Comparative Analysis of Some Homotopy Based Methods

By

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Homotopy theory.

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IN
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We accept this thesis as conforming to the required standard.

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2016

Dedicated
to
my
loving Parents

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Fatira Khalid

Preface

Non-linear differential equations arises in the modeling of many physical problems. The solution of such equation is very tedious job. Over the year, many scientist work in this field to find the solution procedure that helps to solve most of such problems. In pursuit many numerical scheme and analytical scheme are developed [1-10]. All of these technique have there advantages and disadvantages. The analytical solution are more difficult to handle. Right toward the start of twentieth century perturbation methods are developed [11] by A. H. Nafyeh. Which helps to decompose these non-linear problems to system of linear equations which were then solved, but it needs a small parameter on same the same footing many technique were developed. Liao introduced an embedding parameter in the equation and then decompose to linear form about that parameter and uses the concept to homotopy from topology to develop technique,now known as Homotopy Analysis method [12].This method gives series solution of most of non-linear ODE's. Latter, developed on this schemes shows varity of technique emerges to have better solution [13-20]. Optimal techniques are also merged with this scheme to have quick convergence [21-25]. This thesis is design to analysis some of these homotopy base techniques. In this effort five chapters are constructed.

In first chapter, we define some basic definition and methods which are involved in subsequent chapters.

In second chapter, we solve non-linear initial value problem arising in mathematical modeling of free falling body [26]. A first order ordinary differential equation arises along with an initial condition. Five Homotopy base solution are obtained and compared.

In third chapter, we consider a problem from fluid mechanics. In which a power law fluids flows over a streching surface [26]. Some are transformation applied and problem is formed in the domain 0 and 1. The problem is solved using five homotopy base technique and solutions are compared with each other.

In fourth chapter, the problem of micro polar fluid with $n=(k/2)$ is assumed. The equation of micro rotation reduces to zero [27]. Problem is reduced to ordinary differential equation by using similarity transform. Different Homotopy solution are obtained and compared.

Fifth chapter is formulated to discuss the homotopic solution by coupled ordinary differential equations [27]. The problem in previous chapter is repharased with $k=1$ and equation of microrotation appear. The equations are coupled and boundary conditions are described at $x=0$ and $x=\infty$. The problem is solved using varity of homotopy techniques.

Contents

Chapter 1

Introduction

This chapter is oriented to define some basic defination and methods which are involved in subsiquent chapters.

1.1 Basic Definations

1.2 Differential Equation

An equation containing the derivatives or differential of one or more dependent variables with respect to one or more independent variables is said to be a differential equation .

1.3 Classification by Type

1.3.1 Ordinary Differential Equation

An equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is then said to be an ordinary differential equation (ODE).

1.3.2 Partial Differential Equation

An equation involving the partial derivative of one or more dependent variables of two or more independent variables is called a partial differential equation (PDE).

1.4 Classification Linear or Non-Linear

1.4.1 Linear Equation

A differential equation is said to be linear if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{1.1}$$

1.4.2 Non-Linear Equation

An equation which cannot be written in the form of Eq (1.1) is said to be non-linear equation.

1.5 Homogenous and Non-Homogenous Differential Equation

1.5.1 Homogenous Differential Equation

An equation in differential form M(x,y)dx+N(x,y)dy=0, is said to be homogenous when written in derivative form

$$\frac{dy}{dx} = f(x,y) = g(\frac{y}{x}), \tag{1.2}$$

there exist a function g such that f(x,y)=g($\frac{y}{x}$).

1.5.2 Non-Homogenous Differential Equation

A differential equation, which fails the condition (1.2) is called non-homogenous differential equation.

1.6 Initial and Boundary-Value Problem

1.6.1 Initial-Value Problem

An initial value problem has all of the conditions specified at the same value of the independent variable.

1.6.2 Boundary-Value Problem

A boundary value problem has conditions specified at the boundaries of the independent variable in the equation or the condition define at more than one point in the domain.

1.7 Homotopy Analysis Method

Homotopy Analysis method(HAM) allows perturbation solution to be valid for moderate to large value of parameter. HAM has been developed by Liao in 1992 [8]. This method has been successfully applied to solve many types of non-linear problems[12]-[14]. The basic idea of HAM is to produce a succession of approximate solutions tend to the exact solution from any initial guess of the problem. The presence of auxiliary parameter and functions in the approximate solution results in a production of a family of approximation solution rather than the single solution produced by traditional perturbation methods. By varying these auxiliary parameter and functions, it is possible to adjust the region and rate of convergence of series solution.

1.7.1 General Approach of HAM

consider non-linear equation

$$N[u(x)] = 0, \tag{1.3}$$

subject to some initial condition or boundary conditions. The first step in the HAM solution of the equation is construct the homotopy

$$H[\phi(x;q); \phi_0(x), H(x), \hbar, q] = (1-q)L[\phi(x;q) - \phi_0(x)] - q\hbar H(x)N[\phi(x;q)].$$

where $\hbar \neq 0$ is an auxilary parameter , $H(x) \neq 0$ is an auxilary function, $q \in [0,1]$ is an embedding parameter, $\phi_0(x)$ is an initial approximation to the solution that satisfies the given initial condition or boundary conditions, $\phi(x;q)$ satisfies the initial or boundary conditions and L is some linear operator. The linear operator L should normally be of the same order as the non-linear operator N. Setting homotopy equal to zero so that

$$(1-q)L[\phi(x;q) - \phi_0(x)] = q\hbar H(x)N[\phi(x;q)]. \tag{1.4}$$

Eq (1.4) is known as the zero-order deformation equation. By letting q=0 in this equation, we obtain

$$L[\phi(x;0) - \phi_0(x)] = 0. \tag{1.5}$$

it follows from our defination of $L[\phi(x)], \phi(x;q)$and $\phi_0(x)$ that

$$\phi(x;0) = \phi_0(x), \tag{1.6}$$

now letting q=1 then

$$N[\phi(x;1)] = 0.$$

It is clear that $\phi(x;q)$ satisfy the initial or boundry condition of the problem and

$$\phi(x;1) = \phi(x), \tag{1.7}$$

so, $\phi(x;q)$ varies continuously from initial approximation to the required solution $\phi(x)$ as q increases 0 to 1. Now we define the terms

$$\phi_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x;q)}{\partial q^m}\bigg|_{q=0}. \tag{1.8}$$

By Taylor's Theorem we can write

$$\phi(x;q) = \phi(x;0) + \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m \phi(x;q)}{\partial q^m}\bigg|_{q=0} .q^m = \phi_0(x) + \sum_{m=1}^{\infty} \phi_m(x) q^m. \tag{1.9}$$

Now we differentiate Eq (1.4) with respect to q and setting q=0 and finally dividing by m! Then so called mth-order deformation equation become

$$L[\phi_m(x) - \varkappa_m \phi_{m-1}(x)] = \hbar H(x) \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0}, \tag{1.10}$$

where,

$$\varkappa_m = \begin{cases} 0, \dots .\ m \leq 1 \\ 1, \dots \dots\ else \end{cases}.$$

Thus this equation is valid for all $m \geq 1$. The right hand side of Eq (1.10) will depend on term $\phi_m(x)$ with n<m. As a result the terms $\phi_m(x)$ can be obtained in order of increasing m by solving the linear deformation equations in succession. The solution to the $m^{th} order$ deformation equation can be written as,

$$\phi_m(x) = \phi^h(x) + \phi_m^p(x), \tag{1.11}$$

Where $\phi^h(x)$ satisfies the homogenus equation

$$L[\phi^h(x)] = 0, \tag{1.12}$$

and $\phi_m^p(x)$ is a particular solution of Eq (1.9) we can express it as

$$\phi_m^p(x) = \varkappa_m \phi_{m-1}(x) + L^{-1}\left(\frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0} \right), \tag{1.13}$$

L^{-1} is the inverse operator of the linear operator L. The m^{th} partial sum of the terms $\phi_m(x)$ as

$$\phi^m(x) = \sum_{k=0}^{\infty} \phi_k(x), \tag{1.14}$$

thus solution can be expressed as

$$\phi(x) = \phi(x;1) = \sum_{k=0}^{\infty} \phi_k(x) = \lim_{m \longrightarrow \infty} \phi^m(x). \tag{1.15}$$

This solution will be valid wherever the series converges.

1.8 Advantages of Homotopy Analysis Method

HAM provides the liberty in how to develop the solutions to nonlinear problems. This liberty endure several benefits over ordinary perturbation methods such as,

1. It is always valid no matter whether there exist small physical parameter or not.

2. The HAM technique can be used to develop valid solution even to problems that are

highly non-linear.

3. The HAM provides a convenient way to guarantee the convergence of approximation series.

4. The HAM provides great freedom to choose the equation type of linear sub-problems and the base function of solutions.

We can use this freedom to express a solution in terms of base function that mostly closely mirror the behaviour of the actual series. As a result the HAM overcomes the restriction of all other analytic approximation methods is valid for highly nonlinear problems.

1.9 HAM Solution With Different Base Function

HAM provides the choice to choose the linear operator and base function. The different base function have different solution behaviour but all of the solutions are valid over given domain in the region of convergence.

HAM Solution Using Polynomial Function

Firstly to solve the problem in form of polynomial function Non-linear operator is needed to be defined

$$N[\phi(x;q)] = 0. \tag{1.16}$$

Hence the base function is choosen as

$$\{x^{an+b} | a > 0; b \geq 0; n = 0, 1, 2...\}.$$

By rule of approximation together with the initial conditions. The initial approximation which satisfies the initial or boundary conditions are obtained. Then assume a linear operator $L\left[\phi(x;q)\right]$. L is the linear operator, x is the independent variable, ϕ is the unknown function and q is the embedding parameter. To find the inverse linear operator we use the mth-order deformation equation.

$$L[\phi_m(x) - \varkappa_m \phi_{m-1}(x)] = \hbar H(x) \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0}.$$

1.9.1 HAM Solution Using Rational Function

To improve on the polynomial approximation it is essential to use technique Pade approximation. This technique uses the terms of a polynomial approximation to generate a rational approximation to a function $\phi(x)$. The [m,n] pade rational approximation is written as

$$r_{m,n}(x) = \frac{\sum_{k=0}^{m} A_k x^k}{1 + \sum_{k=1}^{n} B_k x^k} \approx \sum_{k=0}^{m+n} C_k x^k, \tag{1.17}$$

where $\sum_{k=0}^{m+n} C_k x^k$ is the $(m+n)^{th}$ degree polynomial approximation to f(x). From this we obtain

$$\sum_{k=0}^{m+n} A_k x^k \approx \sum_{k=0}^{m+n} C_k x^k \left(1 + \sum_{k=1}^{n} B_k x^k\right). \tag{1.18}$$

By equating the coeffiecients of the various powers of x on each side of this equation, system of linear equation is obtained that can be easily solved for the coefficients A_k and B_k. We can improve the effectiveness of the pade technique by combining it with the HAM method . We can then express the [m,n] Homotopy-Pade approximation to the solution of equation

$$\phi_{m,m}(x) = \frac{\sum_{k=0}^{m} A_{m,k}(x) q^k}{1 + \sum_{k=1}^{m} B_{m,k}(x) q^k}|_{q=1}, \tag{1.19}$$

since the actual solution is

$$\phi(x) = \phi(x;1) = \sum_{k=0}^{\infty} \phi_k(x) q^k|_{q=1}, \tag{1.20}$$

we write

$$\frac{\sum_{k=0}^{m} A_{m,k}(x) q^k}{1 + \sum_{k=1}^{m} B_{m,k}(x) q^k} \approx \sum_{k=0}^{2m} \phi_k(x) q^k, \tag{1.21}$$

after solving for the coefficient with the Pade approximation, we set $q = 1$ to obtain the desired Pade approximation. Now

$$\sum_{k=0}^{m} A_{m,k}(x)q^k \approx \left(1 + \sum_{k=1}^{m} B_{m,k}(x)q^k\right) \sum_{k=0}^{2m} \phi_k(x)q^k,$$

$$= \sum_{k=0}^{2m} \phi_k(x)q^k + \sum_{k=1}^{m}\sum_{n=1}^{k} B_{m,n}(x)\phi_{k-n}(x)q^k + \sum_{k=m+1}^{2m}\sum_{n=1}^{m} B_{m,n}(x)\phi_{k-n}(x)q^k + O(q^{2m+1}),$$

$$= \phi_0(x) + \sum_{k=1}^{m}\left[\phi_k(x) + \sum_{n=1}^{k} B_{m,n}(x)\phi_{k-n}(x)\right]q^k + \sum_{k=m+1}^{2m}\left[\begin{array}{c}\phi_k(x)+\\ \sum_{n=1}^{m} B_{m,n}(x)\phi_{k-n}(x)\end{array}\right]q^k. \quad (1.22)$$

By equating the coefficients of the various powers of q on each side of this equation we obtain the following equations:

$$A_{m,0}(x) = \phi_0(x), \quad (1.23)$$

$$A_{m,k}(x) = \phi_k(x) + \sum_{n=1}^{m} B_{m,n}(x)\phi_{k-n}(x), \quad 1 \leq k \leq m, \quad (1.24)$$

$$0 = \phi_k(x) + \sum_{n=1}^{m} B_{m,n}(x)\phi_{k-n}(x), \ m+1 \leq k \leq 2m. \quad (1.25)$$

1.9.2 HAM Solution Using Exponential Functions

The HAM polynomial and rational approximation have provided great improvement over the perturbation Solution. Ideally we would like an approximation that agrees with the exact solution even for large value of x. This can be accomplished by choosing the set of base function

$$\{e^{-anx} | a > 0; n = 0, 1, 2...\}.$$

with these base functions, it is possible to construct an initial approximation that satisfies both, the initial condition and has the asymptotic behaviour. The exponential base function also suggest that we define the linear operator $L[\phi(x;q)]$ such that of it gives exponential function in resultant form. Where ϕ is the unknow function, x is the independent variable L is the linear operator and q is the embedding parameter.and to find the inverse linear operator we can write the mth-order deformation equation as

$$L[\phi_m(x) - \varkappa_m\phi_{m-1}(x)] = \hbar H(x)\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0},$$

we find that this deformation equation has solution of

$$\phi_m(x) = \varkappa_m\phi_{m-1}(x) + e^{-x}h\int e^{x}H(x)\left(\frac{\partial N[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0}\right)dx + c_m e^{-x}, m > 0. \tag{1.26}$$

By the rule of expression we define the auxiliary function as

$$H(x) = e^{-x},$$

using this expression in equation (1.26) we have

$$\phi_m(x) = \varkappa_m\phi_{m-1}(x) + e^{-x}h\int \left(\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0}\right)dx + c_m e^{-x}, m > 0. \tag{1.27}$$

1.10 Optimal-Homotopy Analysis Method

The concept of Optimal Homotopy Analysis Method(OHAM) is on the basis of Homotopy analysis method which we discuss in above. For a given non-linear differential equation

$$N[u(x)] = 0, \tag{1.28}$$

where N is the non-linear operator and u(x) is an unknown function. A one-parameter family of equations in the embedding parameter $q \in [0,1]$. Then so called mth-order deformation equation become

$$(1-q)L[u(x;q) - u_0(x)] + qN[u(x;q)] = 0, \tag{1.29}$$

where L is the linear operator and u_0 is the initial guess. At q=0 and q=1 we have $u(x;1) = u(x)$ and $u(x;0) = u_0$ and

$$u(x) = u_0(x) + \sum_{n=1}^{\infty} u_n(x), \tag{1.30}$$

In 1997, Liao[21] introduced such a nonzero auxilary parameter c_0 to construct two parameter family of equation,so, the Zeroth-order deformation equation becomes

$$(1-q)L[u(x;q)-u_0(x)] = qc_0N[u(x;q)]. \tag{1.31}$$

In this way the homotopy series solution is not depent only x but also depend on the auxilary parameter c_0. This auxilary parameter c_0 can adjust and control the convergence region and rate of homotopy series solution. The use of the convergence control parameter c_0 has a great progress. Thus, in 1999, Liao[21] further introduced the more "artificial" degrees of freedom by using zero order deformation equation in more general form:

$$(1-B(q;c2))L[u(x;q)-u_0(x)] = c_0A(q;c1)N[u(x;q)], \tag{1.32}$$

where A(q) and B(q) is the deformation function satisfying

$$A(0) = B(0) = 0 \text{ and } A(1) = B(1) = 1. \tag{1.33}$$

whose Taylor's series

$$A(q;c_1) = \sum_{m-1}^{+\infty} \mu_m(c_1)q^m, B(q;c_2) = \sum_{m-1}^{+\infty} \sigma_m(c_2)q^m. \tag{1.34}$$

There are infinite number of deformation function. For the sake of simplicity we choose

$$\begin{aligned} \mu_1(c_1) &= 1-c_1, \ \mu_m(c_1) = (1-c_1)c_1^{m-1}, \ \ m>1, \\ \sigma_1(c_2) &= 1-c_2, \ \sigma_m(c_2) = (1-c_2)c_2^{m-1}, m>1, \end{aligned} \tag{1.35}$$

1.11 One-Step Optimal Homotopy Analysis Method (OOHAM)

The idea of One-step Optimal Homotopy Analysis Method is based on the OHAM. For this we define non-linear operator

$$N[u(x)] = 0,$$

We set $H(q) = \hbar A(q)$ and $B(q) = q$ then the zeroth-order deformation equatiom becomes

$$(1-q)L[u(x;q) - u_0(x)] = H(q)N[u(x;q)], \tag{1.36}$$

where L is non-linear operator, $u_0(x)$ is initial guess and q is embedding parameter where $q \in [0,1]$and $H(q)$ is called the convergence control function satisfying $H(0) = 0$ and $H(1) \neq 0$

$$u(x;q) = \sum_{m=0}^{+\infty} u_m(x)q^m and\ H(q) = \sum_{k=0}^{+\infty} h_k q^k. \tag{1.37}$$

so the mth-order deformation equation for unknown $u_m(x)$ becomes

$$L[u_m(x) - \varkappa_m u_m(x)] = \sum_{k=1}^{m} h_k \left(N[u(x;q)]\right). \tag{1.38}$$

In this method h_k is calculated at each step such that the residual is minimum. Different technique employed to find h_k. In this thesis a hybrid Genetic Algorithm and Neldermead is used to solve the problem.

Chapter 2

Homotopy Base Solution of Initial Value Problem

In this chapter we solve non-linear initial value problem arising in mathematical modeling of free falling body. A first order ordinary differential equation arises along with an initial condition. Five Homotopy base solution are obtained and compared.

2.1 Mathematical Modeling

The problem is modeled for body falling under the effect of gravity with an air resistance propotional to the square of the velocity is assumed. A body of mass m falling freely through space with a velocity U(τ)that varies with a time τ under the influence of gravity g and air resistance $aU^2(\tau)$.Newton's second law for this situation takes the form

$$m\frac{dU(\tau)}{d\tau} = mg - aU^2(\tau). \tag{2.1}$$

With the initial condition

$$U(0) = 0, \tag{2.2}$$

as $\tau \to \infty, U(\tau)$ will approach a terminal velocity

$$U_\infty = \sqrt{\frac{mg}{a}}. \tag{2.3}$$

We make the subsitution

$$\tau = \frac{U_\infty}{g} x,\ U(\tau) = U_\infty \phi(x).$$

Using this transformation, we get the equation

$$\frac{\partial \phi(x;q)}{\partial x} + (\phi(x;q))^2 = 1, \tag{2.4}$$

with initial condition

$$\phi(0) = 0, \tag{2.5}$$

and the condition in (2.3) become

$$lim_{x \to \infty} \phi(x) = 1.$$

2.1.1 Close Form Solution

To find the close form solution series of the form

$$\phi(x) = \sum_{n=0}^{\infty} a_n x^n,$$

is assumed using initial condition (2.5). We subsitute the expression of $\phi(x)$ given in Eq (2.4) into the Eq (2.5) to obtain

$$\sum_{n=0}^{\infty} \left((n+1) a_{n+1} + \sum_{r=0}^{n} a_r a_{n-r} \right) x^n = 1. \tag{2.6}$$

The recursive form as

$$a_1 = 1,\ a_{n+1} = \frac{1}{n+1} \sum_{r=0}^{n} a_r a_{n-r}. \tag{2.7}$$

The series solution is

$$\phi(x) = x - \frac{1}{3} x^3 + \frac{2}{15} x^5 - \frac{17}{315} x^7 + \ldots\ldots$$

Which is the series of $tanh(x)$ when t is small. Hence ,

$$\phi(x) = \tanh(x). \tag{2.8}$$

2.2 Homotopy

To solve this problem with Homotopy Analysis method first we introduced the Non-linear operator

$$N[\phi(x;q)] = \frac{\partial \phi(x;q)}{\partial x} + (\phi(x;q))^2 - 1. \tag{2.9}$$

The mth-order deformation equation become

$$L[\phi_m(x) - \varkappa_m \phi_{m-1}(x)] = \hbar H(x) \left[\phi'_{m-1}(x) + \sum_{r=0}^{m-1} \phi_r(x)\phi_{m-r-1}(x) - (1 - \varkappa_m) \right],$$

we require our initial guess to satisfy

$$\phi(0) = 0.$$

2.3 HAM Solution Using Polynomial Function

Assuming the base function of polynomial type in the form

$$\left\{ x^{an+b} | a > 0; b \geq 0; n = 0, 1, 2.... \right\}.$$

In accordance to the initial condition

$$\phi_0(x) = x.$$

Assuming the linear operator

$$L\left[\phi(x;q)\right] = \frac{\partial \phi(x;q)}{\partial x}. \tag{2.10}$$

The mth-order deformation equation becomes

$$\frac{\partial}{\partial x}[\phi_m(x) - \varkappa_m \phi_{m-1}(x)] = \hbar H(x) \left(\phi'_{m-1}(x) + \sum_{r=0}^{m-1} \phi_r(x)\phi_{m-r-1}(x) - (1 - \varkappa_m) \right). \tag{2.11}$$

The constant of integration in each iteration found using

$$\phi_m(0) = 0, m > 0. \tag{2.12}$$

For simplicity auxilary function is taken

$$H(x) = 1.$$

The first three itration yields the solution

$$\begin{aligned}
\phi^{[1]}(x) &= x + \frac{1}{3}\hbar x^3, \\
\phi^{[2]}(x) &= x + \frac{1}{3}\hbar x^3(2+\hbar) + \frac{2}{15}\hbar^2 x^5, \\
\phi^{[3]}(x) &= x + \frac{1}{3}\hbar x^3(3+3\hbar+\hbar^2) + \frac{2}{15}\hbar^2 x^5(2+\hbar) + \frac{17}{315}\hbar^3 x^7.
\end{aligned} \tag{2.13}$$

Table 2 − 1

Analysis of Error

x	**tanh(x)**	**Polynomial**	
	Exact	*Appro*	*Error*
0	0	0	0
0.2	0.19737532	0.19737532	0
0.4	0.379948962	0.379948962	0
0.6	0.537049567	0.537049567	0
0.8	0.66403677	0.664036954	-1.84×10^{-7}
1	0.761594156	0.761622106	−0.00002795
1.5	0.905148254	1.135710851	−0.230563
2	0.96402758	126.6691007	−125.705

2.4 HAM Solution Using Rational Function

To find the better solution the the polynomial function, the solution can be found in form of rational function using Homotopy-Pade approximation as described in section 1.11.2. Here Pade approximation is applied on embedding parameter 'q', and then putting q=1 as defined in (1.17-1.25) using Homotopy-Pade technique on Eq (2.1). The solution obtained in section 2.2 is used and found the coeffiecient at each step. The first three approximations are stated as,

$$\begin{aligned} \phi_{1,1}(x) &= \frac{x(15+x^2)}{3(5+2x^2)}, \\ \phi_{2,2}(x) &= \frac{x(945+10x^2+x^4)}{15(63+28x^2+x^4)}, \\ \phi_{3,3}(x) &= \frac{x(135135+17325x^2+378x^4+x^6)}{7(19305)+8910x^2+450x^4+4x^6}. \end{aligned} \tag{2.14}$$

Table 2 − 2

Analysis of Error

x	**tanh(x)**	**Rational**	
	Exact	*Appro*	*Error*
0	0	0	0
0.2	0.19737532	0.19737532	0
0.4	0.379948962	0.379948962	0
0.6	0.537049567	0.537049567	0
0.8	0.66403677	0.664036775	-5×10^{-9}
1	0.761594156	0.761594203	−0.0975574
1.5	0.905148254	0.905150215	-1.961×10^{-6}
2	0.96402758	0.964048866	−0.000021286

2.5 HAM Solution Using Exponential Function

As time varies from 0 to ∞ the polynomial solution also tends to infinity. To get the better approximation assymptotic solution should be taken. This can accomplished by choosing the set of base function in the form of

$$\{e^{-anx}|a > 0; n = 0, 1, 2...\}. \tag{2.15}$$

The initial guess which satisfy the initial condition with exponential function can be

$$\phi_0(x) = 1 - e^{-x}. \tag{2.16}$$

The exponential base function also suggest that we define the linear operator by

$$L[\phi(x;q)] = \frac{\partial\phi(x;q)}{\partial x} + \phi(x;q), \tag{2.17}$$

it follows that the inverse linear operator is

$$L^{-1}[\phi(x;q)] = e^{-x}\int e^{x}\phi(x;q)dx. \tag{2.18}$$

The mth-order deformation equation is written as

$$\left(\frac{\partial}{\partial x}+1\right)[\phi_m(x) - \varkappa_m\phi_{m-1}(x)] = \hbar H(x)\begin{pmatrix} \phi'_{m-1}(x) + \sum_{r=0}^{m-1}\phi_r(x)\phi_{m-r-1}(x) \\ -(1-\varkappa_m) \end{pmatrix}, \tag{2.19}$$

$$\phi_m(x) = \varkappa_m\phi_{m-1}(x) + e^{-x}\hbar\int e^{x}H(x)\begin{pmatrix} \phi'_{m-1}(x) + \sum_{r=0}^{m-1}\phi_r(x)\phi_{m-r-1}(x) \\ -(1-\varkappa_m) \end{pmatrix}dx. \tag{2.20}$$

We find that the deformation equation has the solution in the form of To simplify this equation the auxilary function is defined as

$$H(x) = e^{-x}. \tag{2.21}$$

Eq (2.20) becomes

$$\phi_m(x) = \varkappa_m\phi_{m-1}(x) + e^{-x}\hbar\int\left(\begin{array}{c}\phi'_{m-1}(x) + \sum_{r=0}^{m-1}\phi_r(x)\phi_{m-r-1}(x)- \\ (1-\varkappa_m)\end{array}\right)dx + C_m e^{-x}, m > 0. \tag{2.22}$$

where C_m is constant of integration satisfies the Eq (2.12). The first three approximation obtained are

$$\begin{aligned}
\phi^{[0]}(x) &= 1 - e^{-x}, \\
\phi^{[1]}(x) &= 1 - \frac{1}{2}(2+\hbar)e^{-x} + \hbar e^{-2x} - \frac{1}{2}\hbar e^{-3x}, \\
\phi^{[2]}(x) &= 1 - \frac{1}{4}(4+4\hbar+\hbar^2)e^{-x} + \frac{1}{2}\hbar(4+\hbar)e^{-2x} - \frac{1}{2}\hbar(2+\hbar)e^{-3x} + \frac{1}{2}\hbar^2 e^{-4x} - \frac{1}{4}\hbar^2 e^{-5x}.
\end{aligned} \tag{2.23}$$

Table 2 – 3

Analysis of Error

x	tanh(x)	Exponential	
	Exact	**Appro**	**Error**
0	0	0	0
0.2	0.19737532	0.19737532	0
0.4	0.379948962	0.379948836	1.26×10^{-7}
0.6	0.537049567	0.537047239	2.382×10^{-6}
0.8	0.66403677	0.66402516	0.00001161
1	0.761594156	0.761564619	0.000029537
1.5	0.905148254	0.90507308	0.000075174
2	0.96402758	0.963946923	0.000080657

2.6 HAM Solution Using Optimal Homotopy Analysis Method

For the solution of problem (2.1) with initial condition (2.2) by the OHAM. The initial guess is choosen by polynomial type

$$\phi_0(x) = x,$$

and linear operator is

$$L\left[\phi(x_n;q)\right] = \frac{\partial \phi(x;q)}{\partial x},$$

by the homotopy series solution

$$\phi(x) = \phi_0(x) + \sum_{k=1}^{\infty} \phi_k(x).$$

The mth-order deformation equation becomes

$$L[\phi_m(x) - \sum_{k=1}^{m-1} \sigma_{m-k}(c_2)u_k(x)] = c_0 \sum_{k=1}^{m-1} \mu_{m-k}(c_1)R_k(x).$$

subject to the condition $u(0) = 0$

where

$$R_k(x) = \frac{1}{k!}\frac{\partial^k N[\phi(x;q)]}{\partial q^k}\bigg|_{q=0} = \phi_k'(x) + \sum_{j=0}^{k} \phi_j(x)\phi_{k-j}(x) - 1(1 - \varkappa_{k+1}).$$

we can obtain the first three itration for solution of (2.1) is

$$\begin{aligned}
\phi_0(x) &= x, \\
\phi_1(x) &= x + (1 - c_2)\left(2xc_0 + c_0\frac{x^3}{3}\right), \\
\phi_2(x) &= x + (1 - c_2)\left(2xc_0 + \frac{c_0 x^3}{3}\right) + \\
&\quad (1 - c_2)c_2\left(2c_0 x + 2c_0^2 x - 2xc_0^2 c1 + \frac{c_0 x^3}{3} + \frac{5x^3 c_0^2}{3} - \frac{5}{3}c_0^2 c_1 x^3 + \frac{2c_0^2 x^5}{15} - \frac{2}{15}c_0^2 c_1 x^5\right).
\end{aligned}$$

The values of $c_{0,}c_1, c_2$ are calculated after sixth iteration. which are

$$c_0 = -0.99999999999941, c_1 = 0.8803960001, c_2 = 0.973410425.$$

Table 2 − 4

Analysis of Error

x	tanh(x)	Optimal	
	Exact	*Appro*	*Error*
0	0	0	0
0.2	0.197375	0.197381	0.0004943
0.4	0.3799489	0.379937	0.0008912
0.6	0.5370495	0.537162	0.0068876
0.8	0.6640367	0.664384	0.0046528
1	0.7615941	0.761651	−0.00005684
1.5	0.9051482	0.905272	−0.0000238
2	0.9640275	0.9641376	−0.0000101

2.7 HAM Solution Using One-Step Optimal Homotopy Analysis Method

Using the same linear operator and initial guess as in section 2.5. The mth-order deformation equation

$$L[\phi_m(x) - \varkappa_m\phi_{m-1}(x)] = \sum_{k=1}^{n} h_k R_{n-k}(k),$$

where

$$R_n(x) = \frac{1}{n!}\frac{\partial^n N[\phi[x;q]}{\partial q^n}\bigg|_{q=0} = \phi_n'(x) + \sum_{j=0}^{n}\phi_j(x)\phi_{n-j}(x) - 1(1-\varkappa_{n+1}),$$

The first three approximation obtained are

$$\begin{aligned}
\phi_0[x] &= x, \\
\phi_1(x) &= x + x^3h[0] \quad \text{and} \quad h[0] = -0.211439, \qquad (2.29)\\
\phi_2[x] &= x - 0.422877x^3 - 0.634316x^3h[1] - 0.422877x^5h[1] \quad \text{and} \quad h[1] = -0.172246.
\end{aligned}$$

Table 2 – 5

Analysis of Error

x	tanh(x)	One – stepoptimal	
	Exact	*Appro*	*Error*
0	0	0	0
0.2	0.197375	0.197384	0.00199132
0.4	0.3799489	0.379149	0.01479996
0.6	0.5370495	0.537824	0.0432256
0.8	0.6640367	0.660193	0.0788438
1	0.7615941	0.766705	0.0918892
1.5	0.9051482	0.9061324	0.0000158
2	0.9640275	0.9650213	6.2×10^{-6}

Chapter 3

Homotopy Base Solution of Boundary Value Problem with Finite Domain

In this chapter we consider a problem from fluid mechanics. In which a power law fluids flows over a streching surface. Some transformation are applied and problem is formed in the domain 0 and 1. The problem is solved using five homotopy base technique and solutions are compared with each other.

3.1 Mathematical Model

The equation is taken from the flow problem of power law fluid past streching sheet [26]. It is assumed that the flow is axisymmetric. The problem is reduced to ordinary deformation equation and finite domain by using similarity transfroms. The ODE is given as

$$u\frac{d^2u}{dx^2} - \frac{1}{k}\left(\frac{du}{dx}\right)^2 + \frac{x^2}{k}\frac{du}{dx} + \frac{(k-2)}{k}xu = 0\ , \tag{3.1}$$

$$u(0) = 0, \frac{du(1)}{dx} = 1. \tag{3.2}$$

3.2 Homotopy

To solve this problem we introduced the Non-linear operator

$$N[u(x;q)] = u(x;q)\frac{\partial^2 u(x;q)}{\partial x^2} - \frac{1}{k}\left(\frac{\partial u(x;q)}{\partial x}\right)^2 + \frac{x^2}{k}\frac{\partial u(x;q)}{\partial x} + \frac{(k-2)}{k}xu(x;q). \qquad (3.3)$$

The mth-order deformation equation become

$$L[u_m(x) - \varkappa_m u_{m-1}(x)] = \hbar H(x)\left[\sum_{r=0}^{m-1} u_r u''_{m-r-1} - \frac{1}{k}\sum_{r=0}^{m-1} u'_r u'_{m-r-1} + \frac{x^2}{k}u'_{m-1} + \frac{k-2}{k}xu_{m-1}\right]. \qquad (3.4)$$

we require our initial approximation to satisfy

$$u_0(0) = 0, \ u'_0(1) = 1. \qquad (3.5)$$

and subsequient term to satisfy

$$u_m(0) = 0 = u'_m(1) \ \text{ m>0}. \qquad (3.6)$$

3.3 HAM Solution Using Polynomial Function

To find the solution of (3.1) we use the base function of polynomial type,

$$\{x^{an+b} | a > 0, b \geq 0, n = 0, 1, 2....\}.$$

$$u_0(x) = x. \qquad (3.7)$$

which satisfies our boundary condition (3.2) and linear operator is defined by

$$L[u(x;q)] = \frac{\partial^2 u(x;q)}{\partial x^2}. \qquad (3.8)$$

The mth-term in the Polynomial HAM solution is given by

$$u_m(x) = \varkappa_m u_{m-1}(x) + h\int\int H(x)\left(\frac{\partial^{m-1}N[\phi(x;q)]}{\partial q^{m-1}}\bigg|_{q=0}\right)dxdx + c_{0,m} + c_{1,m}.$$

choose $H(x) = 1$, then

$$u_m(x) = \varkappa_m u_{m-1}(x) + h \int\int \left(\begin{array}{c} \sum_{r=0}^{m-1} u_r u''_{m-r-1} - \frac{1}{k} \sum_{r=0}^{m-1} u'_r u'_{m-r-1} \\ + \frac{x^2}{k} u'_{m-1} + \frac{k-2}{k} x u_{m-1} \end{array} \right) dx dx + c_{0,m} + c_{1,m}. \quad (3.9)$$

where $c_{0,m}$ and $c_{1,m}$ are chosen so that Eq (3.6) is satisfied for each value of m>0. so we get the first few terms in the polynomial approximation.

$$\begin{aligned} u^{[1]}(x) &= x + \frac{\hbar(4-k)}{3k} x - \frac{\hbar}{2k} x^2 + \frac{\hbar(k-1)}{12k} x^4, \\ u^{[2]}(x) &= x + \frac{\hbar(142\hbar - 14\hbar k - 11\hbar k^2 + 96k - 24k^2)}{72k^2} x - \frac{\hbar(8\hbar - 2\hbar k + 3k)}{6k^2} x^2 + \frac{\hbar^2(2-k)}{6k^2} x^3 - \\ & \frac{\hbar(4\hbar - 5\hbar k + \hbar k^2 + 3k - 3k^2)}{36k^2} x^4 + \frac{\hbar(4 - 13k + 6k^2)}{120k^2} x^5 - \frac{\hbar^2(2 - k - k^2)}{504k^2} x^7. \end{aligned} \quad (3.10)$$

Table 3 − 1

Analysis of Error

x	Numerical	Polynomial approx	Error
0	0	0	0
0.2	0.060335	0.060335	0
0.4	0.146390	0.146383	7×10^{-6}
0.6	0.262502	0.262497	5×10^{-6}
0.8	0.411729	0.411727	2×10^{-6}
1	0.595021	0.595020	1×10^{-6}

3.4 HAM Solution Using Rational Function

Now we use the Homotopy-Pade technique to produce a rational approximation on the terms which we obtain by the polynomial approximation. In the Eq (1.19). The coefficient of $A_{m,k}$ and $B_{m,k}$ can be obtained by solving the 2m+1 linear equation given by (1-23),(1-24) and (1-25) using the term calculated with Eq (3.10). The first two rational approximation calculated are

$$u_{0,0} = x, \tag{3.11}$$

$$u_{1,1} = \frac{x\left[\begin{array}{c}(55-80k+25k^2)x^6+(252+126k-252k^2)x^4-(560-700k+140k^2)x^3-\\(420-840k)x^2-(980+3500k-1330k^2)\end{array}\right]}{2\left[\begin{array}{c}(-10+5k+5k^2)x^6+(84-273k+126k^2)x^4-(280-350k+70k^2)x^3+\\(840-420k)x^2-(3360-840k)x+(4970-490k-385k^2)\end{array}\right]}.$$

Table 3 − 2

Analysis of Error

x	Numerical	Rational approx	Error
0	0	0	0
0.2	0.060335	0.060335	0
0.4	0.146390	0.146383	7×10^{-6}
0.6	0.262502	0.262496	6×10^{-6}
0.8	0.411729	0.411725	4×10^{-6}
1	0.595021	0.594994	0.000027

3.5 HAM Solution Using Exponential Function

To find the exponential solution we choose the base function in the form of

$$\left\{ e^{-ant} \middle| \, a > 0; n = 0, 1, 2... \right\}.$$

We choose

$$u_0(x) = e^{-1}(e^x - 1), \tag{3.12}$$

and linear operator

$$L[u(x;q)] = \frac{\partial^2 u(x;q)}{\partial x^2} - \frac{\partial u(x;q)}{\partial x}. \tag{3.13}$$

It follows that the inverse linear operator

$$L^{-1}[u] = \int e^{x} \int e^{-x} L(u) dx dx.$$

The mth-order deformation equation becomes

$$\begin{aligned} u_m(t) &= \varkappa_m u_{m-1}(t) + \int e^{x} h \int e^{-x} H(x) \\ & \left(\sum_{r=0}^{m-1} u_r u''_{m-r-1} - \frac{1}{k} \sum_{r=0}^{m-1} u'_r u'_{m-r-1} + \frac{x^2}{k} u'_{m-1} + \frac{k-2}{k} x u_{m-1} \right) dx dx. \end{aligned}$$

For simplicity we choose

$$H(x) = e^{x}.$$

so we get first few approximation

$$\begin{aligned} u_0[x] &= e^{-1}(e^{x} - 1), \\ u_1(x) &= \frac{\hbar\left((2-k)e^{x} + e^{2x}\right)}{2ke} x^2 - \frac{\hbar\left((4-2k)e^{x} + (5-k)e^{2x}\right)}{2ke} x + \\ & \frac{\left[\begin{array}{c} (-12ek - 51\hbar e + 6\hbar e^2 k + 4\hbar k + 3\hbar ek + 6\hbar e^2 + 2\hbar) + e^{x}(12ek - 6\hbar e^2 k + 6\hbar ek - \\ 6\hbar e^2 + 12 e\hbar) + e^{2x}(39\hbar e - 6\hbar k - 9\hbar ek) + e^{3x} 2\hbar(k-1) \end{array}\right]}{12ke^2}. \end{aligned} \tag{3.14}$$

Table 3 − 3

Analysis of Error

x	Numerical	Exponential approx	Error
0	0	0	0
0.2	0.060335	0.0373888	0.0229462
0.4	0.146390	0.104468	0.041922
0.6	0.262502	0.205304	0.057198
0.8	0.411729	0.344195	0.067534
1	0.595021	0.523811	0.07121

3.6 HAM Solution Using Optimal Homotopy Analysis Method

To find the solution from Optimal Homotopy Analysis Method, we choose polynomial type base function with initial guess

$$u_0(x) = x,$$

and linear operator

$$L[u(x;q)] = \frac{\partial^2 u(x;q)}{\partial x^2}.$$

By the homotopy series solution

$$u(x) = u_0 + \sum_{k=1}^{\infty} u_k(x).$$

The mth-order deformation equation becomes

$$L[u_m(x) - \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) u_k(x)] = c_0 \sum_{k=1}^{m-1} \mu_{m-k}(c_1) R_k(x).$$

The deformation function σ_m and μ_m are defined above. Now

$$R_k(x) = \sum_{r=0}^{m-1} u_r u''_{m-r-1} - \frac{1}{k} \sum_{r=0}^{m-1} u'_r u'_{m-r-1} + \frac{x^2}{k} u'_{m-1} + \left(\frac{k-2}{k}\right) x u_{m-1}.$$

So we obtain the first two terms of approximation

$$\begin{aligned} u_0[x] &= x, \\ u_1[x] &= x + (1 - c_2)\left(\frac{4c_0 x}{3k} - \frac{c_0 k x}{3} - \frac{c_0 x^2}{2k} - \frac{c_0 x^4}{12k} + \frac{1}{12} c_0 x^4\right). \end{aligned} \tag{3.15}$$

The values of c_0, c_1, c_2 are calculated after fifth iteration that are

$$c_0 = -1, c_1 = 0.626816, c_2 = -0.0967632.$$

Table 3 − 4

Analysis of Error

x	Numerical	Optimal approx	Error
0	0	0	0
0.2	0.060335	0.060336	-1×10^{-6}
0.4	0.146390	0.146388	2×10^{-6}
0.6	0.262502	0.262508	-6×10^{-6}
0.8	0.411729	0.411727	2×10^{-6}
1	0.595021	0.595022	-1×10^{-6}

3.7 HAM Solution Using One-Step Optimal Homotopy Analysis Method

Now we find the solution of the Eq (3.1) by the one step-optimal HAM.firstly we choose the initial guess polynomial type which we define above in Eq (3.7) and same linear operator as in Eq (3.8)

$$u_0(x) = x.$$

$$L[u(x;q)] = \frac{\partial^2 u(x;q)}{\partial x^2}.$$

The mth-order deformation equation become

$$L[u_m(x) - \varkappa_m u_{m-1}(x)] = \sum_{k=1}^{m} h_k R_{m-k}(k).$$

where

$$R_m(x) = \sum_{r=0}^{m-1} u_r u''_{m-r-1} - \frac{1}{k}\sum_{r=0}^{m-1} u'_r u'_{m-r-1} + \frac{x^2}{k} u'_{m-1} + \frac{k-2}{k} x u_{m-1},$$

$$u_m(x) = \varkappa_m u_{m-1}(x) + \int\int h_k \left(\sum_{r=0}^{m-1} u_r u''_{m-r-1} - \frac{1}{k}\sum_{r=0}^{m-1} u'_r u'_{m-r-1} + \frac{x^2}{k} u'_{m-1} + \frac{k-2}{k} x u_{m-1}\right) dx dx.$$

Here we choose $k = 1$. So we get the first few approximation solution

$$\begin{aligned} u_1(x) &= x + xh[0] - \frac{1}{2}x^2h[0] \quad \text{and} \quad \mathrm{h}[0] = -0.571491, \\ u_2(x) &= -0.142982x + 0.571491x^2 - 1.14298xh[1] + 1.14298x^2h[1] - \\ &\quad 0.285746x^3h[1] - 0.285746x^4h[1] + 0.142873x^5h[1] \quad \text{and} \quad \mathrm{h}[1] = -0.91655. \end{aligned} \tag{3.16}$$

Table 3 − 5

Analysis of Error

x	Numerical	One − StepOptimal approx	Error
0	0	0	0
0.2	0.060335	0.060361	−0.000026
0.4	0.146390	0.146381	9×10^{-6}
0.6	0.262502	0.262522	−0.00002
0.8	0.411729	0.411732	-3×10^{-6}
1	0.595021	0.595025	-4×10^{-6}

Chapter 4

Homotopy Base Solution of Boundary Value Problem with Infinite Domain

In this chapter, the problem of micro polar fluid with $n = k/2$ is assumed. The equation of micro rotation reduces to zero. Problem is reduced to ordinary differential equation by using similarity transform. Different Homotopy solution are obtained and compared.

4.1 Mathematical Formulation

The stagnation point flow of micropolar fluid past a streching sheet is defined. The partial differential equation in flow problem transformed to ordinary differential equation [27].

$$\left(1+\frac{k}{2}\right) f''' + ff'' + 1 - f'^2 = 0, \tag{4.1}$$

with boundary condition

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1. \tag{4.2}$$

The subsequent term is

$$f_m(0) = 0 = \ f'_m(0) \ f'(\infty) = 1 \ m > 0. \tag{4.3}$$

4.2 Homotopy

we now solve this problem first by using HAM. To solve this problem we introduced the Non-linear operator

$$N[f(x;q)] = (1+\frac{k}{2})\frac{\partial^3 f(x;q)}{\partial x^3} + f(x;q)\frac{\partial^2 f(x;q)}{\partial x^2} + 1 - \left(\frac{\partial f(x;q)}{\partial x}\right). \tag{4.4}$$

mth-order deformation equation becomes

$$L[f_m(x) - \varkappa_m f_{m-1}(x)] = \hbar H(x)\left[\begin{array}{c} (1+\frac{k}{2})f_{m-1}^{'''}(x) + \sum_{r=0}^{m-1} f_r(x)f_{m-r-1}''(x) - \\ \sum_{r=0}^{m-1} f_r'(x)f_{m-r-1}'(x) + 1(1-\varkappa_m) \end{array}\right].$$

we require our initial approximation to satisfy

$$f(0) = f^{'}(0) = 0, \; f^{'}(\infty) = 1.$$

4.3 HAM Solution Using Polynomial Function

To solve the above problem (4.1) by the polynomial approximation we first choose the initial guess which satisfies our boundary condition (4.2)

$$f_0 = x - \frac{x}{1+x}, \tag{4.5}$$

and we choose our linear operator is

$$L[f(x;q)] = \frac{\partial^3 f(x;q)}{\partial x^3}. \tag{4.6}$$

mth-order deformation equation becomes

$$\frac{\partial}{\partial x}[f_m(x) - \varkappa_m f_{m-1}(x)] = \hbar H(x)\left(\begin{array}{c} (1+\frac{k}{2})f_{m-1}^{'''}(x) + \sum_{r=0}^{m-1} f_r(x)f_{m-r-1}''(x) - \\ \sum_{r=0}^{m-1} f_r'(x)f_{m-r-1}'(x) + 1(1-\varkappa_m) \end{array}\right), \tag{4.7}$$

$$f_m(x) = \varkappa_m f_{m-1}(x) + \hbar \iiint H(x) \begin{pmatrix} (1+\frac{k}{2}) f'''_{m-1}(x) + \sum_{r=0}^{m-1} f_r(x) f''_{m-r-1}(x) - \\ \sum_{r=0}^{m-1} f'_r(x) f'_{m-r-1}(x) + 1(1-\varkappa_m) \end{pmatrix} dxdxdx,$$

we choose $H(x) = 1$ so the equation become

$$f_m(x) = \varkappa_m f_{m-1}(x) + \hbar \iiint \begin{pmatrix} (1+\frac{k}{2}) f'''_{m-1}(x) + \sum_{r=0}^{m-1} f_r(x) f''_{m-r-1}(x) - \\ \sum_{r=0}^{m-1} f'_r(x) f'_{m-r-1}(x) + 1(1-\varkappa_m) \end{pmatrix} dxdxdx.$$

using all the values in above equation we get few terms of approximation

$$\begin{aligned} f_0(x) &= x - \frac{x}{1+x}, \\ f_1(x) &= x - \frac{x}{1+x} + \frac{6\hbar x}{1+x} + \frac{41\hbar x^2}{6(1+x)} + \frac{\hbar k x^2}{2(1+x)} - \\ & \frac{6\hbar \ln(1+x)}{1+x} - \frac{10\hbar x \ln(1+x)}{1+x} - \frac{4\hbar x \ln(1+x)}{1+x}. \end{aligned} \tag{4.8}$$

Table 4 − 1

Analysis of Error

x	Numerical	Polynomial approx	Error
0	0	0	0
0.3	0.0433709	1.22474	−1.18137
0.5	0.115047	1.33141	−1.21636
0.8	0.274385	1.61573	−1.34135
1	0.408703	2.12279	−1.71409
1.5	0.816017	2.34076	−1.52474
2	1.29187	2.11967	−0.8278

4.4 HAM Solution Using Exponential Function

To obtain the solution of the Eq (4.1) by the exponential approximation we choose our initial guess in the form of exponential which satisfies boundary condition (4.2). By the rule of expression, we choose

$$f_0 = x - 1 + e^{-x}, \tag{4.9}$$

and linear operator will be

$$L[f(x;q)] = \frac{\partial^3 f(x;q)}{\partial x^3} + \frac{\partial^2 f(x;q)}{\partial x^2}. \tag{4.10}$$

The mth-order deformation equation become

$$\left(\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right)[f_m(x) - \varkappa_m f_{m-1}(x)] = \hbar H(x) \left(\begin{array}{c} (1 + \frac{k}{2}) f'''_{m-1}(x) + \sum_{r=0}^{m-1} f_r(x) f''_{m-r-1}(x) - \\ \sum_{r=0}^{m-1} f'_r(x) f'_{m-r-1}(x) + 1(1 - \varkappa_m) \end{array} \right).$$

Choose $H(x) = e^{-x}$ and $k = 1$ so we get first few approximation

$$\begin{aligned} f_0(x) &= x - 1 + e^{-x}, \\ f_1(x) &= -1 + e^{-x} + \frac{3\hbar}{8} - \frac{3}{8}e^{-2x}\hbar + x - \frac{\hbar x}{2} - \frac{1}{4}e^{-2x}\hbar x. \end{aligned} \tag{4.11}$$

Table 4 – 2

Analysis of Error

x	Numerical	Exponential approx	Error
0	0	0	0
0.3	0.0433709	0.0432467	0.0001242
0.5	0.115047	0.113249	0.001798
0.8	0.274385	0.258496	0.015889
1	0.408703	0.295639	0.113064
1.5	0.816017	1.07845	−0.262433
2	1.29187	1.69799	−0.40612

4.5 HAM Solution Using Rational Function

Now we use the Homotopy-Pade technique to produce a rational approximation on the terms which we obtain by the polynomial approximation. Eq (1.19) becomes

$$f_{m,m}(x) = \left. \frac{\sum_{k=0}^{m} A_{m,k}(x) q^k}{1 + \sum_{k=1}^{m} B_{m,k}(x) q^k} \right|_{q=1} . \tag{4.12}$$

The coefficient of $A_{m,k}$ and $B_{m,k}$ can be obtained by solving the 2m+1 linear equation given by (1-23),(1-24) and (1-25) using the term calculated in previous section Eq (4.8). The first two rational approximation with choosing k=1, calculated are

$$\begin{aligned} f_{0,0} &= x - \frac{x}{1+x}, \\ f_{1,1} &= \frac{0.33333(3+\hbar)x^2}{1 + \frac{(3+4\hbar)x}{3+\hbar} + \frac{0.5(15\hbar+23\hbar^2)x^2}{(3+\hbar)^2}}. \end{aligned} \tag{4.13}$$

and further terms are calulated easily.

Table 4 − 3

Analysis of Error

x	Numerical	Rational approx	Error
0	0	0	0
0.3	0.0433709	0.0432455	0.0001254
0.5	0.115047	0.115859	0.000812
0.8	0.274385	0.253574	0.020811
1	0.408703	0.303026	0.105677
1.5	0.816017	1.642708	−0.826691
2	1.29187	2.15337	−0.86150

4.6 HAM Solution Using Optimal-Homotopy Analysis Method

To find the solution from Optimal homotopy analysis method we choose exponential type base function with initial guess

$$f_0 = x - 1 + e^{-x}, \tag{4.14}$$

with linear operator

$$L[f(x;q)] = \frac{\partial^3 f(x;q)}{\partial x^3} + \frac{\partial^2 f(x;q)}{\partial x^2}. \tag{4.15}$$

By the homotopy series solution

$$f(x) = f_0(x) + \sum_{k=1}^{\infty} f_k(x). \tag{4.16}$$

The mth-order deformation equation becomes

$$L[f_m(x) - \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) f_k(x)] = c_0 \sum_{k=1}^{m-1} \mu_{m-k}(c_1) R_k(x), \tag{4.17}$$

where the deformation function σ_m and μ_m are defined above.and

$$R_k(x) = (1 + \frac{k}{2}) f'''_{m-1}(x) + \sum_{r=0}^{m-1} f_r(x) f''_{m-r-1}(x) - \sum_{r=0}^{m-1} f'_r(x) f'_{m-r-1}(x) + 1(1 - \varkappa_m),$$

so we obtain the first few terms of approximation

$$\begin{aligned} f_0(x) &= x - 1 + e^{-x}, \\ f_1(x) &= x - 1 + e^{-x} + (1 - c_2)(-3c_0 + 3c_0 e^{-x} + c_0 k - c_0 k e^{-x} + c_0 x \\ &\quad + 2c_0 x e^{-x} - \frac{k x c_0}{2} - \frac{1}{2} c_0 e^{-x} k x + \frac{1}{2} c_0 e^{-x} x^2). \end{aligned} \tag{4.18}$$

The values of c_0, c_1, c_2 are calculated after fifth term that are

$$c_0 = -1, c_1 = 0.409607, c_2 = 0.443246$$

so we get first few approximated terms

$$\begin{aligned} f_0(x) &= x - 1 + e^{-x}, \\ f_1(x) &= x - 1 + e^{-x} - \frac{1}{4}e^{-x}x^2h[0] + \frac{1}{2}e^{-x}x^3h[0] \quad \text{and} \quad \text{h[0]=0.080483.} \end{aligned} \tag{4.22}$$

Further terms are easily calculated.

Table 4 – 5

Analysis of Error

x	**Numerical**	**One – Step approx**	**Error**
0	0	0	0
0.3	0.0433709	0.0414325	0.0019384
0.5	0.115047	0.11311	0.0019389
0.8	0.274385	0.27216	0.002225
1	0.408703	0.40642	0.002283
1.5	0.816017	0.81351	0.002507
2	1.29187	1.28914	0.00273

Chapter 5

Homotopy Base Solutions of Coupled Equation with Infinite Domain

This chapter is formulated to discuss the homotopic solution by coupled ordinary differential equations. The problem in previous chapter is repharased with k=1 and equation of micro-rotation appear. The equations are coupled and boundary conditions are described at $x = 0$ and $x = \infty$. The problem is solved using varity of homotopy techniques. The solutions are compared.

5.1 Mathematical Formulation

The problem of stagnation point flow of micropolar fluid was discussed in previous chapter now the problem is remodeled for orthognal stagnation point and considering with n=1 [27].

$$
\begin{aligned}
(1+k)\frac{d^3f}{dx^3} + f(x)\frac{d^2f}{dx^2} - \left(\frac{df}{dx}\right)^2 + k\frac{dg}{dx} + 1 &= 0, \qquad (5.1)\\
\left(1+\frac{k}{2}\right)\frac{d^2g}{dx^2} + f(x)\frac{dg}{dx} - g(x)\frac{df}{dx} - k\left(2g + \frac{d^2f}{dx^2}\right) &= 0,
\end{aligned}
$$

with condition

$$
\begin{aligned}
f(0) &= 0 = f^{'}(0),\ f^{'}(\infty) = 1, \\
g(0) &= 0, g(\infty) = 0.
\end{aligned} \tag{5.2}
$$

5.2 Homotopy

To solve this problem with Homotopy Analysis method first we introduced the Non-linear operator

$$
\begin{aligned}
N[f(x;q)] &= (1+k)\frac{d^3 f}{dx^3} + f(x)\frac{d^2 f}{dx^2} - \left(\frac{df}{dx}\right)^2 + k\frac{dg}{dx} + 1, \\
N[g(x;q)] &= \left(1+\frac{k}{2}\right)\frac{d^2 g}{dx^2} + f(x)\frac{dg}{dx} - g(x)\frac{df}{dx} - k\left(2g + \frac{d^2 f}{dx^2}\right).
\end{aligned} \tag{5.3}
$$

The mth order deformation equations becomes

$$
\begin{aligned}
L[f_m(x) - \varkappa_m f_{m-1}(x)] &= \hbar H(x)\left[\begin{array}{c} (1+k)f_{m-1}^{'''}(x) + \sum_{r=0}^{m-1} f_r(x) f_{m-r-1}^{''}(x) - \\ \sum_{r=0}^{m-1} f_r^{'}(x) f_{m-r-1}^{'}(x) + kg_{m-1}^{'} + 1(1-\varkappa_m) \end{array}\right]. \\
L[g_m(x) - \varkappa_m g_{m-1}(x)] &= \hbar H(x)\left[\begin{array}{c} (1+\frac{k}{2})g_{m-1}^{''}(x) + \sum_{r=0}^{m-1} f_r(x) g_{m-r-1}^{'}(x) - \\ \sum_{r=0}^{m-1} g_r(x) f_{m-r-1}^{'}(x) + kg_{m-1}^{'} - 2kg_{m-1}(x) - kf_{m-1}^{''}(x) \end{array}\right].
\end{aligned}
$$

we require our initial guess to satisfy

$$
\begin{aligned}
f_0(0) &= 0 = f_0^{'}(0), f_0^{'}(\infty) = 1, \\
g_0(0) &= 0, g_0(\infty) = 0.
\end{aligned} \tag{5.4}
$$

and subsequent terms to satisfy are

$$
\begin{aligned}
f_m(0) &= 0 = f_m^{'}(0), f_m^{'}(\infty) = 1\ ,\ \text{m>0} \\
g_m(0) &= 0, g_m(\infty) = 0\ \ ,\ \text{m>0}
\end{aligned} \tag{5.5}
$$

5.3 HAM Solution Using Polynomial Function

To find the solution of Eq (5.1) by polynomial we choose our initial approximation which satisfy our boundary condition (5.2)

$$\begin{aligned} f_0 &= x - \frac{x}{1+x}, \\ g_0 &= \frac{1}{x+1} + \frac{1}{x-1}, \end{aligned} \tag{5.6}$$

and choose linear operator in the form of

$$\begin{aligned} L[f(x;q)] &= \frac{\partial^3 f(x;q)}{\partial x^3}, \\ L[g(x;q)] &= \frac{\partial^2 g(x;q)}{\partial x^3}. \end{aligned} \tag{5.7}$$

The mth-order deformation equations becomes

$$\begin{aligned} f_m(x) &= \varkappa_m f_{m-1}(x) + \hbar \int\int\int H(x) \left(\begin{array}{c} (1+k) f_{m-1}^{'''}(x) + \sum_{r=0}^{m-1} f_r(x) \\ f_{m-r-1}''(x) - \sum_{r=0}^{m-1} f_r'(x) \\ f_{m-r-1}'(x) + k g_{m-1}' + 1(1-\varkappa_m) \end{array} \right) dx dx dx. \quad (5.8) \\ g_m(x) &= \varkappa_m g_{m-1}(x) + \hbar \int\int H(x) \left(\begin{array}{c} (1+\frac{k}{2}) g_{m-1}^{''}(x) + \sum_{r=0}^{m-1} f_r(x) \\ g_{m-r-1}'(x) - \sum_{r=0}^{m-1} g_r(x) f_{m-r-1}'(x) + k g_{m-1}' \\ -2k g_{m-1}(x) - k f_{m-1}''(x) \end{array} \right) dx dx. \end{aligned}$$

Choose $H(x) = 1$ in both equations.so we get first few approximation

$$
\begin{aligned}
f_0(x) &= x - \frac{x}{1+x}, \\
g_0(x) &= \frac{1}{x+1} + \frac{1}{x-1}. \\
f_1(x) &= x - \frac{x}{1+x} + \frac{6\hbar x}{1+x} + \frac{47x^2\hbar}{6(1+x)} - \frac{6\hbar\ln(x+1)}{1+x} - \frac{10\hbar x\ln(x+1)}{1+x} - \frac{4\hbar x^2\ln(x+1)}{1+x}, \\
g_1(x) &= \frac{1}{x+1} + \frac{1}{x-1} - \frac{6\hbar x}{-1+x^2} + \frac{\hbar x^2}{-1+x^2} + \frac{8\hbar x^3}{-1+x^2} - \frac{4\hbar\ln(1-x)}{-1+x^2} - \\
&\quad \frac{7\hbar x\ln(1-x)}{2(-1+x^2)} + \frac{4x^2\hbar\ln(1-x)}{(-1+x^2)} - \frac{7\hbar x^3\ln(1-x)}{2(-1+x^2)} + \frac{6\hbar\ln(1+x)}{-1+x^2} + \frac{9\hbar x\ln(1+x)}{2(-1+x^2)} - \\
&\quad \frac{6x^2\hbar\ln(1+x)}{2(-1+x^2)} - \frac{9\hbar x^3\ln(1+x)}{2(-1+x^2)}.
\end{aligned} \tag{5.9}
$$

and further calculations can be easily calculated.

Table 5 − 1 for f(x)

Analysis of Error

x	Numerical	Polynomial Approxi	Error
0	0	0	0
0.3	0.0444	0.042795	0.001605
0.7	0.1676	0.16532	0.002279
1	0.3542	0.35150	0.002697
1.7	0.8595	0.83161	0.02789
2	1.1541	1.12361	0.03049
4	3.1066	2.01453	1.09207

Table 5 − 1 for g(x)

Analysis of Error

x	Numerical	Polynomial Approxi	Error
0	0	0	0
0.3	0.0679	0.06561	0.00229
0.7	0.0937	0.08912	0.00458
1.3	0.0803	0.07524	0.00506
1.7	0.0619	0.05674	0.00516
2	0.0438	0.0325	0.01131
4	1.0313	0.01345	1.01785

5.4 HAM Solution Using Exponential Function

To obtain the solution of the Eq (5.1) by the exponential approximation we choose our initial guess which satisfy our boundary conditions (5.2)

$$\begin{aligned} f_0 &= -1 + x + e^{-x}, \\ g_0 &= e^{-x} - e^{-2x}, \end{aligned} \tag{5.10}$$

and linear operator

$$\begin{aligned} L[f(x;q)] &= \frac{\partial^3 f(x;q)}{\partial x^3} + \frac{\partial^2 f(x;q)}{\partial x^2}, \\ L[g(x;q)] &= \frac{\partial^2 g(x;q)}{\partial x^2} + g(x;q). \end{aligned} \tag{5.11}$$

Then the mth-order deformation equation becomes

$$\left(\frac{\partial^3}{\partial x^3}+\frac{\partial^2}{\partial x^2}\right)[f_m(x)-\varkappa_m f_{m-1}(x)] \;=\; \hbar H(x)\begin{pmatrix}(1+k)f'''_{m-1}(x)+\sum_{r=0}^{m-1} f_r(x)\\ f''_{m-r-1}(x)-\sum_{r=0}^{m-1} f'_r(x)\\ f'_{m-r-1}(x)+kg'_{m-1}+1(1-\varkappa_m)\end{pmatrix} \quad (5.12)$$

$$\left(\frac{\partial^2}{\partial x^2}+1\right)[g_m(x)-\varkappa_m g_{m-1}(x)] \;=\; \hbar H(x)\begin{pmatrix}(1+\frac{k}{2})g''_{m-1}(x)+\sum_{r=0}^{m-1} f_r(x)\\ g'_{m-r-1}(x)-\sum_{r=0}^{m-1} g_r(x)f'_{m-r-1}(x)\\ +kg'_{m-1}-2kg_{m-1}(x)-kf''_{m-1}(x)\end{pmatrix}.$$

and we choose

$$H(x)=e^{-x},$$

so we get first few terms

$$\begin{aligned}
f_0(x) &= -1+x+e^{-x},\\
g_0(x) &= e^{-x}-e^{-2x}.\\
f_1(x) &= -1+e^{-x}+\frac{\hbar}{9}-\hbar\frac{1}{9}e^{-3x}-\frac{\hbar x}{12}-\frac{1}{4}\hbar xe^{-2x}+x, \qquad (5.13)\\
g_1(x) &= e^{-x}-e^{-2x}+\frac{1}{15}e^{-4x}\hbar-\frac{7}{16}\hbar e^{-3x}-\frac{1}{9}\hbar e^{-2x}+\frac{347}{720}\hbar e^{-x}-\frac{5}{6}\hbar ke^{-2x}+\frac{5}{6}\hbar ke^{-x}\\
&\quad +\frac{1}{4}\hbar xe^{-3x}-\frac{1}{3}\hbar xe^{-2x}.
\end{aligned}$$

Table 5 − 2 for f(x)

Analysis of Error

x	Numerical	Exponential approx	Error
0	0	0	0
0.3	0.0444	0.044157	0.000243
0.7	0.1676	0.167288	0.000312
1	0.3542	0.353812	0.000388
1.7	0.8595	0.85556	0.00394
2	1.1541	1.1496	0.00449
4	1.1066	1.10113	0.00547

Table 5 − 2 for g(x)

Analysis of Error

x	Numerical	Exponential Appro	Error
0	0	0	0
0.3	0.0679	0.062867	0.005033
0.7	0.0937	0.086758	0.006942
1.3	0.0803	0.072521	0.007779
1.7	0.0619	0.05282	0.00908
2	0.0438	0.03474	0.00906
4	0.0523	0.04316	0.00914

5.5 HAM Solution Using Rational Function

Now we use the Homotopy-Pade technique to produce a rational approximation on the terms which we obtain by the exponential approximation.Eq (1.19) becomes

$$\begin{aligned} f_{m,m}(x) &= \left.\frac{\sum_{k=0}^{m} A_{m,k}(x)q^k}{1+\sum_{k=1}^{m} B_{m,k}(x)q^k}\right|_{q=1}, \\ g_{m,m}(x) &= \left.\frac{\sum_{k=0}^{m} A_{m,k}(x)q^k}{1+\sum_{k=1}^{m} B_{m,k}(x)q^k}\right|_{q=1}. \end{aligned} \tag{5.14}$$

The coefficient of $A_{m,k}$ and $B_{m,k}$ can be obtained by solving the 2m+1 linear Eqs (1-23),(1-24) and (1-25) using the term calculated with Eq (5.13). The first two rational approximation calculated are

$$
\begin{aligned}
f_{0,0}(x) &= -1+x+e^{-x}, \\
g_{0,0}(x) &= e^{-x}-e^{-2x}, \\
f_{1,1}(x) &= \frac{0.458333x^2}{1+0.363636x-0.0495868x^2}, \\
g_{1,1}(x) &= \frac{-1.19444x+12.0919x^2}{1-9.07692x-10.6154x^2}.
\end{aligned} \tag{5.15}
$$

Table 5 − 3 for f(x)

Analysis of Error

x	Numerical	Rational Approx	Error
0	0	0	0
0.3	0.0444	0.0425884	0.001812
0.7	0.1676	0.165538	0.002062
1	0.3542	0.350969	0.003231
1.7	0.8595	0.85524	0.00426
2	1.1541	1.14923	0.00487
4	1.1066	1.06723	0.03937

Table 5 – 3 for g(x)

Analysis of Error

x	Numerical	Rational Approx	Error
0	0	0	0
0.3	0.0679	0.0645948	0.0033052
0.7	0.0937	0.08936	0.004336
1.3	0.0803	0.07516	0.00514
1.7	0.0619	0.05641	0.00549
2	0.0438	0.02541	0.01839
4	0.0513	0.01245	0.03885

5.6 HAM Solution Using Optimal Homotopy Analysis Method

To find the solution from Optimal homotopy analysis method we choose exponential type base function with initial guess

$$f_0 = x - 1 + e^{-x}, \quad g_0 = e^{-x} - e^{-2x}.$$

with linear operator

$$\begin{aligned} L[f(x;q)] &= \frac{\partial^3 f(x;q)}{\partial x^3} + \frac{\partial^2 f(x;q)}{\partial x^2}, \\ L[g(x;q)] &= \frac{\partial^2 g(x;q)}{\partial x^2} + g(x;q). \end{aligned} \tag{5.16}$$

By the homotopy series solution

$$\begin{aligned} f(x) &= f_0(x) + \sum_{k=1}^{\infty} f_k(x), \\ g(x) &= g_0(x) + \sum_{k=1}^{\infty} g_k(x). \end{aligned}$$

The mth-order deformation equation becomes

$$\begin{aligned} L[f_m(x) - \sum_{k=1}^{m-1} \sigma_{m-k}(c_2) f_k(x)] &= c_0 \sum_{k=1}^{m-1} \mu_{m-k}(c_1) R_k(x), \\ L[g_m(x) - \sum_{k=1}^{m-1} \upsilon_{m-k}(c_4) g_k(x)] &= c_0 \sum_{k=1}^{m-1} \eta_{m-k}(c_3) \xi_k(x). \end{aligned}$$

where the deformation function σ_m,η_m, $\upsilon_m$and μ_m are defined. and

$$\begin{aligned} \mu_1(c_1) &= 1 - c_1,\ \mu_m(c_1) = (1-c_1)c_1^{m-1} \quad m > 1, \\ \sigma_1(c_2) &= 1 - c_2 \ \ \sigma_m(c_2) = (1-c_2)c_2^{m-1} m > 1, \\ \eta_1(c_3) &= 1 - c_3,\ \mu_m(c_3) = (1-c_3)c_3^{m-1} \quad m > 1, \\ \upsilon_1(c_4) &= 1 - c_4,\ \upsilon_m(c_4) = (1-c_4)c_4^{m-1} \quad m > 1. \end{aligned}$$

and

$$\begin{aligned} R_k(x) &= (1+k)f_{m-1}^{'''}(x) + \sum_{r=0}^{m-1} f_r(x) f_{m-r-1}''(x) - \sum_{r=0}^{m-1} f_r'(x) f_{m-r-1}'(x) + k g_{m-1}' + 1(1-\varkappa_m), \\ \xi_k(x) &= (1+\frac{k}{2}) g_{m-1}^{''}(x) + \sum_{r=0}^{m-1} f_r(x) g_{m-r-1}'(x) - \sum_{r=0}^{m-1} g_r(x) f_{m-r-1}'(x) + \\ & \quad k g_{m-1}' - 2k g_{m-1}(x) - k f_{m-1}''(x). \end{aligned}$$

so we obtain the first few terms of approximation

$$\begin{aligned} f_0(x) &= x - 1 + e^{-x}, \\ g_0(x) &= e^{-x} - e^{-2x}, \\ f_1(x) &= x - 1 + e^{-x} + (1-c_2)\begin{pmatrix} -3c_0 + 3c_0 e^{-x} + \frac{c_0 k}{2} - \frac{1}{2}c_0 k e^{-2x} + \\ c_0 x + 2c_0 x e^{-x} - c_0 x k + \frac{1}{2}c_0 e^{-x} x^2 \end{pmatrix}, \\ g_1(x) &= e^{-x} - e^{-2x} + (1-c_4)\begin{pmatrix} \frac{1}{8}c_0 e^{-3x} - \frac{7}{9}c_0 e^{-2x} + \frac{47 c_0 e^{-x}}{72} + \frac{2}{3}c_0 x e^{-2x} - \\ \frac{1}{4}c_0 x e^{-x} + \frac{5}{4}c_0 e^{-x} k x + \frac{1}{4}c_0 e^{-x} x^2 \end{pmatrix}. \end{aligned} \tag{5.17}$$

The values for c_0, c_1, c_2, c_3, c_4 are obtained after third iteration which are

$$c_0 = -0.229192, c_1 = -1, c_2 = -0.736885, c_3 = -1, c_4 = -0.805561.$$

Table 5 – 4 for f(x)

Analysis of Error

x	Numerical	Optimal Approx	Error
0	0	0	0
0.3	0.0444	0.04387	0.00053
0.7	0.1676	0.166974	0.00062
1	0.3542	0.353548	0.000652
1.7	0.8595	0.858752	0.000748
2	1.1541	1.15321	0.0008923
4	1.1066	1.105701	0.0008999

Table 5 – 4 for g(x)

Analysis of Error

x	Numerical	OptimalApprox	Error
0	0	0	0
0.3	0.0679	0.067721	0.000179
0.7	0.0937	0.093481	0.000219
1.3	0.0803	0.080061	0.000239
1.7	0.0619	0.06164	0.000252
2	0.0438	0.043506	0.000294
4	0.0513	0.05101	0.000298

5.7 HAM Solution Using One-Step Optimal Homotopy Analysis Method

To obtain the solution by the One-step homotopy analysis method of Eq (5.1) we define the non-linear operator

$$\begin{aligned} N[f(x;q)] &= (1+k)\frac{d^3 f}{dx^3} + f(x)\frac{d^2 f}{dx^2} - \left(\frac{df}{dx}\right)^2 + k\frac{dg}{dx} + 1, \\ N[g(x;q)] &= \left(1+\frac{k}{2}\right)\frac{d^2 g}{dx^2} + f(x)\frac{dg}{dx} - g(x)\frac{df}{dx} - k\left(2g + \frac{d^2 f}{dx^2}\right). \end{aligned} \tag{5.18}$$

with initial guess

$$\begin{aligned} f_0 &= x - 1 + e^{-x}, \\ g_0 &= 1 - e^{-x}, \end{aligned} \tag{5.19}$$

and linear operator

$$\begin{aligned} L[f(x;q)] &= \frac{\partial^3 f(x;q)}{\partial x^3} + \frac{\partial^2 f(x;q)}{\partial x^2}, \\ L[g(x;q)] &= \frac{\partial^2 g(x;q)}{\partial x^2} + g(x;q). \end{aligned}$$

The mth-order deformation equation

$$\begin{aligned} L[f_m(x) - \varkappa_m f_{m-1}(x)] &= \sum_{k=1}^{m} h_k R_{m-k}(k), \\ L[g_m(x) - \varkappa_m g_{m-1}(x)] &= \sum_{k=1}^{m} h_k \xi_{m-k}(k). \end{aligned}$$

where

$$
\begin{aligned}
R_k(x) &= (1+k)f'''_{m-1}(x) + \sum_{r=0}^{m-1} f_r(x)f''_{m-r-1}(x) - \\
&\quad \sum_{r=0}^{m-1} f'_r(x)f'_{m-r-1}(x) + kg'_{m-1} + 1(1-\varkappa_m), \\
\xi_k(x) &= (1+\frac{k}{2})g''_{m-1}(x) + \sum_{r=0}^{m-1} f_r(x)g'_{m-r-1}(x) - \\
&\quad \sum_{r=0}^{m-1} g_r(x)f'_{m-r-1}(x) + kg'_{m-1} - 2kg_{m-1}(x) - kf''_{m-1}(x).
\end{aligned}
$$

so we get first few approximated terms

$$
\begin{aligned}
f_0(x) &= x - 1 + e^{-x}, \\
g_0(x) &= e^{-x} - e^{-2x}, \\
f_1(x) &= x - 1 + e^{-x} - \frac{1}{2}x^2h[0] + e^{-2x}x^2h[0] - e^{-x}x^2h[0] + \frac{1}{2}e^{-x}x^3h[0], \quad h[0] = -0.185689. \\
g_1(x) &= e^{-x} - e^{-2x} - e^{-4x}h[0] - 2e^{-3x}h[0] + \qquad (5.20) \\
&\quad \frac{3}{2}e^{-2x}h[0] + \frac{3}{2}e^{-x}h[0] - 2xe^{-3x}h[0] + 3xe^{-2x}h[0] - xe^{-x}h[0], \\
h[0] &= -0.185689.
\end{aligned}
$$

Table 5 – 5 for f(x)

Analysis of Error

x	Numerical	One – Step Approx	Error
0	0	0	0
0.3	0.0444	0.044185	0.000215
0.7	0.1676	0.16738	0.000218
1	0.3542	0.35392	0.00028
1.7	0.8595	0.85916	0.00034
2	1.1541	1.15373	0.00037
4	1.1066	1.1059	0.0007

Table 5 – 5 for g(x)

Analysis of Error

x	Numerical	One – Step Approx	Error
0	0	0	0
0.3	0.0679	0.0679	0.000216
0.7	0.0937	0.093458	0.000242
1.3	0.0803	0.07991	0.00039
1.7	0.0619	0.06149	0.00041
2	0.0438	0.043382	0.000418
4	0.1243	0.12386	0.00044

Chapter 6

Results and Conclusion

This thesis is designed to discuss and analysis some homtopy base analytical solutions of differential equations. The thesis consists of four numerical examples, all of which are solved using five homtopy base techniques, namely Homtopy Analysis Method with polynimal type base functions, Homtopy Analysis Method with Pade approximation, Homtopy Analysis Method with exponential type base functions, Optimal Homtopy Analysis Method, One step Optimal Homtopy Analysis Method. The problem chosen are non-linear in nature and have diverse boundaries. First order non-linear, non-homogenous intial value problem due to Newton's law for freely falling body was chosen as first problem. From intial value problem next aim is boundary vale problems. Second problem was Non-linear homogenous problem with bounded domain. This problem was modeled for Power law fluid on streching sheet and uses similarty transform to reduce the problem to finite domain and ODEs. Third and fourth problem is for infinite domian. In third problem a non-linear,non- homogenous differential equation was solved, whereas, two coupled ODEs are solved in chapter five. The solutions obtained are compared with exact or numerical solutions for different values of domain. It is observed that

1. Homtopy Analysis Method with polynimal type base functions gives good and quick approximation in bounded domain and for infinte domain the results close to the boundaries as x increases the solution starts to diverge, also choice of intial guess is very difficult if domain is between zero and infinite. The solution might tends to convert to logrithmic form in such cases.

2. Homtopy Analysis Method with exponetial type base functions gives good approximation in infinite domain and for bounded domain it is hard to find expontial type guess also solutions are not as accurate.

3. Homtopy Analysis Method with Pade approximation gives excellent approximation in bounded domain, also, Results are better for infinte domain then polynimial function infact results gives good approximation for larger values of x, but these results are two large and take a lot of computation time.

4. Optimal Homtopy Analysis Method gives excellent solutions. It helps to attain convergent solutions at smaller number of itrations which reduces the computational time. But, finding the approximate values of convergence control parameter is a difficult job. Here, we use hybrid genetic algorithms and Nelder mead to find the value of these parameter.

5. One step Optimal Homtopy Analysis Method also provide excellent solutions and helps to attain convergent solutions at smaller number of itrations. But, it each itration it need to find approximate values of convergence control parameter which increase computation, again we hybrid genetic algorithms and Nelder mead to find the value of these parameter.

Every techniques we use have there advantages we recomend Polynomial type base functions either HAM is applied or any optimal scheme is used if domain is bounded and expontiential type base function for infinte domain. we could use Pade of any optimal technique to reduce the error by minimizing Residual.

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