Magnetohydrodynamic Boundary Layer Flow: Perturbation and Numerical solution

By

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بسم الله الرحمن الرحيم

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A Thesis
Submitted in the Partial Fulfillment of the
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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF PHILOSOPHY IN MATHEMATICS***

We accept this thesis as conforming to the required standard.

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2012

Dedication

I dedicate this work to
My Taya G and My Respected
teacher Ch. Sada Husain

Declaration

I hereby declare that this thesis, neither as a whole nor as a part thereof, has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

Signature: ____________

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Preface

Study of boundary layer flows of an incompressible fluid induced by a continuously stretching sheet maintained at a constant temperature is an active area of research. This area has wide range of applications in industry. Examples may include extrusion of polymers, the cooling of metallic plates, the aerodynamic extrusion of plastic sheets etc. The theoretical analysis of such flows helps in improving the quality of the final product of such processes. Since the first investigation of crane [1], several researchers devoted their efforts to analyze the boundary layer flow and heat transfer over a stretching sheet under various assumptions [2-10]. Recently Mahapatra and Gupta [11] reported the study of heat transfer in a steady two-dimensional stagnation-point flow of a viscous incompressible fluid toward a stretching sheet surface. In their analysis fluids is considered as hydrodynamic fluid and Joule's heating effects are not taken into account. The aim of this dissertation is to present a perturbation and numerical analysis of Magnetohydrodynmic boundary layer flow of a viscous incompressible fluid over a stretching sheet in the presence of Joule's heating. The layout of the dissertation is as follows. Chapter 1 presents basic concepts and fundamental equations. A brief introduction to solutions techniques is also included.

In chapter 2 an analysis is performed for Magnetohydrodynamic boundary layer flow over a heated stretched plate. Asymptotic solutions for small and large values of Hartmann are established and compared with the numerical solutions.

Chapter 3 extends the analysis of chapter 2 for a porous stretching sheet. Solutions are reported using the same methodology as discussed in chapter 2. The suction and injection on the velocity and temperature fields are illustrated through several graphs.

Contents

Chapter 1

Basic definitions

In this chapter we begin with some of the basic definitions and equations of fluids mechanics. The reader is exposed to boundary layer equations for a two dimensional flow along with energy equation. Later in the text, the techniques used to solve the governing equations are illustrated.

1.1 Flow

In the presence of different forces, a material or substance goes under deformation. In many cases this deformation is change of position of its particles. If this deformation increases continuously without limit, then the phenomenon is known as flow.

1.2 Fluid

A fluid is a material/substance that flows under the action of shearing forces.

1.3 Fluid mechanics

Fluid mechanics deals with the study of all fluids under static and dynamic situations. It is a branch of continuum mechanics which deals with a relationship between forces, motions, and statical conditions in a continuous material. This study area deals with many diversified problems such as surface tension, fluid statics, flow in enclose bodies, flow round bodies (solid or otherwise), flow stability, etc.

1.4 Deformation

The relative change in position or length of the fluid particles is known as deformation (strain).

1.5 Density

The density of a fluid is defined as its amount of mass per unit volume. It is denoted by the Greek symbol ρ. Mathematically, it can be written as

$$\rho = \frac{m}{V} \tag{1.1}$$

where m is the mass and V is the volume.

1.6 Pressure

Pressure is defined as the magnitude of force per unit area and it can be written as

$$p = \frac{F}{A} \tag{1.2}$$

where p is pressure, F is the magnitude of normal force and A is the area.

1.7 Viscosity

Viscosity of the fluid is defined as the property of the fluid that tends to resist the movement of one layer of the fluid over adjacent layer of the fluid. It is denoted by the symbol μ and is defined as

$$\mu = \frac{shear\ stress}{rate\ of\ shear\ strain}. \tag{1.3}$$

μ is also called as dynamic viscosity.

1.8 Kinematic viscosity

It is defined as the ratio of dynamic viscosity to fluid density. It is denoted by ν. Mathematically, it is defined as

$$\nu = \frac{\mu}{\rho}. \tag{1.4}$$

1.9 Electrical conductivity

It is a measures of material's ability to conduct an electric current. It is denoted by symbol σ_e.

1.10 Magnetic permeability

A measure of the ability of a substance to sustain a magnetic field. It is equal to the ratio between magnetic flux density and magnetic field strength. It is denoted by μ_e.

1.11 Thermal conductivity

It is measure of the ability of a substance to conduct heat and it is denoted by symbol κ.

1.12 Dimensionless numbers

A dimensionless number is the number without any unit associated with it. It is the ratio of quantities having same unit. It is usually used to simplify our procedure and various quantities are replaced by a single number saving a lot of time and work. There are a number of dimensionless numbers but here we mention only those being used in our work.

1.12.1 Hartmann number

Hartmann number is the ratio of electromagnetic force to the viscous force. It was first introduced by Hartmann. It is defined as

$$M = B_0\sqrt{\frac{\sigma_e}{\rho c}} \tag{1.5}$$

where B_0 is magnetic field and c is a proportionality constant of the velocity of stretching sheet.

1.12.2 Prandtl number

It is the ratio of the product of dynamic viscosity and specific heat to the thermal conductivity. It is denoted by symbol Pr and its mathematical form is given by

$$\mathrm{Pr} = \frac{\mu c_p}{\kappa}. \tag{1.6}$$

1.12.3 Eckert number

The ratio of the characteristic velocity to the product of the specific heat and temperature difference between the body and surface is recognized as Eckert number and is denoted by Ec, i.e.

$$Ec = \frac{v_0^2}{c_{p(\theta_0 - \theta_\infty)}} \tag{1.7}$$

where v_0 is the characteristic velocity and $\theta_0 - \theta_\infty$ represents the temperature difference.

1.12.4 Nusselt number

In heat transfer at a boundary (surface) within a fluid, the Nusselt number is the ratio of convective to conductive heat transfer across (normal to) the boundary. It is a dimensionless number.

1.13 Types of flow

1.13.1 Uniform flow

A flow in which the velocities of fluid particles are same at each point.

1.13.2 Non-uniform flow

A flow in which the velocities of fluid particles are different at different point.

1.13.3 Steady flow

It is a flow in which fluid properties do not depend on the time 't'. If ξ is any fluid property then for steady flow

$$\frac{\partial \xi}{\partial t} = 0. \tag{1.8}$$

1.13.4 Unsteady flow

It is a flow in which fluid properties depend on the time 't'. Mathematically, for an unsteady flow

$$\frac{\partial \xi}{\partial t} \neq 0. \tag{1.9}$$

1.13.5 Compressible flow

A flow in which the density of the fluid is not constant is called compressible flow.

1.13.6 Incompressible flow

A flow in which the density of the fluid is constant throughout the flow is called an incompressible flow.

1.14 Classification of fluids

There are two main types of fluids.

1.14.1 Ideal fluid

A non-existent, assumed fluid without either viscosity or compressibility is called an ideal or perfect fluid . It is the hypothetical form of fluids. However, the fluid with negligible viscosity may be considered as an ideal fluid.

1.14.2 Real fluid

Real fluids are those in which fluid friction has significant effects on the fluid motion. In other words, we cannot neglect the viscosity effects on the motion of the fluid. Real fluids are further classified into two classes on the basis of Newton's law of viscosity. According to this law "shear stress is directly proportional to the rate of deformation ". For one dimensional flow, it can be written as

$$\tau_{yx} = \mu \frac{du}{dy}. \tag{1.10}$$

where τ_{yx} is the shear stress and du/dy is the rate of deformation. Here u is the x-component of velocity of fluid and y is the direction normal to the flow.

Newtonian fluid

Newtonian fluid is a fluid whose stress versus strain (deformation) rate curve is linear and passes through the origin, i.e., Newtonian fluid obeys *Newton's* law of viscosity. Water, gasoline and mercury are some examples of Newtonian fluids.

Non-Newtonian fluid

Fluids in which shear stress is not linearly proportional to the deformation rate are known as non-Newtonian. A non-Newtonian fluid is a fluid whose flow properties are not described by a single constant value of viscosity, i.e., it does not obey *Newton's law of viscosity*. For non-Newtonian fluids

$$\tau_{yx} = \kappa\left(\frac{du}{dy}\right)^n, \qquad n \neq 1 \tag{1.11}$$

or

$$\tau_{yx} = \eta\left(\frac{du}{dy}\right), \tag{1.12}$$

where

$$\eta = \kappa(\frac{du}{dy})^{n-1}, \tag{1.13}$$

is called the apparent viscosity. Examples of non-Newtonian fluids are toothpaste, ketchup, gel, shampoo, blood, soaps etc.

1.15 Heat

The total molecular kinetic energy in a system is called the heat of the system. In thermodynamics, heat is the process of energy transfer from one body or system to another due to the thermal contact.

1.16 Temperature

The average kinetic energy of the particles in a substance is called the temperature of the substance.

1.17 Specific heat

It is amount of heat required to raise the temperature of one gram of a substance to 1^0C. The relationship between amount of heat transferred, specific heat and change in temperature is defined as

$$C_p = \frac{Q}{m\Delta T}. \tag{1.14}$$

where Q is heat transferred, m is mass and ΔT is change in temperature.

1.18 Heat transfer mechanism

1.18.1 Conduction

Conduction is the transfer of energy through matter from particle to particle. It is the transfer and distribution of heat energy from atom to atom within a substance. For example, a spoon in a cup of hot tea becomes warmer because the heat from the tea is conducted along the spoon. This phenomenon occurs usually in solids but it can be happen in fluids.

1.18.2 Convection

Convection is the transfer of heat by the actual movement of the warmed matter. Heat leaves the hot cup of tea as the currents of steam and air rise. Convection usually occurs in liquids and gases.

1.18.3 Radiation

Radiation is the transfer of heat from one object to another by means of electromagnetic waves. Radiative heat transfer does not require that objects be in contact or that a fluid flow between those objects. Radiative heat transfer occurs in the void of space (that's how the sun warms us).

1.19 Skin friction

When a fluid moves across a surface, a certain amount of friction called skin friction occurs between the fluid and the surface which tends to slow down the motion of fluid.

1.20 Joule's heating

Joule heating, also known as ohmic heating and resistive heating, is the process by which the passage of an electric current through a conductor releases heat. It was first studied by James Prescott Joule in 1841. Joule immersed a length of wire in a fixed mass of water and measured the temperature rise due to a known current flowing through the wire for a 30 minute period. By varying the current and the length of the wire he deduced that the heat produced was proportional to the square of the current multiplied by the electrical resistance of the wire.

$$Q \propto I^2.R \tag{1.15}$$

This relationship is known as Joule's First Law. The SI unit of energy was subsequently named the joule and given the symbol J. The commonly known unit of power, the watt, is equivalent to one joule per second.

1.20.1 Joule's heating effect

When a potential difference is applied across the ends of a conductor, the free electrons are accelerated and acquire kinetic energy. As the electrons move through, they collide with the positive ions and atoms of the conductor and transfer their kinetic energy to them. Between two collisions, the electrons again pick up kinetic energy from the electric field. As a result, the kinetic energy of vibration of these lattice ions or atoms increases. This increases the thermal energy of the lattice, which means that the temperature of the conductor increases. Since the source of emf (e.g., a battery) is maintaining current in the conductor, the electric energy supplied by the battery is converted into heat in the conductor.

1.21 Boundary layer concept in the study of fluid flow

When fluids flow over surfaces, the molecules near the surface are brought to rest due to the viscosity of the fluid. The adjacent layers also slow down, but to a lower and lower extent. This slowing down is found limited to a thin layer near the surface. The fluid beyond this layer is not affected by the presence of the surface. The fluid layer near the surface in which there is a general slowing down is defined as boundary layer. The velocity of flow in this layer increases from zero at the surface to free stream velocity at the edge of the boundary layer.

1.22 Boundary layer equations

The discovery of the boundary layer equations can be considered as one of the more important advances in the fluids mechanics. The use of an order of magnitude analysis results in the simplified form of governing Navier-Strokes equations of viscous flow within the boundary layer. Indeed , the partial differential equations become parabolic. This greatly enhances the solution procedure for the equations. The flow is divided into inviscid portion (which is easy to solve by a number of approaches) and the boundary layer (which is governed by an easy partial differential equation). Navier-Strokes equations for an incompressible two-dimensional flow are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}), \tag{1.16}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}), \tag{1.17}$$

and continuity equation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.18}$$

In above expressions x and y are the horizontal and vertical coordinates and u, v are the velocity components parallel to x and y axes. A wall is considered at $y = 0$. The non-dimensional quantities are defined as

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{\delta_1}, \quad u^* = \frac{u}{U}, \quad v^* = \frac{v}{U}\frac{L}{\delta_1}, \quad p^* = \frac{p}{\rho U^2}. \tag{1.19}$$

Here L indicates the horizontal length scale and δ_1 the boundary layer thickness. Equations (1.15) to (1.17) in non-dimensional variables are

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\nu}{UL}\frac{\partial^2 x^*}{\partial x^{*2}} + \frac{\nu}{UL}(\frac{L}{\delta_1})\frac{\partial^2 u^*}{\partial y^{*2}}, \tag{1..20}$$

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -(\frac{L}{\delta_1})^2\frac{\partial p^*}{\partial y^*} + \frac{\nu}{UL}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\nu}{UL}(\frac{L}{\delta_1})^2\frac{\partial^2 v^*}{\partial y^{*2}}. \tag{1.21}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{1.22}$$

in which the Reynold number is written as

$$R = \frac{UL}{\nu}. \tag{1.23}$$

Inside the boundary layer the inertial and viscous forces are of the same and hence

$$\frac{\nu}{UL}(\frac{L}{\delta_1})^2 = O(1), \tag{1.24}$$

or

$$\delta_1 = O(R^{-1/2}L). \tag{1.25}$$

Dropping asterisks and utilizing above equations one obtains

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \tag{1.26}$$

$$\frac{1}{R}(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \frac{1}{R^2}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}), \tag{1.27}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.28}$$

For $R \to \infty$ we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}, \tag{1.29}$$

$$-\frac{\partial p}{\partial y} = 0, \tag{1.30}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.31}$$

In which Eq. (1.29) shows that pressure is constant across the boundary layer. In dimensional form, Eqs. (1.28) to (1.30) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{1.32}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \tag{1.33}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.34}$$

The set of equations given by (1.31)-(1.33) are known as boundary layer equations for a two dimensional incompressible flow.

1.23 Law of conservation of energy

The law of conservation of energy states that energy may neither be created nor destroyed. Therefore, the sum of all the energies in the system is a constant. The laws of conservation of energy which is also called the energy equation is described as

$$\rho c_p \frac{D\theta}{Dt} = \mathbf{T} \cdot \mathbf{L} - \boldsymbol{\nabla} \cdot \mathbf{q}, \tag{1.35}$$

in which

$$\mathbf{L} = \boldsymbol{\nabla}\mathbf{V}. \tag{1.36}$$

1.24 Solution techniques

1.24.1 Perturbation method

The governing equations of physical, biological and economic models often involve features which make it impossible to obtain their exact solutions. Examples of such features are:

- The occurrence of a complicated algebraic equation
- The occurrence of a complicated integral
- Varying coefficients in a differential equation
- An awkwardly shaped boundary
- A nonlinear term in a differential equation

When a large or small parameter occurs in a mathematical model there are various methods of constructing perturbation expansions for the solution of the governing equations. Often the terms in the perturbation expansions are governed by simpler equations for which exact techniques are available. Even if exact solutions cannot be obtained, the numerical methods used to solve the perturbation equations approximately are often easier to construct than the numerical methods for the original governing equations.

1.24.2 Runge-Kutta method

There are many different methods for solving initial value problems relating to ordinary differential equations numerically. Amongst these Runge-Kutta method of order four is preferred because of its higher order accuracy i.e. of O(4.)

The general equation of second order initial value problem can be written as

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}), \tag{1.37}$$

subject to the initial conditions

$$y(x_0) = y_0, \quad \frac{dy}{dx}(x_0) = a. \tag{1.38}$$

In order to solve the problem, we need to convert second order initial value problem to the system of first order initial value problems by defining

$$\frac{dy}{dx} = z = f(x, \ y, \ z), \tag{1.39}$$

and

$$\frac{dz}{dx} = g(x, \ y, \ z), \tag{1.40}$$

with initial conditions

$$y(x_0) = y_0, \quad z(x_0) = a. \tag{1.41}$$

Now the Runge-Kutta method of order 4 for the above system of first order differential Eqs. (1.39) and (1.40) is defined as

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{1.42}$$

and

$$z_{n+1} = z_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4). \tag{1.43}$$

where

$$k_1 = hf(x_n,\ y_n,\ z_n), \qquad l_1 = hg(x_n,\ y_n,\ z_n), \tag{1.44}$$

$$k_2 = hf(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2},\ z_n + \frac{l_1}{2}), \qquad l_2 = hg(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2},\ z_n + \frac{l_1}{2}), \tag{1.45}$$

$$k_3 = hf(x_n + \frac{h}{2},\ y_n + \frac{k_2}{2},\ z_n + \frac{l_2}{2}), \qquad l_3 = hg(x_n + \frac{h}{2},\ y_n + \frac{k_2}{2},\ z_n + \frac{l_2}{2}), \tag{1.46}$$

$$k_4 = hf(x_n + h,\ y_n + k_3,\ z_n + l_3), \qquad l_4 = hg(x_n + h,\ y_n + k_3,\ z_n + l_3). \tag{1.47}$$

where h is uniform step size defined as

$$h = \frac{x_n - x_0}{n}. \tag{1.48}$$

and n is number of steps.

1.24.3 Shooting method

Shooting method is iterative technique which is very popular for the two points boundary value problems. In this method, the boundary value problem of higher order is reduced to the system of first order initial value problems by letting the missing condition. Then the goal is to find the solution of initial value problem instead of given boundary value problem directly. For this purpose, any scheme for the solution of initial value problem can be used. Runge-Kutta method of order 4 is often used for this purpose. For illustration, lets consider a second order boundary value problem

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}), \tag{1.49}$$

with boundary value problem

$$y(0) = 0, \qquad y(L) = A, \tag{1.50}$$

where f is a arbitrary function and data is prescribed at $x = 0$ and $x = L$. The same differential equation describes an initial value problem if data is prescribed as

$$y(0) = 0, \quad y'(0) = s. \tag{1.51}$$

To solve the boundary value problem we reduce it into a system of two first order differential equations as

$$\frac{dy}{dx} = u, \quad \frac{du}{dx} = f(x, y, u), \tag{1.52}$$

and initial conditions are

$$y(0) = 0, \quad y'(0) = u(0) = s, \tag{1.53}$$

where 's' the missing initial condition which will be assigned an initial value. The problem is to find s such that solution of equation (1.49) subject to the initial conditions (1.51) satisfies the boundary conditions (1.50). In other words, if the solutions of the initial value problem are denoted by $y(x, s)$ and $u(x, s)$, one searches for the value of s such that

$$y(L, s) - A = 0 = \phi(s). \tag{1.54}$$

To find the appropriate value of s which satisfy (1.53) one can used Newton's method.

For Newton's method, the iteration formula for s is given by

$$s^{(n+1)} = s^{(n)} - \frac{\phi(s^{(n)})}{d\phi(s^{(n)})/ds}, \tag{1.55}$$

or

$$s^{(n+1)} = s^{(n)} - \frac{y(L,\ s^{(n)} - A)}{\partial y(L, s^{(n)})/\partial s}. \tag{1.56}$$

To find the derivatives of y with respect to s equations (1.51) and (1.52) are differentiated with respect to s, we get

$$\frac{dY}{dx} = U, \quad \frac{dU}{dx} = \frac{\partial f}{\partial y}Y + \frac{\partial f}{\partial u}U, \tag{1.57}$$

where

$$Y = \frac{\partial y}{\partial s}, \quad U = \frac{\partial u}{\partial s}, \tag{1.58}$$

and initial conditions becomes

$$Y(0) = 0, \qquad U(0) = 1. \tag{1.59}$$

Chapter 2

Magnetohydrodynamic boundary layer flow over a heated stretched plate

This chapter presents an analysis of magnetohydrodynamic (MHD) boundary layer flow of a viscous incompressible fluid over a heated plate which is continuously stretched in its own plane. The Joule's heating term is incorporated in the energy equation to take care of heat generation due to electric current. The equations governing the flow are reduced to a set of ordinary differential equations using a similarity transformation. An asymptotic solution of the equations for small and large values of Hartmann number is established and compared with the exact numerical solution. Several graphs are plotted in order to illustrate the effects of emerging parameters on the velocity and temperature fields.

2.1 Formulation of the problem

Let us consider the two-dimensional steady boundary layer flow of a viscous incompressible electrically conducting fluid over a flat sheet such that the sheet is stretched in its own plane with velocity proportional to the distance from the origin. The flow is in presence of an externally applied magnetic field of strength B_0 normal to the flow. The stretching surface has uniform temperature T_w and a linear velocity u_w while the velocity of the flow external to the boundary layer is $u_e(x)$, which is assumed to be zero here. The governing equations for the flow problem under considerations are:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{du_e}{dx} + v\frac{\partial^2 u}{\partial y^2} + \frac{\sigma_e B_0^2(u_e - u)}{\rho}, \tag{2.2}$$

$$\rho C_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}) = \kappa\frac{\partial^2 T}{\partial y^2} + \mu(\frac{\partial u}{\partial y})^2 + \sigma_e B_0^2 u^2. \tag{2.3}$$

The boundary conditions are:

$$y = 0: \; u = u_w = cx, \; v = 0; \; T = T_w, \tag{2.4a}$$

$$y = \infty: \; u = u_e(x) = 0; \; T = T_\infty. \tag{2.4b}$$

where c is a proportionality constant of the velocity of stretching sheet and T_∞ is the temperature of the ambient fluid.

2.2 Analysis

The continuity equation (2.1) is identically satisfied by stream function $\psi(x, y)$, defined as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{2.5}$$

For the solution of the momentum and energy equation (2.2) and (2.3) respectively, the following dimensionless variables are defined:

$$\psi(x, y) = x\sqrt{c\nu}f(\eta), \tag{2.6}$$

$$\eta = y\sqrt{\frac{c}{\nu}}, \tag{2.7}$$

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \tag{2.8}$$

Substituting (2.6) - (2.8) into Eqs. (2.2) and (2.3) we obtain

$$f''' + ff'' - f'^2 - M^2 f' = 0, \tag{2.9}$$

$$\frac{1}{\Pr}\theta'' + f\theta' + Ecf''^2 + M^2 Ecf'^2 = 0. \tag{2.10}$$

The corresponding boundary conditions in non-dimensional form are:

$$\eta = 0: \ f = 0, \quad f' = 1; \quad \theta = 1, \tag{2.11a}$$

$$\eta = \infty: \ f' = 0; \ \theta = 0. \tag{2.11b}$$

where, $M = B_0\sqrt{\sigma_e/\rho c}$ is the Hartmann number, $\Pr = \mu C_p/\kappa$ is the Prandtl number, $Ec = u_w^2/Cp(T_w - T_\infty)$ is the Eckert number and prime denotes differentiation with respect to η. In the next to come we present the solution of Eqs. (2.9) and (2.10) subject to the boundary conditions (2.11). We first present the solution for small values of Hartmann number. For this we write

$$f(\eta) = \sum_{i=0}^{\infty}(M^2)^i f_i(\eta), \tag{2.12}$$

$$\theta(\eta) = \sum_{j=0}^{\infty}(M^2)^j \theta_j(\eta). \tag{2.13}$$

Substituting Eqs. (2.12) and (2.13) and its derivatives into Eqs. (2.9) and (2.10) and then equating the coefficients of like powers of M^2, we get the following set of equations and boundary conditions.

$$f_0''' + f_0 f_0'' - f_0'^2 = 0, \tag{2.14}$$

$$\theta_0'' + \Pr f_0 \theta_0' = -\Pr Ec f_0''^2, \tag{2.15}$$

$$f_1''' + f_0 f_1'' - 2 f_0' f_1' + f_0'' f_1 = f_0', \tag{2.16}$$

$$\theta_1'' + \Pr f_0 \theta_1' = -\Pr f_1 \theta_0' - \Pr Ec(2 f_0'' f_1'' + f_0'^2), \tag{2.17}$$

$$f_2''' + f_0 f_2'' - 2 f_0' f_2' + f_0'' f_2 = -f_1 f_1'' + f_1'^2 + f_1', \tag{2.18}$$

$$\theta_2'' + \Pr f_0 \theta_2' = -\Pr f_1 \theta_1' - \Pr f_2 \theta_0' - \Pr Ec(2 f_0'' f_2'' + f_1''^2 + 2 f_0' f_1'). \tag{2.19}$$

$$\eta = 0: \ f_i = 0, \ f_0' = 1, \quad f_j' = 0; \ \theta_0 = 1, \ \theta_j = 0; \tag{2.20a}$$

$$\eta = \infty: \ f_i' = 0; \ \theta_i = 0; \qquad\qquad i \geq 0, \quad j > 0. \tag{2.20b}$$

The set of Eqs. (2.14)-(2.19) along with boundary conditions (2.20) are solved numerically using shooting method combined with fourth-order Runge-Kutta method.

To obtain an asymptotic solution for large values of M, we first define new functions $F(z)$ and $\phi(z)$ through the following transformations

$$F(z) = M f(\eta), \tag{2.21}$$

$$\phi(z) = \theta(\eta). \tag{2.22}$$

where

$$z = M\eta. \tag{2.23}$$

Substitution of (2.21) - (2.23) into Eqs. (2.9) and (2.10) yield

$$F''' - F' + \epsilon^2(FF'' - (F')^2) = 0, \tag{2.24.}$$

$$\frac{1}{\mathrm{Pr}}\phi'' + Ec(F''^2 + F'^2) + \epsilon^2 F\phi' = 0, \tag{2.25}$$

The boundary conditions (2.11) takes the following form:

$$z = 0;\ F = 0,\ F' = 1;\ \phi = 1, \tag{2.26a}$$

$$z = \infty;\ F' = 0;\ \phi = 0. \tag{2.26b}$$

where $\epsilon = 1/M$ is a small parameter.

Now we expand $F(z)$ and $\phi(z)$ in ascending powers of ϵ as

$$F(z) = \sum_{m=0}^{\infty}(\epsilon^2)^m F_m(z), \tag{2.27}$$

$$\phi(z) = \sum_{n=0}^{\infty}(\epsilon^2)^n \phi_n(z). \tag{2.28}$$

Substitute Eqs. (2.27) and (2.28) and its derivatives in Eqs. (2.24) and (2.25) and then equate the coefficients of like powers of ϵ^2. Thus we get the following set of equations.

$$F_0''' - F_0' = 0, \tag{2.29}$$

$$\phi_0'' = -\mathrm{Pr}\, Ec(F_0''^2 + F_0'^2), \tag{2.30}$$

$$F_1''' - F_1' = F_0'^2 - F_0F_0'', \tag{2.31}$$

$$\phi_1'' = -\mathrm{Pr}\, F_0\phi_0' - \mathrm{Pr}\, Ec(2F_0''F_1'' + 2F_0'F_1'), \tag{2.32}$$

$$F_2''' - F_2' = 2F_0'F_1' - F_0F_1'' - F_0''F_1, \tag{2.33}$$

$$\phi_2'' = -\Pr(F_0\phi_1' + \phi_0'F_1) - \Pr Ec(F_1''^2 + F_1'^2 + 2F_0''F_2'' + 2F_0'F_2'), \tag{2.34}$$

Similarly the boundary conditions at different order become

$$z = 0: \;\; F_i = 0, \;\; F_0' = 1, \;\; F_j' = 0; \;\; \phi_0 = 1, \;\; \phi_j = 0; \tag{2.35a}$$

$$z = \infty: \;\; F_i' = 0; \;\; \phi_j = 0, \qquad\qquad i \geq 0, \;\; j > 0. \tag{2.35b}$$

It is noted that following exact expressions for F_0, F_1 and F_2 exist

$$F_0(z) = e^{-z}(-1 + e^z), \tag{2.36}$$

$$F_1(z) = -\frac{e^{-z}(-1 + e^z - z)}{2}, \tag{2.37}$$

$$F_2(z) = \frac{e^{-z}(-3 + 3e^z - 3y - z^2)}{8}. \tag{2.38}$$

Therefor the exact expression for $F(z)$ for large values of M up to second order become

$$F(z) = F_0(z) + \epsilon^2 F_1(z) + \epsilon^2 F_2(z)... \tag{2.39}$$

It is further noted from Eqs. (2.29)-(2.35) that an asymptotic solution for $\phi(z)$ does not exist.

A numerical solution of the governing Eqs. (2.9)-(2.11) is also obtained using shooting method with Runge-Kutta algorithm.

2.3 Results and discussion

In this section our intention is to present a parametric study in order to illustrate the effects of Hartmann number M, Prandtl number Pr and Eckert number Ec on velocity and temperature distributions. The perturbation solution for small and large values of M is also compared with exact numerical solution. Moreover, the values of skin friction $f''(0)$ and Nusselt number $-\theta'(0)$ are also tabulated in the end.

We start with Figs. 2.1-2.3. Fig. 2.1 shows comparison of numerical solution and perturbation solution (2.12) for velocity distribution $f(\eta)$. It shows that the perturbation solution and numerical solution for fixed values of Ec and Pr exactly match for small values of M. In Fig. 2.2 the temperature field $\theta(\eta)$ obtained via Eq. (2.13) and numerical solution is plotted against η for $M = 0.01$. Here again Pr and Ec are assumed fixed. Again this figure shows a perfect match between both the solutions. Fig. 2.3 present the comparison of asymptotic solution for large values of M and numerical solution for $f(\eta)$. Here again a perfect match between both the solution is observed. The asymptotic solution for $\theta(\eta)$ does not exist and therefore its comparison with the numerical solution is not shown.

Fig. 2.4 is prepared to see the variation of velocity distribution $f(\eta)$ for different values of M. The figure reveals that velocity decreases with an increase in M. The thickness of the boundary layer also decreases for large values of M. However, the temperature $\theta(\eta)$ and thermal boundary layer thickness increase by increasing M. This fact is evident from Fig. 2.5.

The effect of Prandtl number Pr and Eckert number Ec on temperature field $\theta(\eta)$ are illustrated through Figs. 2.6 and 2.7. These figures depict that $\theta(\eta)$ decreases/increases by increasing $\Pr / Ec$

The values of the skin friction coefficient $f''(0)$ and local Nusselt $-\theta'(0)$ are also obtained for fixed values of Pr and Ec. The values of the skin friction coefficient $f''(0)$ are obtained using perturbation solution for small and large values of M and direct numerical solution. The values of Local Nusselt number $-\theta'(0)$ are obtained using perturbation solution for small values of M and direct numerical solution.

These values of $f''(0)$ and $-\theta'(0)$ are presented in **Tables 2.1** and **2.2**. From these tables one can see that $f''(0)$ and $-\theta'(0)$ obtained by perturbation solution are in good agreement with the numerical solution.

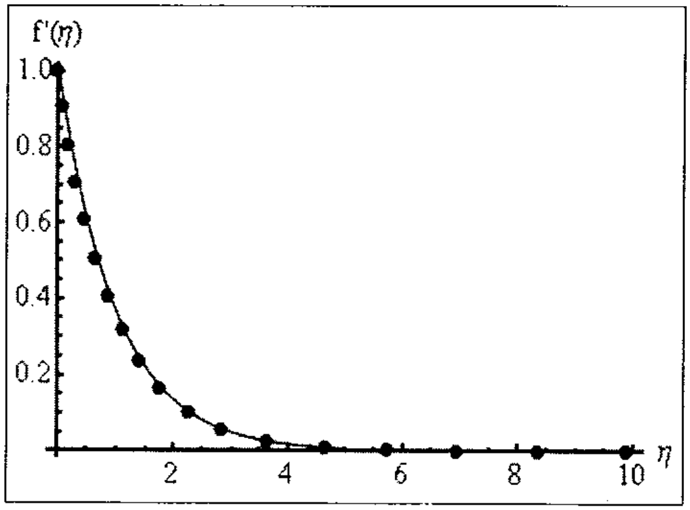

Fig. 2.1: Perturbation solution (dotted line) and numerical solution (solid line) for $f'(\eta)$ with $M = 0.01$, $Ec = 0.2$ and $Pr = 0.7$.

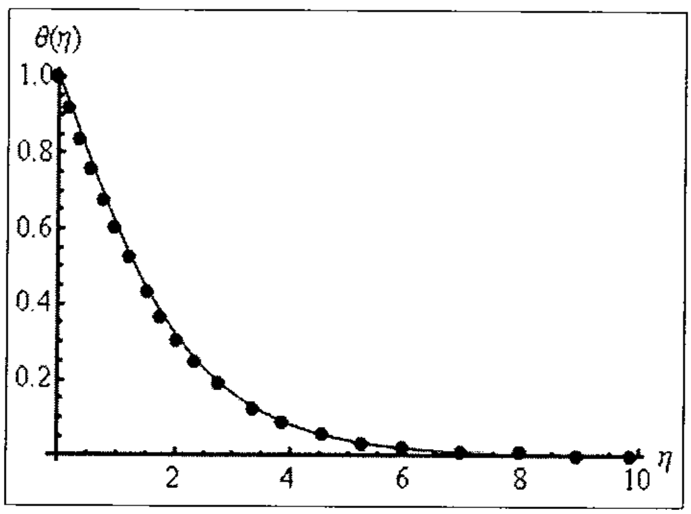

Fig. 2.2: Perturbation solution (dotted line) and numerical solution (solid line) for $\theta(\eta)$ with $M = 0.01$ $Ec = 0.2$ and $Pr = 0.7$.

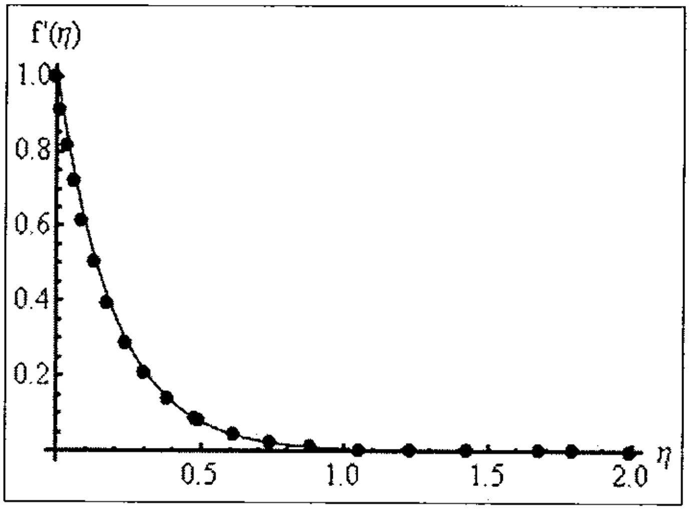

Fig. 2.3: Perturbation solution (dotted line) and numerical solution (solid line) for $f'(\eta)$ with $M = 5$, $Ec = 0.2$ and $Pr = 0.7$.

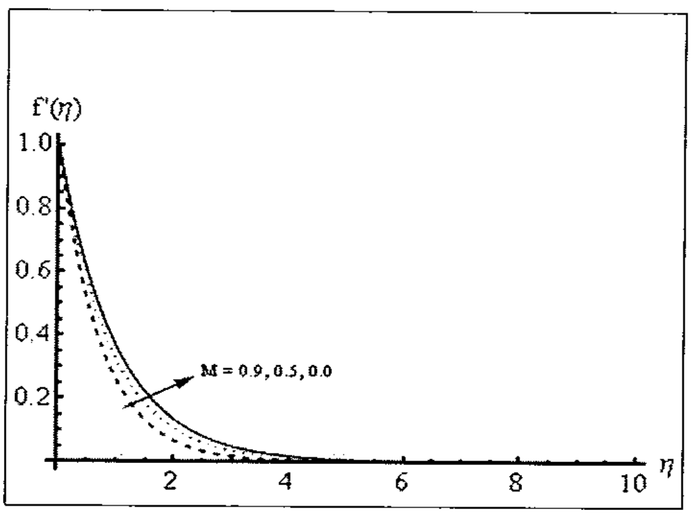

Fig. 2.4: Velocity distribution $f'(\eta)$ against η for various of M with $Ec = 0.2$ and $Pr = 0.7$.

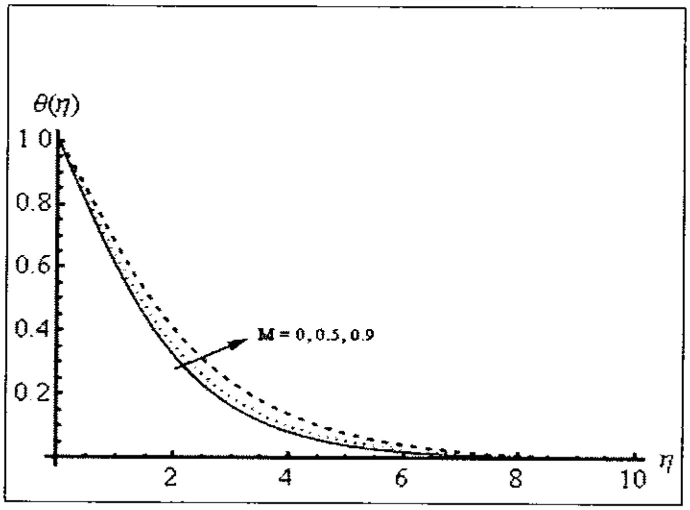

Fig. 2.5: Temperature distribution $\theta(\eta)$ against η for various of M with $Ec = 0.2$ and $Pr = 0.7$.

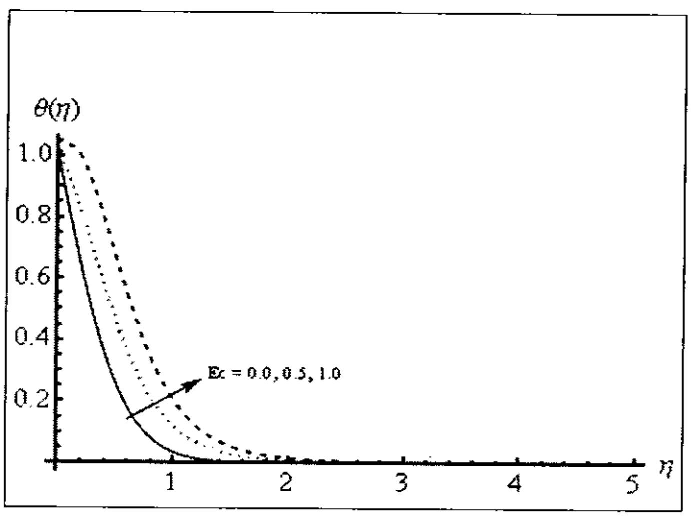

Fig. 2.6: Temperature distibution $\theta(\eta)$ against η for various of Ec with $M = 0.5$ and $Pr = 7.0$.

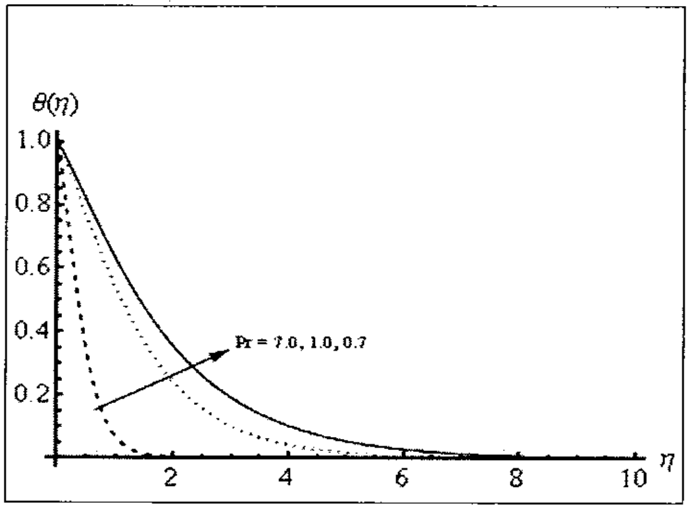

Fig. 2.7: Temperature distibution $\theta(\eta)$ against η for various of Pr with $M = 0.5$ and $Ec = 0.2$.

Table 2.1: Values of $f''(0)$ when $Pr = 7$ and $Ec = 0.2$.

M	Numerical sol.	Small M	Large M
0	-1.0014	-1.0014	
0.1	-1.0063	-1.0063	
0.2	-1.0210	-1.0210	
0.3	-1.0449	-1.0449	
0.4	-1.0777	-1.0776	
0.5	-1.1185	-1.1179	
1	-1.1442		
2	-2.2370		
4	-4.1231		
10	-10.0499		-10.0375
100	-100.005		-100.004
1000	-1000.00		-1000.00
10000	-10000.00		-10000.00

Table 2.2: Values of $-\theta'(0)$ when $Pr = 7$ and $Ec = 0.2$.

M	Numerical sol.	Small M
0	1.45524	1.45524
0.1	1.4469	1.4469
0.2	1.4221	1.4221
0.3	1.3815	1.3816
0.4	1.3261	1.3264
0.5	1.2571	1.2581
0.6	1.1760	
0.8	0.9828	
1	0.7569	
2	-0.6389	
4	-3.7883	
6	-6.7351	
8	-9.4010	
10	-12.0717	

Chapter 3

An analysis of magnetohydrodynamic boundary layer flow and heat transfer over a porous stretching sheet

The aim of this chapter is to extend the analysis performed in chapter 2 for a porous stretching plate. The solution of the governing problem is presented using perturbation and numerical techniques. A comparison of perturbation solution and numerical solution is made. Finally the effects of suction and injection on velocity and temperature fields are illustrated through graphs.

3.1 Problem description

The geometry and the underlying assumptions for the flow under consideration are almost same as described in chapter 2 except that the stretching sheet here is assumed to be porous so that suction or injection is possible.

We start with the same governing equations as used in chapter 2 i.e., Eqs. (2.1)-(2.3). The second boundary condition of (2.4a) is modified to take into account suction and injection at the plate surface i.e.,

$$y = 0: \quad v = -V_0, \tag{3.1}$$

Where $V_0 > 0$ corresponds to suction and $V_0 < 0$ represents injection. Rest of boundary conditions remains unaltered.

3.2 Solution of the problem

The transformed problem after using the stream function and dimensionless parameters defined through Eqs. (2.5)-(2.8) remain the same as described in Eqs. (2.9)-(2.11) except that the first boundary condition of (2.11a) take the following form

$$\eta = 0: \quad f = \gamma, \tag{3.2}$$

Where $\gamma = v_0/\sqrt{c\nu}$ is the dimensionless suction/injection parameter. Therefore, the governing equations and boundary conditions for the flow under consideration read

$$f''' + ff'' - f'^2 - M^2 f' = 0, \tag{3.3}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + f\theta' + Ecf''^2 + M^2 Ecf'^2 = 0, \tag{3.4}$$

$$\eta = 0: \quad f = \gamma, \; f' = 1; \theta = 1, \tag{3.5a}$$

$$\eta = \infty: \; f' = 0; \; \theta = 0. \tag{3.5b}$$

In the coming part of the chapter we explain the solution methodology for the problem consisting of Eqs. (3.3)-(3.5) without referring back to chapter 2 and making the reader inconvenient.

To obtain the perturbation solution for small values of M we write

$$f(\eta) = \sum_{i=0}^{\infty} (M^2)^i f_i(\eta), \tag{3.6}$$

$$\theta(\eta) = \sum_{j=0}^{\infty} (M^2)^j \theta_j(\eta). \tag{3.7}$$

Substitution of above expansions in (3.3)-(3.5) leads to following equations at zeroth order, first order and second order respectively

$$f_0''' + f_0 f_0'' - f_0'^2 = 0, \tag{3.8}$$

$$\theta_0'' + \Pr f_0 \theta_0' = -\Pr Ec f_0''^2, \tag{3.9}$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = f_0', \tag{3.10}$$

$$\theta_1'' + \Pr f_0 \theta_1' = -\Pr f_1 \theta_0' - \Pr Ec(2f_0'' f_1'' + f_0'^2), \tag{3.11}$$

$$f_2''' + f_0 f_2'' - 2f_0' f_2' + f_0'' f_2 = -f_1 f_1'' + f_1'^2 + f_1', \tag{3.12}$$

$$\theta_2'' + \Pr f_0 \theta_2' = -\Pr f_1 \theta_1' - \Pr f_2 \theta_0' - \Pr Ec(2f_0'' f_2'' + f_1''^2 + 2f_0' f_1'), \tag{3.13}$$

Similarly boundary condition at various order become

$$\eta = 0: \; f_i = \gamma, \; f_0' = 1, \quad f_j' = 0; \; \theta_0 = 1, \; \theta_j = 0; \tag{3.14a}$$

$$\eta = \infty: \; f_i' = 0; \; \theta_i = 0; \qquad\qquad i \geq 0, \quad j > 0. \tag{3.14b}$$

The solution of equations at various order is obtained by shooting method with Runge-Kutta algorithm.

In order to obtain solution for large value of M, we first define the new function

$$F(z) = Mf(\eta), \tag{3.15}$$

$$\phi(z) = \theta(\eta), \tag{3.16}$$

where

$$z = M\eta. \tag{3.17}$$

Eqs. (3.3)-(3.5) in terms of new functions become

$$F''' - F' + \epsilon^2(FF'' - (F')^2) = 0, \tag{3.18}$$

$$\frac{1}{\mathrm{Pr}}\phi'' + Ec(F''^2 + F'^2) + \epsilon^2 F\phi' = 0, \tag{3.19}$$

$$z = 0;\ F = \frac{\gamma}{\epsilon},\ F' = 1;\ \phi = 1, \tag{3.20a}$$

$$z = \infty;\ F' = 0;\ \phi = 0. \tag{3.20b}$$

where $\epsilon = 1/M$. For convenience we put $\gamma = \epsilon = 1/M$ and thus (3.20a) and (3.20b) become

$$z = 0;\ F = 1,\ F' = 1;\ \phi = 1, \tag{3.21a}$$

$$z = \infty;\ F' = 0;\ \phi = 0. \tag{3.21b}$$

Now expanding $F(z)$ and $\phi(z)$ in powers of ϵ^2 as

$$F(z) = \sum_{m=0}^{\infty}(\epsilon^2)^m F_m(z), \tag{3.22}$$

$$\phi(z) = \sum_{n=0}^{\infty} (\epsilon^2)^n \phi_n(z), \tag{3.23}$$

and inserting them in Eqs. (3.18), (3.19) and (3.21) yield following equations and boundary conditions.

$$F_0''' - F_0' = 0, \tag{3.24}$$

$$\phi_0'' = -\Pr Ec(F_0''^2 + F_0'^2), \tag{3.25}$$

$$F_1''' - F_1' = F_0'^2 - F_0 F_0'', \tag{3.26}$$

$$\phi_1'' = -\Pr F_0 \phi_0' - \Pr Ec(2F_0'' F_1'' + 2F_0' F_1'), \tag{3.27}$$

$$F_2''' - F_2' = 2F_0' F_1' - F_0 F_1'' - F_0'' F_1, \tag{3.28}$$

$$\phi_2'' = -\Pr(F_0 \phi_1' + \phi_0' F_1) - \Pr Ec(F_1''^2 + F_1'^2 + 2F_0'' F_2'' + 2F_0' F_2'). \tag{3.29}$$

$$z = 0: \; F_0 = 1, \; F_j = 0, \; F_0' = 1, \; F_j' = 0; \; \phi_0 = 1, \; \phi_j = 0, \tag{3.30a}$$

$$z = \infty: \; F_i' = 0; \; \phi_j = 0 \qquad\qquad i \geq 0, \; j > 0. \tag{3.30b}$$

Solving the above equation yield the following values of $F_0(z)$, $F_1(z)$ and $F_2(z)$

$$F_0(z) = e^{-z}(-1 + 2e^z). \tag{3.31}$$

$$F_1(z) = e^{-z}(-1 + e^z - z). \tag{3.32}$$

$$F_2(z) = \frac{e^{-z}(-2 + 2e^z - 2z - z^2)}{2}. \quad (3.33)$$

and hence

$$F(z) = e^{-z}(-1 + 2e^z) + \epsilon^2[e^{-z}(-1 + e^z - z)] + \epsilon^4[\frac{e^{-z}(-2 + 2e^z - 2z - z^2)}{2}] \ldots \quad (3.34)$$

The temperature field however is singular for large values of M which is evident from (3.25), (3.27) and (3.29). A numerical solution of the consisting of Eqs. (3.3)-(3.5) valid for all values of M is also obtained using shooting method and compared with the approximate solution.

3.3 Results and discussion

In this section we give a comparison of numerical solution and approximate solution for non zero values of suction/injection parameter γ. We further illustrate effects of various parameters of interest on velocity and temperature fields through graphs. In the end we tabulate the values of skin friction and local Nusselt number for different values of γ and M.

Fig. 3.1-3.3 shows that approximate solution for $f(\eta)$ and $\theta(\eta)$ is in excellent agreement with the numerical solution and thus approximate solution can be used with a full confidence. Fig. 3.4 presents the variation of $f(\eta)$ for different values of γ. It is noted from this figure that suction reduces the velocity and boundary layer thickness. However, injection has effects opposite to that of suction on velocity boundary layer thickness. The variation of $\theta(\eta)$ for various values of γ can be seen through Fig. 3.5. It is observed that the effects of γ on $\theta(\eta)$ and $f(\eta)$ are similar.

The values of skin friction $f''(0)$ and local Nusselt number $-\theta(\eta)$ are tabulated for various values γ and M in **Tables 3.1-3.4.** These tables on one hand demonstrate the validity of approximate solution and on the other hand give information about behavior of skin friction and local Nusselt number. One can see from these tables that magnitude of skin friction and local Nusselt number increases for large values of γ.

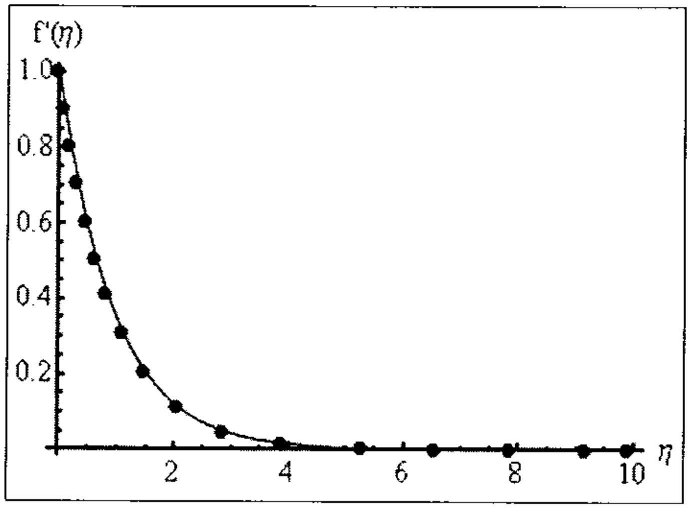

Fig. 3.1: Perturbation solution (dotted line) and numerical solution (solid line) for $f'(\eta)$ with $M = 0.01$, $Ec = 0.2$, $Pr = 0.7$ and $\gamma 0.1$

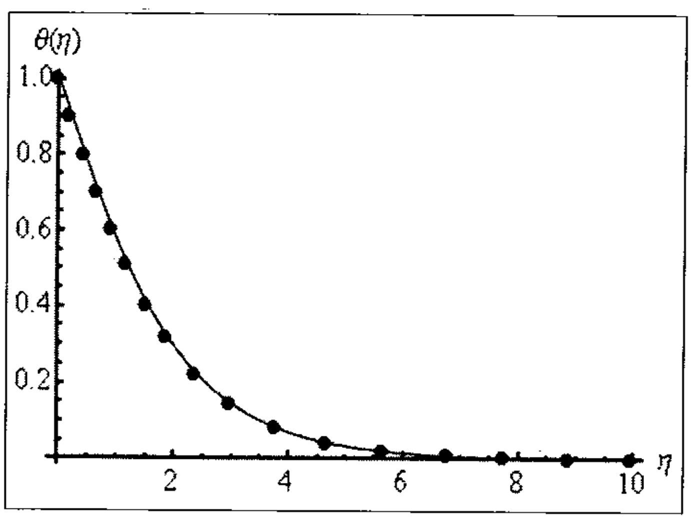

Fig. 3.2: Perturbation solution (dotted line) and numerical solution (solid line) for $\theta'(\eta)$ with $M = 0.01$, $Ec = 0.2$, $Pr = 0.7$ and $\gamma = 0.1$.

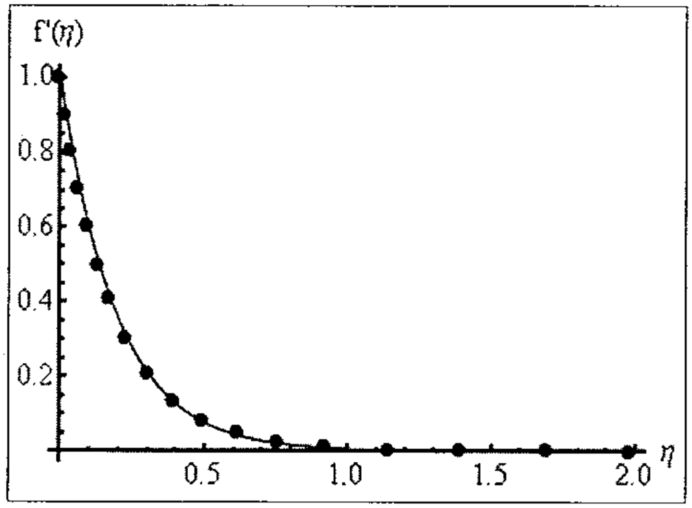

Fig. 3.3: Perturbation solution (dotted line) and numerical solution (solid line) for $f'(\eta)$ with $M = 5$, $Ec = 0.2$, $Pr = 0.7$ and $\gamma = 0.1$

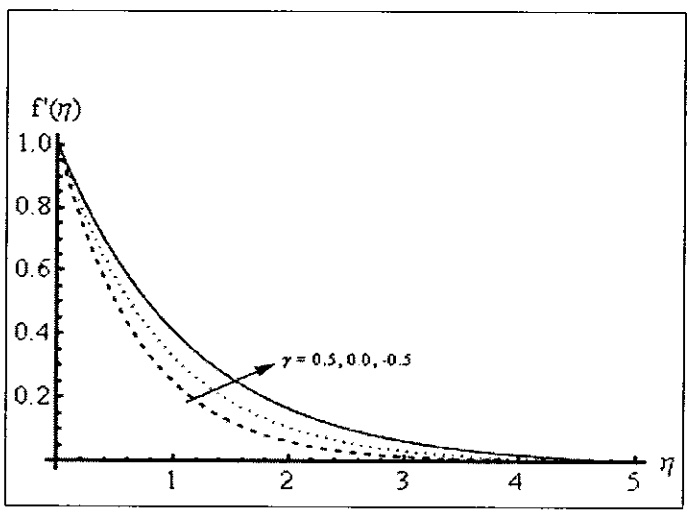

Fig. 3.4: Velocity distribution against η for different γ with $Ec = 0.2$, $Pr = 0.7$ and $M = 0.5$

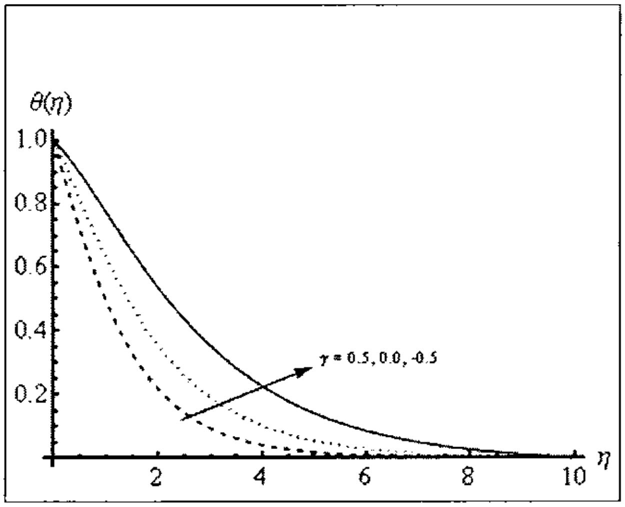

Fig. 3.5: Temperature distribution against η for different γ with $Ec = 0.2$, $Pr = 0.7$ and $M = 0.5$

Table 3.1: Values of $f''(0)$ when $Pr = 7$, $Ec = 0.2$.

M	γ	Numerical sol.	Small M	Large M
0	0.1	-1.0525	-1.0525	
0.1	0.1	-1.0574	-1.0574	
0.2	0.1	-1.0721	-1.721	
0.3	0.1	-1.0961	-1.0961	
0.4	0.1	-1.1288	-1.1287	
0.5	0.1	-1.1696	-1.1690	
1	0.1	-1.4651		
2	0.1	-2.2866		
4	0.1	-4.1734		
10	0.1	-10.1		-10.1
100	0.01	100.01		-100.01
1000	0.001	-1000.00		-1000.00
10000	0.0001	-10000.00		-10000.00

Table 3.2: Values of $-\theta'(0)$ when $Pr = 7$, $Ec = 0.2$ and $\gamma = 0.1$.

M	Numerical sol.	Small M
0	1.8777	1.8777
0.1	1.8693	1.8693
0.2	1.8443	1.8443
0.3	1.8035	1.8035
0.4	1.7478	1.7481
0.5	1.6785	1.6796
0.6	1.5972	
0.8	1.4039	
1	1.1785	
2	-0.2014	
4	-3.3008	
6	-6.3051	
8	-9.1235	
10	-11.7786	

Table 3.3: Values of $f''(0)$ and $\theta'(0)$ when $\Pr = 7,\ Ec = 0.2$ and $M = 0.9$ for different γ.

γ	$f''(0)$	$-\theta(0)$
-0.5	-1.1193	-0.3513
-0.4	-1.1612	-0.2435
-0.3	-1.2047	-0.0652
-0.2	-1.2500	0.1839
-0.1	-1.2972	0.4989
0.0	-1.3463	0.8715
0.1	-1.3972	1.2929
0.2	-1.4499	1.7545
0.3	-1.5045	2.2488
0.4	-1.5609	2.7694
0.5	-1.6191	3.3112

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