Series Solution of Nonlinear Hydroelastic Waves Equation in a Fluid of Finite Depth

By

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
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IN
MATHEMATICS

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MASTER OF SCIENCE IN MATHEMATICS***

We accept this dissertation as conforming to the required standard

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2016

Dedication:

This thesis is dedicated to my parents for their kind guidance, endless love, support and encouragement to make my future bright.

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Muhammad Imran

DECLARATION

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision of my supervisor **Dr Ahmed Zeeshan**. No portion of the work, presented in this dissertation, has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

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Preface

In recent decades, the ice-cover in the polar region has attracted more and more attention in the field of ocean engineering and polar engineering in view of their practical importance and theoretical investigations. One of the most important problems in this field would appear to be the accurate measurement of the characteristics of **nonlinear hydroelastic** waves traveling beneath a floating ice sheet. And such waves may have been generated in the ice cover itself by the wind, or may have originated by a moving load on the ice sheets. The nonlinear hydroelastic waves propagating beneath floating ice sheet on an inviscid fluid of finite depth were first investigated analytically by A.G. Greenhill[1]

The equation that governs the motion of nonlinear hydroelastic waves in incompressible fluid under an elastic sheet is **nonlinear hydroelastic wave equation.** The propagation of waves of finite amplitude on the surface of an ocean under ice, regarding the ice sheet as an elastic shell. And when we studied it is assumed throughout that there are no frictional forces between the sheet and the fluid beneath. Hydroelastic waves are the waves propagating on sheets of fluid of finite depth that are bounded by elastic plates. The fluid motion is assumed to be both inviscid and irrotational. Two elastic plates sandwich a layer of moving fluid and deform according to the dynamic pressure exerted by the fluid. A comprehensive summary on mathematical method and modeling for the problem can be found in some review articles such as Squire et al [2]. Motivated by the above facts the aim of the present dissertation is to find the series solution of nonlinear hydroelastic waves equation in a fluid of finite depth. The dissertation is structured as follows:

Chapters 1 is introductory and provide reader the basic terminology and equations of fluid flow. The results of Ping Wang [3,4] are reproduced with full mathematical details in chapter 2 and Chapter 3. In these chapters we investigate the motion of nonlinear hydroelastic waves under an ice sheet lying over an incompressible inviscid fluid of finite uniform depth by the regular perturbation and Homotopy analysis method (HAM). Graphical results are presented in order to see the that how Young's modulus of the plate increases, the wave elevation becomes lower, and the increasing thickness of the plate flattens the crest and sharpens the trough of the wave profile.The results obtained here demonstrate that Young's modulus and the thickness of the sheet have important effects on the energy and the profile of nonlinear hydroelastic waves under an ice sheet floating on a fluid of finite depth.

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Chapter 1

Preliminaries

This chapter includes some basic definitions and governing equations relevant to the material presented in the subsequent chapters and idea of Homotopy analysis method is presented for the better understanding of readers.

1.1 Fluid

Fluid is a substance or material that deforms or flows continously when shear stress applied to it, no matter how small the stress may be, fluids include liquids and gases. For example water, milk and blood.

1.2 Fluid mechanics

Fluid mechanics is a well known branch of continuum mechanics. It is usually deals with the behavior of fluids.in the states of rest and motion and its sffects on boundaries is known as fluid mechanics. Fluid mechanics has mainly three types.

Fluid statics: It is the study of fluids at rest.

Fluid kinematics: The study of fluids which are in motion.

Fluid dynamics: The study of the effect of forces on the fluid motion. which deals with the properties of stationary and moving fluids.

1.3 Fluid dynamics

Fluid dynamics is a sub discipline of fluid mechanics that deals with fluid flow, the natural science of fluids (liquids and gases) in motion. It has several sub disciplines itself those are

Aerodynamics: (the study of air and other gases in motion)

Hydrodynamics: (the study of liquids in motion).

1.3.1 Hydrodynamics

It is the study of liquids in motion. Specifically, it looks at the ways different forces affect the movement of liquids. A series of equations explain how the conservation laws of mass, energy, and momentum apply to liquids, particularly those that are not compressed.

1.3.2 Nonlinear hydroelastic waves

One of the most important problems in this field would appear to be the accurate measurement of the characteristics of nonlinear hydroelstic waves traveling beneath a floating ice sheet. And such waves may have been generated in the ice cover itself by the wind, or may have originated by a moving load on the ice sheets. The nonlinear hydroelastic waves propagating beneath floating ice sheet on an inviscid fluid of finite depth were first investigated analytically by A.G. Greenhill [1].

1.3.3 Nonlinear hydroelastic waves equation

The equation that govern the motion of nonlinear hydroelastic waves in incompressible fluid under an elastic sheet is called nonlinear hydroelastic wave equation.

1.4 Characteristics of fluid

1.4.1 Pressure

The amount of force per unit area is known as pressure. If P is the pressure then mathematicaly it can be written as

$$P = \frac{F}{A}, \tag{1.1}$$

1.4.2 Density

The mass per unit volume of the fluid is known as density of that fluid. It is denoted by ρ and mathematicaly we can express it as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}, \tag{1.2}$$

1.4.3 Viscosity

Viscosity is defined as the measure of resistance of a fluid to being deformed by external stresses or either by shear stresses It is usually taken as "thickness or resistance to flow". It is denoted by μ and defined as

$$\mu = \frac{shear\ stress}{rate\ of\ shear\ strain}, \tag{1.3}$$

where μ has the dimension $[M/LT]$.

1.4.4 Kinematic viscosity

kinematic viscosity is stated as the ratio of absolute viscosity to density and is given as

$$\nu = \frac{\mu}{\rho}, \tag{1.4}$$

The units of kinematic viscosity is m^2/s or Stoke (St) and the dimension of kinematic viscosity is $[L^2T^{-1}]$.

1.4.5 Dynamic viscosity

Absolute viscosity or dynamic viscosity is a measure of the internal resistance. Dynamic (absolute) viscosity is the tangential force per unit area required to move one horizontal plane with respect to the other at unit velocity when maintained a unit distance apart by the fluid. Mathematically, it can be written as

$$\mu = \frac{\tau}{du/dy} \tag{1.5}$$

The dynamic viscosity units in SI system are Ns/m^2 or kg/ms, i.e.,

$$1Pas = 1Ns/m^2 = 1kg/ms.$$

In CGS system it can be described as g/cms, $dyne\ s/cm^2$ or $Poise(p)$, i.e.,

$$1Poise = dynes/cm^2 = g/cms = 1/10Ps = 1/10Ns/m^2.$$

1.4.6 Shear stress

A shear stress is defined as the component of stress coplanar with a material cross section.

1.5 Types of fluids

Fluids are iexdt in six main types which can be expressed as following

1.5.1 Ideal fluid

The fluid with zero viscosity $(\mu = 0)$, is generally considered as an ideal fluid and the motion of it is called as ideal or inviscid. In an ideal flow, there is no existence of shear force beacause of vanishing viscosity, i.e.

$$\tau = \mu\frac{du}{dy} = 0, \quad as \quad \mu = 0, \tag{1.6}$$

1.5.2 Real fluids

Those fluids which possess some viscosity $(\mu \neq 0)$, in known as real fluids. Since by newton,s law of viscosity, we have

$$\tau_{yx} = \mu\frac{du}{dy}, \tag{1.7}$$

where τ_{yx} is the shear stress on a fluid surface in the x direction at a distance y from the origin, μ is the viscosity of fluid and $\frac{du}{dy}$ is the rate of deformation.

1.5.3 Newtonian fluid

Newtonian fluid is the fluid which have linear relation between shear stress and rate of strain. It can also be defined as "Fluid which holds Newton,s law of viscosity" is called Newtonian fluid. Mathematically it can be described as

$$\tau_{xy} = \mu\frac{du}{dx}, \tag{1.8}$$

where τ_{xy} is the shear stress, μ is the viscosity of the fluid, x is the direction of the flow and y is perpendicular to the flow. Water, gasoline, air and glecrine exhibits Newtonian behavior.

1.5.4 Non-Newtonian fluid

Non-newtonian fluids are thos fluids in which shear stress is directly but non linearly proportional to the rate of deformation. It can also be stated as "Fluid which obey power law model". Mathematically it can be represented as

$$\tau_{xy} = \left(\mu \frac{du}{dx}\right)^n, \quad n \neq 1 \tag{1.9}$$

or

$$\tau_{xy} = \eta \left(\frac{du}{dy}\right), \tag{1.10}$$

where $\eta = \left(\frac{du}{dy}\right)^{n-1}$ is the viscosity which is the function of deformation. Examples of Non-Newtonian fluids.are toothpaste, blood, ketchup, paint, drilling muds and biological fluids.

1.5.5 Compressible fluids

Compressible fluids are those in which fluid density changes with the change in pressure or temperature. In general, all gasses are treated as compressible fluids.

1.5.6 Incompressible fluids

Incompressible fluids are those in which fluid density remains independent of the pressure or temperature.

1.6 Types of flow

1.6.1 Steady flow

Steady flow is defined as the type of flow in which fluid characteristics like velocity, pressure,density etc at a point do not change with respect to time.

1.6.2 Unsteady flow

If at any point in the fluid, the conditions change with respect to time, the flow is known as unsteady.

1.6.3 Uniform flow

Uniform flow can be defined as if the velocity of the fluid has the same magnitude and direction at every point in the fluid.

1.6.4 Non-uniform flow

If the velocity of the fluid does not have the same magnitude and direction at every point in the fluid is called as non-uniform flow.

1.6.5 Laminar flow or Stream flow

Laminar flow is defined as when fluid flows in parallel layers such that there is no disruption. In laminar flow, the velocity of the fluid at each point does not change in magnitude as well as in direction. Examples include flow of air over an aircraft wing.

1.6.6 Turbulent flow

It is a flow in which fluid undergoes irregular fluctuations as compared to laminar flow. In turbulent flow, the velocity of fluid at each point continuously changes both in magnitude and direction. Examples are flow over a golf ball and smoke rising from cigeratte.

1.6.7 Compressible flow

Compressible flow is that flow in which the density of the fluid changes during the flow and viscosity of the fluid increases with temperature. All gases are compressible fluids.

1.6.8 Incompressible flow

The flow in which the density of the fluid does not change during the flow and viscosity of fluid decreases with temperature is known as incompressible flow. All liquids are incompressible fluids.

1.6.9 Rotational flow

Flow of a fluid in which the curl of the fluid velocity is not zero, so that each minute particle of fluid rotates about its own axis. Also known as rotational motion. Mathematically it can be desribed as

$$\nabla \times V \neq 0, \tag{1.11}$$

1.6.10 Irrotational flow

Flow of a fluid in which the curl of the fluid velocity is zero is known as irrotational flow of the fluid.

Mathematically it can be desribed as

$$\nabla \times V = 0, \tag{1.12}$$

1.6.11 Vorticity

In simple words, vorticity is the rotation of the fluid. The rate of rotation of fluid can be expressed various ways.

Mathematicaly

$$\omega = \nabla \times V, \tag{1.13}$$

1.7 Basic Governing equations

In this section the general form of equations governing the flow of a fluid are presented in usual notations, These include

1.7.1 The general problem of wave motion

The problem which we have to solve, in all studies of waves on irrotational and incompressible flows, whether studies of propagating waves or standing waves or considering aspects of propagation, diffraction, reflection or refraction is to solve Laplace equation. Then the governing equations for a velocity potential $\phi(x, z, t)$ can be written as

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial^2 x} + \frac{\partial^2\phi}{\partial^2 z} = 0, (-h \leq z \leq \zeta(x,t)), \tag{1.14}$$

where $\zeta(x,t)$ is wave surface elevation.

1.7.2 The continuity equation:

Continuity equation is the mathematical expression for law of conservation of mass and mathematically it is described as

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho \mathbf{V}) = 0, \tag{1.15}$$

where $\mathbf{V}$ is the velocity field. If density "ρ" remains constant with respect to time and space then for incompressible flow, we have

$$\mathrm{div}\,\mathbf{V} = 0. \tag{1.16}$$

1.8 Boundary Conditions

1.8.1 Kinematic boundary condition

If a fluid particle is adjacent to a boundary then we must impose a condition which links the velocity of the boundary to that of the particle on the un known surface $z = \zeta(x,t)$. This is known as the kinematic boundary condition.which is

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} - \frac{\partial\phi}{\partial z} = 0. \tag{1.17}$$

1.8.2 Dynamic boundary condition

The dynamic boundary condition at the free surface is that the pressure equals the exterior athmospheric pressure: $p = patm(const)$. on $z = \zeta(x,t)$, so the dynamic boundary condition becomes

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla^2\phi + \frac{p_e}{\rho} + g\zeta = 0. \tag{1.18}$$

1.9 Homotopy

A homotopy between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0,1] \to Y$ from the product of the space X with the unit interval $[0,1]$ to Y such that, if $x \in X$ then $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

1.9.1 Homotopy analysis method (HAM)

It is a general analytical approach for obtaining approximate series solutions to nonlinear di®erential Equation. Based on the homotopy analysis method (HAM) which has been successfully applied to solve many types of problems The homotopy analysis method (HAM) was first described by Liao [5] in his PhD dissertation in 1992. For a given nonlinear differential equation.

$$N[u(x)] = 0, x\epsilon\Omega. \tag{1.19}$$

where N is a nonlinear operator and $u(x)$ is a unknown function, Liao [7] constructed a one-parameter family of equations in the embedding parameter $q \in [0,1]$, called the zeroth-order deformation equation

$$(1-q)L[(x;q) - u_o(x)] + qN[(x;q)] = 0, x \in \Omega, q \in [0,1]. \tag{1.20}$$

where L is an auxiliary linear operator and $u_0(x)$ is an initial guess.In theory, the homotopy provides us much larger freedom to choose both of the auxiliary linear operator L and the intial guess At $q = 0$ and$q = 1$, we have$(x;0) = u_0(x)$ and $(x;1) = u(x)$, respectively.So, as the embedding parameter$q \in [0,1]$increases from0 to 1, the solution $(x;q)$ of the zeroth-order deformation equations varies (or deforms) from the initial guess $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear differential equation$N[u(x)] = 0$.

Since $(x;q)$ is also dependent upon the embedding parameter $q \in [0,1]$, we can expand it into the Maclaurin series with respect to

$$\phi(x;q) = u_0(x) + \sum u_n(x)q^n. \tag{1.21}$$

called the homotopy-Maclaurin series. Note that we have extremely large freedom to choose the auxiliary linear operator L and the initial guess $u_0(x)$. Assuming that, the auxiliary linear operator L and the initial guess $u_0(x)$ are so properly chosen that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy-series solution

$$u(x) = u_0(x) + \sum u_n(x). \tag{1.22}$$

which satisfies the original equation $N[u(x)] = 0$, as proved by Liao [19, 20] in general Here,$u_n(x)$ is governed by the so-called high-order deformation equation

$$L[u_n(x) - \chi_n u_{n-1}(x)] = -\delta_{n-1}(x). \tag{1.23}$$

where χ_k equals to 1 when $k \geq 2$ but zero otherwise, and

$$\delta_k(x) = \frac{1}{ki}\frac{\partial^k N[\phi(x;q)}{\partial q^k}. \tag{1.24}$$

The high-order deformation equation (1.10) is always linear with the known term on the right-hand side, therefore is easy to solve, as long as we choose the auxiliary linear operator L properly.

1.9.2 Homotopy perturbation method (HPM)

Consider the following nonlinear differential equation

$$L(v(r)) + N(v(r)) = 0 \; r \in \Omega. \tag{1.25}$$

with the boundary condition

$$B\left(v, \frac{\partial v}{\partial n}, ...\right) = 0, \; r \in \Gamma. \tag{1.26}$$

where L is a linear operator, N is a nonlinear operator, Γ is the boundary of domain Ω, B is a boundary operator, and $\frac{\partial}{\partial n}$ denotes differential along the normal drawn outwards from Ω.

By means of HPM, a homotopy for equation (1.12) is constructed as follows:

$$\left.\begin{aligned} N(v) &= N(v)\mid_{p=0} + \tfrac{\partial N(v)}{\partial p}\mid_{p=0}\ p + ... \\ &= N(v_0) + \left(\tfrac{\partial N(v)}{\partial v}\tfrac{\partial v}{\partial p}\right)\mid_{p=0} p + ... \\ &= N(v_0) + \tfrac{\partial N(v)}{\partial v}\mid_{v=v_0} v_1\ p + ... \end{aligned}\right\}. \tag{1.32}$$

We construct a new homotopy for equation (1.12) as follows:

$$\begin{aligned} H(v,p) = L(v) - L(v_0) + p\left(L(v_0) + K_0(r,C_0)N(v_0)\right) \\ +p\left(K_1(r,C_1)\tfrac{\partial N(v)}{\partial v}\mid_{v=v_0} v_1\right) + ... = 0. \end{aligned} \tag{1.33}$$

where $K_i(r, C_i)$ for $i = 0, 1, ...$ is an auxiliary function, and C_i is a vector of unknown constants. By equating the coefficients of the same powers of p in equation (1.20), we obtain

$$p^0 : L(v_0) - L(v_0) = 0. \tag{1.34}$$

$$p^1 : L(v_1) + L(v_0) + K_0(r,\ C_0)\ N(v_0) = 0. \tag{1.35}$$

$$p^2 : L(v_2) + K_1(r,\ C_1)\frac{\partial N(v)}{\partial v}\mid_{v=v_0} v_1 = 0. \tag{1.36}$$

and so on. The functions $K_0, K_1, ...$are not unique and can be chosen as the same form of nonlinear operator N. The constant C_i that appears in the function $K_i(r, C_i)$ can be optimally determined by minimizing the following residual functional

$$I = \int_a^b \left(L(v_M) + N(v_M)\right)^2 dr. \tag{1.37}$$

where a and b are two values depending on the given problem, and $v_{(M)}$ is the Mth-order approximate solution, which can be written as

$$v = v_{(0)} + v_{(1)} + ... + v_{(M)}. \tag{1.38}$$

Once the parameter C_i is known, the solution of nonlinear differential equation in equation (1.12) subject to the boundary condition given in equation (1.13) can be immediately determined.

1.9.4 Optimal homotopy asymptotic method (OHAM)

We apply OHAM to the following differential equation

$$A(u(x)) + g(x) = 0, \ x \in \Re. \tag{1.39}$$

where $\Re$ is real number and the corresponding boundary conditions are:

$$B\left(u, \frac{\partial u}{\partial x}, \ldots\right) = 0. \tag{1.40}$$

where A is a general differential operator, $g(x)$ is a known analytical function, $u(x)$ is an unknown function. equation (1.26) can therefore be written as follows:

$$L\left(u\left(x\right)\right) + g\left(x\right) + N\left(u\left(x\right)\right) = 0. \tag{1.41}$$

Construct a homotopy $u = \phi(x,p) : \Re \times [0,1] \to \Re$ which satisfies

$$\begin{gathered} H(\phi(x,p),p) = (1-p)[L(\phi(x,p)) + g(x)] + H\left(p\right) \\ [A(\phi(x,p) + g(x))] = 0, \ p \in [0,1]. \end{gathered} \tag{1.42}$$

$$B\left(\phi(x,p), \frac{\partial \phi(x,p)}{\partial x}\right) = 0. \tag{1.43}$$

$H\left(p\right)$ is a nonzero auxiliary function for $p \neq 0$, $H(0) = 0$, $\phi(x,p)$ is an unknown function and p varies from 0 to 1. The solution $\phi(x,p)$ varies from $\phi(x,0) = u_0(x)$ to the solution $\phi(x,1) = u(x)$ equation (1.29) is called optimal homotopy equation. Clearly, we have

$$p = 0 \Rightarrow H(\phi(x,0),0) = L(\phi(x,0)) + g(x) = 0. \tag{1.44}$$

$$p = 1 \Rightarrow H(\phi(x,1),1) = H(1)\left[A(\phi(x,1)) + g(x)\right] = 0. \tag{1.45}$$

We choose auxiliary function $H(p)$ in the form

$$H(p) = pD_1 + pD_2 + \ldots. \tag{1.46}$$

where $D_1, D_2, \ldots$are constants which can be determined latter. Expanding $\phi(x,p,D_i)$ in Taylor's series about p, we obtain

$$\phi(x,p,D_i) = u_0(x) + \sum_{k=1}^{\infty} u_k(x, D_1 + D_2 + \ldots D_k)\, p^k. \tag{1.47}$$

Now substituting equation (1.34) into equations. (1.29) & (1.30) and then equating the coefficient of like powers of p, we obtained the solutions of zeroth order, first order and second order problems. It has been observed that the convergence of series (1.34) depends upon the auxiliary constants $D_1, D_2, \ldots$, we obtain the governing equation of $u_0(x)$, given by equation (1.29), and the governing equation of $u_k(x)$ i. e,

$$L(u_1(x)) = D_1 N_0(u_0(x)),\ B\left(u_1, \frac{du_1}{dx}\right) = 0. \tag{1.48}$$

$$\begin{gathered} L(u_k(x) - u_{k-1}(x)) = D_k N_0(u_0(x)) \\ + \sum_{i=1}^{k-1} D_i \left[L u_{k-1}(x) + N_{k-1} \left(\begin{array}{c} u_0(x), \\ u_1(x), \ldots, u_{k-1}(x) \end{array} \right) \right]. \end{gathered} \tag{1.49}$$

Corresponding boundary conditions are

$$B\left(u_k, \frac{du_k}{dx}, \ldots\right) = 0,\ k = 2, 3, \ldots \tag{1.50}$$

$$N(\phi(x,p,D_i)) = N_0(u_0(x)) + \sum_{m\geq 1}^{\infty} N_m((u_0, u_1, \ldots, u_m))\, p^m,\ i = 1, 2, \ldots \tag{1.51}$$

where $N_m(u_0(x), u_1(x), \ldots, u_m(x))$ is obtained by expanding $N(\phi(x,p,D_i))$ in series with respect to the embedding parameter p and $\phi(x,p,D_i)$ is given in equation (1.31). It should be emphasized that u_k for $k \geq 0$ are governed by the linear equations (1.29), (1.32) & (1.34) with the linear boundary conditions that come from original problem, which can be easily solved.

The convergence of the series in equation (1.31) depends upon the auxiliary constants $D_1, D_2, \ldots$. If it is convergent at $p = 1$, we get

$$u(x, D_i) = u_0(x) + \sum_{k\geq 1}^{m} u_k(x, D_i). \tag{1.52}$$

The solution of equation (1.26) can be determined approximately in the form

$$u^{(m)}(x, D_i) = u_0(x) + \sum_{k=1}^{m} u_k(x, D_i),\ i = 1, 2, ..., m. \tag{1.53}$$

Substituting equation (1.38) into equation (1.26), yields the following residual

$$R(x, D_i) = L\left(u^{(m)}(x, D_i)\right) + g(x) + N\left(u^{(m)}(x, D_i)\right),\ i = 1, 2, ..., m. \tag{1.54}$$

If $R(x, D_i) = 0$ then $u^{(m)}(x, D_i)$ happens to be the exact solution. Generally such case will not arise for nonlinear problems, but we can minimize the functional

$$J(D_i) = \int_a^b R^2(x, D_i)\, dx. \tag{1.55}$$

The unknown constants D_i $(i = 1, 2, \ldots m)$ can be optimally identified from the following conditions

$$\frac{\partial J}{\partial D_1} = \frac{\partial J}{\partial D_2} = ... = \frac{\partial J}{\partial D_m} = 0. \tag{1.56}$$

With these constants known, the approximate solution (of order m) in equation (1.38) is well-determined. The constants D_i can be determined in another forms, for example, if $k_i \in (a, b), i = 1, 2, \ldots m$ and substituting k_i into equation (1.39), we obtain the equation

$$R(k_1, D_i) = R(k_2, D_i) = ... = R(k_m, D_i) = 0,\ i = 1, 2, ..., m. \tag{1.57}$$

1.10 Genetic algorithm and Nelder mead method(GA & NM).

Genetic Algorithm is an optimization tool based on Darwinian evolution which has been developed in 1976, but its utilization in heat transfer problems is not been tested. In fact Genetic Algorithm plays an important role when multiple parameters are involved. The main procedure is inspired by the Darwinian theory of evolution "The survival of the fittest." The Genetic Algorithm is a random search technique. Major advantage of Genetic algorithm is that the demand about computer memory for nonlinear problems is minimum. Genetic Algorithm will be helpful for future even to get minimum and maximum solutions to satisfy inequality relationships as well. There are five main decision points in the procedure given below:

(1) Encoding technique (chromosome structure)

(a) Mechanism to encode solution

(2) Evaluation function (environment)

(a) Fitness function

(3) Selection procedure (creation)

(4) Generating chromosome diversity (evolution)

(a) Crossover, mutation

(5) Parameter settings (practice and art)

(a) Termination condition

(b) (Random) initialization of population

There are several techniques for optimization like analytical approach, downhill simplex method, gradient descent, Newton's method and so on. Moreover, the Nelder Mead method is direct search simplex algorithm published in 1965 and is one of the most widely used methods for nonlinear unconstrained optimization. The Nelder-Mead method minimizes a nonlinear function of n real variables without taking any derivative. The function is evaluated at each point of the simplex structure formed by (n+1) points and the vertex with highest value is replaced by a new point with a lower value. It continues until the minimum value of function is achieved. Furthermore, in topological approach the non-zero auxiliary parameter which can adjust and control the convergence of the series solutions.[6,11]. The Genetic Algorithm and Nelder Mead method is used in order to find the optimum value of c_0 .Also it minimize the residual square error ε_m^T .Which shows its validity and great potential to solve the nonlinear problems in science and engineering [25]. In the forthcoming section we used this method to illustrate the significance of optimal convergence contril parameter c_0 on the velocity potential and wave deflection.

Chapter 2

Series solution of nonlinear hydroelastic waves equation in a thin elastic plate floating on a fluid

2.1 Introduction

The aim of this chapter is to revisit the work of Ping Wang and LU Dong Qiang [3]. In this chapter behaviour of the progressive waves is discussed with HAM. A convergent homotopy series solution for the nonlinear hydroelastic waves is calculated with the help of least squared residual. Also the dynamic effects of plate such as Young's modulus, thickness and density are studied.

2.2 Mathametical formulation

Let us assume an infinite plate floating on an infinitely deep water having thickness d which produces nonlinear hydroelastic waves. Cartesian coordinates OXZ are usually selected so that the plate spread out to the infinity along $x-axis$ and $z-axis$ and $z = 0$ shows the uninterrupted plate water boundary. It is supposed that there is no cavitation between water and plate .And $z = \zeta(x,t)$ is the deflection of plate. It is assumed fluid is inviscid , incompressible and irrotational.$\phi(x,z,t)$ is velocity potential which satisfies the Laplace equation.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=0,(z\leq\zeta(x,t)). \tag{2.1}$$

At deep water the boundary condition is

$$\frac{\partial\phi}{\partial z}=0,(z=-\infty). \tag{2.2}$$

By the supposition that any fluid particle which is in between elastic plate and water surface will remain on it.On the unknown plate water interface $z=\zeta(x,t)$ the kinematic boundary condition is

$$\frac{\partial\zeta}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}-\frac{\partial\phi}{\partial z}=0. \tag{2.3}$$

and dynamic boundary condition is

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+\frac{p_e}{\rho}+g\zeta=0. \tag{2.4}$$

where $p_e(x,t)$, ρ and g are plate water interface pressure ,fluid density and gravitational acceleration respectively.By the Kirchhoff beam theory .For constant thickness d and uniform mass density ρ_e of the plate the relationship between plate deflection $\zeta(x,t)$ and pressure $p_e(x,t)$ in view of Kirchhoff (Euler Bernoulli) beam theory is

$$p_e=D\frac{\partial^4\zeta}{\partial x^4}+m_e(\frac{\partial^2\zeta}{\partial t^2}+g). \tag{2.5}$$

where $m_e=p_e d$; $D=\frac{ed^3}{[12(1-\nu^2)]}$. By substituting equation (2.5) into equation (2.4) gives the full form of dynamic boundary condition as

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}|\nabla\phi|^2+g\zeta+\frac{1}{\rho}\left[D\frac{\partial^4\phi}{\partial x^4}+m_e\left(\frac{\partial^2\zeta}{\partial t^2}+g\right)\right]=0. \tag{2.6}$$

By the concept of traveling wave method an independent variable transformation is introduced as

$$X=kx-\omega t. \tag{2.7}$$

where k is wave number and ω is angular frequency of incident wave.Now velocity potential function $\phi(x,z,t)=\phi(X,z)$ and the hydroelastic wave profile $\zeta(x,t)=\zeta(X)$ are used.For simplificationn by putting all equations into dimensionless form following dimensionless quantities are used

$$\begin{aligned} x^* &= kx,\ z^*=kz, t^*=t(gk)^{\frac{1}{2}},\ d^*=kd,\ \phi^*=\frac{k^2\phi}{(gk)^{\frac{1}{2}}}. \\ \zeta^* &= k\zeta,\ \omega^*=\frac{\omega}{(gk)^{\frac{1}{2}}},\ D^*=\frac{k^4D}{(\rho g)},\ E^*=\frac{kE}{(\rho g)}. \\ \rho_e^* &= \frac{\rho_e}{\rho},\ m_e^*=\frac{km_e}{\rho}. \end{aligned} \tag{2.8}$$

In the succeeding formulae the asterisks denoting dimensionless quantities will be replaced. Then the dimensionless equations for the velocity potential are

$$\frac{\partial^2\phi}{\partial X^2}+\frac{\partial^2\phi}{\partial z^2}=0,(z\le\zeta(X)). \tag{2.9}$$

$$\frac{\partial\phi}{\partial z}=0,(z=-\infty). \tag{2.10}$$

In view of (2.7) on $z=\zeta(X)$, (2.3) and (2.6) are transformed into

$$-\omega\frac{d\zeta}{dX}+\frac{\partial\phi}{\partial X}\frac{d\zeta}{dX}-\frac{\partial\phi}{\partial z}=0. \tag{2.11}$$

$$-\omega\frac{\partial\phi}{\partial X}+f+\zeta+\left[Dk^4\frac{d^4\zeta}{dX^4}+m_e\left(\omega^2\frac{d^2\zeta}{dX^2}+1\right)\right]=0. \tag{2.12}$$

respectively where

$$f=\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial X}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]. \tag{2.13}$$

A partial combination of equations. (2.11) and (2.12) gives the boundary conditions on $z=\zeta(X)$ as follows

$$\omega^2\frac{\partial^2\phi}{\partial X^2}+\frac{\partial\phi}{\partial z}-\omega\frac{\partial f}{\partial X}-\omega D\frac{d^5\zeta}{dX^5}-\omega^3m_e\frac{d^3\zeta}{dX^3}-\frac{\partial\phi}{\partial X}\frac{d\zeta}{dX}=0. \tag{2.14}$$

The velocity potential $\phi(X,z)$ and the plate deflection $\zeta(X)$ are derived by equations (2.9), (2.10), (2.12), and (2.14) in form of Series solutions for $\varphi(X,z)$ and $\zeta(X)$ will be derived based on the HAM in the subsequent section.

2.3 Analytic approach based on the homotopy analysis method

2.3.1 Zeroth-order deformation equations

In view of the homotopy analysis method first of all let us assume a set of base functions and solution expressions as it seems impossible to presume the expression forms for unknown potential function and plate deflection. By physical background of progressive gravity wave elevation on free surface, $\zeta(X)$ can be written as

$$\zeta(X) = \sum_{i=0}^{+\infty} \beta_i \cos(iX). \tag{2.15}$$

with a set of base functions $\{cos(iX), i \geq 0\}$ where βi is an unknown coefficient. Since it is supposed that there is no gap between the bottom surface of thin elastic plate and top surface of the fluid layer. In view of linear wave theory solutions to the Laplace equation (2.9) can be derived by the separation of variables method. Therefore the plate deflection $\zeta(X)$ can also be expressed in the form as equation (2.15). Since the solution expression of the potential function is.

$$\phi(X,z) = \sum_{i=1}^{+\infty} \alpha_i \exp(kz) \sin(iX). \tag{2.16}$$

In view of the solution expression (2.16) and the boundary condition (2.10) with a set of base functions $\{exp(kz)sin(iX), i \geq 0\}$, where αi is an unknown coefficient.The initial approximation for potential function is given by.

$$\phi_0(X,z) = \alpha_{0,1} \exp(z) \sin(X). \tag{2.17}$$

where $\alpha_{0,1}$ is an unknown coefficient. since

$$\zeta_0(X) = 0. \tag{2.18}$$

In view of [8] the initial approximation for $\zeta(X)$ to simplify the subsequent solution procedure. Although the initial guess $\zeta_0(X)$ is zero. Based on the nonlinear boundary condition for equations (2.12) and (2.14), two nonlinear operators N_1 and N_2 are defined

$$\begin{aligned} N_1\left[\Phi(X,z;q),\eta(X;q)\right] &= \omega^2\frac{\partial^2\Phi(X,z;q)}{\partial X^2}+\frac{\partial\Phi(X,z;q)}{\partial z}-\omega\frac{\partial F}{\partial X}-\omega D\frac{\partial^5\eta(X;q)}{\partial X^5} \\ &\quad -\omega^3 m_e\frac{\partial^3\eta(X;q)}{\partial X^3}-\frac{\partial\Phi(X,z;q)}{\partial X}\frac{\partial\eta(X;q)}{\partial X}. \end{aligned} \tag{2.19}$$

$$N_2\left[\eta(X;q),\Phi(X,z;q)\right] = -\omega\frac{\partial\Phi(X,z;q)}{\partial X}+F+D\frac{\partial^4\eta(X;q)}{\partial X^4}-m_e[\omega^2\frac{\partial^2\eta(X;q)}{\partial X^2}+1]. \tag{2.20}$$

where

$$F=\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial X}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]. \tag{2.21}$$

Here an auxiliary linear differential operator $\overline{\mathcal{L}}_1$ is chosen and $q\in[0,1]$ is the embedding parameter in the HAM. Here nonlinear operator N_1 holds a linear operator of $\Phi(X,z;q)$ as given below.

$$\overline{\mathcal{L}}_1\left[\Phi(X,z;q)\right]=\omega^2\frac{\partial^2\Phi(X,z;q)}{\partial X^2}+g\frac{\partial\Phi(X,z;q)}{\partial z}. \tag{2.22}$$

In view of [6, 7] the angular frequency based on the linear wave theory is approximately equal to one. i.e

$$\omega\approx 1 \tag{2.23}$$

By the simplification of equation (2.22), the auxiliary linear operator takes the form as

$$\mathcal{L}_1\left[\Phi\left(X,z;q\right)\right]=\frac{\partial^2\Phi\left(X,z;q\right)}{\partial X^2}+\frac{\partial\Phi\left(X,z;q\right)}{\partial z}. \tag{2.24}$$

Now the linear operator for the wave function $\eta(X;q)$ in the nonlinear operator N_2, another auxiliary linear operator is as follows where $\mathcal{L}_1[0]=0$

$$\mathcal{L}_2\left[\eta\left(X;q\right)\right]=\frac{\partial^4\Phi\left(X,z;q\right)}{\partial X^4}+\frac{\partial^2\eta\left(X;q\right)}{\partial X^2}+\eta(X;q). \tag{2.25}$$

where $\mathcal{L}_2[0]=0$.

Now for the zeroth order deformation equation the equations (2.9), (2.10), (2.12),and (2.14), takes the form as

$$\frac{\partial^2\Phi\left(X,z;q\right)}{\partial X^2}+\frac{\partial^2\Phi\left(X,z;q\right)}{\partial z^2}=0,\left(z\leq\eta\left(X;q\right)\right). \tag{2.26}$$

$$\frac{\partial\Phi\left(X,z;q\right)}{\partial z}=0,\left(z=-\infty\right). \tag{2.27}$$

$$(1-q)\,\mathcal{L}_1\left[\Phi\left(X,z;q\right)-\phi_0\left(X,z\right)\right]=qc_0N_1\left[\Phi\left(X,z;q\right),\eta\left(X;q\right)\right],\left(z=\eta\left(X;q\right)\right). \tag{2.28}$$

$$(1-q)\,\mathcal{L}_2\left[\eta(X;q)-\zeta_0\left(X\right)\right]=qc_0N_2\left[\eta\left(X;q\right),\Phi\left(X,z;q\right)\right],\left(z=\eta\left(X;q\right)\right). \tag{2.29}$$

By the help of Taylor series for $\Phi(X,z;q)$ and $\eta(X;q)$ at $q=0$, the exact solutions $\phi(X,z)$ and $\zeta(X)$ from initial approximation $\phi_0(X,z)$ and $\zeta_0(X)$ and from equations (2.28) and (2.29) can be found.

$$\Phi\left(X,z;q\right)=\phi_0\left(X,z\right)+\sum_{m=1}^{+\infty}\phi_m\left(X,z\right)q^m. \tag{2.30}$$

$$\eta\left(X;z\right)=\zeta_0\left(X\right)+\sum_{m=1}^{+\infty}\zeta_m\left(X,z\right)q^m. \tag{2.31}$$

$$\{\phi_m(X,z),\zeta_m(X)\} = \frac{1}{m!}\frac{\partial^m}{\partial q^m}\{\Phi(X,z;q),\eta(X;q)\} \text{ at } q=0 \tag{2.32}$$

Assuming that c_0 is right chosen in the series of equations (2.30) and (2.31) converges at $q = 1$, since by formal homotopy series solutions

$$\Phi(X;z) = \Phi(X;z,1) = \phi_0(X,z) + \sum_{m=1}^{+\infty}\phi_m(X,z). \tag{2.33}$$

$$\zeta(X) = \eta(X;1) = \zeta_0(X) + \sum_{m=1}^{+\infty}\zeta_m(X). \tag{2.34}$$

And for the nth order approximation

$$\phi(X,z) = \phi_0(X,z) + \sum_{m=1}^{+n}\phi_m(X,z). \tag{2.35}$$

$$\zeta(X) = \zeta_0(X) + \sum_{m=1}^{+\infty}\zeta_m(X). \tag{2.36}$$

2.3.2 Deformation equations of high order

Here PDEs for the unknown functions $\phi_m(X,z)$ and $\zeta_m(X)$ are calculated from the zeroth order deformation equations. Substituting (2.30) and (2.31) into (2.26) and (2.27), and then equating likepowers of the embedding parameter q.

$$\frac{\partial^2\phi_m(X,z)}{\partial X^2} + \frac{\partial^2\phi_m(X,z)}{\partial z^2} = 0, (z \leq 0). \tag{2.37}$$

$$\frac{\partial\phi_m(X,z)}{\partial X} = 0, (z = -\infty). \tag{2.38}$$

where $m \geq 1$

By putting the suitable series into boundary conditions (2.28) and (2.29), two linear BCs are as follows on $z = 0$

$$|\mathcal{L}_1(\phi_m)|_{z=0} = c_0\Delta^{\phi}_{m-1} + \chi_m S_{m-1} - \overline{S_m}. \tag{2.39}$$

$$\mathcal{L}_2(\zeta_m) = c_0\Delta^{\zeta}_{m-1} + \chi_m\left(\frac{d^4\zeta_{m-1}}{dX^4} + \frac{d^2\zeta_{m-1}}{dX^2} + \zeta_{m-1}\right). \quad (2.40)$$

where

$$S_{m-1} = \sum_{i=0}^{m-2}\left(\frac{d^2\psi_{m-1-i,i}}{dX^2} + \gamma_{m-1-i,i}\right)$$

$$\overline{S}_m = \sum_{i=1}^{m-1}\left(\frac{d^2\psi_{m-i,i}}{dX^2} + \gamma_{m-i,i}\right)$$

$$\Delta^{\zeta}_{m-1} = -\omega\frac{d\varphi_{m-1}}{dX} + \frac{1}{2}\sum_{n=0}^{m-1}\left(\frac{d\varphi_n}{dX}\frac{d\varphi_{m-1-n}}{dX} + \overline{\varphi}_n\overline{\varphi}_{m-1-n}\right) + \zeta_{m-1} + D\frac{d^4\zeta_{m-1}}{dX^4} + m_e\omega^2\frac{d^2\zeta_{m-1}}{dX^2}$$

and

$$\Delta^{\phi}_{m-1} = \omega^2\frac{d^2\varphi_m}{dX^2} + \overline{\varphi}_m - \omega\sum_{n=0}^{m}\left(\frac{d\varphi_n}{dX}\frac{d^2\varphi_{m-n}}{dX^2} + \overline{\varphi}_n\frac{d\overline{\varphi}_{m-n}}{dX}\right) - \omega D\frac{d^5\zeta_m}{dX^5} - \omega^3 m_e\frac{d^3\zeta_m}{dX^3} - \sum_{n=0}^{m}\frac{d\varphi_n}{dX}\frac{d\zeta_{m-n}}{dX}$$

for

$$\chi_m = \begin{cases} 0, m \le 1 \\ 1, m > 1 \end{cases} \quad (2.41)$$

we introduce an additional equation for the wave height H

$$\zeta_1(m\pi) - \zeta_1(m\pi) = H = 2a. \quad (2.42)$$

Here m and n are even and odd integers respectively and a is the dimensionless amplitude of the plate deflection $\zeta(X)$. As it is clear that equations (2.28) and (2.29) hold on the unknown boundary function $z = \eta(X;q)$ while equations (2.39) and (2.40) hold on $z = 0$.Hence the equations (2.37) to (2.42) can be solved.

2.3.3 Approximation and iteration of solutions

By applying the inverse linear operator $\pounds_2$ on equation (2.40) $\zeta_1(X)$ can be calculated as follows:

$$\zeta_1(X) = \frac{1}{2}(2d\rho c_0 + c_0\alpha_{0,1}^2) - \omega c_0\alpha_{0,1}\cos(X). \tag{2.43}$$

Here $\alpha_{0,1}$ is still unknown which can be determined by equation (2.42) Now by the inverse linear operator $\pounds_1$ in equation (2.39), $\phi_1(X,z)$ can easily be derived. Since

$$\alpha_{0,1} = \frac{-a}{\omega c_0}. \tag{2.44}$$

$$\phi_1(X,z) = \alpha_{1,1}\exp(z)\sin(X). \tag{2.45}$$

As $\alpha_{1,1}$ is still unknown which can be calculated with the help of (2.46) by eliminating the secular term $sin(X)$. Now with the aid of first order approximations equations (2.39) and (2.40) takes the form as

$$\begin{aligned}
\zeta_2(X) &= \frac{\alpha^2 + \alpha^2 c_0 + 2d\sigma\omega^2 c_0^2 + 2d\sigma\omega^2 c_0^3 - 2\alpha\omega c_0\alpha_{1,1}}{2\omega^2 c_0} + (\alpha + \alpha c_0 + D\alpha c_0 - d\alpha\rho\omega^2 c_0 - \omega c_0\alpha_{1,1})\cos(X) \\
\alpha_{1,1} &= \frac{\alpha\omega(D - d\rho\omega^2)}{\omega^2 - 1}. \\
\phi_2(X,z) &= \frac{\alpha^2 - 3\alpha\omega\alpha_{1,1}}{4\omega}\exp(2z)\sin(2X) + \alpha_{2,1}\exp(z)\sin(X).
\end{aligned} \tag{2.46}$$

Now for higher order unknown functions $\phi_m(X,z)$ and $\zeta_m(X)$ by following this approach infinite order solutions can be obtained .It is also valuable to point out that these solutions will keep the convergence control parameter c_0.

2.3.4 Optimal convergence control parameter

Here two residual square errors of BCs (2.28) and (2.29) are defined, according to Liao [7] because optimal value of parameter c_0 is required.

$$\varepsilon_m^{\phi} = \frac{1}{1+M}\sum_{i=0}^{M}\left(N_1\left[\phi\left(X,z\right),\zeta\left(X\right)\right] \text{ at } X = i\Delta X\right)^2. \tag{2.47}$$

$$\varepsilon_m^{\zeta} = \frac{1}{1+M}\sum_{i=0}^{M}\left(N_2\left[\phi\left(X,z\right),\zeta\left(X\right)\right] \text{ at } X = i\Delta X\right)^2. \tag{2.48}$$

For $X = \frac{\pi}{M}$, M is the number of the discrete points.

Since total residual square error will be.

$$\varepsilon_m^T = \varepsilon_m^{\phi} + \varepsilon_m^{\zeta}. \tag{2.49}$$

For generality $\frac{d\varepsilon_m^T}{dc_0} = 0$ the optimal convergence control parameter c_0 by the minimum of the squared residual ε_m^T is obtained.

2.4 Results and analysis

In figures 1 and 2 the effects of Young's modulus E of the plate on the wave elevation $\zeta(X)$ under a floating elastic plate are studied.which shows the change in $\zeta(X)$ for different values of $E = 12822.7, 12822.8$,and 12822.9.

As it is clear from figures 1 and 2 that the nonlinear hydroelastic response of the waves becomes flatter at the crest and steeper at the trough due to the larger value of Young's modulus E .

And in figures 3 and 4 the effects of plate thickness d on the several displacements $\zeta(X)$ under a floating elastic plate are studied.which shows the change in $\zeta(X)$ for different values of d.It is observed that by increasing d from 0.005 to 0.02 the nonlinear hydroelastic response of the waves becomes flatter at the crest and steeper at the trough due to increase in plate thickness d .

These figures indicates that the results are very similar to the theory of nonlinear hydroelastic waves beneath a floating ice sheet.Which further shows the validity of results.

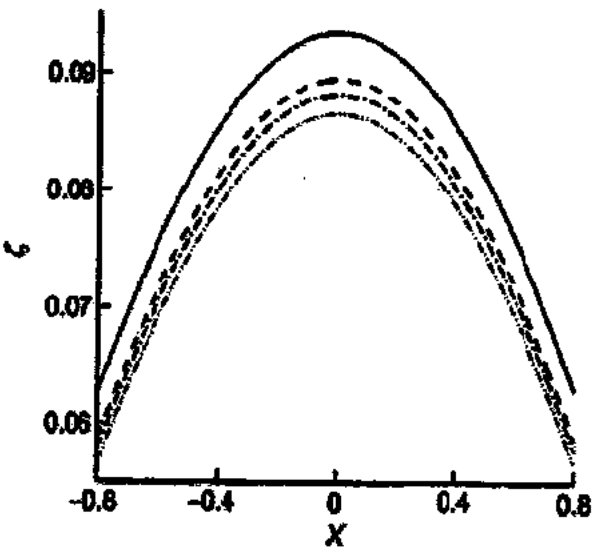

Figure 2.1 Change of the plate deflection $\zeta(X)$ near the crest against X for different values of Young's modulus of the plate E. Solid line, no plate condition, dashed line, $E = 12822.7$, dashdotted line, $E = 12822.8$, dashdot dotted line, $E = 12822.9$.

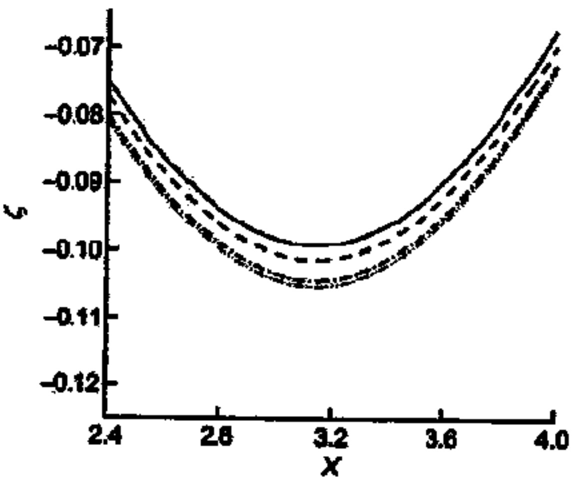

Figure 2.2 Change of the plate deflection $\zeta(X)$ near the trough against X Young's modulus of the plate E. Solid line, no plate condition, dashed line, $E = 12822.7$, dash dotted line, $E = 12822.8$, dashdot-dotted line, $E = 12822.9$.

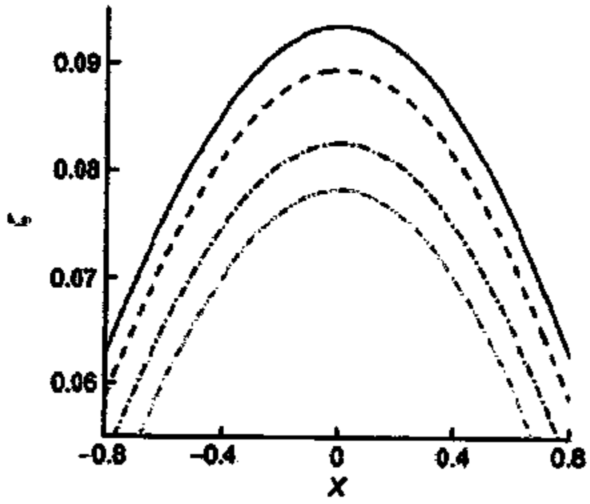

Figure 2.3 Change of the plate deflection $\zeta(X)$ near the crest against X for different plate thicknesses d. Solid line, no plate condition, dashed line, $d = 0.005$, dash dotted line, $d = 0.01$, dashdot-dotted line, $d = 0.02$.

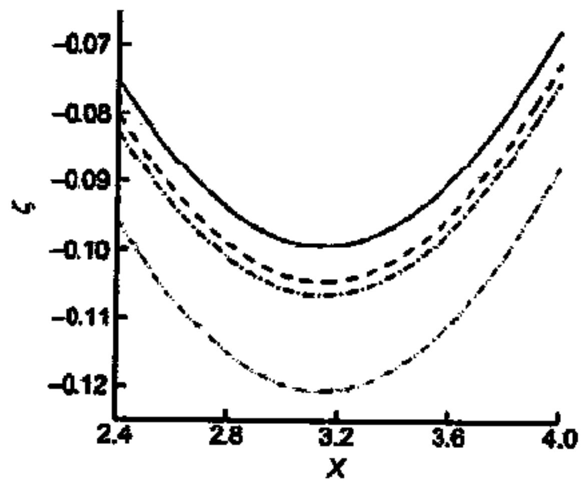

Figure 2.4 Change of the plate deflection $\zeta(X)$ near the trough against X for different plate thicknesses d. Solid line, no plate condition, dashed line, $d = 0.005$, dashdotted line, $d = 0.01$, dashdot-dotted line, $d = 0.02$.

2.5 Conclusions

In this chapter nonlinear hydroelastic waves traveling in a thin elastic plate floating on a fluid of finite depth is investigated analytically by the HAM. Mathematically. Both equations (2.19) and (2.20) there are linear operators for $\zeta(X)$ and $\phi(X,z)$ As HAM gives us with great option for the auxiliary linear operators. So the auxiliary linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are chosen containing the derivatives of $\phi(X,z)$ and $\zeta(X)$ respectively.By these auxiliary linear operators calculation of nonlinear hydroelastic wave propagation can be solved easily. Also influences of the Young's modulus E and plate thickness d on the plate deflection $\zeta(X)$ are investigated .The plate deflections become lower as the Young's modulus E of the plate increases.The hydroelastic response of the plate is greatly affected by large plate thickness d. The results obtained here demonstrate that the thickness d of the plate and Young's modulus E of the incident wave have major effects on the hydroelastic response of an ice sheet.

Chapter 3

Series solution of nonlinear hydroelastic waves equation in a fluid of finite depth

3.1 Introduction

The purpose of this chapter is to revisit the work of Ping Wang and Zunshui Cheng [4]. In this chapter the motion of nonlinear hydroelastic waves under an ice sheet lying over an incompressible inviscid fluid of finite depth is discussed by regular perturbation and Homotopy analysis method.The nonlinear partial differential equations (3.1) to (3.5) are composed of the Laplace equation taken as the main equation. The convergent homotopy series solutions for the velocity potential and the wave surface elevation are formally derived by means of HAM under the consideration of minimizing the squared residual.The effects of the water depth and two important physical parameters including Young's modulus and the thickness of the ice sheet on the wave energy and its elevation are shown graphically. Discussion and conclusions are made in Sections 3.4 and 3.5 respectively.

3.2 Mathematical formulation

Let us assume nonlinear hydroelastic waves traveling in an infinite elastic plate of thickness d floating on a fluid of finite depth h and. A rectangular coordinate OXZ is used, as the $z-axis$ points vertically upward, while $z=0$ denotes the undisturbed surface. By following Greenhill model [1] It is assumed that the fluid is inviscid , incompressible and irrotational.$\phi(x,z,t)$ is velocity potential which satisfies the Laplace equation.

$$\frac{\partial^2\phi}{\partial^2 x}+\frac{\partial^2\phi}{\partial z^2}=0,(-h\leq z\leq\zeta(x,t)). \tag{3.1}$$

The boundary condition is

$$\frac{\partial\phi}{\partial z}=0,(z=-h). \tag{3.2}$$

By the supposition that any fluid particle which is in between elastic plate and water surface will remain on it.On the unknown plate water interface $z=\zeta(x,t)$ the kinematic boundary condition is

$$\frac{\partial\zeta}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}-\frac{\partial\phi}{\partial z}=0. \tag{3.3}$$

and dynamic boundary condition is

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\nabla^2\phi+\frac{p_e}{\rho}+g\zeta=0. \tag{3.4}$$

where $p_e(x,t),\rho$ and g are plate water interface pressure ,fluid density and gravitational acceleration respectively.By the Kirchhoff beam theory .For constant thickness d and uniform mass density ρ_e of the plate the relationship between plate deflection $\zeta(x,t)$ and pressure $p_e(x,t)$ in view of Kirchhoff (Euler Bernoulli) beam theory is

$$p_e=D\frac{\partial^4\zeta}{\partial x^4}+m_e(\frac{\partial^2\zeta}{\partial t^2}+g). \tag{3.5}$$

where $m_e=p_e d$, $D=\frac{ed^{\,3}}{[12(1-v^2)]}$. By substituting equation (3.5) into equation (3.4) gives the full form of dynamic boundary condition as

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left|\nabla\phi\right|^2+g\zeta+\frac{1}{\rho}\left[D\frac{\partial^4\phi}{\partial x^4}+m_e\left(\frac{\partial^2\zeta}{\partial t^2}+g\right)\right]=0. \tag{3.6}$$

By the concept of traveling wave method an independent variable transformation is introduced as

$$X = kx - \omega t. \tag{3.7}$$

where k is wave number and ω is angular frequency of incident wave.Now velocity potential function $\phi(x, z, t) = \phi(X, z)$ and the hydroelastic wave profile $\zeta(x, t) = \zeta(X)$ are used.Then the governing equation and the bottom boundary condition for the velocity potential are transformed by

$$k^2 \frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, (-h \leq z \leq \zeta(X)). \tag{3.8}$$

$$\frac{\partial \phi}{\partial z} = 0, (z = -h). \tag{3.9}$$

In view of (3.7) on $z = \zeta(X)$, (3.3) and (3.6) are transformed into

$$-\omega \frac{d\zeta}{dX} + k^2 \frac{\partial \phi}{\partial X} \frac{d\zeta}{dX} - \frac{\partial \phi}{\partial z} = 0. \tag{3.10}$$

$$-\omega \frac{\partial \phi}{\partial X} + f + g\zeta + \frac{1}{\rho}\left[Dk^4 \frac{d^4\zeta}{dX^4} + m_e\left(\omega^2 \frac{d^2\zeta}{dX^2} + g\right)\right] = 0. \tag{3.11}$$

respectively, where

$$f = \frac{1}{2}\left[k^2\left(\frac{\partial \phi}{\partial X}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]. \tag{3.12}$$

A partial combination of equations. (3.10) and (3.11) gives the boundary conditions on $z = \zeta(X)$ as follows

$$\omega^2 \frac{\partial^2 \phi}{\partial X^2} + g\frac{\partial \phi}{\partial z} - \omega \frac{\partial f}{\partial X} - \frac{\omega}{\rho}\left(Dk^4 \frac{d^5\zeta}{dX^5} + m_e\omega^2 \frac{d^3\zeta}{dX^3}\right) - k^2 g \frac{\partial \phi}{\partial X}\frac{d\zeta}{dX} = 0. \tag{3.13}$$

The velocity potential $\phi(X, z)$ and the wave surface elevation $\zeta(X)$ are derived by equations (3.8), (3.9), (3.11), and (3.13) in form of Series solutions for $\varphi(X, z)$ and $\zeta(X)$ will be derived in the subsequent section based by homotopy analysis method.

3.3 Analytic approach based on the homotopy analysis method

3.3.1 Solution expression and initial approximation.

First of all in homotopy analysis method, set of base functions and solution expression are assumed.Which are used for unknown solutions of the nonlinear hydroelastic waves problem. As it is very difficult to deals with the expression forms for unknown potential function and plate deflection. Since in view of physical background of the pure water waves, the progressive wave elevation $\zeta(X)$ can be written as

$$\zeta(X) = \sum_{n=0}^{+\infty} \beta_n \cos(nX). \tag{3.14}$$

By a set of base functions $\{\cos(nX), n \geq= 0\}$, where β_n are unknown coefficients. In the case of plate covered surface, since it is assumed that there is no space between bottom surface of plate and top surface of fluid layer.The upright displacement of plate is periodic in the X direction.Therefore, it is clear that $\zeta(X)$ can be expressed in the above form (3.14).In view of linear wave theory, the solutions of the Laplace equation (3.8) by the separation of variables method can be found.Here kinematic, dynamic and boundary condition in finite water depth are used to obtain these solutions. Since $\phi(X,z)$ becomes

$$\phi(X,z) = \sum_{n=1}^{+\infty} \alpha_n \frac{\cosh[nk(z+h)]}{\cosh(nkh)} \sin(nX). \tag{3.15}$$

Now consider a set of base functions $\{\cosh[nk(z+h)]/\cosh(nkh)\sin(nX), n \geq 0\}$, where α_n are unknown coefficients. Here potential function $\phi(X,z)$ defined by (3.15) automatically satisfies the governing equation (3.8) and the bottom boundary condition (3.9). The equations (3.14) and (3.15) are the solution expressions of $\phi(X,z)$ and $\zeta(X)$ respectively.Which is important in homotopy analysis method. In view of equations (3.9) and (3.15), the initial approximation for potential function is given by

$$\phi_0(X,z) = \alpha_{0,1} \frac{\cosh[k(z+h)]}{\cos(kh)} \sin(X). \tag{3.16}$$

where $\alpha_{0,1}$is an unknown coefficient, since

$$\zeta_0(X) = 0. \tag{3.17}$$

In view of [8, 9] the initial approximation for $\zeta(X)$ is zero. According to the equations (3.11) and (3.13), two nonlinear operators N_1 and N_2 are defined in the subsequent section for analytic series solution.

3.3.2 Continuous variation

The HAM depends on an initial approximation to the exact solution. Since based on the nonlinear boundary conditions (3.11) and (3.13), two nonlinear operators N_1 and N_2 are defined as

$$\begin{aligned} N_1\left[\Phi\left(X,z;q\right),\eta\left(X;q\right)\right] &= \omega^2\frac{\partial^2\Phi\left(X,z;q\right)}{\partial X^2}+g\frac{\partial\Phi\left(X,z;q\right)}{\partial z}-\omega\frac{\partial F}{\partial X} \\ &-\frac{\omega}{\rho}\left(Dk^4\frac{\partial^5\eta\left(X;q\right)}{\partial X^5}+\omega^2 m_e\frac{\partial^3\eta\left(X;q\right)}{\partial X^3}\right) \\ &-k^2 g\frac{\partial\Phi\left(X,z;q\right)}{\partial X}\frac{\partial\eta\left(X;q\right)}{\partial X}. \end{aligned} \tag{3.18}$$

$$\begin{aligned} N_2\left[\eta\left(X;q\right),\Phi\left(X,z;q\right)\right] &= -\omega\frac{\partial\Phi\left(X,z;q\right)}{\partial X}+F+g\eta\left(X;q\right) \\ &+\frac{1}{\rho}\left[Dk^4\frac{\partial^4\eta\left(X;q\right)}{\partial X^4}+m_e\left(\omega^2\frac{\partial^2\eta\left(X;q\right)}{\partial X^2}+g\right)\right]. \end{aligned} \tag{3.19}$$

where

$$F=\frac{1}{2}\left[k^2\left(\frac{\partial\Phi}{\partial X}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right]. \tag{3.20}$$

Here $q \in [0,1]$ is the embedding parameter of the homotopy analysis method. As explained by Liao, Cheung and Tao et al [9, 10], in homotopy analysis method the auxiliary linear operator and the initial guess can be chosen by extremely large freedom. It is noted that both linear terms of $\Phi(X,z;q)$ and linear terms of $\eta(X,q)$ are all contained in (3.18).Now based on the homotopy analysis method, by neglecting the linear terms in equation (3.13) and auxiliary linear operator of $\Phi(X,z;q)$ is so properly chosen,by means of the solution expression (3.15), which is obtained as given below.

$$\overline{\mathcal{L}}_1\left[\Phi\left(X,z;q\right)\right]=\omega^2\frac{\partial^2\Phi\left(X,z;q\right)}{\partial X^2}+g\frac{\partial\Phi\left(X,z;q\right)}{\partial z}. \tag{3.21}$$

If angular frequency ω is given so an approximation can be chosen based on the linear wave theory to simplify the subsequent resolution of the nonlinear PDEs as follows:

$$\omega=\sqrt{gk\tan\left(kh\right)}. \tag{3.22}$$

Since the auxiliary linear operator in (3.21) can be simplified as

$$\mathcal{L}_1\left[\Phi\left(X,z;q\right)\right]=gk\tanh\left(kh\right)\frac{\partial^2\Phi\left(X,z;q\right)}{\partial X^2}+g\frac{\partial\Phi\left(X,z;q\right)}{\partial z}. \tag{3.23}$$

Here $\mathcal{L}_1(0)=0$.

Since due to the weakly nonlinear effects there is a difference between the actual frequency ω and linear dispersion relation $\omega_0=\sqrt{gk\tanh(kh)}$ upto some extent.Results are compared with those obtained by the perturbation method. In view of linear operator of the wave profile function $\eta(X;q)$ and the nonlinear operator N_2, another auxiliary linear operator may be chosen as

$$\mathcal{L}_2\left[\eta\left(X;q\right)\right]=\frac{\partial^4\Phi\left(X,z;q\right)}{\partial X^4}+\frac{\partial^2\eta\left(X;q\right)}{\partial X^2}+\eta(X;q). \tag{3.24}$$

Here $\mathcal{L}_2(0)=0$.

Now let c_0 be a nonzero convergence control parameter. It is noted that both c_0 and q in the HAM are auxiliary parameters. Instead of the nonlinear PDEs (3.8), (3.9), (3.11), and (3.13) the zeroth order deformation equations are constructed as

$$\frac{\partial^2\Phi\left(X,z;q\right)}{\partial X^2}+\frac{\partial^2\Phi\left(X,z;q\right)}{\partial z^2}=0,\left(\left(-h\leq z\leq\eta\left(X;q\right)\right)\right). \tag{3.25}$$

$$\frac{\partial\Phi\left(X,z;q\right)}{\partial z}=0,\left(z=-h\right). \tag{3.26}$$

$$(1-q)\mathcal{L}_1\left[\Phi(X,z;q)-\phi_0(X,z)\right]=qc_0N_1\left[\Phi(X,z;q),\eta(X;q)\right],(z=\eta(X;q)). \quad (3.27)$$

$$(1-q)\mathcal{L}_2\left[\eta(X,z;q)-\zeta_0(X,z)\right]=qc_0N_2\left[\Phi(X,z;q),\eta(X;q)\right],(z=\eta(X;q)). \quad (3.28)$$

It is clear that two mapping functions $\Phi(X,z;q)$ and $\eta(X;q)$ of the original problem vary from initial approximation $\phi_0(X,z)$ and $\zeta_0(X)$ to the exact solutions $\phi(X,z)$ and $\zeta(X)$. Since in view of equations (3.27) and (3.28) the Taylor series of functions $\Phi(X,z;q)$ and $\eta(X;q)$ at $q=0$ are as follows

$$\Phi(X,z;q)=\Phi_0(X,z)+\sum_{m=1}^{+\infty}\phi_m(X,z)q^m. \quad (3.29)$$

$$\eta(X;z)=\zeta_0(X)+\sum_{m=1}^{+\infty}\zeta_m(X,z)q^m. \quad (3.30)$$

$$\{\phi_m(X,z),\zeta_m(X)\}=\frac{1}{m!}\frac{\partial^m}{\partial q^m}\{\Phi(X,z;q),\eta(X;q)\}\ \text{at}\ q=0. \quad (3.31)$$

As it is assumed that c_0 is chosen so properly that the series in (3.29) and (3.30) converges at $q=1$,since homotopy series solutions will be as

$$\begin{aligned}\Phi(X;z) &= \Phi(X;z,1)=\phi_0(X,z)+\sum_{m=1}^{+\infty}\phi_m(X,z).\\ \zeta(X) &= \eta(X;1)=\zeta_0(X)+\sum_{m=1}^{+\infty}\zeta_m(X).\end{aligned} \quad (3.32)$$

since at the nth order approximations

$$\begin{aligned}\phi(X,z) &= \phi_0(X,z)+\sum_{m=1}^{+n}\phi_m(X,z).\\ \zeta(X) &= \zeta_0(X)+\sum_{m=1}^{+\infty}\zeta_m(X).\end{aligned} \quad (3.33)$$

As shown later in the following section, the unknown terms $\phi_m(X,z)$ and $\zeta_m(X)$ are governed by the linear PDEs (3.34) to (3.36).

3.3.3 Deformation equations of high order

By putting the homotopy Maclaurin series (3.29) and (3.30) into equations (3.25) and (3.26) the deformation equations of high order for the unknown functions $\phi_m(X,z)$ and $\zeta_m(X)$ are derived directly from the deformation equations of zeroth order,and then equating the like powers of embedding parameter q it follows that

$$\begin{aligned} k^2\frac{\partial^2\phi_m(X,z)}{\partial X^2}+\frac{\partial^2\phi_m(X,z)}{\partial z^2} &= 0,(-h\le z\le 0). \\ \frac{\partial\phi_m(X,z)}{\partial X} &= 0,(z=-h). \end{aligned} \tag{3.34}$$

where $m\ge 1$.Note that, $\Phi(X,z;q)$ at the unknown surface $z=\eta(X;q)$ may be expressed in terms of the Taylor expansion at $z=0$ instead of $z=\eta(X;q)$ since two linear boundary conditions on $z=0$ are as follows

$$\mathcal{L}_1(\phi_m) \text{ at } z=0=c_0\Delta^m_{m-1}+\chi_m S_{m-1}-\overline{S_m}. \tag{3.35}$$

$$\mathcal{L}_2(\zeta_m)=c_0\Delta^{\zeta}_{m-1}+\chi_m\left(\frac{d^4\zeta_{m-1}}{dX^4}+\frac{d^2\zeta_{m-1}}{dX^2}+\zeta_{m-1}\right). \tag{3.36}$$

where

$$S_{m-1}=\sum_{i=0}^{m-2}\left(\frac{d^2\psi_{m-1-i,i}}{dX^2}+\gamma_{m-1-i,i}\right)$$

$$\overline{S}_m=\sum_{i=1}^{m-1}\left(\frac{d^2\psi_{m-i,i}}{dX^2}+\gamma_{m-i,i}\right)$$

$$\Delta^{\zeta}_{m-1}=-\omega\frac{d\varphi_{m-1}}{dX}+\frac{1}{2}\sum_{n=0}^{m-1}\left(\frac{d\varphi_n}{dX}\frac{d\varphi_{m-1-n}}{dX}+\overline{\varphi}_n\overline{\varphi}_{m-1-n}\right)+\zeta_{m-1}+\frac{Dk^4}{\rho}\frac{d^4\zeta_{m-1}}{dX^4}+\frac{m_e\omega^2}{\rho}\frac{d^2\zeta_{m-1}}{dX^2}$$

and

$$\Delta_{m-1}^{\phi} = \omega^2 \frac{d^2\varphi_m}{dX^2} + g\overline{\varphi}_m - \omega \sum_{n=0}^{m} \left(\frac{d\varphi_n}{dX} \frac{d^2\varphi_{m-n}}{dX^2} + \overline{\varphi}_n \frac{d\overline{\varphi}_{m-n}}{dX} \right) - \frac{\omega}{\rho} Dk^4 \frac{d^5\zeta_m}{dX^5} - \frac{\omega^3 m_e}{\rho} \frac{d^3\zeta_m}{dX^3} - k^2 g \sum_{n=0}^{m} \frac{d\varphi_n}{dX} \frac{d\zeta_{m-n}}{dX}$$

for

$$\chi_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1 \end{cases}. \tag{3.37}$$

It should be noted that (3.27) and (3.28) holds on the unknown boundary,$z = \eta(X; q)$ while (3.35) and (3.36) hold on $z = 0$. Furthermore, the original nonlinear PDEs (3.1) to (3.5) are transferred into an infinite number of linear decoupled high order deformation equations (3.34) to (3.36). Namely, given $\phi_{m-1}, \zeta_{m-1}, \phi_m$ and ζ_m can be obtained easily by means of the inverse operators of the right hand sides of (3.35) and (3.36), respectively.The resulting expressions for ϕ_m and ζ_m are presented to the second order in the subsequent subsection.

3.3.4 First order and second order approximations.

$$\begin{aligned} \zeta_1(X) &= \frac{1}{4}\left[4dgc_0 + c_0\alpha_{0,1}^2 + k^2 c_0 \alpha_{0,1}^2 \tanh^2(hk)\right] - \omega c_0 \alpha_{0,1} \cos(X) \\ &\quad + \frac{1}{52}\left[c_0\alpha_{0,1}^2 - k^2 c_0 \alpha_{0,1}^2 \tanh^2(hk)\right] \cos(2X). \end{aligned} \tag{3.38}$$

But now the coefficient $\alpha_{0,1}$ in (3.16) is still unknown. So an additional equation to relate the solutions with the wave height is introduced.

$$\zeta_1(m\pi) - \zeta_1(m\pi) = H. \tag{3.39}$$

Here m and n are even and odd integers respectively and H is the wave height to the first order based on the HAM.The solution of $\alpha_{0,1}$ can be determined by the relation (3.39) for the wave height and its vertical displacement. Now by using the inverse linear operator $\mathcal{L}$ in (3.35), it is easy to get the solution of $\phi_1(X,z)$.

$$\begin{aligned}\alpha_{0,1} &= -\frac{H}{2\omega c_0}.\\ \phi_1(X,z) &= \alpha_{1,1}\frac{\cosh\left[k(z+h)\right]}{\cos(kh)}\sin(X) + \frac{-H^2+H^2k^2\tanh^2(hk)}{16gk\omega c_0\left[2\tanh(hk)-\tanh(2hk)\right]}\\ &\quad\times\frac{\cosh\left[2k(z+h)\right]}{\cos(2kh)}\sin(2X).\end{aligned} \tag{3.40}$$

Now the solution of $\phi_1(X,z)$ has one unknown coefficient $\alpha_{1,1}$, which can be determined by avoiding the secular term $sin(X)$ in $\phi_2(X,z)$. It is noted that all subsequent functions occur recursively. Since in view of the linear equations (3.35) and (3.36) to continue with the first order approximations

$$\begin{aligned}\zeta_2(X) &= \beta_{2,0}+\beta_{2,1}\cos(X)+\beta_{2,2}\cos(2X)+\beta_{2,3}\cos(3X)+\beta_{2,4}\cos(4X).\\ \phi_2(X,z) &= \alpha_{2,1}\frac{\cosh\left[k(z+h)\right]}{\cos(kh)}\sin(X)+\alpha_{2,2}\frac{\cosh\left[2k(z+h)\right]}{\cos(2kh)}\sin(2X)+\alpha_{2,3}\frac{\cosh\left[3k(z+h)\right]}{\cos(3kh)}\sin(3X)\\ &\quad+\alpha_{2,4}\frac{\cosh\left[4k(z+h)\right]}{\cos(4kh)}\sin(4X)+\alpha_{2,5}\frac{\cosh\left[5k(z+h)\right]}{\cos(5kh)}\sin(5X).\end{aligned} \tag{3.41}$$

where $\alpha_{i,j}$ is the jth unknown coefficient of $\phi_i(X,z)$ and $\beta_{i,j}$ is the jth unknown coefficient of $\zeta_i(X)$.In order to obtain higher order functions $\phi_m(X,z)$ and $\zeta_m(X)$, the infinite order solutions for physical model can be acquired by continuing this approach.

3.3.5 Optimal convergence control parameter.

As all model parameters in approximate series solutions are fixed, since there is still an unknown convergence control parameter c_0 which is used to guarantee the convergence of approximation solutions. According to Liao [7], it is the convergence control parameter c_0 that essentially differs the homotopy analysis method from all other analytic methods. And the optimal value of c_0 is determined by the minimum of the total squared residual ε_m^T of our nonlinear problem, defined as

$$\varepsilon_m^T = \varepsilon_m^\phi + \varepsilon_m^\zeta. \tag{3.42}$$

where

$$\varepsilon_m^{\phi} = \frac{1}{1+M}\sum_{i=0}^{M}(N_1\left[\phi(X,z),\zeta(X)\right] \text{ at } X = i\Delta X)^2.$$
$$\varepsilon_m^{\zeta} = \frac{1}{1+M}\sum_{i=0}^{M}(N_2\left[\phi(X,z),\zeta(X)\right] \text{ at } X = i\Delta X)^2. \qquad (3.43)$$

Here M is the number of the discrete points and $X = \frac{\pi}{M}$.

For generality $\frac{d\varepsilon_m^T}{dc_0} = 0$ the optimal convergence control parameter c_0 by the minimum of the squared residual ε_m^T is obtained.

3.4 Solutions by Genetic algorithm and Nelder mead method.

As in Homotopy Analysis Method we need optimal convergence control parameter to select the values and confirm the convergence of the problem.Figures 3.1 to 3.4 shows the effects of plate deflection $\zeta(X)$ at different number of terms used in series solution.For range of M the total squared residual ε_m^T is found by using equation (3.42) in view of (3.43).

Where i is the number of terms or iterations of the obtained solution.The total squared residual ε_m^T and optimal convergence control parameter c_0 is indicated in table 3.1.The analytical solutions given in equations (3.35) and (3.36) are obtained by using Homotopy Analysis Method and the embedding parameter is found using Genetic Algorithm and Nelder Mead method as shown in Table 3.1.

Following are the parameters used for Genetic Algorithm.

Population: Population type: Double vector

Creation function: Uniform

Initial population: 100

Initial Range: [0; 1]

Scaling function: Rank

Selection function: Stochastic uniform.

Reproduction: Elite Count: 10

Crossover fraction: 0.8

Mutation function: Uniform

Rate: 0.01

Crossover function: Arithmetic.

Migration: Direction: Both

Fraction: 0.2

Interval: 20

Hybrid function: None.

Stopping Criteria: Generation: 100

Time limit: inf

Fitness limit: -inf

Tolerance: 10^{-6}.

Table 3.1 Comparison of homotopic series solution and optimal series solution using Genetic Algorithm and Nelder Mead method.

		Series solution Homotopy analysis method				Optimal series solution using Genetic algorithm and Nelder mead method			
d	E	Iteration	Time	Range/value	Residual Error	Iteration	Time	Range/value	Residual Error
0.005	12822.7	5	1.263	-0.8 to -1.1	2.1×10^{-1}	2	0.093	-0.8 to -1.1	5.2×10^{-3}
0.01	12822.8	10	95.443	-0.7 to -1.2	4.1×10^{-3}	5	1.263	-0.7 to -1.2	3.0×10^{-4}
0.02	12822.9	16	296.182	-1.3 to -0.5	7.9×10^{-5}	7	5.362	-1.3 to -0.5	1.9×10^{-6}
0.001	10^{8}	5	1.123	-0.8 to -1.1	2.5×10^{-1}	2	0.11	-0.8 to -1.1	1.19×10^{-3}
0.005	10^{9}	10	109.543	-0.7 to -1.2	7.1×10^{-3}	5	1.123	-0.7 to -1.2	1.5×10^{-5}
0.01	10^{10}	16	309.426	-1.3 to -0.5	1.6×10^{-5}	7	5.469	-1.3 to -0.5	1.2×10^{-6}

Table 3.1 Signify that by using the proposed hybrid scheme the time and obtained total squared residual ε_m^T can be considerably reduced. We can introduce further embedding parameters in these solutions to gain more efficient results.

3.5 Results and analysis

In figures 1 and 2 the effects of Young's modulus E of on the wave elevation $\zeta(X)$ under a floating elastic plate are studied.which shows the change in $\zeta(X)$ for different values of $E = 10^8, 10^9$,and 10^{10}.

As it is clear from figures 1 and 2 that the nonlinear hydroelastic response of the waves becomes flatter at the crest and steeper at the trough due to the larger value of Young's modulus E .It is clear that larger E reduces the plate deflection $\zeta(X)$.

And in figures 3 and 4 the effects of plate thickness d on the several displacements $\zeta(X)$ under a floating elastic plate are studied.which shows the change in $\zeta(X)$ for different values of d.It is observed that by increasing d from 0.001 to 0.01. the nonlinear hydroelastic response of the waves becomes flatter at the crest and steeper at the trough due to increase in plate thickness d .Particularly when the plate thickness is nearly equal to zero, the wave becomes pure gravity wave as observed in [9].

Let P.E. be the mean potential density per unit length in the X-axis. In terms of the wave surface elevation function, the energy density can be written as

$$P.E = \frac{1}{4\pi^2}\int_0^{2\pi} \zeta^2(X)\,dX. \tag{3.44}$$

These figures indicates that the results are very similar to the theory of nonlinear hydroelastic waves beneath a floating ice sheet. Also by Genetic algorithm and Nelder mead method results are compared as shown in Table 3.1.

Which further shows the validity of results.

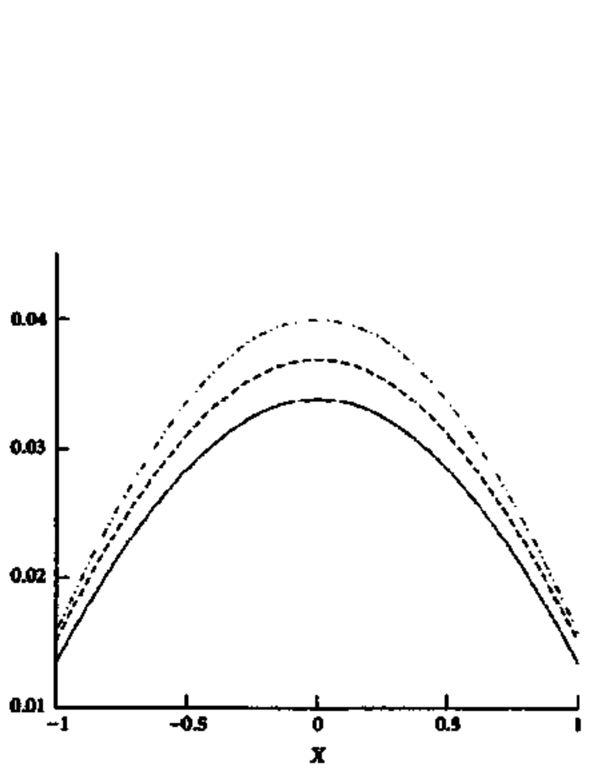

Figure 3.1 Change of the plate deflection $\zeta(X)$ near the crest against X for different Young's modulus of the plate E. Solid line $E = 10^8$ dashed line $E = 10^9$ dashdot-dotted line $E = 10^{10}$.

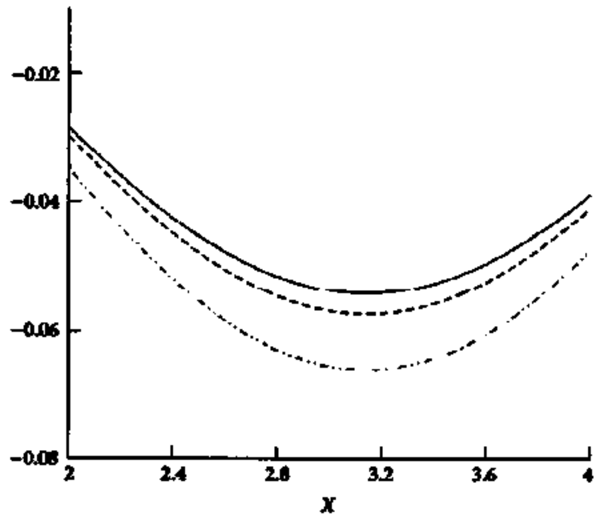

Figure 3.2 Change of the plate deflection $\zeta(X)$ near the trough against X for different Young's modulus of the plate E. Solid line $E = 10^8$ dashed line $E = 10^9$ dashdot-dotted line $E = 10^{10}$.

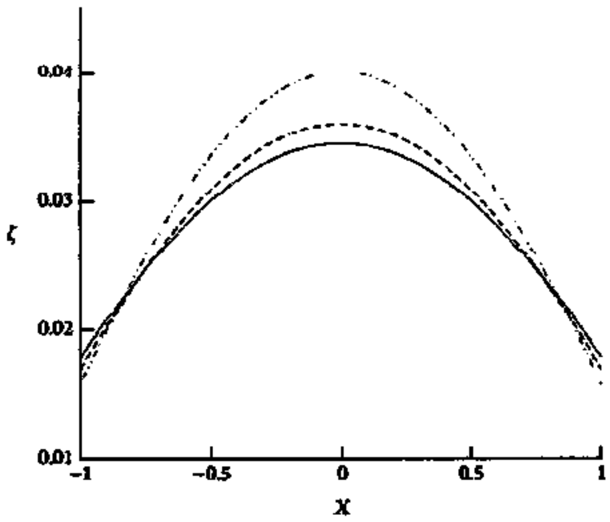

Figure 3.3 Change of the plate deflection $\zeta(X)$ near the crest against X for different plate thicknesses d .Solid line $d = 0.001$, dashed line $d = 0.005$, dashdot-dotted line $d = 0.01$.

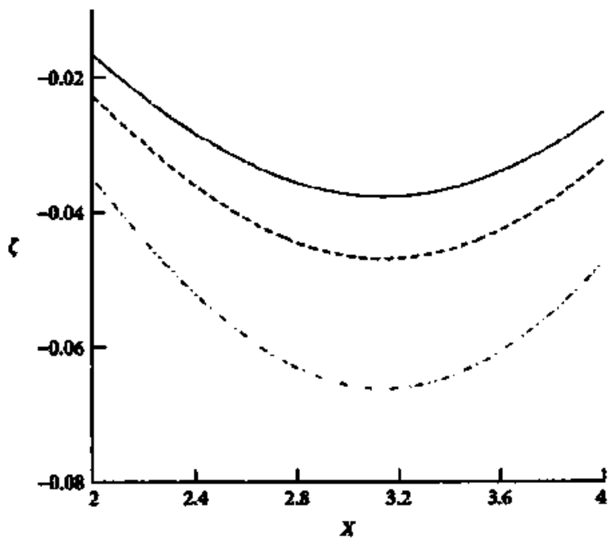

Figure 3.4 Change of the plate deflection $\zeta(X)$ near the trough against X for different plate thicknesses d .Solid line $d = 0.001$, dashed line, $d = 0.005$, dashdot-dotted line $d = 0.01$.

3.6 Conclusions

In this chapter the nonlinear hydroelastic waves propagating beneath a two dimensional infinite elastic plate floating on a fluid of finite depth are investigated analytically by the HAM.

From kinematic and dynamic boundary conditions at a constant velocity in a fluid of finite depth the PDE's in (3.20), (3.21) and (3.22) are obtained by simple elimination of the time dependent terms.

Here for a general case it should be noted that ,when traveling wave method is directly applied to transfer the temporal differentiation into the spatial one in a fixed Cartesian coordinate OXZ PDE's are constructed. Furthermore, the convergent homotopy series solutions for the PDE's are derived by the HAM with the optimal convergence control parameter.

Also influences of the Young's modulus E and thickness d of the plate on the plate deflection $\zeta(X)$ are investigated. The plate deflections become lower by the increase in Young's modulus E of the plate.The plate thickness d greatly effects the hydroelasticity of the plate. The results obtained here express that the hydroelasticity of ice sheet effected by the thickness d of the plate and Young's modulus E of the incident wave.Which is proved in the theory of nonlinear hydroelastic waves beneath a floating ice sheet in a fluid of finite depth [10].

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