# Shock Wave Phenomena in Supersonic Flow

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**A Dissertation**
**Submitted in the Partial Fulfillment of the Requirements for the Degree of**
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**In**
**MATHEMATICS**

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# Certificate

## Shock Wave Phenomena in Supersonic Flow

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## Iqrar Raza

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN MATHEMATICS

**We accept this dissertation as conforming to the required standard.**

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**2016**

# DECLARTION

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*Dedicated*

*To*

*My parents,*

*My*

*Loving brothers and sisters,*

*My*

*Nephew and niece*

Acknowledgments

I begin in the name of **Allah** Almighty, the creator of the whole universe and the source of knowledge. I thank **Allah** Almighty for giving me the strength and energy to complete this dissertation. I also thank the **Holy Prophet Muhammad (PBUH)**, who is forever a torch of guidance for humanity. Without his goodwill nothing can be achieved.

I also wish to express my gratitude to my research supervisor **Dr. Ahmer Mehmood**, for initiating the present topic and for his continuing interest and guidance throughout the course of this work.

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**Iqrar Raza**

# Preface

Abundant of flows exist in nature for which the fluid properties change continuously. Flows are also possible for which the behaviour of fluid properties vary discontinuously. Such discontinuities occur in explosions, detonations, etc. which is actually caused by shock wave. Many scientists and engineers had faced such discontinuous phenomena and used different names in order to elaborate it. Euler (1759) talked about the "size of disturbance of sound wave meaning its amplitude". Poisson (1808) mentioned about an instance sound wave as the case "where the molecular velocities can no longer be regarded as very small". Stokes (1848) used term "surface of discontinuity" and Reimann (1859) used the term "shock compression" and "compression wave" to describe the jump-like steepening of the wave front. Toepler (1864) was the first to use the term shock wave, he originated shock wave from a spark discharge and for the first time visualized it using stroboscopic method. Rankine (1870) used the terms "abrupt disturbance and wave of finite longitudinal disturbance" and Hugoniot (1885) used the term "discontinuity". Ernst Mach and Coworkers (1875-1885) used the terms "shock wave, Riemann wave, bang wave and explosion wave". Earnst and Ludwig also designated shock wave as "a sound wave of large amplitude". Von Oettingen and Von Gernet (1888) used the term "detonation wave". Vieille and Hadamard (1898), Duhem (1901) and Jouguet (1904) used the term shock wave. Lord Rayleigh (1910) used the term "aerial waves of finite amplitude" while discussing the characteristics of shock wave in the air.

The visualization of shock wave over different objects was made possible through different visualization techniques. Schliren and Shadowgraph techniques were applied by different researchers in order to visualize the shock wave. August Toepler, the inventor of schliren technique, used it to visualize the shock wave in 1864. Ernst Mach [6] had also used schliren technique and proved the existence of shock wave through photographs. Shadowgraph technique [9] was also used by many researchers for the visualization of shock wave.

After the successful and history making experiment of Bell-X1 in 1947, the researchers has started to calculate the shock wave and it's properties over the different parts of the jet. Belotserkovskii and O. M [12] worked on the flow past a circular cylinder with detached shock wave in 1958. Van Dyke and M. D [13] reviewed and extended the supersonic blunt body problem in 1958. Garabedian, P. R and Lieberstein, H. M. [14] worked on the numerical calculation of the bow shock waves in hypersonic flow on the same year. Gino Moretti and

Michael Abbett [15] worked on the time dependent computational method for blunt body problem in 1966. The shock wave phenomena may also occur in supersonic convergent-divergent (CD) nozzle [10] which is infact the subject of the current study. The dissertation is arranged as follow:

Chapter 1 includes the basic definitions of fluid flow, thermodynamic properties and governing equations of fluid flow. A brief history of shock wave, its characteristics and the visualization technique is elaborated. The numerical scheme for the simulation of the problem is also described in this chapter.

In chapter 2, governing equations of the normal shock wave are derived. The physical behaviour of the normal shock is described by using Rankine-Hugoniot relations. The physical behaviour of the flow through a CD nozzle is described and it is also examined that the shock wave changes its location with the reduction of exit to inlet pressure ratio. Under expansion and over expansion processes are also illustrated in this chapter.

Quasi one dimensional flow through a CD nozzle is discussed in chapter 3. The location of the shock wave have been captured successfully with great accuracy which has been shown by the appearance of abrupt changes to the flow properties.

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# Chapter 1

## 1.1 Introduction

Fluid flows are present everywhere in nature and are widely experienced by those people who observe the nature with open eyes. As 75% of earth is covered with water and 100% with air, the scope of the fluid is vast and has numerous application in human life. Fluid mechanics, as with many other fields of scientific studies, is rooted in the history of humanity. Throughout its history, fluid mechanics is a field that has been constantly advancing. As with other engineering fields, it has now reached the point of scientific maturity with most of the fundamentals clearly understood. As such, it has become a vital component for many engineering curricula. It is therefore useful exercise to examine the historical development for this field.

Ancient civilization had enough knowledge to solve certain problems of their time. Archmedes (285-212 B.C) formulated the buoyancy law and applied it to floating and submerged bodies. Leonardo da Vinci (1459-1519) derived the equation of conservation of mass in one dimensional steady flow. He experimented with waves, hydraulic jumps, eddy formations, etc. A French man Edme Mariotte (1620-1684) built the wind tunnel and tested different models in it. Sir Isaac Newton (1642-1727) postulated his laws of motion and laws of viscosity of linear fluids, now called Newtonian fluids after his name. Leonhard Euler (1707-1783) developed both differential and integral forms of the equations of motion known as Bernoulli equation now a days. William Froude (1810-1879) and his son developed laws of model testing. Lord Rayleigh (1822-1919) proposed the dimensional analysis of physical problems. Osborne Reynolds (1842-1912) worked on the classical pipe experiments and showed the importance of dimensionless "Reynolds number". Navier (1785-1836) and Stokes (1875-1953) independently added Newtonian viscous term to equation of motion known as Navier-Stokes equation. Ludwig Prandtl (1875-1953) gave the idea of boundary layer which has become the most important tool in modern viscous flow analysis.

Beside the study of low speed flows, high speed flows such as flow over the bodies and inside the ducts were also investigated by the researchers and engineers. The importance of such high speed flows over the bodies was enhanced when first successful airplane flight conducted by Wright's brothers in 1903. This was really a great achievement in the history of engineering science. After this experiment engineers had started to investigate the effects by which the speed of plane might increase. The study of air over the wings and other parts of planes was conducted extensively in order to reduce the resistance of air in turn to increase the speed of the plane. The speed of airplane was increased successfully but not as fast as the speed of sound. Many experiments were conducted to cross the speed of sound but whenever the planes approached it, the drag became more sever. At that time, the piolets and some aeronautical engineers felt that airplanes could never move faster than the speed of sound. But aerodynamicists believed that planes could move faster than the speed of sound. Adolf Busemann [1] presented his work on the swept wings and discussed how swept wings would have less drag at high speeds than straight wings. Eastman Jacobs [2] proposed wind-tunnel test results for compressibility effects on airfoils at high subsonic speed. Jakob Ackeret presented work on the design of supersonic wind-tunnels. Ludwig Prandtl also worked on supersonic aerodynamics. The myth (sound barrier) was broken in 1947 when Captain Chuck Yeager flew Bell-X1; approached the speed of sound and thus broke so-called "sound barrier". After breaking the sound barrier, he entered into the new era of flow, the "supersonic flow". The history was made with this experiment but with the generation of nonlinear high altitude wave over the jet named as "Shock wave". The rapid changes across this wave had been experienced that gave a new direction to researchers to investigate about its mechanism. The location of shock wave had also become a new challenge for engineers and scientists.

The existence of shock wave had already been visualized over the solid objects and inside the ducts by number of researchers but not over the jets. Knowledge of shock wave is not unique to the twentieth century. Its existence was recognized in the early nineteenth century. The German Mathematician G. F. Bernhard Riemann first attempted to calculate shock waves properties in 1858. But he neglected an essential physical feature and hence obtained incorrect results. Twelve years later, William John Rankine [3] derived proper equations for the flow across a normal shock wave. The French ballistician Pierre Hugoniot [4] rediscovered the normal shock wave equations in 1887. The German aerodynamicist Ludwig Prandtl and his student Theodor Meyer [5] discovered oblique shock waves in 1908. The nineteenth century was also the time of experimental work on supersonic flows. The most important event was the proof of the existence of shock waves. Ernst Mach [6] proved the existence of shock waves in 1887. He took the photographs of shock wave on a moving body at supersonic speeds.

The existence of shock wave had also been investigated mathematically since long. Number of techniques have been developed to investigate it theoretically. In this dissertation, we intend to study the existence of shock wave as well as it effects in the supersonic flow of air through CD nozzle. The second order finite difference scheme [7] has been utilized in capturing and studying the normal shock wave.

## 1.2 Basic Definitions

### 1.2.1 Fluid

Fluid is a substance that deforms continuously under the action of shear stress, no matter however small. It can also be defined as a substance which is capable to flow.

### 1.2.2 Fluid Mechanics

The branch of applied mechanics in which the behavior of fluid is examined under the action of forces. It may be classified as following

i. Fluid statics
ii. Fluid kinematics
iii. Fluid dynamics

The study of fluid at rest is known as fluid statics whereas the study of fluid in motion without considering the cause of motion (external forces) is known as fluid kinematics. The study of fluid in motion with the consideration of cause of motion is known as fluid dynamics.

The fluid dynamics may also be classified as hydrodynamics and gas dynamics. Hydrodynamics contains the study of liquids whereas gas dynamics contains the study of gases. Usually air is assumed for the experimental study of gas dynamics, therefore it may be named as "Aerodynamics".

## 1.3 Physical Properties of Fluid

### 1.3.1 Density

The density is defined as mass per unit volume and is used to characterize the mass of fluid system. It is designated by Greek symbol ρ (rho); SI unit of density is $\frac{kg}{m^3}$.

### 1.3.2 Pressure

The pressure exerted on or by the fluid is a force applied normal to the surface of an object per unit area over which the force is distributed. It is designated by $p$ and has SI unit Pascal (pa). If the distribution of applied force over the surface area is uniform, then the mathematical formulation is given as

$$p = \frac{|F|}{A}.$$

If the distribution of applied force over the surface area is not uniform then we have

$$p = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA},$$

where $\Delta A$ is smallest element of the surface area over which $\Delta F$ acts.

### 1.3.3 Temperature

Temperature is defined as the measure of the intensity of heat in certain body of mass. Different scales are used to measure the temperature. Three commonly used scales are given as follow:

i. Celsius (°C)
ii. Fahrenheit (°F)
iii. Kelvin ($K$)

### 1.3.4 Total Temprerature

Total temperature at a given point in a flow is the temperature that exists if the flow is slowed down adiabatically to zero velocity. It is designated by $T_0$.

### 1.3.5 Total Pressure

Total pressure at a given point in a flow is the pressure that exists if the flow is brought to rest isentropically. It is designated by $p_0$.

### 1.3.6 Total Density

Total density at a given point in a flow is the density that exists if the flow is brought to rest isentropically. It is designated by $\rho_0$.

## 1.4 Types of Fluid

### 1.4.1 Incompressible and Compressible Fluids

The fluid having constant density with the change in temperature and pressure is known as incompressible whereas the fluid whose density changes with temperature and pressure is known as compressible.

### 1.4.2 Inviscid and Viscous Fluids

The fluid whose viscosity is zero is known as invicid fluid whereas the fluid having finite viscosity is known as viscous.

### 1.4.3 Ideal and Real Fluids

An inviscid, incompressible fluid is known as ideal whereas a viscous fluid is known as real fluid.

## 1.5 Types of Flows

### 1.5.1 Uniform and Non-Uniform Flows

A flow is said to be uniform if the fluid properties such as velocity, pressure, temperature, etc. do not change from point to point during the whole course of flow, i.e.

$$\frac{\partial V}{\partial r} = 0, \ \frac{\partial P}{\partial r} = 0, \ \frac{\partial T}{\partial r} = 0, \text{ etc.}$$

A flow is said to be non-uniform if the fluid properties change from point to point during the flow, i.e.

$$\frac{\partial V}{\partial r} \neq 0, \ \frac{\partial P}{\partial r} \neq 0, \ \frac{\partial T}{\partial r} \neq 0, \text{ etc.}$$

### 1.5.2 Steady and Unsteady Flows

A flow is said to be steady if the fluid properties don't change with respect to time, i.e.

$$\frac{\partial V}{\partial t} = 0, \ \frac{\partial P}{\partial t} = 0, \ \frac{\partial T}{\partial t} = 0, \text{ etc.}$$

A flow is said to be unsteady if the fluid properties change with respect to time, i.e.

$$\frac{\partial V}{\partial t} \neq 0, \ \frac{\partial P}{\partial t} \neq 0, \ \frac{\partial T}{\partial t} \neq 0, \text{ etc.}$$

### 1.5.3 Internal and External Flows

The flows through confined mediums such as ducts are known as internal flows whereas the flows which occur over the objects immersed in an unbounded fluid are known as external flows. The flows over airplanes, missiles and submarines are the examples of external flows.

### 1.5.4 Subsonic and Transonic Flows

A flow is said to be subsonic if it is moving with the speed less than 273 $ms^{-1}$ whereas the flow moving with the speed in between 273-409 $ms^{-1}$ is said to be transonic flow.

### 1.5.5 Supersonic and Hypersonic Flows

A flow is said to be supersonic if it is moving with the speed in between 409-1,702 $ms^{-1}$ whereas the flow moving with the speed higher than 1,702 $ms^{-1}$ is said to be hypersonic flow.

### 1.5.6 One-, Two-, and Three-Dimensional Flows

A flow can be classified as one-, two-, and three- dimensional flow depending on the number of space coordinates. A flow is said to be one-dimensional flow if the fluid properties are depending only on one space coordinate. The flow across a duct is an example of one-dimensional flow. A flow is said to be two-dimensional flow if the fluid properties are depending on two space coordinates. The flow over a plate is an example two-dimensional. A flow is said to be three-dimensional flow if the fluid properties are depending only on all three space coordinates. The flow over a wing is an example of three-dimensional flow.

## 1.6 Thermodynamic Properties

Thermodynamics is defined as the study of the relationship between the heat and other forms of energy. It plays an important role in the flow of gas, particularly for high speed flows i.e. flows around high speed flight of aircrafts and missiles.

### 1.6.1 Ideal Gas

A gas in which intermolecular forces are neglected is known as an ideal gas. It is also known as a perfect gas. The equation of an ideal gas is given as

$$p = \rho R T,$$

where $R$ is the specific gas constant. It has different values for different gases.

It is sometime convenient to define the values of thermodynamic properties per unit mass; such values are distinguished by the word "specific".

### 1.6.2 Specific Volume

It is defined as the volume occupied by a unit mass of the fluid and is denoted by $V_s$.

$$V_s = \frac{1}{\rho}.$$

### 1.6.3 Specific Weight

It is defined as the weight per unit volume and is defined as

$$\gamma = \frac{W}{V} = \frac{m\,g}{V},$$

where $W$ is the weight per unit volume.

### 1.6.4 Specific Internal Energy

It defined as the energy per unit mass of the fluid due to molecular activity and is denoted by U.

### 1.6.5 Specific Enthalpy

It is defined as the total heat introduced per unit mass of the fluid and is denoted by $H$.

$$H = U + p\,V_s.$$

### 1.6.6 Specific Heat

It is defined as the amount of heat required to raise the temperature of a unit mass of fluid by one degree. It is denoted by $C$ and is defined as

$$C = \frac{\partial Q}{d\,T}.$$

### 1.6.7 Ratio of Specific Heats

It is defined as a measure of the relative internal complexity of the molecules of the fluid and is denoted by $\gamma$,

$$\gamma = \frac{C_p}{C_v}.$$

For air, $\gamma \doteq 1.4$ .

## 1.7 Types of Thermodynamic Processes

When gases are expanded or compressed, the relationship between the pressure, temperature, and density depends on the nature of the process. The types of thermodynamic processes are discussed in the coming subsections.

### 1.7.1 Reversible Process

The process in which no dissipative phenomena occurs, i.e. the process occurs in such a manner that it can be returned to its original state.

### 1.7.2 Irreversible Process

The process that is not reversible is known as irreversible process.

### 1.7.3 Adiabatic Process

A process in which no heat is added to or taken away from a gas during expansion or compression. In this case

$$\partial Q = 0.$$

### 1.7.4 Isentropic Process

A process which is reversible and adiabatic is said to be isentropic process.

### 1.7.5 First Law of Thermodynamics

It states that the heat added and work done on a gas causes a change in energy of the gas. Its mathematical form is

$$\partial q + \partial w = de,$$

where $\partial q$ is the heat added, $\partial w$ is the work done and $de$ is the change in energy of the gas.

### 1.7.6 Second Law of Thermodynamics

It states that the entropy change is greater than the heat transferred to the system divided by the temperature, that is

$$dS \geq \frac{\partial Q}{T}.$$

For an adiabatic process, $\partial Q = 0$, then $dS = 0$.

## 1.8 Speed of Sound

The distance travelled by a sound wave through a medium in a unit time is known as the speed of sound and is designated by $a$. The mathematical expression for the speed of sound is given as

$$a = \sqrt{\gamma RT}.$$

At standard sea level, the value of speed of sound is

$$a = 340.9\, ms^{-1}.$$

## 1.9 Mach Number

The Mach number is a dimensionless number and is defined as the ratio of the speed of an object $V$ to the speed of sound $a$ in the same medium. It is designated by $M$ and has expression

$$M = \frac{V}{a}.$$

The flows can also be classified with the help of Mach number $M$. The types of flow are given in the coming subsections. The aforementioned subsonic, transonic, supersonic and supersonic flows can also be characterized through Mach number which is usually being practiced by the aerodynamists. In the following, the said flows have been defined in terms of Mach ranges.

### 1.9.1 Subsonic Flow

A flow is said to be subsonic if the Mach value $M$ is less than 1.0, i.e. $M < 0.8$.

### 1.9.2 Transonic Flow

A flow is said to be transonic if the Mach value $M$ is in between 0.8 – 1.2, i.e. $0.8 < M < 1.2$.

### 1.9.3 Supersonic Flow

A flow is said to be supersonic if the Mach value $M$ is in between 1.2 – 5.0, i.e. $1.2 < M < 5.0$.

### 1.9.4 Hypersonic Flow

A flow is said to be hypersonic if the Mach value $M$ is higher than 5.0, i.e. $M > 5.0$.

## 1.10 Shock Wave

There are flows for which the fluid properties change continuously. The flows are also possible in nature for which these quantities vary discontinuously. Such discontinuities occur in explosions, detonations, supersonic movements, powerful electric discharge and such other phenomena's that create extreme changes in fluid properties.

### 1.10.1 Definition

Shock wave is a surface of discontinuity propagating in a gas at which temperature, pressure, density and velocity experience abrupt changes.

### 1.10.2 History of Shock Wave

Many scientists and engineers had faced such discontinuous phenomena (shock wave phenomena) in their research and they gave different names to this phenomena. Without mentioning the name of shock wave, Euler (1759) talked about the "size of disturbance of sound wave meaning its amplitude". Poisson (1808) mentioned about an instance sound wave as the case "where the molecular velocities can no longer be regarded as very small". Stokes (1848) used term "surface of discontinuity" and Reimann (1859) used the term "shock compression" and "compression wave" to describe the jump-like steepening of the wave front. Toepler (1864) was the first to use the term shock wave, he originated shock wave from a spark discharge and for the first time visualized it using stroboscopic method. Rankine (1870) used the terms "abrupt disturbance and wave of finite longitudinal disturbance" and Hugoniot (1885) used the term "discontinuity". Ernst Mach and Coworkers (1875-1885) used the terms "shock wave, Riemann wave, bang wave and explosion wave". Earnst and Ludwig also designated shock wave as "a sound wave of large amplitude". Von Oettingen and Von Gernet (1888) used the term "detonation wave". Vieille and Hadamard (1898), Duhem (1901) and Jouguet (1904) used the term shock wave. Lord Rayleigh (1910) used the term "aerial waves of finite amplitude" while discussing the characteristics of shock wave in the air.

### 1.10.3 Characteristics of Shock Wave

Physically the generation of shock waves is always characterized in the fluid flow by instantaneous changes in temperature, pressure, density and velocity. This means that, in comparisons with free-stream flow conditions, the region between the body and shock wave will be a region of high temperature, pressure and density. In other words, whenever a flow passes through a shock wave, instantaneous increase in pressure, temperature and density is observed with the decrease in velocity of the flow. This instantaneous change in the flow properties is one of the unique feature that characterizes the presence of shock wave.

### 1.10.4 Types of Shock Wave

A shock wave may occur inside a nozzle or over a solid body like flat plate, wedge or blunt body, etc. when such bodies are inserted in a supersonic flow. A shock wave perpendicular to the direction of flow is named as normal shock wave whereas a shock wave inclined at some angle to the direction of flow is named as oblique shock wave. Another form of shock wave which is perpendicular as well as inclined to the direction of flow is named as bow shock wave. The normal shock wave may form inside a nozzle whereas an oblique shock wave may form over a wedge or a concave corner. A bow shock wave may form over a blunt body. The aforementioned types of shock waves are shown in the figures (1.1), (1.2) and (1.3).

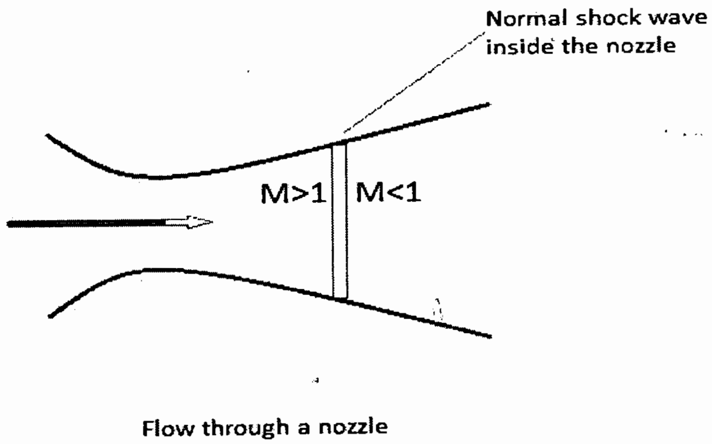


**Fig. 1.1:** Normal shock wave inside a nozzle.

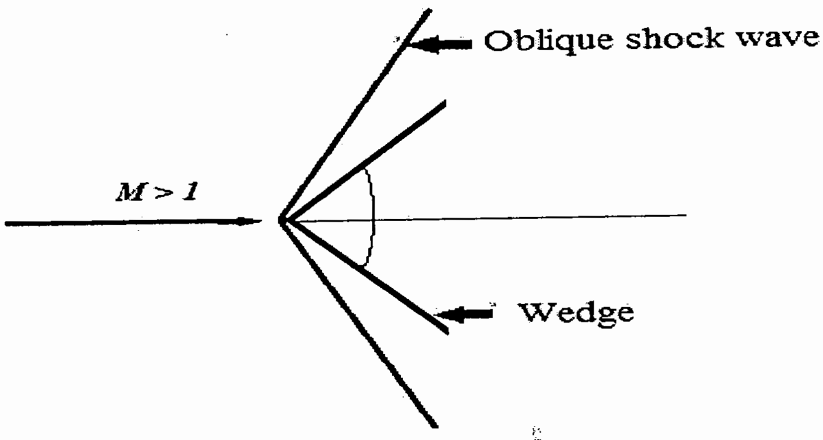


**Fig. 1.2:** Oblique Shock wave over a wedge.

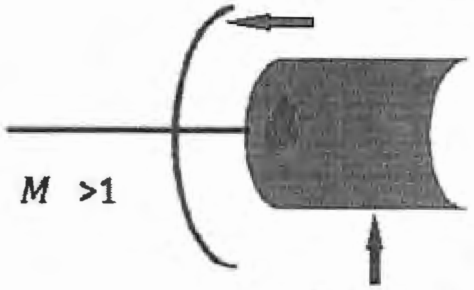


**Fig. 1.3:** Bow shock wave over a blunt body.

## 1.11 Governing Equations for Compressible Flow

The governing equations for an inviscid, compressible flow in their integral form are given as follow

Continuity:

$$\frac{\partial}{\partial t}\oiiint \rho dV + \oiint \rho \boldsymbol{V}.\boldsymbol{dS} = 0 \tag{1.1}$$

Momentum:

$$\frac{\partial}{\partial t}\oiiint \rho \boldsymbol{V} dV + \oiint \rho(\boldsymbol{V}.\boldsymbol{dS})\boldsymbol{V} = -\oiint p\boldsymbol{dS} + \oiiint \rho \boldsymbol{f} dV \tag{1.2}$$

Energy:

$$\frac{\partial}{\partial t}\oiiint \rho(e + \frac{V^2}{2})dV + \oiint \rho\left(e + \frac{V^2}{2}\right)\boldsymbol{V}.\boldsymbol{dS} \tag{1.3}$$
$$= \oiiint \dot{q}\rho dV - \oiint p\boldsymbol{V}.\boldsymbol{dS} + \oiiint \rho(\boldsymbol{f}.\boldsymbol{V})dV$$

## 1.12 Shock Wave Visualization Techniques

The most common techniques used for the visualization of compressible flows are the shadowgraph and the schliren techniques [9]. The working principle of these techniques is illustrated as the change in density of a gas produces corresponding change in the index of refraction of the gas i.e. the refractive index $n$ of a gas is a strong function of the density of that gas. The greatest advantage of these techniques is that only a single beam of light is used during the experiment. If the density of the gas is uniform in the flow direction, the beam of light crossing the test section remains parallel whereas if the density of the gas is non-uniform in the flow direction, the light rays crossing the test section are refracted. The simplest one of these techniques is the shadowgraph technique. It is quick in setup and adaptable to large fields of view. The images obtained from this technique may be cast on projection screens, ground glass, photoelectric film or on any reasonable flat reflecting surface.

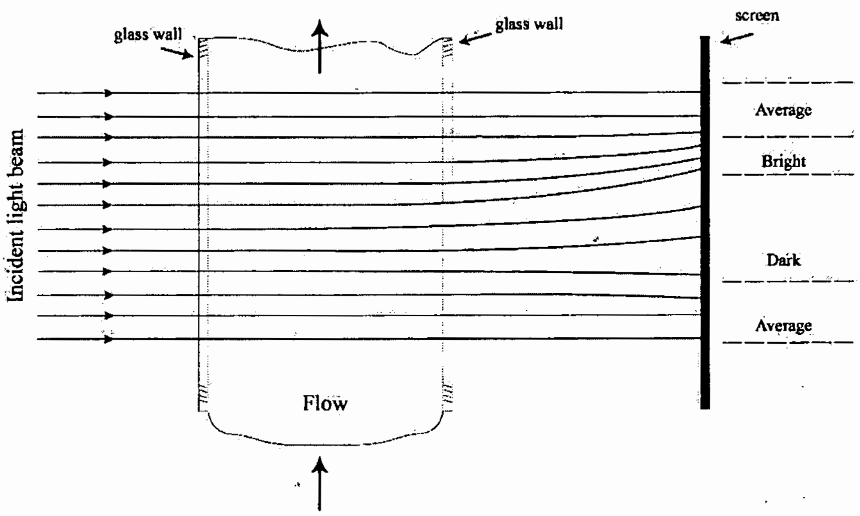


**Fig. 1.4:** Test section used for visualization techniques.

### 1.12.1 Shadowgraph Technique

Shadowgraph technique is applied to obtain the direct view of the flow phenomena. A screen is placed at some distance opposite to the light source and effect of refracted rays are made visible. When there is no flow across the test section, the distribution of light rays remains uniform and hence, illumination of the light is average. On the other hand, when flow passes the test section, the distribution of the light becomes non uniform. Dark and bright spots appear on the screen. Dark spots mark the region where the light rays diverge whereas the bright spots mark the region where the light rays culminate togather. There are places on the screen where the intensity of light rays remain uniform, the brightness is average. These effects produce shadows on the screen called shadowgram. For example, the visualization of shock waves due to the deflection of light rays i.e. due to the change in density is given in the figures (1.5) and (1.6).

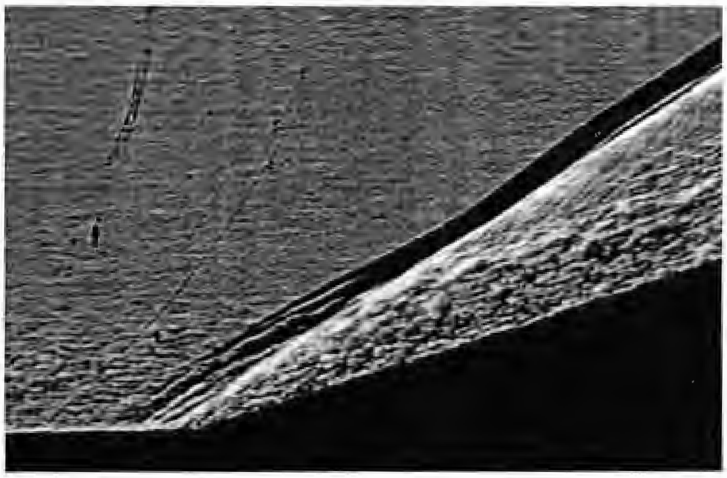

**Fig. 1.5:** Shadowgram of Mach 3 airflow over an 18 degree compression corner which shows the shock wave (from G. Settles PhD thesis, Princeton, 1975). Prof. Gary Settles of Gas Dynamics Lab, Penn State University.

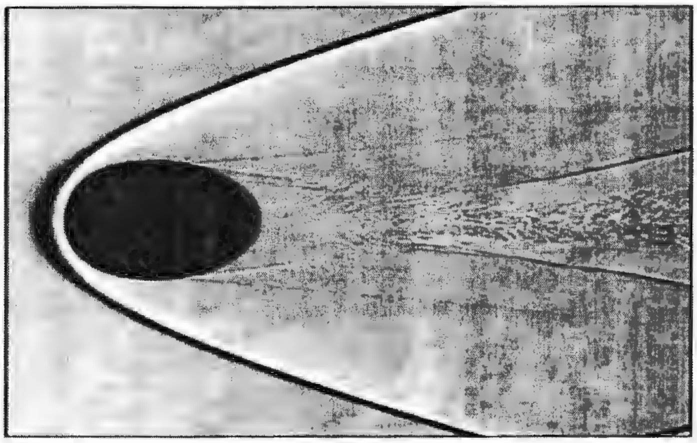

**Fig. 1.6:** Shadowgram of Mach 3 airflow around a sphere which shows a bow wave.

## 1.13 MacCormack Numerical Scheme

The MacCormack scheme is an explicit second-order finite difference numerical scheme introduced by Robert W. MacCormack in 1969 [7]. It is a discretization scheme used for solving hyperbolic partial differential equations. This scheme is good enough to implement over the nonlinear equations such as Euler's equations. The MacCormack scheme contains predictor and corrector steps. In the predictor step, forward difference is applied whereas backward difference is applied in the corrector step. This order can be reversed for the time steps. In order to describe this scheme, let us take a first order hyperbolic partial differential equation

$$\frac{\partial E}{\partial t} + k\frac{\partial E}{\partial x} = 0, \tag{1.4}$$

where k is a constant.

### 1.2.3 Predictor Step

In this step, the value of the dependent variable $E$ is predicted at time $t + \Delta t$. This value is designated and is expressed as

$$\overline{(E)}_i^{t+\Delta t} = (E)_i^t + (\frac{\partial E}{\partial t})_i^t \Delta t, \tag{1.5}$$

where

$$(\frac{\partial E}{\partial t})_i^t = -k\left[\frac{(E)_{i+1}^t - (E)_i^t}{\Delta x}\right]. \tag{1.6}$$

### 1.2.4 Corrector Step

In the corrector step, the predicted value $\overline{(E)}_i^{t+\Delta t}$ is corrected as follows:

$$(E)_i^{t+\Delta t} = (E)_i^t + (\frac{\partial E}{\partial t})_{ave} \Delta t, \tag{1.7}$$

where

$$(\frac{\partial E}{\partial t})_{ave} = \frac{1}{2}\left[\left(\frac{\partial E}{\partial t}\right)_i^t + \overline{\left(\frac{\partial E}{\partial t}\right)_i^{t+\Delta t}}\right], \tag{1.8}$$

in which

$$\overline{\left(\frac{\partial E}{\partial t}\right)_i^{t+\Delta t}} = -k\left[\frac{\overline{(E)}_i^{t+\Delta t} - \overline{(E)}_{i-1}^{t+\Delta t}}{\Delta x}\right]. \tag{1.9}$$

Equation (1.9) is evaluated by using the predicted value $\bar{E}$ obtained from equation (1.5) whereas equation (1.8) represents the average value of $E$ obtained from equations (1.6) and (1.9) at time steps $\Delta t$ and $t + \Delta t$. The final value of the dependent variable $E$ is obtained from equation (1.7).

# Chapter 2

# Physical Description of Flow Properties Across Normal Shock Wave

## 2.1 Introduction

Since the physical properties such as Mach number, density, temperature, pressure, etc. experience abrupt changes across a normal shock wave, therefore the behaviour of these properties is described with the help of graphs [6]. In the second part, the governing equations for quasi one-dimensional flow through a variable area duct are derived first [6]. Then the construction of a CD nozzle is elaborated briefly. Next, the behavior of flow through a CD nozzle is examined. As the pressure ratio is a key factor in the nozzle flow, so the flow for different pressure ratios with the corresponding changes in the Mach values is described. The generation of normal shock wave, "under expansion" and "over expansion" phenomenon are also described briefly [9] in this chapter.

## 2.2 Governing Equations for Normal Shock Wave

Consider a normal shock wave shown in Fig. 2.1. Assume that flow is steady, one dimensional and adiabatic with no body forces. The area is assumed to be constant throughout the normal shock wave. The region in front of the upstream shock is a uniform flow, and the region behind the downstream shock is a different uniform flow. The velocity, temperature, pressure, density, Mach number, total pressure, total temperature, total enthalpy, and entropy in front of the shock are $u_1$, $T_1$, $p_1$, $\rho_1$, $M_1$, $p_{0,1}$, $T_{0,1}$, $h_{0,1}$ and $s_1$ respectively. The corresponding quantities behind the shock are $u_2$, $T_2$, $p_2$, $\rho_2$, $M_2$, $p_{0,2}$, $T_{0,2}$, $h_{0,2}$ and $s_2$ respectively.

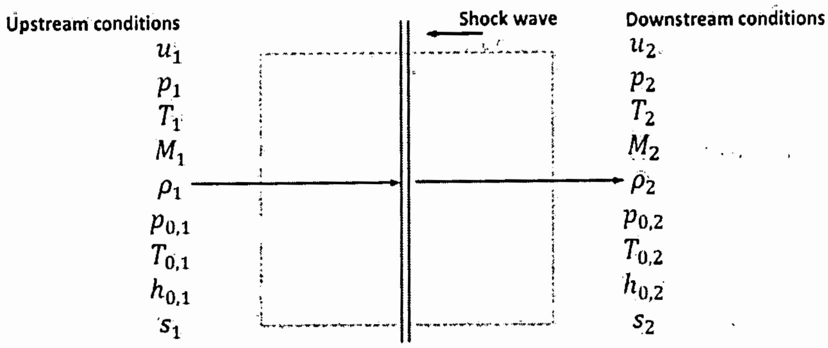


**Fig. 2.1:** Sketch of normal shock wave.

The continuity, momentum and energy equations (1.1), (1.2) and (1.3) for an inviscid, compressible flow under the above assumptions are

$$\rho_1 u_1 = \rho_2 u_2, \tag{2.1}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2, \tag{2.2}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{2.3}$$

The quantities upstream of the shock wave $u_1$, $p_1$, $\rho_1$, etc. are known. We have a system of three algebraic equations with four unknowns $u_2$, $p_2$, $\rho_2$ and $h_2$. If we add the following thermodynamics relations

$$h_2 = c_p T_2, \tag{2.4}$$

$$p_2 = \rho_2 R T_2, \tag{2.5}$$

To the governing system, we have five equations with five unknowns, namely, $u_2$, $T_2$, $p_2$, $\rho_2$, and $h_2$. The above equations can explicitly be solved to obtain the values of unknowns.

## 2.3 Special Forms of Energy Equation

The energy equation (2.3) for an inviscid, steady, one dimensional and adiabatic flow is given as

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}.$$

For a calorically perfect gas, the ratios of total to static temperature, pressure and density are the function of Mach number $M$ only. Their relations are given as

$$\frac{T_0}{T} = 1 + \frac{\gamma+1}{2} M^2, \tag{2.6}$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\gamma/(\gamma-1)}, \tag{2.7}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{1/(\gamma-1)}. \tag{2.8}$$

From these ratios, the quantities $T_0$, $p_0$, $and\, \rho_0$ can be calculated from the actual conditions $T$, $p$, $and\, \rho$.

Consider the case when flow is exactly sonic i.e. $M = 1$. The static temperature, pressure, and density, at sonic condition, are denoted by $T^*$, $p^*$, and $\rho^*$. Then the above relations become

$$\frac{T^*}{T_0} = \frac{2}{\gamma + 1}, \tag{2.9}$$

$$\frac{p^*}{p_0} = (\frac{2}{\gamma + 1})^{\gamma/(\gamma-1)}, \tag{2.10}$$

$$\frac{\rho^*}{\rho_0} = (\frac{2}{\gamma + 1})^{1/(\gamma-1)}. \tag{2.11}$$

From the definition of Mach number, $= \frac{V}{a}$, where $a$ is the speed of sound. Let us introduce a "characteristic" Mach number $M^*$ defined by

$$M^* = \frac{V}{a^*}, \tag{2.12}$$

where $a^*$ is the speed of sound at sonic condition. The relationship between the actual Mach number $M$ and the characteristic Mach number $M^*$ is given as

$$M^{*^2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)\, M_1^2}. \tag{2.13}$$

## 2.4 Calculation of Normal Shock Wave Properties

The flow properties across a normal shock wave can be obtained from the following useful relations:

$$M_2^2 = \frac{1 + \left[\frac{\gamma - 1}{2}\right] M_1^2}{\gamma\, M_1^2 - (\gamma - 1)/2}, \tag{2.14}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1)M_1^2}{2 + (\gamma - 1)\, M_1^2}, \tag{2.15}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\,(M_1^2 - 1), \tag{2.16}$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)\, M_1^2}. \tag{2.17}$$

From the above results, note that the Mach number $M_1$ is the determining parameter for changes across a normal shock wave in a calorically perfect gas.

The shock wave occurs in supersonic flows ($M_1 > 1$) and never occurs in subsonic flows ($M_1 < 1$). Note that in equations (2.16), (2.17), (2.18), and (2.19), $M_1 \geq 1$. However, these equations are also solvable for $M_1 \leq 1$ (on mathematical basis). This issue can be resolved with the help of second law of thermodynamics. Apply the second law to the flow across a normal shock wave. The relation of the entropy changes across a normal shock wave is given as

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\,\ln\frac{p_2}{p_1}. \tag{2.18}$$

Using equations (2.16) and (2.17), we get

$$s_2 - s_1 = c_p \ln\left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right]\frac{2 + (\gamma - 1)M_1^2}{(\gamma + 1)\, M_1^2} - R\,\ln\left[1 + \frac{2\gamma}{\gamma + 1}(M_1^2 - 1)\right] \tag{2.19}$$

Note that the entropy change $s_2 - s_1$ across the shock is a function of $M_1$ only. The second law of thermodynamics states that

$$s_2 - s_1 \geq 0. \tag{2.20}$$

In the above equation, if $M_1 = 1$, we have

$$s_1 = s_2. \tag{2.21}$$

If $M_1 > 1$, we have

$$s_2 - s_1 > 0. \tag{2.22}$$

Both the above results obey second law of thermodynamics. If $M_1 < 1$, we have

$$s_2 - s_1 < 0, \tag{2.23}$$

Which clearly disobeys the second law. Hence, in nature a normal shock wave occurs only in supersonic flows ($M_1 \geq 1$).

The behaviour of the total temperature and pressure across a normal shock wave can be observed as follow. From equation (2.3) and using $h = C_p T$, we have

$$C_p T_1 + \frac{u_1^2}{2} = C_p T_2 + \frac{u_2^2}{2}. \tag{2.24}$$

By the definition of total temperature $T_0$, we have from equation (2.24) that

$$C_p T + \frac{u^2}{2} = C_p T_0. \tag{2.25}$$

Combining equations (2.24) and (2.25), we get

$$T_{0,1} = T_{0,2}. \tag{2.26}$$

Equation (2.26) states that for a normal shock wave, the total temperature is constant. The equation (2.26) can be used in equation (2.18) in order to examine the behaviour of the total pressure $p_0$. We have

$$\frac{p_{0,2}}{p_{0,1}} = e^{-(s_2 - s_1)/R}. \tag{2.27}$$

From equation (2.22), the equation (2.27) gives the result $p_{0,2} < p_{0,1}$ which states that the total pressure decreases across a normal shock wave. The following figures show the variation in the behaviour of flow properties across a normal shock wave.

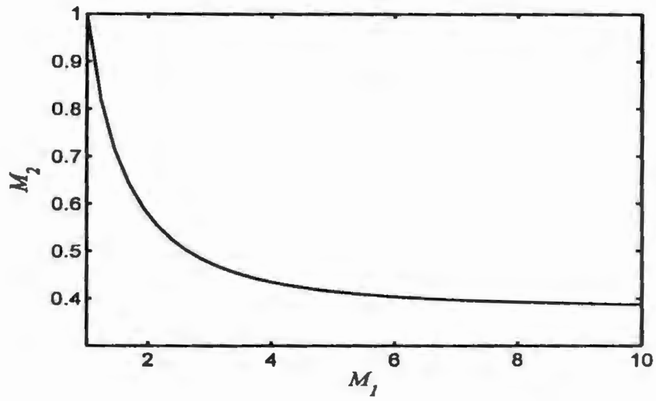


**Fig.2.2:** The variation of downstream Mach number $M_2$ across a normal shock wave as a function of upstream Mach number $M_1$: for $\gamma$=1.4.

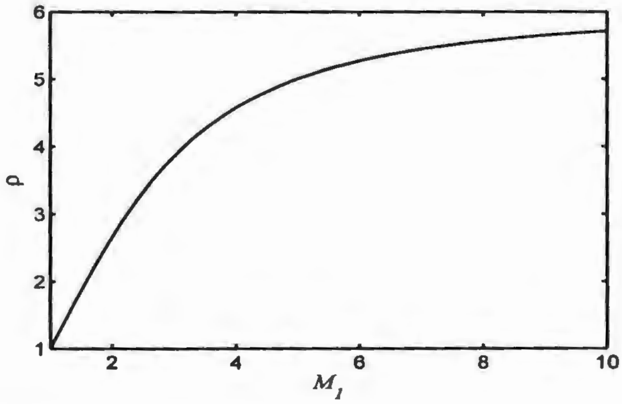


**Fig. 2.3:** The variation of density $\rho$ across a normal shock wave as a function of upstream Mach number $M_1$: for $\gamma$=1.4.

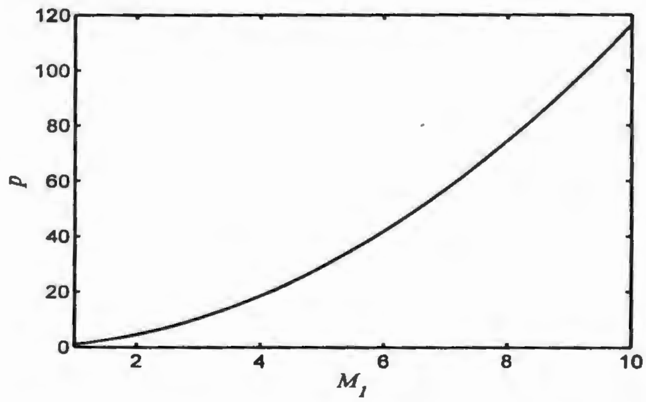


**Fig. 2.4:** The variation of pressure $p$ across a normal shock wave as a function of upstream Mach number $M_1$: for $\gamma$=1.4.

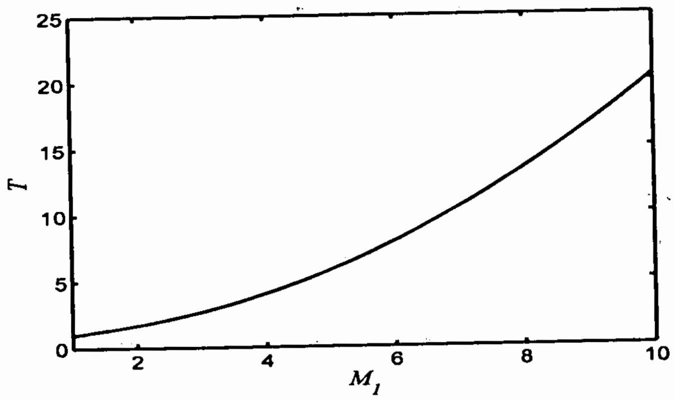


**Fig. 2.5:** The variation of static temperature T across a normal shock wave as a function of upstream Mach number $M_1$: for $\gamma$=1.4.

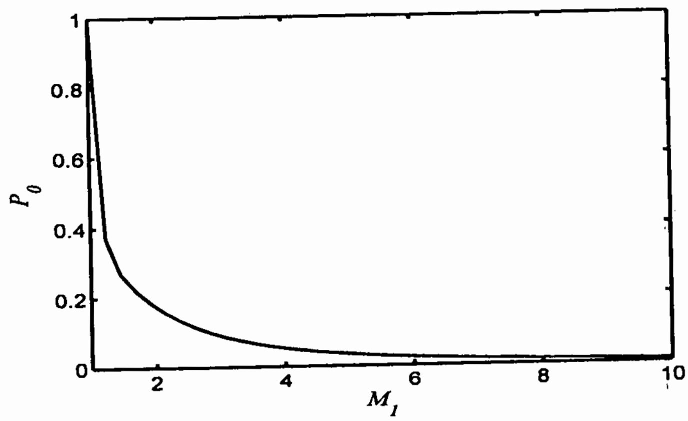


**Fig. 2.6:** The variation of total pressure $p_0$ across a normal shock wave as a function of upstream Mach number $M_1$: for $\gamma$=1.4.

It can be observed that as long as the upstream Mach value $M_1$ increases, the downstream values of Mach number $M_2$ and total pressure $p_0$ decrease. On the other hand, the values of density $\rho$, static pressure $p$ and temperature $T$ downstream of the normal shock wave increase with the increase of upstream Mach value $M_1$.

## 2.5 Governing Equations for Quasi One-Dimensional Flow

The quasi one-dimensional flow is defined as the flow in which properties of the flow variables changes in one direction only by changing the flow area. A schematic of the flow is shown is shown in Fig. (2.7). The governing equations for quasi one-dimensional flow [11] obtained from conservation equations (1.1), (1.2) and (1.3) take the new form

Continuity:

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2, \tag{2.29}$$

Momentum:

$$p_1 A_1 + \rho_1 u_1{}^2 A_1 + \int_{A1}^{A2} P\, dA = p_2 A_2 + \rho_2 u_2{}^2 A_2, \tag{2.30}$$

Energy:

$$h_1 + \frac{u_1{}^2}{2} = h_2 + \frac{u_2{}^2}{2}. \tag{2.31}$$

For a calorically perfect gas, we have from equations (2.4) and (2.5)

$$p_2 = \rho_2 R T_2,$$

$$h_2 = C_p T_2.$$

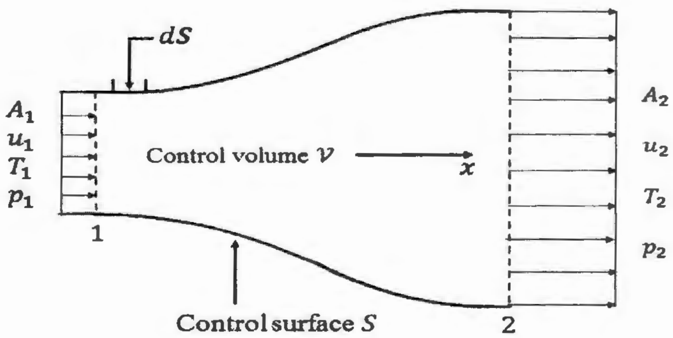


**Fig. 2.7:** Quasi-one-dimensional flow in a control volume $\mathcal{V}$.

### 2.5.1 Differential Form

The differential form of the governing equations of quasi one-dimensional flow are given as

Continuity:

$$d(\rho u A) = 0, \tag{2.32}$$

Momentum:

$$dp = -\rho u \, du, \tag{2.33}$$

Energy:

$$dh + u \, du = 0. \tag{2.34}$$

Equation (2.33) is known as Euler's equation.

### 2.5.2 Area-velocity Relation

The relation which relates the nozzle area to the flow velocity obtained from the differential form of the equations of quasi one-dimensional flow is given as

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}. \tag{2.35}$$

The equation (2.35) is used to construct the of flow passage required for a nozzle in subsonic and supersonic flow.

## 2.6 The Nozzle

Nozzle is a device such as pipe or duct designed to control the flow rate of a fluid. There exist various forms of a nozzle like convergent nozzle, divergent nozzle or convergent-divergent nozzle (CD nozzle).

### 2.6.1 Convergent Nozzle

Convergent nozzle is a nozzle that has wide diameter from entrance as compared to the exit diameter. It is used to accelerate a subsonic flow as well as to decelerate a supersonic flow.

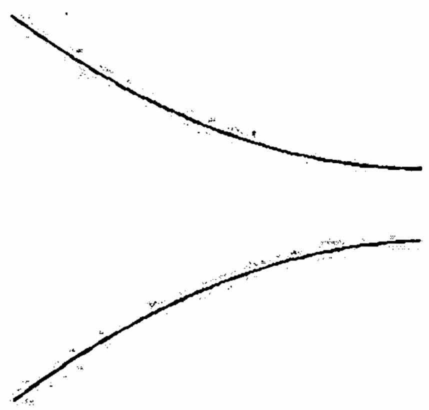

**Fig. 2.8:** Shape of convergent nozzle.

### 2.6.2 Divergent Nozzle

A divergent nozzle is a type of nozzle that has small diameter from entrance as compared to the exit diameter. It is used to decelerate a subsonic flow as well as to accelerate a supersonic flow.

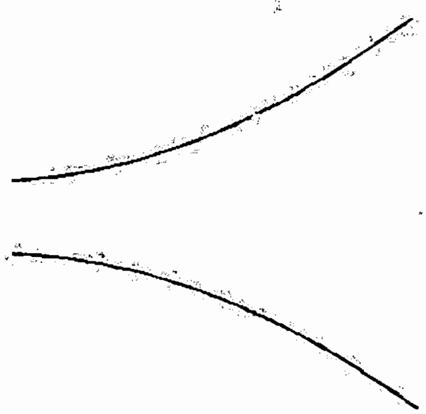

**Fig. 2.9:** Shape of the divergent nozzle.

### 2.6.3 Convergent-Divergent Nozzle

A convergent-divergent nozzle [11] is a device used to increase the velocity of a fluid flow as its pressure is reduced in order to produce thrust. It is shown in Fig 2.12.

### 2.6.4 Construction of CD Nozzle

The shape of a nozzle is an important factor in the flow rate phenomena. The area-velocity relation describes about the formation of a CD nozzle. The relation is given as

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}, \qquad (2.36)$$

This relation gives the following information:

1. For a subsonic flow, i.e. $0 \leq M < 1$, the R.H.S of the relation becomes negative. Note that if velocity increases, the area decreases. Similarly, if velocity decreases, the area increases. The conclusion is that for a subsonic flow, we must have a convergent nozzle in order to increase the velocity whereas a divergent nozzle is needed to decrease the velocity. These results are described in Fig. 2.10.
2. For a supersonic flow, i.e. $M > 1$, the R.H.S of the relation becomes positive. Note that if velocity increases, the area increases and if velocity decreases the area also decreases. Hence, for supersonic flow, a divergent nozzle is needed to increase the velocity and a

convergent nozzle is needed to decrease the velocity. These results are described in Fig. 2.11.

3. For the sonic flow, i.e. $M = 1$ we have

$$dA = 0, \tag{2.37}$$

This shows the area location where the Mach number is unity. At this value, the flow area is either minimum or maximum. Physically, it corresponds to the minimum flow area.

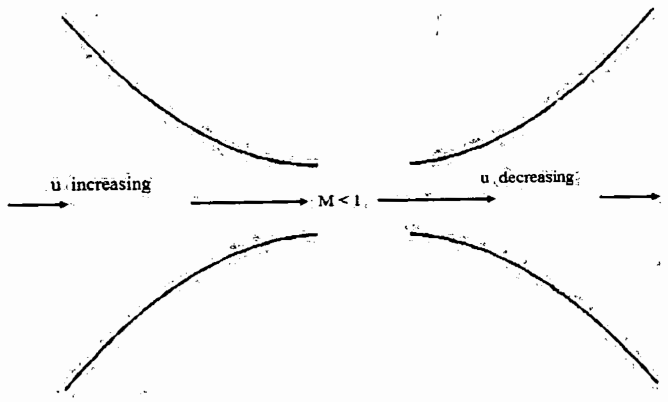


**Fig. 2.10:** The results for the case $0 \leq M < 1$.

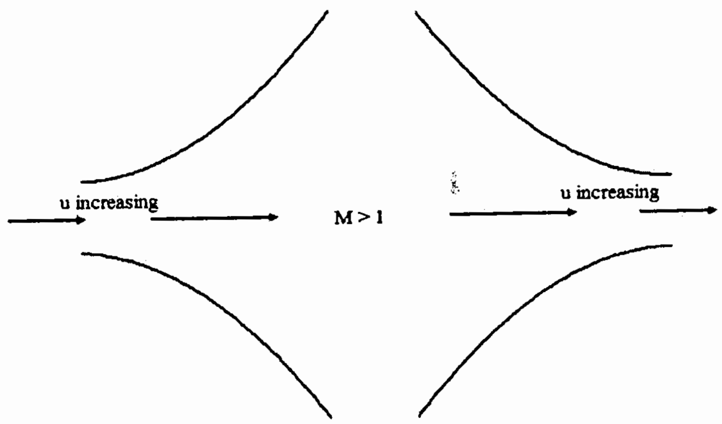


**Fig. 2.11:** The results for the case $M > 1$.

From the above discussion, one may conclude that for a flow to expand it from subsonic speed to supersonic speed, it must first accelerate sub-sonically through a convergent nozzle. As soon as the flow approaches the sonic speed, it must be accelerated through a divergent nozzle in order to

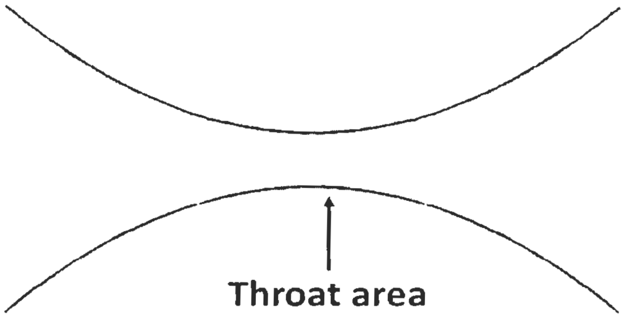


**Fig. 2.12:** Shape of a convergent-divergent nozzle.

achieve the supersonic flow. Hence, a nozzle designed to achieve supersonic flow at its exit is a "convergent-divergent nozzle" as shown in Fig. 2.12. The minimum area that divides the convergent and divergent portion of the nozzle is named as throat area.

## 2.7 Flow through Convergent-Divergent Nozzle

Consider a convergent-divergent nozzle as shown in Fig. 2.13. The pinched area of the nozzle is known as throat. At this location, the flow must be sonic. The Mach number, velocity and area associated to this location are denoted by $M^*$, $u^*$ and $A^*$, respectively. The Mach number, velocity and area associated to any other location are denoted by $M$, $u$ and $A$, respectively.



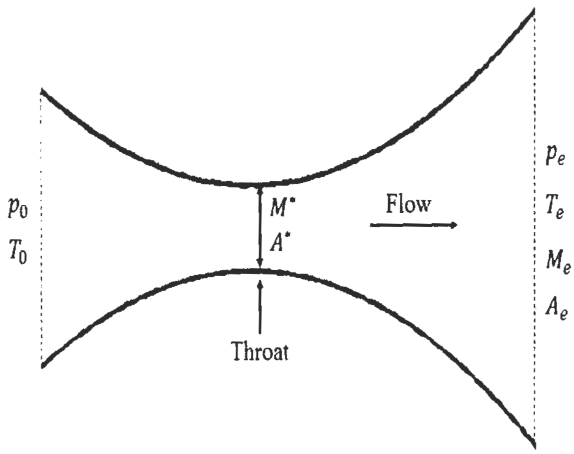


**Fig. 2.13:** Flow through a CD nozzle.

The area-velocity relation for a convergent-divergent nozzle is given below

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{1+r}\left(1+\frac{r-1}{2}M^2\right)\right]^{\frac{r+1}{r-1}}. \tag{2.38}$$

The relation shows that for a given area ratio $A/_{A^*}$, there exists two values of Mach number M. One for subsonic flow and other for supersonic flow.

The inlet temperature and pressure are denoted by $T_0$ and $p_0$. The corresponding values of temperature and pressure at the exit of nozzle are denoted by $T_e$ and $p_e$ whereas the Mach number at the exit is denoted as $M_e$. The flow through a CD nozzle depends on the exit to inlet pressure ratio $^{Pe}/_{P_o}$. If the pressure ratio $^{Pe}/_{P_o}$ is unity, i.e. $p_e = p_0$, there would be no flow through the nozzle. The flow through the nozzle exists only when the exit pressure is small as compared to the inlet pressure, i.e. $p_e < p_0$. For a small pressure ratio, say $\frac{P_e}{P_o} = 0.99$, there exists a very low speed flow through the nozzle. The Mach number increases slightly in the convergent portion of the nozzle and reaches to its maximum value at throat. Since flow speed is low and the sonic condition is not achieved, it would be some subsonic value. Hence, it decreases to a minimum Mach value while passing through the divergent portion of the nozzle. The minimum value of Mach number at the nozzle exit is denoted by $M_{e,1}$ and the corresponding value of the Pressure at the nozzle exit is denoted by $p_{e,1}$. This phenomena is shown in Fig. 2.14(a, b).

Let us further reduce the pressure ratio $^{Pe}/_{P_o}$, as a result the flow moves a bit faster than the first case through the convergent portion of the nozzle and reaches to its maximum Mach value. But the sonic condition is not achieved at the throat. The flow is subsonic at the throat and downstream of the throat, it passes through the divergent portion of the nozzle and attain the Mach value $M_{e,2}$ at the exit with the corresponding exit pressure $p_{e,2}$. This phenomena is shown in Fig. 2.14(a, b).

The further reduction in pressure ratio $^{Pe}/_{P_o}$ helps to move the flow more faster than first two cases through the convergent section and will attain the sonic conditions at the throat, i.e. $M = 1$. Downstream of the throat, the Mach value at the exit becomes $M_{e,3}$ with the corresponding exit pressure $p_{e,3}$. This phenomena is shown in Fig. 2.14(a, b).

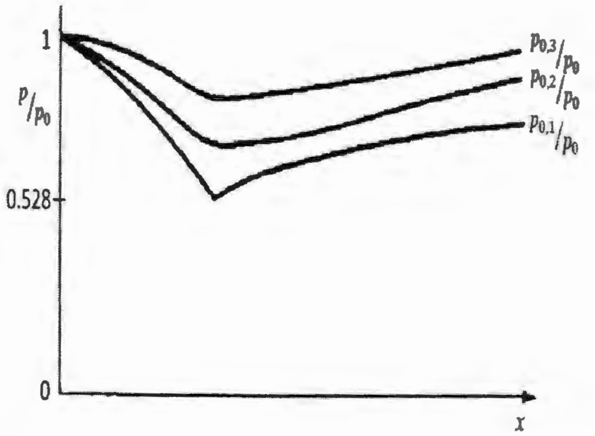


**Fig. 2.14(a):** Variation of pressure through CD nozzle.

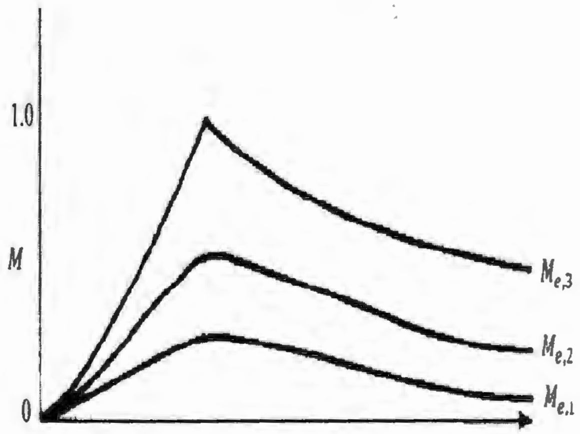


**Fig. 2.14(b):** Variation of Mach number through CD nozzle.

One may note that increase in pressure ratio causes to increase the flow velocity. Once, the sonic conditions are achieved at the nozzle throat, the further reduction of the exit pressure $p_e$ below $p_{e,3}$ will not change these conditions. Hence, the flow rate remains constant as the exit pressure $p_e$ is reduced below $p_{e,3}$. The condition when the flow becomes sonic at the throat and flow rate remains unchanged no matter how low the exit pressure $p_e$ is reduced, known as "chocked flow".

The reduction of the exit pressure $p_e$ below $p_{e,3}$ does not change the flow properties in the convergent section of the nozzle but a lot happens in the divergent section of the nozzle. As $p_e < p_{e,3}$, the flow becomes supersonic in the divergent section .If the exit pressure $p_e$ is reduced still further, the flow rapidly changes from supersonic to subsonic which is followed by a rapid change in pressure. A shock wave normal to the flow direction appears in the divergent section of the

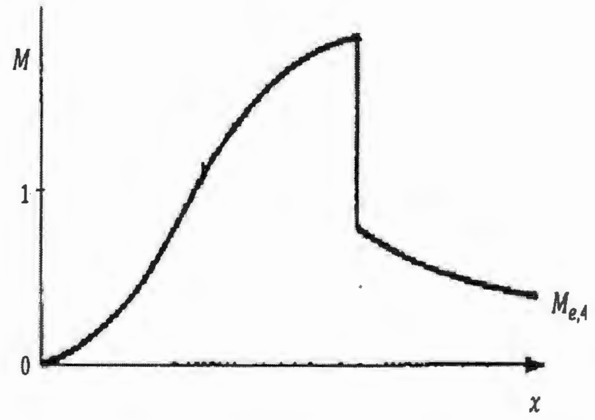


**Fig. 2.15(c):** Abrupt change in Mach number due the formation of normal shock.

As the exit pressure $p_e$ is further reduced below $p_{e,4}$, location of the shock wave moves toward the nozzle exit and at a certain value of the pressure $p_{e,5}$, the shock is located exactly at the exit with the corresponding Mach value $M_{e,5}$. This phenomena is shown in Fig. 2.16(a,b,c).

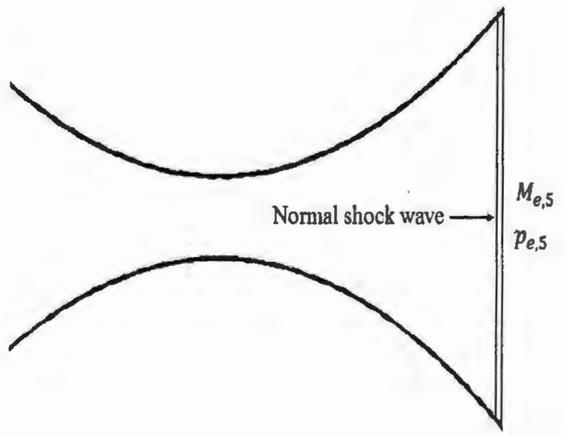


**Fig. 2.16(a):** Movement of shock wave with the reduction of exit pressure.

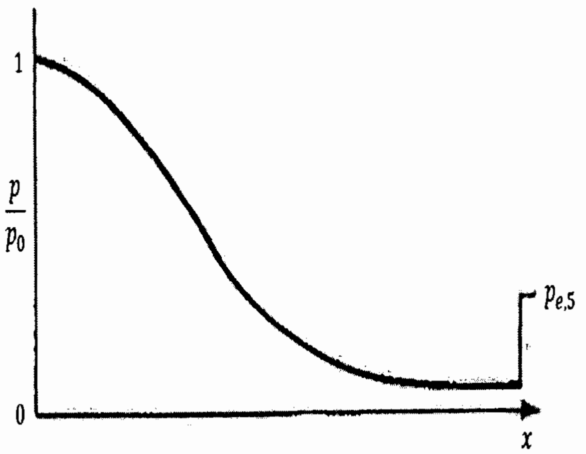


**Fig. 2.36(b):** Abrupt change in pressure due to flow through normal shock wave.

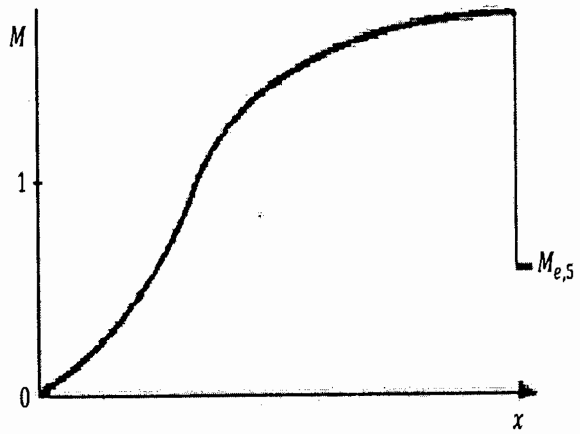


**Fig. 2.16(c):** Abrupt change in Mach number due to flow through normal shock wave.

When the atmospheric pressure also known as back pressure $p_B$ is higher than $p_{e,5}$, i.e. $p_B > p_{e,5}$, oblique shock waves occur attached to the tip of the nozzle in order to adjust the pressure with the back pressure. This situation is known as over expansion and is shown in Fig. 2.17. When the back pressure $p_B$ is reduced such that it becomes equal to the exit pressure say $p_{e,B}$, i.e. $p_B = p_{e,B}$, no wave is formed and the flow will be uniformly supersonic in the divergent section of the nozzle. This pressure is known as the design pressure and is shown in Fig. 2.18. If the back pressure is reduced to the design pressure the flow must expand in order to adjust the back pressure $p_B$. The expansion waves occur attached to the exit. This situation is known as under expansion and is shown in Fig. 2.19.

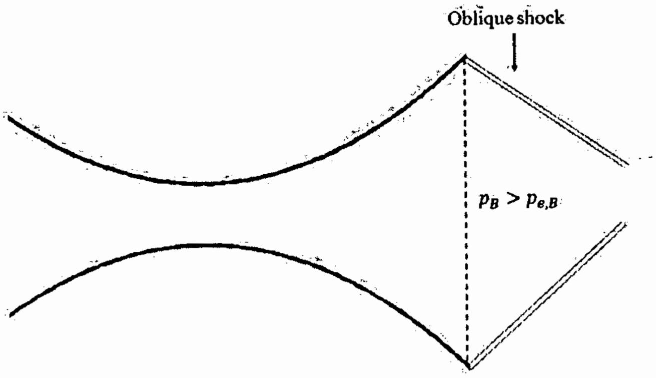


**Fig. 2.17:** Formation of oblique shock waves at nozzle exit when $p_B > p_{e,5}$.

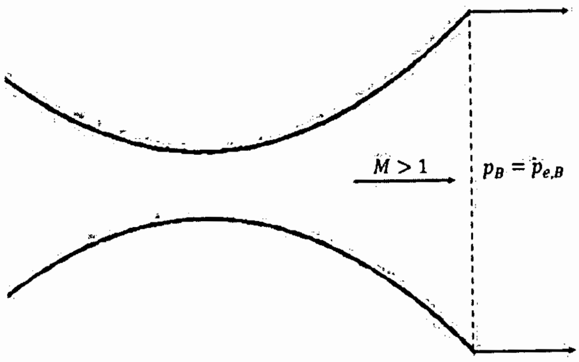


**Fig. 2.18:** Situation of nozzle flow at design pressure, i.e. $p_B = p_{e,5}$.

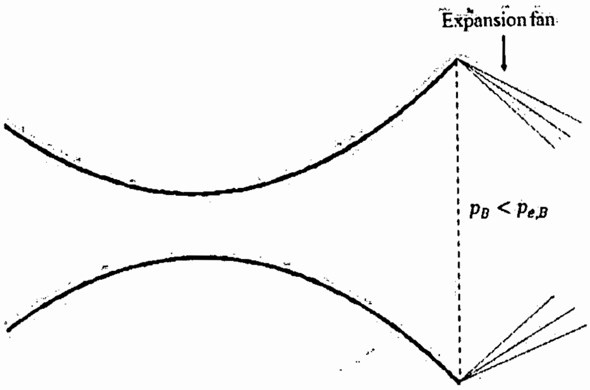


**Fig. 2.19:** Formation of expansion waves at nozzle exit when $p_B < p_{e,5}$.

### 2.7.1 Under Expansion

Under expansion is a phenomena [9] that occurs when the back pressure $p_B$ is lower than the design pressure. Such kinds of phenomena occur in propulsive devices such as rocket engines or jet engines when these devices are operated at high altitudes. As the atmospheric pressure (back pressure) is very low at high altitudes, therefore high pressure ratio (which requires high area ratio) is required in order to get very low pressure ratio at the exit of the nozzle which is practically impossible due to structural and aerodynamic limitations. When the jet engine is operated at such a low atmospheric pressure region where the design pressure is higher than the back pressure, expansion takes place outside the nozzle exit to the atmospheric region. Same situation arises when a rocket engine is operated in space where the atmospheric pressure is zero. The under expansion process outside the nozzle is shown in Fig. 2.19.

### 2.7.2 Over Expansion

Over expansion phenomena [9] occurs when the atmospheric pressure $p_B$ is higher than the design pressure. Such situation arises when a high altitude jet or rocket engine nozzle is operated at low altitudes or at sea level. In such situation, there occurs an oblique shock wave attached to the exit of nozzle. Since the back pressure is higher than the design pressure, the flow has to get compressed to the back pressure $p_B$. The oblique shock wave helps to compress the exit flow and increases its pressure to the required back pressure $p_B$. The over expansion process is shown in Fig. 2.17.

## Discussion

1. As discussed before, the flow properties experience abrupt changes across the normal shock wave. Here, the changes in these properties are examined with the help of graphs. It is observed that density $\rho$, pressure $p$ and temperature $T$ increase whereas Mach number $M_2$ and total pressure $p_0$ decrease across the shock wave. The total temperature $T_0$ remains constant due to the isentropic condition. It is also examined that a normal shock wave can only occur in the supersonic flow with the help of second law of thermodynamic.
2. It is observed that how a CD nozzle is formed by using area velocity relation. The physical aspects of the generation of normal shock wave are examined and also observed that it changes its location when the pressure ratio is decreased. It is also observed that under expansion process occurs at high altitudes whereas over expansion process occurs at low altitudes.

# Chapter 3

# Quasi One Dimensional Flow Through CD Nozzle

## 3.1 Introduction

The shock wave phenomena may arise in nozzle flows or over wedge or blunt body flows as mentioned before. Different types of shock waves may occur in nature but the topic of our concentration is the normal shock wave. The normal shock wave phenomena may occur inside a CD nozzle used in turbine engines, ramjets and scramjets which generate thrust. A Swedish inventor Gustaf de Laval (1888) was the first who developed a CD nozzle [11] and used it in a steam turbine. The nozzle flow phenomena [10] solved in this chapter is a very important and interesting problem in aerodynamics. It has established in chapter 2 that the generation as well as the location of a normal shock wave depends strongly on the pressure ratio. In this chapter, the location of normal shock wave will be determined for the given conditions. The changes in flow properties will also be examined. A finite-difference MacCormack's scheme is used to simulate this flow problem.

## 3.2 The Setup

Consider a CD nozzle whose area is defined by the following relation

$$A' = 1 + 2.2(x' - 1.5)^2. \tag{3.1}$$

The area $A_1$ describes the location of the shock wave and $A^*$ is the throat area. The nozzle area is shown in the following Fig. 3.1.

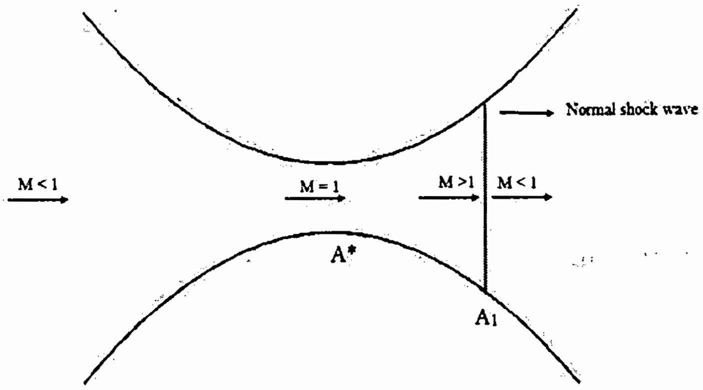


**Fig. 3.1:** Area of the convergent-divergent nozzle.

The continuity, momentum and energy equations for quasi-one-dimensional flow are given below

$$\frac{\partial(\rho' A')}{\partial t'} + \frac{\partial(\rho' A' V')}{\partial x'} = 0, \tag{3.2}$$

$$\frac{\partial(\rho' A' V')}{\partial t'} + \frac{\partial(\rho' A' V'^2)}{\partial x'} = -A' \frac{\partial P'}{\partial x'}, \tag{3.3}$$

$$\frac{\partial}{\partial t'}\left[\rho'\left(e' + \frac{V'^2}{2}\right)A'\right] + \frac{\partial}{\partial x'}\left[\rho'\left(e' + \frac{V'^2}{2}\right)A'V' + P'A'V'\right] = 0. \tag{3.4}$$

The above governing equations can be written in dimensionless from with the help of the following dimensionless parameters.

$$T = \frac{T'}{T_0}, \ \rho = \frac{\rho'}{\rho_0}, \ V = \frac{V'}{a_0}, \ P = \frac{P'}{P_0}, \ t = \frac{t'}{L/a_0}, \ x = \frac{x'}{L}, \ A = \frac{A'}{A^*}, \ e = \frac{e'}{e_0}, \tag{3.5}$$

where

$$a_0 = \sqrt{R\gamma T_0}, \qquad e_0 = C_v T_0 = \frac{R T_0}{\gamma - 1}.$$

By using these dimensionless parameters, the governing equations take the new form

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho A V)}{\partial x} = 0, \tag{3.6}$$

$$\frac{\partial(\rho A V)}{\partial t} + \frac{\partial[\rho A V^2 + \frac{1}{\gamma} P A]}{\partial x} = \frac{1}{\gamma} P \frac{\partial A}{\partial x}, \tag{3.7}$$

$$\frac{\partial[\rho\left(\frac{e}{\gamma - 1} + \frac{2}{\gamma} V^2\right)A]}{\partial t} + \frac{\partial\left[\rho\left(\frac{e}{\gamma - 1} + \frac{2}{\gamma} V^2\right)VA + PVA\right]}{\partial x} = 0, \tag{3.8}$$

The equations (3.6), (3.7) and (3.8) are the dimensionless forms of the continuity, momentum and energy equations for quasi-one-dimensional flow. These equations can be expressed in a generic form. let us define the elements of the solution vector by $E$, the flux vector by $F$ and the source term by $J$ as follows:

$$E_1 = \rho A, \tag{3.9}$$

$$E_2 = \rho A V, \tag{3.10}$$

$$E_3 = \rho\left(\frac{e}{\gamma - 1} + \frac{2}{\gamma}V^2\right)A, \tag{3.11}$$

$$F_1 = \rho AV, \tag{3.12}$$

$$F_2 = \rho AV^2 + \frac{1}{\gamma}PA, \tag{3.13}$$

$$F_3 = \rho\left(\frac{e}{\gamma - 1} + \frac{2}{\gamma}V^2\right)VA + PVA, \tag{3.14}$$

$$J_2 = \frac{1}{\gamma}P\frac{\partial A}{\partial x}. \tag{3.15}$$

Note that $J_1$ and $J_3$ are zero. Using these elements, the system (3.6), (3.7) and (3.8) takes the form

$$\frac{\partial E_1}{\partial t} = -\frac{\partial F_1}{\partial x}, \tag{3.16}$$

$$\frac{\partial E_2}{\partial t} = -\frac{\partial F_2}{\partial x} + J_2, \tag{3.17}$$

$$\frac{\partial E_3}{\partial t} = -\frac{\partial F_3}{\partial x}. \tag{3.18}$$

It should be remembered that the above equations will result in the form of solution vectors $E_1$, $E_2$ and $E_3$. In order to obtain the results in the form of the primitive variables $(p, V, T, P)$, etc. we are to decode the elements $E_1$, $E_2$ and $E_3$ as follows

$$\rho = \frac{E_1}{A}, \tag{3.19}$$

$$V = \frac{E_2}{E_1}, \tag{3.20}$$

$$e = T = (\gamma - 1)\left[\frac{E_3}{E_1} - \frac{\gamma}{2}V^2\right], \tag{3.21}$$

$$P = \rho T, \tag{3.22}$$

where

$$e = \frac{e'}{e_0} = \frac{C_v T'}{C_v T_0} = T.$$

Hence, after obtaining all $E's$, the primitive variables can be obtained.

The flux terms $F_1$, $F_2$ and $F_3$ can be expressed in terms of the dependent variables $E_1$, $E_2$ and $E_3$, respectively. Substituting equations (3.19) and (3.20) in (3.12) for $F_1$, we get

$$F_1 = E_2. \tag{3.23}$$

Substituting equations (3.19), (3.20) and (3.22) for $F_2$, we have

$$F_2 = \frac{E_2^2}{E_1} + \frac{\gamma - 1}{\gamma}\left(E_3 - \frac{\gamma}{2}\frac{E_2^2}{E_1}\right). \tag{3.24}$$

For the flux term $F_3$, substituting (3.19), (3.20), (3.21) and (3.22) result in

$$F_3 = \gamma\frac{E_2 E_3}{E_1} - \frac{\gamma(\gamma - 1)}{\gamma}\frac{E_2^3}{E_1^2}. \tag{3.25}$$

Similarly, the source term $J_2$ takes the new form

$$J_2 = \frac{\gamma - 1}{\gamma}\left(E_3 - \frac{\gamma}{2}\frac{E_2^2}{E_1}\right)\frac{\partial(\ln A)}{\partial x}. \tag{3.26}$$

### 3.2.1 Boundary Conditions

At the inflow boundary, two dependent flow-field variables are held fixed and one other variable is allowed to float, i.e. density $\rho$ and temperature $T$ are held fixed at the inflow boundary and velocity $V$ is allowed to float. Note that by holding density $\rho$ fixed at point 1, $E_1$ is also fixed. The value of $E_2$ is linearly extrapolated at grid point 1 from the known value at grid points 2 and 3 respectively,i.e.

$$E_{2(i=1)} = 2E_{2(i=2)} - E_{2(i=3)}. \tag{3.27}$$

The floating value of $V$ is calculated from equation (3.27). Since $V$ is floating at the inflow boundary, so is $E_3$.

$$E_3 = E_1\left(\frac{T}{\gamma - 1} + \frac{\gamma}{2}V^2\right). \tag{3.28}$$

At the out flow boundary which is also subsonic, all the flow-field variables are allowed to float instead of the exit pressure $P_e$ which must be specified. The values of $E_1$ and $E_2$ at the outflow boundary are calculated by linear extrapolation from their adjacent points.

$$(E_1)_N = 2(E_1)_{N-1} - (E_1)_{N-2}, \tag{3.29}$$

$$(E_2)_N = 2(E_2)_{N-1} - (E_2)_{N-2}. \tag{3.30}$$

The value of $V$ at the outflow boundary is obtained from equation (3.20)

$$V_N = \frac{(E_2)_N}{(E_1)_N}. \tag{3.31}$$

The value of $E_3$ at the outflow boundary is obtained by using $V_N$ and the specified value of the pressure $P_N$ in the way

$$(E_3)_N = \frac{1}{\gamma - 1} P_N A + \frac{\gamma}{2}(E_2)_N V_N. \tag{3.32}$$

### 3.2.2 Initial Conditions

Since the dependent variables $E_1$, $E_2$ and $E_3$ are in the form of primitive variables $\rho, P, T,$ etc. Therefore, initial conditions at time $t = 0$ are needed to form the values of $E_1$, $E_2$ and $E_3$.

$$\left.\begin{aligned} \rho &= 1.0 \\ T &= 1.0 \end{aligned}\right\} \quad 0 \le x \le 0.5, \tag{3.33}$$

$$\left.\begin{aligned} \rho &= 1.0 - 0.366(x - 0.5) \\ T &= 1.0 - 0.167(x - 0.5) \end{aligned}\right\} \quad 0.5 \le x \le 1.5, \tag{3.34}$$

$$\left.\begin{aligned} \rho &= 0.634 - 0.702(x - 1.5) \\ T &= 0.833 - 0.4908(x - 1.5) \end{aligned}\right\} \quad 1.5 \le x \le 2.1, \tag{3.35}$$

$$\left.\begin{aligned} \rho &= 0.5892 + 0.10228(x - 2.1) \\ T &= 0.93968 + 0.0622(x - 2.1) \end{aligned}\right\} \quad 2.1 \le x \le 3.0. \tag{3.36}$$

The initial conditions for pressure $P$ at time $t = 0$ can be obtained from the equation of state

$$P = \rho T. \tag{3.37}$$

Note that $E_2 = \rho A V$ is physically the local mass flow and is assumed constant 0.59 for the sake of initial condition. The value of $V$ is calculated as

$$V = \frac{E_2}{\rho A} = \frac{0.59}{\rho A}. \tag{3.38}$$

Finally, the initial conditions for $E_1$, $E_2$ and $E_3$ are calculated by using equations (3.33) to (3.38).

### 3.2.3 Calculation of Time Step

As the governing equations (3.6), (3.7) and (3.8) for unsteady, quasi-one dimensional flow are hyperbolic partial differential equations. Therefore, for an explicit finite-difference solution of such equations, the stability criterion for time step calculation $\Delta t$ is specified by the CFL criterion. The stability criterion for hyperbolic system of equations is given as

$$\Delta t = C\frac{\Delta x}{a+V}, \tag{3.39}$$

where $C$ is the courant number, $\Delta x$ is the step size, $a$ is the speed of sound and $V$ is the velocity. For the stability of solution, $C \leq 1$.

Equation (3.39) states that $\Delta t$ must be less than or equal to the time it takes a sound wave to move from one grid point to the next. Since $a$ and $V$ are variables, therefore for the fixed $\Delta x$ the values of $\Delta t$ at different grids would be different.

$$(\Delta t)_i^t = C\frac{\Delta x}{a_i^t + V_i^t}, \tag{3.40}$$

$$(\Delta t)_{i+1}^t = C\frac{\Delta x}{a_{i+1}^t + V_{i+1}^t}. \tag{3.41}$$

Since the values of $\Delta t$ are different at different grids points, therefore the value of $\Delta t$ would be the minimum of all calculated values of $\Delta t$ over the grids, i.e.

$$\Delta t = \min(\Delta t_1^t, \Delta t_2^t, \dots \Delta t_i^t, \dots \Delta t_N^t). \tag{3.42}$$

## 3.3 Numerical Scheme

The MacCormack's scheme [7], a second order finite-difference scheme, is used for the solution of the problem. It is based on the predictor and corrector steps. The description of the scheme for the system of equations (3.16), (3.17) and (3.18) is given as follow.

### 3.3.1 Predictor Step

In the predictor step, the values of the solution vectors $E's$ are predicted for the next time step $t + \Delta t$ by the following manner:

$$\overline{(E_1)}_i^{t+\Delta t} = (E_1)_i^t + \left(\frac{\partial E_1}{\partial t}\right)_i^t . \Delta t, \tag{3.43}$$

$$\overline{(E_2)}_i^{t+\Delta t} = (E_2)_i^t + (\frac{\partial E_2}{\partial t})_i^t.\Delta t, \tag{3.44}$$

$$\overline{(E_3)}_i^{t+\Delta t} = (E_3)_i^t + (\frac{\partial E_3}{\partial t})_i^t.\Delta t, \tag{3.45}$$

where

$$(\frac{\partial E_1}{\partial t})_i^t = -\frac{(F_1)_{i+1}^t - (F_1)_i^t}{\Delta x}, \tag{3.46}$$

$$(\frac{\partial E_2}{\partial t})_i^t = -\frac{(F_2)_{i+1}^t - (F_2)_i^t}{\Delta x} + J_2, \tag{3.47}$$

$$(\frac{\partial E_3}{\partial t})_i^t = -\frac{(F_3)_{i+1}^t - (F_3)_i^t}{\Delta x}. \tag{3.48}$$

The flux terms $F$'s used in this step are obtained from equations (3.23), (3.24) and (3.25), respectively.

### 3.3.2 Corrector Step

In the corrector step, the predicted values are corrected for the time step $t + \Delta t$ by the following way:

$$\overline{(\frac{\partial E_1}{\partial t})}_i^{t+\Delta t} = -\frac{\overline{(F_1)}_i^{t+\Delta t} + \overline{(F_1)}_{i-1}^{t+\Delta t}}{\Delta x}, \tag{3.49}$$

$$\overline{(\frac{\partial E_2}{\partial t})}_i^{t+\Delta t} = -\frac{\overline{(F_2)}_i^{t+\Delta t} + \overline{(F_2)}_{i-1}^{t+\Delta t}}{\Delta x} + \overline{(J_2)}_i^{t+\Delta t}, \tag{3.50}$$

$$\overline{(\frac{\partial E_3}{\partial t})}_i^{t+\Delta t} = -\frac{\overline{(F_3)}_i^{t+\Delta t} + \overline{(F_3)}_{i-1}^{t+\Delta t}}{\Delta x}, \tag{3.51}$$

where the values of $\overline{F,s}$ are calculated from equations (3.23), (3.24) and (3.25), respectively. Then the average values of the dependent variables are obtained as

$$(\frac{\partial E_1}{\partial t})_{ave} = \frac{1}{2}\left[\left(\frac{\partial E_1}{\partial t}\right)_i^t + \overline{(\frac{\partial E_1}{\partial t})}_i^{t+\Delta t}\right], \tag{3.52}$$

$$(\frac{\partial E_2}{\partial t})_{ave} = \frac{1}{2}\left[\left(\frac{\partial E_2}{\partial t}\right)_i^t + \overline{(\frac{\partial E_2}{\partial t})}_i^{t+\Delta t}\right], \tag{3.53}$$

$$(\frac{\partial E_3}{\partial t})_{ave} = \frac{1}{2}\left[\left(\frac{\partial E_3}{\partial t}\right)_i^t + \overline{(\frac{\partial E_3}{\partial t})}_i^{t+\Delta t}\right]. \tag{3.54}$$

Finally, the final values of the dependent variables $E_1$, $E_2$ and $E_3$ at the next time step $t + \Delta t$ are calculated as

$$(E_1)_i^{t+\Delta t} = (E_1)_i^t + (\frac{\partial E_1}{\partial t})_{ave}\Delta t, \tag{3.55}$$

$$(E_2)_i^{t+\Delta t} = (E_2)_i^t + (\frac{\partial E_2}{\partial t})_{ave}\Delta t, \tag{3.56}$$

$$(E_3)_i^{t+\Delta t} = (E_3)_i^t + (\frac{\partial E_3}{\partial t})_{ave}\Delta t. \tag{3.57}$$

The above equations (3.55), (3.56) and (3.57) can be used to calculate the values of the primitive variables $\rho, P, T$ and $V$ from equations (3.19), (3.20), (3.21) and (3.22). The value of Mach number $M$ can be obtained by the following equation:

$$M = \frac{V}{\sqrt{T}}. \tag{3.58}$$

## 3.4 Artificial Viscosity

Artificial viscosity is introduced in order to remove oscillations occurred in the solution [10]. The implementation of the artificial viscosity is described by the following way. The expression for artificial viscosity is

$$S_i^t = \frac{C_x|(P)_{i+1}^t - 2(P)_i^t + (P)_{i-1}^t|}{(P)_{i+1}^t + 2(P)_i^t + (P)_{i-1}^t}[(P)_{i+1}^t + 2(P)_i^t + (P)_{i-1}^t]. \tag{3.59}$$

As before, the predicted values of the dependent variables are obtained as

$$\overline{(E)}_i^{t+\Delta t} = (E)_i^t + (\frac{\partial E}{\partial t})_i^t\Delta t, \tag{3.60}$$

with the addition of the artificial viscosity, the predicted values are obtained as

$$\overline{(E_1)}_i^{t+\Delta t} = (E_1)_i^t + (\frac{\partial E_1}{\partial t})_i^t\Delta t + (S_1)_i^t, \tag{3.61}$$

$$\overline{(E_2)}_i^{t+\Delta t} = (E_2)_i^t + (\frac{\partial E_2}{\partial t})_i^t \Delta t + (S_2)_i^t, \tag{3.62}$$

$$\overline{(E_3)}_i^{t+\Delta t} = (E_3)_i^t + (\frac{\partial E_3}{\partial t})_i^t \Delta t + (S_3)_i^t, \tag{3.63}$$

where the values of $S's$ are calculated from equation (58) with their respective values $E, s$. Similarly, the corrected values of the dependent variables are calculated as

$$(E)_i^{t+\Delta t} = (E)_i^t + (\frac{\partial E}{\partial t})_{ave} \Delta t. \tag{3.64}$$

But with the addition of artificial viscosity, the corrected values of the dependent variables are calculated as

$$(E_1)_i^{t+\Delta t} = (E_1)_i^t + (\frac{\partial E_1}{\partial t})_{ave} \Delta t + \overline{(S_1)}_i^t, \tag{3.65}$$

$$(E_2)_i^{t+\Delta t} = (E_2)_i^t + (\frac{\partial E_2}{\partial t})_{ave} \Delta t + \overline{(S_2)}_i^t, \tag{3.66}$$

$$(E_3)_i^{t+\Delta t} = (E_3)_i^t + (\frac{\partial E_3}{\partial t})_{ave} \Delta t + \overline{(S_3)}_i^t, \tag{3.67}$$

where

$$\overline{(S)}_i^{t+\Delta t} = \frac{C_x \left| \overline{(P)}_{i+1}^{t+\Delta t} - 2\overline{(P)}_i^{t+\Delta t} + \overline{(P)}_{i-1}^{t+\Delta t} \right|}{\overline{(P)}_{i+1}^{t+\Delta t} + 2\overline{(P)}_i^{t+\Delta t} + \overline{(P)}_{i-1}^{t+\Delta t}} \left[ \overline{(E)}_{i+1}^{t+\Delta t} - 2\overline{(E)}_i^{t+\Delta t} + \overline{(E)}_{i-1}^{t+\Delta t} \right]. \tag{3.68}$$

## 3.5 Results and Discussion

1. The following numerical results are obtained with 31 grid points that can be increased in order to obtain the precise location of shock wave. The grid points are not increased because increment in grid points gives slight changes in the solution which are acceptable. Therefore, solution is grid independent.
2. It is seen that artificial viscosity is added with adjustable constant 0.1 in order to avoid oscillations in the solution. Also, a courant number of 0.5 is employed for the obtained results shown in figures 3.2 and 3.3. Notice that shock wave is located at point $x = 2.1$ and across this location, the Mach value $M$ decreases suddenly whereas pressure $p$

increases. The results of velocity $V$ and density $\rho$ are not shown because of the similar behavior of velocity $V$ with Mach number $M$ as well as of density $\rho$ with pressure $p$.

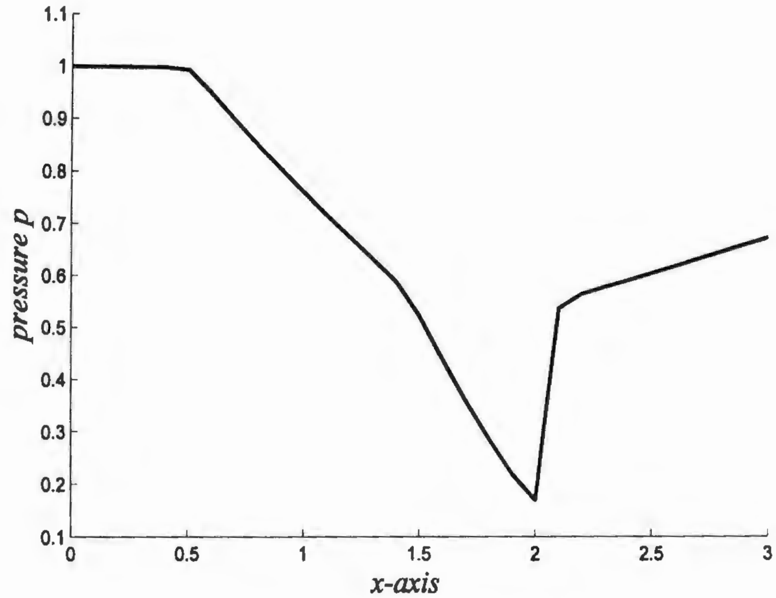


**Fig. 3.2:** Numerical results of shock capturing for the pressure distribution across nozzle with artificial viscosity $Cx = 0.1$.

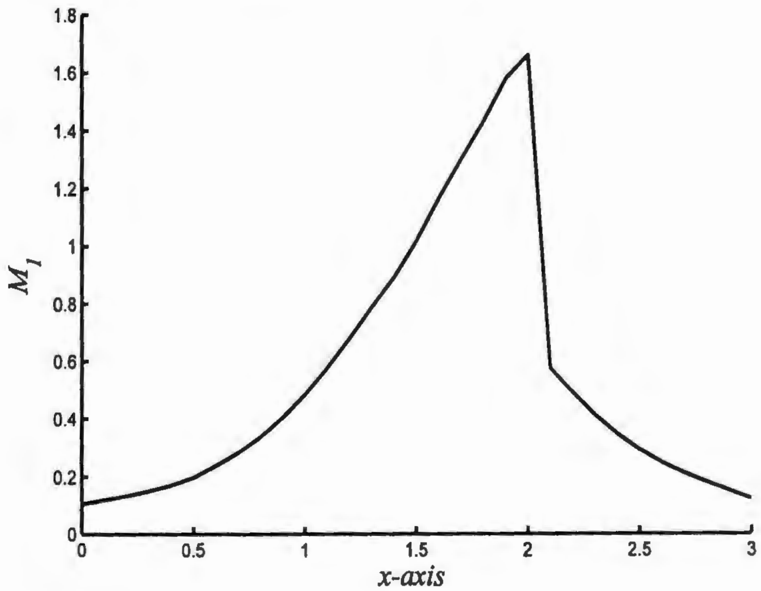


**Fig. 3.3:** Numerical results of shock capturing for the Mach number distribution across nozzle with artificial viscosity $Cx = 0.1$.

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