Impact of Human Capital Upon Economic Growth in Pakistan; Bayesian Econometric Analysis

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بسم الله الرحمن الرحيم

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
MASTER OF SCIENCE IN STATISTICS

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Certificate

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF SCIENCE IN STATISTICS

We accept this dissertation as conforming to the required standard.

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2017

Dedication

To my loving Parents Father and Mother,
For the endless support and patience.

To my Teachers,
For the constant source of Knowledge and Inspiration.

To my friends,
The ones that are close and the ones that are far.

Forwarding Sheet by Research Supervisor

The thesis entitled **"Impact of Human Capital Upon Economic Growth in Pakistan; Bayesian Econometric Analysis"** submitted by **Muhammad Tariq** (Registration # 58-FBAS/MSST/F14) in partial fulfillment of M S degree in Statistics has been completed under my guidance and supervision I am satisfied with the quality of his research work and allow him to submit this thesis for further process to graduate with Master of Science degree from Department of Mathematics and Statistics, as per IIU Islamabad rules and regulations

Dated ____________________

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At this occasion, I am thankful to all my teachers who taught me at any stage of my studies Especially, I express my sincere appreciation to all the helpful staff of the Mathematics and Statistics Department of IIUI I would like to thanks to everybody who was important to successful realization of this thesis as well as expressing my apology to those that I could not mention I am thank full to Prof Dr **Zahid Iqbal**, Dr. **Ishfaq Ahmad** and Dr **Baber Waseem** for enlighten my statistics learning during my course work My great thanks are extended to all the respected teachers specially **Kousar Abbas** (G.P G C Chakwal) I am also thankful from the core of my heart to youngest my brothers, **Muhammad Bilal, Muhammad Shoukat, Muhammad Abid** and **Muhammad Sajid** who always pray for my success. I apologize to the people whose company and moral support I enjoyed but their names I forgot to include in this document I especially thank my father for always believing me and my mother for her non ending prayers for me. Writing the thesis without my parents support was really unthinkable for me

Muhammad Tariq RajpooT

DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor **Dr. Muhammad Akbar** No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute

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TABLE OF CONTENTS

List of Abbreviations

GDP	Gross Domestic Product
ACF	Auto Correlation Function
MCMC	Markov Chain Monte Carlo
MC error	Monte Carlo Error
VECM	Vector Error Correction Model
GLM	Generalized Linear Model
ADF	Augmented Dicky Fuller Test
UR Test	Unit Root Test
SD	Standard Deviation
CUSUM	Cumulative Sum Plot
ECM	Error Correction Model
BGR Test	Brooks Gelman Rubin Test

LIST OF TABLES

LIST OF FIGURES

ABSTRACT

Application of Bayesian econometric techniques to analyze real economic phenomena are not common and limited in the literature This study is conducted to investigate the impact of human capital trade openness and physical capital on economic growth of Pakistan by using Bayesian econometric techniques Data is covering 50 years from 1965 to 2014 Human capital is represented by education and health capital Model is estimated using non-informative priors, i e Jeffrey's priors, uniform priors and weekly informative priors as well as informative priors elicited on the basis of informative priors Results of the four estimated models are compared Three major conclusions are drawn Firstly, use of Jeffrey's priors gives larger precision as compared to uniform priors and weekly informative priors Secondly, incorporation of prior information on the basis of experts' knowledge increases precision of estimates as well as overall model Thirdly, coefficients' estimates of all explanatory variables show significant positive impact on economic growth in Pakistan Hence, human capital can be considered as a vital factor to achieve long-run economic growth in Pakistan

However, all these empirical studies contain econometric analysis based on classical regression techniques To the best of our knowledge, there is not a single study in the literature that analyses impact of human capital upon economic growth by using Bayesian regression techniques Bayesian inference is considered as superior to the Classical inferential approach Bayesian approach considered parameters as random variables and hence are estimated on the basis of posterior probability distributions Posterior distribution is constructed on the basis of sample data as well as prior information other than sample information. The Bayesian methodology adopts that the useful information on the sample is constant and that the model parameters are stochastic The posterior distribution of parameters is constructed on the real data and the prior distribution of parameters On the hand, Classical approach considered parameters as constants and are estimated on the basis of sample data The frequentist methodology adopts that the applied data is a random sample and parameters are unknown but stable and constant through the repeated samples Estimates are obtained on the basis of sampling distribution of the information

Bayesian analysis responses problems constructed on the distribution of constraints conditional on the detected sample, but frequentist examination replies complications based on the distribution of information attained from frequent hypothetical samples. which would be produced by the identical procedure that formed the detected sample specified that parameters are unidentified but fixed This supposition may not constantly be achievable Frequentist examination is fully data-driven but Bayesian analysis implements a further robust approximation methodology by using not simply the data at the indicator, but also about present evidence or information about model limitations Bayesian inference is constructed on the basis of posterior distribution of the parameters and gives summaries of the distribution, containing posterior means along with MC standard errors and credible intervals Credible intervals are interpreted with probability statement. However, Frequentist confidence intervals cannot be interpreted with probabilistic explanations as do Bayesian credible intervals

The discussion may be summarized that the Bayesian inferential approach has some superior features as compare to Classical Inferential approach Hence, application of Bayesian technique to analyse the impact of human capital along with other components of Cob-Douglas production function on economic growth may be a significant contribution in the literature of economic growth as well as Applied Statistics.

1.1. Objectives of the study

Keeping in view of the above discussions, objectives of the study are specified as follows

1. Specification of model to analyze the impact of human capital on economic growth
2. Elicitation of hyper parameters on the basis of experts (Dummy) information
3. Bayesian estimation of the model under Jeffery's priors, Uniform priors, Weekly informative priors and Informative priors
4. Comparison of the results obtained under different priors
5. Policy implications

1.2. Plan of the Study

To achieve the above specified objectives this study has been divided into five chapters After introduction of the topic in first chapter a detailed review of literature is presented in Chapter 2 It follows Chapter 3 which contains material and methodology

This chapter presents a detailed discussion about specification of the model, data to be used in the study, derivation of posterior distributions, elicitation of hyper parameters and procedure of Markov Chain Monte Carlo (MCMC) simulations Chapter 4 presents estimation results along with discussion Chapter 5 contains summary of the study, concluding remarks, policy implications on the basis of findings and future research work on the topic

CHAPTER 2

LITERATURE REVIEW

2.1. Introduction

In this chapter, we briefly explain the previous accessible studies related to our study Section 2 contains review of studies analyzing impact of human capital on economic growth Section 3 presents some basic concepts related to Bayesian inference A review of empirical studies regarding application of Bayesian econometric techniques is presented in Section 4. The last Section 5 contains concluding remarks of the chapter

2.2. Review of the Literature

Shah *et al* (2015) investigated the impact of human investment on economic growth using the data of selected Asian countries Human capital consists of government expenditures on health and gross school enrolment Besides human capital, the study uses labour force, Gross Foxed Capital Formation and personal remittances as explanatory variables The study contains descriptive analysis, correlation analysis and estimated fixed as well as random effect models The study concludes that human capital significantly and positively affects economic growth in the countries

Karimzadeh and Karimzadeh (2013) presents a model of economic growth where trade openness, human capital, foreign direct investment, exchange rate and domestic investment are taken as the explanatory variables Data for the study has been taken from a hand book of India economy and economy survey for 1980-2011 Using Ordinary Least Squares (OLS) technique, the study conclude that trade openness and human capital significantly and positively affect economic growth

Jadoon *et al* (2015) analyses the impact of trade liberalization and human capital on economic growth This study utilizes time series data of Pakistan economy for the span of 1971-72 to 2009-10 This study used the classical econometric technique (ARDL) for both long run and short run The study concludes that education effects has significant positive but poverty has significant inverse impact on economic growth Almasi *et al* (2015) estimates a model of emerging economies taking GDP growth as a dependent variable while production gap, stability of the human capital, the structure of human capital, physical capital, and the rate of productivity as explanatory variables Classical econometric techniques have applied and it is concluded that all the explanatory variables have positive and significant impact on economic growth Javed *et al* (2013) examined the impact of human capital development on economic growth of Pakistan The study estimates error correction model (ECM) on the basis of classical approach Public expenditures on health, education, primary and secondary school memberships and labour force are taken as explanatory variables All the explanatory variables show positive and significant impact on economic growth Asghar *et al* (2012) conducted co-integration and causality analysis in order to examine the impact of human capital on economic growth Classical approach is followed by using the data from 1974-2009 of Pakistan economy Education index and health index represents human capital Vector error correction model (VECM) based on classical approach is followed The study determines significant and positive role of human capital for economic growth in Pakistan Maria *et al* (2013) estimates ECM based on classical approach by taking economic growth as the dependent variable and public expenditures on health, education, primary and secondary school enrolments, total investment, and labour force as explanatory variables The study uses time series data from

1978 to 2008 of Pakistan economy The results show that all explanatory variables have significant and positive impact on economic growth in Pakistan Abbas and Foreman (2008) analysed the association regarding human capital and economic growth in Pakistan with the time series data for the period of 1960-2003 ECM is estimated based on the basis of classical approach and the results show that human capital significantly and positively affect economic growth in Pakistan

Review of above studies show that all studies regarding human capital and economic growth have been conducted using classical econometric techniques No such study is available in the literature that has been estimated on the basis of Bayesian approach Moreover, all studies show that human capital has significant and positive impact on economic growth

2.3. Introduction to Bayesian Analysis

Bayesian technique based on Bayes Theorem and is considered as superior to classical technique due to the combination of prior information The following points illustrate the essential elements of Bayesian inference

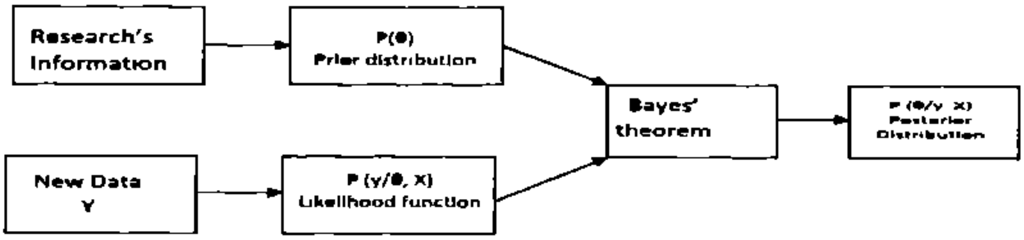
1. Construction of likelihood function
2. Formulation of prior distribution of unknown θ, P(θ)
3. Derivation of posterior distribution P(θ/Y) by updating our beliefs about θ by combining information from prior distribution and likelihood function

Bayes' theorem enables us to combine the prior distribution and the model information in the following way

$$P(\theta/Y)=P(Y,\theta)/(p(Y)=\frac{p(Y/\theta)*p(\theta)}{p(Y)}=\frac{p(Y/\theta,x)\,p(\theta)}{\int P(\frac{Y}{\theta})p(\theta)d\theta}\propto p(\theta)\,p(Y/\theta,X)$$

Where the quantity

$$P(Y) = \int p(Y/\theta)\, p(\theta)$$



2.3.1. Meanings and Objectives of Priors

A prior distribution represents vagueness about the parameter before the current data are studied Multiplying the prior distribution and the likelihood function gives posterior distribution of the parameter The posterior distribution used to bring out all inferences Priors may be of two types, i e non-informative priors and informative priors When no information are available about unknown parameters, then non-informative priors like Jeffrey's or uniform priors are used to apply Bayesian inference

An informative prior is a prior that is not dominated by the likelihood and that has a control on the posterior distribution An informative prior is a precise demonstration of prior beliefs when we are concerned in a prior existence part of some conjugate family Elicitation, in this situation, is the method of converting someone's beliefs into a prior distribution of parameters These are sensible priors to use if one has real prior information from an earlier comparable study Informative priors must be identified with care in actual practice Otherwise, we can get misleading results Sometimes it is superior to avoid completely flat/smooth priors, especially if flat priors lead to effectively improper posterior densities or poor identification of the parameters The key source of informative prior are preceding studies, published work, interviewing crucial experts Also nonparametric and other data derived sources, which can obviously be overlapping definitions

A non-informative prior or diffuse prior express vague or general information about the parameters A prior is non informative if it is flat to the likelihood function A non-informative prior P(θ) is non informative when it has minimal impact on the posterior of there is absence of prior beliefs in Bayesian estimation The estimator converted a function of the likelihood only A frequent non informative uniform prior π (θ) ∝ 1, which assigns equal likelihood to all possible values of the parameter Similarly other non-informative priors like reference priors, Jeffrey prior and weak information prior are called diffuse prior

Jeffery prior is invariant to a parameter transformation and are chosen in such a way that the prior is proportional to the square root of the Fisher information matrix representatively, the rule is taken as

$$P(\theta) \propto I(\theta)$$

Where I(θ) is the Fisher information function The Fisher information is a very significant conception introduced by Fisher in 1922 This rule turns out to be transformation invariant There are various reasons for thinking that this prior might be a useful prior but

$$I(\theta) = -E_\theta\left[\frac{d^2 \log p(\frac{X}{\theta})}{d\theta^2}\right]$$

$$\text{Hence, } p(\theta) \propto \sqrt{I(\theta)}$$

A prior is said to be a conjugate prior for a family distribution, if the prior and posterior distributions are from the same family In Conjugate priors, prior and posterior both have same class of distributions A prior is conjugate for a family of distributions if the prior and the posterior are of the same family The conjugate prior for normal is gamma for variance unknown case and for means normal, that is called normal gamma conjugate priors

2.3.2. Posterior Distribution

Posterior distribution is derived by multiplying the likelihood function with prior distribution i.e

$$Posterior \propto Prior \times Likelihood$$

The posterior distribution is used for estimation of parameters and for prediction All Bayesian inference is based on the posterior distribution Bayesian inference is achieved through Monte Carlo integration where samples of posterior distribution can be obtained from MCMC simulation

2.4. Review of Bayesian Analysis

Tiao and Zellner (1964) demonstrates that prior knowledge can be incorporated with sample information constructing inferences about the parameters of the model of regression The main concern of the study is to make improvement in precision by incorporating prior information The two samples are used are drawn from the normal population with bias variances The posterior distribution formed in section two is a product of multivariate normal and multivariate-t form

Zellner (1983) presents the applications of Bayesian Analysis in Econometrics This study points out that diffuse prior are very useful and also suggests that reference informative Priors (RIPs) will probably be useful too The study describes the procedure for formulating RIPs for regression models. For complicated likelihood functions, numerical regression techniques have been very helpful in analysing PDF and checking the legality of asymptotic and other approximations

Chen and Deely (1996) applied Bayesian model for linear regression for problem of estimation The constraints arise naturally in the context of expecting the coming crop of apples for the year onward In this study, the Bayesian method with Gibbs sampler to determine the solution of joining issues connected with Bayesian examination The Bayesian methodology with the Gibbs sampling is shown to be particularly suited to the constrained problem. Using just one Gibbs sample, it is possible to obtain the Bayesian estimates of model parameters, marginal posterior density estimates and Bayesian predictions Alternative methods such as ordinary and inequality constrained least square estimations are investigated, and comparison among Bayesian, ordinary, and inequality least square estimations are also used

Sinay and Hsu (2014) studied the specification of flexible prior for the covariance structure of multivariate regression model The study discussed the Bellman's solution of linear Volterra integral equation The study discusses the posterior mean of covariance structure and prior information in the restricted data Posterior estimates calculated by numerical results of MHWG procedure MH algorithm applied to take sampling from posterior distribution of the covariance structure Rashwan and Salem (2014) present the Bayesian method to estimate the parameters of regression instead of classical method The study concluded that the standard deviation is less than of regression estimates in case of non-informative prior Song and Xia (2016) studies the generalized framework of Bayesian linear regression with Gaussian assumptions to student-t linear regression assumptions In the frame work, both conjugate prior and expectation maximum algorithms are generalized The study concluded that Bayesian linear regression with Gaussian estimates are identical to student-t estimates

Sereno (2016) studied the Bayesian linear regression model that involve a multiple number of parameters In this study the analysis of the hierarchical models performed with the help of MCMC simulations and all the models are demonstrated as conditional probabilities, the posterior estimates can be efficiently explored with Gibbs sampling The degree to select the best hierarchical can be effected by current and future samples information. The R-package LIRA is used for analysis of the hierarchical analysis

CHAPTER 3

Material and Methodology

3.1. Introduction

This chapter contains specification of the model along with data sources in section 2 Section 3 presents derivation of likelihood function Posterior distributions under different priors are derived in Section 4 For the purpose of estimation of the model under Bayesian framework, elicitation of hyper parameters is presented in Section 5 Section 6 contains discussion about methodology of estimation and diagnostic tests used in the study The last section presents conclusion of the chapter

3.2. Model's specification

This section contains discussion about model's specification and data to be used in the study Model is specified on the basis of extended Solow growth model by introducing human capital along with physical capital in the model Final form of the model is derived like as Rahman (2011) Assuming Cobb-Douglass production function i e

$$Y(t) = K(t)^{a} A(t) L(t)^{1-a}$$

where $Y(t)$ represent GDP at factor cost, $K(t)$represent physical capital, L(t) represent Labor force available and A(t) represent technology at different time periods Dividing L(t) on both sides gives us the following equation for output per unit of labor

$$y(t) = k(t)^{a} \ A(t)$$

Since 1990, country-wise Human development index has been constructing by UNDP on the basis of three indicators, i e "average life" of a new-born person, literacy rate and living standard According to (Keskin 2011·128), the concept of human capital includes the information, skills, abilities, experiences as well as physical and mental fitness or strength of the individuals These views are supported by many other studies in the literature, (e.g., Bloom/Canning/Sevilla 2001) Hence, Human capital can be included by introducing education capital and health capital in the above production function That is,

$$y(t) = k(t)^{\alpha}E(t)^{\beta}h(t)^{\gamma}.A(t) \text{--} (3\ 1)$$

Making transformation and adding trade openness as control variable to capture the effect of foreign sector, the above function can be written as follows after some simplifications

$$ln\big(y(t)\big) = \beta_0 + \beta_1 ln\,[k(t)] + \beta_2 ln[E(t)] + \beta_3 ln[h(t)] + \beta_4 Q(t) + \in (t) \text{--}(3\ 2)$$

The above theoretical specification may be written as follows

$$Y_t = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \in (t) \quad \text{--------------------------------}$$

(3 3)

Here, GDP per worker = Y_t, Trade openness = X_1, Education capital = X_2 Physical capital = X_3 and Health capital = X_4

The study uses time series annual data of Pakistan's Economy from years 1965 to 2014 The secondary data used in the analysis was obtained from *Pakistan Economic Survey* Data of Real GDP at factor cost and total population is taken to obtain GDP per worker Total trade as percentage of GDP is taken as proxy for trade openness Literacy rate is used as proxy for education capital Health expenditure is taken as proxy for health capital where as physical capital is derived from the data of investment.

3.3. Derivations of Likelihood

Consider the above specified model to derive likelihood function

$$Y_t = \beta_0 + \beta_2 X_{t1} + \beta_3 X_{t2} + \beta_4 X_{t3} + \beta_5 X_{t4} + \varepsilon_t, \qquad \text{Where } t=1,2,3, \quad ,n$$

The model can be written more compactly in matrix representation as follows

$$Y_t = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \\ y_n \end{bmatrix} \quad X=\begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} & x_{41} \\ 1 & x_{12} & x_{22} & x_{32} & x_{42} \\ 1 & x_{13} & x_{23} & x_{33} & x_{43} \\ & & & & \cdot \\ \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 1 & x_{1N} & x_{2N} & x_{3N} & x_{4N} \end{bmatrix} \beta = \begin{bmatrix} \beta_0 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} \quad \varepsilon = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \cdot \\ \\ e_n \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \\ \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_{11} & x_{21} & x_{31} & x_{41} \\ 1 & x_{12} & x_{22} & x_{32} & x_{42} \\ 1 & x_{13} & x_{23} & x_{33} & x_{43} \\ \cdot & & & & \cdot \\ & \cdot & \cdot & & \\ & & & \cdot & \\ 1 & x_{1N} & x_{2N} & x_{3N} & x_{4N} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \\ \cdot \\ e_n \end{bmatrix}$$

$$Y = X\beta + \varepsilon$$

Where Y=n×1 vector of explanations

X=n×5 matrix, with rank of observation on 4 independent variables

β=5×1 vector of regression of co-efficient

ε= n×1 vector of disturbance terms

Assuming that $\epsilon_t \sim N(0, \sigma^2)$

$$P\left({}^{\varepsilon}/_{Y,X}\right) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp[\frac{-1}{2\sigma^2}(\epsilon'\epsilon)]$$

$$P(Y/X, \beta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp[\frac{-1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)]$$

The likelihood function will be as follows

$$L(Y) = P(Y/X, \beta, \tau) = (\frac{\sqrt{\tau}}{\sqrt{2\pi}})^n \exp[\frac{-\tau}{2}(Y - X\beta)'(Y - X\beta)]$$

Considering the exponent term while adding and subtracting $\hat{Y}$.

$$(Y - X\beta)'(Y - X\beta) = (Y - \hat{Y} + \hat{Y} - X\beta)'(Y - \hat{Y} + \hat{Y} - X\beta)$$

$$= (Y - X\hat{\beta} + X\hat{\beta} - X\hat{\beta})'(Y - X\hat{\beta} + X\hat{\beta} - X\beta)$$

$$= [(Y - X\hat{\beta}) - X(\beta - \hat{\beta})]'[(Y - X\hat{\beta}) - X(\beta - \hat{\beta})]$$

$$= (Y - X\hat{\beta})'(Y - X\hat{\beta}) - (Y - X\beta)'X(\beta - \hat{\beta}) - (\beta - \hat{\beta})'X'(Y - X\hat{\beta}) + (\beta - \hat{\beta})'X'X(\beta - \hat{\beta}) \text{------------------------------------- } (3\ 4)$$

$$(Y - X\beta)'(Y - X\beta) = (n - k)s^2 + (\beta - \hat{\beta})'X'X(\beta - \hat{\beta}) + 0$$

$$(Y - X\beta)'(Y - X\beta) = vs^2 + (\beta - \hat{\beta})'X'X(\beta - \hat{\beta})$$

where $v = n - k$ and $vs^2 = (Y - X\hat{\beta})'(Y - X\hat{\beta})$

In the above equation (3 4) the zero terms are such as equivalent to zero

$$(\beta - \hat{\beta})'X'(Y - X\hat{\beta}) = (\beta - \hat{\beta})'(X'Y - X'X\hat{\beta})$$

$$= (\beta - \hat{\beta})'(X'Y - X'X(X'X)^{-1}X'Y)$$

$$= (\beta - \hat{\beta})'(X'Y - X'Y) = 0$$

Similarly, other term $(Y - X\hat{\beta})'X(\beta - \hat{\beta}) = (Y' - \hat{\beta}'X')X(\beta - \hat{\beta})$

$$= (Y'X - \hat{\beta}'X'X)(\beta - \hat{\beta})$$

$$= \{Y'X - [(X'X)^{-1}X'Y]'X'X\}(\beta - \hat{\beta})$$

$$= [Y'X - (X'Y)'((X'X)^{-1})'X'X](\beta - \hat{\beta})$$

$$=(Y'X - Y'X)\ (\beta - \hat{\beta}) = 0$$

Hence, replacing the above results into equation (3 4) the likelihood can be written as follows

$$P(Y/X,\beta,\tau) \propto \left(\frac{\sqrt{\tau}}{\sqrt{2\pi}}\right)^n \exp\left[\frac{-\tau}{2}\left(vs^2 + (\beta - \hat{\beta})X'X(\beta - \hat{\beta})\right)\right]$$

Or

$$P(Y/X,\beta,\tau) \propto \tau^{\frac{n}{2}} \exp\left[\frac{-\tau}{2}\left(vs^2 + (\beta - \hat{\beta})X'X(\beta - \hat{\beta})\right)\right] \text{---------------------} (3\ 5)$$

3.4. Derivation of posterior distributions

Posterior distribution is derived by multiplying likelihood function with prior distributions There are two types of priors, i e non-informative priors and informative priors An informative prior is important type of Bayesian analysis that have much influence for parameters of interests Informative prior bases on previous studies, published work, research intuition, interviewing experts Jeffery priors, Uniform prior and weekly informative priors are considered as important non-informative priors The following four sections contain derivations of posterior distributions under informative and non-informative priors

3.4.1. Derivation of Posterior Distribution under informative priors

We assume that β follows multivariate normal distribution i e

$$\beta \sim MN(\beta_0, \Omega_0)$$

$$P(\beta) = |2\pi\Omega_0|^{-1}\exp[\tfrac{-1}{2}(\beta - \beta_0)'\Omega_0(\beta - \beta_0)] \text{----------------} (3\ 6)$$

Assuming that precision follows gamma distribution i e

$$\tau \sim G(a, b) \quad \text{where } a = \frac{v_0}{2} \quad and \quad b = \frac{v_0\sigma_0^2}{2}$$

$$P(\tau) = \frac{(v_0\sigma_0^2)^{\frac{v_0}{2}}}{\Gamma\frac{v_0}{2}}(\tau)^{\frac{v_0}{2}-1}\exp\left[-\tau\frac{v_0\sigma_0^2}{2}\right] \quad \text{for } \tau > 0 \text{------------} (3\ 7)$$

$$Posterior = \frac{prior \times likelihood}{marginal} = \frac{P(\theta) \times P(x|\theta)}{\int P(\theta) \times P(x|\theta)d\theta}$$

Posterior is proportional to prior multiplied by likelihood function

$$P(\beta, \tau/Y, X) \propto P(Y/\beta, \tau)P(\beta)P(\tau)$$

Using Equations (3 5), (3 6), (3 7) get the following results

$$(\beta,\tau/Y,X) \propto (\tau)^{\frac{v_0}{2}-1}\exp\left(-\tau\frac{v_0\sigma_0^2}{2}\right) \times \exp[\frac{-1}{2}(\beta-\beta_0)'\Omega_0(\beta-\beta_0)]$$

$$\times (\tau)^{\frac{n}{2}}\exp(\frac{-\tau}{2}(vs^2)\exp\left[\frac{-\tau}{2}(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})\right]$$

$$P(\beta,\tau/Y,X) \propto (\tau)^{\left(\frac{v_0}{2}+\frac{n}{2}\right)-1}\exp\left(-\tau(\frac{v_0\sigma_0^2+vs^2}{2})\right)$$

$$\times \exp\left(\frac{-1}{2}(\beta-\beta_0)'\Omega_0(\beta-\beta_0)+(\beta-\hat{\beta})'\tau(X'X)(\beta-\hat{\beta})\right)$$

$$P(\beta,\tau/Y,X) \propto \tau^{a^*-1}\exp(-\tau b^*) \times \exp\left(\frac{-1}{2}(\beta-\beta_0)'\Omega_0(\beta-\beta_0)+(\beta-\hat{\beta})'\tau(X'X)(\beta-\hat{\beta})\right) \qquad (3\ 8)$$

Solving exponent power

$$(\beta-\beta_0)'\Omega_0(\beta-\beta_0)+(\beta-\hat{\beta})'\tau(X'X)(\beta-\hat{\beta})=\left(\beta'\Omega_0\beta-\beta'\Omega_0\beta_0-\beta_0'\Omega_0\beta+\beta_0'\Omega_0\beta_0+\beta'\tau(X'X)\beta-\beta'\tau(X'X)\hat{\beta}-\hat{\beta}'\tau(X'X)\beta+\hat{\beta}'\tau(X'X)\hat{\beta}\right)$$

$$=\beta'\Omega_0\beta-2\beta'\Omega_0\beta_0+\beta_0'\Omega_0\beta_0+\beta'\tau(X'X)\beta-2\beta'\tau(X'X)\hat{\beta}+\hat{\beta}'\tau(X'X)\hat{\beta}$$

$$=\beta'\Omega_0\beta+\beta'\tau(X'X)\beta-2\beta'\Omega_0\beta_0-2\beta'\tau(X'X)\hat{\beta}+\beta_0'\Omega_0\beta_0+\hat{\beta}'\tau(X'X)\hat{\beta}$$

$=\beta'(\Omega_0+\tau(X'X))\beta-2\beta'(\Omega_0\beta_0+\tau(X'X)\hat{\beta})+$ stuff that does not depend on β

Completing the square root by add and subtract

$$=\left(\sqrt{(\Omega_0+\tau(X'X))}\right)^2\beta^2-2\beta\sqrt{(\Omega_0+\tau(X'X))}\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}}+\left(\frac{(\Omega_0\beta_0-\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}}\right)^2-\left(\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}}\right)^2$$

$$\left[\sqrt{(\Omega_0+\tau(X'X))}\,\beta-\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}}\right]^2-\left[\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}}\right]^2$$

Taking common $\sqrt{(\Omega_0+\tau(X'X))}$ from square root we get

$$=(\Omega_0+\tau(X'X))\left[\beta-\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{(\Omega_0+\tau(X'X))}\right]^2 + \text{stuff that does not depend on } \beta$$

$$P(\beta/Y,X) \propto \exp(-0.5V_n^{-1}(\beta-M^*)^2)$$

$$V_n^{*-1}=(\Omega_0+\tau(X'X))\text{And}M^*=\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{(\Omega_0+\tau(X'X))}$$

$$P(\beta,\tau/Y,X) \propto (\tau)^{a^*-1}\exp[-\tau b^*]\times\exp\left(\frac{-1}{2V^*}(\beta-M^*)^2\right)$$

The posterior estimates of the parameters become as

$$a^*=\left(\frac{v_0}{2}+\frac{n}{2}\right) \qquad \text{And} \qquad b^*=\left(\frac{v_0\sigma_0^2+vs^2}{2}\right)$$

$$V^*=(\Omega_0+\tau X'X)^{-1} \text{ and } M^*=V^{*-1}(\Omega_0\beta_0+\tau X'X\hat{\beta}) \text{ or } M^*=\frac{(\Omega_0\beta_0+\tau X'X\hat{\beta})}{(\Omega_0+\tau X'X)}$$

3.4.2 Derivation of Posterior using Jeffrey Priors

A number of non-informative priors are used. Jeffrey priors and uniform priors are considered in this study for comparison. According to Jeffrey priors, prior could be achieved by up-to-date the square root of the determinant of Fisher information matrix in multivariate case. So the Jeffrey prior is taken as fellows

$$P(\theta)\propto\sqrt{I(\theta)}$$

Where $\quad I(\theta)=-E_\theta\left[\frac{d^2\log p(\frac{Y}{\theta})}{d\theta^2}\right]$

Considering the likelihood function presented as equation 3 5

$$L= L(Y)=P(Y/X,\beta,\sigma^2)=(\frac{1}{\sqrt{2\tau\sigma^2}})^n \exp\left[\frac{-1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)\right]$$

$$\ln L(Y)=\ln P(Y/X,\beta,\sigma)=\ln\{(\frac{1}{\sqrt{2\tau\sigma^2}})^n \exp[\frac{-1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)]\}$$

$$\ln L(Y)=c-\frac{n}{2}\ln\sigma^2-\frac{-1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)$$

$$\frac{\partial \ln L(Y)}{\partial\beta}=0-\frac{2\acute{X}(Y-X\beta)}{2\sigma^2}=0$$

$$\frac{\partial^2 \ln L(Y)}{\partial\beta^2}=-\frac{\acute{X}X}{\sigma^2}$$

$$\frac{\partial^2 \ln L(Y)}{\partial\sigma^2\partial\beta}=-\frac{\acute{X}(Y-X\beta)}{2\sigma^4}=0$$

$$\frac{\partial \ln L(Y)}{\partial\sigma^2}=-\frac{n}{2\sigma^2}+\frac{(Y-X\beta)'(Y-X\beta)}{2\sigma^4}$$

$$\frac{\partial^2 \ln L(Y)}{\partial\sigma^4}=\frac{n}{2\sigma^4}-\frac{(Y-X\beta)'(Y-X\beta)}{\sigma^6}$$

$$\frac{\partial^2 \ln L(Y)}{\partial\sigma^2\partial\beta}=-\frac{\acute{X}(Y-X\beta)}{2\sigma^4}=0$$

$$I(\theta)=-E\begin{bmatrix}\frac{\partial^2 \ln L(Y)}{\partial\beta^2} & \frac{\partial^2 \ln L(Y)}{\partial\sigma^2\partial\beta}\\ \frac{\partial^2 \ln L(Y)}{\partial\sigma^2\partial\beta} & \frac{\partial^2 \ln L(Y)}{\partial\sigma^4}\end{bmatrix}$$

$$I(\theta)=-E\begin{bmatrix}-\frac{\acute{X}X}{\sigma^2} & 0\\ 0 & \frac{n}{2\sigma^4}-\frac{(Y-X\beta)'(Y-X\beta)}{\sigma^6}\end{bmatrix}$$

$$I(\theta)=\begin{vmatrix}\frac{\acute{X}X}{\sigma^2} & 0\\ 0 & \frac{n-2k}{\sigma^4}\end{vmatrix}=\frac{XX(n-2k)}{2\sigma^6}$$

$$P(\theta) \propto [I(\theta)]^{\frac{1}{2}}$$

$$P(\beta.\sigma^2) \propto c * [\frac{1}{\sigma^2}]^{1\,5}$$

Jeffrey prior distribution for the k parameters is taken as follows

$$P(\beta.\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^k \text{-----------}(3\,9)$$

$$Posterior = \frac{prior \times likelihood}{marginal} = \frac{P(\theta) \times P(x|\theta)}{\int P(\theta) \times P(x|\theta)d\theta}$$

Posterior ∝ prior × likelihood

$$P(\beta,\sigma^2/Y,X) = P(Y,X/\beta,\sigma^2)P\left({}^{Y}/_{\beta,\sigma^2}\right)P(\beta)P(\sigma^2)$$

Putting the results from (3 5) and (3 9)

$$P(\beta,\sigma^2/Y,X) \propto (\sigma^2)^{-k} \times (\frac{1}{\sqrt{2\pi\sigma^2}})^n \exp[\frac{-1}{2\sigma^2}(vs^2 + (\beta - \hat{\beta})'X'X(\beta - \hat{\beta})]$$

$$P(\beta,\sigma^2/Y,X) \propto (\sigma^2)^{-k} \times (\sigma^2)^{\frac{-n}{2}} \exp(\frac{-vs^2}{2\sigma^2}) \times \exp[\frac{-1}{2}(\beta - \hat{\beta})'\sigma^{-2}X'X(\beta - \hat{\beta})]$$

Which becomes marginal t distribution and inverse gamma distribution in unknown case of beta and variance with parameters

$$\beta \sim N(\hat{\beta}, \sigma^2(X'X)^{-1}) \qquad \sigma^2 \sim IG(\frac{v}{2}, \frac{vs^2}{2})$$

Results are approximately equivalent to the classical results of the maximum likelihood estimates (MLE) and where E (β) = $(X'X)^{-1} X'Y$ and variance of β is var (β) =$\sigma^{-2}(X'X)$

3.4.3. Posterior Derivation using Uniform prior

Define the non-informative uniform prior $P(\beta,\sigma^2) \propto c$ over the limits for β from $[-\infty,\infty]$ and σ^2 from $[0,\infty]$, individually The joint posterior from the likelihood function is derived as follows

$$P(Y/X,\beta,\sigma) \propto \frac{1}{\sigma^n}\exp\left(-\frac{1}{2\sigma^2}\left((n-k)s^2+(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})\right)\right)$$

Posterior $\propto$ prior $\times$ likelihood

$$P(\beta,\sigma^2/Y,X) \propto \frac{1}{\sigma^n}\exp\left(-\frac{1}{2\sigma^2}\left((n-k)s^2+(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})\right)\right)\times c$$

$$P(\beta,\sigma^2/Y,X) \propto \frac{1}{\sigma^n}\exp\left(-\frac{1}{2\sigma^2}\left(\upsilon s^2+(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})\right)\right)$$

$$P(\beta,\sigma^2/Y,X) \propto \frac{1}{\sigma^n}\exp(-\frac{\upsilon s^2}{2\sigma^2})\exp\left(-\frac{1}{2\sigma^2}\left((\beta-\hat{\beta})'X'X(\beta-\hat{\beta})\right)\right)$$

$$P(\beta,\sigma^2/Y,X) \propto (\sigma^2)^{-\frac{n}{2}}\exp(-\frac{\upsilon s^2}{2\sigma^2})\exp\left(-\frac{1}{2\sigma^2(X'X)^{-1}}\left((\beta-\hat{\beta})'(\beta-\hat{\beta})\right)\right)$$

$$P(\beta,\sigma^2/Y,X) \propto (\sigma^2)^{-\frac{n}{2}+1-1}\exp(-\frac{\upsilon s^2}{2\sigma^2})\exp\left(-\frac{1}{2\sigma^2(X'X)^{-1}}\left((\beta-\hat{\beta})'(\beta-\hat{\beta})\right)\right)$$

$$P(\beta,\sigma^2/Y,X) \propto (\sigma^2)^{-(\frac{n}{2}-1)-1}\exp(-\frac{\upsilon s^2}{2\sigma^2})\exp\left(-\frac{1}{2\sigma^2(X'X)^{-1}}\left((\beta-\hat{\beta})'(\beta-\hat{\beta})\right)\right)$$

$$P(\beta,\sigma^2/Y,X) \propto (\sigma^2)^{-a^*-1}\exp(-\frac{b^*}{\sigma^2})\exp\left(-\frac{1}{2V^*}\left((\beta-\hat{\beta})'(\beta-\hat{\beta})\right)\right)$$

Posterior parameters are as follows

$$a^{*} = \frac{n}{2} - 1 \quad b^{*} = \frac{\nu s^2}{2} \qquad V^{*} = \sigma^2 (X'X)^{-1} \quad \hat{\beta} = (X'X)^{-1} X'Y$$

3.4.4. Posterior Derivation using Weekly Non-Informative Priors

In the case of weekly non-informative priors, parameters of the model are assumed to follow normal distribution with zero means and large variances Precision is assumed to follow gamma distribution with scale parameter (a) as well as shape parameter (b) is equal to 0 01 Derivation of posterior distribution as follows

The conjugate normal distribution for β is as stated in Equation (3 6) follows $\beta \sim MN(\beta_0, \Omega_0)$

$$P(\beta) = |2\pi\Omega_0|^{-1} \exp[\frac{-1}{2}(\beta - \beta_0)'\Omega_0(\beta - \beta_0)]$$

For precision $\tau \sim G(a, b)$

Where $a = scale \quad and \quad b = shape$ parameters respectively

$$P(\tau) = \frac{a)^b}{\Gamma a} (\tau)^{a-1} \exp[-\tau b] \quad \text{for } \tau > 0 \text{---------------}(3\ 10)$$

$$Posterior = \frac{prior \times likelihood}{marginal} = \frac{P(\theta) \times P(x|\theta)}{\int P(\theta) \times P(x|\theta) d\theta}$$

Posterior is proportional to prior and likelihood function

$$P(\beta, \tau / Y, X) \propto P(Y / \beta, \tau) P(\beta) P(\tau)$$

Mathematically combing the Equation (3 5), (3 6) and (3 10)

$$(\beta,\tau/Y,X) \propto (\tau)^{a-1}\exp(-\tau b) \times \exp[\frac{-1}{2}(\beta-\beta_0)'\Omega_0(\beta-\beta_0)]$$

$$\times (\tau)^{\frac{n}{2}}\exp(\frac{-\tau}{2}(vs^2)) \exp[\frac{-\tau}{2}(\beta-\hat{\beta})'X'X(\beta-\hat{\beta})]$$

$$P(\beta,\tau/Y,X) \propto (\tau)^{\left(a+\frac{n}{2}\right)-1} \exp\left(-\tau(b+\frac{vs^2}{2})\right)$$

$$\times \exp\left(\frac{-1}{2}(\beta-\beta_0)'\Omega_0(\beta-\beta_0) + (\beta-\hat{\beta})'\tau(X'X)(\beta-\hat{\beta})\right)$$

$$P(\beta,\tau/Y,X) \propto \tau^{a^*-1} \exp(-\tau b^*)$$

$$\times \exp\left(\frac{-1}{2}(\beta-\beta_0)'\Omega_0(\beta-\beta_0) + (\beta-\hat{\beta})'\tau(X'X)(\beta-\hat{\beta})\right)$$

Solving exponent power $(\beta-\beta_0)'\Omega_0(\beta-\beta_0) + (\beta-\hat{\beta})'\tau(X'X)(\beta-\hat{\beta})$

Solving the exponent power for beta's

$$\exp\left(\frac{-1}{2}(\beta-\beta_0)'\Omega_0(\beta-\beta_0) + (\beta-\hat{\beta})'\tau(X'X)(\beta-\hat{\beta})\right)$$

Solving only the exponent power for β

$$=\Big(\beta'\Omega_0\beta - \beta'\Omega_0\beta_0 - \beta_0'\Omega_0\beta + \beta_0'\Omega_0\beta_0 + \beta'\tau(X'X)\beta - \beta'\tau(X'X)\hat{\beta} - \hat{\beta}'\tau(X'X)\beta +$$

$$\hat{\beta}'\tau(X'X)\hat{\beta}\Big)$$

$$=\beta'\Omega_0\beta - 2\beta'\Omega_0\beta_0 + \beta_0'\Omega_0\beta_0 + \beta'\tau(X'X)\beta - 2\beta'\tau(X'X)\hat{\beta} + \hat{\beta}'\tau(X'X)\hat{\beta}$$

$$=\beta'\Omega_0\beta + \beta'\tau(X'X)\beta - 2\beta'\Omega_0\beta_0 - 2\beta'\tau(X'X)\hat{\beta} + \beta_0'\Omega_0\beta_0 + \hat{\beta}'\tau(X'X)\hat{\beta}$$

$= \beta'(\Omega_0 + \tau(X'X))\beta - 2\beta'(\Omega_0\beta_0 + \tau(X'X)\hat{\beta})$ + stuff that does not depend on β

To complete the square root by add and subtract

$$=\left(\sqrt{(\Omega_0 + \tau(X'X))}\right)^2 \beta^2 - 2\beta\sqrt{(\Omega_0 + \tau(X'X))}\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}} + \left(\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}}\right)^2 - \left(\frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0+\tau(X'X))}}\right)^2$$

$$\left[\sqrt{(\Omega_0 + \tau(X'X))}\,\beta - \frac{(\Omega_0\beta_0 + \tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0 + \tau(X'X))}}\right]^2 - \left[\frac{(\Omega_0\beta_0 + \tau(X'X)\hat{\beta})}{\sqrt{(\Omega_0 + \tau(X'X))}}\right]^2$$

Taking common $\sqrt{(\Omega_0 + \tau(X'X))}$ from square root we get

$$=(\Omega_0 + \tau(X'X))\left[\beta - \frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{(\Omega_0+\tau(X'X))}\right]^2 + \text{stuff that does not depend on } \beta$$

$$P(\beta/Y, X) \propto \exp(-0.5V_n^{-1}(\beta - M^*)^2)$$

$$V_n^{*-1} = (\Omega_0 + \tau(X'X))\text{And}M^* = \frac{(\Omega_0\beta_0+\tau(X'X)\hat{\beta})}{(\Omega_0+\tau(X'X))}$$

$$P(\beta,\tau/Y,X) \propto (\tau)^{a^*-1}\exp[-\tau b^*] \times \exp\left(\frac{-1}{2V^*}(\beta - M^*)^2\right)$$

$$a^* = \left(a + \frac{n}{2}\right) \qquad \text{And} \qquad b^* = \left(b + \frac{vs^2}{2}\right)$$

$$V^* = (\Omega_0 + \tau X'X)^{-1} \text{ and } \quad M^* = V^{*-1}(\Omega_0\beta_0 + \tau X'X\hat{\beta}) \text{ or } M^* = \frac{(\Omega_0\beta_0+\tau X'X\hat{\beta})}{(\Omega_0+\tau X'X)}$$

The required final posterior estimates become above The mean of conditional distribution is the weighted average of prior mean and MLE estimator with the weight given by the reciprocal of the variances of prior and MLE The large value of prior variance will imply that weak weight on the prior and approximately posterior s demonstrated by OLS estimates

3.5. Elicitation of Hyper Parameters

Estimation of the model under Bayesian framework requires knowledge about parameters of prior distribution which are called hyper parameters When informative priors are used, hyper parameters are elicitated on the basis experts' information Hence, Elicitation is the procedure of transforming someone's views into a distribution as some prior parameters PV method of elicitation is used in this study where five experts are asked five quantile values for each parameter and then a simple regression is run by taking experts' values as the dependent variable and Z values for the quantiles as explanatory variable Estimates of intercept are taken as elicitated mean value while slope is considered as the standard error of the parameter Experts' guesses about parameters are given Table 3 1 whereas elicitated results are given in Table-3 2

Besides parameters of the model, precision of the model follows gamma distribution with "a" and "b" hyper parameters i.e $\tau \sim G(a, b)$ where $a = \frac{v_0}{2}$ *and* $b = \frac{v_0 \sigma_0^2}{2}$ Here σ_0^2= 0 024781 which is from the classical estimate of the model where as v_0=48 as the degrees of freedom Hence, a=24 and b=0 01473835 are taken as values of hyper parameters of the gamma prior distribution for precision

Table 3.1: Experts Guesses about Parameters

Parameters	Experts No	Probabilities at the guess is taken at different quantile points				
		P=0 1	P=0 25	P=0 5	P=0 75	P=0 95
β_0	Expert 1st guess	-2.5	-2	-1 5	-1	-0.5
β_0	Expert 2nd guess	-2 3	-1 9	-1 4	-0 9	-0 4
β_0	Expert 3rd guess	-2 25	-1 8	-1 3	-0 8	-0 3
β_0	Expert 4th guess	-2 45	-1 85	-1 45	- 95	-0 45
β_0	Expert 5th guess	-2 40	-1 99	-1 35	-0 85	-0 35
β_1	Expert 1st guess	0 0076	0.0082	0 0088	0 094	0 01
β_1	Expert 2nd guess	0 00761	0 0081	0 00882	0 00942	0 0114
β_1	Expert 3rd guess	0 0075	0 00815	0 00885	0 00945	0 0113
β_1	Expert 4th guess	0.00752	0.00822	0 00877	0 00948	0 0101
β_1	Expert 5th guess	0.00755	0 00825	0 00888	0 00939	0 0103
β_2	Expert 1st guess	0.00087	0 00388	0 006106	0 008724	0 01134
β_2	Expert 2nd guess	0.00871	0 003485	0 006109	0 008719	0 01130
β_2	Expert 3rd guess	0 000874	0.003481	0 006111	0 008712	0 01125
β_2	Expert 4th guess	0 000869	0 00349	0.00609	0.008709	0 01122
β_2	Expert 5th guess	0 0008755	0.003492	0 006088	0 00872	0 1120
β_3	Expert 1st guess	0.7685	0 8349	0.90125	0 96762	1 034*
β_3	Expert 2nd guess	0 7680	0 8345	0 9012	0 96765	1 030*
β_3	Expert 3rd guess	0 7575	0 8342	0 9018	0 9676	1 032*
β_3	Expert 4th guess	0 7688	0 8352	0 9009	0 96758	1 028*
β_3	Expert 5th guess	0 7678	0.8355	0 90128	0 96757	1.035*
β_4	Expert 1st guess	0 1388	0 156075	0 173350	0 190625	0 2079*
β_4	Expert 2nd guess	0 1385	0.1560	0 1730	0 19060	0 2078*
β_4	Expert 3rd guess	0 1382	0 1558	0 1735	0 19062	0 2075*
β_4	Expert 4th guess	0 1380	0 1562	0 1732	0 1905	0 2080*
β_4	Expert 5th guess	0 1392	0 1565	0 1729	0 19063	0 2082*

*denotes that these values are taken at P=0 99 instead of 0 95

Table 3.2.Elicitated Values of Hyper Parameters

Parameters	Mean	Variance	Precision
Intercept	-1 44564	$(0\ 68315)^2$	2 143
Literacy rate	0 008812	$(0\ 0009688)^2$	1065446 727
Trade openness	0 0054685	$(0\ 002973)^2$	113138 4387
Physical capital	0 885508	$(0\ 072864)^2$	188 3536
Health capital	0 1692637	$(0\ 019146)^2$	2728 567

3.6. Estimation Methodology and Diagnostic Tests

After derivation of posterior distribution, the next step is to obtain estimates of parameters along with their standard errors For this purpose, MCMC simulations are conducted using Gibbs sampling to simulate estimates from posterior distributions Procedure of MCMC simulations under Gibbs sampling is as follows

Gibbs sampling is one member of a family of algorithms from the Markov Chain Monte Carlo (MCMC) framework The MCMC algorithms aims to construct a Markov chain that has the target posterior distribution as its stationary distribution In simple words, after a number of iterations of stepping through the chain, sampling from the distribution should converge to be close to sampling from the desired posterior Gibbs sampling is based on sampling from conditional distributions of the variables of the posterior

Thinning is a process used to make the explanations more nearly independent, hence more nearly a random sample from the posterior distribution of the parameters The purpose of expedition approaches is to decrease the autocorrelations A basic deception to lower the autocorrelation contains in retaining only every M^{th} value in the chain, which is so-called thinning the chain In detail, the thinned Markov chain can be even being turned into a chain with all autocorrelations equal to zero for lags greater than one This is completed through taking the thinning factor equal the lag for which the original Markov chain has autocorrelation equal to zero Though, the thinned chain has a higher Monte Carlo error than the original one In detail, in practically all circumstances, the improvement is only with respect to computer loading programs The burn-in period is the number of repetitions it taking for an MCMC arrangement to reach stationary distribution Identifies the number of replications for the burn-in period of MCMC The values of parameters simulated during burn-in are used for adjustment determinations just and are not used for estimation In burn-in period, the first B repetitions are removed starting the sample in order to avoid by the effect of the preliminary values If the produced sample is large enough, the effect of this period on the estimation of posterior summaries is insignificant Dropping the early observations is referred to as using a burn-in period

The formula for remaining total MCMC size in Win Bugs is as the total number of MCMC iterations is calculated

$$\left(\frac{burnin(iterations) - mcmcsize(iterations)}{thinning(iterations)}\right) \times chain(iterations)$$

The formula for remaining total MCMC size in STATA 14 is as the total number of MCMC iterations is calculated by way of burnin $(iterations)$ + (mcmcsize $(iterations)$ − 1) ∗ thinning$(iterations)$ + 1

Large numbers of simulations are conducted while burning and thinning some of the estimates Averages and standard deviations are obtained from the final selected simulation results These estimates are used to construct credible intervals which are used to test significance of parameters Some diagnostic tests are also applied to establish validity of simulated estimates These include Cusum chart for convergence of the results, Trace plots which expresses that how a longer burn-in periods is required, a chain is mixing well, and gives us awareness about the stationary formal of the chain Autocorrelation plots are used to identify non-randomness Since, MCMC simulations may consist of various chains and hence, Gelman-Rubin diagnostic test are applied to test convergence of each of the chain

3.7. Summary

This chapter presents specification of econometric model where GDP per worker depends upon education capital, health capital, physical capital, trade openness The likelihood and full derivation of posterior distribution by using different prior's distributions are obtained The different prior distribution like informative, uniform, Jeffrey and weakly non-informative are also used for the estimation of model parameters. The elicitation of unknown parameters of prior distribution This chapter also discussed the different diagnostic test for the convergences of posterior distribution

CHAPTER 4

RESULTS AND DISCUSSIONS

4.1 Introduction

This chapter contains four major sections After introduction, estimation results under non-informative i e Jeffrey's priors, Uniform priors and weekly non informative priors are presented and compared Section 3 contains estimation results on the basis of informative priors These results are compared with the best model under non-informative priors. Hence, selection of the best model among all the estimated models under non-informative and informative priors is followed by discussion about results and policy implications The last section presents concluding remarks

4.2. Results under non-informative Priors

The model is estimated using three types of non-informative priors These priors include Jeffrey priors, Uniform priors and weekly informative priors

Hyper parameters of Jeffrey priors include the square root of the determinant of Fisher information criteria, i e $P(\beta,\sigma^2) \propto (\sigma^2)^{-k}$ for prior k parameters Variance of residuals is obtained on the basis of classical estimates One hundred thousand MCMC simulations are conducted First five thousand simulations are discarded whereas thinning is fixed as 35 The diagnostics tests of estimated model under Jeffrey priors are presented in Appendix-A Diagnostic tests include Cusum plots, Autocorrelation function plots, Kernel densities plots and trace plots Trace plots of all parameters presented in Figure A 1 which show convergence of estimates Autocorrelation plots presented in Figure A 2 show that simulated estimates are free of autocorrelation Kernel densities are presented in Figure

A 3 show that all parameters follow normal distribution Cusum plot presented in Figure A.4 show that the graphical summaries of detecting the persistence trends in MCMC and faster mixing of chains in simulation because the cu sum curve consistently crosses the x axis several times Hence, these diagnostic tests establish validity of the estimated model

For Uniform priors, Hyper parameters are assumed to follow uniform distribution with a=0 and b=1 One hundred thousand MCMC simulations are conducted First five thousand simulations are discarded whereas thinning is fixed as 35. The diagnostics tests of Jeffrey priors are presented in Appendix-B Trace plots of all parameters presented in Figure B 1 show that convergence of estimates Autocorrelation plots presented in Figure B 2 shows that simulated estimates are free of autocorrelation Kernel densities presented in Figure B.3 show that all parameters follow normal distribution Cusum plot presented in Figure B 4 show that the graphical summaries of detecting the persistence trends in MCMC and faster mixing of chains in simulation because the cu sum curve consistently crosses the x axis several times Hence, these diagnostic test establish validity of the estimated model

For weekly informative prior's, hyper parameters are taken as follows

$$\beta_{0_i} \sim N(0{,}10000)$$

$$\text{and} \quad \tau \sim G(0\ 01{,}0\ 01)$$

Gibbs sampling is used to conduct ten million MCMC simulations with five hundred thousand burning and nine hundred fifty thinning The diagnostics tests of weakly non-informative prior's autocorrelation function plot, kernel densities, quantile plots and trace plots are presented in appendix D Trace plots of all Parameters presented in Figure

C 1 show that convergence of estimates Kernel densities presented in Figure D 2 show that all parameters follow normal

Table 4.1: Estimation Results under Non-informative Priors

Parameters	Results under Jeffrey Prior	Results under Uniform Priors	Results under Weakly non-Informative Priors
	Estimates (S E) [Credible Interval]	Estimates (S E) [Credible Interval]	Estimates (S.E.) [Credible Interval]
β_0	-1 3766 (0 8426)* [-3 04072, 0.27382]**	-1 3774 (0 86075)* [-3 0788, 0 31791]**	-1 307 (0 8624)* [-3 053, 0 3488]**
β_1	0 009048 (0 00067)* [0 00772, 0 010362]**	0 009043 (0.000685)* [0 00769, 0.01039]**	0 009055 (0 00073)* [0 007634, 0 0105]**
β_2	0 0060989 (0 00265)* [0 000859, 0 01133]**	0 0061067 (0 002726)* [0.00072, 0 01149]**	0 006045 (0 002894)* [0 000369, 0 0118]**
β_3	0 90156 (0 06773)* [0 76854, 1 03556]**	0 90155 (0 06918)* [0.76559, 1 03855]**	0 8958 (0 06935)* [0 7621, 1 036]**
β_4	0 1732751 (0 01752)* [0 1391, 0 208]**	0 1734278 (0 0180187)* [0 13798, 0 20891]**	0 1741 (0 0192)* [0 1363, 0 2117]**
R^2	------	-------	0 9752 (0.005398)* [0 9626, 0.9835]**
Sig^2 Y	0 0022949 (0 00051)* [0 00151, 0 0035]**	0 0024087 (0 000549)* [0.00157, 0 0037]**	0 002747 (0 000597)* [0 001825, 0 0041]**
Tau Y	------	------	380 4 (78 5)* [241 60, 547 9]**

* denotes the SD of the posterior parameters and ** denotes the 95% credible interval

Distributions Autocorrelation plots presented in Figure C 3 shows that simulated estimates are free of autocorrelation Cusum plot presented in Figure C.4 show that the graphical summaries of detecting the persistence trends in MCMC and faster mixing of chains in simulation because the cu sum curve consistently crosses the x axis several times Hence, these diagnostic test establish validity of the estimated model Estimation results under above three types of priors are presented in Table 4.1

The posterior estimates under three types of non-informative prior show that all parameters are significant as simulated estimates lie within the credible intervals Moreover, estimates show that all the explanatory variables have positive impact on the economic growth in Pakistan However, the above three estimated models are compared on the basis of standard errors of estimates It shows that estimated model by using Jeffrey priors contains smaller standard errors for all parameters as compare to the standard error of estimates under uniform priors and weekly informative priors It implies that Jeffrey's priors give more precise estimates as compare to other two types of priors Hence, the estimated model under Jeffrey's priors may be considered as the best model among the three estimated model presented in Table 4 1

4.3. Results under Informative Priors

Using updated information is considered as one of the major advantage of Bayesian econometrics Hence, Bayesian estimation of the model is done under informative priors on the basis of experts' information For this purpose, elicited values of hyper parameters presented in Table 3 1 are taken Estimates are obtained from the posterior distribution by conduction six millions MCMC simulations with four chains First five hundred thousand estimates are discarded whereas thinning is fixed as five hundred fifty Estimation results are presented in Table 4 2 which contains posterior mean, standard errors and 95% credible interval of posterior estimates of parameters

Different diagnostics tests are applied to establish validity of simulated estimates Kernel densities presented in Figure 4 1 show that all parameters follow normal distribution Trace plots of all parameters are presented in Figure 4.2 where each colour represents a different chain The trace plot of all the parameters displays well mixing and convergence of estimates Quantile plots of all parameters are presented in Figure 4 3 These plots specify that algorithms converge in condition of posterior These all quantile points indicate the convergence of the parameters of all posterior distribution Moreover, these quantile plots lie within a 95% credible interval for all the periods which shows that they reach their stationary of the distribution of the parameters Figure 4 4 contains graphical presentation of BGR test of convergence BGR test statistics lie on the central line in all graphs which shows the convergence of the parameters In our model we have used three chains for convergence and all the diagnostic graphs of posterior parameters shows a good convergence to own target distribution. Autocorrelation plots of all parameters are presented in Figure 4 5 which shows that there is no autocorrelation in the simulated estimates of all parameters It is observed that autocorrelation for all parameters come to an end at 15^{th} lag length This graph of autocorrelation indicates that the parameters converge to a stationary target distribution Hence, diagnostic tests establish validity of the estimated model under informative priors Estimation results under informative priors are presented in Table 4 2

Table 4.2: Results under informative Priors

Parameters	Estimates	Significance
Intercept (β_0)	-1 373(0 4372)*[-2 216 -0 5085]**	Significant
Education capital (β_1)	0.0090(0 00049)*[0 0081 0 0099]**	Significant
Trade Openness (β_2)	0 0058(0 0019)*[0 0020 0 0095]**	Significant
Physical Capital (β_3)	0.9025(0 0352)*[0 8328 0 9701]**	Significant
Health Capital (β_4)	0 1725(0 0121)*[0 1484 0 1963]**	Significant
R^2	0 9803(0 00297)*[0 9737 0 9853]**	Significant
Sig2 Y	0 002175	-----------
Tau Y	470 0	------------

* denotes the SD of the posterior parameters and ** denotes the 95% credible interval

Figure 4.1: Kernel densities of parameters

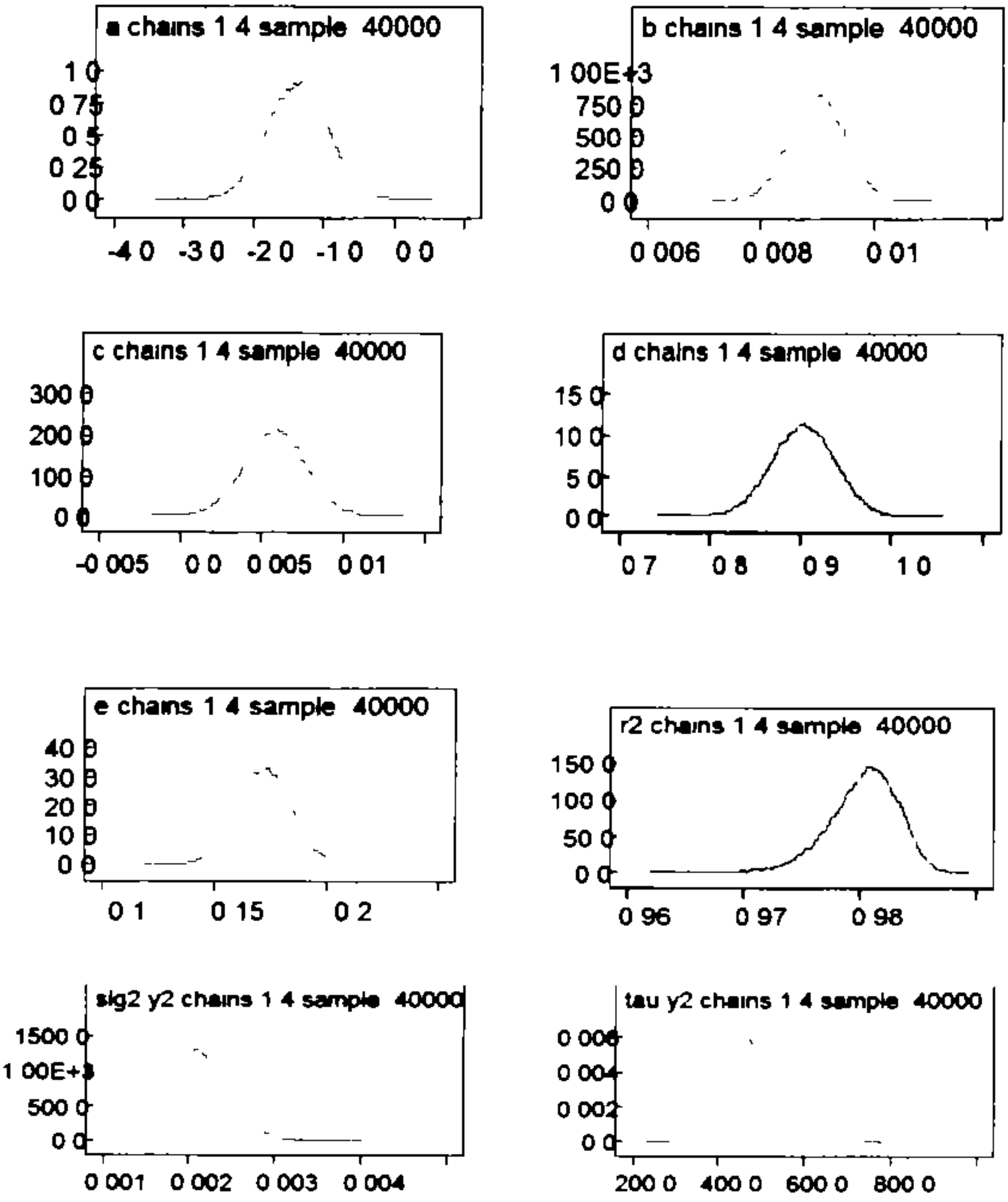


Figure 4.2: Trace Plots of the Parameters

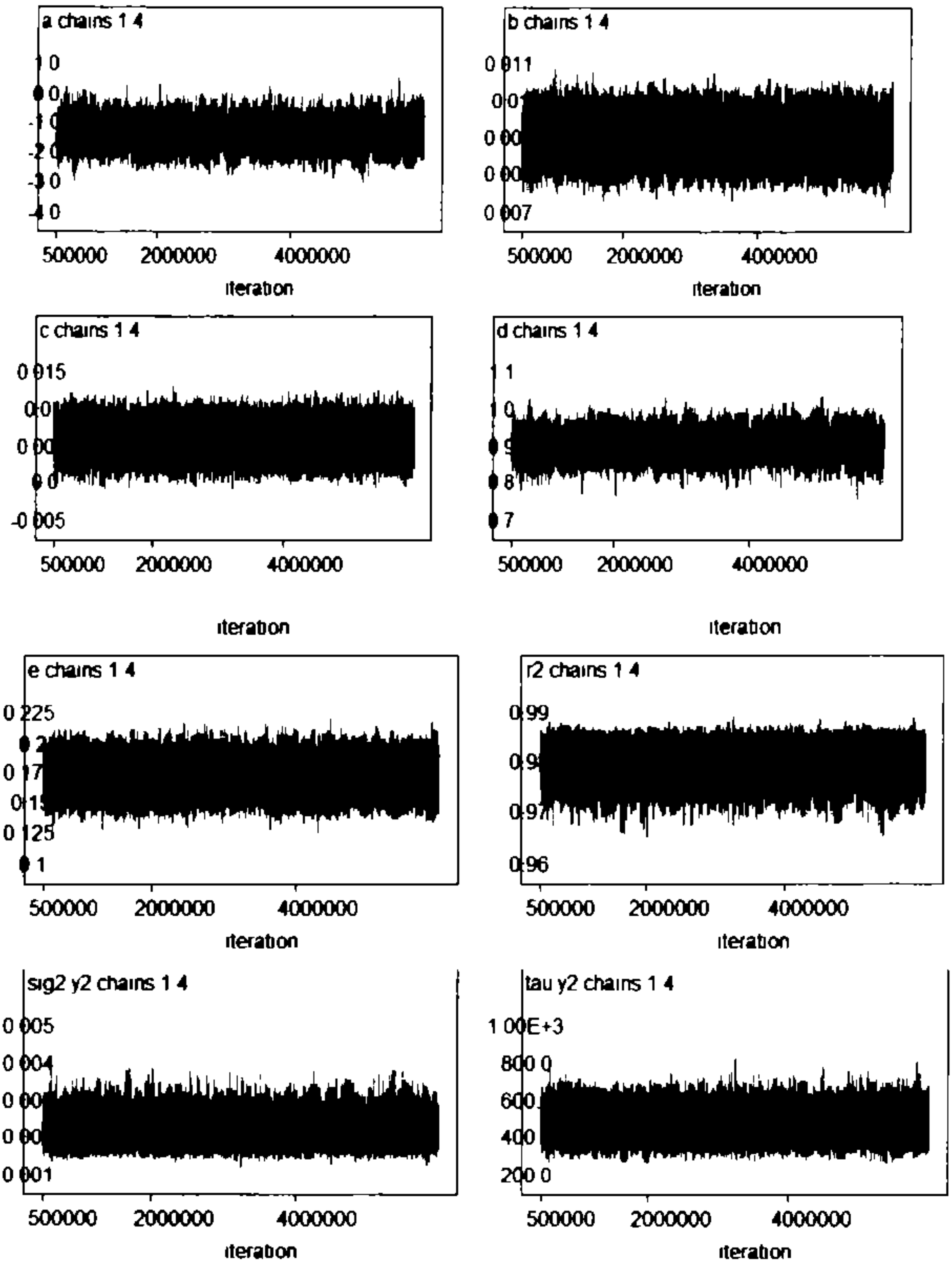


Figure 4.3: Quantile Points

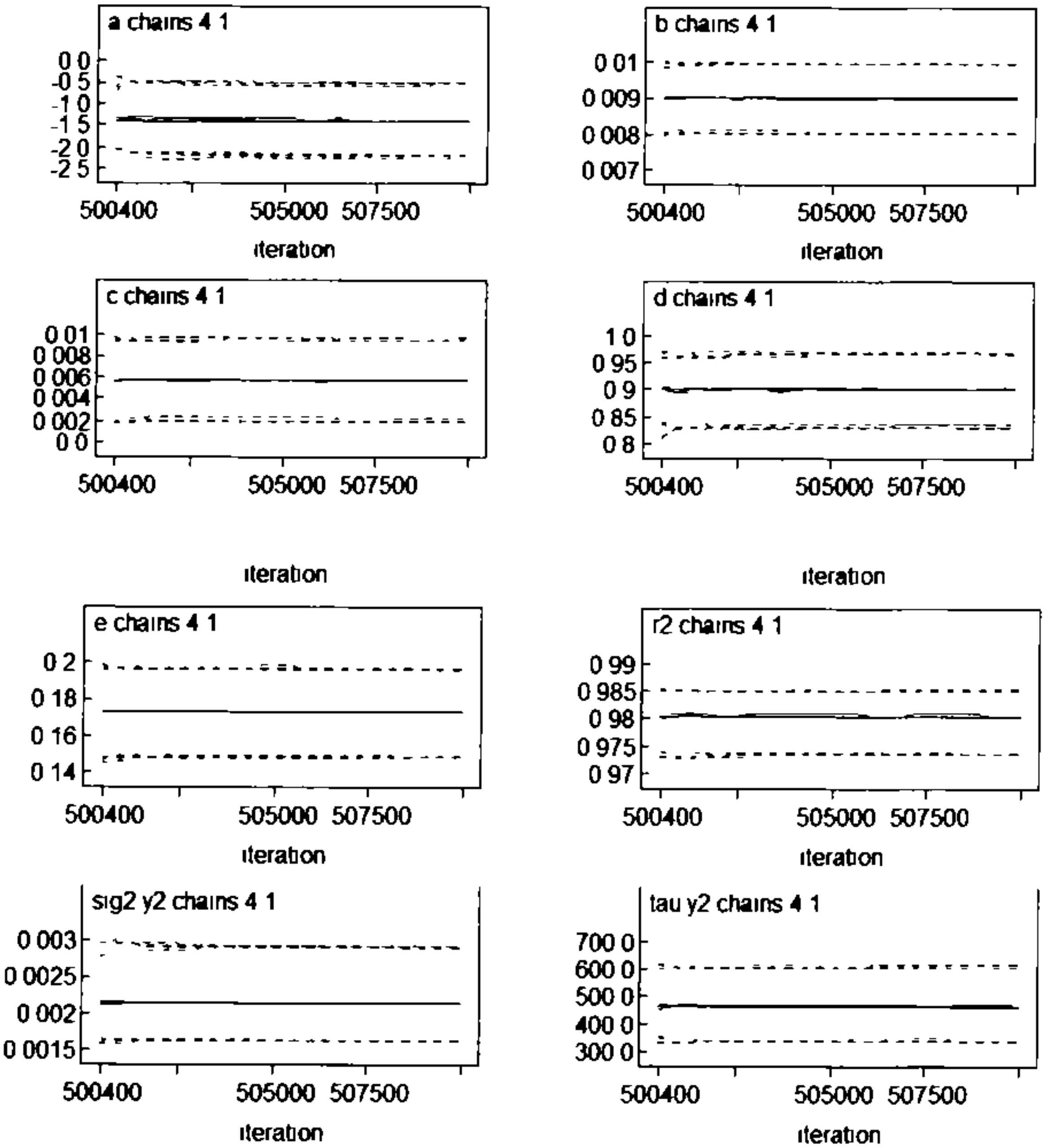


Figure 4.4: Graphical Presentation of BROOKS GELMAN Rubin Test

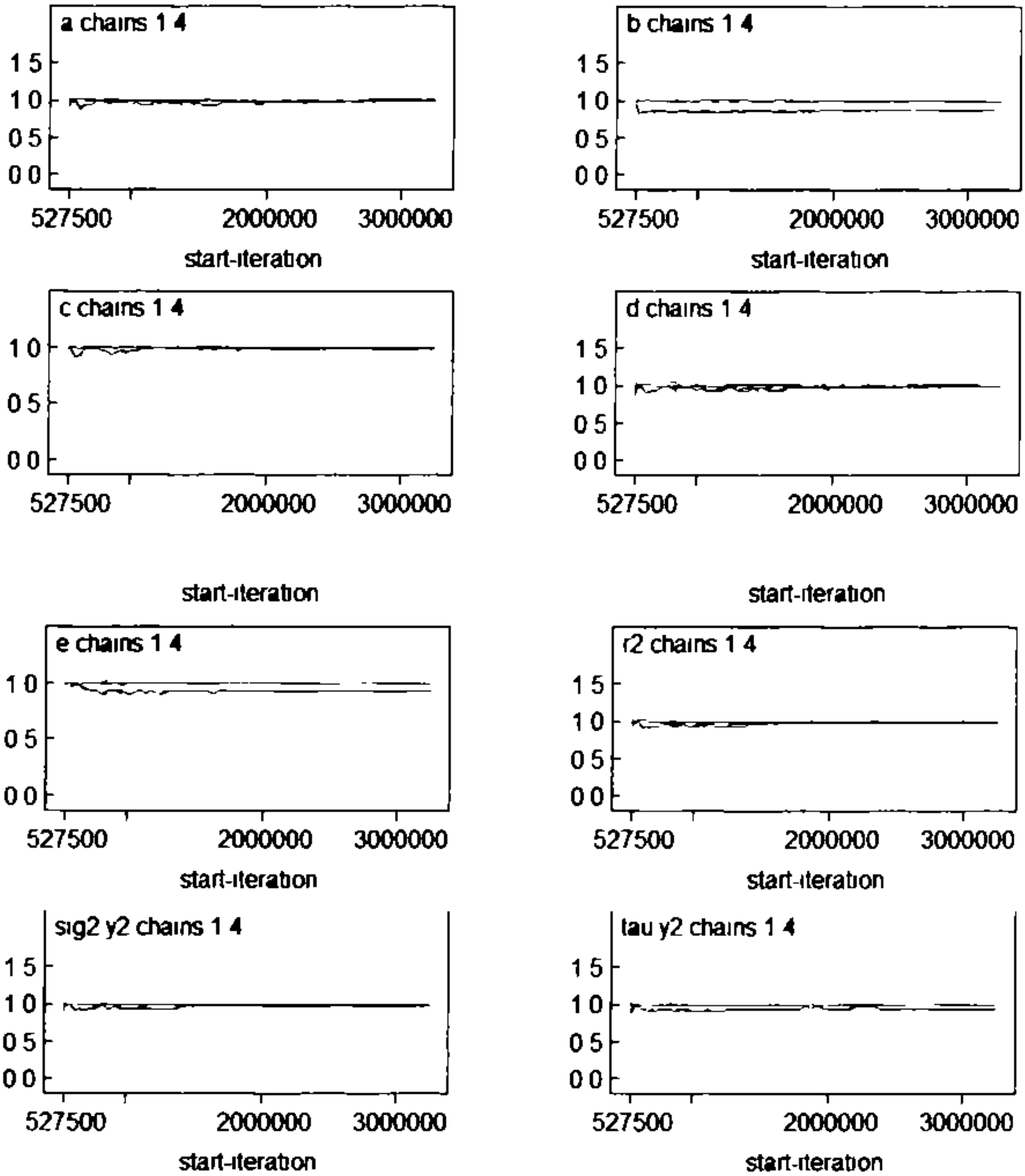


Figure 4.5: Autocorrelation Plots

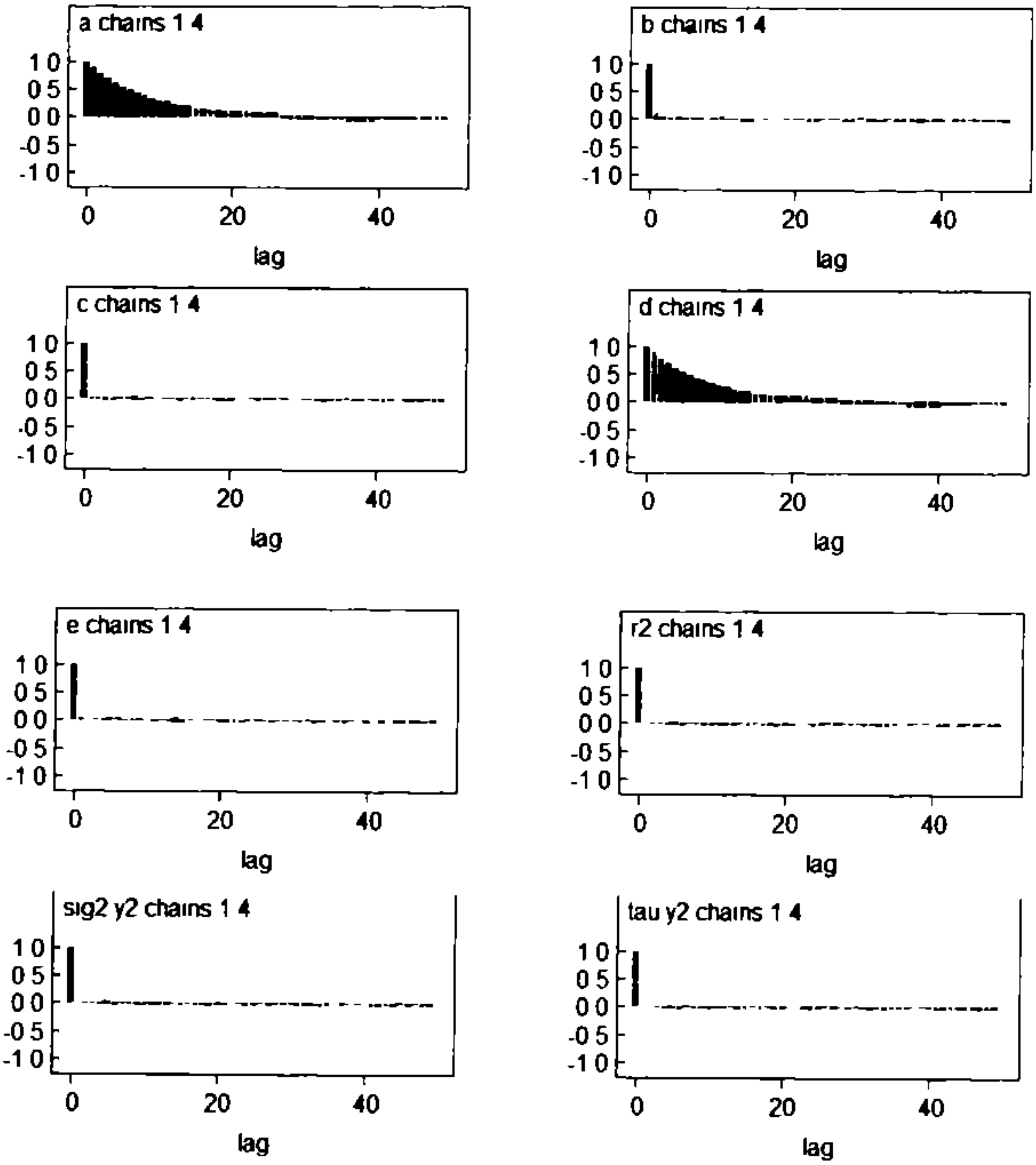


Table 4 2 contains parameters' estimates, along with standard errors and 95% credible intervals Residual variance of the model is 0 002175 which is smaller than the residual variance of the estimated models using non-informative priors including Jeffrey's priors Moreover, standard errors of all estimates obtained by using informative priors are smaller than that of standard errors of all estimates obtained by employing Jeffrey's priors Hence, the model estimated under informative priors is the best model among the four estimated models

Results show that education capital significantly and positively affect economic growth in Pakistan. It is because estimate of education capital lies within 95% credible interval with positive sign The coefficient of the education capital demonstrates that 1% increase in the literacy rate can raise 0 009% posterior mean growth in GDP per worker with standard error 0 00049 and having a 95% Credible Interval as (0 0081 0 0099) It is because more educated people are more efficient to allocate resources that are important to increase economic growth The economist accepts that investment on education is the main source of economic growth because education increases the growth of labour productivity The educated Labour power is extra adaptive and informal to transportable

Coefficient estimate of trade openness indicates that trade openness has positive and significant impact on GDP output per worker It is determined that if trade openness rises, the situation be able to similarly improve the economic growth The co-efficient of the trade openness specifies that if the 1 % change in trade openness, it causes a direct change in output per worker as 1% increase in trade openness causes a direct change in posteriori mean as 0 0058% with SD (0 0019) and having a [95% CI (0 0020), (0 0095)]

Co-efficient estimate of physical capital has a positive and significant impact on economic growth It implies that a rise in physical capital causes an increase in GDP per worker. The co-efficient of physical capital specifies that 1% rise in physical capital raises posterior mean of GDP per worker by 0 9025% with SD (0 0352), and having a [95% CI (0 8328), (0 9701)] In case that there is development in constructions similar manufacturing of railways, comprising transportations highways, administrative centres, manufacturing and commercial structures, equipment's of obtaining and technology, its concerns similarly will increase (enhancement) in monetary growth

Co-efficient estimate of health capital indicates statistically significant and positive impact upon output GDP per worker This shows that extension in health expenditures also cause the rise in GDP output per worker The co-efficient of health capita indicates that 1% increase in health expenses increases the GDP output per worker by 0 1725% with SD (0 0121), and having a [95% CI (0 1484), (0 1963)] It means that whenever people are healthy, they are much effective in their creative, which will lead towards the increase in economic output growth per worker Healthy labour stands spiritually and substantially additional forceful and aggressive Healthy workers get higher earnings and extra creative, smaller expected to vague as of their work

As estimates of education capital as well as health capital is significant with positive sign, it implies that role of human capital to raise output per worker in Pakistan is vital To achieve sustainable economic growth, human capital may play an important role in the coming years Hence, government must have to divert its resources in order to compete the other regional economies like as India

4.4. Summary

This chapter contains estimation results along with discussion. comparison and policy implications Section 2 presents results of estimated models under non-informative priors, i e Jeffrey's priors, Uniform priors and weekly informative priors The three estimated models are compared on the basis of significance and precision of parameters' estimates Hence, the estimated model using Jeffrey's' priors is taken as the best among the three models on the basis of precision Section 3 presents results of estimated model on the basis of informative priors elicited on the basis of experts' knowledge and the model is compared with the estimated model based on Jeffrey's priors It proves that incorporation of experts' information through priors improves precision of the estimates as well as of overall model Hence. the model based on informative priors is declared as the best model According to estimation results, physical capital, trade openness, education capital and health capital significantly and positively affect output per worker in Pakistan Hence, two important points are concluded here Firstly, incorporation of priors' information besides current data may improve precision of estimates as well as overall model. Secondly, human capital can be considered as significant factor to obtain economic growth in Pakistan and hence. government must have to divert significant resources to improve human resources in Pakistan

CHAPTER 5

SUMMARY AND CONCLUSIONS

The study contains five chapters Chapter 1 presents an introduction to the topic, Bayesian versus Classical description and the goals of the study Main objective of the study are set as estimation of model to analyze impact of human capital on economic growth by using Bayesian inferential approach. Chapter 2 comprises an overview of literature, and hypothetical material of formerly studies originated in literature associated to area of study The chapter concluded that the literature does not contain any study containing Bayesian inferential approach in order to analyze the impact of human capital on economic growth for any country. Chapter 3 presents specification of model, discussion about data and methodology of analysis We try to elaborate the basic Bayes theorem, types of different prior's, likelihood, elicitation methods, diagnostics tests of the posterior distribution estimation In this chapter diagnostic tests are illuminated trace plots, Gibbs sampling, MCMC, ACF plots, BGR plot for posterior distribution convergence and stationary Chapter 4 presents results and discussion about results Section 4 2 contains estimation results under non-informative priors, i e Jeffreys' priors, uniform priors and weekly informative priors Coefficients' estimates of all parameters are significant with positive sign in all the three estimated models Whereas, precisions of coefficient estimates of the model estimated under Jeffrey's priors are larger than other two models The following section contains estimation results of the model using informative priors elicited on the basis of experts' knowledge and its comparison with other estimated models Precision of coefficient estimates as well as overall precision of the model under informative priors are better than that of all other models It implies that incorporation of prior information through Bayesian inference may be effective to improve precision of the model Three major conclusions are drawn on the basis results in this study Firstly, use of Jeffrey's prior increases precision of estimates as compare to uniform priors and weekly informative priors Secondly, use of prior information through Bayesian inference is an effective methodology to improve parameters' estimates in regression model Thirdly, human capital has significant and positive impact on economic growth in Pakistan

In the light of the empirical findings, this study recommends that the government should increase the education capital by increasing the education facilities in the country and making the good standard of the education in the institutes Investment in physical capital can also increase the output of the country like roads, railways, buildings, industries etc. Expanding the investment in human resources fields will take an immense increment in the output of the country per worker and talent also boost up the efficiency of labour and their experiences It will improve the value of the output of goods By these all increasing the trade also increase, which is very beneficial for a developing country According to the results of the study a developing country should increase the investment in the human capital like education sector as well as health capital and physical capital to increase his economic growth

The current study uses the Bayesian method for estimation to elaborate the influence of Education capital, trade openness, Health expenditures and physical capital This study can also estimate the Bayesian with different prior's specification like graphical method, empirical method and prior predictive methods also different methods for prior parameter estimation This study has further extended by estimating the influence of human capital upon economic growth by adding others independent variables in the model of the human capital like life expectancy, labour force, poverty, personal remittances and much more variables for analysis to increase the model specification Other specification of prior methods is also applied for getting the appropriate results of Bayesian Econometrics

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APPDENDIX-A

Diagnostics of Jeffrey Prior Graphs

Figure A.1 Trace Plot of parameters

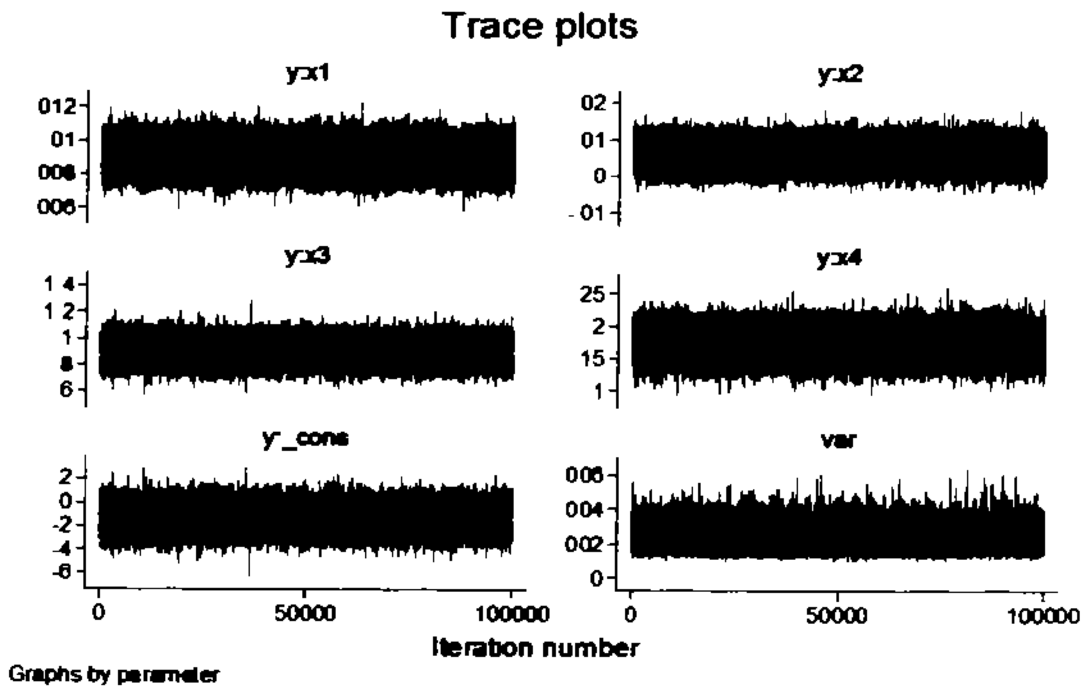


Figure A.2 Autocorrelation Plot

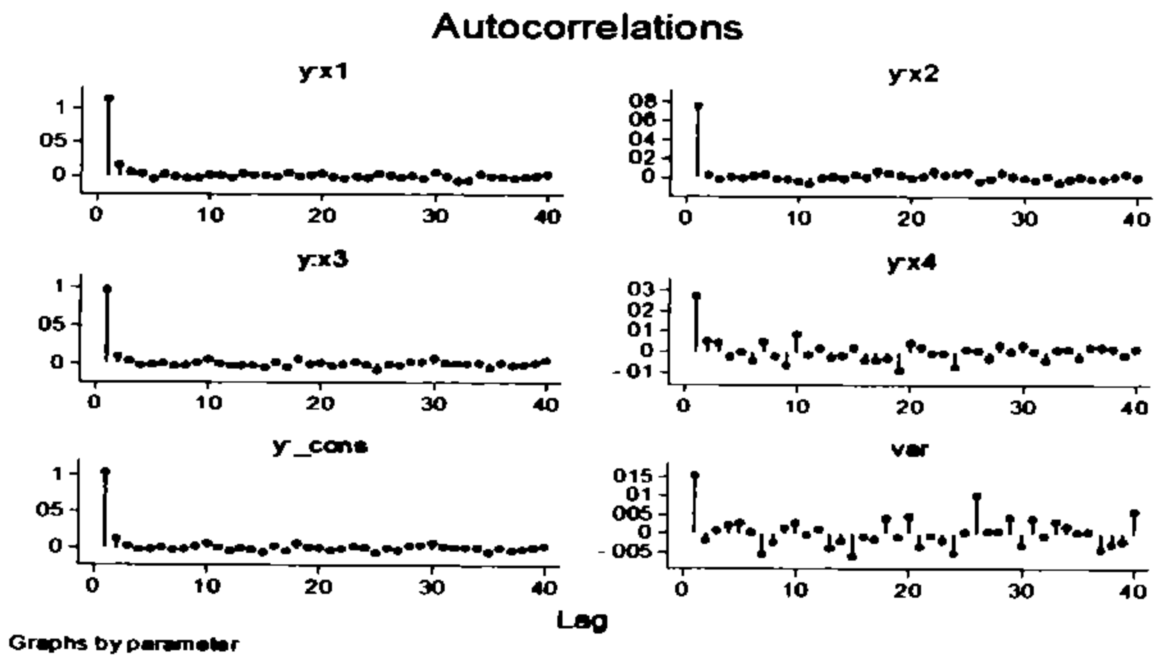


Figure A.3 <u>**Kernel Densities**</u>

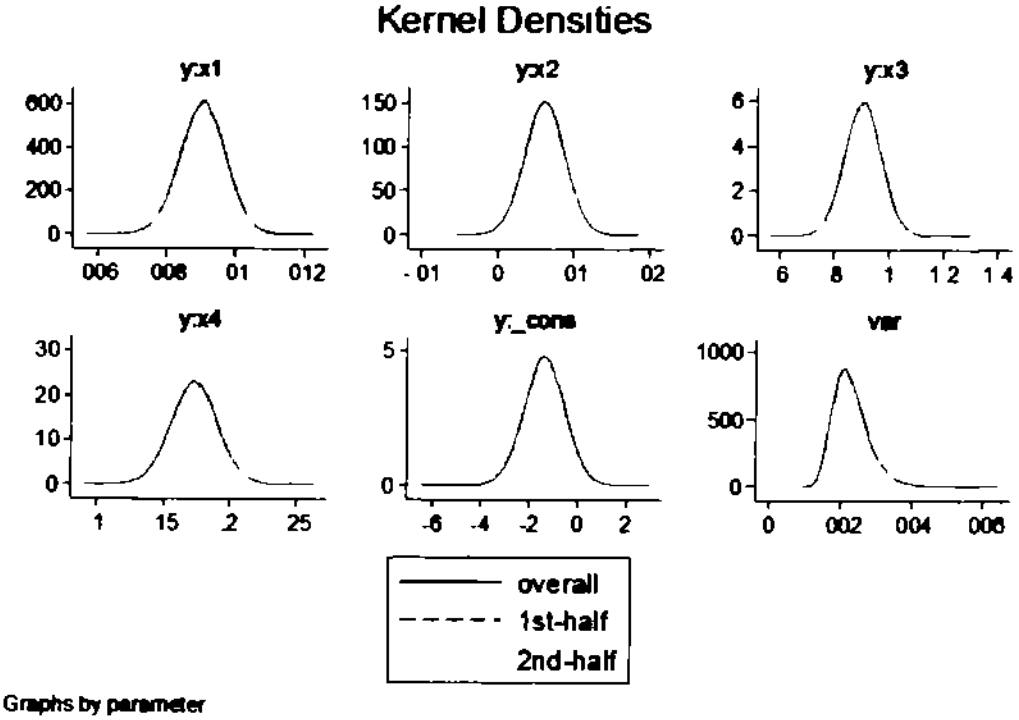
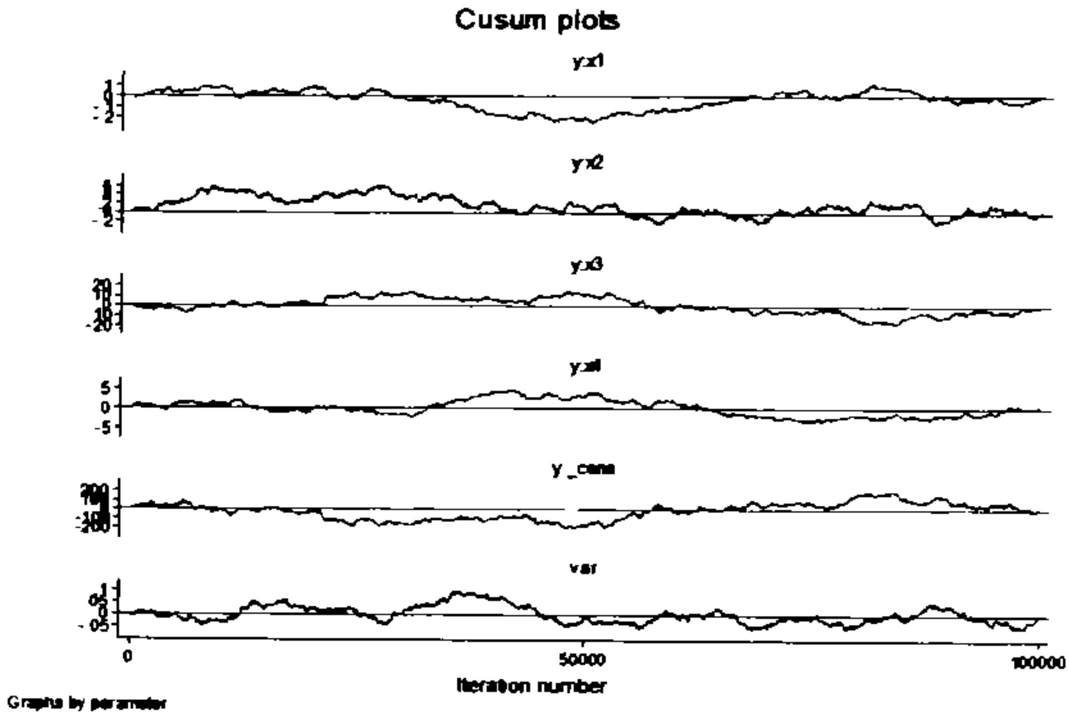


Figure A.4 <u>**Cu sum Plot**</u>



APPDENDIX-B

Diagnostics of Uniform Prior

Figure B.1Trace Plot

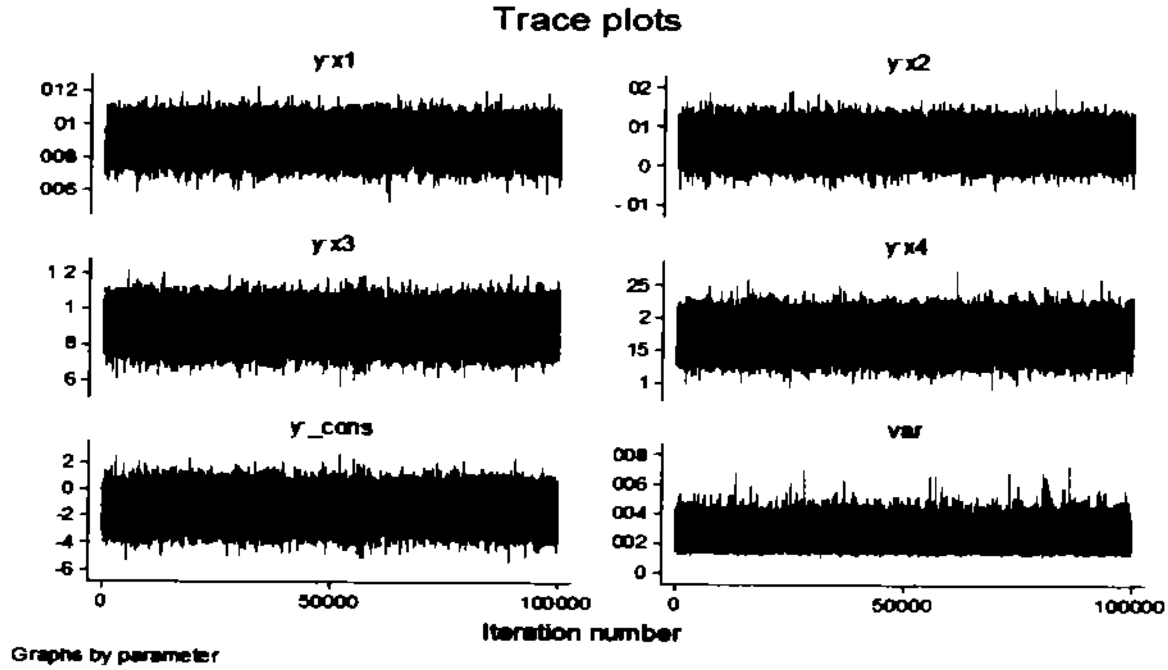


Figure B.2Autocorrelation Plot

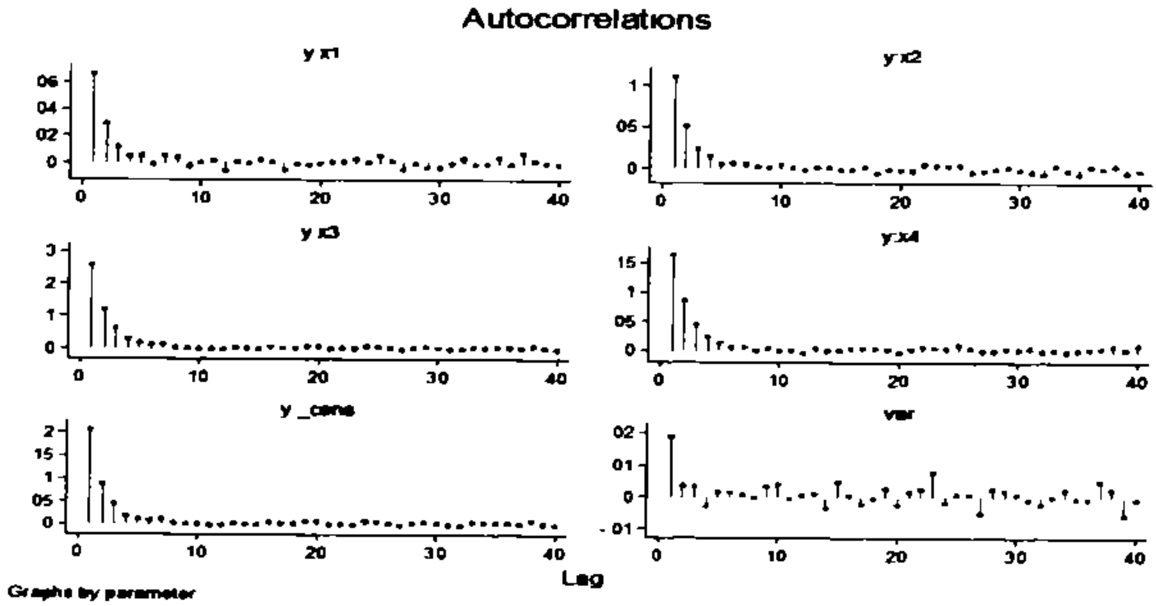


Figure B.3. Kernel Densities

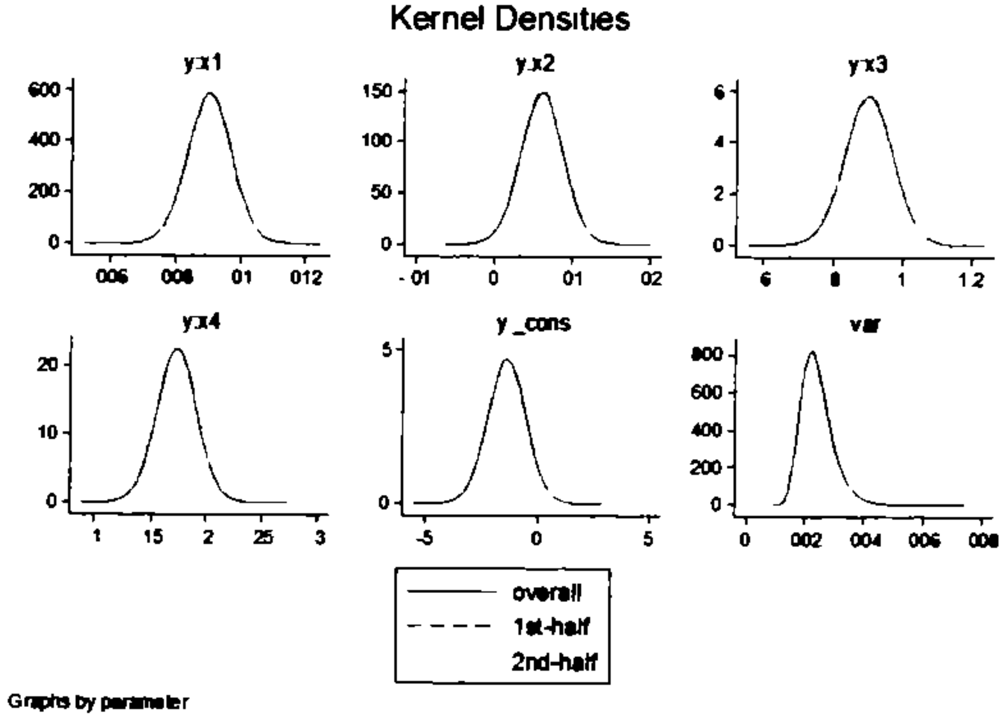
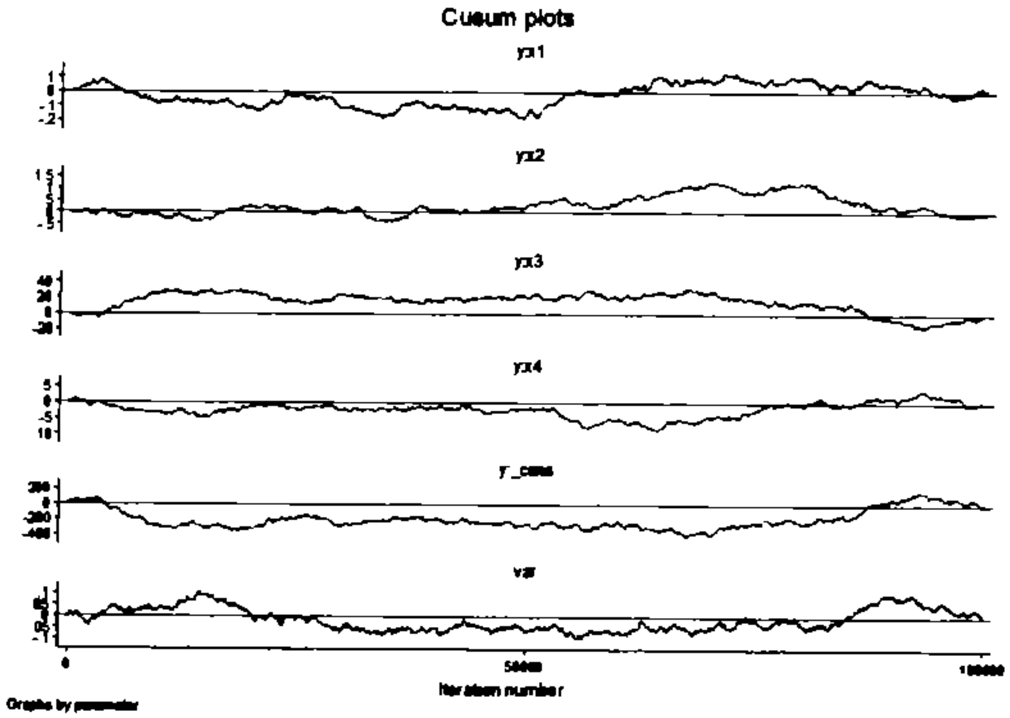


Figure B.4. CuSum Plot



APPDENDIX-C

Diagnostics Weakly Non Informative Prior

Figure C.1. Trace Plots

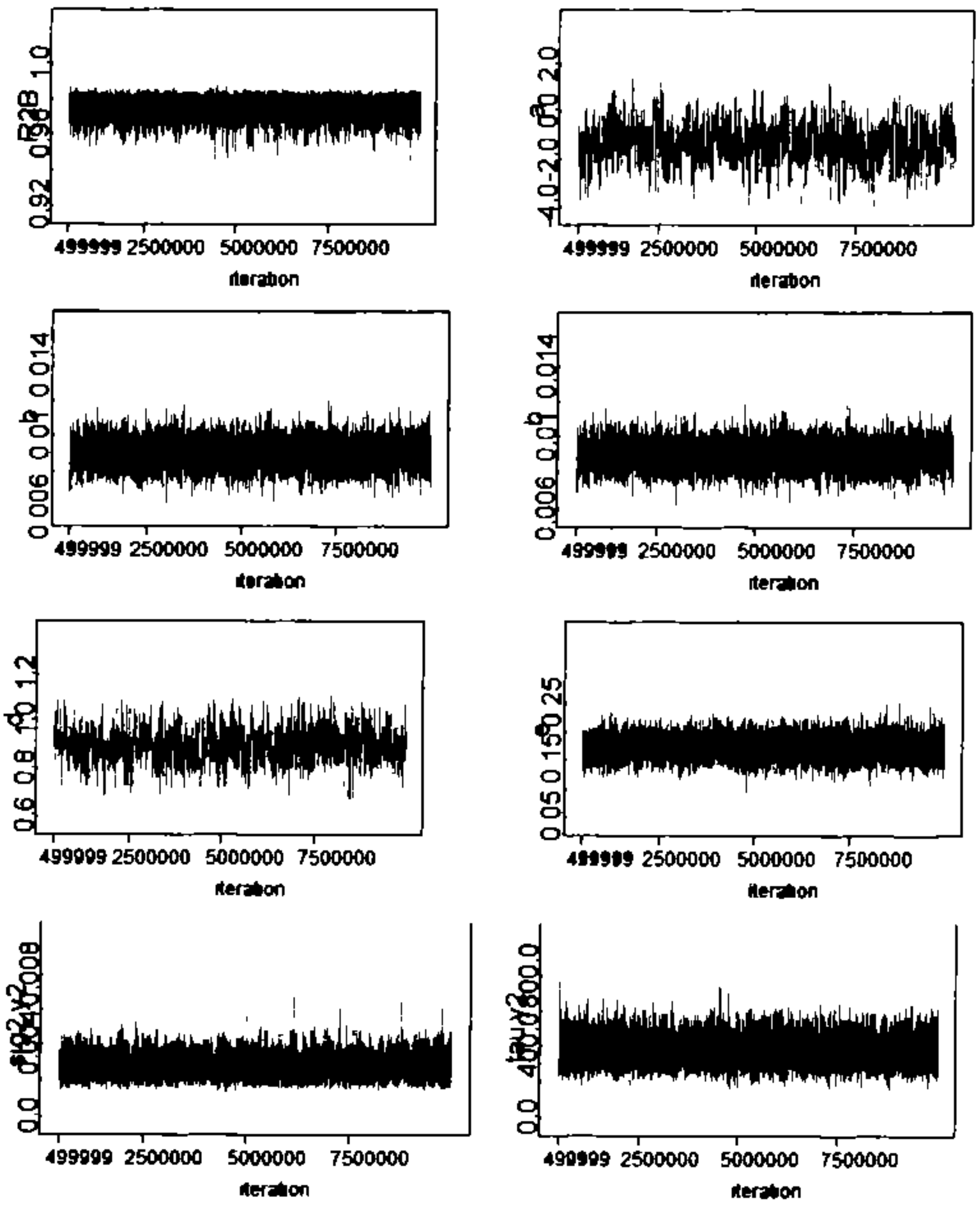


Figure C.2. Kernel Densities

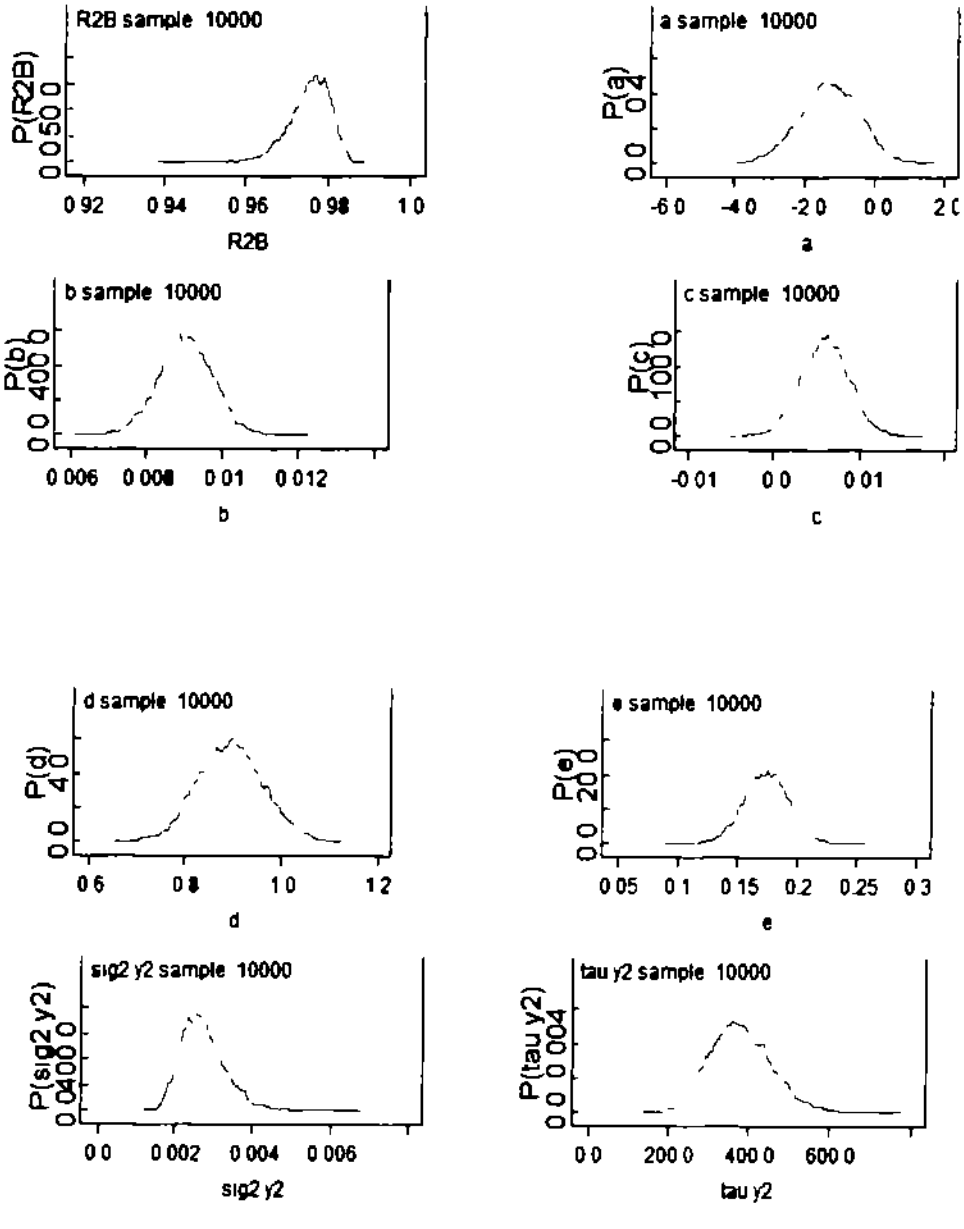


Figure C.3. Autocorrelation Function Plot

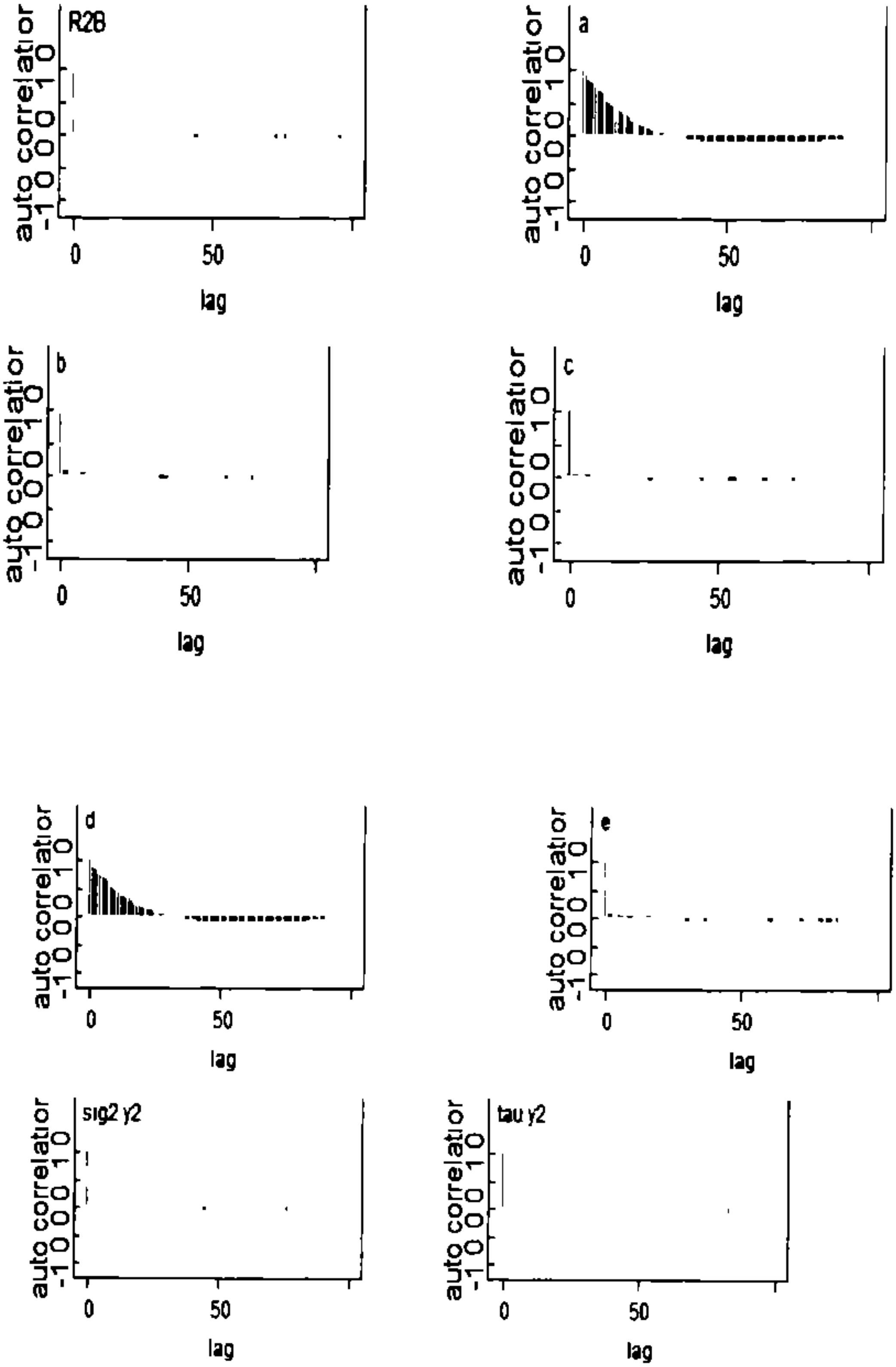


Figure C.4. Quantile Plots