \mathcal{L}_1\left(\mu_2\right)\right) \rho\left(\mu_2, \mathcal{L}_1\left(\mu_2\right)\right)\right),
\end{aligned}
$$

for μ_1, $\mu_2 \in \Im$ and η_1, η_2, η_3, $\eta_4 \geq 0$, such that $\eta_1 + \eta_2 + \eta_3 + 2\eta_4 = 1$, then $\mathcal{L}_1$ has fixed point in $\Im$.

Proof Putting $\eta_5 = 0$ and $\mathcal{L} = I_{\mathcal{L}}$ (Identity mapping) in Theorem 5.2.4 and the result follows.

Corollary 5.2.10. If $\pounds_1 : \Im \to \Im$ is mapping of the compact metric space and satisfy

$$
\begin{aligned}
\left(\rho\left(\pounds_1\left(\mu_1\right), \pounds_1\left(\mu_2\right)\right)\right)^2 \leq\ & \eta_1\left(\rho\left(\mu_1, \pounds_1\left(\mu_1\right)\right) \rho\left(\mu_2, \pounds_1\left(\mu_2\right)\right)\right)+ \\
& \eta_2\left(\rho\left(\mu_2, \pounds_1\left(\mu_2\right)\right)^2\left(1+\rho\left(\mu_2, \pounds_1\left(\mu_1\right)\right)\right)\right),
\end{aligned}
$$

for $\mu_1, \mu_2 \in \Im$ such that $\eta_1 + 2\eta_2 = 1$, then $\pounds_1$ has a unique fixed point in $\Im$.

Proof Putting $\eta_3 = \eta_4 = \eta_5 = \eta_6 = 0$, in Theorem 5.2.7 and the result follow.

Remark 5.2.11. Corollary 5.2.10 is the result of Fisher [34].

Now, we give an example to support the validity of the above Theorem 5.2.4.

Example 5.2.14. If $\Im = \{1, 5, 9\}$ is a finite set and ρ be the metric with ordinary distance. If $\pounds$ and $\pounds_1$ on $\Im$ be defined by

$$
\begin{aligned}
\pounds(1) &= 5, \pounds(5) = 1, \pounds(9) = 9 \text{ and} \\
\pounds_1(1) &= \pounds_1(5) = \pounds_1(9) = 9,
\end{aligned}
$$

Then, it is clear that $\pounds_1(\Im) \subset \pounds(\Im)$ with $\pounds$ and $\pounds_1$ commute, continuous and $(\Im, \rho)$ is a compact metric space. Then, this example satisfy all the conditions of Theorem 5.2.7 with 9 as the only common fixed point in $\Im$.

In the third and last section of this chapter, the third theorem deals with common fixed point of two continuous mappings defined on a hausdorff space.

Theorem 5.2.15. If $\pounds$ and $\pounds_1$ are continuous mappings of a hausdorff space $\Im$ with $\pounds$ and $\pounds_1$ commute with each other such that $\pounds_1(\Im) \subset \pounds(\Im)$. If $\psi : \Im \times \Im \to R^+$ is continuous and for each pair of elements $\mu_1, \mu_2 \in \Im$ with $\pounds(\mu_1) \neq \pounds(\mu_2)$ and satisfy

$$\begin{aligned}((\pounds_1(\mu_1), \pounds_1(\mu_2)))^2 \leq\; & \eta_1 (\psi(\pounds(\mu_1), \pounds_1(\mu_1)) \psi(\pounds(\mu_2), \pounds_1(\mu_2))) + \\ & \eta_2 ((\psi(\pounds(\mu_2), \pounds_1(\mu_2)))^2 (1 + \psi(\pounds(\mu_2), \pounds_1(\mu_1)))) + \\ & \eta_3 \begin{pmatrix} \psi(\pounds(\mu_2), \pounds_1(\mu_2)) \psi(\pounds_1(\mu_1), \pounds_1(\mu_2)) \times \\ (1 + \psi(\pounds(\mu_2), \pounds_1(\mu_1))) \end{pmatrix} + \\ & \eta_3 \begin{pmatrix} \psi(\pounds(\mu_2), \pounds_1(\mu_2)) \psi(\pounds_1(\mu_1), \pounds_1(\mu_2)) \times \\ (1 + \psi(\pounds(\mu_2), \pounds_1(\mu_1))) \end{pmatrix} + \\ & \eta_4 (\psi(\pounds(\mu_1), \pounds_1(\mu_2)) \psi(\pounds(\mu_2), \pounds_1(\mu_2))) + \\ & \eta_5 \left(\frac{(1 + \psi(\pounds(\mu_2), \pounds(\mu_1))) \psi(\pounds(\mu_2), \pounds_1(\mu_2))}{1 + \psi(\pounds(\mu_1), \pounds_1(\mu_1))}\right)^2 + \\ & \eta_6 \left(\frac{\psi(\pounds(\mu_2), \pounds_1(\mu_1)) \psi(\pounds(\mu_2), \pounds_1(\mu_2))}{1 + \psi(\pounds(\mu_1), \pounds_1(\mu_1))}\right)^2,\end{aligned}$$

for $\eta_i \geq 0$, such that $\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + \eta_5 < 1$. If for some $\mathcal{L}_0 \in \Im$, the sequence $\{\mathcal{L}_n\}$ in $\Im$ has a convergent subsequence.Then $\pounds$ and $\pounds_1$ admits a common fixed point.

Proof Since $\pounds_1(\Im) \subset \pounds(\Im)$. So, for $\mathcal{L}_0 \in \Im$, we choose $\mathcal{L}_1 \in \Im$ such that $\pounds_1(\mathcal{L}_0) = \pounds(\mathcal{L}_1)$,with the sequence $\{\mathcal{L}_n\}$ defined by $\pounds_1\mathcal{L}_0 = \pounds\mathcal{L}_1$, $\pounds_1\mathcal{L}_1 = \pounds\mathcal{L}_2, ..., \pounds_1\mathcal{L}_{n-1} = \pounds\mathcal{L}_n$, $\pounds_1\mathcal{L}_n = \pounds\mathcal{L}_{n+1}$where $(\psi(\pounds(\mathcal{L}_{n+1}), \pounds(\mathcal{L}_{n+2})))^2 = (\psi(\pounds_1(\mathcal{L}_n), \pounds_1(\mathcal{L}_{n+1})))^2$ and using (5.10) to obtain

$$\begin{aligned}(\psi(\pounds(\mathcal{L}_{n+1}), \pounds(\mathcal{L}_{n+2})))^2 \leq\; & \eta_1 (\psi(\pounds(\mathcal{L}_n), \pounds_1(\mathcal{L}_n)) \psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_{n+1}))) + \\ & \eta_2 ((\psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_{n+1})))^2 (1 + \psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_n)))) + \\ & \eta_3 \begin{pmatrix} \psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_{n+1})) \psi(\pounds_1(\mathcal{L}_n), \pounds_1(\mathcal{L}_{n+1})) \times \\ (1 + \psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_n))) \end{pmatrix} + \\ & \eta_4 (\psi(\pounds(\mathcal{L}_n), \pounds_1(\mathcal{L}_{n+1})) \psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_{n+1}))) + \\ & \eta_5 \left(\frac{(1 + \psi(\pounds(\mathcal{L}_{n+1}), \pounds(\mathcal{L}_n))) \psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_{n+1}))}{1 + \psi(\pounds(\mathcal{L}_n), \pounds_1(\mathcal{L}_n))}\right)^2 + \\ & \eta_6 \left(\frac{\psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_n)) \psi(\pounds(\mathcal{L}_{n+1}), \pounds_1(\mathcal{L}_{n+1}))}{1 + \psi(\pounds(\mathcal{L}_n), \pounds_1(\mathcal{L}_n))}\right)^2,\end{aligned}$$

or

$$
\begin{aligned}
\left(\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\right)^2 \leq\; & \eta_1\left(\psi\left(\pounds\left(\mathcal{L}_n\right),\pounds\left(\mathcal{L}_{n+1}\right)\right)\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\right)+\\
& \eta_2\left(\left(\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\right)^2\left(1+\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+1}\right)\right)\right)\right)+\\
& \eta_3\left(\begin{array}{c}\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\\ \left(1+\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+1}\right)\right)\right)\end{array}\right)+\\
& \eta_4\left(\psi\left(\pounds\left(\mathcal{L}_n\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\right)+\\
& \eta_5\left(\frac{\left(1+\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_n\right)\right)\right)\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)}{1+\psi\left(\pounds\left(\mathcal{L}_n\right),\pounds\left(\mathcal{L}_{n+1}\right)\right)}\right)^2+\\
& \eta_6\left(\frac{\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+1}\right)\right)\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)}{1+\psi\left(\pounds\left(\mathcal{L}_n\right),\pounds\left(\mathcal{L}_{n+1}\right)\right)}\right)^2,
\end{aligned}
$$

or

$$
\begin{aligned}
\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right) \leq\; & \eta_1\psi\left(\pounds\left(\mathcal{L}_n\right),\pounds\left(\mathcal{L}_{n+1}\right)\right)+\eta_2\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)\\
& +\eta_3\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)+\eta_4\psi\left(\pounds\left(\mathcal{L}_n\right),\pounds\left(\mathcal{L}_{n+2}\right)\right)+\\
& \eta_5\psi\left(\pounds\left(\mathcal{L}_{n+1},\pounds\left(\mathcal{L}_{n+2}\right)\right)\right),\\
\psi\left(\pounds\left(\mathcal{L}_{n+1}\right),\pounds\left(\mathcal{L}_{n+2}\right)\right) \leq\; & \frac{\left(\eta_1+\eta_4\right)}{\left(1-\left(\eta_2+\eta_3+\eta_4+\eta_5\right)\right)}\psi\left(\pounds\left(\mathcal{L}_n\right),\pounds\left(\mathcal{L}_{n+1}\right)\right). \qquad (5.11)
\end{aligned}
$$

If $\mathcal{L}_n = \pounds\left(\gamma_n\right)$ for some $n \in N$, by (5.11), we get

$$
\psi\left(\mathcal{L}_1,\mathcal{L}_2\right)=\psi\left(\pounds\left(\gamma_1\right),\pounds\left(\gamma_2\right)\right)\leq\frac{\left(\eta_1+\eta_4\right)}{\left(1-\left(\eta_2+\eta_3+\eta_4+\eta_5\right)\right)}\psi\left(\pounds\left(\gamma_0\right),\pounds\left(\gamma_1\right)\right)<\psi\left(\mathcal{L}_0,\mathcal{L}_1\right).
$$

Similarly,

$$
\psi\left(\mathcal{L}_2,\mathcal{L}_3\right)=\psi\left(\pounds\left(\gamma_2\right),\pounds\left(\gamma_3\right)\right)\leq\frac{\left(\eta_1+\eta_4\right)}{\left(1-\left(\eta_2+\eta_3+\eta_4+\eta_5\right)\right)}\psi\left(\pounds\left(\gamma_1\right),\pounds\left(\gamma_2\right)\right)<\psi\left(\mathcal{L}_1,\mathcal{L}_2\right).
$$

Since, $\left(\eta_1+\eta_2+\eta_3+2\eta_4+\eta_5\right)<1$, repeting the above process, we get

$$
\psi\left(\mathcal{L}_0,\mathcal{L}_1\right)\geq\psi\left(\mathcal{L}_2,\mathcal{L}_3\right)\geq\ldots,\geq\psi\left(\mathcal{L}_n,\mathcal{L}_{n+1}\right)\geq\ldots.
$$

This shows that the sequence $\psi(\mathcal{L}_n, \mathcal{L}_{n+1})$ is bounded which converges along with all its subsequences to some positive real number ω. If $\{\mathcal{L}_n\}$ has a convergent subsequence of $\{\mathcal{L}_{n_k}\}$ which converges to the real number ω. Then,

$$\begin{aligned}
\psi(\omega, \mathcal{L}_1(\omega)) &= \psi\left(\lim_{k\to\infty}(\mathcal{L}_{n_k}), \mathcal{L}_1\left(\lim_{k\to\infty}(\mathcal{L}_{n_k})\right)\right) = \psi\left(\lim_{k\to\infty}(\mathcal{L}_{n_k}), \lim_{k\to\infty}(\mathcal{L}_{n_{k+1}})\right), \\
&= \psi \lim_{k\to\infty}\left((\mathcal{L}_{n_{k+1}}), (\mathcal{L}_{n_{k+2}})\right) = \psi\left(\lim_{k\to\infty}(\mathcal{L}_{n_{k+1}}), \lim_{k\to\infty}(\mathcal{L}_{n_{k+2}})\right), \\
&= \psi\left(\mathcal{L}_1 \lim_{k\to\infty}(\mathcal{L}_{n_k}), \mathcal{L}_1\left(\mathcal{L}_1 \lim_{k\to\infty}(\mathcal{L}_{n_k})\right)\right) = \psi(\mathcal{L}_1(\omega), \mathcal{L}_1(\mathcal{L}_1(\omega))).
\end{aligned}$$

Next, to show ω is fixed point of $\mathcal{L}$ and $\mathcal{L}_1$. First, we show that $\mathcal{L}_1$ admit a fixed point ω. Suppose, for contradiction that $\mathcal{L}_1(\omega) \neq \omega$, then by (5.10) we have $(\psi(\omega, \mathcal{L}_1(\omega)))^2 = (\psi(\mathcal{L}_1(\omega), \mathcal{L}_1(\mathcal{L}_1(\omega))))^2$ and hence,

$$\begin{aligned}
(\psi(\omega, \mathcal{L}_1(\omega)))^2 \leq\ & \eta_1(\psi(\mathcal{L}(\omega), \mathcal{L}_1(\omega))\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))) \\
& \eta_2\left((\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega))))^2(1+\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\omega)))\right) + \\
& \eta_3\left(\begin{array}{c} \psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))\psi(\mathcal{L}_1(\omega), \mathcal{L}_1(\mathcal{L}_1(\omega))) \\ (1+\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\omega))) \end{array}\right) + \\
& \eta_4(\psi(\mathcal{L}(\omega), \mathcal{L}_1(\mathcal{L}_1(\omega)))\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))) + \\
& \eta_5\left(\frac{(1+\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}(\omega)))\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))}{1+\psi(\mathcal{L}(\omega), \mathcal{L}_1(\omega))}\right)^2 + \\
& \eta_6\left(\frac{\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\omega))\psi(\mathcal{L}(\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))}{1+\psi(\mathcal{L}(\omega), \mathcal{L}_1(\omega))}\right)^2, \\
\leq\ & \eta_1(\psi(\mathcal{L}_1(\omega), \mathcal{L}_1(\omega))\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))) + \\
& \eta_2\left((\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega))))^2(1+\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\omega)))\right) + \\
& \eta_3\left(\begin{array}{c} \psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))\psi(\mathcal{L}_1(\omega), \mathcal{L}_1(\mathcal{L}_1(\omega))) \\ (1+\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\omega))) \end{array}\right) + \\
& \eta_4(\psi(\mathcal{L}_1(\omega), \mathcal{L}_1(\mathcal{L}_1(\omega)))\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))) + \\
& \eta_5\left(\frac{(1+\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\omega)))\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))}{1+\psi(\mathcal{L}_1(\omega), \mathcal{L}_1(\omega))}\right)^2 + \\
& \eta_6\left(\frac{\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\omega))\psi((\mathcal{L}_1(\omega)), \mathcal{L}_1(\mathcal{L}_1(\omega)))}{1+\psi(\mathcal{L}(\omega), \mathcal{L}_1(\omega))}\right)^2,
\end{aligned}$$

Hence,

$$\begin{aligned}(\psi(\omega,\mathcal{L}_1(\omega)))^2 &\leq \eta_1(\psi(\omega,\omega)\psi(\omega,\mathcal{L}_1(\omega))) + \eta_2\psi((\omega,\mathcal{L}_1(\omega)))^2(1+\psi(\omega,\omega)) + \\ &\quad \eta_3(\psi(\omega,\mathcal{L}_1(\omega))\psi(\omega,\mathcal{L}_1(\omega))(1+\psi(\omega,\omega))) + \\ &\quad \eta_4(\psi(\omega,\mathcal{L}_1(\omega))\psi(\omega,\mathcal{L}_1(\omega))) + \eta_5(\psi(\omega,\mathcal{L}_1(\omega)))^2,\end{aligned}$$

implies

$$\begin{aligned}\psi(\omega,\mathcal{L}_1(\omega)) &\leq \eta_2\psi(\omega,\mathcal{L}_1(\omega)) + \eta_3\psi(\omega,\mathcal{L}_1(\omega)) + \eta_4\psi(\omega,\mathcal{L}_1(\omega)) + \eta_5\psi(\omega,\mathcal{L}_1(\omega)), \\ &\Rightarrow (1-(\eta_2+\eta_3+\eta_4+\eta_5))\psi(\omega,\mathcal{L}_1(\omega)) \leq 0,\end{aligned}$$

which is contradiction, because $\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + \eta_5 < 1$. Hence, $\psi(\omega,\mathcal{L}_1(\omega)) = 0 \Rightarrow \omega = \mathcal{L}_1(\omega)$. Thus, $\omega \in \Im$ is a fixed point of $\mathcal{L}_1$. Since, $\mathcal{L}$ and $\mathcal{L}_1$ commute and are continuous so, $\mathcal{L}(\mathcal{L}_1(\mathcal{L}_{n_k})) \to \mathcal{L}(\omega)$ and $\mathcal{L}_1(\mathcal{L}(\mathcal{L}_{n_k})) \to \mathcal{L}_1(\omega)$ as $k \to \infty$, which implies that $\mathcal{L}(\mathcal{L}_1(\mathcal{L}_{n_k})) = \mathcal{L}_1(\mathcal{L}(\mathcal{L}_{n_k}))$ as $n_{k\to\infty}$ and by the uniqueness of limit we have $\mathcal{L}(\omega) = \mathcal{L}_1(\omega) = \omega$.

Now, we claim that ω is the unique common fixed point of $\mathcal{L}$ and $\mathcal{L}_1$. Suppose, for contradiction that ω_1 is another fixed point of $\mathcal{L}$ and $\mathcal{L}_1$ so that $\mathcal{L}(\omega_1) = \mathcal{L}_1(\omega_1) = \omega_1$, then by (5.10) we obtain

$$\begin{aligned}
(\psi(\omega,\omega_1))^2 &\leq \eta_1 (\psi(\pounds(\omega),\pounds_1(\omega))\psi(\pounds(\omega_1),\pounds_1(\omega_1))) + \\
&\quad \eta_2 \left((\psi(\pounds(\omega_1),\pounds_1(\omega_1)))^2 (1+\psi(\pounds(\omega_1),\pounds_1(\omega)))\right) + \\
&\quad \eta_3 \left(\begin{array}{c} \psi(\pounds(\omega_1),\pounds_1(\omega_1))\psi(\pounds(\omega),\pounds_1(\omega_1)) \\ (1+\psi(\pounds(\omega_1),\pounds_1(\omega))) \end{array}\right) + \\
&\quad \eta_4 (\psi(\pounds(\omega),\pounds_1(\omega_1))\psi(\pounds(\omega_1),\pounds_1(\omega_1))) + \\
&\quad \eta_5 \left(\frac{(1+\psi(\pounds(\omega_1),\pounds(\omega)))\psi(\pounds(\omega_1),\pounds_1(\omega_1))}{1+\psi(\pounds(\omega),\pounds_1(\omega))}\right)^2 + \\
&\quad \eta_6 \left(\frac{\psi(\pounds(\omega_1),\pounds_1(\omega))\psi(\pounds(\omega_1),\pounds_1(\omega_1))}{1+\psi(\pounds(\omega),\pounds_1(\omega))}\right)^2, \\
&\leq \eta_1 (\psi(\omega,\omega)\psi(\omega_1,\omega_1)) + \eta_2 \left((\psi(\omega_1,\omega_1))^2 (1+\psi(\omega_1,\omega))\right) + \\
&\quad \eta_3 (\psi(\omega_1,\omega_1)\psi(\omega,\omega_1)(1+\psi(\omega_1,\omega))) + \\
&\quad \eta_5 \left(\frac{(1+\psi(\omega_1,\omega))\psi(\omega_1,\omega_1)}{1+\psi(\omega,\omega)}\right)^2 + \eta_6 \left(\frac{\psi(\omega_1,\omega)\psi(\omega_1,\omega_1)}{1+\psi(\omega,\omega)}\right) \\
&\leq 0.
\end{aligned}$$

This implies that $(\psi(\omega,\omega_\cdot))^2 \leq 0$, which is contradiction because $(\eta_1+\eta_2+\eta_3+2\eta_4+\eta_5) < 1$. Hence, $(\psi(\omega,\omega_1))^2 = 0 \Rightarrow \omega = \omega_1$. Hence, ω is unique.

Corollary 5.2.16. If $\pounds_1 : \Im \to \Im$ is continuous on a hausdorff space $\Im$ and $\psi : \Im \times \Im \to R^+$is continuous so that for each pair of elements $\mu_1, \mu_2 \in \Im$, $\pounds(\mu_1) \neq \pounds(\mu_2)$ and satisfy

$$\begin{aligned}
(\psi(\pounds_1(\mu_1),\pounds_1(\mu_2)))^2 &\leq \eta_1 (\psi(\mu_1,\pounds_1(\mu_1))\psi(\mu_2,\pounds_1(\mu_2))) + \\
&\quad \eta_2 \left((\psi(\mu_2,\pounds_1(\mu_2)))^2 (1+\psi(\mu_2,\pounds_1(\mu_1)))\right) + \\
&\quad \eta_3 (\psi(\mu_2,\pounds_1(\mu_2))\psi(\pounds_1(\mu_1),\pounds_1(\mu_2))(1+\psi(\mu_2,\pounds_1(\mu_1)))) + \\
&\quad \eta_4 (\psi(\mu_1,\pounds_1(\mu_2))\psi(\mu_2,\pounds_1(\mu_2))) + \\
&\quad \eta_5 \left(\frac{(1+\psi(\mu_2,\mu_1))\psi(\mu_2,\pounds_1(\mu_2))}{1+\psi(\mu_1,\pounds_1(\mu_1))}\right)^2 + \\
&\quad \eta_6 \left(\frac{\psi(\mu_2,\pounds_1(\mu_1))\psi(\mu_2,\pounds_1(\mu_2))}{1+\psi(\mu_1,\pounds_1(\mu_1))}\right)^2,
\end{aligned}$$

where $\eta_i \geq 0$ satisfy the condition $\eta_1 + \eta_2 + \eta_3 + 2\eta_4 + \eta_5 < 1$. If for some $\mu_0 \in \Im$, the sequence $\{\mu_n\}$ in $\Im$ has a convergent subsequence.Then $\pounds_1$admits a unique fixed point in $\Im$.

Proof Put $\pounds = I_{\pounds}$ (Identity mapping) in Theorem 5.2.15 and the result follow.

Corollary 5.2.17. If $\pounds$ and $\pounds_1$ are continuous mappings of a hausdorff space $\Im$ and if $\pounds$ and $\pounds_1$ commute with each others with the condition $\pounds_1(\Im) \subset \pounds(\Im)$. If $\psi : \Im \times \Im \to R^+$ is a continuous function and for each pair of $\mu_1,\ \mu_2 \in \Im,\ \pounds(\mu_1) \neq \pounds(\mu_2)$ satisfy

$$\begin{gathered}(\psi(\pounds_1(\mu_1), \pounds_1(\mu_2)))^2 \leq \eta_1 (\psi(\pounds(\mu_1), \pounds_1(\mu_1))\, \psi(\pounds(\mu_2), \pounds_1(\mu_2))) + \\ \eta_2 \left((\psi(\pounds(\mu_2), \pounds_1(\mu_2)))^2 (1 + \psi(\pounds(\mu_2), \pounds_1(\mu_1)))\right) +,\end{gathered}$$

where $\eta_1, \eta_2 \geq 0$, such that $\eta_1 + \eta_2 < 1$. If for some $\mu_0 \in \Im$, the sequence $\{\mu_n\}$ in $\Im$ has a convergent subsequence.Then $\pounds$ and $\pounds_1$ admit a common fixed point.

Proof Put $\eta_3 = \eta_4 = \eta_5 = \eta_6 = 0$, in Theorem 5.2.15 we get Corollary 5.2.17.

Remark 5.2.18. Corollary 5.2.16 is the result of [55].

Remark 5.2.19. Corollary 5.2.17 is the result of [23].

Example 5.2.20. If $\Im = \{3, 4, 5\}$and define by $\psi : \Im \times \Im \to [0, \infty)$ and $\pounds,\ \pounds_1 : \Im \to \Re$ by

$$\begin{aligned} \psi(\mathcal{L}_1, \mathcal{L}_1) &= 0 \text{ and } \psi(\mathcal{L}_1, \mathcal{L}_2) = \psi(\mathcal{L}_2, \mathcal{L}_1) \text{ for } \mathcal{L}_1, \mathcal{L}_2 \in \Im \text{ with} \\ \psi(3,4) &= 1,\ \psi(3,5) = \psi(4,5) = 2 \text{ and} \\ \pounds(3) &= 4, \pounds(4) = 3, \pounds(5) = 5, \pounds_1(3) = \pounds_1(4) = \pounds_1(5) = 5. \end{aligned}$$

Now, it is clear that Theorem 5.2.15 is satisfied and 5 is the common fixed point of $\pounds$ and $\pounds_1$.

Conclusion

This research deals with function spaces in general and fixed point theory in particular. In this thesis, we have mainly focused on the generalizations of certain fixed theorems available in the literature on fixed point theory, which include generalizations of fixed results in complete, compact, pseudo-compact, hausdorff and b-metric spaces using the notions of continuous, commuting and weakly commuting mappings. We used the methods adopted by Bailey, Edelstein, Fisher and Jungch etc. The applications of fixed point theory is to seek unique solution of linear algebraic, differential and integral equations reduced to functional equations.

List of Publications

1. Mumtaz ali and Muhammad Arshad, b-Metric Generalization of some fixed point theorems, Journal of Function spaces Vol. 2018, Article ID: 2658653, 9 pages http://doi. org/10. 1155/2018/2658653, Impact factor 0.639.

2. Mumtaz Ali and Muhammad Arshad, Generalizations of Fixed Point Theorems in Pseudo-compact Tichnov Spaces, Turkish Journal of Analysis and Number Theory, Vol. 5, 159-164 (2017).

3. Mumtaz Ali and Muhammad Arshad, Fixed Point Theorems in Complete and Compact Metric Spaces, South Asian Journal of Mathematics, Vol. 7(2): 81-87(2017).

4. Mumtaz Ali and Muhammad Arshad, Generalized Fixed Point Results in Compact Metric Spaces, Journal of Mathematics and Computer Science, Accepted.

5. Mumtaz Ali and Muhammad Arshad, Related Fixed Point Theorems in Compact Metric Spaces, Journal of Advances in Pure Mathematics Paper ID: 5301133, Accepted June, 2016.

6. Mumtaz Ali and Muhammad Arshad, Unique Fixed Point Theorems in Compact Metric Spaces, American Scientific Research Journal for Engineering, Technology and Science Acceptance August, 2016.

7. Mumtaz Ali and Muhammad Arshad, Fixed Point Results and Pachpatte Theorem, South Asian Journal of Mathematics Vol. 7(4), 209-213(2017).

8. Mumtaz Ali and Muhammad Arshad, Generalization of Fixed Point Theorems for Two Mappings, Turkish journal of Analysis and Number Theory Vol. 5(6), 230-239(2017).

9. Mumtaz Ali and Muhammad Arshad, Common Fixed Point Theorems for four Weakly Compatible Mappings in Complete Metric Spaces, South Asian Journal of Mathematics Vol. 8(2), 48-63(2018).

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