Analysis of Generalizations of Intuitionistic Fuzzy Graphs

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy
In
MATHEMATICS

Supervised by

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DECLARATION

I hereby, declare, that this thesis neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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Pakistan.

<u>Certificate</u>

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***DOCTOR OF PHELOSIPHY in MATHEMATICS***

We accept this dissertation as conforming to the required standard.

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3. ____________________

(External Examiner)

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Department of Mathematics and Statistics
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Pakistan
2020

Dedicated
To
My
Mother

Acknowledgement

All praises to almighty **"ALLAH"** the creator of the universe, who blessed me with the knowledge and enabled me to complete the dissertation. All respects to **Holy Prophet MUHAMMAD (S.A.W),** who is the last messenger, whose life is a perfect model for the whole humanity.

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Naeem Jan

0. Research Profile

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0.1. Literature Review

In real life situations, we often face imprecision or uncertainties due to several measures. To cope with such imprecise events, Zadeh introduced fuzzy set (FS) [1] in which an object or element of a set is assigned a membership grade in unit interval [0, 1]. The membership grade 0 points towards no satisfaction at all while membership grade 1 means full satisfaction. The partial satisfaction is denoted by other values in the unit interval based on their degree of satisfaction. FS theory has been applied to many situations like in intelligent systems by Yager and Zadeh [2], pattern recognition by Pedrycz [3], soft sets by Maji et al. [4], traffic and transportation by Trabia et al. [5], clustering by Xie and Beni [6] and many other areas. For some recent developments in FS theory and other tools of uncertainty and their applications one is referred to [7-9].

Zadeh's FS was a success indeed but there were some events which could not be deal with using ordinary FSs and therefore Atanassov [10] developed the concept of intuitionistic fuzzy set (IFS) as an extension of FS that deals with uncertain situations in a better way as its structure is not limited to membership grade only. The concept of IFSs is a better tool to use due to its diverse structure describing membership as well as non-membership grades of an element. The theory of IFSs have been remarkably used in some areas so far. In [11] medical diagnosis is discussed based on IFSs by De et al. and in [12], Xu defined some aggregation operators for IFSs which have been applied to multi-attribute decision making (MADM) by Li in [13]. Some similarity measures for IFSs are discussed in [14] by Szmidt and Kacprzyk and applied to medical diagnostics problems.

There is a limitation exists in Atanassov's structure of IFS that it restricts the sum of membership and non-membership grades on a scale of $[0, 1]$. In some situation, the IFS cannot describe effectively the opinion of human being due to weak limitations. For handling such types of problems, Yager [15] proposed the notion of Pythagorean fuzzy set (PyFS), as an extension of IFS to deal with uncertainty. The constraint of PyFS is that the sum of the square of membership grade and the square of non-membership grade is restricted to [0,1]. Further, Fei and Deng [16] applied Pythagorean fuzzy sets in multi criteria decision-making problems. The study of PyFS is very rich due to its larger domain as [17-21] focused on the aggregation theory of PyFS and its application in MADM. In [22] by Garg, the concept of linguistic PyFSs is developed and a MADM problem is discussed in such environment. In [23] by Garg, the strategic DM with some probability is examined using the framework of PyFSs and in [24] by Garg the famous TOPSIS method is performed in the environment of PyFSs.

When a decision maker provides $(0.85,0.82)$ for membership and non-membership grades, the IFSs and PyFSs cannot deal effectively i.e. $0.85 + 0.82 = 1.67 \geq 1$ and $0.85^2 + 0.82^2 = 0.72 + 0.67 = 1.39 \geq 1$. For handling such kinds of problems, Yager [25] again initiated the notion of q-rung orthopair fuzzy set (q-ROFS). The q-ROFS is a generalization of PyFS to deal with unknown and unpredictable information. The limitation of q-ROFS is that the sum of the n-power of membership grade and the n-power of non-membership grade is restricted to [0,1]. The q-ROFS is more general and more feasible than PyFS, IFS and FS.The geometrical interpretation of q-ROFS and their existing approaches are captured in Figure 1.

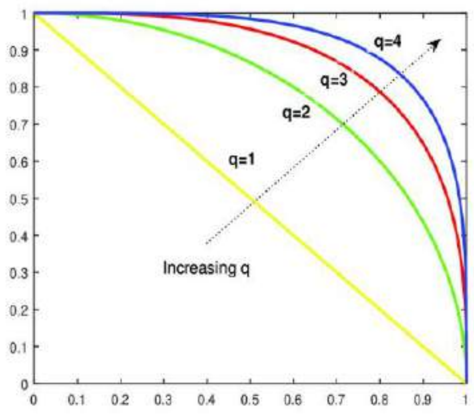


Figure 1(Geometrical interpretation of q-rung orthopair fuzzy set)

FS and IFS assigned single values to objects from unit interval which proved to be handful in most of the places, yet these concepts are further improved and the concepts of interval-valued FS (IVFS) in [26] by Zadeh and interval-valued intuitionistic fuzzy set (IVIFS) by Atanassov and Gargov [27] are proposed. The theory of aggregation of IVIFSs and their applications in MADM is discussed in [28-31]. These new concepts of IVFSs and IVIFSs are improved than FSs and IFSs because these described the membership/non-membership grades in terms of closed interval [0, 1] instead of a single value. Furthermore, the concept of interval valued Pythagorean fuzzy set (IVPyFS) and interval valued q-rung ortho pair fuzzy set (IVq-ROFS) are introduced by [32,33]. For some other relevant work one may refer to [34-36].

The concepts of FS, IFS,PyFS and q- ROFS couldn't be applied in circumstances where human opinion isn't of the type yes and no but it has some sort of abstinence or refusal degree as well. To agree with such demands and to model a concept close to human nature, Cuong [37] proposed the concept of PFS based on four possible situations like satisfaction, abstinence, dissatisfaction and refusal degree. The geometrical representation of all picture fuzzy numbers is demonstrated in Figure 2. Some other basic study about PFSs can be found in [38, 39]. The framework of PFS has be greatly utilized in several real-life problems as the aggregation operators of PFSs have been developed and utilized in multi-attribute decision making (MADM) by Wei [40] and Garg [41] respectively. The similarity (distance) measures of PFSs and their applications are comprehensively discussed by Son in [42]. Thong [43] developed a new computational intelligence-based method for picture fuzzy clustering. For some other relevant work one may refer to [44-45].

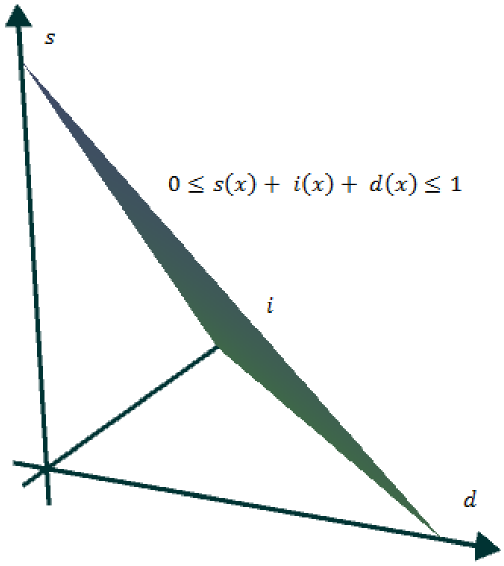


Figure 2 (Space of all picture fuzzy numbers)

By studying the structure of PFSs, it is derived that it generalizes FSs, IFSs and can manage the data or situations that FSs and IFSs could not. But there is a restriction in the structure of PFS that the sum of membership, abstinence and non-membership grade must be less than 1. Because of the condition on PFSs, it is difficult to give values to its membership, abstinence and non- membership tasks by their self-choice. Mahmood et al. [46] introduced the notion of spherical fuzzy sets (SFSs) and (T-SFSs) that make better the construction of PFSs. Such type of structure of T-SFSs, shapes both human attitude and opinion along with yes/no, and can manage any sort of data with no constraints. For instance, if we observe the limitations of PFSs and T-SFSs, then it becomes obvious that the structure of T-SFS has no constraints. The limitations of IFS, PFSs and T-SFSs are as follows:

- for IFS $A = \left\{ t_i, \left(\hat{S}(t), Đ(t_i) \right) \right\}$, we have $0 \leq \hat{S}(t_i) + Đ(t_i) \leq 1$;
- for PFSs $A = \left\{ x_i, \left(\hat{S}(t_i),\ \hat{I}(t_i), Đ(t_i) \right) \right\}$, we have $0 \leq \hat{S}(t_i) + \hat{I}(t_i) + Đ(t_i) \leq 1$;
- for T-SFSs $A = \left\{ x_i, \left(\hat{S}(t_i),\ \hat{I}(t_i), Đ(t_i) \right) \right\}$, we have $0 \leq \hat{S}^n(t_i) + \hat{I}^n(t_i) + Đ^n(t_i) \leq 1$ for some $n \in \mathbb{Z}^+$.

From the relationship with existing structures and its constraints the comprehensive structure and originality of T-SFSs is obvious. Some other useful work on T-SFSs can be found in [47-51].

Dealing with uncertain situations and insufficient information required some high potential tools. Graph is one such mathematical tool which effectively deals with large data. When there is some sort of uncertainty factors, FG is a tool that needs to be used. Due to its capacity of handling large data, graph theory is of some special interest as it can be applied to many problems. The theory of fuzzy graphs (FGs) has its own significance as application of fuzzy set (FS) theory has no limits. Zadeh's FS provided a solid ground for the theory of FGs which have been introduced by Kauffman [52] in 1973. After that, FGs have been comprehensively studied by Rosenfeld [53]. The study of FGs lead many scientists to contribute in this field such as Bhattacharya [54] discussed several graph theoretic results for FGs. Bhutani [55] worked on automorphisms of FGs and Mordeson and Chang-Shyh [56] developed operations on FGs. FGs have been applied to many practical situations like optimization problems by Kóczy [57], clustering by Yeh and Bang [58], shortest path problem by Klein [59] and social networks Nair and Sarasamma [60] etc. For some other work on FGs one may refer to [61-64] etc.

The concept of Intuitionistic fuzzy graphs (IFGs) is introduced in [65] by Atanassov. Like IFSs, some quality works on the theory of IFGs are also being done. Parvathi, Karunambigai and Atanassov [66] studied operations on IFGs, Gani and Begum [67] discussed the size, order and degree of IFGs, Akram and Davvaz [68] investigated strong IFGs, Parvathi and Thamizhendhi [69] discussed domination in IFGs, Akram and Dudek [70] proposed intuitionistic fuzzy hypergraphs, Karunambigai, Akram et al. [71] presented the concept of balanced IFGs, Karunambigai, Parvathi et al. [72] studied constant IFGs and Chountas et al. [73] discussed intuitionistic fuzzy trees. For some other noteworthy work on IFGs one may refer to [74, 75] etc. The concept of PyFSs leads to the development of Pythagorean fuzzy graphs (PyFGs) [76]. For more related concept we may refer to [77, 78].

A generalization of FG known as interval valued fuzzy graph (IVFG) was introduced by Akram and Dudek in [79] where the nodes and edges are in the form of interval valued fuzzy numbers (IVFNs). Some other aspects of IVFG were discussed in [80-85]. The concept of interval valued IFG (IVIFG) was proposed by Mishra and Pal in [86] and some other terms and notions related to IVIFG can be found in [87,88]. The concept of single valued neutrosophic graph (SVNG) introduced by Broumi et al. in [89], has been extensively used in several problems such as shortest path problem [90, 91], communication problem [92], decision making [93] etc. The concept of SVNG was further generalized to interval valued neutrosophic graph (IVNG) proposed by Broumi in [94] where the membership, abstinence and non-membership grades are described by closed subintervals of the unit interval. The framework of IVNG is the most sophisticated one among all existing graph structures. For some other work on IVNG, one may refer to [95-97].

Shortest path problem is one of the famous problems that have been greatly discussed in different generalized fuzzy structures. In [98] by Okada and Soper studied the shortest path problem using fuzzy arcs and in [99] by Deng et al. proposed fuzzy Dijkstra algorithm for finding shortest path. In [100-102] some good work on fuzzy shortest path problems was provided. In [103] by Gani, and Jabarulla proposed a way of finding shortest path in intuitionistic fuzzy environment and in [104] by Mukherjee used Dijkstra algorithm to find shortest path in IFG. For some other works on shortest path problems, one may refer to [105-107].

Clustering is a technique in which objects of similar nature are grouped together into clusters. The objects that are in a cluster have some common attributes different from those which are not in the cluster. Cluster analysis is widely used in different fields such as data mining [108], recommender systems [109], image segmentations [110] and wireless sensor networks [111] etc. The concept of FSs and its different extensions have been extensively applied in clustering problems for example [112] is based on some techniques of fuzzy clustering, [113] provided the application of hesitant fuzzy sets in clustering, [114] is about j-divergence of IFSs and its application in clustering, [115] is based on some hierarchal clustering of IFSs and [116] is about a new clustering algorithm choquet aggregation operators.

Further, Clustering in the environment of FG has been studied in [117], while the same algorithm is studied for IFG in [118]. The algorithm is used in a decision-making problem in [118] for both fuzzy and intuitionistic fuzzy environment.

0.2. Chapter wise study

In chapter 1, we describe some very pioneer ideas of FS, IFSs, PyFSs, q-ROFSs, PFSs and TSFSs etc. The relationship between the definitions of each concept is demonstrated. Further, the ideas of FGs, IFGs and PyFGs are also demonstrated with the help of some examples. These notions are helpful in establishing new studies.

In chapter 2, a new concept of intuitionistic fuzzy graph of n^{th} type (IFGNT) is proposed as a generalization of intuitionistic fuzzy graphs (IFGs) and intuitionistic fuzzy graphs of second type (IFGST). Some light has also been shed upon the concepts of constant intuitionistic fuzzy graphs of second type (CIFGST) and constant intuitionistic fuzzy graphs of n^{th} type (CIFGNT). Moreover, some basic definitions of and results from IFGNT have been developed, supported with examples. Besides, the advantages of proposed new concept over the existing concepts have been highlighted and a comparative study of new and existing works made. Further, an application of IFGNT has been demonstrated in social networks context.

In chapter 3, a novel clustering algorithm in the environment of picture fuzzy graphs is developed. The proposed clustering algorithm is an improved version of clustering algorithm of fuzzy graphs and intuitionistic fuzzy graphs. Chapter 3 thoroughly investigated the existing clustering algorithms proposed in the frameworks of intuitionistic fuzzy graphs by pointing out the deficiencies and suggesting a solution which is applicable in handling real-life scenarios. The proposed clustering algorithm is supported with the help of a numerical problem discussing those cases which have not been discussed in the existing algorithms and the results are examined. To develop the new algorithm a study of picture fuzzy graphs along with some interesting result is established. A comparative study of the new work with that of existing work is established proving the worth of the proposed new work. Some drawbacks of the existing concepts and advantages of new theory are also discussed. Chapter ended with a summary of proposed work and possibly related future work in these directions.

In chapter 4, a social network and a wifi network using the concept of picture fuzzy graph (PFG). For this purpose, the concept of PFG and some basic terms are demonstrated including complement, degree and bridges. The main advantage of the proposed PFG is that it describes the uncertainty in any real-life event with the help of four membership degrees where the traditional FG and IFG fails to be applied. The viability of PFG is shown by utilizing the concept in demonstrating two real-life problems including a social network and a Wi-Fi-network. A comparison of PFG with existing notions is established showing its superiority over the existing frameworks.

In chapter 5, some developments based on fuzzy graph theory are discussed in detail. A notion of a T-spherical fuzzy graph (T-SFG, for short) is presented as a generalization of fuzzy graph, an intuitionistic fuzzy graph and a picture fuzzy graph. The originality, the imperativeness and the importance of this notion is discussed by showing some results, giving examples and a graphical analysis. Some theoretical terms of graphs such as a T-spherical fuzzy sub-graph, a complement of T-SFG, degree of T-SFG are clarified and their attributes and aspects are analyzed. The main goal of this chapter is to study two types of decision-making problems using the framework of T-SFGs. These two problems include the problem of the shortest path and a safe root for an airline journey in a T-spherical fuzzy network. The comparison of this new approach towards these problems with existing approaches is also established. A new algorithm is put forward in the event of T-SFGs and is used to seek out the shortest path problem. The overall analysis of the suggested notion under the prevailing theory is conducted. The advantages of the proposed approach were discussed based on the existing tools and a short comparison of the new with existing tools was established.

In chapter 6, some serious flaws in the existing definitions of several root level generalized FG structures with the help of some counter examples are shown and these issues are fixed. First, to improve the existing definition for interval valued FG, interval valued intuitionistic FG and their complements as these existing definitions are not well-defined i.e. one can obtain some senseless intervals using the existing definitions. The limitation of the existing definitions and the validity of new definitions is supported with some examples. It is also observed that the notion of single valued neutrosophic graph (SVNG) is not well-defined either. The consequences of the existing definition of SVNG are discussed with the help of examples. A new definition for SVNG is developed and its improvement is demonstrated with some examples. The definition of interval valued neutrosophic graph is also modified due to the shortcomings existed in the current definition and the validity of new definition is proved. An application of proposed work in decision making is solved in the framework of SVNG where the failure of existing definitions and effectiveness of new definitions is demonstrated.

In chapter 7, the notion of graph for newly established concept of interval-valued PyFS (IVPyFS). Introduced the concept of interval-valued Pythagorean fuzzy graphs (IVPyFGs) and discussed its related ideas with the help of examples. The significance of using interval valued fuzzy concepts over non-interval valued fuzzy concepts are demonstrated numerically. Further the advantages of using the new approach are over the pre-existing ideas is demonstrated with the help of numerical examples. The main goal of this chapter is to study three types of decision-making problems using the framework of IVPyFGs. These three problems include the problem of selection of best university in a network of universities, supply chain management problem, and shortest path problem. The comparison of new approach towards these problems with existing approaches is also established.

In chapter 8, a new notion of interval valued q-rung ortho pair fuzzy graph (IVq-ROFG) and to study the related graphical terms such as subgraph, complement, degree of vertices and path etc. Each of the graphical concept is demonstrated with an example. Another valuable contribution of this chapter is the modeling of some traffic networks, telephone networks and social networks using the concepts of IVq-ROFGs. First, the famous problem of finding a shortest path in a traffic network is studied using two different approaches. A study of social network describing the co-authorship between different researchers from several countries is also established using the concept of IVq-ROFGs. Finally, a telephone networking problem is demonstrated showing the calling ratios of incoming and outgoing calls among a group of people. Through comparative study, the advantages of working in the environment of IVq-ROFG are specified.

Chapter 1

Preliminaries

In this chapter we describe some very pioneer ideas of FSs, IFSs, PyFSs, q-ROFSs, PFSs and TSFSs etc. The relationship between the definitions of each concept is demonstrated. Further, the ideas of FGs, IFGs, PyFGs are also demonstrated with the help of some examples. These notions are helpful in establishing new studies.

1.1. Fuzzy Sets and Their Generalizations

In this section the notion of FSs and their generalizations are discussed.

1.1.1 Definition [1]

Let $\dot{\mathrm{X}}$ be a universal set. Then FS in $\dot{\mathrm{X}}$ is defined as

$$A = \left\{\langle ṭ, \hat{\mathrm{S}}_A(ṭ)\rangle / ṭ \in \dot{\mathrm{X}}\right\}$$

Here $\hat{\mathrm{S}}_A : \dot{\mathrm{X}} \longrightarrow [0, 1]$ is the membership function of the FS on A.

1.1.2 Definition [10]

Let $\dot{\mathrm{X}}$ be a universal set. Then IFS is defined as

$$Ḅ = \{< ṭ, \hat{\mathrm{S}}_{Ḅ}(ṭ), Đ_{Ḅ}(ṭ) >: ṭ \in \dot{\mathrm{X}}\}$$

Here $\hat{\mathrm{S}}_{Ḅ} : \dot{\mathrm{X}} \to [0, 1]$ represents membership degree and $Đ_{Ḅ} : \dot{\mathrm{X}} \to [0, 1]$ represents non-membership degree of $ṭ \in \dot{\mathrm{X}}$ with a condition $0 \leq \hat{\mathrm{S}}_{Ḅ}(ṭ) + Đ_{Ḅ}(ṭ) \leq 1$. Furthermore, $\left(\hat{\mathrm{S}}_{Ḅ}(ṭ), Đ_{Ḅ}(ṭ)\right)$ is known as a intuitionistic fuzzy number (IFN) denoted as $IFN = (\hat{\mathrm{S}}_{Ḅ}, Đ_{Ḅ})$.

1.1.3 Definition [15]

Let $\dot{\mathrm{X}}$ be a universal set. Then PyFS is defined as

$$\text{Þ} = \{< \text{ṭ}, \hat{\text{S}}_{\text{Þ}}(\text{ṭ}), \text{Đ}_{\text{Þ}}(\text{ṭ}) >: \text{ṭ} \in \dot{\text{X}}\}$$

Here $\hat{\text{S}}_{\text{Þ}}: \dot{\text{X}} \to [0, 1]$ represents membership degree and $\text{Đ}_{\text{Þ}}: \dot{\text{X}} \to [0, 1]$ represents non-membership degree of $\text{ṭ} \in \dot{\text{X}}$ to set Þ provided that $0 \le \hat{\text{S}}_{\text{Þ}}^2(\text{ṭ}) + \text{Đ}_{\text{Þ}}^2(\text{ṭ}) \le 1$. The extent of indeterminacy here is defined by

$$\pi_{\text{Þ}}(\text{ṭ}) = \sqrt[2]{\left(1 - \hat{\text{S}}_{\text{Þ}}^2(\text{ṭ}) - \text{Đ}_{\text{Þ}}^2(\text{ṭ})\right)}$$

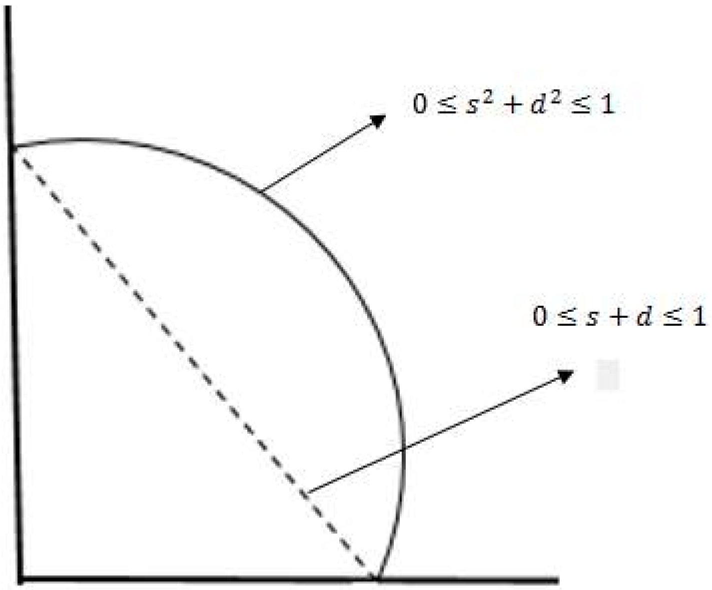


Figure 3 (A comparison of spaces of IFSs and PyFSs)

Furthermore, $\left(\hat{\text{S}}_{\text{Þ}}(\text{ṭ}), \text{Đ}_{\text{Þ}}(\text{ṭ})\right)$ is known as a Pythagorean fuzzy number (PyFN) denoted as $\text{Þ} = (\hat{\text{S}}_{\text{Þ}}, \text{Đ}_{\text{Þ}})$. All the above discussed theory and geometrical compression clearly shows the significance of PyFS as it is the generalization of FS and IFS. The above Figure 3 shows that every fuzzy number and IFN is PyFN but converse is not true.

1.1.4 Definition [25]

Let $\dot{\text{X}}$ be a universal set. Then q-ROFS is defined as

$$\check{\text{r}} = \{< \text{ṭ}, \hat{\text{S}}_{\check{\text{r}}}(\text{ṭ}), \text{Đ}_{\check{\text{r}}}(\text{ṭ}) >: \text{ṭ} \in \dot{\text{X}}\}$$

Here $\hat{S}_{ř}: \dot{X} \to [0,1]$ represents membership degree and $Đ_{ř}: \dot{X} \to [0,1]$ represents non-membership degree of $ț \in \dot{X}$ to set ř provided that $0 \leq \hat{S}_{ř}^{q}(ț) + Đ_{ř}^{q}(ț) \leq 1$ for $q \in Z^{+}$.The extent of indeterminacy here defined by

$$\pi_{ř}(ț) = \sqrt[ǹ]{\left(1 - \hat{S}_{ř}^{q}(ț) - Đ_{ř}^{q}(ț)\right)}$$

Furthermore, $\left(\hat{S}_{ř}(ț), Đ_{ř}(ț)\right)$ is known as a q-rung orthopair fuzzy number (q-ROFN) denoted as $ř = (\hat{S}_{ř}, Đ_{ř})$.

1.1.5 Definition [37]

A PFS on a universal set $\dot{X}$ is characterized by three functions $\hat{S}$, $\hat{I}$ and Đ on [0,1] fulfilling $0 \leq \hat{S}(ț) + \hat{I}(ț) + Đ(ț) \leq 1$. The values of $\hat{S}$, $\hat{I}$ and Đ in the unit interval describe the membership degree, the abstinence and the non-membership degrees of ț in $\dot{X}$. Also $1 - \left(\hat{S}(ț) + \hat{I}(ț) + Đ(ț)\right)$ denotes the refusal degree of $ț \in \dot{X}$. The triplet $\left(\hat{S}, \hat{I}, Đ\right)$ is known as a picture fuzzy number (PFN).

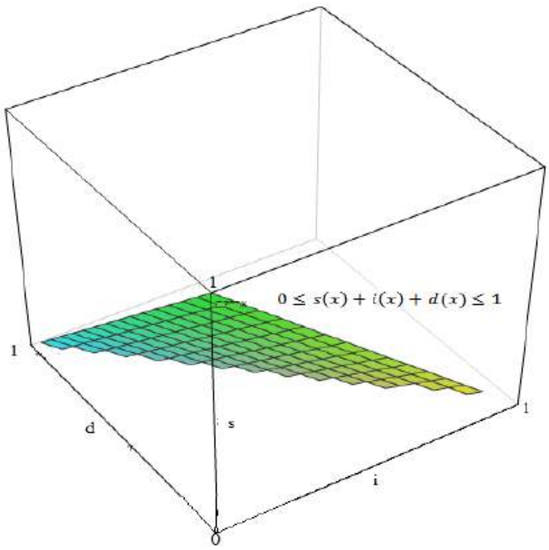


Figure 4 (Picture fuzzy space)

The problem with PFSs and its constraint is depicted in figure 4. Realizing this issue Mahmood et al. in [100] proposed a new concept of SFSs and consequently T-SFSs. The following definitions are described along their geometrical representation in order to make the point clear that SFSs and T-SFSs generalize IFSs and PFSs.

1.1.6 Definition [46]

A SFS on a universal set $\dot{\mathrm{X}}$ is characterized by three functions $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ on $[0,1]$ with the condition $0 \leq \hat{\mathrm{S}}^2(\d{t}) + \hat{\mathrm{I}}^2(\d{t}) + \mathrm{Đ}^2(\d{t}) \leq 1$. The value of $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ in the unit interval describe the membership, the abstinence and the non-membership degree of $\d{t}$ in $\dot{\mathrm{X}}$. Also $\d{R}(\d{t}) = \sqrt{1 - \left(\hat{\mathrm{S}}^2(\d{t}) + \hat{\mathrm{I}}^2(\d{t}) + \mathrm{Đ}^2(\d{t})\right)}$ denote the refusal degree of $\d{t} \in \dot{\mathrm{X}}$. The triplet $\left(\hat{\mathrm{S}}, \hat{\mathrm{I}}, \mathrm{Đ}\right)$ is known as a spherical fuzzy number (SFN).

1.1.7 Definition [46]

A T-SFS on a universal set $\dot{\mathrm{X}}$ is characterized by three functions $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ on $[0,1]$ provided that $0 \leq \hat{\mathrm{S}}^{\dot{n}}(\d{t}) + \hat{\mathrm{I}}^{\dot{n}}(\d{t}) + \mathrm{Đ}^{\dot{n}}(\d{t}) \leq 1$ for some $\dot{n} \in \mathbb{Z}^+$. The values of $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ in the unit interval describes the membership, the abstinence and the non-membership degrees of $\d{t}$ in $\dot{\mathrm{X}}$. Also $\d{R}(\d{t}) = \sqrt[n]{1 - \left(\hat{\mathrm{S}}^{\dot{n}}(\d{t}) + \hat{\mathrm{I}}^{\dot{n}}(\d{t}) + \mathrm{Đ}^{\dot{n}}(\d{t})\right)}$ denotes the refusal degree of $\d{t} \in \dot{\mathrm{X}}$. The triplet $\left(\hat{\mathrm{S}}, \hat{\mathrm{I}}, \mathrm{Đ}\right)$ is known as a T-spherical fuzzy number (TSFN).

The following figures describe SFSs and T-SFSs, showing its novelty and diversity in structure. Moreover, Figures 5, 6 and 7 show that T-SFSs have no limitation.

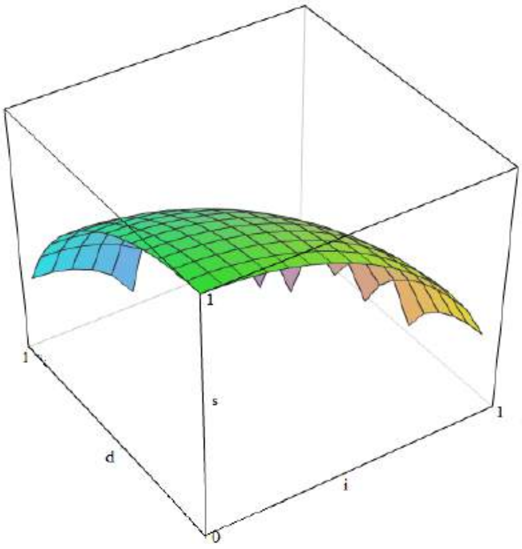


Figure 5 (Space of spherical fuzzy sets)

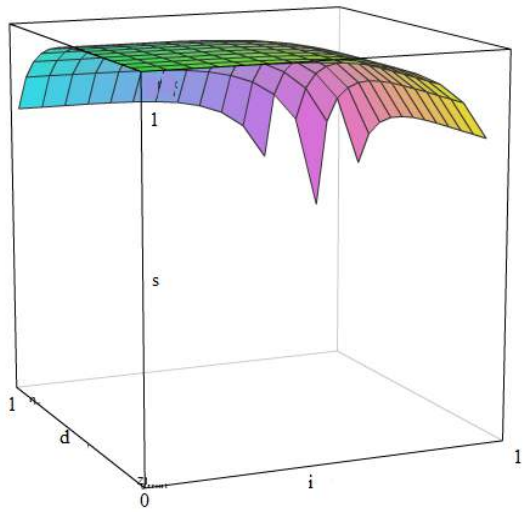


Figure 6 (T-SFS (n=5))

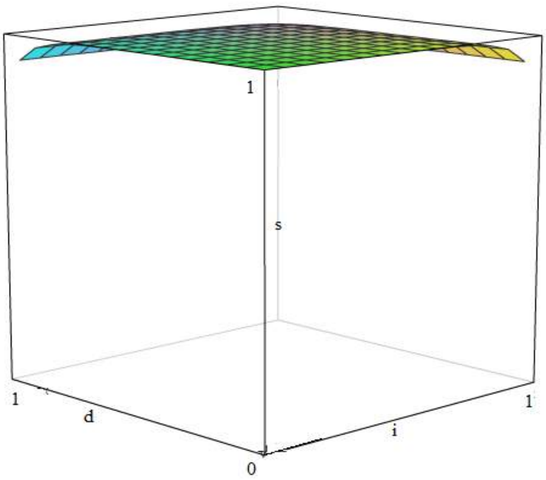


Figure 7. T-SFS (n=10)

It is easy to conclude that the concept of T-SFS is a generalization of the concept of FS, IFS, PyFS, q-ROFS, PFS and also SFS, without the limitation in its structure.

1.2 Score Functions

In this section we discussed the score function of IFSs, PyFSs, q-ROFSs, PFS, SFSs and T-SFs.

1.2.8 Definition [11]

The Score function of an IFN $(\hat{S}_{Ḅ}(ṭ), D_{Ḅ}(ṭ))$ is defined as $Sc_{IFN} = \hat{S}_{Ḅ}(ṭ) - D_{Ḅ}(ṭ)$. Here $\hat{S}_{Ḅ}: \dot{X} \to [0,1]$ represents membership degree and $Đ_{Ḅ}: \dot{X} \to [0,1]$ represents non-membership degree of $ṭ \in \dot{X}$ with a condition $0 \leq \hat{S}_{Ḅ}(ṭ) + Đ_{Ḅ}(ṭ) \leq 1$.

1.2.9 Definition [49]

The Score function of PyFN $(\hat{S}_{Þ}(ṭ), D_{Þ}(ṭ))$ is defined as $Sc_{PyFN} = \hat{S}_{Þ}^2(ṭ) - Đ_{Þ}^2(ṭ)$. Here $\hat{S}_{Þ}: \dot{X} \to [0,1]$ represents membership degree and $Đ_{Þ}: \dot{X} \to [0,1]$ represents non-membership degree of $ṭ \in \dot{X}$ to set Þ provided that $0 \leq \hat{S}_{Þ}^2(ṭ) + Đ_{Þ}^2(ṭ) \leq 1$. The extent of indeterminacy here defined by

$$\pi_{Þ}(ṭ) = \sqrt{\left(1 - \hat{S}_{Þ}^{2}(ṭ) - Đ_{Þ}^{2}(ṭ)\right)}$$

1.2.10 Definition [25]

The Score function of a q-ROFNs $(\hat{S}_{ř}(ṭ), D_{ř}(ṭ))$ is defined as $Sc_{q-ROFN} = \hat{S}_{ř}^{q}(ṭ) - Đ_{ř}^{q}(ṭ)$. Here $\hat{S}_{ř}: \dot{X} \rightarrow [0,1]$ represents membership degree and $Đ_{ř}: \dot{X} \rightarrow [0,1]$ represents non-membership degree of $ṭ \in \dot{X}$ to set Þ provided that $0 \leq \hat{S}_{ř}^{q}(ṭ) + Đ_{v}^{q}(ṭ) \leq 1$ for $q \in Z^{+}$. The extent of indeterminacy here defined by

$$\pi_{ř}(ṭ) = \sqrt[q]{\left(1 - \hat{S}_{ř}^{q}(ṭ) - Đ_{ř}^{q}(ṭ)\right)}$$

1.2.11 Definition [38]

The score function of a PFN is defined by $P_{s} = \hat{S} - Ṛ.Đ$, where $\hat{S}$, Đ represents the membership, non- membership degrees and $Ṛ = 1 - \left(\hat{S}(ṭ) + \hat{I}(ṭ) + Đ(ṭ)\right)$ represents the refusal degree for PFN.

1.2.12 Definition [46]

The score function for a SFN is defined by $Sc_{s} = \hat{S}^{2} - Ṛ^{2}.Đ^{2}$, where $\hat{S}$, Đ represents the membership, non- membership degrees and $Ṛ(ṭ) = \sqrt[2]{1 - \left(\hat{S}^{2}(ṭ) + \hat{I}^{2}(ṭ) + Đ^{2}(ṭ)\right)}$ represents the refusal degree for SFN.

1.2.13 Definition [46]

The score function for a T-SFN is defined by $TS_{s} = \hat{S}^{ǹ} - Ṛ^{ǹ}.Đ^{ǹ}$, where $\hat{S}$, Đ represents the membership, non- membership degrees and $Ṛ(ṭ) = \sqrt[ǹ]{1 - \left(\hat{S}^{ǹ}(ṭ) + \hat{I}^{ǹ}(ṭ) + Đ^{ǹ}(ṭ)\right)}$ represents the refusal degree for T-SFN.

1.3 Aggregation Operators

In this section the aggregation operators of IFNs, PyFNs, q-ROFNs, PFNs, SFNs and TSFNs are discussed.

1.3.1 Definition [12]

The intuitionistic fuzzy weighted averaging (IFWA) operators of an IFNs $ţ_{i1}, ţ_{i2}, \ldots, ţ_{in}$ is denoted and defined by

$$ţ_i = IFWA(ţ_{i1}, ţ_{i2}, \ldots, ţ_{iǹ}) = \left(1 - \prod_{j=1}^{ǹ}\left(1 - \hat{\text{S}}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{ǹ} \text{Đ}\right)^{w_j}\right)$$

1.3.2 Definition [12]

The intuitionistic fuzzy weighted geometric (IFWG) operator of an IFN $ţ_{i1}, ţ_{i2}, \ldots, ţ_{iǹ}$ is denoted and defined by

$$ţ_i = IFWG(ţ_{i1}, ţ_{i2}, \ldots, ţ_{iǹ}) = \left(\left(\prod_{j=1}^{ǹ} \hat{\text{S}}\right)^{w_j}, 1 - \prod_{j=1}^{ǹ}\left(1 - \text{Đ}_{ij}\right)^{w_j}\right)$$

1.3.3 Definition [15]

The Pythagorean fuzzy weighted averaging (PyFWA) operator of PyFN $ţ_{i1}, ţ_{i2}, \ldots, ţ_{iǹ}$ is denoted and defined by

$$ţ_i = PyFWA(ţ_{i1}, ţ_{i2}, \ldots, ţ_{iǹ}) = \left(\sqrt{1 - \prod_{j=1}^{ǹ}\left(1 - \left(\hat{\text{S}}_{ij}\right)^2\right)^{w_j}}, \left(\prod_{j=1}^{ǹ} \text{Đ}_{ij}\right)^{w_j}\right)$$

1.3.4 Definition [15]

The Pythagorean fuzzy weighted geometric (PyFWG) operator of PyFN $ţ_{i1}, ţ_{i2}, \ldots, ţ_{iǹ}$ is denoted and defined by

$$ţ_i = PyFWG(ţ_{i1}, ţ_{i2}, \dots, ţ_{iǹ}) = \left(\left(\prod_{j=1}^{ǹ} \hat{S}_{ij} \right)^{w_j}, \sqrt{1 - \prod_{j=1}^{ǹ} \left(1 - \left(Đ_{ij} \right)^2 \right)^{w_j}} \right)$$

1.3.5 Definition [34]

The q-Rung orthopair fuzzy weighted averaging (q-ROFWA) operator of q-ROFN $ţ_{i1}, ţ_{i2}, \dots, ţ_{iǹ}$ is denoted and defined by

$$ţ_i = q - ROFWA(ţ_{i1}, ţ_{i2}, \dots, ţ_{iǹ}) = \left(\sqrt[q]{1 - \prod_{j=1}^{ǹ} \left(1 - \left(\hat{S}_{ij} \right)^q \right)^{w_j}}, \left(\prod_{j=1}^{ǹ} Đ_{ij} \right)^{w_j} \right)$$

1.3.6 Definition [34]

The q-Rung orthopair fuzzy weighted geometric (q-ROFWG) operator of q-ROFN $ţ_{i1}, ţ_{i2}, \dots, ţ_{iǹ}$ is denoted and defined by

$$ţ_i = q - ROFWG(ţ_{i1}, ţ_{i2}, \dots, ţ_{iǹ}) = \left(\left(\prod_{j=1}^{ǹ} \hat{S}_{ij} \right)^{w_j}, \sqrt[q]{1 - \prod_{j=1}^{ǹ} \left(1 - \left(Đ_{ij} \right)^q \right)^{w_j}} \right)$$

1.3.7 Definition [40]

The Picture fuzzy weighted averaging (PFWA) operator of an PFN $ţ_{i1}, ţ_{i2}, \dots, ţ_{iǹ}$ is denoted and defined by

$$ţ_i = PFWA(ţ_{i1}, ţ_{i2}, \dots, ţ_{iǹ}) = \left(\begin{array}{c} 1 - \prod_{j=1}^{ǹ} \left(1 - \hat{S}_{ij} \right)^{w_j}, \\ \left(\prod_{j=1}^{ǹ} \hat{I}_{ij} \right)^{w_j}, \left(\prod_{j=1}^{ǹ} Đ_{ij} \right)^{w_j} \end{array} \right)$$

1.3.8 Definition [40]

The Picture fuzzy weighted geometric (PFWG) operator of a PFN $ṭ_{i1}, ṭ_{i2}, \ldots, ṭ_{iǹ}$ is denoted and defined by

$$ṭ_i = PFWG(ṭ_{i1}, ṭ_{i2}, \ldots, ṭ_{iǹ}) = \begin{pmatrix} \left(\prod_{j=1}^{ǹ} \hat{S}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{ǹ} \hat{I}_{ij}\right)^{w_j}, \\ 1 - \prod_{j=1}^{ǹ} \left(1 - Đ_{ij}\right)^{w_j} \end{pmatrix}$$

1.3.9 Definition [48]

The spherical fuzzy weighted averaging (SFWA) operator of SFN $ṭ_{i1}, ṭ_{i2}, \ldots, ṭ_{iǹ}$ is denoted and defined by

$$ṭ_i = SFWA(ṭ_{i1}, ṭ_{i2}, \ldots, ṭ_{iǹ}) = \begin{pmatrix} \sqrt{1 - \prod_{j=1}^{ǹ} \left(1 - \left(\hat{S}_{ij}\right)^2\right)^{w_j}}, \\ \left(\prod_{j=1}^{ǹ} \hat{I}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{ǹ} Đ_{ij}\right)^{w_j} \end{pmatrix}$$

Where w_j be the weight vector with a condition that is $\sum_{j=1}^{ǹ} w_j = 1$.

1.3.10 Definition [48]

The spherical fuzzy weighted geometric (SFWG) operator of SFN $ṭ_{i1}, ṭ_{i2}, \ldots, ṭ_{iǹ}$ is denoted and defined by: $ṭ_i = SFWG(ṭ_{i1}, ṭ_{i2}, \ldots, ṭ_{iǹ}) =$

$$\left(\left(\prod_{j=1}^{ǹ} \hat{S}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{ǹ} \hat{I}_{ij}\right)^{w_j}, \sqrt{1 - \prod_{j=1}^{ǹ} \left(1 - \left(Đ_{ij}\right)^2\right)^{w_j}}\right).$$

1.3.11 Definition [48]

The T- spherical fuzzy weighted averaging (T-SFWA) operator of TSFN $ţ_{i1}, ţ_{i2}, \dots, ţ_{i\grave{n}}$ is denoted and defined by

$$ţ_i = TSFWA(ţ_{i1}, ţ_{i2}, \dots, ţ_{i\grave{n}}) = \left(\begin{array}{c} \sqrt[q]{1 - \prod_{j=1}^{\grave{n}} \left(1 - \left(\hat{S}_{ij}\right)^q\right)^{w_j}}, \\ \left(\prod_{j=1}^{\grave{n}} \hat{I}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{\grave{n}} Đ_{ij}\right)^{w_j} \end{array} \right)$$

1.3.12 Definition [48,50]

The T- spherical fuzzy weighted geometric (TSFWG) operator of TSFN $ţ_{i1}, ţ_{i2}, \dots, ţ_{i\grave{n}}$ is denoted and defined by

$$ţ_i = TSFWG(ţ_{i1}, ţ_{i2}, \dots, ţ_{i\grave{n}}) = \left(\begin{array}{c} \left(\prod_{j=1}^{\grave{n}} \hat{S}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{\grave{n}} \hat{I}_{ij}\right)^{w_j}, \\ \sqrt[q]{1 - \prod_{j=1}^{\grave{n}} \left(1 - \left(Đ_{ij}\right)^q\right)^{w_j}} \end{array} \right)$$

1.4 Addition and Multiplication Operations

In this section operations of addition and multiplications of q-ROFN and TSFNs are deliberated.

1.4.1 Definition [34]

The addition and multiplication of q- ROFNs is denoted and defined as

1) $\left(\hat{S}_1, Đ_1\right) + \left(\hat{S}_2, Đ_2\right) = \left(\sqrt[\grave{n}]{\left(\hat{S}_1\right)^{\grave{n}} + \left(\hat{S}_2\right)^{\grave{n}} - \left(\hat{S}_1\right)^{\grave{n}}\left(\hat{S}_2\right)^{\grave{n}}}, Đ_1 Đ_2\right)$

2) $\left(\hat{S}_1, Đ_1\right) \times \left(\hat{S}_2, Đ_2\right) = \left(\hat{S}_1 \hat{S}_2, \sqrt[\grave{n}]{(Đ_1)^{\grave{n}} + (Đ_2)^{\grave{n}} - (Đ_1)^{\grave{n}}(Đ_2)^{\grave{n}}}\right)$

1.4.2 Remark

Replacing $\grave{n} = 1$ reduces the above defined equations in the environment of IFNs and for $\grave{n} = 2$ reduces defined equations in the environment of PyFNs.

1.4.3 Definition [48,50]

The addition and multiplication of TSFNs are denoted and defined as:

1) $(\hat{S}_1, \hat{I}_1, Đ_1) + (\hat{S}_2, \hat{I}_2, Đ_2) = \left(\sqrt[\grave{n}]{(\hat{S}_1)^{\grave{n}} + (\hat{S}_2)^{\grave{n}} - (\hat{S}_1)^{\grave{n}}}, \hat{I}_1\hat{I}_2, Đ_1Đ_2\right)$

2) $(\hat{S}_1, \hat{I}_1, Đ_1) \times (\hat{S}_2, \hat{I}_2, Đ_2) =$

$$\left(\hat{S}_1\hat{S}_2, \sqrt[\grave{n}]{(\hat{I}_1)^{\grave{n}} + (\hat{I}_2)^{\grave{n}} - (\hat{I}_1)^{\grave{n}}(\hat{I}_2)^{\grave{n}}}, \sqrt[\grave{n}]{(Đ_1)^{\grave{n}} + (Đ_2)^{\grave{n}} - (Đ_1)^{\grave{n}}(Đ_2)^{\grave{n}}}\right)$$

1.4.4 Remark

Replacing $\grave{n} = 1$ reduces the above defined equations in the environment of PFNs and for $\grave{n} = 2$ reduces defined equations in the environment of SFNs.

1.5 Fuzzy Graphs and Their Generalizations

In this section, we recall definitions related to graphs of FSs, IFSs and PyFSs underlying the main considerations of this Thesis.

1.5.1 Definition [52]

A FG is a pair $\dot{G} = (\underset{.}{V}, \dot{E})$ where $\underset{.}{V}$ is the set of nodes and $\dot{E}$ is the collection of edges between these nodes such that

1) Every $\underset{.}{t} \in \underset{.}{V}$ is characterized by a function $\hat{S}: \underset{.}{V} \to [0, 1]$ denoting the degree of membership of $\underset{.}{t} \in \underset{.}{V}$.

2) Every $\d{t} \in \dot{\acute{E}}$ is characterized by a function $\hat{S}: \d{V} \times \d{V} \rightarrow [0,1]$ denoting the degree of membership $\d{t} \in \d{V} \times \d{V}$ satisfying the condition $\hat{S}(\d{t}_i, \d{t}_j) \leq \min\left(\hat{S}(\d{t}_i), \hat{S}_j(\d{t}_j)\right)$.

1.5.2 Definition [65]

An IFG is a duplet $\dot{G} = (\d{V}, \dot{\acute{E}})$ where the set of nodes is denoted by $\d{V}$ and $\dot{\acute{E}}$ is the collection of edges between these nodes such that

1) Every $\d{t} \in \d{V}$ is characterized by two functions $\hat{S}: \d{V} \rightarrow [0,1]$ and $Đ: \d{V} \rightarrow [0,1]$ denoting the membership and non-membership degree of $\d{t} \in \d{V}$which satisfies the condition that $0 \leq \hat{S} + Đ \leq 1$. Moreover, the term $\d{R}$ defined by $\d{R} = 1 - \hat{S} - Đ$, denotes the hesitancy level of $\d{t} \in \dot{\acute{E}}$.
2) Every $\d{t} \in \dot{\acute{E}}$ is characterized by two functions $\hat{S}: \d{V} \times \d{V} \rightarrow [0,1]$ and $Đ: \d{V} \times \d{V} \rightarrow [0,1]$ denoting the membership and non-membership degree of $\d{t} \in \d{V} \times \d{V}$ satisfying the conditions:

$$\hat{S}(\d{t}_i, \d{t}_j) \leq \min\left(\hat{S}(\d{t}_i), \hat{S}(\d{t}_j)\right)$$

$$Đ(\d{t}_i, \d{t}_j) \leq \max\left(Đ(\d{t}_i), Đ(\d{t}_j)\right)$$

with a condition that $0 \leq \hat{S} + Đ \leq 1$. Moreover, the term $\d{R}$ denotes the hesitancy level of $\d{t} \in \dot{\acute{E}}$such that $\d{R} = 1 - \hat{S} - Đ$.

1.5.3 Example

The following Figure 8 is an example of IFG.

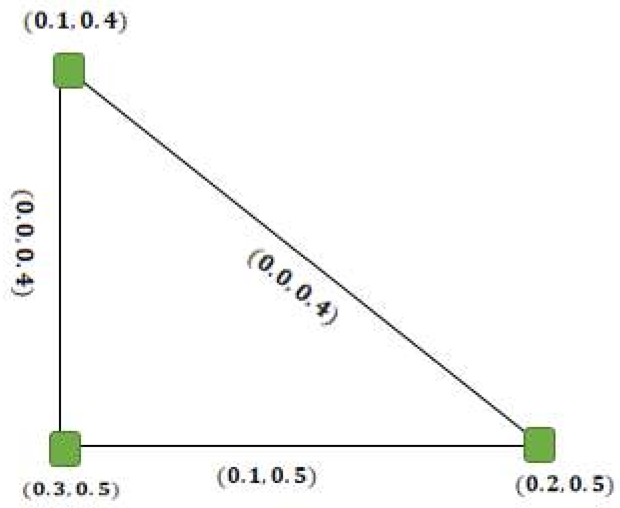


Figure 8 (Intuitionistic fuzzy graph compatible with definition 1.5.2)

1.5.4 Definition [76]

A PyFG is denoted and defined as $\dot{G} = (\underset{\cdot}{V}, \dot{E})$ where

1. $\underset{\cdot}{V} = \{t_1, t_2, t_3, \ldots, t_{\grave{n}}\}$ such that $\hat{S}: \underset{\cdot}{V} \to [0,1]$ represents membership degree and $Đ: \underset{\cdot}{V} \to [0,1]$ represents non-membership degree of $t_i \in \tilde{V}$ respectively provided that $0 \leq \hat{S}^2 + Đ^2 \leq 1$ For all $t_i \in \underset{\cdot}{V}, (i = 1, 2, 3, \ldots \grave{n})$
2. $\dot{E} \subseteq \underset{\cdot}{V} \times \underset{\cdot}{V}$ where $\hat{S}: \underset{\cdot}{V} \times \underset{\cdot}{V} \to [0,1]$ and $Đ: \underset{\cdot}{V} \times \underset{\cdot}{V} \to [0,1]$ are such $\hat{S}(t_i, t_j) \leq \min[\hat{S}(t_i), \hat{S}(t_j)]$ $Đ(t_i, t_j) \leq \max[Đ(t_i), Đ(t_j)]$with the condition $0 \leq \hat{S}^2(t_i, t_j) + Đ^2(t_i, t_j) \leq 1$

   for all $(t_i, t_j) \in \dot{E}$.

1.5.5 Definition [52]

A pair $H = (\underset{.}{\mathrm{V}}', \dot{\mathrm{E}}')$ is considered as fuzzy subgraph (FSG) of FG $\dot{\mathrm{G}} = (\underset{.}{\mathrm{V}}, \dot{\mathrm{E}})$ if $\underset{.}{\mathrm{V}}' \subseteq \underset{.}{\mathrm{V}}$ and $E' \subseteq \dot{\mathrm{E}}$ i.e. $\hat{\mathrm{S}}'_{1i} \leq \hat{\mathrm{S}}_{1i}$ and $\hat{\mathrm{S}}'_{2ij} \leq \hat{\mathrm{S}}_{2ij}$ $(i, j = 1,2, \ldots, n)$.

1.5.6 Definition [65]

A pair $H = (\underset{.}{\mathrm{V}}', \dot{\mathrm{E}}')$ is considered as an intuitionistic fuzzy subgraph (IFSG) of IFG $\dot{\mathrm{G}} = (\underset{.}{\mathrm{V}}, \dot{\mathrm{E}})$ if $\underset{.}{\mathrm{V}}' \subseteq \underset{.}{\mathrm{V}}$ and $E' \subseteq \dot{\mathrm{E}}$ i.e. $\hat{\mathrm{S}}'_{1i} \leq \hat{\mathrm{S}}_{1i}, \mathrm{Đ}'_{1i} \geq \mathrm{Đ}_{1i}$ and $\hat{\mathrm{S}}'_{2ij} \leq \hat{\mathrm{S}}_{2ij}, \mathrm{Đ}'_{2ij} \geq \mathrm{Đ}_{2ij}$ $(i, j = 1,2, \ldots, \grave{\mathrm{n}})$.

1.5.7 Example

An IFSG H of IFG of Figure 8 is shown in Figure 9.

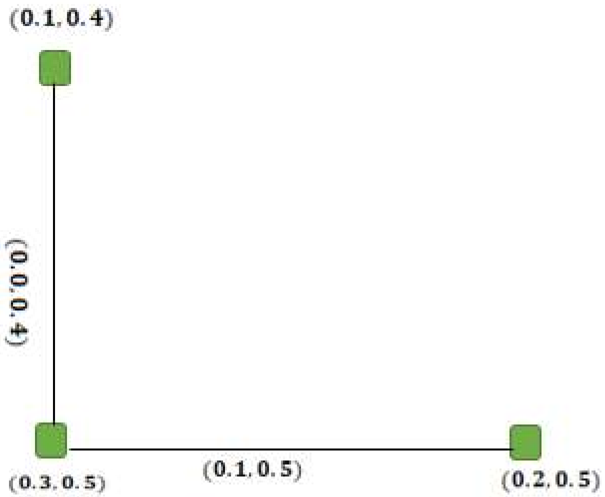


Figure 9 (Intuitionistic fuzzy sub graph of Figure 8)

1.6 Complement, Degree, Density and Path of FGs and IFGs

In this section, the complement, degree, density and path of FGs and IFGs are discussed.

1.6.1 Definition [53]

The complement of FG $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ is defined by $\dot{\mathrm{G}}^c = \left(\underset{\cdot}{\mathrm{V}}^c, \dot{\mathrm{E}}^c\right)$ where $\underset{\cdot}{\mathrm{V}}^c = \underset{\cdot}{\mathrm{V}}$ and the membership grade of $\dot{\mathrm{E}}$ is defined by $(\hat{\mathrm{S}})^c(\underset{\cdot}{\mathrm{t}}_i, \underset{\cdot}{\mathrm{t}}_j) = \min\left(\hat{\mathrm{S}}(\underset{\cdot}{\mathrm{t}}_i), \hat{\mathrm{S}}(\underset{\cdot}{\mathrm{t}}_j)\right) - \hat{\mathrm{S}}\left((\underset{\cdot}{\mathrm{t}}_i, \underset{\cdot}{\mathrm{t}}_j)\right)$.

1.6.2 Example

The following Figure 10 is an example of an FG while Figure 11 represents the complement of the FG depicted in Figure 10.

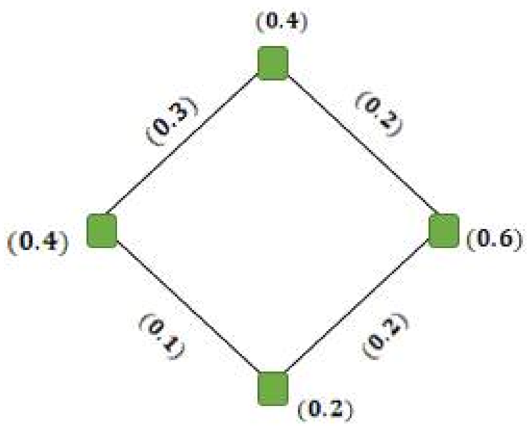


Figure 10 (Fuzzy graph compatible with definition 1.5.1)

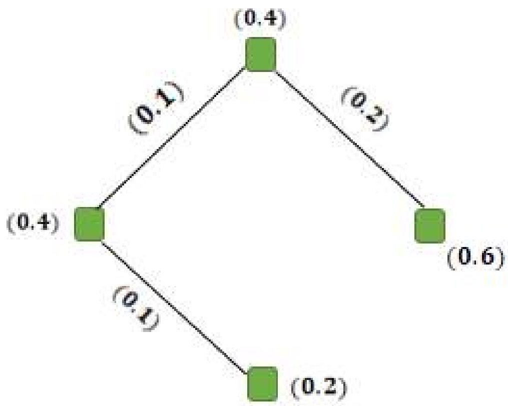


Figure 11 (Complement of fuzzy graph depicted in Figure 10)

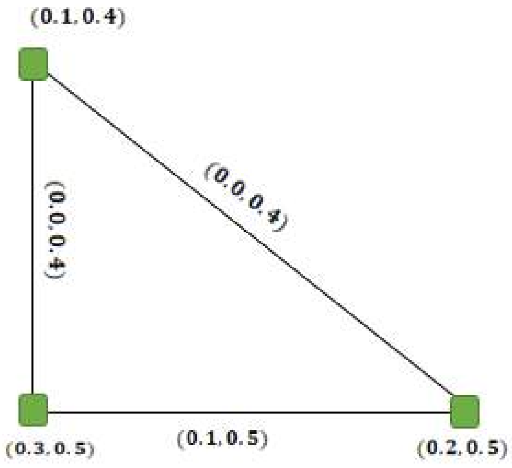
1.6.3 Definition [66]

The complement of an IFG $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ is defined as

1. $\underset{\cdot}{\mathrm{V}}^c = \underset{\cdot}{\mathrm{V}}$.
2. $\hat{\mathrm{S}}_{1i}{}^c = \hat{\mathrm{S}}_{1i}$ and $\mathrm{Đ}_{1i}{}^c = \mathrm{Đ}_{1i}$ for all $i = 1,2, \ldots n$.
3. $\hat{\mathrm{S}}_{2ij}{}^c = min(\hat{\mathrm{S}}_{1i}, \hat{\mathrm{S}}_{1j}) - \hat{\mathrm{S}}_{2ij}$ and $\mathrm{Đ}_{2ij}{}^c = max(\mathrm{Đ}_{1i}, \mathrm{Đ}_{1j}) - \mathrm{Đ}_{2ij}$ for every $i.j = 1,2, \ldots n$.

1.6.4 Example

The following Figure 12 is an example of IFG while Figure 13 represents the complement of the IFG depicted in Figure 12.



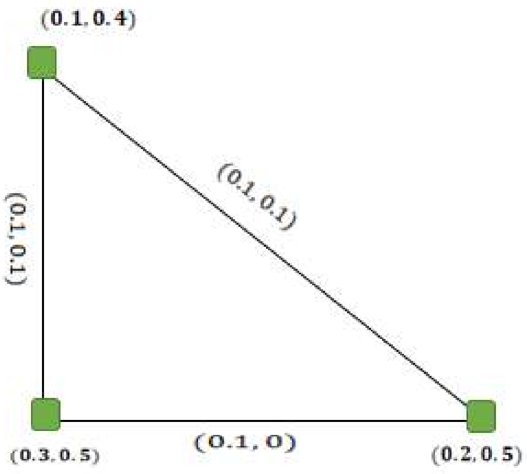


Figure 12 (Intuitionistic fuzzy graph compatible with definition 1.5.2)

Figure 13 (Complement of Intuitionistic fuzzy graph depicted in Figure 12)

1.6.5 Definition [67]

The degree of any vertex of an IFG is denoted and defined by

$$d(\mathrm{t}) = \left(d_{\hat{S}}(\mathrm{t}), d_{D}(\mathrm{t})\right) \text{ where } d_{\hat{S}}(\mathrm{t}) = \sum_{i\neq j} \hat{S}_2(\mathrm{t}_i, \mathrm{t}_j) \text{ and } d_d(\mathrm{t}) = \sum_{i\neq j} d_2(\mathrm{t}_i, \mathrm{t}_j).$$

1.6.6 Example

Consider a graph $\dot{G} = (V, \dot{E})$ where $V = \{\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3\}$ be the set of vertices and $\dot{E}$ be the set of edges. Then the degree of vertices of an IFG in Figure 14 is given below.

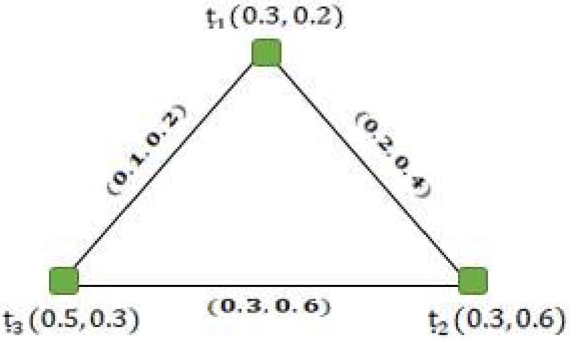


Figure 14 (Intuitionistic fuzzy graph)

Degree of vertices of the above Figure 14 is

$d(\d{t}_1) = (0.3, 0.6), d(\d{t}_2) = (0.5,\ 1.0), d(\d{t}_3) = (0.4, 0.8)$.

1.6.7 Definition [56]

Let $\dot{G} = (\d{V}, \dot{E})$ be a FG. Then its density is defined as

$$DN(G) = \left(DN_{\hat{S}}(\dot{G})\right) = \left(\frac{2\sum_{\d{t},w\in V}(\hat{S}_2(\d{t},w))}{\sum_{(\d{t},w)\in E}(\hat{S}_1(\d{t})\wedge\hat{S}_1(w))}\right)$$

1.6.8 Definition [71]

Let $\dot{G} = (\d{V}, \dot{E})$ be an IFG. Then its density is defined as

$$DN(G) = \left(DN_{\hat{S}}(\dot{G}), DN_d(\dot{G})\right) = \left(\begin{array}{c}\dfrac{2\sum_{\d{t},w\in V}\left(\hat{S}_2(\d{t},w)\right)}{\sum_{(\d{t},w)\in E}\left(\hat{S}_1(\d{t})\wedge\hat{S}_1(w)\right)}, \\ \dfrac{2\sum_{\d{t},w\in V}(Đ_2(\d{t},w))}{\sum_{(\d{t},w)\in E}(Đ_1(\d{t})\vee Đ_1(w))}\end{array}\right)$$

1.6.9 Example

The density of an IFG depicted in Figure 12 is calculated as

The density of $DN_{\hat{S}}(\dot{G}) = \left(\frac{2(0.2+0.3+0.1)}{0.3+0.3+0.3}\right) = 1.3$ and $DN_{Đ}(\dot{G}) = \left(\frac{2(0.4+0.6+0.2)}{0.6+0.6+0.3}\right) = 1.6$

1.6.10 Definition [66]

An arrangement of distinct vertices $\d{t}_1, \d{t}_2, \dots, \d{t}_{\dot{n}}$ is called a path in an IFG if one of the following conditions is satisfied:

1. $\hat{S}_{2ij} > 0$ *and* $Đ_{2ij} = 0$
2. $\hat{S}_{2ij} = 0$ *and* $Đ_{2ij} > 0$

3. $\hat{S}_{2ij} > 0 \; and \; Đ_{2ij} > 0.$

1.6.11 Example

The following Figure 15 is an example of IFG which is explain below.

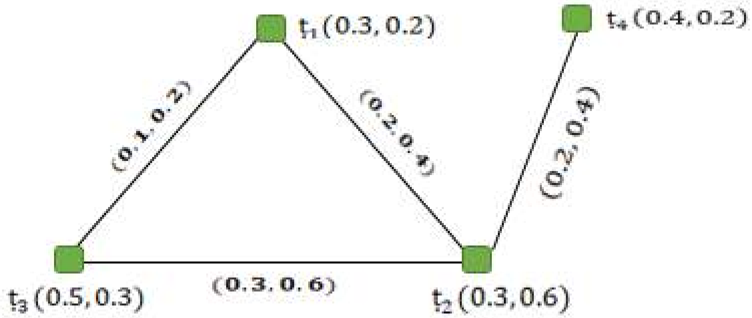


Figure 15 Intuitionistic fuzzy graph

In the above Figure 15, $ṭ_1, ṭ_2$ and $ṭ_2, ṭ_3$ is a path.

1.7 Aggregation Operators of IVIFNs and Their Generalizations

The aggregation operators of IVIFNs and their generalizations discussed in this section.

1.7.1 Definition [28]

The Interval valued intuitionistic fuzzy weighted averaging (IVIFWA) operators of IVIFNs $ṭ_{i1}, ṭ_{i2}, \dots, ṭ_{in}$ is denoted and defined by

$$ṭ_i = IVIFWA(ṭ_{i1}, ṭ_{i2}, \dots, ṭ_{in}) = \begin{pmatrix} \begin{bmatrix} 1 - \prod_{j=1}^{n} \left(1 - \hat{S}^L{}_{ij}\right)^{w_j}, \\ 1 - \prod_{j=1}^{n} \left(1 - \hat{S}^U{}_{ij}\right)^{w_j} \end{bmatrix}, \\ \left[\left(\prod_{j=1}^{n} Đ^L{}_{ij} \right)^{w_j}, \left(\prod_{j=1}^{n} Đ^U{}_{ij} \right)^{w_j} \right] \end{pmatrix}$$

1.7.2 Definition [28]

The Interval valued intuitionistic fuzzy weighted geometric (IVIFWG) operators of IVIFNs $ṭ_{i1}, ṭ_{i2}, \dots, ṭ_{in}$ is denoted and defined by

$$ṭ_i = IVIFWG(ṭ_{i1}, ṭ_{i2}, \dots, ṭ_{in}) = \left(\begin{array}{c} \left(\prod_{j=1}^{n} \hat{S}^{L}{}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{n} \hat{S}^{U}{}_{ij}\right)^{w_j}, \\ 1 - \prod_{j=1}^{n} \left(1 - Đ^{L}{}_{ij}\right)^{w_j}, \\ 1 - \prod_{j=1}^{n} \left(1 - Đ^{U}{}_{ij}\right)^{w_j} \end{array} \right)$$

1.7.3 Definition [29]

The Interval valued Pythagorean fuzzy weighted averaging (IVPyWA) operators of IVIFNs $ṭ_{i1}, ṭ_{i2}, \dots, ṭ_{in}$ is denoted and defined by

$$ṭ_i = IVPFWA(ṭ_{i1}, ṭ_{i2}, \dots, ṭ_{in}) = \left(\begin{array}{c} \left[\begin{array}{c} \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\hat{S}^{L}{}_{ij}\right)^2\right)^{w_j}}, \\ \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\hat{S}^{U}{}_{ij}\right)^2\right)^{w_j}} \end{array} \right], \\ \left[\left(\prod_{j=1}^{n} Đ^{L}{}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{n} Đ^{U}{}_{ij}\right)^{w_j} \right] \end{array} \right)$$

1.7.4 Definition [29]

The Interval valued Pythagorean fuzzy weighted geometric (IVPyWG) operators of IVIFNs $ţ_{i1}, ţ_{i2}, \ldots, ţ_{in}$ is denoted and defined by

$$ţ_i = IVPFWG(ţ_{i1}, ţ_{i2}, \ldots, ţ_{in}) = \left(\begin{array}{c} \left(\prod_{j=1}^{n} \hat{S}^L{}_{ij}\right)^{w_j}, \left(\prod_{j=1}^{n} \hat{S}^U{}_{ij}\right)^{w_j}, \\ \sqrt{1-\prod_{j=1}^{n}\left(1-\left(Đ^L{}_{ij}\right)^2\right)^{w_j}}, \\ \sqrt{1-\prod_{j=1}^{n}\left(1-\left(Đ^U{}_{ij}\right)^2\right)^{w_j}} \end{array} \right)$$

1.8 Addition and Multiplication of IVPyFNs

In this section the addition and multiplication of IVPyFNs are discussed.

1.8.1 Definition [29]

The addition and multiplication in an IVPyFNs is denoted and defined as

3) $\left([\hat{S}^L{}_1, \hat{S}^U{}_1], [Đ^L{}_1, Đ^U{}_1]\right) + \left([\hat{S}^L{}_2, \hat{S}^U{}_2], [Đ^L{}_2, Đ^U{}_2]\right) =$

$$\left(\begin{array}{c} \left[\begin{array}{c} \sqrt{\left(\hat{S}^L{}_1\right)^2 + \left(\hat{S}^L{}_2\right)^2 - \left(\hat{S}^L{}_1\right)^2\left(\hat{S}^L{}_2\right)^2}, \\ \sqrt{\left(\hat{S}^U{}_1\right)^2 + \left(\hat{S}^U{}_2\right)^2 - \left(\hat{S}^U{}_1\right)^2\left(\hat{S}^U{}_2\right)^2} \end{array} \right], \\ [Đ^L{}_1 Đ^L{}_2, Đ^U{}_1 Đ^U{}_2] \end{array} \right)$$

4) $\left([\hat{S}^L{}_1, \hat{S}^U{}_1], [Đ^L{}_1, Đ^U{}_1]\right) \times \left([\hat{S}^L{}_2, \hat{S}^U{}_2], [Đ^L{}_2, Đ^U{}_2]\right) =$

$$\left(\begin{array}{c} [\hat{S}^L{}_1 \hat{S}^L{}_2, \hat{S}^U{}_1 \hat{S}^U{}_2], \\ \left[\begin{array}{c} \sqrt{(Đ^L{}_1)^2 + (Đ^L{}_2)^2 - (Đ^L{}_1)^2(Đ^L{}_2)^2}, \\ \sqrt{(Đ^U{}_1)^2 + (Đ^U{}_2)^2 - (Đ^U{}_1)^2(Đ^U{}_2)^2} \end{array} \right] \end{array} \right)$$

1.8.2 Remark

Replacing $n = 1$ reduces the defined score function in the environment of IVIFSs.

Chapter 2

Intuitionistic Fuzzy Graphs of n^{th} Type with Applications

In this chapter, a new concept of IFGNT is proposed as a generalization of IFGs and IFGST. Some light has also been shed upon the concepts of constant intuitionistic fuzzy graphs of second type (CIFGST) and constant intuitionistic fuzzy graphs of n^{th} type (CIFGNT). Moreover, some basic definitions of and results from IFGNT have been developed, supported with examples. Besides, the advantages of proposed new concepts over the existing concepts have been highlighted and a comparative study of new and existing work is established. Further, an application of IFGNT has been demonstrated in social networks context.

2.1 Intuitionistic fuzzy graphs of n^{th} type

In this section, the notion of IFGNT is discussed with examples. The concepts of subgraph and complement of IFGNT are also discussed and exemplified. The degree of IFGNT is defined and illustrated with an example.

2.1.1 Definition

A Pair $\dot{\mathrm{G}} = (\mathrm{Ṿ}, \mathrm{Ẻ})$ is known as IFGNT if

1) $\mathrm{Ṿ} = \{\mathrm{ṭ}_1, \mathrm{ṭ}_2, \mathrm{ṭ}_3, \ldots \mathrm{ṭ}_n\}$ is the set of vertices such that $\hat{\mathrm{S}}_1 \colon \mathrm{Ṿ} \longrightarrow [0,1]$ and $\mathrm{Đ}_1 \colon \mathrm{Ṿ} \longrightarrow [0,1]$ represents the degrees of membership and non-membership of the element $\mathrm{ṭ}_i \in \mathrm{Ṿ}$ respectively with a condition that $0 \leq {\hat{\mathrm{S}}_1}^n(\mathrm{ṭ}_i) + {\mathrm{Đ}_1}^n(\mathrm{ṭ}_i) \leq 1$ for all $\mathrm{ṭ}_i \in \mathrm{Ṿ}$ $(i \in I)$.
2) $\mathrm{Ẻ} \subseteq \mathrm{Ṿ} \times \mathrm{Ṿ}$ where $\hat{\mathrm{S}}_2 \colon \mathrm{Ṿ} \times \mathrm{Ṿ} \longrightarrow [0,1]$ and $\mathrm{Đ}_2 \colon \mathrm{Ṿ} \times \mathrm{Ṿ} \longrightarrow [0,1]$ represent the degrees of membership and non-membership of the element $(\mathrm{ṭ}_i, \mathrm{ṭ}_j) \in \mathrm{Ẻ}$ such that $\hat{\mathrm{S}}_2(\mathrm{ṭ}_i, \mathrm{ṭ}_j) \leq$

$min\{\hat{S}_1(ṭ_i), \hat{S}_1(ṭ_j)\}$ and $Đ_2(ṭ_i, ṭ_j) \leq max\{Đ_1(ṭ_i), Đ_1(ṭ_j)\}$ with a condition that $0 \leq \hat{S}_2{}^n(ṭ_i, ṭ_j) + Đ_2{}^n(ṭ_i, ṭ_j) \leq 1$ for all $(ṭ_i, ṭ_j) \in \dot{Ė}$ $(i \in I)$.

2.1.2 Example

Consider the following graphs where Figure 16 represents an IFGNT while Figure 17 is not an IFGNT because of the edge $(ṭ_3, ṭ_4)$. The vertices in the below figures are purely intuitionistic fuzzy numbers (IFNs) of n-type for $n = 4$.

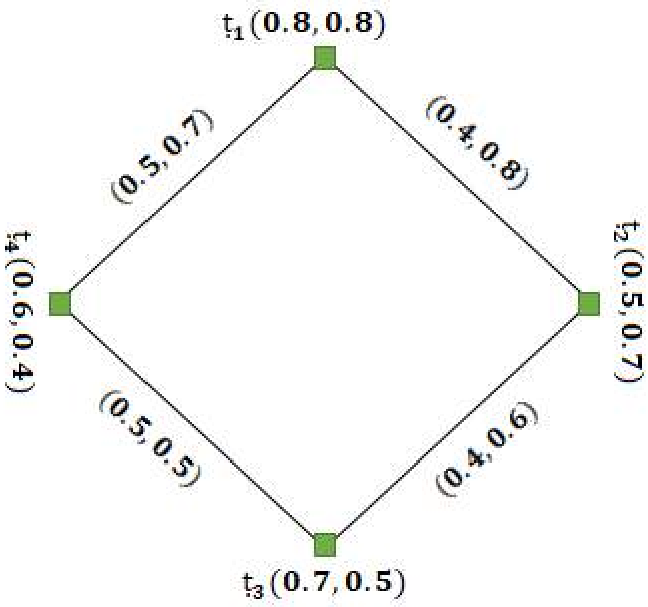


Figure 16 (Intuitionistic fuzzy graph of nth type)

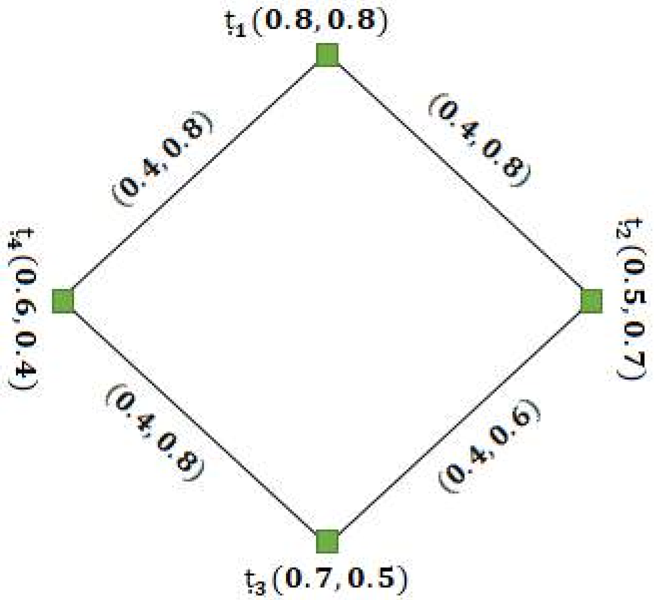


Figure 17 (Not an intuitionistic fuzzy graph of nth type)

2.1.3 Remark

An IFG and IFGST are surely IFGNT but converse is not true in general.

2.1.4 Example

For $n = 4$, the graph in Figure 16 is an IFGNT but the graph in Figure 17 is not an IFGNT and clearly shows that none of them is either IFGST nor IFG. Because, for $ț_1 = (0.8, 0.8)$, we have $0.8 + 0.8 = 1.6 \nleq 1$ also $0.8^2 + 0.8^2 = 1.28 \nleq 1$.

2.1.5 Definition

A pair $\dot{H} = (\underset{.}{V}^{\circ}, \dot{E}^{\circ})$ is known as intuitionistic fuzzy subgraph of n-type of $\dot{G} = (\underset{.}{V}, \dot{E})$ if $\underset{.}{V}^{\circ} \subseteq \underset{.}{V}$ and $\dot{E}^{\circ} \subseteq \dot{E}$ that is $\hat{S}_{1i}{}^{\circ} \leq \hat{S}_1$;$Đ_1{}^{\circ} \geq Đ_1$ and $\hat{S}_{2ij}{}^{\circ} \leq \hat{S}_{2ij}$;$Đ_{2ij}{}^{\circ} \geq Đ_{2ij}$ for all $i, j = 1,2, \dots n$.

2.1.6 Example

The graph depicted in Figure 18 is an intuitionistic fuzzy subgraph of n-type of IFGNT portrayed in Figure 16. Moreover, the vertices in the below Figure 18 are purely IFNs of n-type for $n = 4$.

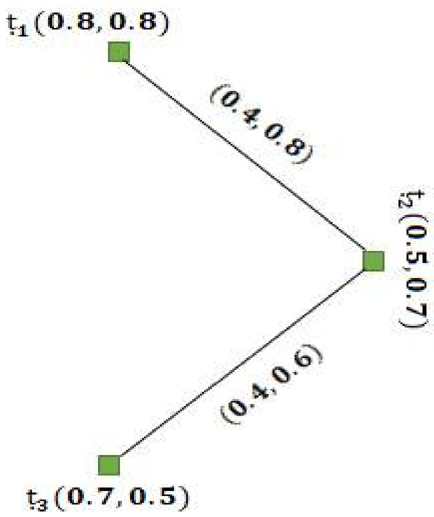


Figure 18 (Intuitionistic fuzzy subgraph of Figure 16)

2.1.7 Definition

The set of all triplets for vertices is denoted by $(\underset{.}{V}_t, \hat{S}_{1t}, Đ_{1t})$ in an IFGNT $\dot{G} = (\underset{.}{V}, \dot{E})$ where $\hat{S}_{1t} = \{\underset{.}{t}_i \in \underset{.}{V} : (\hat{S}_{1i})^2 \geq t\}$ and $Đ_{1t} = \{\underset{.}{t}_i \in \underset{.}{V} : (Đ_{1i})^2 \leq t\}$ for some $i.j = 1, 2, \dots n$ is a subset of $\underset{.}{V}$ for $0 \leq t \leq 1$ and the set of all triplets for edges is denoted by $(\dot{E}_t, \hat{S}_{2t}, Đ_{2t})$ where $\hat{S}_{1t} = \{(\underset{.}{t}_i, \underset{.}{t}_j) \in \underset{.}{V} \times \underset{.}{V} : (\hat{S}_{2ij})^2 \geq t\}$ and $Đ_{1t} = \{(\underset{.}{t}_i, \underset{.}{t}_j) \in \underset{.}{V} \times \underset{.}{V} : (Đ_{2ij})^2 \leq t\}$ for some $i.j = 1, 2, \dots n$ is a subset of $\dot{E}$ for $0 \leq t \leq 1$. Here $(\underset{.}{V}_t, \dot{E}_t)$ is a subgraph of $\dot{G}$.

2.1.8 Theorem

If $\dot{G} = (\underset{.}{V}, \dot{E})$ is an IFGNT. Then $(\underset{.}{V}_{\dot{x}}, \dot{E}_{\dot{x}})$ is an intuitionistic fuzzy subgraph of $(\underset{.}{V}_{\underset{.}{y}}, \dot{E}_{\underset{.}{y}})$ for any $\dot{x}, \underset{.}{y}$ such that $0 \leq \dot{x} \leq \underset{.}{y} \leq 1$.

Proof: Consider $\underset{.}{t}_i \in \underset{.}{V}_{\dot{x}}$ then ${\underset{.}{t}_{1i}}^n \leq \dot{x}$ so ${\underset{.}{t}_{1i}}^n \leq \underset{.}{y}$ implies $\underset{.}{t}_i \in \underset{.}{V}_{\underset{.}{y}}$. Therefore, $\underset{.}{V}_{\dot{x}} \subseteq \underset{.}{V}_{\underset{.}{y}}$. Now, consider $(\underset{.}{t}_i, \underset{.}{t}_j) \in \dot{E}_{\dot{x}}$ then ${\underset{.}{t}_{2ij}}^n \leq \dot{x}$ so ${\underset{.}{t}_{2ij}}^n \leq \underset{.}{y}$. Therefore, $\dot{x} \leq \underset{.}{y}$ implies $(\underset{.}{t}_i, \underset{.}{t}_j) \in \dot{E}_{\underset{.}{y}}$ and hence $\dot{E}_{\dot{x}} \subseteq \dot{E}_{\underset{.}{y}}$. So, $(\underset{.}{V}_{\dot{x}}, \dot{E}_{\dot{x}})$ is an intuitionistic fuzzy subgraph of $(\underset{.}{V}_{\underset{.}{y}}, \dot{E}_{\underset{.}{y}})$.

2.1.9 Theorem

Let $H = (Ṿ_{ẋ}^{\circ}, Ẻ_{ẋ}^{\circ})$ be an intuitionistic fuzzy subgraph of an IFGNT $\dot{G} = (Ṿ, Ẻ)$. Then for any $0 \leq ẋ \leq 1$ $(Ṿ_{ẋ}^{\circ}, Ẻ_{ẋ}^{\circ})$ is an intuitionistic fuzzy subgraph of $(Ṿ_{ẋ}, Ẻ_{ẋ})$.

Proof: Given that $Ṿ_{ẋ}^{\circ} \subseteq Ṿ$ and $Ẻ_{ẋ}^{\circ} \subseteq Ẻ$. To prove that $(Ṿ_{ẋ}^{\circ}, Ẻ_{ẋ}^{\circ})$ is an intuitionistic fuzzy subgraph of $(Ṿ_{ẋ}, Ẻ_{ẋ})$. For this we have to show that $Ṿ_{ẋ}^{\circ} \subseteq Ṿ_{ẋ}$ and $Ẻ_{ẋ}^{\circ} \subseteq Ẻ_{ẋ}$. Assume that $ṭ_i \in Ṿ_{ẋ}^{\circ}$ implies $(\hat{S}_{1i}^{\circ})^n \geq ẋ$ implies $\hat{S}_{1i}^{\,n} \geq ẋ$. Therefore, $(\hat{S}_{1i}^{\circ})^n \leq \hat{S}_{1i}^{\,n}$ implies $ṭ_i \in Ṿ_{ẋ}$ implies $Ṿ_{ẋ}^{\circ} \subseteq Ṿ_{ẋ}$.

Now consider $(ṭ_i, ṭ_j) \in Ẻ_{ẋ}'$ implies $(\hat{S}_{2ij}^{\circ})^n \geq ẋ$ implies $(\hat{S}_{2ij})^n \geq ẋ$. Therefore, $(\hat{S}_{2ij}^{\circ})^n \leq \hat{S}_{2ij}^{\,n}$ implies $(ṭ_i, ṭ_j) \in Ẻ_{ẋ}$ implies $Ẻ_{ẋ}^{\circ} \subseteq Ẻ_{ẋ}$. Therefore $\left(Ṿ_{ẋ}^{\circ}, Ẻ_{ẋ}^{\circ}\right)$ is an intuitionistic fuzzy subgraph of $(Ṿ_{ẋ}, Ẻ_{ẋ})$.

2.1.10 Definition

The complement of an IFGNT $\dot{G} = (Ṿ, Ẻ)$ is defined as

1. $\overline{Ṿ} = Ṿ$.
2. $\overline{\hat{S}_{1i}} = \hat{S}_{1i}$ and $\overline{Đ_{1i}} = Đ_{1i}$ for all $i = 1,2, \dots n$.
3. $\overline{\hat{S}_{2ij}} = min(\hat{S}_{1i}, \hat{S}_{1j}) - \hat{S}_{2ij}$ and $\overline{Đ_{2ij}} = max(Đ_{1i}, Đ_{1j}) - Đ_{2ij}$ for every $i.j = 1,2, \dots n$.

2.1.11 Example

Consider the graphs in Figure 19 and Figure 20, where graph in Figure 20 represents the complement of graph depicted in Figure 19. Moreover, the vertices in these graphs are purely IFNs of nth type for $n = 3$.

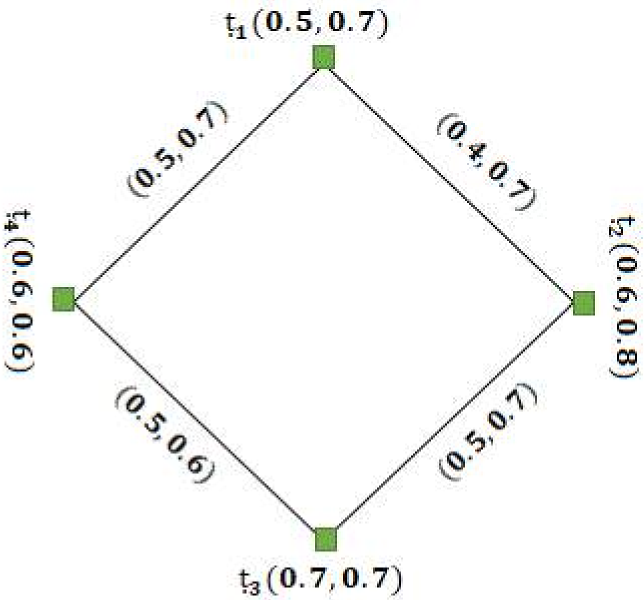


Figure 19 (Intuitionistic fuzzy graph of nth type)

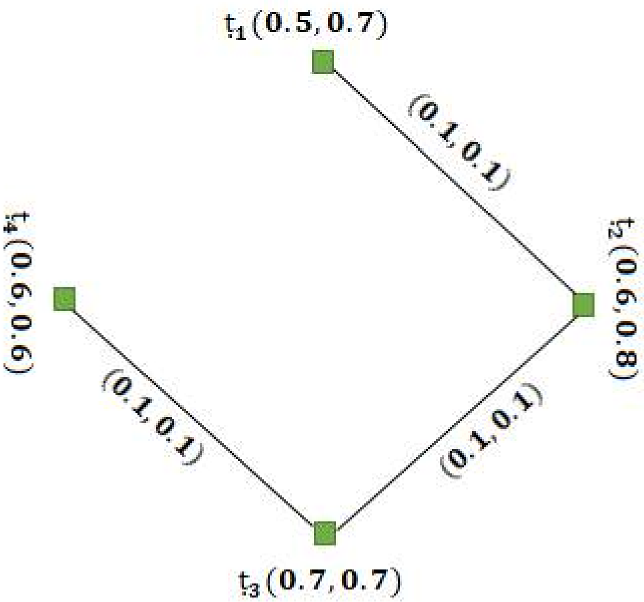


Figure 20 (Complement of Figure 19)

2.1.12 Definition

Let $\dot{G} = (\underset{.}{V}, \dot{E})$ be an IFGNT. Then the degree of vertex $\underset{.}{t}$ is defined by $d(\underset{.}{t}) = (d_{\hat{S}}(\underset{.}{t}), d_{Đ}(\underset{.}{t}))$ where $d_{\hat{S}}(\underset{.}{t}) = \sum_{u \neq \underset{.}{t}} \hat{S}_2\left(\underset{.}{t}, u\right)$ and $d_{Đ}(\underset{.}{t}) = \sum_{u \neq \underset{.}{t}} Đ_2\left(\underset{.}{t}, u\right)$.

2.1.13 Example

Consider an IFGNT depicted in Figure 21 where all the nodes are purely IFNs of nth type for $n = 4$. The degrees of all vertices are determined below using Definition 2.1.12.

$d(t_1) = (1.1, 1.6)$, $d(t_2) = (0.9, 1.4)$,

$d(t_3) = (0.8, 1.3)$, $d(t_4) = (1.0, 1.5)$.

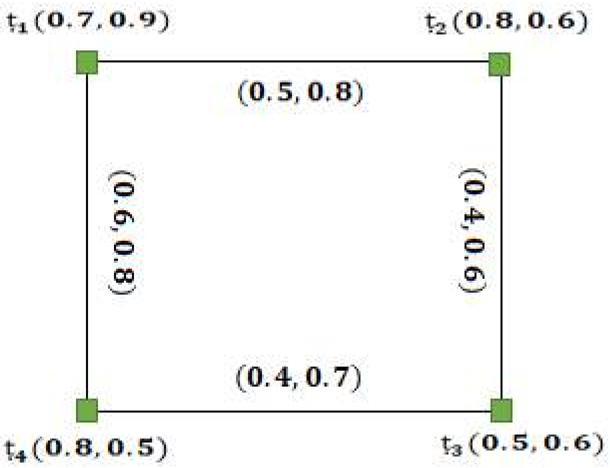


Figure 21(Intuitionistic fuzzy graph of nth type)

2.2 Constant Intuitionistic Fuzzy Graphs of n-Type

This section is based on the novel concept of CIFGNT and CIFGST. These concepts are illustrated with the help of examples. The notion of total degree and constant function are also studied and supported with examples.

2.2.1 Definition

An IFGNT $\dot{G} = (V, \dot{E})$ is said to be CIFGNT of degree (k_i, k_j) or (k_i, k_j) −IFGNT if $d_{\hat{S}}(t_i) = k_i$ and $d_{Đ}(t_j) = k_j$ for all $t_i, t_j \in V$.

2.2.2 Example

The graph portrayed in Figure 22 is an example of CIFGNT where all nodes are IFNs of nth type for $n = 4$. The degree of all vertices is same i.e. $(0.9, 1.6)$.

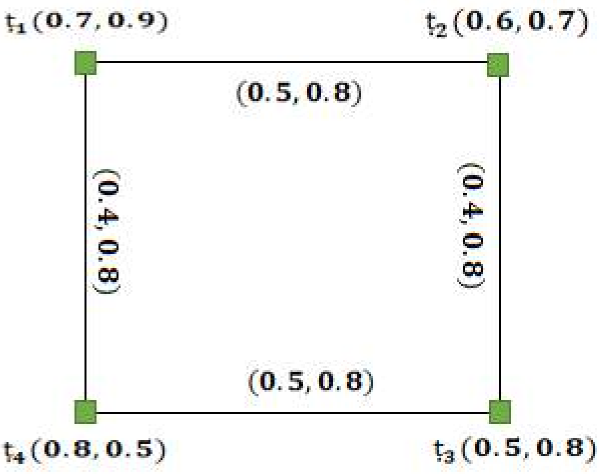


Figure 22 (Constant Intuitionistic fuzzy graph of nth type)

2.2.3 Remark

Definition 2.2.2 reduces to the definition of CIFGST if every element of $\underset{\cdot}{V}$ and $\dot{E}$ are purely IFNs of second type.

2.2.4 Remark

A complete IFGNT need not be CIFGNT.

This remark is demonstrated by the following example.

2.2.5 Example

Consider the following IFGNT depicted in Figure 23 where all nodes are purely IFNs for $n = 3$. Further, this IFGNT is complete but not constant.

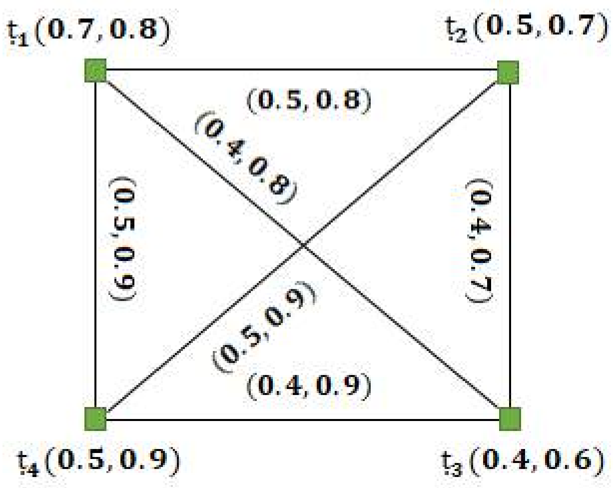


Figure 23(Complete intuitionistic fuzzy of nth type)

2.2.6 Definition

For an IFGNT $\dot{G} = (\underset{\cdot}{V}, \dot{E})$. The total degree (τ_1, τ_2) of a vertex ţ is defined as:

$$td(ţ) = \left[\sum_{ţ \in \dot{E}} d_{\hat{S}_2}(ţ) + \hat{S}_1(ţ), \sum_{ţ \in \dot{E}} d_{Đ_2}(ţ) + Đ_1(ţ)\right]$$

If total degree of each vertex is same, then $\dot{G}$ is called IFGNT of total degree (τ_1, τ_2) or (τ_1, τ_2)-totally CIFGNT.

2.2.7 Example

Consider an IFGNT in Figure 24 where all the nodes are purely IFNs for $n = 2$. Further, total degree of each vertex is $(1.5, 2.3)$.

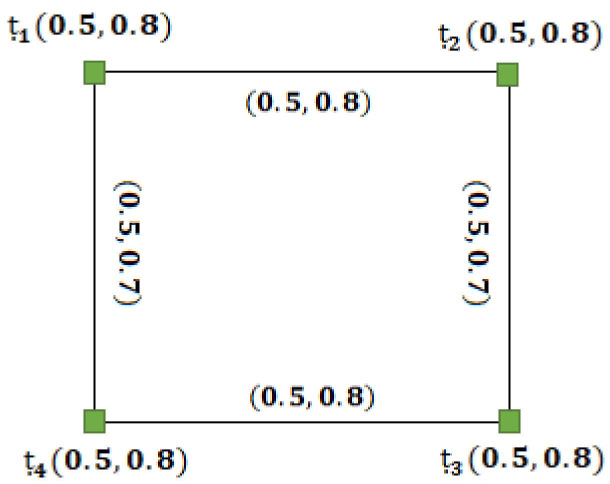


Figure 24 (Intuitionistic fuzzy graph of nth type)

2.2.8 Example

Consider an IFGNT in Figure 25 where all the nodes are purely IFNs for $n = 3$. Further, Ġ is totally constant.

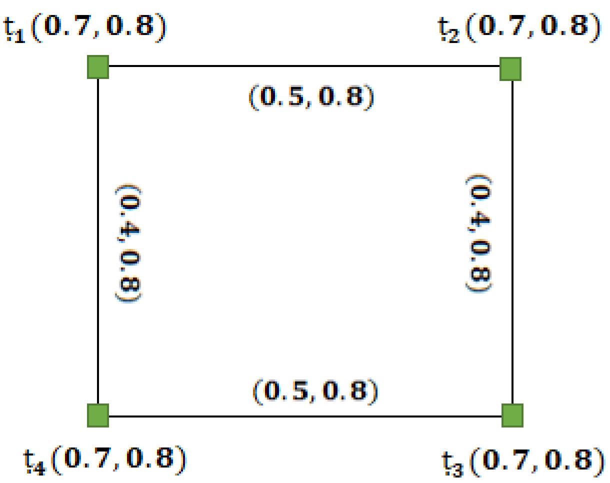


Figure 25 (Intuitionistic fuzzy graph of nth type)

2.2.9 Theorem

Let Ġ be an IFGNT. Then $(\hat{S}_1, Đ_1)$ is a constant function iff the following are equivalent.

1) is CIFGNT.

2) is totally CIFGNT.

Proof. Assume that $(\hat{S}_1, Đ_1)$ is a constant function. Consider $\hat{S}_1(ṭ_i) = ć_1$ and $Đ_1(ṭ_i) = ć_2$ for all $ṭ_i \in Ṿ$ where $ć_1$ and $ć_2$ are constants. Suppose that $\dot{G}$ is a $(ƙ_i, ƙ_j)$ −IFGTN.Then $d_{\hat{S}}(ṭ_i) = ƙ_1$ and $d_{Đ}(ṭ_i) = ƙ_2$ for all $ṭ_i \in Ṿ$. So, $td_{\hat{S}}(ṭ_i) = d_{\hat{S}}(ṭ_i) + \hat{S}_1(ṭ_i)$, $td_{Đ}(ṭ_i) = d_{Đ}(ṭ_i) + Đ_1(ṭ_i)$, $td_{\hat{S}}(ṭ_i) = ƙ_1 + ć_1$, $td_{Đ}(ṭ_i) = ƙ_2 + ć_2$ for all $ṭ_i \in Ṿ$. Therefore $\dot{G}$ is totally CIFGNT. Hence $(1) \implies (2)$ is proved. Now to prove $(2) \implies (1)$. Suppose $\dot{G}$ is totally CIFGNT to prove $\dot{G}$ is CIFGNT. As $\dot{G}$ is totally CIFGNT then $td_{\hat{S}}(ṭ_i) = \hat{r}_1$, $td_{Đ}(ṭ_i) = \hat{r}_2$ for all $ṭ_i \in Ṿ$. $d_{\hat{S}}(ṭ_i) + \hat{S}_1(ṭ_i) = \hat{r}_1$, $d_{Đ}(ṭ_i) + Đ_1(ṭ_i) = \hat{r}_2$, $d_{\hat{S}}(ṭ_i) + ć_1 = \hat{r}_1$, $d_{\hat{S}}(ṭ_i) = \hat{r}_1 - ć_1$. Likewise $d_{Đ}(ṭ_i) + Đ_1(ṭ_i) = \hat{r}_2$, $d_{Đ}(ṭ_i) + ć_1 = \hat{r}_1$, $d_{Đ}(ṭ_i) = \hat{r}_1 - ć_1$. Therefor (1) and (2) are equivalent. Conversely, suppose that (1) and (2) are equivalent i.e. $\dot{G}$ is CIFGNT iff $\dot{G}$ is totally CIFGNT. Assume that $(\hat{S}_1, Đ_1)$ is not a constant function. Then $\hat{S}_1(ṭ_1) \neq \hat{S}_1(ṭ_2)$, $Đ_1(ṭ_1) \neq Đ_1(ṭ_2)$ for at least $ṭ_1, ṭ_2 \in Ṿ$. Let $\dot{G}$ is totally CIFGNT. Then $d_{\hat{S}}(ṭ_1) = d_{\hat{S}}(ṭ_2) = ƙ_1$, $d_{Đ}(ṭ_1) = d_{Đ}(ṭ_2) = ƙ_2$. So, $td_{\hat{S}}(ṭ_1) = d_{\hat{S}}(ṭ_1) + \hat{S}_1(ṭ_1) = ƙ_1 + \hat{S}_1(ṭ_1)$ and $td_{\hat{S}}(ṭ_2) = ƙ_2 + \hat{S}_1(ṭ_2)$. Likewise $td_{Đ}(ṭ_1) = d_{Đ}(ṭ_1) + Đ_1(ṭ_1) = ƙ_1 + Đ_1(ṭ_1)$ and $td_{Đ}(ṭ_2) = ƙ_2 + Đ_1(ṭ_2)$. Therefore, $\hat{S}_1(ṭ_1) \neq \hat{S}_1(ṭ_2)$, $Đ_1(ṭ_1) \neq Đ_1(ṭ_2)$. We have $td_{\hat{S}}(ṭ_1) \neq td_{\hat{S}}(ṭ_2)$, $td_{Đ}(ṭ_1) \neq td_{Đ}(ṭ_2)$. So, $\dot{G}$ is not totally CIFGNT which is contradiction to our supposition. Now, consider $\dot{G}$ is totally CIFGNT. Then

$td_{\hat{S}}(ṭ_1) = td_{\hat{S}}(ṭ_2), d_{\hat{S}}(ṭ_1) + \hat{S}_1(ṭ_1)$

$= d_{\hat{S}}(ṭ_2) + \hat{S}_1(ṭ_2), d_{\hat{S}}(ṭ_1) - d_{\hat{S}}(ṭ_2)$

$= \hat{S}_1(ṭ_1) - \hat{S}_1(ṭ_2)\ (i.e. \neq 0)\ d_{\hat{S}}(ṭ_1) \neq d_{\hat{S}}(ṭ_2)$

Likewise, $d_{Đ}(ṭ_1) \neq d_{Đ}(ṭ_2)$. So, $\dot{G}$ is not constant which contradiction to our supposition. Therefore $(\hat{S}_1, Đ_1)$ is a constant function.

2.2.10 Theorem

If an IFGNT $\dot{G}$ is constant and totally constant. Then $(\hat{S}_1, Đ_1)$ is constant function.

Proof. Consider $\dot{\mathrm{G}}$ is $(ƙ_1, ƙ_2)$ −constant and (τ_1, τ_2)-totally CIFGNT. Then by definitions $d_{\hat{S}}(ṭ_1) = ƙ_1$ and $d_{Đ}(ṭ_1) = ƙ_2$ for $ṭ_1 \in Ṿ$ and $td_{\hat{S}}(ṭ_1) = \tau_1$, $td_{Đ}(ṭ_1) = \tau_2$ for $ṭ_1 \in Ṿ$, $d_{\hat{S}}(ṭ_1) + \hat{S}_1(ṭ_1) = \tau_1$ for all $ṭ \in Ṿ$. $ƙ_1 + \hat{S}_1(ṭ_1) = \tau_1$ implies that $\hat{S}_1(ṭ_1) = (\tau_1 - ƙ_1)$, for all $ṭ \in Ṭ$. Hence $\hat{S}_1(ṭ_1)$ is constant function. Similarly, $Đ_1(ṭ_1) = (\tau_2 - ƙ_2)$, for all $ṭ \in Ṿ$.

2.2.11 Remark

Converse of the Theorem 2.2.10 does not hold. We show this by the following example.

2.2.12 Example

The following IFGNT depicted in Figure 26 where each node is purely IFNs for $n = 3$ support the above Remark 2.2.11 i.e. $(\hat{S}_1, Đ_1)$ is constant function, but neither CIFGNT nor totally CIFGNT.

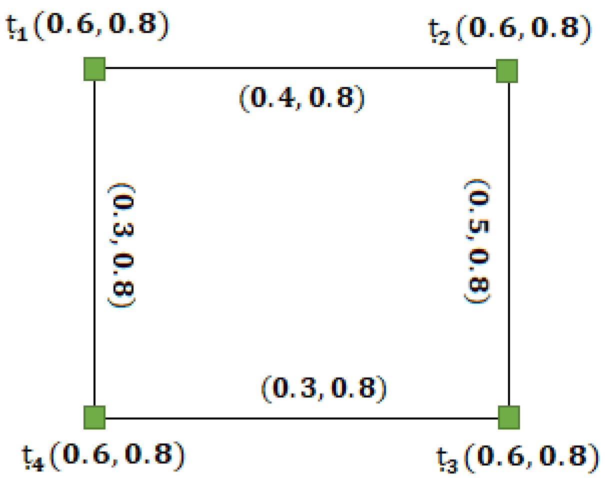


Figure 26 (Intuitionistic fuzzy graph of nth type)

2.3 Application

In this section, we discussed the application of IFGNT in social networks. It is discussed that the framework of IFGNT is diverse in nature than that of IFS and IFSST. It allows the membership and non-membership values to be chosen from anywhere in the interval $[0, 1]$

regardless of any condition. So, this type of structure can be applied to many real-life problems with no limitations.

2.3.1 Social networks

We discussed the application of IFGNT in social networks where the relationship between different countries based on different matters has been studied. We used IFGNT to determine the level of relationship. For this purpose, we consider different matters that are important in the relationship of different countries with each other. These includes culture, area, religion, behavior of peoples, budget for defense expenditure, visa policy, trade, political, border management and media.

Consider Figure 27 where a list of countries is taken into account and their relations are studied keeping in mind the above discussed matters in the environment of IFGNT. The countries are India, Saudi-Arab, Iran, Pakistan, and China. The following graph gives us a brief information about the relationship of these countries with each other.

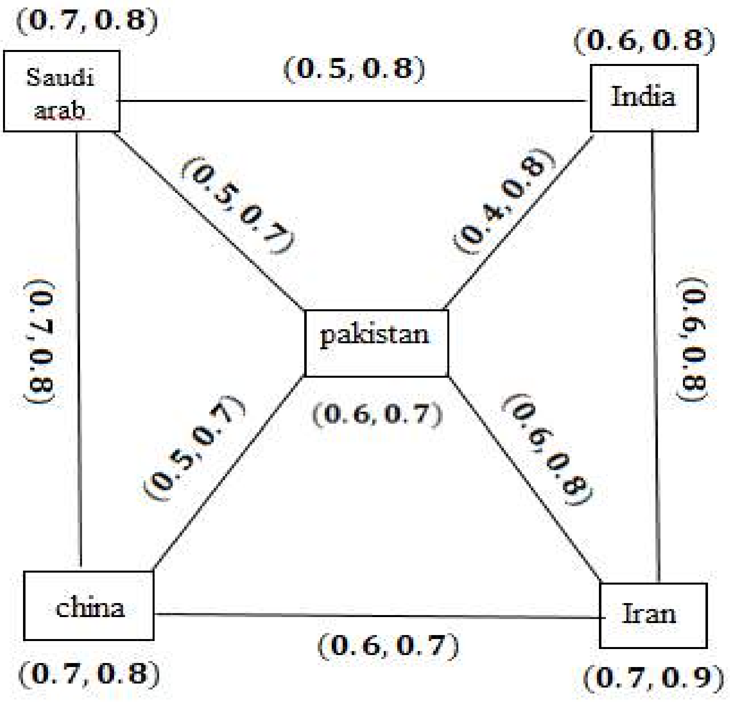


Figure 27 (Intuitionistic fuzzy network of nth type)

The edge between two countries represent their relation. We list the edge values of each pair of countries in Table 1. These edges are in the form of IFNNT having a membership and non-membership grade which mean that if the membership degree is greater compared to the non-membership degree then the relationship is considered as strong otherwise weak. The degree of each vertex will give us the strength of relationship of a country with all other countries.

Country	**Country**	**Relationship**
Pakistan	India	(0.4, 0.8)
Iran	India	(0.6, 0.8)
China	Saudi-Arab	(0.7, 0.8)
Pakistan	China	(0.5, 0.7)
India	Saudi-Arab	(0.5, 0.8)
Pakistan	Iran	(0.6, 0.8)
Pakistan	Saudi-Arab	(0.5, 0.7)
China	Iran	(0.6, 0.7)

Table 1(relation of every pair of countries in the form of IFNs of nth type)

Now the degree of relation of each country is calculated based on Definition 2.1.12. The high degree of membership shows the good relation of it with other countries and vice versa. The degree of relation of each country is listed in Table 2.

Country	Degree
Iran	(1.8, 2.3)
Pakistan	(2.0,3.0)
Saudi-Arab	(1.1, 1.6)
India	(1.5, 2.4)
China	(1.8, 2.1)

Table 2 (Degree of relation of each country determined from Figure 27)

For simplicity, the strength of each vertex is measured by using the effective degree which is defined as the difference of membership degree and non-membership degree. The effective degree of each vertex is calculated in Table 3.

Country	Score
India	-0.9
Saudi-Arab	-0.5
Pakistan	-1
China	-0.3
Iran	-0.5

Table 3 (Effective degree of relation of each country)

From the calculation in Table 3 it is clear that on different matters, china has better relationship with other countries. The effective degree of Saudi-Arab and Iran shows that their relationship is not as much stronger compared to china but better than Pakistan and India.

2.4 Advantages

The proposed framework IFGNT generalizes both IFG and IFGST. The main advantage of proposed framework is that the space of IFGNT is much larger than that of IFS and IFSST and is free of any barriers i.e. it allows the decision makers to assign membership and non-membership values from anywhere in the interval [0, 1]. All the works done so far in IFG and IFGST can be done in the proposed structure of IFGNT. On the other hand, the work information in the form of IFNNT could not be processed using the notions of IFSs or IFSSTs because of their limited structures. For example, if we look at Figure 27 of social networks, all the information is in the form of IFNs of nth type, hence IFGs and IFGST are failed to describe it.

2.5 Conclusion

In this chapter, the theory of IFGNT, CIFGST and CIFGNT have been proposed. Some basic graph theoretic concepts are defined for IFGNT and their properties are investigated. The structure of IFS, IFSST and IFSNT are compared and it is proved that IFSNT generalizes IFS and IFSST which also prove the generalization of IFGNT over IFG and IFGST. A real-life application of proposed IFGNT is discussed showing is worth. In future some further contribution to this theory could be made such as the concepts of cycles, tree could be defined in this frame work. The minimum spinning tree problems could be discussed along with some other real-life problems. Further this study could be extended to the directions of soft set theory and rough set theory to deal with some multi attribute decision making problems.

Chapter 3

An Improved Clustering Algorithm for Picture Fuzzy Graphs and its Applications in Human Decision Making

In this chapter, we developed a novel clustering algorithm in the environment of PFGs. The proposed clustering algorithm is an improved version of clustering algorithm of FGs and IFGs. We thoroughly investigated the existing clustering algorithms proposed in the frameworks of IFG by pointing out the deficiencies and suggesting a solution which is applicable in handling real-life scenarios. The proposed clustering algorithm is supported with the help of a numerical problem discussing those cases which have not been discussed in the existing algorithms and the results are examined. To develop the new algorithm a study of PFGs along with some interesting result is established. A comparative study of the new work with that of existing work is established proving the worth of the proposed new work. Some drawbacks of the existing concepts and advantages of new theory are also discussed. We ended with a summary of proposed work and possibly related future work in these directions.

3.1 A Note on Picture Fuzzy Graphs

In this section, the concepts PFG, subgraph of PFG, density of PFG, balanced PFG and single-valued edge density PFG (SEDPFG) are developed. The defined concepts are supported with the help of examples and some results based on balanced PFGs are also studied. These concepts are the essential parts of clustering algorithm proposed in Section 3.4.

3.1.1 Definition

A pair $\dot{G} = (\underset{\cdot}{V}, \dot{\dot{E}})$ is known as PFG if

i. $\underset{\cdot}{V}$ denotes the set of vertices such that $\hat{S}_1 : \underset{\cdot}{V} \longrightarrow [0,1], \hat{I}_1 : \underset{\cdot}{V} \longrightarrow [0,1]$ and $Đ_1 : \underset{\cdot}{V} \longrightarrow [0,1]$denote the degrees of membership, abstinence and non-membership of vertex $ṭ_i \in \underset{\cdot}{V}$ respectively with a condition $0 \le \hat{S}_1 + \hat{I}_1 + Đ_1 \le 1$ for any $ṭ_i \in \underset{\cdot}{V}$ $(\, i \in I)$.

ii. $\dot{\dot{E}} \subseteq \underset{\cdot}{V} \times \underset{\cdot}{V}$ where $\hat{S}_2 : \underset{\cdot}{V} \times \underset{\cdot}{V} \longrightarrow [0,1]$, $\hat{I}_2 : \underset{\cdot}{V} \times \underset{\cdot}{V} \longrightarrow [0,1]$ and $Đ_2 : \underset{\cdot}{V} \times \underset{\cdot}{V} \longrightarrow [0,1]$ denote the degree of membership, abstinence and non-membership of edge$(ṭ_i, ṭ_j) \in \underset{\cdot}{V} \times \underset{\cdot}{V}$ such that $\hat{S}_2(ṭ_i, ṭ_j) \le \min[\hat{S}_1(ṭ_i), \hat{S}_1(ṭ_j)], \hat{I}_2(ṭ_i, ṭ_j) \le \min[\hat{I}_1(ṭ_i), \hat{I}_1(ṭ_j)]$ and $Đ_2(ṭ_i, ṭ_j) \le \max[Đ_1(ṭ_i), Đ_1(ṭ_j)]$ with a condition that $0 \le \hat{S}_2 + \hat{I}_2 + Đ_2 \le 1$. Moreover, $1 - (\hat{S}_{1i} + \hat{I}_{1i} + Đ_{1i})$ denotes the refusal degree.

3.1.2 Example

Consider the two graphs below where the one depicted in Figure 28 is a PFG while the one depicted in Figure 29 is not a PFG.

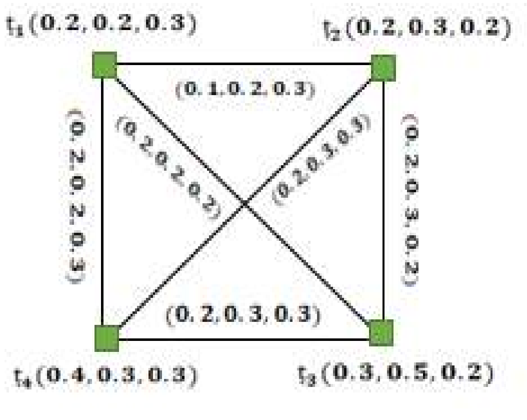


Figure 28 (picture fuzzy graph)

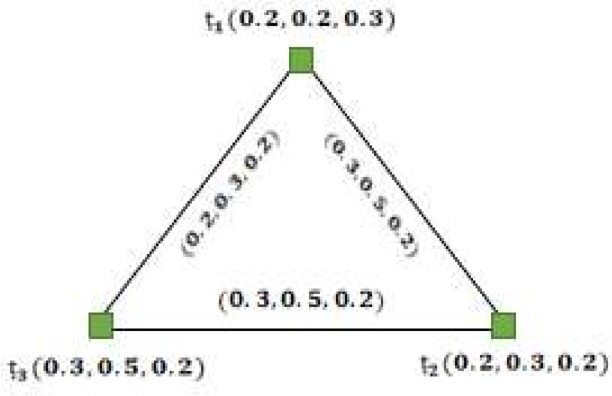


Figure 29 (Not a picture fuzzy graph)

3.1.3 Definition

A pair $\mathrm{H} = (\text{Ṿ}', \text{Ẻ}')$ is said to be picture fuzzy subgraph (PFSG) of a PFG $\dot{\mathrm{G}} = (\text{Ṿ}, \text{Ẻ})$ if $\text{Ṿ}' \subseteq \text{Ṿ}$ and $\text{Ẻ}' \subseteq \text{Ẻ}$ that is $\hat{\mathrm{S}}_{1i}{}' \leq \hat{\mathrm{S}}_{1i}$,$\hat{\mathrm{I}}_{1i}{}' \leq \hat{\mathrm{I}}_{1i}$, $\text{Đ}_{1i}{}' \geq \text{Đ}_{1i}$ and $\hat{\mathrm{S}}_{2ij}{}' \leq \hat{\mathrm{S}}_{2ij}$, $\hat{\mathrm{I}}_{2ij}{}' \leq \hat{\mathrm{I}}_{2ij}$,$\text{Đ}_{2ij}{}' \geq \text{Đ}_{2ij}$ for all $i, j = 1, 2, \dots, n$.

3.1.4 Definition

The density of a PFG $\dot{\mathrm{G}} = (\text{Ṿ}, \text{Ẻ})$ is denoted by $D(\dot{\mathrm{G}}) = \left(D_{\hat{\mathrm{S}}}(\dot{\mathrm{G}}), D_{\hat{\mathrm{I}}}(\dot{\mathrm{G}}), D_{\text{Đ}}(\dot{\mathrm{G}})\right)$ and defined as:

$$D(\dot{\mathrm{G}}) = \left(D_{\hat{\mathrm{S}}}(\dot{\mathrm{G}}), D_{\hat{\mathrm{I}}}(\dot{\mathrm{G}}), D_{\text{Đ}}(\dot{\mathrm{G}})\right) = \begin{pmatrix} \dfrac{2\sum_{\text{ṭ}, w \in \text{Ṿ}}\left(\hat{\mathrm{S}}_2(\text{ṭ}, w)\right)}{\sum_{(\text{ṭ}, w) \in \text{Ẻ}}\left(\hat{\mathrm{S}}_1(\text{ṭ}) \wedge \hat{\mathrm{S}}_1(w)\right)}, \\ \dfrac{2\sum_{\text{ṭ}, w \in \text{Ṿ}}\left(\hat{\mathrm{I}}_2(\text{ṭ}, w)\right)}{\sum_{(\text{ṭ}, w) \in \text{Ẻ}}\left(\hat{\mathrm{I}}_1(\text{ṭ}) \wedge \hat{\mathrm{I}}_1(w)\right)}, \\ \dfrac{2\sum_{\text{ṭ}, w \in \text{Ṿ}}(\text{Đ}_2(\text{ṭ}, w))}{\sum_{(\text{ṭ}, w) \in \text{Ẻ}}(\text{Đ}_1(\text{ṭ}) \vee \text{Đ}_1(w))} \end{pmatrix} \qquad (1)$$

3.1.5 Example

The density of PFG depicted in Figure 28 is calculated as:

$$D_{\hat{\mathrm{S}}}(\dot{\mathrm{G}}) = \frac{2(0.2 + 0.2 + 0.2 + 0.1 + 0.2 + 0.2)}{0.2 + 0.3 + 0.2 + 0.2 + 0.2 + 0.2} = 1.69$$

$D_{\hat{\mathrm{I}}}(\dot{\mathrm{G}}) = \dfrac{2(0.3 + 0..3 + 0.2 + 0.2 + 0.3 + 0.2)}{0.3 + 0.3 + 0.2 + 0.2 + 0.3 + 0.2} = 2.0$

$D_{\mathrm{Đ}}(\dot{\mathrm{G}}) = \dfrac{2(0.2 + 0.3 + 0.3 + 0.3 + 0.3 + 0.2)}{0.2 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3} = 1.88$

$D(\dot{\mathrm{G}}) = (1.69,2.0,.1.88).$

3.1.6 Definition

A PFG $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ is balanced if $D(H) \leq D(\dot{\mathrm{G}})$ that is $D_{\hat{\mathrm{S}}}(H) \leq D_{\hat{\mathrm{S}}}(\dot{\mathrm{G}}), D_{\hat{\mathrm{I}}}(H) \leq D_{\hat{\mathrm{I}}}(\dot{\mathrm{G}})$ and $D_{\mathrm{Đ}}(H) \leq D_{\mathrm{Đ}}(\dot{\mathrm{G}})$ for all subgraphs H of $\dot{\mathrm{G}}$.

3.1.7 Definition

A PFG $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ is said to be complete PFG if $\hat{\mathrm{S}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) = [\hat{\mathrm{S}}_1(\underset{\cdot}{t}_i) \wedge \hat{\mathrm{S}}_1(\underset{\cdot}{t}_j)]$, $\hat{\mathrm{I}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) = [\hat{\mathrm{I}}_1(\underset{\cdot}{t}_i) \wedge \hat{\mathrm{I}}_1(\underset{\cdot}{t}_j)]$ and $\mathrm{Đ}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) = [\mathrm{Đ}_1(\underset{\cdot}{t}_i) \vee \mathrm{Đ}_1(\underset{\cdot}{t}_j)]$ for every $\underset{\cdot}{t}_i, \underset{\cdot}{t}_j \in \dot{\mathrm{E}}$ and a PFG $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ is said to be strong PFG if $\hat{\mathrm{S}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) = [\hat{\mathrm{S}}_1(\underset{\cdot}{t}_i) \wedge \hat{\mathrm{S}}_1(\underset{\cdot}{t}_j)]$, $\hat{\mathrm{I}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) = [\hat{\mathrm{I}}_1(\underset{\cdot}{t}_i) \wedge \hat{\mathrm{I}}_1(\underset{\cdot}{t}_j)]$ and $\mathrm{Đ}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) = [\mathrm{Đ}_1(\underset{\cdot}{t}_i) \vee \mathrm{Đ}_1(\underset{\cdot}{t}_j)]$ for every $\underset{\cdot}{t}_i, \underset{\cdot}{t}_j \in \dot{\mathrm{E}}$**.** Further, the complement of a PFG $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ is represented by $\dot{\mathrm{G}}^c = (\underset{\cdot}{\mathrm{V}}^c, \dot{\mathrm{E}}^c)$ and defined by

1. $\underset{\cdot}{\mathrm{V}}^c = \underset{\cdot}{\mathrm{V}}$.
2. $\hat{\mathrm{S}}_1(\underset{\cdot}{t}_i)^c = \hat{\mathrm{S}}_1(\underset{\cdot}{t}_i), \hat{\mathrm{I}}_1(\underset{\cdot}{t}_i)^c = \hat{\mathrm{I}}_1(\underset{\cdot}{t}_i), \mathrm{Đ}_1(\underset{\cdot}{t}_i)^c = \mathrm{Đ}_1(\underset{\cdot}{t}_i), \forall\, \underset{\cdot}{t}_i \in \underset{\cdot}{\mathrm{V}}$.
3. $\hat{\mathrm{S}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j)^c = [\hat{\mathrm{S}}_2(\underset{\cdot}{t}_i) \wedge \hat{\mathrm{S}}_2(\underset{\cdot}{t}_j)] - \hat{\mathrm{S}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j), \hat{\mathrm{I}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j)^c = [\hat{\mathrm{I}}_2(\underset{\cdot}{t}_i) \wedge \hat{\mathrm{I}}_2(\underset{\cdot}{t}_j)] - \hat{\mathrm{I}}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j)$ and $\mathrm{Đ}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j)^c = [\mathrm{Đ}_2(\underset{\cdot}{t}_i) \vee \mathrm{Đ}_2(\underset{\cdot}{t}_j)] - \mathrm{Đ}_2(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) \,\forall\, \underset{\cdot}{t}_i, \underset{\cdot}{t}_j \in \dot{\mathrm{E}}$.

3.1.8 Theorem

Every complete PFG is balanced.

Proof. Let $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ be a complete PFG. Then we have

$\hat{\mathrm{S}}_2(\underset{\cdot}{t}, w) = \hat{\mathrm{S}}_1(\underset{\cdot}{t}) \wedge \hat{\mathrm{S}}_1(w)$

$\hat{\mathrm{I}}_2(\d{t}, w) = \hat{\mathrm{I}}_1(\d{t}) \wedge \hat{\mathrm{I}}_1(w)$

$\mathrm{Đ}_2(\d{t}, w) = \mathrm{Đ}_1(\d{t}) \vee \mathrm{Đ}_1(w)$.

Because,

$\sum_{\d{t},w\in \d{V}}(\hat{\mathrm{S}}_2(\d{t}, w)) = \sum_{(\d{t},w)\in \dot{\mathrm{E}}}(\hat{\mathrm{S}}_1(\d{t}) \wedge \hat{\mathrm{S}}_1(w))$

$\sum_{\d{t},w\in \d{V}}(\hat{\mathrm{I}}_2(\d{t}, w)) = \sum_{(\d{t},w)\in \dot{\mathrm{E}}}(\hat{\mathrm{I}}_1(\d{t}) \wedge \hat{\mathrm{I}}_1(w))$ and

$\sum_{\d{t},w\in \d{V}}(\mathrm{Đ}_2(\d{t}, w)) = \sum_{(\d{t},w)\in \dot{\mathrm{E}}}(\mathrm{Đ}_1(\d{t}) \vee \mathrm{Đ}_1(w))$

Now

$$D(\dot{\mathrm{G}}) = \left(\frac{2\sum_{\d{t},w\in \d{V}}\left(\hat{\mathrm{S}}_2(\d{t}, w)\right)}{\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\hat{\mathrm{S}}_1(\d{t}) \wedge \hat{\mathrm{S}}_1(w)\right)}, \frac{2\sum_{\d{t},w\in \d{V}}\left(\hat{\mathrm{I}}_2(\d{t}, w)\right)}{\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\hat{\mathrm{I}}_1(\d{t}) \wedge \hat{\mathrm{I}}_1(w)\right)}, \frac{2\sum_{\d{t},w\in \d{V}}\left(\mathrm{Đ}_2(\d{t}, w)\right)}{\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\mathrm{Đ}_1(\d{t}) \vee \mathrm{Đ}_1(w)\right)} \right)$$

$$= \left(\frac{2\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\hat{\mathrm{S}}_1(\d{t}) \wedge \hat{\mathrm{S}}_1(w)\right)}{\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\hat{\mathrm{S}}_1(\d{t}) \wedge \hat{\mathrm{S}}_1(w)\right)}, \frac{2\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\hat{\mathrm{I}}_1(\d{t}) \wedge \hat{\mathrm{I}}_1(w)\right)}{\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\hat{\mathrm{I}}_1(\d{t}) \wedge \hat{\mathrm{I}}_1(w)\right)}, \frac{2\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\mathrm{Đ}_1(\d{t}) \vee \mathrm{Đ}_1(w)\right)}{\sum_{(\d{t},w)\in \dot{\mathrm{E}}}\left(\mathrm{Đ}_1(\d{t}) \vee \mathrm{Đ}_1(w)\right)} \right)$$

$= (2,2,2)$

Let H be a PFSG of $\dot{\mathrm{G}}$. Then $D(H) = (2,2,2)$ and hence $\dot{\mathrm{G}}$ is balanced.

3.1.9 Remark

For the above Theorem, converse statement is not true.

3.1.10 Example

The graph in Figure 30 is balanced but not complete.

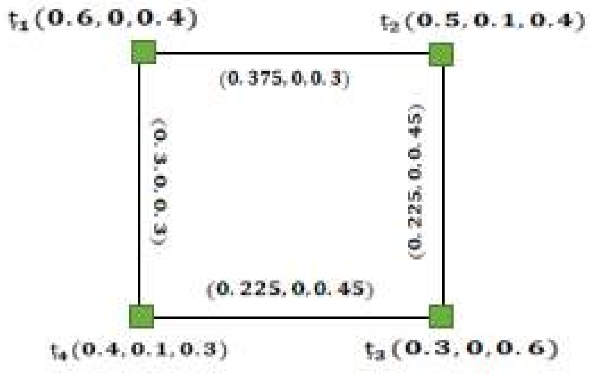


Figure 30 (Balanced picture fuzzy graph but not complete)

$D(\dot{G}) = \left(D_{\hat{S}}(\dot{G}), D_{\hat{I}}(\dot{G}), D_{Đ}(\dot{G})\right) = (1.5,0,1.5)$

Let $H_1 = \{ṭ_1, ṭ_2\}, H_2 = \{ṭ_1, ṭ_3\}, H_3 = \{ṭ_1, ṭ_4\}, H_4 = \{ṭ_2, ṭ_3\}, H_5 = \{ṭ_2, ṭ_4\}, H_6 = \{ṭ_3, ṭ_4\}, H_7 = \{ṭ_1, ṭ_2, ṭ_3\}, H_8 = \{ṭ_1, ṭ_3, ṭ_4\}, H_9 = \{ṭ_1, ṭ_2, ṭ_4\}, H_{10} = \{ṭ_2, ṭ_3, ṭ_4\}, H_{11} = \{ṭ_1, ṭ_2, ṭ_3, ṭ_4\}$ be a non-empty subgraphs of Ġ.

Now we find the densities of all subgraphs:

$D(H_1) = (1.5,0,1.5)$, $D(H_2) = (0,0,0)$, $D(H_3) = (1.5,0,1.5)$, $D(H_4) = (1.5,0,1.5)$, $D(H_5) = (0,0,0)$, $D(H_6) = (1.5,0,1.5)$, $D(H_7) = (1.5,0,1.5)$, $D(H_8) = (1.5,0,1.5)$, $D(H_9) = (1.5,0,1.5)$, $D(H_{10}) = (1.5,0,1.5)$ and $D(H_{11}) = (1.5,0,1.5)$.

Hence, Ġ is balanced PFG as $D(H_i) \leq D(\dot{G})$ for $i = 1,2, \ldots n$. So, it is observed from above graph that

$\hat{S}_2(ṭ, w) \neq \hat{S}_1(ṭ) \wedge \hat{S}_1(w)$

$\hat{I}_2(ṭ, w) \neq \hat{I}_1(ṭ) \wedge \hat{I}_1(w)$

$Đ_2(ṭ, w) \neq Đ_1(ṭ) \vee Đ_1(w)$

Which shows that $\dot{G}$ is not complete. Hence, we proved that $\dot{G}$ is not complete but balanced.

3.1.11 Corollary

All strong PFGs are balanced.

3.1.12 Definition

If $\dot{G} = (\dot{G}^c)^c$. Then $\dot{G}$ is considered as self-complementary PFG.

3.1.13 Theorem

For a self-complementary PFG, the density of $\dot{G}$ is $D(\dot{G}) = (1,1,1)$.

Proof. Straightforward.

3.1.14 Theorem

Let $\dot{G} = (\underset{.}{V}, \dot{E})$ be a strictly balanced PFG and $\dot{G}^c = (\underset{.}{V}^c, \dot{E}^c)$ be its complement. Then $D(\dot{G}) + D(\dot{G}^c) = (2, 2, 2)$.

Proof. Proof is straightforward.

3.1.15 Definition

Let $\dot{G}_1=(\underset{.}{V}_1, \dot{E}_1)$ and $\dot{G}_2 = (\underset{.}{V}_2, \dot{E}_2)$ be two PFGs. A homomorphism $f: \dot{G}_1 \longrightarrow \dot{G}_2$ is a bijective mapping $f: \underset{.}{V}_1 \longrightarrow \underset{.}{V}_2$ which satisfies the following conditions:

1. $\hat{S}_1(\underset{.}{t}_1) \leq \hat{S}_2(f(\underset{.}{t}_1))$, $\hat{I}_1(\underset{.}{t}_1) \leq \hat{I}_2(f(\underset{.}{t}_1))$ and $Đ_1(\underset{.}{t}_1) \geq Đ_2(f(\underset{.}{t}_1))$
2. $\hat{S}_1(u_1\underset{.}{t}_1) \leq \hat{S}_2(f(u_1), f(\underset{.}{t}_1))$, $\hat{I}_1(u_1\underset{.}{t}_1) \leq \hat{I}_2(f(u_1)f(\underset{.}{t}_1))$ and $Đ_1(u_1\underset{.}{t}_1) \geq Đ_2(f(u_1), f(\underset{.}{t}_1))$ for all $\underset{.}{t}_1 \in \underset{.}{V}_1$ and $u_1, \underset{.}{t}_1 \in \dot{E}_1$.

3.1.16 Definition

Let $\dot{G}_1=(\underset{.}{V}_1,\dot{\mathring{E}}_1)$ and $\dot{G}_2=(\underset{.}{V}_2,\dot{\mathring{E}}_2)$ be two PFGs. An isomorphism $f:\dot{G}_1\longrightarrow\dot{G}_2$ is a bijective mapping $f:\underset{.}{V}_1\longrightarrow\underset{.}{V}_2$ which satisfies the following conditions:

1. $\hat{S}_1(\underset{.}{t}_1)=\hat{S}_2(f(\underset{.}{t}_1)),\hat{I}_1(\underset{.}{t}_1)=\hat{I}_2(f(\underset{.}{t}_1))$ and $Đ_1(\underset{.}{t}_1)=Đ_2(f(\underset{.}{t}_1))$
2. $\hat{S}_1(u_1\underset{.}{t}_1)=\hat{S}_2(f(u_1)\,f(\underset{.}{t}_1)),\hat{I}_1(u_1\underset{.}{t}_1)=\hat{I}_2(f(u_1)f(\underset{.}{t}_1))$ and $Đ_1(u_1\underset{.}{t}_1)=Đ_2(f(u_1)f(\underset{.}{t}_1))$

   for all $\underset{.}{t}_1\in\underset{.}{V}_1$ and $u_1\underset{.}{t}_1\in\dot{\mathring{E}}_1$.

3.1.17 Theorem

Let $\dot{G}_1=(\underset{.}{V}_1,\dot{\mathring{E}}_1)$ and $\dot{G}_2=(\underset{.}{V}_2,\dot{\mathring{E}}_2)$ be two isomorphic PFGs. If $\dot{G}_2$ is balanced, then $\dot{G}_1$ is balanced.

Proof. Let $f:\underset{.}{V}_1\rightarrow\underset{.}{V}_2$ be a bijection such that for every $u,\underset{.}{t}\in\underset{.}{V}_1$

$\hat{S}_1(\underset{.}{t})=\hat{S}_1^*(f(\underset{.}{t}))$

$\hat{I}_1(\underset{.}{t})=\hat{I}_1^*(f(\underset{.}{t}))$

$Đ_1(\underset{.}{t})=Đ_1^*(f(\underset{.}{t}))$

$\hat{S}_2(\underset{.}{t},w)=\hat{S}_2^*(f(\underset{.}{t}),f(w))$

$\hat{I}_2(\underset{.}{t},w)=\hat{I}_2^*(f(\underset{.}{t}),f(w))$

$Đ_2(\underset{.}{t},w)=Đ_2^*(f(\underset{.}{t}),f(w))$

Then

$\sum_{\underset{.}{t},w\in\underset{.}{V}_1}\hat{S}_1(\underset{.}{t})=\sum_{\underset{.}{t},w\in\underset{.}{V}_2}\hat{S}_1^*(\underset{.}{t})$

$\sum_{\underset{.}{t},w\in\underset{.}{V}_1}\hat{I}_1(\underset{.}{t})=\sum_{\underset{.}{t},w\in\underset{.}{V}_2}\hat{I}_1^*(\underset{.}{t})$

$\sum_{\underset{.}{t},w\in\underset{.}{V}_1}Đ_1(\underset{.}{t})=\sum_{\underset{.}{t},w\in\underset{.}{V}_2}Đ_1^*(\underset{.}{t})$

and

$\sum_{ṭ,w\in Ṿ_1} \hat{S}_2(ṭ,w) = \sum_{ṭ,w\in Ṿ_2} \hat{S}_2^*(ṭ,w)$

$\sum_{ṭ,w\in Ṿ_1} \hat{I}_2(ṭ,w) = \sum_{ṭ,w\in Ṿ_2} \hat{I}_2^*(ṭ,w)$

$\sum_{ṭ,w\in Ṿ_1} Đ_2(ṭ,w) = \sum_{ṭ,w\in Ṿ_2} Đ_2^*(ṭ,w)$

If $H_1 = \dot{G}\left(Ṿ(H_1), \dot{E}(H_1)\right)$ is a PFSG of $\dot{G}_1$ and $H_2 = \dot{G}\left(Ṿ(H_2), \dot{E}(H_2)\right)$ is a PFSG of $\dot{G}_2$ where for every $ṭ, w \in Ṿ(H_1)$ we have

$\hat{S}_1(ṭ) = \hat{S}_1^*\big(f(ṭ)\big)$

$\hat{I}_1(ṭ) = \hat{I}_1^*\big(f(ṭ)\big)$

$Đ_1(ṭ) = Đ_1^*\big(f(ṭ)\big)$

and

$\hat{S}_2(ṭ,w) = \hat{S}_2^*\big(f(ṭ), f(w)\big)$

$\hat{I}_2(ṭ,w) = \hat{I}_2^*\big(f(ṭ), f(w)\big)$

$Đ_2(ṭ,w) = Đ_2^*\big(f(ṭ), f(w)\big)$

Since $\dot{G}_2$ is balanced so $D(H_2) \leq D(\dot{G}_2)$ i. e. $D_{\hat{S}}(H_2) \leq D_{\hat{S}}(\dot{G}_2), D_{\hat{I}}(H_2) \leq D_{\hat{I}}(\dot{G}_2)$ and $D_{Đ}(H_2) \leq D_{Đ}(\dot{G}_2)$. Now

$$\frac{2\sum_{ṭ,w\in Ṿ(H_1)} \hat{S}_2^*(ṭ,w)}{\sum_{(ṭ,w)\in \dot{E}(H_1)} \hat{S}_1^*(ṭ) \wedge \hat{S}_1^*(w)} \leq \frac{2\sum_{ṭ,w\in Ṿ_2} \hat{S}_2^*(ṭ,w)}{\sum_{(ṭ,w)\in \dot{E}_2} \hat{S}_1^*(ṭ) \wedge \hat{S}_1^*(w)}$$

$$\frac{2\sum_{ṭ,w\in Ṿ(H_1)} \hat{I}_2^*(ṭ,w)}{\sum_{(ṭ,w)\in \dot{E}(H_1)} \hat{I}_1^*(ṭ) \wedge \hat{I}_1^*(w)} \leq \frac{2\sum_{ṭ,w\in Ṿ_2} \hat{I}_2^*(ṭ,w)}{\sum_{(ṭ,w)\in \dot{E}_2} \hat{I}_1^*(ṭ) \wedge \hat{I}_1^*(w)}$$

and

$$\frac{2\sum_{ṭ,w\in Ṿ(H_1)} Đ_2^*(ṭ,w)}{\sum_{(ṭ,w)\in Ė(H_1)} Đ_1^*(ṭ)\wedge Đ_1^*(w)} \leq \frac{2\sum_{ṭ,w\in Ṿ_2} Đ_2^*(ṭ,w)}{\sum_{(ṭ,w)\in Ẻ_2} Đ_1^*(ṭ)\wedge Đ_1^*(w)}$$

$$\frac{2\sum_{ṭ,w\in Ṿ(H_1)} \hat{S}_2(ṭ,w)}{\sum_{(ṭ,w)\in Ẻ(H_1)} \hat{S}_1^*(ṭ)\wedge \hat{S}_1^*(w)} \leq \frac{2\sum_{ṭ,w\in Ṿ_1} \hat{S}_1(ṭ,w)}{\sum_{(ṭ,w)\in Ẻ_1} \hat{S}_1^*(ṭ)\wedge \hat{S}_1^*(w)}$$

$$\frac{2\sum_{ṭ,w\in Ṿ(H_1)} \hat{I}_2(ṭ,w)}{\sum_{(ṭ,w)\in Ẻ(H_1)} \hat{I}_1^*(ṭ)\wedge \hat{I}_1^*(w)} \leq \frac{2\sum_{ṭ,w\in Ṿ_1} \hat{I}_1(ṭ,w)}{\sum_{(ṭ,w)\in Ẻ_1} \hat{I}_1^*(ṭ)\wedge \hat{I}_1^*(w)}$$

and

$$\frac{2\sum_{ṭ,w\in Ṿ(H_1)} Đ_2(ṭ,w)}{\sum_{(ṭ,w)\in Ẻ(H_1)} Đ_1^*(ṭ)\wedge Đ_1^*(w)} \leq \frac{2\sum_{ṭ,w\in Ṿ_1} Đ_1(ṭ,w)}{\sum_{(ṭ,w)\in Ẻ_1} Đ_1^*(ṭ)\wedge Đ_1^*(w)}$$

i.e. $D_{\hat{S}}(H_1) \leq D_{\hat{S}}(Ġ_1), D_{\hat{I}}(H_1) \leq D_{\hat{I}}(Ġ_1)$ and $D_{Đ}(H_1) \leq D_{Đ}(Ġ_1)$ because $D(H_1) \leq D(Ġ_1)$. Hence $Ġ_1$ is balanced.

3.1.18 Definition

Let $Ġ = (Ṿ, Ẻ)$ be a PFG. The index matrix representation of PFG is of the form $[Ṿ, Ẻ_M(Ġ) \subset Ṿ \times Ṿ]$, where $Ṿ = \{ṭ_1, ṭ_2, ṭ_3, \ldots ṭ_n\}$ and

$$Ẻ_M(Ġ) = \{(\hat{S}_{ij}, \hat{I}_{ij}, Đ_{ij})\} = \begin{array}{c c} & \begin{array}{cccc} ṭ_1 & ṭ_2 & \cdots & ṭ_n \end{array} \\ \begin{array}{c} ṭ_1 \\ ṭ_2 \\ \cdots \\ ṭ_n \end{array} & \begin{bmatrix} (\hat{S}_{11}, Đ_{11}) & (\hat{S}_{12}, Đ_{12}) & \cdots & (\hat{S}_{1n}, Đ_{1n}) \\ (\hat{S}_{21}, Đ_{21}) & (\hat{S}_{22}, Đ_{22}) & \cdots & (\hat{S}_{2n}, Đ_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ (\hat{S}_{n1}, Đ_{n1}) & (\hat{S}_{n2}, Đ_{n2}) & \cdots & (\hat{S}_{nn}, Đ_{nn}) \end{bmatrix} \end{array}$$

Where $(\hat{S}_{ij}, \hat{I}_{ij}, Đ_{ij}) \in [0,1] \times [0,1] \times [0,1]$ and $(i, j = 1, 2, \ldots, n)$. An edge between two vertices $ṭ_i$ and $ṭ_j$ is indexed by $(\hat{S}_{ij}, \hat{I}_{ij}, Đ_{ij})$.

3.1.19 Definition

Let $Ġ = (Ṿ, Ẻ)$ be a PFG. Then the edge density of an edge e of $Ġ$ is defined as

$$D_{\dot{G}}(e) = \left(\frac{2\left(\hat{S}_2(ṭ, w)\right)}{\sum_{(ṭ,w)\in\dot{E}}(\hat{S}_1(ṭ)\wedge\hat{S}_1(w)}, \frac{2\left(\hat{I}_2(ṭ, w)\right)}{\sum_{(ṭ,w)\in\dot{E}}(\hat{I}_1(ṭ)\wedge\hat{I}_1(w)}, \frac{2(Đ_2(ṭ, w))}{\sum_{(ṭ,w)\in\dot{E}}(Đ_1(ṭ)\vee Đ_1(w)} \right) \quad (2)$$

3.1.20 Definition

A PFG $\dot{G} = (Ṿ, \dot{E})$ with edge density on its each edge is called edge density PFG of $\dot{G}$ and is denoted by EDPF$(\dot{G})$.

3.1.21 Example

The edge density PFG of the graph depicted in Figure 28 is given in Figure 31.

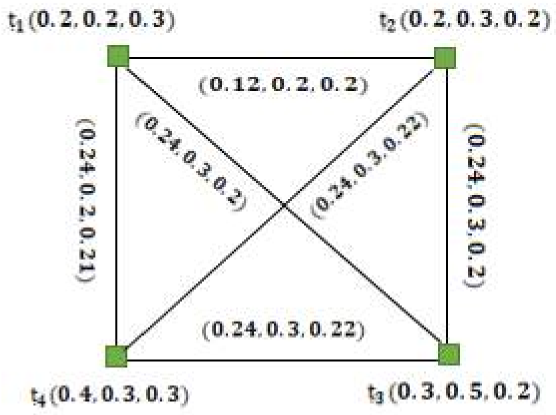


Figure 31(Edge density picture fuzzy graph)

3.1.22 Definition

The single valued edge density of an edge e of a PFG $\dot{G} = (Ṿ, \dot{E})$ is defined as:

$$S\dot{E}D(e) = \frac{\left(\frac{2\left(\hat{S}_2(ṭ, w)\right)}{\sum_{(ṭ,w)\in\dot{E}}\left(\hat{S}_1(ṭ)\wedge\hat{S}_1(w)\right)} \right)}{\left(\frac{2(Đ_2(ṭ, w))}{\sum_{(ṭ,w)\in\dot{E}}(Đ_1(ṭ)\vee Đ_1(w))} \right) + \left(\frac{2\left(\hat{I}_2(ṭ, w)\right)}{\sum_{(ṭ,w)\in\dot{E}}\left(\hat{I}_1(ṭ)\wedge\hat{I}_1(w)\right)} \right)} \quad (3)$$

3.1.23 Definition

A PFG $\dot{\mathrm{G}} = (\underset{.}{\mathrm{V}}, \dot{\mathrm{E}})$ with single valued edge density on its each edge is called single valued edge density PFG and is denoted by SEDPFG.

3.1.24 Definition

The single valued edge density of picture fuzzy matrix $\left(S\dot{\mathrm{E}}DPF_M\right)$ is defined as

$S\dot{\mathrm{E}}DPF_M(\dot{\mathrm{G}}) = [\mathrm{Đ}_{ij}]$ where

$$\mathrm{Đ}_{ij} = \begin{cases} \dfrac{\left(\dfrac{2\left(\hat{\mathrm{S}}_2(\underset{.}{\mathrm{t}}, w)\right)}{\sum_{(\underset{.}{\mathrm{t}}, w)\in\dot{\mathrm{E}}}\left(\hat{\mathrm{S}}_1(\underset{.}{\mathrm{t}})\wedge\hat{\mathrm{S}}_1(w)\right)}\right)}{\left(\dfrac{2\left(\mathrm{Đ}_2(\underset{.}{\mathrm{t}}, w)\right)}{\sum_{(\underset{.}{\mathrm{t}}, w)\in\dot{\mathrm{E}}}\left(\mathrm{Đ}_1(\underset{.}{\mathrm{t}})\vee\mathrm{Đ}_1(w)\right)}\right) + \left(\dfrac{2\left(\hat{\mathrm{I}}_2(\underset{.}{\mathrm{t}}, w)\right)}{\sum_{(\underset{.}{\mathrm{t}}, w)\in\dot{\mathrm{E}}}\left(\hat{\mathrm{I}}_1(\underset{.}{\mathrm{t}})\wedge\hat{\mathrm{I}}_1(w)\right)}\right)} & \text{for } \underset{.}{\mathrm{t}} \neq w \\ 0 \quad \text{for } \underset{.}{\mathrm{t}} = w \end{cases} \tag{4}$$

3.1.25 Example

The $S\dot{\mathrm{E}}DPF_M$ of PFG defined in Figure 28 is given by:

$$S\dot{\mathrm{E}}DPF_M(\dot{\mathrm{G}}) = \begin{array}{c} \\ \underset{.}{\mathrm{t}}_1 \\ \underset{.}{\mathrm{t}}_2 \\ \underset{.}{\mathrm{t}}_3 \\ \underset{.}{\mathrm{t}}_4 \end{array} \begin{array}{cccc} \underset{.}{\mathrm{t}}_1 & \underset{.}{\mathrm{t}}_2 & \underset{.}{\mathrm{t}}_3 & \underset{.}{\mathrm{t}}_4 \end{array} \\ \begin{bmatrix} 0 & 0.23 & 0.59 & 0.47 \\ 0.23 & 0 & 0.47 & 0.39 \\ 0.59 & 0.47 & 0 & 0.39 \\ 0.47 & 0.39 & 0.39 & 0 \end{bmatrix}$$

The next section briefly illustrated the clustering algorithm for PFGs and its applications in human decision-making.

3.2 Pre-Existing Study on Clustering in the Environment of FG and IFG and Some Challenges

This section is divided into three subsections. In the first subsection, the clustering algorithms of FG and IFG proposed by [117, 118], are discussed followed by numerical examples. In the second subsection, some lapses are pointed out in while working in the environment of FG and PFG and their solutions are proposed.

3.2.1 Fuzzy Clustering Algorithm [117]

In this subsection, the clustering algorithm in the environment of FGs followed by an example for demonstration of the algorithm.

1. Establish the edge density fuzzy matrix $EDF_M(\dot{G})$ with m vertices.
2. Based on step one, establish the single-valued fuzzy matrix $SEDF_M(\dot{G})$.
3. Now for the $SEDF(\dot{G})$, construct the narrow slicing.

Here arise two cases:

Case 1: If $SEDF(\dot{G})$ is balanced. Then $SEDF(\dot{G}) = SEDF(\dot{G}_1)$.

Case 2: If $SEDF(\dot{G})$ is not balanced. Then proceed as follows:

a) For every row of $SEDF_M(\dot{G})$, find the sum of its entries.
b) Chose the least value corresponding to a vertex ṭ in $SEDF_M(\dot{G})$.
c) Develop an induced subgraph i.e. $SEDF(\dot{G}_{\hat{\imath}})$ with the help of remaining vertices.
d) Repeat step (a), (b) and (c) and continue doing so until the selection of $m - 1$ vertices.
e) Arrange the vertices obtained in each step into groups.
f) Thus, we have obtained narrow slicing of $SEDF(\dot{G})$.

4. In order to compute the τ-edge components of $SEDF(\dot{G})$, we proceed as follows:

a) From $SEDF(\dot{G})$ obtained a $SEDF_M(\dot{G})$.
b) Utilizing the concept of edge cohesiveness and by grouping the vertices into clusters, a minimal τ-edge connected subgraph is obtained.

Now an example is presented in support of the above algorithm in which a human decision-making problem is illustrated.

3.2.2 Numerical Example

Consider a FG having five vertices and each vertex is connected with the other in Figure (32). The five vertices represent five different attributes among which a group of human needs to decide which of them are substantial for assigning a good brand or object. Here we used the notions of FG to solve the problem of determining best attributes among the list of attributes. In our case, the five vertices $ṭ_1, ṭ_2, ṭ_3, ṭ_4$ and $ṭ_5$ represents the attributes *Quality*, *Service*, *Price*, *Technology*, and *Advertisement* respectively. The fuzzy clustering algorithm is demonstrated stepwise below.

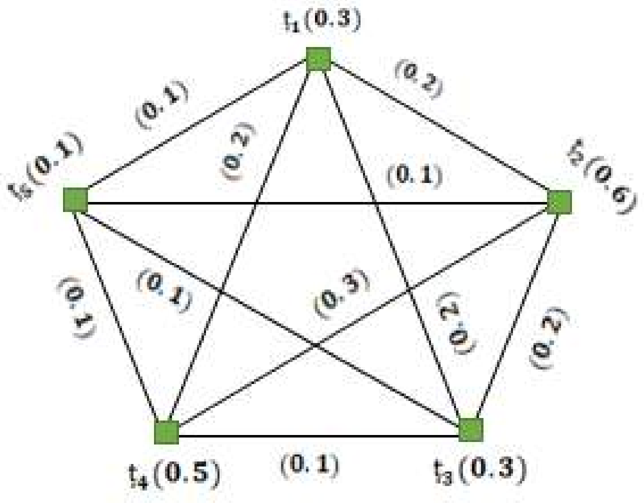


Figure 32 (fuzzy graph based on information of human opinion)

Step 1. First, we used the concept of edge density of FGs [117] to obtain an $EDF_M(\dot{G})$.

$$ED_M(\dot{G}) = \begin{array}{c} \\ ṭ_1 \\ ṭ_2 \\ ṭ_3 \\ ṭ_4 \\ ṭ_5 \end{array} \begin{array}{ccccc} ṭ_1 & ṭ_2 & ṭ_3 & ṭ_4 & ṭ_5 \\ \end{array}$$

$$\begin{array}{c|ccccc} & ṭ_1 & ṭ_2 & ṭ_3 & ṭ_4 & ṭ_5 \\ \hline ṭ_1 & 0 & 0.167 & 0.167 & 0.167 & 0.083 \\ ṭ_2 & 0.167 & 0 & 0.167 & 0.25 & 0.083 \\ ṭ_3 & 0.167 & 0.167 & 0 & 0.083 & 0.083 \\ ṭ_4 & 0.167 & 0.25 & 0.083 & 0 & 0.083 \\ ṭ_5 & 0.083 & 0.083 & 0.083 & 0.083 & 0 \end{array}$$

Step 2. The sum of every row is calculated and listed in Table 4

t_1	t_2	t_3	t_4	t_5
0.584	0.167	0.5	0.583	0.332

Table 4(Sum of values in SEDF matrix)

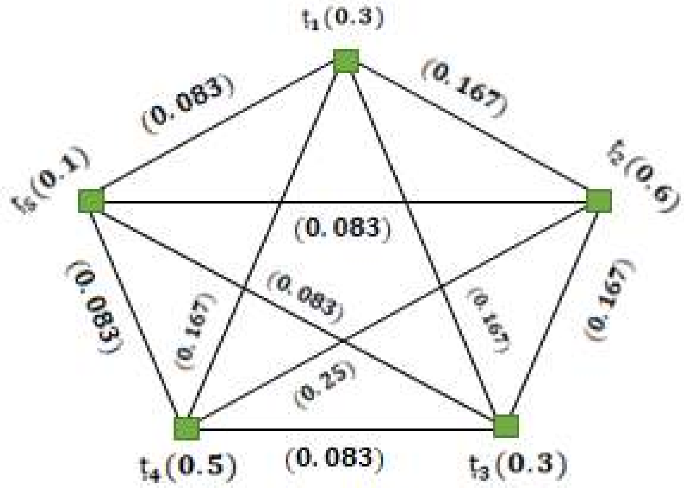


Figure 33 (Single-valued edge density fuzzy graph)

Based on calculations in Table 4, the least value occurs for row 1. Hence, we have the first cluster $C_1 = (\{ṭ_1\}, \{ṭ_2, ṭ_3, ṭ_4, ṭ_5\})$. Now the $SEDF_M(\dot{G})$induced by the remaining vertices is denoted by $SEDF_M(\dot{G}_1)$ and is given by:

$$SEDF_M(\dot{G}_1) = \begin{array}{c} \\ ṭ_2 \\ ṭ_3 \\ ṭ_4 \\ ṭ_5 \end{array} \begin{bmatrix} ṭ_2 & ṭ_3 & ṭ_4 & ṭ_5 \\ 0 & 0.167 & 0.25 & 0.083 \\ 0.167 & 0 & 0.083 & 0.083 \\ 0.25 & 0.083 & 0 & 0.083 \\ 0.083 & 0.083 & 0.083 & 0 \end{bmatrix}$$

Proceeding similarly, we have

t_2	t_3	t_4	t_5
0.5	0.33	0.42	0.25

Table 5 (Sum of values in SEDF matrix)

Based on calculations in Table 5, the least value occurs for row 2. Hence, we have the second cluster $C_2 = (\{t_2\}, \{t_3, t_4, t_5\})$. Now the $SEDF_M(\dot{G})$induced by the remaining vertices is denoted by $SEDF_M(\dot{G}_2)$ and is given by:

$$SEDIF_M(\dot{G}_2) = \begin{array}{c} \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{ccc} t_3 & t_4 & t_5 \end{array} \\ \begin{bmatrix} 0 & 0.083 & 0.083 \\ 0.083 & 0 & 0.083 \\ 0.083 & 0.083 & 0 \end{bmatrix} \end{array}$$

Proceeding similarly, we have $C_2 = (\{t_3\}, \{t_4\}, \{t_5\})$ and $\{(\{t_1\}, \{t_2, t_3, t_4, t_5\}), (\{t_2\}, \{t_3, t_4, t_5\}), (\{t_3\}, \{t_4\}, \{t_5\})\}$is the required narrow slicing. It is clear that $SEDIF_M(\dot{G}_2)$ contains $C_{345} = 0.083$. Hence, the parameters *service* and *Technology* corresponding to t_3, t_4 and t_5 respectively are the factors that influenced the consumer decision for a brand.

Now we moved towards step 3 where τ-edge components of $SEDF(\dot{G})$ are evaluated.

Step 3. For various values of τ, the τ-edge components of $SEDF_M(\dot{G})$are shown in Table 6.

τ	τ −edge components
$(0.25, \infty]$	$(\{t_1\}, \{t_2\}, \{t_3\}, \{t_4\}, \{t_5\})$
$(0.167, 0.25]$	$(\{t_3, t_4, t_5\}, \{t_1\}, \{t_2\})$
$(0.083, 0.167]$	$(\{t_2, t_3, t_4, t_5\}, \{t_1\})$
$(0, 0.083]$	$(\{t_1, t_2, t_3, t_4, t_5\})$

Table 6 (t-edge component of fuzzy graph)

Now we discussed the clustering algorithm for IFGs proposed by [118].

3.2.3 Intuitionistic Fuzzy Clustering Algorithm [118]

In this subsection, the clustering algorithm in the environment of FGs followed by an example for demonstration of the algorithm.

1. Establish the edge density intuitionistic fuzzy matrix $EDIF_M(\dot{G})$ with m vertices.
2. Based on step one, establish the single-valued intuitionistic fuzzy matrix $SEDIF_M(\dot{G})$.
3. Now for the $SEDIF(\dot{G})$, construct the narrow slicing.

Here arise two cases:

Case 1: If $SEDIF(\dot{G})$ is balanced. Then $SEDIF(\dot{G}) = SEDIF(\dot{G}_1)$.

Case 2: If $SEDF(\dot{G})$ is not balanced. Then proceed as follows:

a) For every row of $SEDIF_M(\dot{G})$, find the sum of its entries.
b) Chose the least value corresponding to a vertex ṭ in $SEDIF_M(\dot{G})$.
c) Develop an induced subgraph i.e. $SEDIF(\dot{G}_{\hat{I}})$ with the help of remaining vertices.
d) Repeat step (a), (b) and (c) and continue doing so until the selection of $m - 1$ vertices.
e) Arrange the vertices obtained in each step into groups.
f) Thus, we have obtained narrow slicing of $SEDIF(\dot{G})$.

4. In order to compute the τ-edge components of $SEDIF(\dot{G})$, we proceed as follows:

a) From $SEDIF(\dot{G})$ obtained a $SEDIF_M(\dot{G})$.
b) Utilizing the concept of edge cohesiveness and by grouping the vertices into clusters, a minimal τ-edge connected subgraph is obtained.

Now an example is presented in support of the above algorithm for IFGs in which a human decision-making problem is illustrated.

3.2.4 Numerical Example

In this numerical example, we consider the same problem as discussed in FGs but with intuitionistic fuzzy information. Consider the graph in Figure 34.

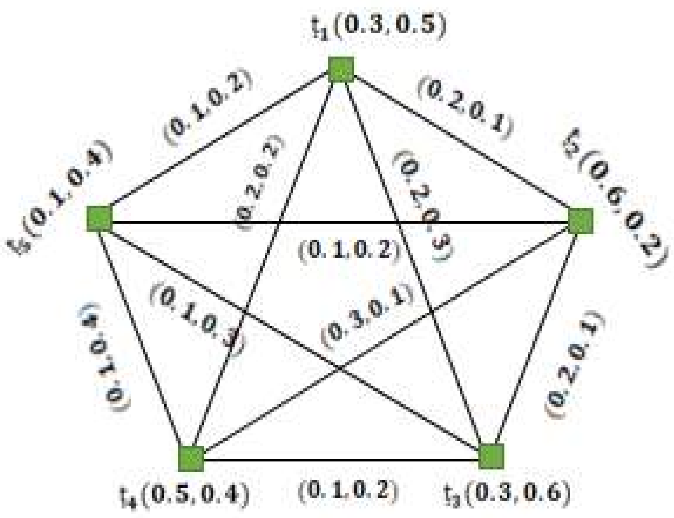


Figure 34 (Intuitionistic fuzzy graph based on information of human opinion)

Step 1. First, we used the concept of edge density of IFG [118] to obtain an $EDIF_M(\dot{G})$.

$$ED_M(\dot{G}) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ (0,1) & (0.167,0.039) & (0.167,0.12) & (0.167,0.078) & (0.083,0.078) \\ (0.167,0.039) & (0,1) & (0.167,0.039) & (0.25,0.039) & (0.083,0.078) \\ (0.167,0.12) & (0.167,0.039) & (0,1) & (0.083,0.078) & (0.083,0.12) \\ (0.167,0.078) & (0.25,0.039) & (0.083,0.078) & (0,1) & (0.083,0.16) \\ (0.083,0.078) & (0.083,0.078) & (0.083,0.12) & (0.083,0.16) & (0,1) \end{bmatrix}$$

And based on $SEDIF(\dot{G})$ depicted in Figure 34, the $EDIF_M(\dot{G})$ is developed.

$$SEDIF_M(\dot{G}) = \begin{array}{c} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ 0 & 4.28 & 1.39 & 2.14 & 1.064 \\ 428 & 0 & 4.28 & 6.41 & 1.064 \\ 1.39 & 4.28 & 0 & 1.064 & 0.69 \\ 2.14 & 6.41 & 1.064 & 0 & 0.52 \\ 1.064 & 1.064 & 0.69 & 0.52 & 0 \end{bmatrix}$$

Step 2. The sum of every row is calculated and listed in Table 7.

t_1	t_2	t_3	t_4	t_5
8.87	16.03	7.42	10.13	3.34

Table 7 (Sum of values in SEDIF matrix)

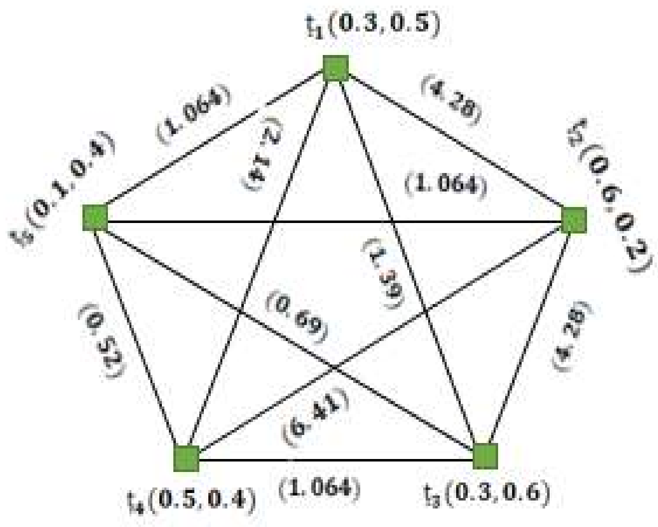


Figure 35(Single-valued edge density intuitionistic fuzzy graph)

Based on calculations in Table 7, the least value occurs for row 2. Hence, we have the first cluster $C_1 = (\{t_2\},\{t_1,t_3,t_4,t_5\})$. Now the $SEDIF_M(\dot{G})$ induced by the remaining vertices is denoted by $SEDIF_M(\dot{G}_1)$ and is given by:

$$SEDIF_M(\dot{G}_1) = \begin{array}{c} \\ t_1 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_3 & t_4 & t_5 \end{array} \\ \begin{bmatrix} 0 & 1.39 & 2.14 & 1.064 \\ 1.39 & 0 & 1.064 & 0.69 \\ 2.14 & 1.064 & 0 & 0.52 \\ 1.064 & 0.69 & 0.52 & 0 \end{bmatrix} \end{array}$$

Proceeding similarly, we have

t_1	t_3	t_4	t_5
4.59	3.15	3.73	2.28

Table 8 (Sum of values in SEDIF matrix)

Based on calculations in Table 8, the least value occurs for row 1. Hence, we have the second cluster$C_2 = (\{t_1\}, \{t_3, t_4, t_5\})$. Now the $SEDIF_M(\dot{G})$induced by the remaining vertices is denoted by $SEDIF_M(\dot{G}_2)$ and is given by:

$$SEDIF_M(\dot{G}_2) = \begin{array}{c c} & \begin{array}{ccc} t_3 & t_4 & t_5 \end{array} \\ \begin{array}{c} t_3 \\ t_4 \\ t_5 \end{array} & \begin{bmatrix} 0 & 1.064 & 0.69 \\ 1.064 & 0 & 0.52 \\ .69 & 0.52 & 0 \end{bmatrix} \end{array}$$

Proceeding similarly, we have

t_3	t_4	t_5
1.75	1.59	1.21

Table 9 (Sum of values in SEDIF matrix)

Based on calculations in Table 9, the least value occurs for row 3. Hence, we have the third cluster$C_3 = (\{t_3\}, \{t_4, t_5\})$. Now the $SEDIF_M(\dot{G})$induced by the remaining vertices is denoted by $SEDIF_M(\dot{G}_3)$ and is given by:

$$SEDIF_M(\dot{G}_3) = \begin{array}{c c} & \begin{array}{cc} t_4 & t_5 \end{array} \\ \begin{array}{c} t_4 \\ t_5 \end{array} & \begin{bmatrix} 0 & 0.52 \\ 0.52 & 0 \end{bmatrix} \end{array}$$

Proceeding similarly, we have $C_4 = (\{t_4\}, \{t_5\})$ and $\{(\{t_2\}, \{t_1, t_3, t_4, t_5\}), (\{t_1\}, \{t_3, t_4, t_5\}), (\{t_3\}, \{t_4, t_5\}), (\{t_4\}, \{t_5\})\}$ is the required narrow slicing. It is clear that $SEDIF_M(\dot{G}_3)$ contains $C_{45} = 0.52$. Hence, the parameters *service* and

Technology corresponding to $ṭ_4$ and $ṭ_5$ respectively are the factors that influenced the consumer decision for a brand.

Now we moved towards step 3 where τ-edge components of $SEDIF(\dot{G})$ are evaluated.

Step 3. For various values ofτ, the τ-edge components of $SEDIF_M(\dot{G})$are shown in Table 10.

τ	τ −edge components
$(6.41, \infty]$	$(\{ṭ_1\},\{ṭ_2\},\{ṭ_3\},\{ṭ_4\},\{ṭ_5\})$
$(4.28,6.41]$	$(\{ṭ_4,ṭ_5\},\{ṭ_1\},\{ṭ_3\},\{ṭ_2\})$
$(2.41,4.28]$	$(\{ṭ_3,ṭ_4,ṭ_5\},\{ṭ_1\},\{ṭ_2\})$
$(1.39,2.41]$	$(\{ṭ_1,ṭ_3,ṭ_4,ṭ_5\},\{ṭ_2\})$
$(0,1.39]$	$(\{ṭ_1,ṭ_2,ṭ_3,ṭ_4,ṭ_5\})$

Table 10 (t-edge components of intuitionistic fuzzy graph)

Now we discussed the parameters of IFG and pointed out the limitation in it.

a. Parameters Assigning for Intuitionistic Fuzzy Graphs

In [118], a human decision-making problem is portrayed using the concept of FGs and IFGs. In the problem that is solved in the environment of IFGs, human opinion was established about 10 brands under some attributes such as *quality (Q), service (S), price (P), technology (T)* and *advertisement (A).* It is discussed that human decision about a brand could be *yes* or *no* or between *yes* or *no*, i.e., *hesitancy*. The human opinion is taken in the form of intuitionistic fuzzy numbers (IFNs). The influence chart of human-decisions is given as follows:

S. No	Brands	Quality (Q)	Service (S)	Price (P)	Technology (T)	Advertisement (A)
1	Y_1	⊕	?	−	⊕	−
2	Y_2	⊖	⊕	⊖	⊖	⊖
3	Y_3	⊖	⊖	⊖	⊖	⊖
4	Y_4	⊕	⊕	⊕	⊕	⊕
5	Y_5	⊖	⊕	?	⊕	⊖
6	Y_6	?	⊕	⊖	⊕	⊖
7	Y_7	?	⊖	⊖	⊖	⊖
8	Y_8	⊖	⊕	⊖	?	⊖
9	Y_9	⊕	⊕	⊕	⊖	⊖
10	Y_{10}	⊖	?	⊖	⊕	⊖

Table 11 (Chart representing human opinion about brands [118])

This problem was modelled using IFGs where vertices are represented by the attributes Q, *S*, *P*, *T* and *A*, and the edges show the relation of these edges. The opinion *yes*, *no* and *hesitancy* are symbolized respectively by ⊕, ⊖ and ?. Some rules were set which are:

- All $+\d{t}e$ values were considered as membership of a vertex.
- All $-\d{t}e$ values were considered as non-membership of a vertex.
- All values marked as ? were considered as hesitancy degree of a vertex.
- All $+\d{t}e$ values of consecutive vertices are termed as membership value of an edge.
- Difference in $-\d{t}e$ values of consecutive vertices is termed as non-membership value of an edge.

b. Limitations Occurred in Assigning Parameter for IFGs and Solution to Overcome these Limitations:

By observing Table 4, it becomes clear that the approach of [117, 118] to assign parameters for FGs and IFGs are good enough, but it could not explain some aspects which somehow effects the reliability of model. Here are some reasons based on which IFG is of less significance.

- The model assigned values to membership, non-membership and hesitancy but refusal degree was ignored.
- The abstinence or neutral value has not been discussed in the model.
- The refusal degree of an edge has been defined.
- For edges $\oplus, \oplus$ is considered as membership degree of an edge while $\ominus, \oplus$ is considered as non-membership degree but the cases like$(\oplus,\ominus)$,$(\ominus,\ominus)$,$(\oplus, ?)$, $(\ominus, ?)$ and $(?, ?)$ have not been given any attention.

The reason behind these types of limitations is the structure of IFSs i.e. in an IFSs one only models the membership, hesitancy and non-membership degree. To overcome the situation, in this article we developed the clustering algorithm in the environment of PFGs. A PFSs is the only framework which described not only the membership and non-membership degree of an element but it also described the abstinence and refusal degree and is considered as very much suitable to model human opinion. We took the problem defined in [118] to the environment of PFGs and solved the decision-making problem. Now we discussed some basic graph theoretic concept in the field of PFGs.

Now we present the algorithm in the environment of PFG and illustrate it with the help of an example.

3.3 Edge Density Picture Fuzzy Clustering Algorithm

In this section, we provide an algorithm for clustering in the environment of PFG, a flowchart to describe the steps of algorithm and demonstration for the parameter of PFGs.

3.3.1 Picture Fuzzy Clustering Algorithm

A detailed picture fuzzy clustering algorithm is described as follows:

c) Establish the edge density picture fuzzy matrix $EDPF_M(\dot{G})$ with m vertices.

d) Based on step one, establish the single-valued picture fuzzy matrix $SEDPF_M(\dot{G})$.

e) Now for the $SEDPF(\dot{G})$, construct the narrow slicing.

Here arise two cases:

Case 1: If $SEDPF(\dot{G})$ is balanced. Then $SEDPF(\dot{G}) = SEDPF(\dot{G}_1)$.

Case 2: If $SEDPF(\dot{G})$ is not balanced. Then we proceed as follows:

g) For every row of $SEDPF_M(\dot{G})$, find the sum of its entries.

h) Chose the least value corresponding to a vertex ṭ in $SEDPF_M(\dot{G})$.

i) Develop an induced subgraph i.e. $SEDPF(\dot{G}_{\hat{I}})$ with the help of remaining vertices.

j) Repeat step (a), (b) and (c) and continue doing so until the selection of $m - 1$ vertices.

k) Arrange the vertices obtained in each step into groups.

l) Thus, we have obtained narrow slicing of $SEDPF(\dot{G})$.

f) In order to compute the τ-edge components of $SEDPF(\dot{G})$, we proceed as follows:

g) From $SEDPF(\dot{G})$ obtained a $SEDPF_M(\dot{G})$.

h) Utilizing the concept of edge cohesiveness and by grouping the vertices into clusters, a minimal τ-edge connected subgraph is obtained.

3.3.2 Demonstration of Parameters for Picture Fuzzy Graphs

Consider the collection of m attributes and n number of people are involved to vote in favor, vote against, remain abstain or did not vote. Such a phenomenon can be modeled using the concept of PFGs where each attribute will denote a vertex of the PFG and the relation between them is considered as an edge of PFG. In the following we demonstrate the meaning of each component of vertices and edges of PFGs.

- A vertex ṭ$_{\hat{I}}$ is of the shape $(\hat{S}, \hat{I}, Đ)$ where $\hat{S}$ & Đ denote the number of people who voted in favor (against) respectively while $\hat{I}$ denote the people whose vote is not counted at all and the term $1 - \hat{S}um(\hat{S}, \hat{I}, Đ) = r$ denote the people who did not vote at all i.e. who refused to vote to any option.
- An edge e_{jk} is having the form $(\hat{S}_{jk}, \hat{I}_{jk}, Đ_{jk})$ where $\hat{S}_{jk}$ denote who vote for both$j$ & k, $\hat{I}_{jk}$ denote the people whose remain abstained for jk while $Đ_{jk}$ denote the people who voted against jk. The term $r_{jk} = 1 - \hat{S}um(\hat{S}_{jk}, \hat{I}_{jk}, Đ_{jk})$ denote the people who did not vote to any jk.

A flow chart of the algorithm proposed is depicted in the next subsection.

3.3.3 Flow Chart:

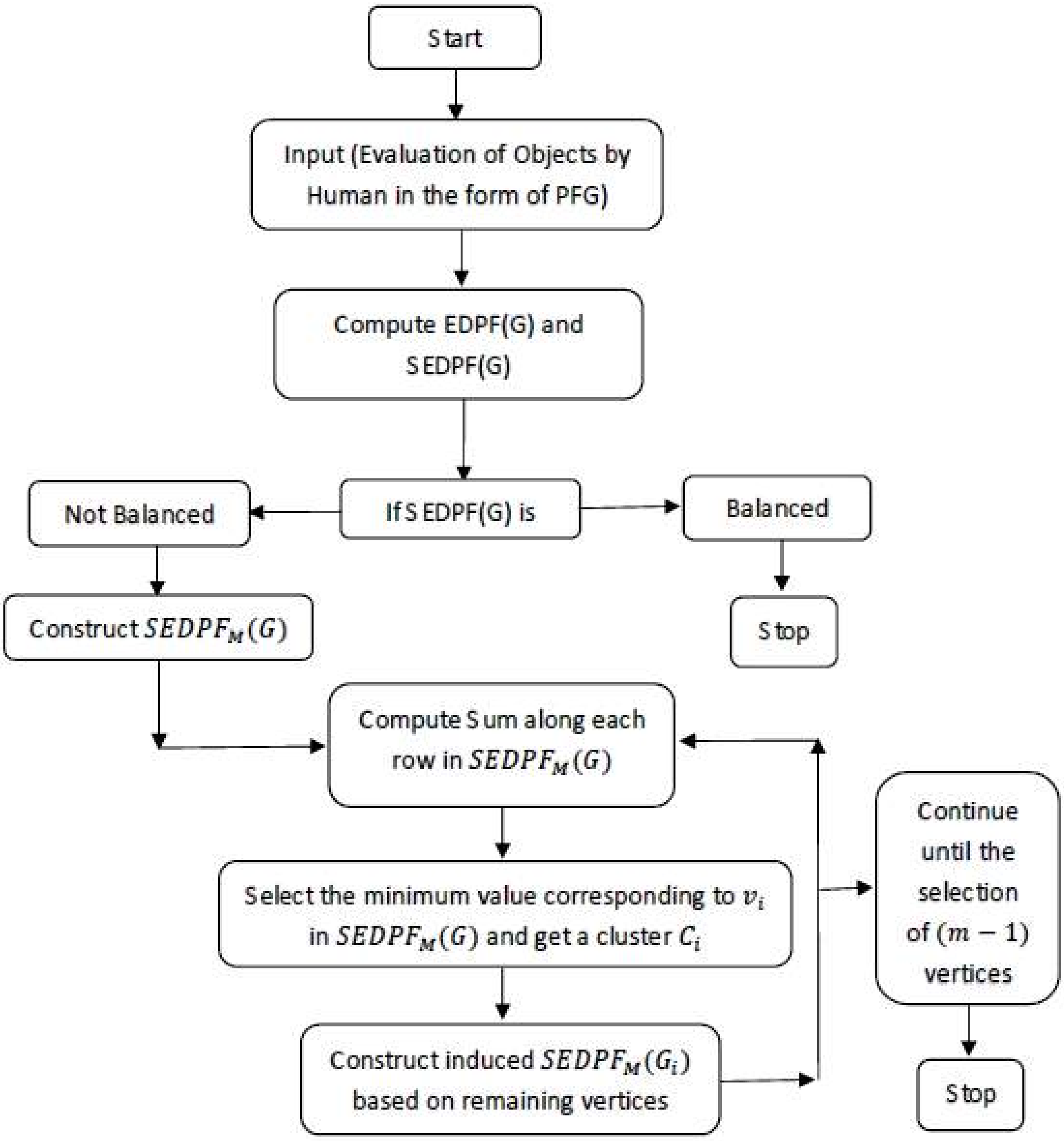


Figure 36 (Flow chart of picture fuzzy clustering algorithm)

Now we illustrate the proposed algorithm with the help of a numerical example where we assumed a picture fuzzy decision graph based on the demonstrated theory.

3.3.4 Numerical Example:

Consider a PFG having five vertices and each vertex is connected with the other. The five vertices represent five different attributes among which a group of human needs to decide which of them are substantial for assigning a good brand or object. We use the approach of clustering algorithm for PFGs to solve the problem of determining best attributes among the list provided. In our case, the five vertices t_1, t_2, t_3, t_4 and t_5 represents the attributes *Quality, Service, Price, Technology* and *Advertisement* respectively. The picture fuzzy clustering algorithm is demonstrated stepwise below.

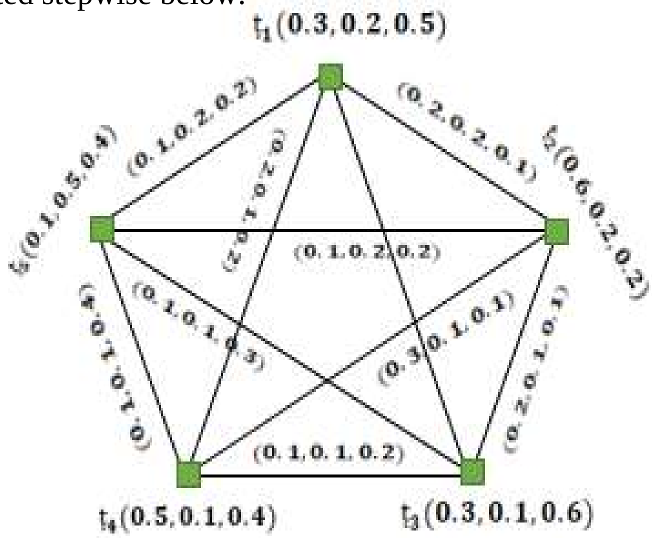


Figure 37 (Picture fuzzy graph based on information of human opinion)

Step 1. First, we used Eq. (1) to obtain an $EDPF_M(\dot{G})$.

$$ED_M(\dot{G}) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \\ \end{array}$$

$$\begin{bmatrix} (0,0,1) & (0.167,0.29,0.039) & (0.167,0.14,0.12) & (0.167,0.14,0.078) & (0.083,0.29,0.078) \\ (0.167,0.29,0.039) & (0,0,1) & (0.167,0.14,0.039) & (0.25,0.14,0.039) & (0.083,0.29,0.078) \\ (0.167,0.14,0.12) & (0.167,0.14,0.039) & (0,0,1) & (0.083,0.14,0.078) & (0.083,0.14,0.12) \\ (0.167,0.14,0.078) & (0.25,0.14,0.039) & (0.083,0.14,0.078) & (0,0,1) & (0.083,0.14,0.16) \\ (0.083,0.29,0.078) & (0.083,0.29,0.078) & (0.083,0.14,0.12) & (0.083,0.14,0.16) & (0,0,1) \end{bmatrix}$$

And based on $SEDPF(\dot{G})$ depicted in Figure 37, the $EDPF_M(\dot{G})$ is developed.

$$SEDPF_M(\dot{G}) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{array} \begin{array}{c} \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array} \\ \begin{bmatrix} 0 & 0.51 & 0.64 & 0.77 & 0.23 \\ 0.51 & 0 & 0.93 & 1.40 & 0.23 \\ 0.64 & 0.93 & 0 & 0.38 & 0.022 \\ 0.77 & 1.40 & 0.38 & 0 & 0.28 \\ 0.23 & 0.23 & 0.022 & 0.28 & 0 \end{bmatrix} \end{array}$$

Step 2. The sum of every row is calculated and listed in Table 12.

v_1	v_2	v_3	v_4	v_5
2.15	3.07	1.972	2.83	0.762

Table 12 (Sum of values in SEDPF matrix)

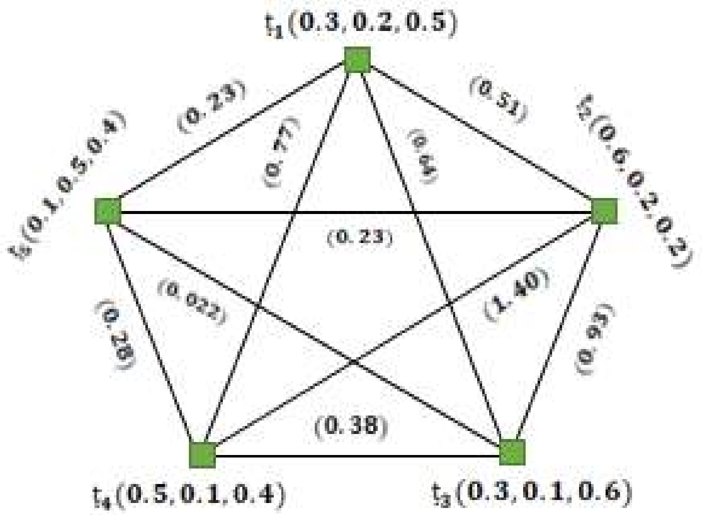


Figure 38 (single-valued edge density picture fuzzy graph)

Based on calculations in Table 12, the least value occurs for row 5. Hence, we have the first cluster $C_1 = (\{t_5\}, \{t_1, t_2, t_3, t_4\})$. Now the $SEDPF_M(\dot{G})$ induced by the remaining vertices is denoted by $SEDPF_M(\dot{G}_1)$ and is given by:

$$SEDPF_M(\dot{G}_1) = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{array}{c} \begin{array}{cccc} t_1 & t_2 & t_3 & t_4 \end{array} \\ \begin{bmatrix} 0 & 0.51 & 0.64 & 0.77 \\ 0.51 & 0 & 0.93 & 1.40 \\ 0.64 & 0.93 & 0 & 0.38 \\ 0.77 & 1.40 & 0.38 & 0 \end{bmatrix} \end{array}$$

Proceeding similarly, we have

t_1	t_2	t_3	t_4
1.92	2.84	1.95	2.55

Table 13 (Sum of values in SEDPF matrix)

Based on calculations in Table 13, the least value occurs for row 1. Hence, we have the second cluster$C_2 = (\{t_1\}, \{t_2, t_3, t_4\})$. Now the $SEDPF_M(\dot{G})$ induced by the remaining vertices is denoted by $SEDPF_M(\dot{G}_2)$ and is given by:

$$SEDPF_M(\dot{G}_2) = \begin{array}{c} \\ t_2 \\ t_3 \\ t_4 \end{array} \begin{bmatrix} t_2 & t_3 & t_4 \\ 0 & 0.93 & 1.40 \\ 0.93 & 0 & 0.38 \\ 1.40 & 0.38 & 0 \end{bmatrix}$$

Proceeding similarly, we have

t_2	t_3	t_4
2.33	1.38	1.78

Table 14 (Sum of values in SEDPF matrix)

Based on calculations in Table 14, the least value occurs for row 3. Hence, we have the third cluster $C_3 = (\{t_3\}, \{t_2, t_4\})$. Now the $SEDPF_M(\dot{G})$ induced by the remaining vertices is denoted by $SEDPF_M(\dot{G}_3)$ and is given by:

$$SEDPF_M(\dot{G}_3) = \begin{array}{c} t_2 \\ t_4 \end{array} \begin{bmatrix} t_2 & t_4 \\ 0 & 0.38 \\ 0.38 & 0 \end{bmatrix}$$

Proceeding similarly, we have $C_4 = (\{t_2\},\{t_4\})$ and

$\{(\{t_5\},\{t_1,t_2,t_3,t_4\}),(\{t_1\},\{t_2,t_3,t_4\}),(\{t_3\},\ \{t_2,t_4\}),(\{t_2\},\{t_4\})\}$is the required narrow slicing. It is clear that $SEDPF_M(\dot{G}_3)$ contains $C_{24} = 0.38$. Hence, the parameters *service* and *Technology* corresponding to t_2 and t_4 respectively are the factors that influenced the consumer decision for a brand.

Now we moved towards step 3 where τ-edge components of $SEDPF(\dot{G})$ are evaluated.

Step 3. For various values ofτ, the τ-edge components of $SEDPF_M(\dot{G})$are shown in Table 15.

τ	τ **−edge components**
$(1.4,\infty]$	$(\{t_1\},\{t_2\},\{t_3\},\{t_4\},\{t_5\})$
$(0.93,1.4]$	$(\{t_2,t_4\},\{t_1\},\{t_3\},\{t_5\})$
$(0.77,0.93]$	$(\{t_2,t_3,t_4\},\{t_1\},\{t_5\})$
$(0.64,0.77]$	$(\{t_1,t_2,t_3,t_4\},\{t_5\})$
$(0,0.64]$	$(\{t_1,t_2,t_3,t_4,t_5\})$

Table 15 (τ-edge components of T-spherical fuzzy graph)

3.4 Advantages of Picture Fuzzy Clustering over Fuzzy and Intuitionistic Fuzzy Clustering

It is already discussed in Section 3.3.2 that a PFG handle model the human opinion with the help of four membership functions denoting membership, abstinence, non-membership and refusal degree. On the other hand, and IFG uses two kinds of values to describe an imprecise event while a FG has only one kind of membership grade. We refer to the PFG depicted in Figure 37. If the abstinence values are dropped, the IFG used in Section 3.2.4 is obtained. Similarly, if we dropped the values of non-membership and abstinence grades both, the FG used in Section 3.2.2 is obtained. Now we analysed the results obtained in case of FG, IFG and PFG to show the superiority of PFG over IFG and FG.

Clustering results using FG obtained in Section 3.2.2 are:

$$\left\{\begin{matrix}(\{t_1\},\{t_2,t_3,t_4,t_5\}),\\ (\{t_2\},\{t_3,t_4,t_5\}),\\ (\{t_3\},\{t_4\},\{t_5\})\end{matrix}\right\}$$

Clustering Results using IFG obtained in Section 3.2.4 are:

$$\left\{\begin{matrix}(\{t_2\},\{t_1,t_3,t_4,t_5\}),\\ (\{t_1\},\{t_3,t_4,t_5\}),\\ (\{t_3\},\{t_4,t_5\}),\\ (\{t_4\},\{t_5\})\end{matrix}\right\}$$

Clustering Results using PFG obtained in Section 3.3.2 are:

$$\left\{\begin{matrix}(\{t_5\},\{t_1,t_2,t_3,t_4\}),\\ (\{t_1\},\{t_2,t_3,t_4\}),\\ (\{t_3\},\{t_2,t_4\}),\\ (\{t_2\},\{t_4\})\end{matrix}\right\}$$

The results clearly show that results obtained using IFG are improved than FG while results obtained using PFGs are improved than FG and IFG. Hence the clustering approach in the environment of PFG is relatively effective than FG and IFGs. Further, information provided in both fuzzy environment and intuitionistic fuzzy environment can be processed using the concept of PFG but not conversely.

3.5 Conclusion

In this chapter, a novel clustering approach in the environment of PFG is proposed due to the shortcomings existed in the clustering algorithms of FGs and IFGs. We have briefly discussed some notions and terms related to IFGs and have explained them with examples. We have studied the drawbacks of intuitionistic fuzzy clustering algorithm and have provided a solution by developing the algorithm to the environment of PFGs. To illustrate the new algorithm, an example is provided where a decision-making problem is solved using picture fuzzy clustering technique. To show the viability of the new picture fuzzy algorithm, it is explained that under certain conditions the new algorithm can be used to solve the problems that lies in the environment of FGs and IFGs. PFG is a useful generalization of FGs and IFG. In the near future, we plan to study some other decision-making problems in the environment of PFGs. The concept of PFG can also be used networking problems and shortest path problems.

Chapter 4

Analysis of Social Networks and Wi-Fi Networks by Using the Concept of Picture Fuzzy Graphs

In this chapter, we analysed a social network and a wife network using the concept of PFG. For this purpose, the concept of PFG is proposed and some basic terms are demonstrated including complement, degree and bridges. The main advantage of the proposed PFG is that it describes the uncertainty in any real-life events with the help of four membership degrees where the traditional FG and IFG fails to be applied. The viability of PFG is shown by utilizing the concept in demonstrating two real-life problems including a social network and a Wi-Fi-network. A comparison of PFG with existing notions is established showing its superiority over the existing frameworks.

4.1 Picture Fuzzy Graph

In this section, the concept of PFG is introduced and several related results are discussed. Further, some basic terms of PFGs are demonstrated including complement, degree and bridges.

4.1.1 Definition

A pair $\dot{\mathrm{G}} = (\underset{.}{\mathrm{V}}, \dot{\acute{\mathrm{E}}})$ is known as PFG if

(i) $\underset{.}{\mathrm{V}} = \{\underset{.}{\mathrm{t}}_1, \underset{.}{\mathrm{t}}_2, \dots, \underset{.}{\mathrm{t}}_n\}$ such that $\hat{\mathrm{S}}_1 : \underset{.}{\mathrm{V}} \longrightarrow [0,1], \hat{\mathrm{I}}_1 : \underset{.}{\mathrm{V}} \longrightarrow [0,1]$ and $\text{Đ}_1 : \underset{.}{\mathrm{V}} \longrightarrow [0,1]$ represents the degrees of truth membership, abstinence membership and false membership of the element $\underset{.}{\mathrm{t}}_i \in \underset{.}{\mathrm{V}}$ respectively with a condition $0 \leq \hat{\mathrm{S}}_1 + \hat{\mathrm{I}}_1 + \text{Đ}_1 \leq 1$

For any $\underset{.}{\mathrm{t}}_i \in \underset{.}{\mathrm{V}}, (\, i \in I)$.

(ii) $\dot{\mathrm{E}} \subseteq \mathrm{V} \times \mathrm{V}$ where $\hat{\mathrm{S}}_2 : \mathrm{V} \times \mathrm{V} \longrightarrow [0,1], \hat{\mathrm{I}}_1 : \mathrm{V} \times \mathrm{V} \longrightarrow [0,1]$ and $\mathrm{Đ}_2 : \mathrm{V} \times \mathrm{V} \longrightarrow [0,1]$ such that $\hat{\mathrm{S}}_2(\mathrm{t}_i, \mathrm{t}_j) \leq \min[\hat{\mathrm{S}}_1(\mathrm{t}_i), \hat{\mathrm{S}}_1(\mathrm{t}_j)]$, $\hat{\mathrm{I}}_2(\mathrm{t}_i, \mathrm{t}_j) \leq \min[\ \hat{\mathrm{I}}_1(\mathrm{t}_i),\ \hat{\mathrm{I}}_1(\mathrm{t})]$ and $\mathrm{Đ}_2(\mathrm{t}_i, \mathrm{t}_j) \leq \max[\ \mathrm{Đ}_1(\mathrm{t}_i),\ \mathrm{Đ}_1(\mathrm{t}_j)]$ with a condition that $0 \leq \hat{\mathrm{S}}_2(\mathrm{t}_i, \mathrm{t}_j) + \hat{\mathrm{I}}_2(\mathrm{t}_i, \mathrm{t}_j) + \mathrm{Đ}_2(\mathrm{t}_i, \mathrm{t}_j) \leq 1$ for any $(\mathrm{t}_i, \mathrm{t}_j) \in \dot{\mathrm{E}}$, $(\ i \in I)$. Moreover, $1 - (\hat{\mathrm{S}}_{1i} + \hat{\mathrm{I}}_{1i} + \mathrm{Đ}_{1i})$ represent refusal degree.

The amount of truth membership, abstinence and false membership of the vertex t_i represented by $(\mathrm{t}_i, \hat{\mathrm{S}}_{1i}, \hat{\mathrm{I}}_{1i}, \mathrm{Đ}_{1i})$ and the degree of truth membership, abstinence and false membership of the edge relation $e_{ij} = (\mathrm{t}_i, \mathrm{t}_j)$ on $\dot{\mathrm{E}}$ represented by $(e_{ij}, \hat{\mathrm{S}}_{2ij}, \hat{\mathrm{I}}_{2ij}, \mathrm{Đ}_{2ij})$.

4.1.2 Example

Consider a graph $\dot{\mathrm{G}} = (\mathrm{V}, \dot{\mathrm{E}})$ where $\mathrm{V} = \{\mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3, \mathrm{t}_4\}$ be the set of vertices and $\dot{\mathrm{E}} = \{\mathrm{t}_1\mathrm{t}_2, \mathrm{t}_2\mathrm{t}_3, \mathrm{t}_2\mathrm{t}_4, \mathrm{t}_3\mathrm{t}_4, \mathrm{t}_4\mathrm{t}_1\}$ be the set of edges. Then the following Figure 39 is an example of PFG.

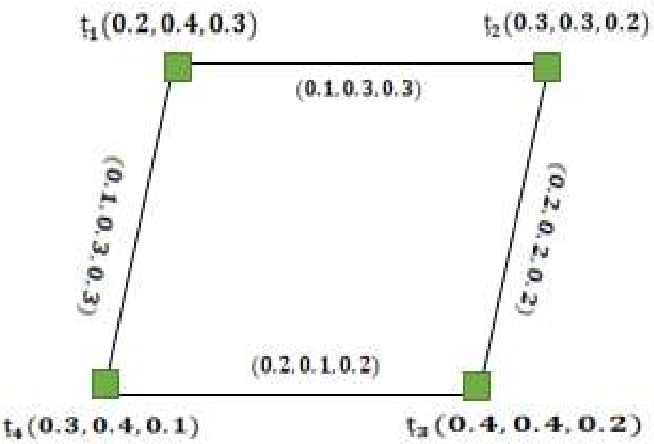


Figure 39 (Picture fuzzy graph)

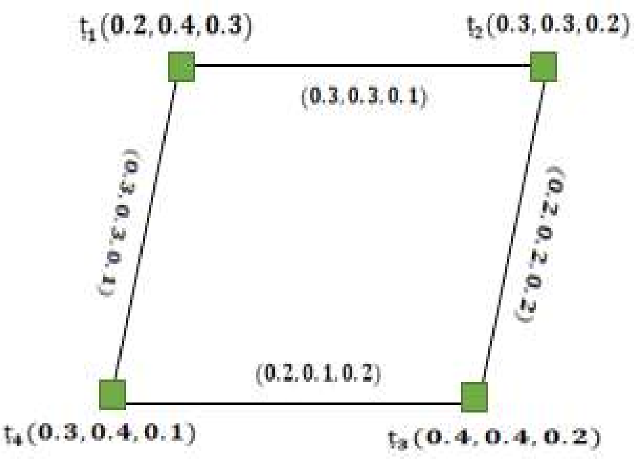


Figure 40 (Not a picture fuzzy graph)

4.1.3 Definition

A pair $H = (\text{Ṿ}', \text{Ė}')$ is called picture fuzzy subgraph (PFSG) of the PFG $\dot{G} = (\text{Ṿ}, \text{Ė})$ if $\text{Ṿ}' \subseteq \text{Ṿ}$ and $\text{Ė}' \subseteq \text{Ė}$ or if $\hat{S}_{1i}{}' \le \hat{S}_{1i}, \hat{I}_{1i}{}' \le \hat{I}_{1i}, Đ_{1i}{}' \ge Đ_{1i}$ and $\hat{S}_{2i}{}' \le \hat{S}_{2i}, \hat{I}_{2i}{}' \le \hat{I}_{2i}$,$Đ_{2i}{}' \ge Đ_{2i}$ for any $i \in$ I.

4.1.4 Definition

The score function of a PFG is defined by $P_s = 1 - r.Đ_1$ where $Đ_1$ represents the false membership and $r = 1 - (\hat{S}_1 + \hat{I}_1 + Đ_1)$ represent the refusal degree in PFG.

4.1.5 Definition

An arrangement of distinct vertices $ṭ_1, ṭ_2, \ldots, ṭ_n$ is called a path in PFG if one of the following conditions is satisfied:

4. $\hat{S}_{2ij} > 0, \hat{I}_{2ij} > 0$ and $Đ_{2ij} = 0$
5. $\hat{S}_{2ij} = 0, \hat{I}_{2ij} = 0$ and $Đ_{2ij} > 0$
6. $\hat{S}_{2ij} > 0, \hat{I}_{2ij} > 0$ and $Đ_{2ij} > 0$.

4.1.6 Example

The following Figure 41 is an example of PFG for a path which is explain below.

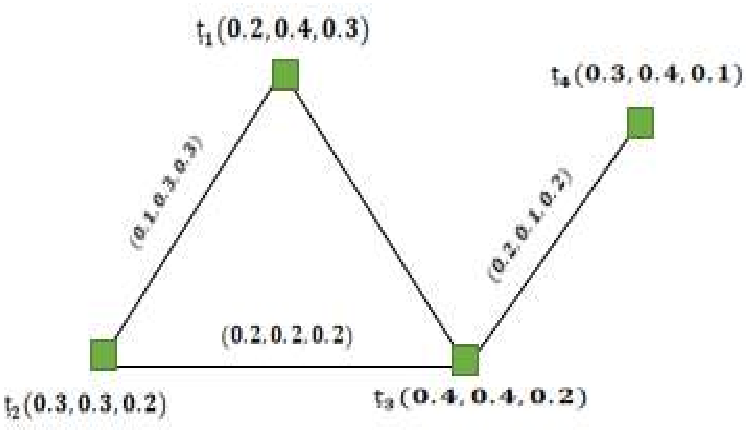


Figure 41 (Picture fuzzy graph)

In the above Figure t_1t_3, t_3t_2 is a path.

4.1.7 Definition

An open walk in no which vertex appear more than one is called path on a graph.

4.1.8 Definition

If $P = t_1t_2, \dots, t_{n+1}$ and $t_1 = t_{n+1}$ for $n \geq 3$, then it is called cycle.

4.1.9 Definition

A path is called connected if two vertices are joined trough this path.

4.1.10 Definition

For a path P

1. The $\hat{S} - strength$ is $S_{\hat{S}} = \min_{i,j} \{ \hat{S}_{2ij}\}$
2. The $\hat{I} - strength$ is $S_{\hat{I}} = \min_{i,j} \{ \hat{I}_{2ij}\}$

3. The $Đ - strength$ is $S_{Đ} = \max_{i,j} \{ Đ_{2ij}\}$

4. The strength of the path $S_P = (S_{\hat{S}}, S_{\hat{I}}, S_{Đ})$.

4.1.11 Definition

If $\dot{G} = (\underset{.}{V}, \dot{E})$ be a PFG. Then the degree of any vertex $\underset{.}{t}$ is denoted and defined by

$deg(\underset{.}{t}) = \left(d_{\hat{S}}(\underset{.}{t}), d_{\hat{I}}(\underset{.}{t}), d_{Đ}(\underset{.}{t})\right)$ where $deg_{\hat{S}}(\underset{.}{t}) = \sum_{\underset{.}{t} \neq u} \hat{S}_2(\underset{.}{t}, u)$, $deg_{Đ}(\underset{.}{t}) = \sum_{\underset{.}{t} \neq u} Đ_2(\underset{.}{t}, u)$ and $deg_{\hat{I}}(\underset{.}{t}) = \sum_{\underset{.}{t} \neq u} \hat{I}_2(\underset{.}{t}, u)$.

4.1.12 Example

Consider a graph $\dot{G} = (\underset{.}{V}, \dot{E})$ where $\underset{.}{V} = \{\underset{.}{t}_1, \underset{.}{t}_2, \underset{.}{t}_3, \underset{.}{t}_4\}$ be the set of vertices and $\dot{E} = \{\underset{.}{t}_1\underset{.}{t}_2, \underset{.}{t}_2\underset{.}{t}_3, \underset{.}{t}_3\underset{.}{t}_4, \underset{.}{t}_4\underset{.}{t}_1\}$ be the set of edges. Then PFG and the degree of its vertices is given below.

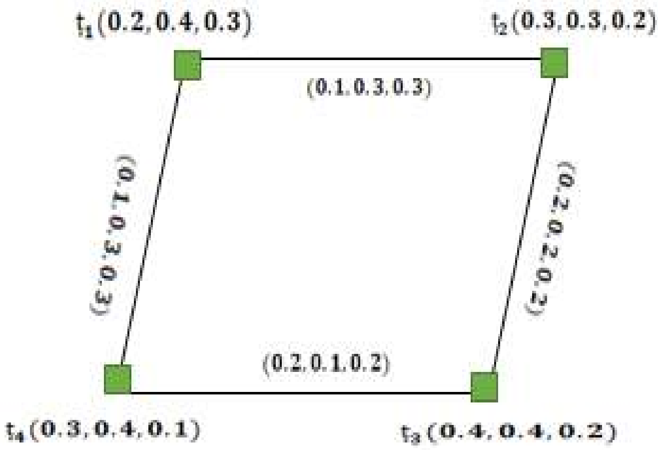


Figure 42 (Picture fuzzy graph)

The degree of vertices are:

$d(\underset{.}{t}_1) = (0.2, 0.6, 0.6)$, $d(\underset{.}{t}_2) = (0.3, 0.5, 0.5)$, $d(\underset{.}{t}_3) = (0.4, 0.3, 0.4)$, $d(\underset{.}{t}_4) = (0.3, 0.4, 0.5)$.

4.1.13 Proposition

A PFG is a generalization of IFG.

Proof. This is obvious as by taking abstinence equal to zero PFG reduces to IFG.

4.1.14 Definition

The complement $\dot{G}^c$ of PFG $\dot{G} = (\underset{\cdot}{V}, \dot{E})$ is

4. $\hat{S}_A(t_i)^c = \hat{S}_A(t_i), \hat{I}_A(t_i)^c = \hat{I}_A(t_i), Đ_A(t_i)^c = Đ_A(t_i), \forall\ t_i \in V$.
5. $\hat{S}_B(t_i, t_j)^c = \min[\hat{S}_B(t_i), \hat{S}_B(t_j)] - \hat{S}_B(t_i, t_j), \hat{I}_B(t_i, t_j)^c = \min[\ \hat{I}_2(t_i),\ \hat{I}_2(t_j)] - \hat{I}_B(t_i, t_j)$ and $Đ_2(t_i, t_j)^c = \max[\ Đ_2(t_i),\ Đ_2(t_j)] - Đ_2(t_i, t_j)\ \forall\ t_i, t_j \in \dot{E}$.

4.1.15 Remark

If $\dot{G} = (\underset{\cdot}{V}, \dot{E})$ is a PFG, then definition 4.1.14 implies that $\dot{G}^{c^c} = (\underset{\cdot}{V}^{c^c}, \dot{E}^{c^c}) = \dot{G}$.

4.1.16 Proposition

$\dot{G} = \dot{G}^{c^c}$ iff $\dot{G}$ is a strong PFG.

Proof. This result is obvious by the definition of $\dot{G}^c$.

The following figures provided a verification of proposition 4.1.16.

4.1.17 Example

Consider a graph $\dot{G} = (\underset{\cdot}{V}, \dot{E})$ where $\underset{\cdot}{V} = \{t_1, t_2, t_3, t_4\}$ be the set of vertices and $\dot{E} = \{t_1t_2, t_2t_3, t_3t_4, t_4t_1\}$ be the set of edges. Then, PFG and its complement is given below.

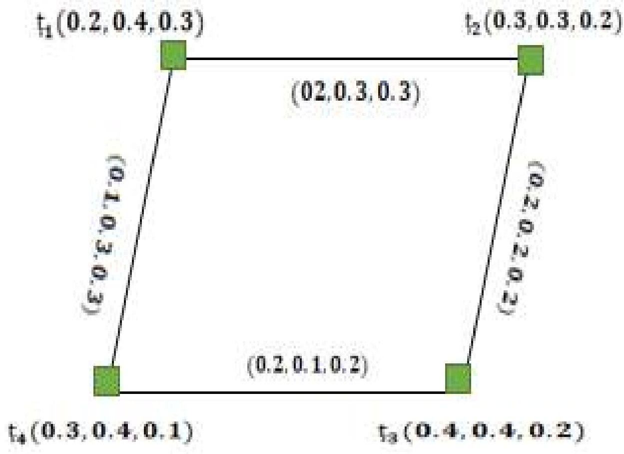


Figure 43 (Picture fuzzy graph)

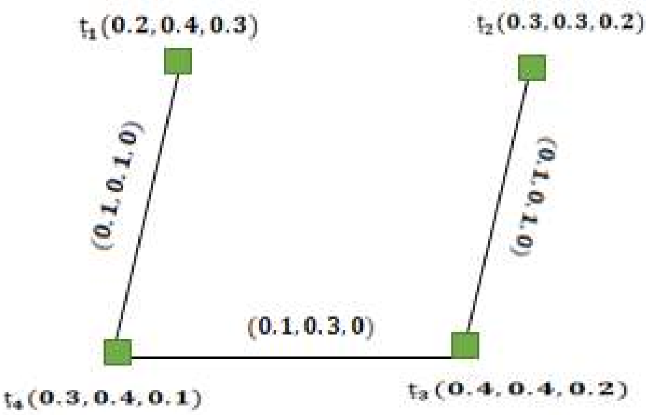


Figure 44 (Complement of Figure 43)

4.1.18 Definition

A PFG $\dot{\mathrm{G}}$ is called self-complementary If $\dot{\mathrm{G}} = \dot{\mathrm{G}}^{c^c}$

4.1.19 Example

From the above Figure 43 and Figure 44, it is clear that $\dot{\mathrm{G}}$ is self -complementary.

4.1.20 Definition

The composition of two edge relations $(e_{ij}, \hat{\mathrm{S}}_{2ij}, \hat{\mathrm{I}}_{2ij}, \mathrm{Đ}_{2ij})$ and $(e_{jk}, \hat{\mathrm{S}}_{2jk}, \hat{\mathrm{I}}_{2jk}, \mathrm{Đ}_{2jk})$ is a PFG represented by $e_{ij}{}^{\circ}e_{jk}$ is of the form $(e_{jk}, \hat{\mathrm{S}}_{2jk}, \hat{\mathrm{I}}_{2jk}, \mathrm{Đ}_{2jk})$ where

$\hat{S}_{2ik} = \max\{^{min}_{j}[\hat{S}_{2ij}, \hat{S}_{2jk}]\}, \hat{I}_{2ik} = \max\{^{min}_{j}[\hat{I}_{2ij}, \hat{I}_{2jk}]$

and $Đ_{2ik}$ $=\min\{^{max}_{j}[Đ_{2ij}, Đ_{2jk}]\}$ $\forall$ $ṭ_i, ṭ_k \in Ṿ$.

4.1.21 Definition

An edge relation $(e_{ij}, \hat{S}_{2ij}, \hat{I}_{2ij}, Đ_{2ij})$ is

1. Reflexive if $(e_{ii}, \hat{S}_{2ii}, \hat{I}_{2ii}, Đ_{2ii}) = (ṭ_i, \hat{S}_{1i}, \hat{I}_{1i}, Đ_{1i})$.
2. Symmetric if $(e_{ij}, \hat{S}_{2ij}, \hat{I}_{2ij}, Đ_{2ij}) = (e_{ji}, \hat{S}_{2ji}, \hat{I}_{2ji}, Đ_{2ji})$.
3. Transitive if $(ṭ_i, ṭ_j)$ and $(ṭ_j, ṭ_k)$ implies the edge relation $(ṭ_i, ṭ_k)$ $\forall$ $ṭ_i, ṭ_j, ṭ_k \in Ṿ$.

4.1.22 Definition

If e_{ij} is the edge relation then powers of e_{ij} is defined as:

$$e_{ij}^{1} = e_{ij} = (e_{ij}, \hat{S}_{2ij}, \hat{I}_{2ij}, Đ_{2ij})$$

$$e_{ij}^{2} = e_{ij} \circ e_{ij} = (e_{ij}, \hat{S}_{2ij}{}^{2}, \hat{I}_{2ij}{}^{2}, Đ_{2ij}{}^{2})$$

$$e_{ij}^{3} = e_{ij} \circ e_{ij} \circ e_{ij} = (e_{ij}, \hat{S}_{2ij}{}^{3}, \hat{I}_{2ij}{}^{3}, Đ_{2ij}{}^{3})$$ and so on.

Also

$$e_{ij}^{\infty} = e_{ij} = (e_{ij}, \hat{S}_{2ij}{}^{\infty}, \hat{I}_{2ij}{}^{\infty}, Đ_{2ij}{}^{\infty})$$

Where $\hat{S}_{2ij}{}^{\infty} = \max_{k=1,2,\ldots n}\{\hat{S}_{2ij}{}^{k}\}$, $\hat{I}_{2ij}{}^{\infty} = \max_{k=1,2,\ldots n}\{\hat{I}_{2ij}{}^{k}\}$, and $Đ_{2ij}{}^{\infty} = \min_{k=1,2,\ldots n}\{Đ_{2ij}{}^{k}\}$, are the $\hat{S} - strength$, $\hat{I} - strength$ and $Đ - strength$ of connectedness between only two vertices $ṭ_i$ and $ṭ_j$.

Also

$$e_{ij}^{0} = \{0, if\ ṭ_i \neq ṭ_j\ and\ (ṭ_i, \hat{S}_{1i}, \hat{I}_{1i}, Đ_{1i})\ if\ ṭ_i \text{ is equal to } ṭ_j\} .$$

The following Theorem 4.1.23 signifies the relation of subgraph and a graph in context of the Definition 4.1.22.

4.1.23 Theorem

If $H = (Ṿ', Ẻ')$ is a PFSG of PFG $\dot{G} = (Ṿ, Ẻ)$. Then $\hat{S}_{2iJ}{}'^{\infty} \leq \hat{S}_{2ij}{}^{\infty}$, $\hat{I}_{2iJ}{}'^{\infty} \leq \hat{I}_{2ij}{}^{\infty}$ and $Đ_{2iJ}{}'^{\infty} \geq Đ_{2ij}{}^{\infty}$ for any (ṭ$_i$, ṭ) ∈ Ẻ.

Proof: As $Ṿ' \subseteq Ṿ$ *and* $Ẻ' \subseteq Ẻ$.

$\Longrightarrow \hat{S}_{1i}{}' \leq \hat{S}_{1i}$, $\hat{I}_{1i}{}' \leq \hat{I}_{1i}$, $Đ_{1i}{}' \geq Đ_{1i}$ for every ṭ$_i$ ∈ Ṿ

and

$$\hat{S}_{2ij}{}' \leq \hat{S}_{2ij}$$

$$\hat{I}_{2ij}{}' \leq \hat{I}_{2ij}$$

$$Đ_{2ij}{}' \geq Đ_{2ij}$$

For every ṭ$_i$, ṭ$_j$ ∈ Ṿ.

Suppose a path ṭ$_1$ṭ$_2$, ... , ṭ$_n$ of H.

Here $\hat{S}_{2ij}{}'^{\infty} = \underset{k=1,2,\ldots n}{min}\{(\hat{S}_{2ij}{}')^{k}\}$

$$\hat{I}_{2ij}{}'^{\infty} = \underset{k=1,2,\ldots n}{min}\{(\hat{I}_{2ij}{}')^{k}\}$$

$$Đ_{2ij}{}'^{\infty} = \underset{k=1,2,\ldots n}{max}\{(Đ_{2ij}{}')^{k}\}$$

and

$$\hat{S}_{2ij}{}^{\infty} = \underset{k=1,2,\ldots n}{min}\{(\hat{S}_{2ij})^{k}\}$$

$$\hat{I}_{2ij}{}^{\infty} = \underset{k=1,2,\ldots n}{min}\{(\hat{I}_{2ij})^{k}\}$$

$$Đ_{2ij}{}^{\infty} = \underset{k=1,2,\dots n}{max}\{(Đ_{2ij})^k\}$$

Therefore,

$$\hat{S}_{2ij}{}'^{\infty} = \underset{k=1,2,\dots n}{min}\{(\hat{S}_{2ij}{}')^k\}$$

$$\leq \underset{k=1,2,\dots n}{min}\{(\hat{S}_{2ij})^k\}$$

$$= \hat{S}_{2ij}{}^{\infty}$$

$$\hat{I}_{2ij}{}'^{\infty} = \underset{k=1,2,\dots n}{min}\{(\hat{I}_{2ij}{}')^k\}$$

$$\leq \underset{k=1,2,\dots n}{min}\{(\hat{I}_{2ij})^k\}$$

$$= \hat{I}_{2ij}{}^{\infty}$$

also

$$Đ_{2ij}{}'^{\infty} = \underset{k=1,2,\dots n}{max}\{(Đ_{2ij}{}')^k\}$$

$$\geq \underset{k=1,2,\dots n}{max}\{(Đ_{2ij})^k\}, \text{ by (12)}$$

$$= Đ_{2ij}{}^{\infty}$$

Hence proved.

4.1.24 Definition

If $\dot{G} = (\underset{.}{V}, \dot{E})$ be a PFG and let $ṭ_i, ṭ_j$ be any two different vertices. If deleting an edge $(ṭ_i, ṭ_j)$ reduces the strength between some pair of vertices then it is called bridge in $\dot{G}$.

4.1.25 Example

In Figure 43 the strength of $(ṭ_1ṭ_2)$ is $(0.1, 0.3, 0.3)$. Here $(ṭ_1, ṭ_2)$ is a bridge, because if an edge $(ṭ_1, ṭ_2)$ remove from $\dot{G}$ in Figure 43 then the strength of the connectedness between $ṭ_1$ and $ṭ_2$ is reduced.

In the following Theorem 4.1.26 it is proved that if an edge is a bridge, then it can never be the part of a cycle and conversely.

4.1.26 Theorem

If $e_{ij} = \left(ṭ_i, ṭ_j\right)$ be an edge. Then for any two vertices in PFG $\dot{G} = \left(Ṿ, \dot{E}\right)$ the following are equivalent:

(i) e_{ij} is a bridge.

(ii) $\hat{S}_{2ij}{}'^{\infty} < \hat{S}_{2ij}, \hat{I}_{2ij}{}'^{\infty} < \hat{I}_{2ij}$and $Đ_{2ij}{}'^{\infty} > Đ_{2ij}$.

(iii) e_{ij} is not an edge of any cycle.

Proof. (ii)⟹(i). Suppose $\hat{S}_{2ij}{}'^{\infty} < \hat{S}_{2ij}, \hat{I}_{2ij}{}'^{\infty} < \hat{I}_{2ij}$and $Đ_{2ij}{}'^{\infty} > Đ_{2ij}$.To show that (i) is true. Suppose on contrary (i) is not true, then

$$\hat{S}_{2ij}{}'^{\infty} = \hat{S}_{2ij}{}^{\infty} \geq \hat{S}_{2ij}, \hat{I}_{2ij}{}'^{\infty} = \hat{I}_{2ij}{}^{\infty} \geq \hat{I}_{2ij} \text{ and } Đ_{2ij}{}'^{\infty} = Đ_{2ij}{}^{\infty} \leq Đ_{2ij}$$

$\Longrightarrow \hat{S}_{2ij}{}'^{\infty} \geq \hat{S}_{2ij}$, $\hat{I}_{2ij}{}'^{\infty} \geq \hat{I}_{2ij}$ and $Đ_{2ij}{}'^{\infty} \leq Đ_{2ij}$, leading to a contradiction. Hence (i) is true.(i)⟹(iii) now if (i) is true. To prove (iii). If e_{ij} is edge of a cycle, then including the edge e_{ij} to any path can be transformed into a path not including $(ṭ_i, ṭ_j)$. As cycle is a path from $ṭ_i$ to $ṭ_j \Rightarrow e_{ij}$ can not be a bridge which is contradiction so our supposition is wrong. Therefor, e_{ij} is not an edge of any cycle. (iii)⟹(ii) is straight forward.

4.1.27 Theorem

If $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ be a PFG. Then

(i) $\hat{\mathrm{S}}_{2ij}$, $\hat{\mathrm{I}}_{2ij}$ and $\mathrm{Đ}_{2ij}$ are constants then no bridge contains in $\dot{\mathrm{G}}$ $\forall$ $\underset{\cdot}{\mathrm{t}}_i, \underset{\cdot}{\mathrm{t}}_j \in \underset{\cdot}{\mathrm{V}}$..

(ii) $\hat{\mathrm{S}}_{2ij}$, $\hat{\mathrm{I}}_{2ij}$ and $\mathrm{Đ}_{2ij}$ are not constants then there is at least one bridge in $\dot{\mathrm{G}}$ for all $(\underset{\cdot}{\mathrm{t}}_i, \underset{\cdot}{\mathrm{t}}_j) \in \dot{\mathrm{E}}$.

Proof. Straight forward.

4.2 Applications of picture fuzzy graph

As studied, in a PFG, the nodes and edges are picture fuzzy numbers (PFNs) and hence described the uncertain information better than discussed in the environment of IFS and FS. Therefore in this section, applications of PFG in social network and Wi-Fi network system are discussed.

4.2.1 Application in social network

A social network basically established the relationship between a group of people or members of a society or organization. Whenever the networks are large, computing the strength of relationship between the members of that network is a challenging task.

Here, a social network of some peoples a a certain group is considered where the relationship of every member with other is defined in terms of a PFN. In such network, vertices represent the people and edges represent the relationship between them. Consider the representation of a social group and its members in the environment of a PGF. Let the given social network is a PFG denoted by $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$, where $\underset{\cdot}{\mathrm{V}} = \{Stone,\ Michal,\ Parera, Mathews, Raina, Julie, Watson. Hissy, Messy, Charly,\ Sting, Shaun\}$. Each vertex represents the social skills of a person in the form of a PFN. The larger the membership grade is the more social a person is and vice versa. In this network, if two peoples share some common characteristics including good personality, good behavior, best skills and social sympathy, then they are connected, and their relation is described in the form of a PFN. The set of all edges of this social network are given in Table 13 followed by the social network $\dot{G} = (\underset{\cdot}{V}, \dot{\grave{E}})$ in Figure 45.

Table 16 shows the relationship of the peoples connected to each other in terms of picture fuzzy numbers. Greater membership grade corresponds to more social values of a person.

Relation	Edges	Values
Raina-Charli	d_1	(0.2,0.1,0.4)
Parera-hissy	d_2	(0.3,0.2,0.3)
Missy-hissy	d_3	(0.3,0.3,0.2)
Shaun-Mathews	d_4	(0.2,0.2,0.3)
Sting-Shaun	d_5	(0.2,0.2,0.3)
Stone-Michal	d_6	(0.2,0.2,0.4)
Mathews-Parera	d_7	(0.2,0.1,0.3)
Mathews-Charli	d_8	(0.3,0.0,0.3)
Stone-sting	d_9	(0.2,0.2,0.5)
Shaun-Charli	d_{10}	(0.1,0.1,0.3)
Stone-Julie	d_{11}	(0.2,0.2,0.5)
Stone-Watson	d_{12}	(0.1,0.1,0.5)

Table 16 (Edge values of social network depicted in Figure 45)

The social network connecting different people is shown in Figure 45 where the edge values are in the form of picture fuzzy number showing their social skills. The larger membership grade shows that a person is more social. The larger non-membership grade shows the low sociality of a person in connection.

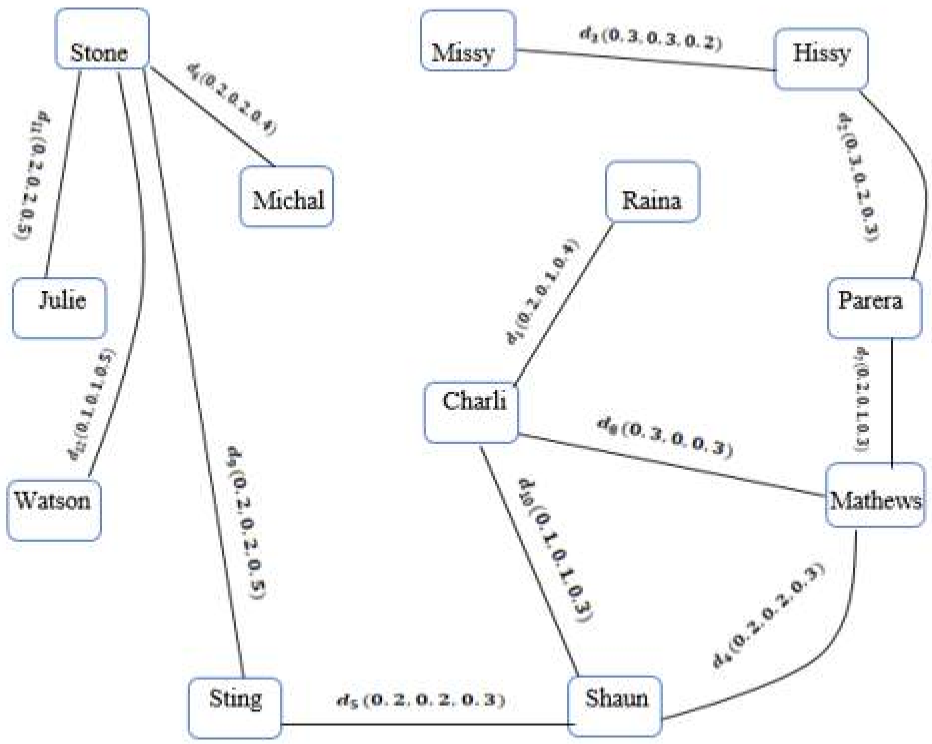


Figure 45 (Social network of a group of people)

In order to compute the strength of the socially connected members, the concept of degree of PFG is utilized. The degree of a node determines how much social a person is. For this purpose, first the degree of each node is computed in Table 17 and then based on the difference of membership and non-membership grade of the degree, the strength is decided.

In the Table 17 below, the degree of each person in social network is calculated. These degrees are utilized to compute the score values of each person in social network. The more is the degree of a person means the more social a person is.

Vertex(candidate)	Degree of vertex
Hissy	(0.6, 0.5, 0.5)
Charli	(0.6, 0.2, 0.1)
Raina	(0.2, 0.1, 0.4)
Parera	(0.5, 0.3, 0.6)
Michal	(0.2, 0.2, 0.4)
Stone	(0.7, 0.7, 1.9)
Mathews	(0.7, 0.3, 0.9)
Julie	(0.2, 0.2, 0.5)
Messy	(0.3, 0.3, 0.2)
Shaun	(0.5, 0.5, 0.9)
Sting	(0.4, 0.4, 0.8)

Table 17 (Degrees of vertices of Figure 45)

To compute the most social person, the difference of the membership and non-membership grade is calculated and shown in Table 18 below.

Vertex	Difference of the membership and non-membership grade
Hissy	0.1
Charli	0.5
Raina	-0.2
Parera	-0.1
Michal	-0.2
Stone	-1.2
Mathews	-0.2
Julie	-0.3
Messy	0.1
Shaun	-0.4
Sting	-0.4
Watson	-0.4

Table 18 (difference of the membership and non-membership grade of degrees given in Table 17)

Results in Table 18 clearly indicates that the difference value of the membership and non-membership grade of Charli is greatest among all values hence making him the most social person in this network. Hissy and Messy stands at No 2 with a difference value of 1. The rest of the ranking is shown in a bar graph depicted in Figure 46.

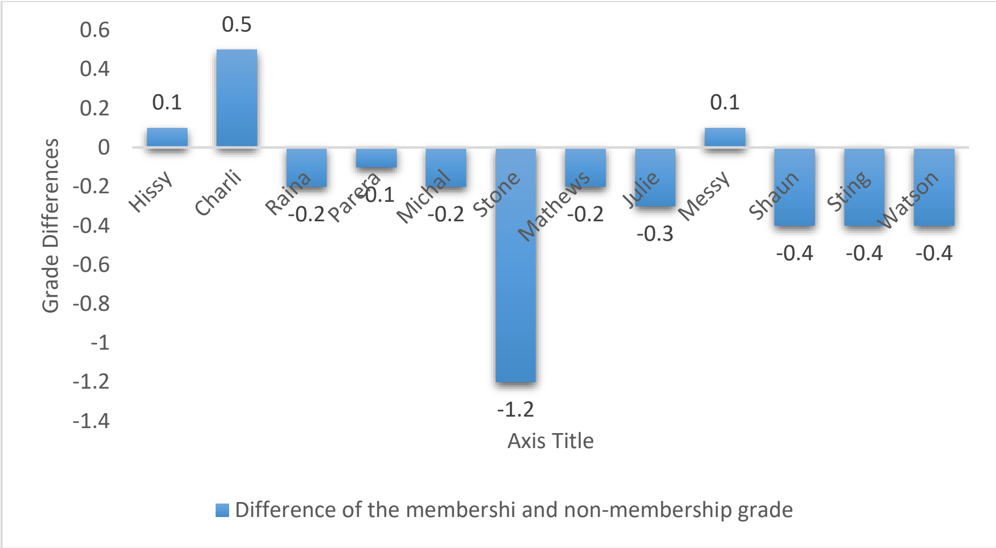


Figure 46 (Analysis of the difference of the membership and non-membership grade)

The observations in Table 15 are shown geometrically in Figure 46. Clearly Charlie has the greatest score value and is declared as most social person in this social network.

4.2.2 Application in Wi-Fi system

Consider a Wi-Fi system containing 12 devices $\{Ç_1, Ç_2, Ç_3, Ç_4, Ç_5, Ç_6, Ç_7, Ç_8, Ç_9, Ç_{10}, Ç_{11}, Ç_{12}\}$ each device connected to the main server ($Ç_0$). The given Wi-Fi system is described using IFG as well as PFG in Figure 47 and Figure 48. The membership function show the connectivity of the device with the main server ($Ç_0$) and the non-membership function shows the dis-connectivity of the device with the main server ($Ç_0$) in IFG.

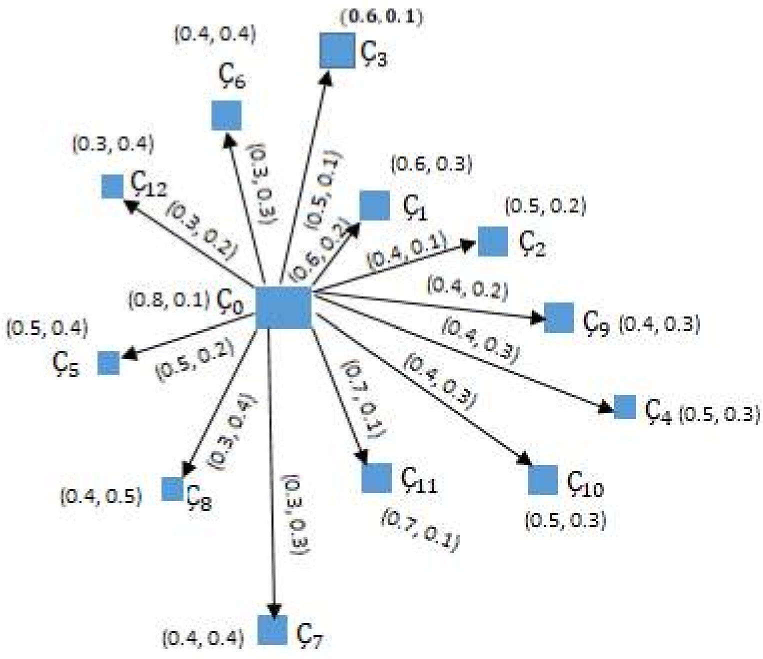


Figure 47(Intuitionistic fuzzy Wi-Fi-network)

Upon observing Figure 48, the connectivity of devices is discussed in terms of picture fuzzy numbers where the membership, abstinence and non-membership grades represent the connectivity, technical error and the dis-connectivity with the main device ($Ç_0$) respectively. Further, the refusal degree represents that the device is not connected at all.

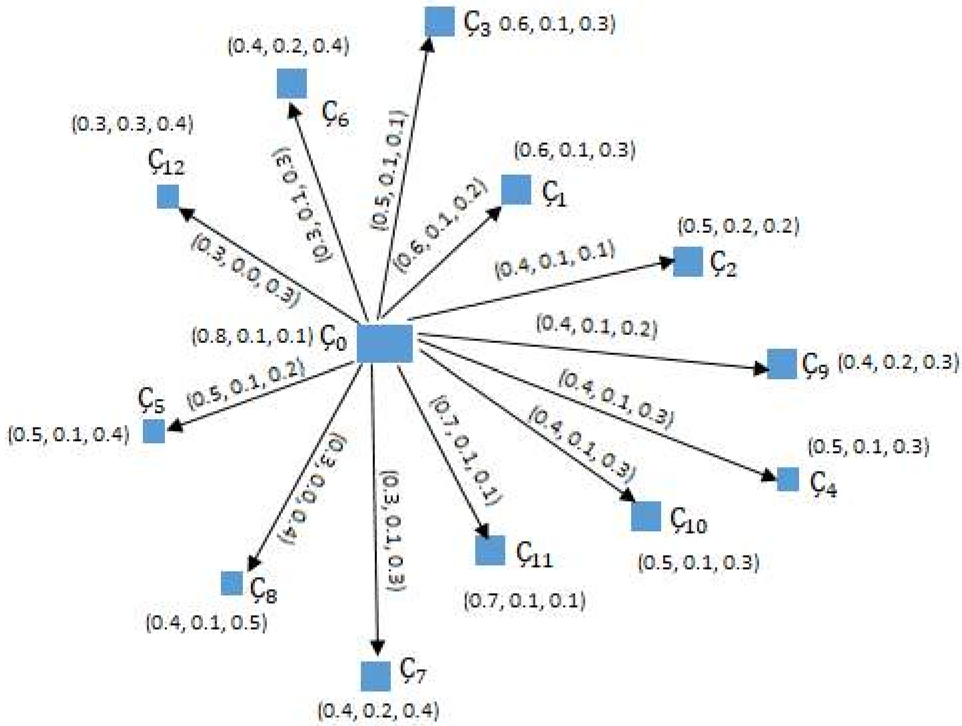


Figure 48(Picture fuzzy Wi-Fi network)

Now, the effective degrees of devices of the Wi-Fi network depicted in Figure 47 are computed and given in Table 19 followed by a bar-graph showing the strength of connectivity of all devices.

Device	Edge	score
$(Ç_0, Ç_1)$	(0.6,0.2)	0.56
$(Ç_0, Ç_2)$	(0.4,0.1)	0.05
$(Ç_0, Ç_3)$	(0.5,0.1)	0.46
$(Ç_0, Ç_4)$	(0.4,0.3)	0.31
$(Ç_0, Ç_5)$	(0.5,0.2)	0.44
$(Ç_0, Ç_6)$	(0.3,0.3)	0.18
$(Ç_0, Ç_7)$	(0.3,0.3)	0.18
$(Ç_0, Ç_8)$	(0.3,0.4)	0.18
$(Ç_0, Ç_9)$	(0.4,0.2)	0.32
$(Ç_0, Ç_{10})$	(0.4,0.3)	0.31
$(Ç_0, Ç_{11})$	(0.7,0.1)	0.68
$(Ç_0, Ç_{12})$	(0.3,0.2)	0.2

Table 19 (Score values of the Wi-Fi network based on intuitionistic fuzzy information)

The results of Table 19 suggest that the strength of connectivity of device $Ç_{11}$ is highest among all. This means that device $Ç_{11}$ gain maximum signals from the server. The connectivity strength of other devices can be observed from the bar-graph given in Figure 49 below.

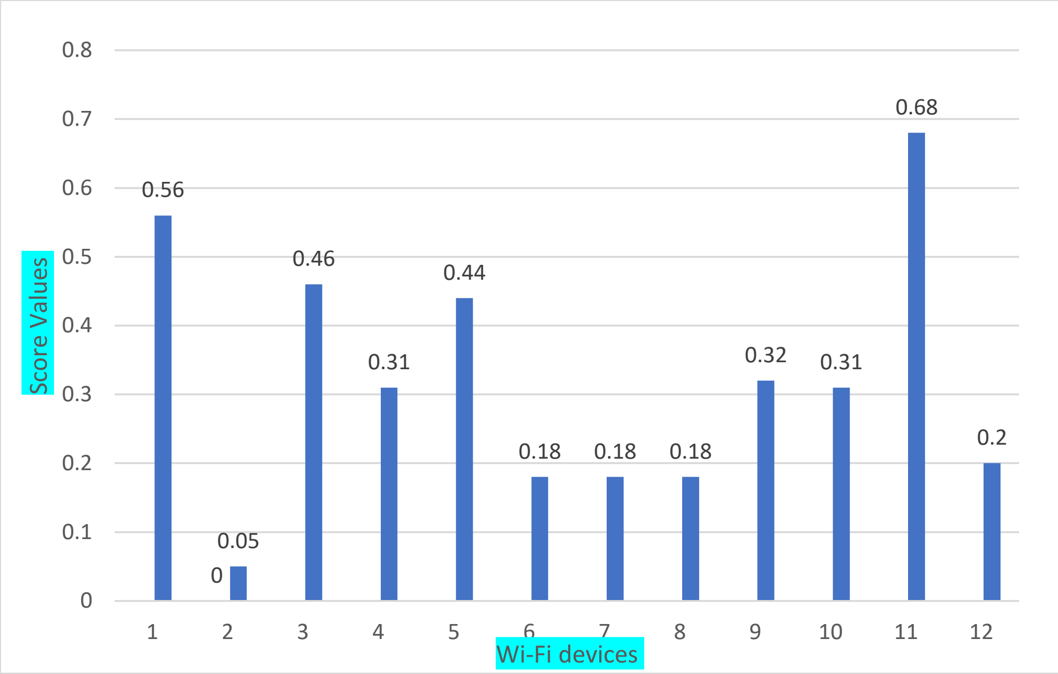


Figure 49 (Ranking of the signal strength of Wi-Fi devices of intuitionistic fuzzy W-Fi network)

Similarly, the effective degrees of devices of the Wi-Fi network depicted in Figure 49 based on picture fuzzy information are computed and given in Table 20 followed by a bar-graph showing the strength of connectivity of all devices.

Device	Edge	Score
$(\c{C}_0, \c{C}_1)$	(0.6,0.1,0.2)	0.58
$(\c{C}_0, \c{C}_2)$	(0.4,0.1,0.1)	0.36
$(\c{C}_0, \c{C}_3)$	(0.5,0.1,0.1)	0.47
$(\c{C}_0, \c{C}_4)$	(0.4,0.1,0.3)	0.34
$(\c{C}_0, \c{C}_5)$	(0.5,0.1,0.2)	0.46
$(\c{C}_0, \c{C}_6)$	(0.3,0.1,0.3)	0.21
$(\c{C}_0, \c{C}_7)$	(0.3,0.1,0.3)	0.21
$(\c{C}_0, \c{C}_8)$	(0.3,0.1,0.4)	0.22
$(\c{C}_0, \c{C}_9)$	(0.4,0.1,0.2)	0.34
$(\c{C}_0, \c{C}_{10})$	(0.4,0.1,0.3)	0.34
$(\c{C}_0, \c{C}_{11})$	(0.7,0.1,0.1)	0.69
$(\c{C}_0, \c{C}_{12})$	(0.3,0.1,0.2)	0.22

Table 20 (Score values of the Wi-Fi network based on picture fuzzy information)

The effective degrees of all devices of Table 20 shows the connectivity strength. The results displayed in Table 20 are more significant than that of provided in Table 19 because of the nature of information of picture fuzzy numbers. Here $\c{C}_{11}$ is the strongest connected device but the score value is different from that in intuitionistic fuzzy environment because of the abstinence grade involved. The ranking of connectivity strength of all devices in picture fuzzy environment is shown using a bar graph depicted in Figure 50 below.

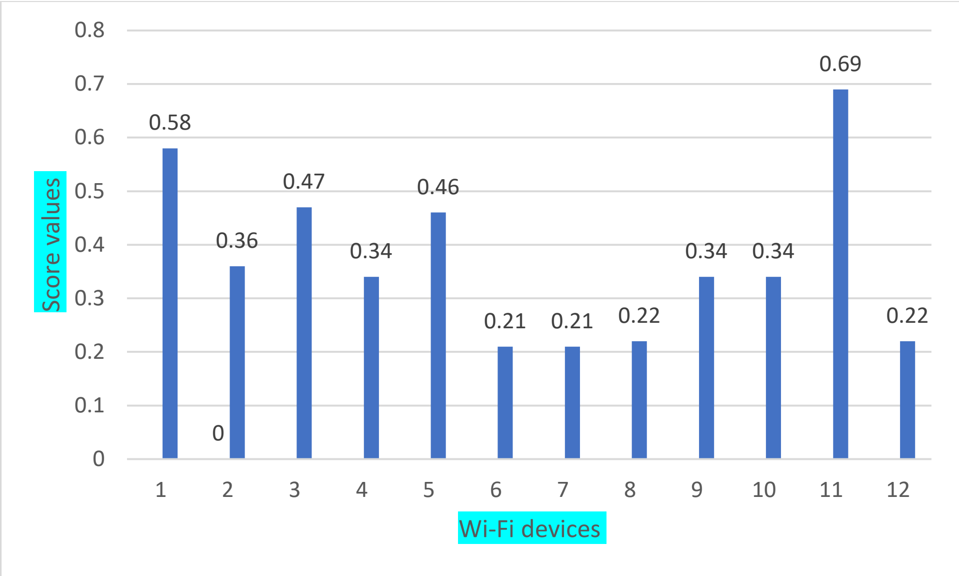


Figure 50 (Ranking of the signal strength of Wi-Fi devices of picture fuzzy W-Fi network)

Figure 50 presents the geometrical representation of the effective degrees of the information provided in Table 20. This analysis clearly indicates that device $Ç_{11}$ has the greatest signal strength while devices $Ç_6$ and $Ç_7$ are the lowest signal providing devices.

4.3 Conclusion

In this chapter, the concept of PFG is introduced and some of its basic terms and notions such as picture fuzzy subgraphs, strength of PFG, degree of PFG and bridges in PFGs are defined are defined followed by numerical demonstration. The novelty of PFGs over IFGs is studied with the help of some results. The concept of degree of a vertex is proposed which shows the strength of connectivity of a vertex with other vertices. The proposed concept of PFG is utilized in describing a social network where the relationships of the people of a certain social group are discussed using the degree of PFG. Further, a Wi-Fi network is also studied where the nodes and edges are based on picture fuzzy information. The strength of signals is briefly described using degree of PFGs. The significance of the using PFGs over IFG and FG is demonstrated numerically via a comparative study. In near future, our aim is to define some other terms of graph theory such as operations on PFGs, modular product on PFGs and picture fuzzy hypergraphs and applied them to practical situation.

Chapter 5

An approach towards decision making and shortest path problems based on T-spherical fuzzy information

In this chapter, we propose some developments in fuzzy graph theory. An original notion of a TSFG is presented as a commonality of FG, an IFG and PFG. The originality, the imperativeness and the importance of this notion is discussed by showing some results, giving examples and a graphical analysis. Some theoretical terms of graphs such as a T-spherical fuzzy sub-graph, a complement of TSFG, degree of TSFG are clarified and their attributes and aspects are analyzed. The main goal of this chapter is to study two types of decision-making problems using the framework of TSFGs. These two problems include the problem of the shortest path and a safe root for an airline journey in a T-spherical fuzzy network. The comparison of this new approach towards these problems with existing approaches is also established. A new algorithm is put forward in the event of T-SFGs and is used to seek out the shortest path problem. The overall analysis of the suggested notion under the prevailing theory is conducted. The advantages of the proposed approach were discussed based on the existing tools and a short comparison of the new with existing tools was established.

5.1 T-Spherical Fuzzy Graphs

The TSFG is an extension of TSFS [46], characterized by membership, abstinence and non-membership grades. The limitations of TSFS is that the sum of n-power of membership, n-power of abstinence and n-power of non-membership grades is less than or equal to 1. The T-SFS is a modified version of the SFS [46], PFS [37], q-ROFS [25], PyFS [15], IFS [10] and FS [1] to cope with complicated and unknown information in the environment of fuzzy set theory. Keeping the advantages of the T-SFS, In this section, we will introduce the concept of TSFG and present its most important properties.

5.1.1 Definition

A graph $\dot{G} = (\underset{.}{V}, \dot{E})$ is called TSFG if

(i) $\underset{.}{V} = \{\underset{.}{t}_1, \underset{.}{t}_2, \underset{.}{t}_3, \dots, \underset{.}{t}_n\}$ be the set of vertices for which $\hat{S}: \underset{.}{V} \longrightarrow [0,1], \hat{I}: \underset{.}{V} \longrightarrow [0,1]$ and $Đ: \underset{.}{V} \longrightarrow [0,1]$represent respectively, the membership degree, the abstinence degree and the non-membership degree of the element $\underset{.}{t}_i \in \underset{.}{V}$ under the condition $0 \leq \hat{S}_1{}^n(\underset{.}{t}_i) + \hat{I}_1{}^n(\underset{.}{t}_{\hat{I}}) + Đ_1{}^n(\underset{.}{t}_i) \leq 1$ for $n \in Z^+$, for all $\underset{.}{t}_i \in \underset{.}{V}(i \in I)$ and $\sqrt[n]{1 - (\hat{S}_1{}^n(\underset{.}{t}_i) + \hat{I}_1{}^n(\underset{.}{t}_i) + Đ_1{}^n(\underset{.}{t}_i))}$ represents the refusal degree of $\underset{.}{t}$ in $\underset{.}{V}$.

(ii) $\dot{E} \subseteq \underset{.}{V} \times \underset{.}{V}$ where $\hat{S}_2: \underset{.}{V} \times \underset{.}{V} \longrightarrow [0,1], \hat{I}_2: \underset{.}{V} \times \underset{.}{V} \longrightarrow [0,1]$ and $Đ_2: \underset{.}{V} \times \underset{.}{V} \longrightarrow [0,1]$ represents the membership degree, the abstinence degree and the non-membership degree of the element $(\underset{.}{t}_i, \underset{.}{t}_j) \in \dot{E}$ such that $\hat{S}_2(\underset{.}{t}_i, \underset{.}{t}_j) \leq \min\{\hat{S}_1(\underset{.}{t}_i), \hat{S}_1(\underset{.}{t}_j)\}$, $\hat{I}_2(\underset{.}{t}_i, \underset{.}{t}_j) \leq \min\{\hat{I}_1(\underset{.}{t}_i), \hat{I}_1(\underset{.}{t}_j)\}$ and $Đ_2(\underset{.}{t}_i, \underset{.}{t}_j) \leq \max\{Đ_1(\underset{.}{t}_i), Đ_1(\underset{.}{t}_j)\}$ under the condition $0 \leq \hat{S}_2{}^n(\underset{.}{t}_i, \underset{.}{t}_j) + \hat{I}_2{}^n(\underset{.}{t}_i, \underset{.}{t}_j) + Đ_2{}^n(\underset{.}{t}_i, \underset{.}{t}_j) \leq 1$ for all $(\underset{.}{t}_i, \underset{.}{t}_j) \in \dot{E}$, and $\sqrt[n]{1 - (\hat{S}_2{}^n(\underset{.}{t}_i, \underset{.}{t}_j) + \hat{I}_2{}^n(\underset{.}{t}_i, \underset{.}{t}_j) + Đ_2{}^n(\underset{.}{t}_i, \underset{.}{t}_j))}$ describes the refusal degree of $(\underset{.}{t}_i, \underset{.}{t}_j)$ in $\dot{E}$.

5.1.2 Example

Consider a graph $\dot{G} = (\underset{.}{V}, \dot{E})$ contains the set of vertices and edges. The following graph shown on Figure 51 is an example of TSFG for $n = 4$. In turn, Figure 52 shows an example of a non-TSFG.

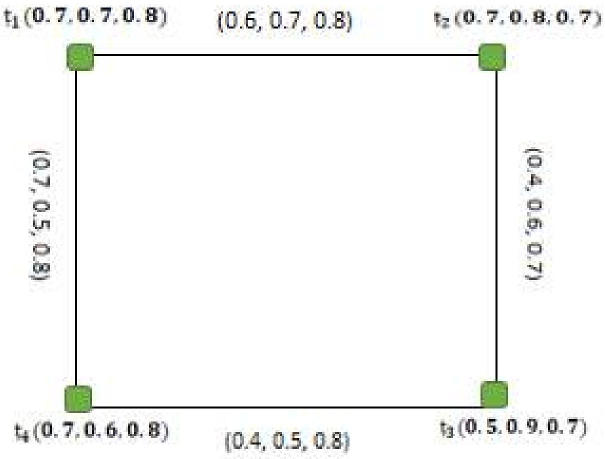


Figure 51 (T-Spherical fuzzy graph)

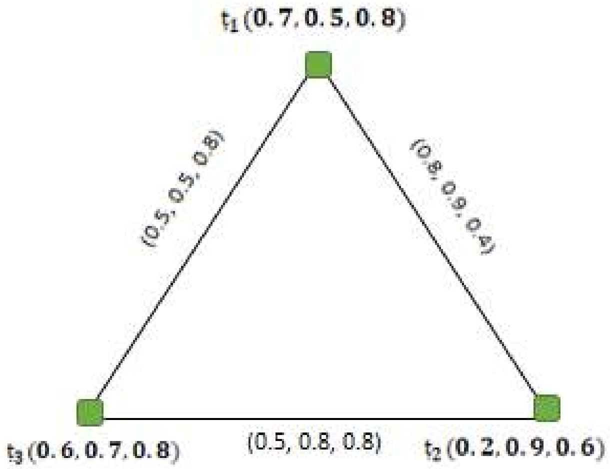


Figure 52 (Not T-spherical fuzzy graph)

5.1.3 Theorem

Every T-SFG is a generalization of PFG, IFG and FG.

Proof: It is easy to verify that

(i) if $n = 1$, then the definition of T-SFG reduces to PFG;

(ii) if $n = 1$ and $\hat{\text{I}} = 0$, then the definition of T-SFG reduces to IFG;

(iii) if $n = 1$, $\hat{\text{I}} = 0$ and $\text{Đ} = 0$, then the definition of T-SFG reduces to FG.

The above result shows the importance of the new concept, because it generalizes all existing structures and may cope with situations in which existing structures fail due to limitations in their structures.

5.1.4 Definition

For T-SFSs $A = \{\hat{S}_A, \hat{I}_A, Đ_A\}$ and $B = \{\hat{S}_B, \hat{I}_B, Đ_B\}$, we define

$$A \oplus B = \left\{\left\{ ț, \left(\begin{array}{c} \sqrt[n]{\hat{S}_A{}^n(ț) + \hat{S}_B{}^n(ț) - \hat{S}_A{}^n(ț).\hat{S}_B{}^n(ț)}, \sqrt[n]{\hat{I}_A{}^n(ț) + \hat{I}_B{}^n(ț) - \hat{I}_A{}^n(ț).\hat{I}_B{}^n(ț)} \\ Đ_A.Đ_B \end{array} \right) \right\}\right\}$$

$$A \otimes B = \left\{\left\{ \varkappa, \left((\hat{S}_A(ț).\hat{S}_B(ț)), (\hat{I}_A(ț).\hat{I}_B(ț)), \sqrt[n]{Đ_A{}^n(ț) + Đ_B{}^n(ț) - Đ_A{}^n(ț).Đ_B{}^n(ț)} \right) \right\}\right\}$$

Comparison rules have always been a challenge in fuzzy environment because at some occasion the function for comparing two values could not differentiate between the numbers due to the ill structure of the score function. If we look at the comparison rules of IFSs, it becomes quite clear that several score functions have been developed from time to time. In [48], an improved score function for IFS is developed and it is discussed that the existing score functions have their limitations, as demonstrated by the examples. In addition, we have significantly fewer results regarding PFSs and there is no score function in the literature. Therefore, here we developed a new score function as a generalization of the score function defined in [48]. This new score function will be used in the shortest path problem (Section 5.2.2).

5.1.5 Definition

The score function for a TSFN $A = (\hat{S}, \hat{I}, Đ)$ is defined as:

$$SC(A) = \frac{(\hat{S})^n \left(1 - (\hat{I})^n - (Đ)^n\right)}{3} \in [0, 1] \qquad (5)$$

5.1.6 Remark

For $\hat{\mathrm{I}} = 0$ and $n = 1$, the defined score function (5) reduces to the case of IFSs.

5.1.7 Definition

A TSFG $\dot{\mathrm{G}}' = (\mathrm{Ṿ}', \dot{\mathrm{E}}')$ is T-spherical fuzzy subgraph (T-SFSG) of the graph $\dot{\mathrm{G}} = (\mathrm{Ṿ}, \dot{\mathrm{E}})$ if $\mathrm{Ṿ}' \subseteq \mathrm{Ṿ}$ and $\dot{\mathrm{E}}' \subseteq \dot{\mathrm{E}}$.

5.1.8 Definition

Let the representation $\bar{G} = (\bar{V}, \bar{E})$ is called the complement of T-SFG, if

(i) $\overline{\mathrm{Ṿ}} = \mathrm{Ṿ}$ i.e. $\overline{\hat{\mathrm{S}}_\iota} = \hat{\mathrm{S}}_{i,}\overline{\hat{\mathrm{I}}_\iota} = \hat{\mathrm{I}}_i$ and $\overline{\mathrm{Đ}_\iota} = \mathrm{Đ}_{i,} i = 1,2, \dots, n;$

(ii) $\overline{\hat{\mathrm{S}}_{2\iota\jmath}} = \min(\hat{\mathrm{S}}_i, \hat{\mathrm{S}}_j) - \hat{\mathrm{S}}_{2ij,}\overline{\hat{\mathrm{I}}_{2\iota\jmath}} = \min(\hat{\mathrm{I}}_i, \hat{\mathrm{I}}_j) - \hat{\mathrm{S}}_{2ij}$and $\overline{\mathrm{Đ}_{2\iota\jmath}} = \max(\mathrm{Đ}_i, \mathrm{Đ}_j) - \mathrm{Đ}_{2ij,} i, j = 1,2, \dots, n.$

5.1.9 Example

In this example, we give a TSFG and its complement.

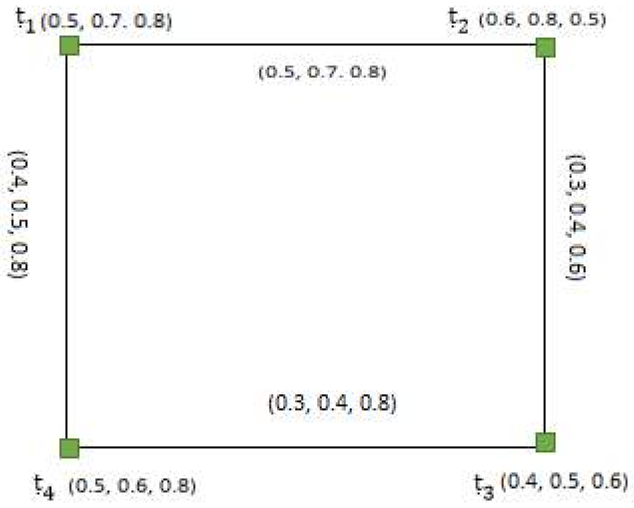


Figure 53 (T-spherical fuzzy graph)

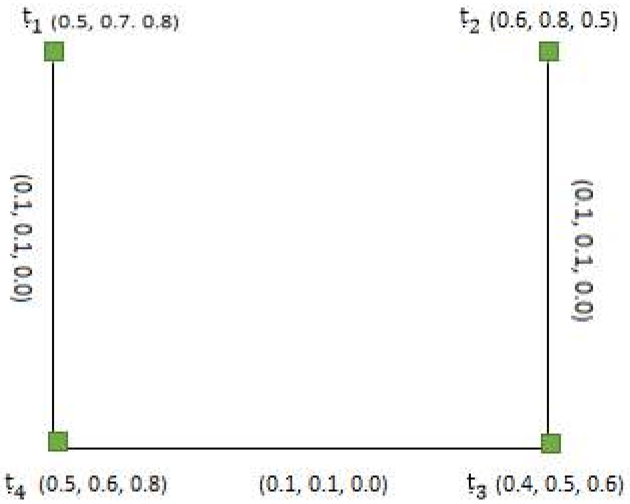


Figure 54 (Complement of the T-spherical fuzzy graph in figure 53)

The vertices of the above Figures 53 and Figure 54 are entirely T-SFNs for $n = 3$.

5.1.10 Definition

The degree of a TSFG $\dot{G} = (\underset{.}{V}, \dot{E})$ is defined by $d(ṭ) = (d_{\hat{S}}(ṭ), d_{\hat{I}}(ṭ), d_{Đ}(ṭ))$, where $d_{\hat{S}}(ṭ) = \sum_{u \neq ṭ} \hat{S}_2(ṭ, u)$, $d_{\hat{I}}(ṭ) = \sum_{u \neq ṭ} \hat{I}_2(ṭ, u)$ and $d_{Đ}(ṭ) = \sum_{y \neq ṭ} d_{Đ_2}(ṭ, u)$ for $ṭ, u \in \underset{.}{V}$.

5.1.11 Definition

A TSFG is said to be

1) ***semi*** $\hat{S}$ ***strong*:** if $\hat{S}(ṭ_i, ṭ_j) = \min\left(\hat{S}(ṭ_i), \hat{S}(ṭ_j)\right)$;

2) ***semi*** $\hat{I}$ ***strong*:** if $\hat{I}(ṭ_i, ṭ_j) = \min\left(\hat{I}(ṭ_i), \hat{I}(ṭ_j)\right)$;

3) ***semi*** Đ ***strong*:** if $Đ(ṭ_i, ṭ_j) = \max\left(Đ(ṭ_i), Đ(ṭ_j)\right)$;

4) ***strong*:** if (1), (2) and (3) hold.

5.1.12 Example

Let $\dot{G} = (\underset{.}{V}, \dot{E})$ be a graph, where $\underset{.}{V}$ is the set of vertices and $\dot{E}$ is the set of edges, which is presented in Figure 55.

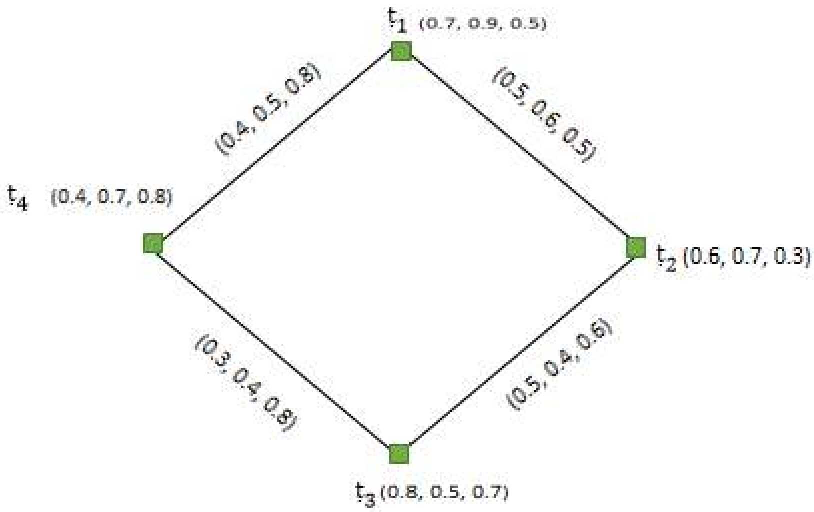


Figure 55 (T-spherical fuzzy graph)

The vertices of the above Figure 55 are entirely T-SFNs for $n = 4$, where the degrees of the vertices are

$$d(\underset{.}{t}_1) = (0.9, 1.1, 1.3),\ d(\underset{.}{t}_2) = (1, 1, 1.1),\ d(\underset{.}{t}_3) = (0.8, 0.8, 1.4),\ d(\underset{.}{t}_4) = (0.7, 0.9, 1.6).$$

5.1.13 Definition

A graph $\dot{G} = (\underset{.}{V}, \dot{E})$ is defined as a strong TSFG, if

1. $\underset{.}{V} = \{\underset{.}{t}_1, \underset{.}{t}_2, \underset{.}{t}_3, \dots, \underset{.}{t}_n\}$ is the set of vertices such that $\hat{S}: \underset{.}{V} \longrightarrow [0,1]$, $\hat{I}: \underset{.}{V} \longrightarrow [0,1]$ and $Đ: \underset{.}{V} \longrightarrow [0,1]$ represent, respectively the membership degree, abstinence and the non-membership degree of the element $\underset{.}{t}_i \in \underset{.}{V}$ under the condition $0 \le \hat{S}_1{}^n(\underset{.}{t}_i) + \hat{I}_1{}^n(\underset{.}{t}_i) + Đ_1{}^n(\underset{.}{t}_i) \le 1$ for $n \in Z^+$, for all $\underset{.}{t}_i \in \underset{.}{V}(i \in I)$ and $\sqrt[n]{1 - (\hat{S}_1{}^n(\underset{.}{t}_i) + \hat{I}_1{}^n(\underset{.}{t}_i) + Đ_1{}^n(\underset{.}{t}_i))}$ is represents the refusal degree of $\underset{.}{t}$ in $\underset{.}{V}$.

2. $\dot{E} \subseteq V \times V$ where $\hat{S}_2 : V \times V \longrightarrow [0,1]$, $\hat{I}_2 : V \times V \longrightarrow [0,1]$ and $Đ_2 : V \times V \longrightarrow [0,1]$ represent, respectively the membership degree, the abstinence degree and the non-membership degree of the element $(t_i, t_j) \in \dot{E}$ such that $\hat{S}_2(t_i, t_j) = \min\{\hat{S}_1(t_i), \hat{S}_1(t_j)\}$, $\hat{I}_2(t_i, t_j) = \min\{\hat{I}_1(t_i), \hat{I}_1(t_j)\}$ and $Đ_2(t_i, t_j) = \max\{Đ_1(t_i), Đ_1(t_j)\}$ under the condition $0 \le \hat{S}_2{}^n(t_i, t_j) + \hat{I}_2{}^n(t_i, t_j) + Đ_2{}^n(t_i, t_j) \le 1$ for all $(t_i, t_j) \in \dot{E}$, and $\sqrt[n]{1 - (\hat{S}_2{}^n(t_i, t_j) + \hat{I}_2{}^n(t_i, t_j) + Đ_2{}^n(t_i, t_j))}$ describes the refusal degree of (t_i, t_j) in $\dot{E}$.

5.1.14 Theorem

If $\dot{G}$ is a strong TSFG, then its complement $\dot{G}$ is also a strong TSFG.

Proof. Two cases to consider. If $tu \in \dot{E}$, then it follows from the fact $\acute{G}$ is strong TSFG that

$$\overline{\hat{S}_2}(tu) = \min(\hat{S}_1(t), \hat{S}_1(u)) - \hat{S}_2(tu) = \min(\hat{S}_1(t), \hat{S}_1(u)) - \min(\hat{S}_1(t), \hat{S}_1(u)) = 0,$$

$$\overline{\hat{I}_2}(tu) = \min(\hat{I}_1(t), \hat{I}_1(u)) - \hat{I}_2(tu) = \min(\hat{I}_1(t), \hat{I}_1(u)) - \min(\hat{I}_1(t), \hat{I}_1(u)) = 0$$

and

$$\overline{Đ_2}(tu) = \max(Đ_1(t), Đ_1(u)) - Đ_2(tu) = \max(Đ_1(t), Đ_1(u)) - \max(Đ_1(t), Đ_1(u)) = 0.$$

If $tu \notin \dot{E}$, then $\overline{\hat{S}_2}(tu) = \min(\hat{S}_1(t), \hat{S}_1(u)) - \hat{S}_2(tu) = \min(\hat{S}_1(t), \hat{S}_1(u))$, $\overline{\hat{I}_2}(tu) = \min(\hat{I}_1(t), \hat{I}_1(u)) - \hat{I}_2(tu) = \min(\hat{I}_1(t), \hat{I}_1(u))$ and $\overline{Đ_2}(tu) = \max(Đ_1(t), Đ_1(u)) - Đ_2(tu) = \max(Đ_1(t), Đ_1(u))$.

5.1.15 Definition

In TSFG a path is a sequence of some distinct vertices $t_i (i = 1, 2, 3 \dots n)$ one of the following conditions hold for some $(i, j = 1, 2, 3, \dots n)$

- $\hat{S}_{2ij} > 0$, $\hat{I}_{2ij} > 0$ and $Đ_{2ij} > 0$;

- $\hat{S}_{2ij} = 0,\ \hat{I}_{2ij} = 0$ and $Đ_{2ij} > 0$;
- $\hat{S}_{2ij} > 0,\ \hat{I}_{2ij} > 0$ and $Đ_{2ij} = 0$.

5.1.16 Definition

Let $t = t_1, t_2, t_3, \dots t_{n+1} (n > 0)$ be a path. Then, its length is n. This path is called a cycle, if $t_1 = t_{n+1}$ for $(n \geq 3)$. Moreover, we say that two vertices combined by a path are connected.

5.1.17 Example

Let $V = \{t_1, t_2, t_3, t_4\}$ and $\tilde{E}$ be the set of edges illustrated by Figure 56.

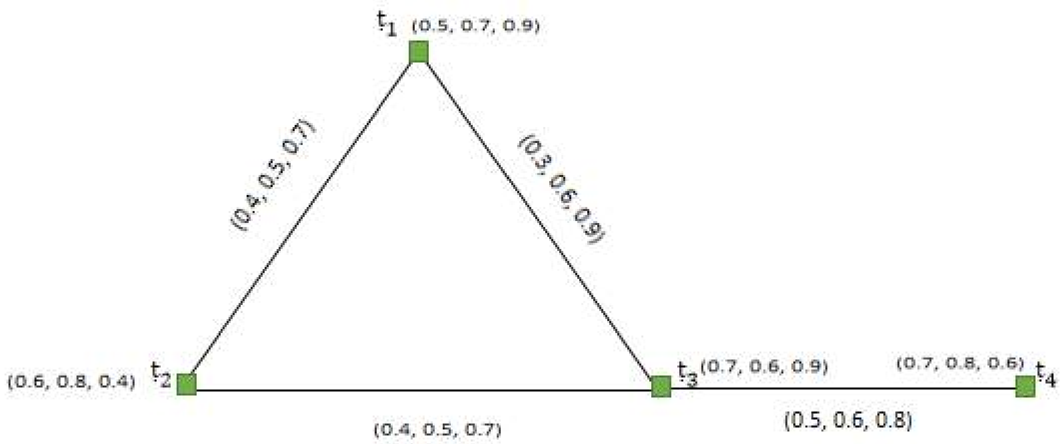


Figure 56 (T-spherical fuzzy graph)

Here $t_1 t_2 t_3 t_4$ is a path, and hence t_1, t_2, t_3, t_4 are connected vertices. The length of this path is 3. Moreover $t_1 t_2 t_3 t_1$ form a cycle.

5.2 Application in Finding Shortest Path in a Network

The most common problem of graph theory is the shortest path problem. It has been extensively tested for almost every fuzzy structure [104-107] with an algorithm, which is relatively easy and thanks to that we obtain the best results expected. The comprehensive steps of the algorithm, are implemented in this section using the numerical example and a series of relative tests.

5.2.1 Dijkstra Algorithm

Dijkstra algorithm is the most widely used algorithm for computing shortest path in a network. From time to time, some modifications have been made in Dijkstra algorithm and several formulae are included to get optimum results. These formulae include distance and similarity measures and aggregation operators etc. Some useful work in this regard have been done in [104, 105].

The steps of the Dijkstra algorithm are as follows.

1. Take the first node as $(0, 0, 1)$ and the distance of every node to itself is zero.
2. Take $\mathrm{i} = 1$.
3. Find j for $\mathrm{P}_{1\mathrm{j}} = \mathrm{P}_{11} \oplus \wedge_{\mathrm{k}\in\mathrm{NP}(1)} \mathrm{P}_{1\mathrm{k}}$ and determine $\mathrm{P}_{1\mathrm{j}}$ where "$\wedge$" denotes the minimum of $\mathrm{P}_{1\mathrm{k}}$ for $\mathrm{k} \in \mathrm{NP}(1)$ which can be calculated using score function. Further $\mathrm{NP}(i)$ denote the collection of all nodes having some relation with i.
4. Put $\mathrm{i} = \mathrm{j}$.
5. Find k for $\mathrm{P}_{\mathrm{jk}} = \mathrm{P}_{1\mathrm{j}} \oplus \wedge_{\mathrm{h}\in\mathrm{NP}(\mathrm{j})} \mathrm{P}_{\mathrm{jh}}$ and determine P_{jk}.
6. This process should be continued until destination node is obtained.
7. When destination node is reached, then the algorithm is stopped.

5.2.2 Example

Let us consider a network of 6 nodes as in Figure 57, where the distance of every two, connected nodes, is provided in by the form of TSFNs. We proceed with the algorithm as follows.

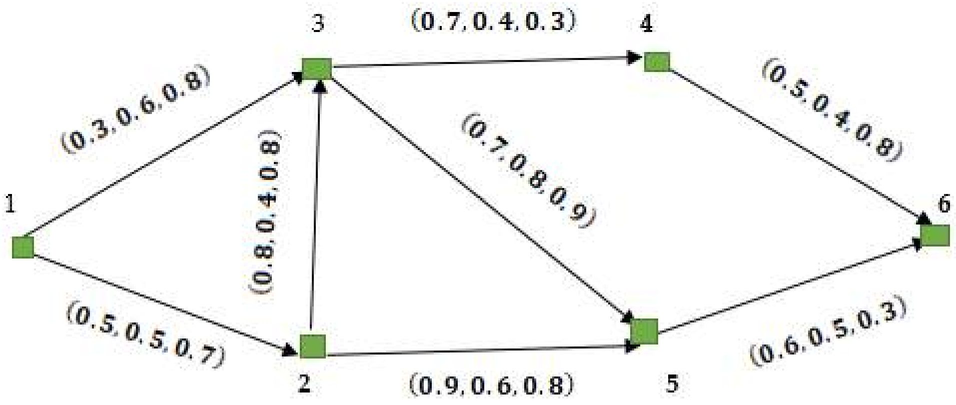


Figure 57 (T-spherical fuzzy network)

In the above graph it is shown that the source node is node 1 and the destination node is node 6, so for $n = 6$ and for the first step $P_{11} = (0,0,1)$. Note that $P_{11} = (0,0,1)$ because the distance at node 1 is zero so its non-membership grade is 1 and membership and abstinence grades are zero. At first $i = 1$, so it needs to find j by the equation

$$P_{1j} = P_{11} \oplus \wedge_{j \in NP(1)} P_{1j} = P_{11} \oplus \wedge_{j \in \{2,3\}} P_{1j} = P_{11} \oplus (P_{12} \wedge P_{13})$$

We are doing the first iteration in details and the rest are similar so we just omit the details there.

Note that $P_{12} \wedge P_{13}$ means the minimum of P_{12} & P_{13}. To compute the minimum between P_{12} & P_{13}, we calculate their score values i.e. $SC(P_{12})$ and $SC(P_{13})$. If

- $SC(P_{12}) < SC(P_{13})$ then $P_{12} \wedge P_{13} = P_{12}$
- $SC(P_{12}) > SC(P_{13})$ then $P_{12} \wedge P_{13} = P_{13}$

By using Eq. (1).

$$SC(P_{12}) = SC(0.5,0.5,0.7) = \frac{0.5^6(1 - 0.5^6 - 0.7^6)}{3} = 0.0045$$

Similarly

$$SC(P_{13}) = SC(0.3,0.6,0.8) = 0.000168$$

As $SC(P_{12}) > SC(P_{13})$. Therefore, $P_{12} \wedge P_{13} = P_{13}$ and

$P_{1j} = P_{11} \oplus \wedge_{j \in NP(1)} P_{1j} = P_{11} \oplus \wedge_{j \in \{2,3\}} P_{1j} = P_{11} \oplus (P_{12} \wedge P_{13}) = P_{11} \oplus P_{13}$

$= (0,0,1) \oplus (0.3,0.6,0.8)$.

Then by using Def. (4.1.8), we have

$$= (0.00405,0.064,0.8)$$

This implies that $j = 2$, $P_{13} = (0.00405,0.064,0.8)$. similarly, we continued the processes.

For the next step, for $i = 2$, we need to find j by the equation

$$P_{1j} = P_{12} \oplus \wedge_{j \in NP(2)} P_{2j} = P_{12} \oplus \wedge_{j \in \{3,5\}} P_{2j} = P_{13} \oplus (P_{34} \wedge P_{35})$$

$$= (0.045,0.18,0.8) \oplus \big((0.7,0.4,0.3) \wedge (0.7,0.8,0.9)\big)$$

$$= \big((0.045,0.18,0.8) \oplus (0.1225) \wedge (0.008095)\big)$$

$$= (0.00405,0.0648,0.8) \oplus (0.7,0.8,0.9) = (0.058825,0.131072,0.72).$$

This implies that $j = 2$, $P_{15} = (0.058825,0.131072,0.72)$.

First subsequently for $i = 3$, we must find j by the equation

$$P_{1j} = P_{13} \oplus \wedge_{j \in NP(3)} P_{3j} = P_{13} \oplus \wedge_{j \in \{4,5\}} P_{1j} = P_{15} \oplus P_{56}$$

$$= (0.058825,0.131072,0.72) \oplus (0.6,0.5,0.3) = (0.181107,0.131442,0.216).$$

This implies that $j = 2$, $P_{16} = (0.181107,0.131442,0.216)$.

So, the shortest path in the above graph is $P_1 \longrightarrow P_3 \longrightarrow P_5 \longrightarrow P_6$. Hence, by using a modified Dijkstra algorithm, we have successfully computed the shortest path from source node to destination node. The results obtained here have greater precision because of its diverse structure and a detailed comparison showing the validity and superiority of the proposed algorithm is discussed in Section 5.3.

5.3 Proportional Study and Merits of Proposed Algorithm

In the current part, we will explain that the proposed algorithm can be applied in intuitionistic fuzzy environment and to Pythagorean fuzzy networks. Consider the intuitionistic fuzzy network illustrated in Figure 58.

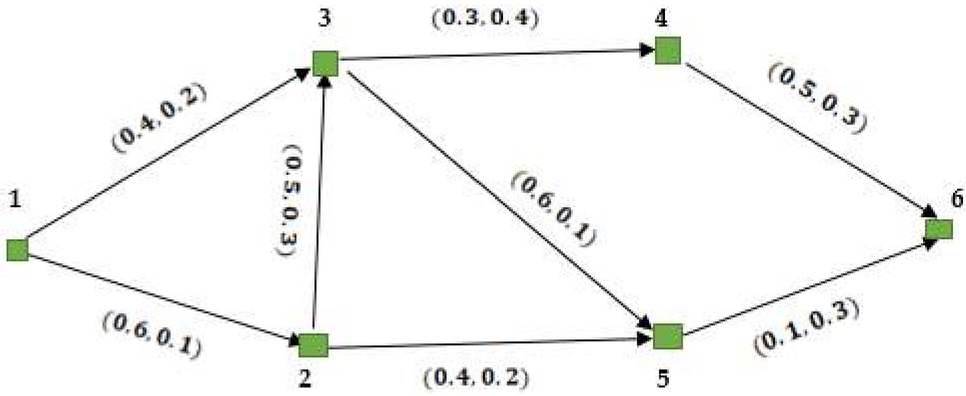


Figure 58 (Intuitionistic fuzzy network)

The proposed algorithm for computing shortest path can be applied in finding the shortest path form node 1 to node 6 by considering the abstinence grade as zero i.e. every duplet given in intuitionistic fuzzy network can be considered as a T-SFN. Hence this overall network based on intuitionistic fuzzy information can be considered as a T-spherical fuzzy network and can be solved accordingly. In addition, if we consider Figure 59 where a network of nodes is given in Pythagorean fuzzy environment. This too can be considered as T-spherical fuzzy environment by the abstinence grade as zero in every duplet and take $n = 2$.

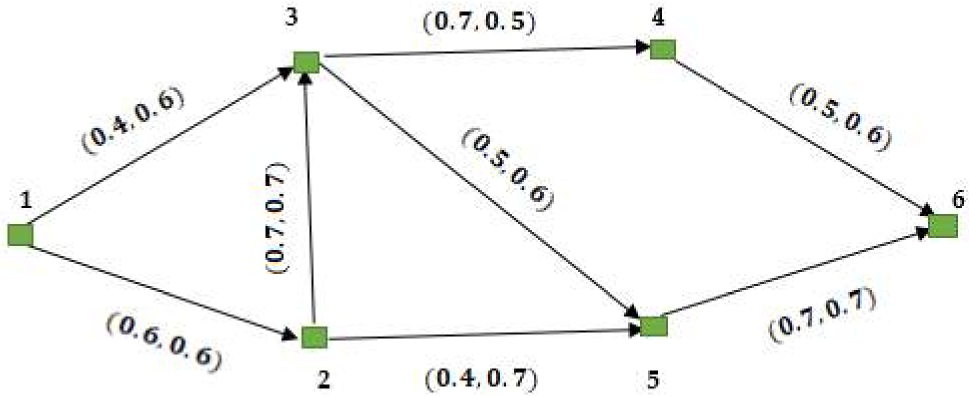


Figure 59 (Pythagorean fuzzy network)

5.4 Observation of a Safe Root for an Airline Journey in a T-Spherical Fuzzy Network

The proposed concept of T-SFG can be applied in important part of mathematics and can be also used as a tool in various fields, including biology, physics, transport networks and social networks. The traffic network of an airline system is always of importance. Usually, those routes are selected for an airline which relatively less expensive and safer.

We assume a set of five different countries at our disposal and we want to travel to and from these countries by plane. Airline companies keep goal to make best amenities available for passengers. Air traffic controllers must strictly follow the schedule of arrivals and departures of company airplanes, and this work can be done by planning efficient routes for airplanes. The efficiency of an airline route can be defined in terms of less expense and safety i.e. an airline is considered as more efficient if it is less expensive, take less time (pick shortest route) and choose a safe route.   A network of T-SFG airlines and five countries is showed in Figure. 60, where vertices and edges represent countries and flights respectively. In Figure 60, the edge values between two countries represent information about the efficiency of the route. The larger value of the membership grade means the more reliable is the airline.

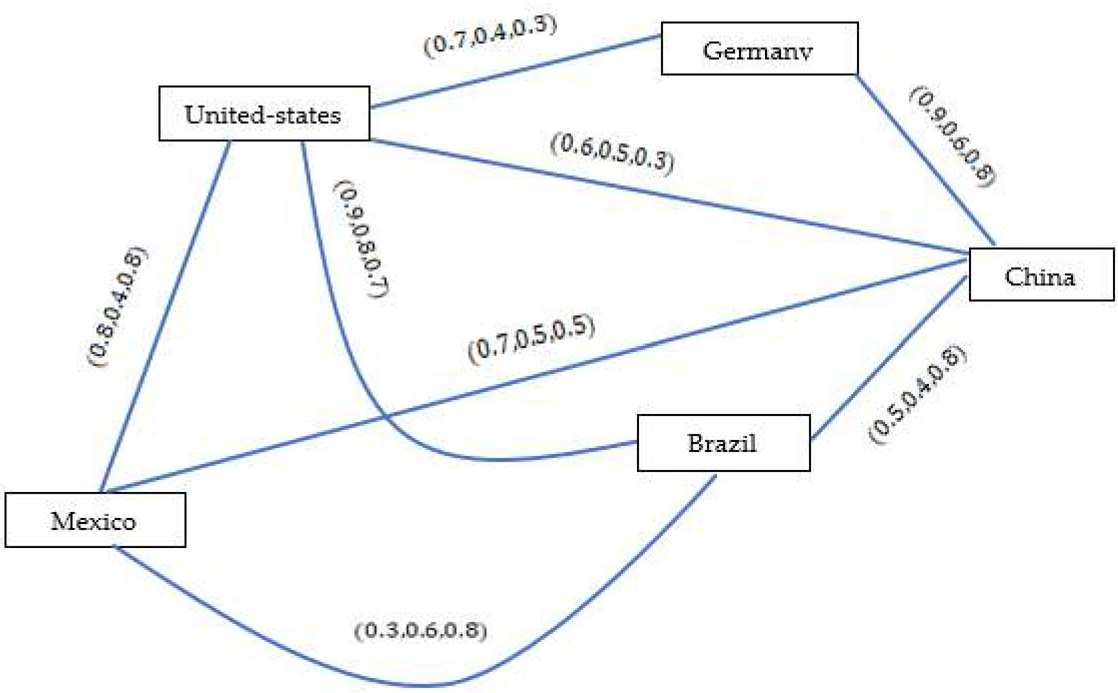


Figure 60 (T-spherical fuzzy airline network)

Country	**Country**	**Edge**
Mexico	China	(0.7,0.5,0.5)
Mexico	Brazil	(0.3,0.6,0.8)
United states	Mexico	(0.8,0.4,0.8)
Brazil	China	(0.5,0.4,0.8)
Germany	China	(0.9,0.6,0.8)
United states	Germany	(0.7,0.4,0.3)
United states	Brazil	(0.9,0.8,0.7)
China	United states	(0.6,0.5,0.3)

Table 18 (T-spherical fuzzy edges of Figure 60)

For convenience, the edges of the above Figure. 60 are summarized in the above Table 18. Note that all the triplets given in Table 18 are purely TSFNs for $n = 6$.

The truth membership degree of each vertex represents the strength of airline route. This means that a larger value of truth membership degree means the more efficient the airline route and vice versa. The indeterminacy membership degree of each denotes the unpredictability in the airline route. The falsity membership degree of each vertex shows the errors of that system and the rejection degree shows the chances of cancelation of the flights. For instance, the edge between Germany and China denotes that flight safety for this travel is 90%, 60% is dependent on unlikely systems and 80% is unsecure. The truth membership degree, the indeterminacy membership degree and the falsity membership degree of each edge is quantified by applying a specified relation or it is determined by some experts in the system.

Now we discuss a special circumstance that often due to weather, technical problems or personal problems, the passenger renounces its direct flight between the two selected countries. So, if he wants to go there straight away, he has to choose an indirect connection, as usually connecting flights exist between these countries.

For instance, if a passenger missed or skipped his flight from Germany to United States then four connecting flights exist which are denoted by routes $\boldsymbol{R_i}$ such as

$\boldsymbol{R_1}: Germany \rightarrow China \rightarrow United\ States$

$\boldsymbol{R_2}: Germany \rightarrow China \rightarrow Mexico \rightarrow United\ States$

$\boldsymbol{R_3}: Germany \rightarrow China \rightarrow Brazil \rightarrow United\ States$

$\boldsymbol{R_4}: Germany \rightarrow China \rightarrow Brazil \rightarrow Mexico \rightarrow United\ States$

By calculating or quantifying the length or duration of all flights, the most appropriate flight can be identified by taking the maximum truth membership value, the minimum indeterminacy membership value, as well as the falsity and rejection membership values. For this purpose, we utilize Definition 5.1.10 by means of which we could get the degree of each vertex i.e. the degree of each route. The lager membership value in the degree of a node will give us the suitable route.

The algorithm for our proposed problem is as follows:

5.4.1 Algorithm

The detailed steps of this algorithm are

Step 1. Enter the truth membership degree, the indeterminacy membership degree and the falsity membership degree of all vertices (countries);

Step 2. Compute all boundaries using subsequent relations;

Step 3. Compute all the possible routes R_i between the countries;

Step 4. Compute the lengths of all the routs R_i applying the subsequent formula given in Definition 5.1.10 i.e. for $(ṭ, u) \in Ṿ$

$$d_{Ŝ}(ṭ) = \sum_{u \neq ṭ} Ŝ_2(ṭ, u)$$

$$d_{Î}(ṭ) = \sum_{u \neq ṭ} Î_2(ṭ, u)$$

$$d_{Đ}(ṭ) = \sum_{y \neq ṭ} Đ_2(ṭ, u)$$

Step 5. Locate the confirmed route with the maximum truth membership level, minimum indeterminacy-membership level and minimum falsity membership level.

Using this proposed algorithm, we have the degrees computed as:

$$d(R_1) = (1.5, 1.1, 1.1)$$

$$d(R_2) = (2.4, 1.5, 2.1)$$

$$d(R_3) = (2.3, 1.8, 2.3)$$

$$d(R_4) = (2.5, 2.0, 3.2)$$

These results show us that R_1 is the best route as the difference of the membership and non-membership values is less comparative to other degrees. However, if we investigate the diagram of the airline route in Figure 60, it seems that the direct route from Germany to United States is the efficient route.

5.5 Application of T-spherical fuzzy graphs in Supply Chain Management (SCM)

In the problem of SCM, the performance of the components which are usually the partners in a supply chain is measured. Companies usually run on several aspects and in SCM problems, we aim to know about the most significant aspect. We adapt the example discussed in [28] where four aspects of a company are assessed. We assume the four aspects as *"standard of the service"*, *"cost & price"*, *"quality"* and *"response time"* which we denoted by $z_i (i = 1, 2, 3, 4)$. Three decision makers gave their preferences in the decision matrices R_1, R_2 and R_3 respectively in the form of TSFNs where the membership, abstinence and non-membership grades of the four aspects are investigated.

$$R_1 = \begin{bmatrix} (0.7,0.8,0.9) & (0.73,0.84,0.95) & (0.74,0.45,0.9) & (0.77,0.56,0.67) \\ (0.55,0.78,0.67) & (0.86,0.77,0.45) & (0.67,0.56,0.6) & (0.67,0.66,0.55) \\ (0.9,0.8,0.7) & (0.8,0.3,0.7) & (0.9,0.8,0.7) & (0.45,0.46,0.47) \\ (0.5,0.4,0.3) & (0.64,0.63,0.62) & (0.8,0.7,0.8) & (0.67,0.87,0.69) \end{bmatrix}$$

$$R_2 = \begin{bmatrix} (0.72,0.82,0.29) & (0.74,0.85,0.91) & (0.71,0.25,0.39) & (0.78,0.56,0.67) \\ (0.65,0.48,0.57) & (0.66,0.67,0.65) & (0.77,0.76,0.76) & (0.69,0.66,0.55) \\ (0.69,0.48,0.57) & (0.38,0.3,0.7) & (0.97,0.87,0.77) & (0.49,0.46,0.47) \\ (0.58,0.48,0.39) & (0.64,0.73,0.72) & (0.77,0.87,0.75) & (0.68,0.87,0.69) \end{bmatrix}$$

$$R_3 = \begin{bmatrix} (0.77,0.78,0.79) & (0.78,0.84,0.85) & (0.54,0.35,0.39) & (0.77,0.56,0.67) \\ (0.65,0.68,0.57) & (0.66,0.67,0.55) & (0.77,0.66,0.63) & (0.67,0.36,0.55) \\ (0.29,0.28,0.27) & (0.68,0.63,0.67) & (0.8,0.7,0.6) & (0.45,0.66,0.47) \\ (0.56,0.46,0.36) & (0.74,0.73,0.72) & (0.8,0.7,0.8) & (0.67,0.77,0.60) \end{bmatrix}$$

Here we utilize the averaging aggregation operators of TSFSs studied in [48] to obtain a single decision matrix R from R_1, R_2 and R_3.

$$TSFWA(\dot{r}_{i1}, \dot{r}_{i2}, \ldots, \dot{r}_{in}) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} (1 - \hat{S}_j^n)^{w_j}}, \prod_{j=1}^{m} (\hat{I}_j)^{w_j}, \prod_{j=1}^{m} (Đ_j)^{w_j} \right)$$

$$R = \begin{bmatrix} (0.73,0.8,0.59) & (0.75,0.84,0.91) & (0.70,0.34,0.55) & (0.77,0.56,0.67) \\ (0.59,0.64,0.61) & (0.79,0.71,0.54) & (0.74,0.65,0.66) & (0.68,0.57,0.55) \\ (0.82,0.51,0.51) & (0.73,0.36,0.69) & (0.93,0.80,0.70) & (0.47,0.50,0.47) \\ (0.55,0.44,0.34) & (0.68,0.69,0.68) & (0.79,0.76,0.78) & (0.67,0.84,0.67) \end{bmatrix}$$

This newly obtained regrouped data is depicted using a T-SFG in Figure 61.

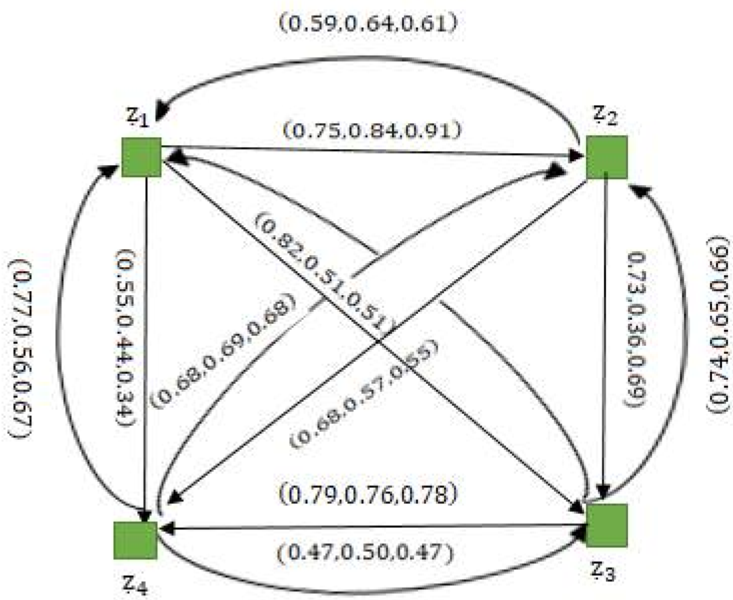


Figure 61 (Directed Network four aspects in the form of T-SFG)

Taking $\hat{S}^{2L} \geq 0.60$ (The highest membership score of every edge in the above figure 61) The T-SFG depicted in Figure 61 converted into another T-SFG depicted in Figure 62 below.

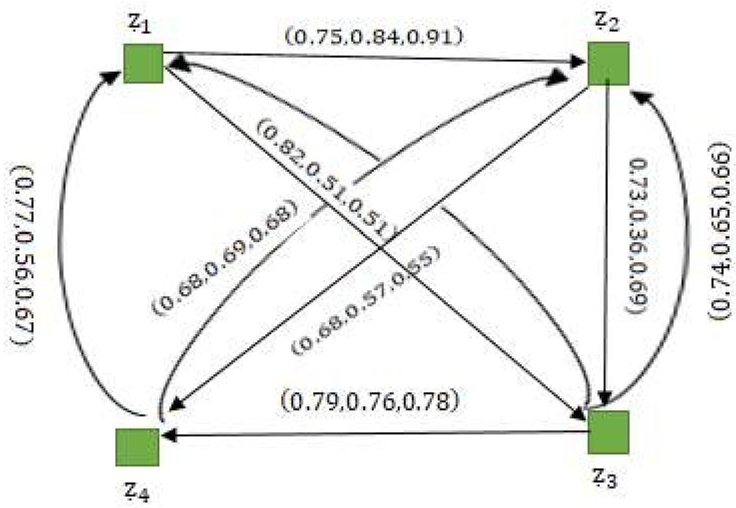


Figure 62 (Partial Directed Network based on T-SFGs)

To compute the out-degree of the four aspects for their ranking, we use the following formula as given below.

$$Out - d(z) =$$

Based on given definition 5.1.10 of out-degree, we have

$$Out - d(z_1) = (0.75,0.84,0.91)$$

$$Out - d(z_2) = (0.68,0.57,0.55))$$

$$Out - d(z_3) = (0.77,0.56,0.67)$$

$$Out - d(z_4) = (0.68,0.69,0.68)$$

After having a look at the membership degree of the out-degrees, we have the following ranking.

$$Out - d(z_3) \geq Out - d(z_1) \geq Out - d(z_2) \geq Out - d(z_4)$$

Implies that $z_3 \geq z_1 \geq z_2 \geq z_4$

Hence, using the concept of T-SFGs, an SCM problem is solved successfully where the most significant factor in a supply chain is determined. The ranking shows that the factor z_3 is most significant factor on which we could count.

5.5 Comparative Study and Advantages

The proposed approaches are more general and more reliable than Existing methods due to its limitations, the sum of the n-power of membership, n-power of abstinence and n-power of non-membership grades are bounded to [0,1]. The characteristics comparison between proposed methods and existing methods are described below in Table 22.

Methods	Whether flexibility to wide range of information	Contains two dimension information	Contains three dimension information	Generalization of FG	Powerful characteristic
FS [8]	Yes	No	No	No	No
IFS [17]	Yes	Yes	No	Yes	No
PyFS [23]	Yes	Yes	No	Yes	No
q-ROFS [27]	Yes	Yes	No	Yes	No
PFS [31]	Yes	Yes	Yes	Yes	No
SFS (proposed)	Yes	Yes	Yes	Yes	No
T-SFS (proposed)	Yes	Yes	Yes	Yes	Yes

Table 22 (Comparison of T-spherical fuzzy networks over other existing networks)

The importance and imperativeness of the proposed approach are due to the fact that the novel approach can solve problems in the environment of PyFSs as well as IFSs. Now we investigate two examples consisting information in the form of PyFNs or IFNs.

Consider the information in the form of PyFNs as follows in Table 23.

Country	Country	Edge
Mexico	China	(0.7,0.5)
Mexico	Brazil	(0.3,0.8)
United states	Mexico	(0.4,0.8)
Brazil	China	(0.5,0.6)
Germany	China	(0.7,0.6)
United states	Germany	(0.7,0.4)
United states	Brazil	(0.7,0.4)
China	United states	(0.6,0.5)

Table 23 (Pythagorean fuzzy edges)

Currently, this type of information can simply be organized using the Definition 2.1.12 with $Ð = 0$. In addition, we have information in the form of IFNs given by Table 24.

Country	Country	Edge
Mexico	China	(0.3,0.5)
Mexico	Brazil	(0.3,0.4)
United states	Mexico	(0.4,0.5)
Brazil	China	(0.5,0.1)
Germany	China	(0.2,0.6)
United states	Germany	(0.3,0.4)
United states	Brazil	(0.2,0.4)
China	United states	(0.1,0.5)

Table 24 (Intuitionistic fuzzy edges)

Therefore, Definition 5.1.10 could employ the proposed approach. Any other way, due to their constraints structures, the tools of PyFSs or IFSs cannot be used to the information of T-SFSs. All this reveals the significance of novel presented approach. When we are considering the value of parameter $n = 1$ and taking the value of abstinence as zero, the proposed work in def. (5.1.10) is reduced for IFG and similarly, when we consider the value of parameter $n = 2$ and taking the value of abstinence is zero, the proposed work in def. (5.1.10) is reduced for PyFG. If we are taking only the abstinence degree will be zero, then the Def. (5.1.10) is converted for q-ROFG.

The proposed methods are more general than existing methods due to some reasons discussed as below:

The T-SFG is an extension of SFG, IFG, PFG and PyFG, characterized by membership, abstinence and non-membership grades. The diversity of T-SFS is that the sum of n-power of membership, n-power of abstinence and n-power of non-membership grades is less than or equal to 1. The T-SFG is an enhanced version of spherical fuzzy graph (SFG), PFG, q-ROFG, PyFG, IFG and FG to cope with complicated and unknown information in the environment of fuzzy set theory.

Due to these all reasons, the T-SFG is proved to be a better tool to deal with all kind of problems in real-life having uncertainty. Therefore, handling problems using T-SFG is more reliable and more feasible than existing methods.

5.6 Conclusion

In this chapter, the notion of T-SFG has initiated based on the new theory of T-SFSs. The significance of T-SFGs has studied in analysis of the novelty of T-SFSs, and it has examined that T-SFGs are generalizations of IFGs and PFGs and can be helpful in those situations where IFGSs and PFGs failed to be useful. Several fundamental theoretical terms of graphs have been defined including T-spherical fuzzy subgraphs, complement and strength of T-SFGs, degree of vertices and bridges in T-SFGs. Several operations have also defined for T-SFGs and related results are studied. For T-SFGs, famous Dijkstra algorithm was created and in a network of T-SFGs the shortest path problem has been solved. The core advantage of the proposed work can be used in practical situations, which can be managed using IFGs or PFGs, but those structures are tunable to manage the data provided in T-spherical fuzzy situation. In a short period, from application point of view, the structure of T-SFGs could be very valuable in some problems including optimization in networks, traffic signal problems, and other problems of engineering and computer sciences.

Chapter 6

Some Root Level Modifications in Interval Valued Fuzzy Graphs and Their Generalizations Including Neutrosophic Graphs

FG and its generalizations played an essential role in dealing with real life problems involving uncertainties. The goal of this chapter is to show some serious flaws in the existing definitions of several root level generalized FG structures with the help of some counter examples. First, we aim to improve the existing definition for IVFG, IVIFG and their complements as these existing definitions are not well-defined i.e. one can obtain some senseless intervals using the existing definitions. The limitation of the existing definitions and the validity of new definitions is supported with some examples. It is also observed that the notion of SVNG is not well-defined either. The consequences of the existing definition of SVNG are discussed with the help of examples. A new definition for SVNG is developed and its improvement is demonstrated with some examples. The definition of interval valued neutrosophic graph is also modified due to the shortcomings existed in the current definition and the validity of new definition is proved. An application of proposed work in decision making is solved in the framework of SVNG where the failure of existing definitions and effectiveness of new definitions is demonstrated.

6.1 Improvements in Interval Valued Fuzzy Graphs and Interval Valued Intuitionistic Fuzzy Graphs

In this section, first, the existing definitions of IVFG and IVIFG are reviewed and the shortcomings of these algebraic structures are pointed out with the help of examples. New definitions for IVFG and IVIFG are developed and their fitness are verified with the help of some examples.

First, we review the definition of IVFG defined in [79] and provide an example to show the limitation of the definitions.

6.1.1 Definition [79]

An IVFG is a duplet $\dot{G} = (\underset{\cdot}{V}, \dot{E})$ where $\underset{\cdot}{V}$ denotes a collection of nodes and $\dot{E}$ denotes the collection of edges between these vertices such that

1) Every $\underset{\cdot}{t} \in \underset{\cdot}{V}$ is characterized by a function $\hat{S}$ representing the membership degree of $\underset{\cdot}{t} \in \underset{\cdot}{V}$. Basically, $\hat{S} = [\hat{S}^L, \hat{S}^U]$ is a closed subinterval [0, 1].
2) Every $e \in \dot{E}$ is characterized by a function $\hat{S}$ denoting the degree of membership $e \in \underset{\cdot}{V} \times \underset{\cdot}{V}$. Basically $\hat{S} = [\hat{S}^L, \hat{S}^U]$ satisfying the conditions:

$$\hat{S}^L(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) \leq \min\left(\hat{S}^L(\underset{\cdot}{t}_i), \hat{S}^L(\underset{\cdot}{t}_j)\right), \hat{S}^U(\underset{\cdot}{t}_i, \underset{\cdot}{t}_j) \leq \min\left(\hat{S}^U(\underset{\cdot}{t}_i), \hat{S}^U(\underset{\cdot}{t}_j)\right)$$

6.1.2 Example

Consider the following Figure 63 where we observed that while determining the edge values, we get undefined intervals as highlighted. This shows that the current notion of an IVFG is not satisfactory and that is leads us to develop a new definition.

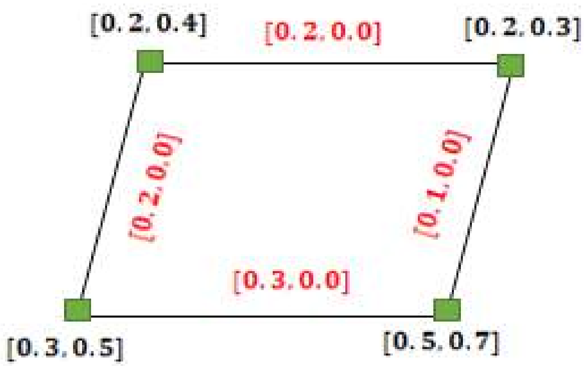


Figure 63 (Interval valued fuzzy graph containing undefined intervals)

6.1.3 Definition

An IVFG is a duplet $\dot{\mathrm{G}} = (\underset{.}{\mathrm{V}}, \dot{\mathrm{E}})$ where $\underset{.}{\mathrm{V}}$ denotes a collection of nodes and $\dot{\mathrm{E}}$ denotes the collection of edges between these vertices such that

1) Every $\underset{.}{\mathrm{t}} \in \underset{.}{\mathrm{V}}$ is characterized by a function $\hat{\mathrm{S}}$ representing the membership grade of $\underset{.}{\mathrm{t}} \in \underset{.}{\mathrm{V}}$. Basically, $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^U]$ is a closed subintervals of the unit interval [0, 1].
2) Every $e \in \dot{\mathrm{E}}$ is characterized by a function $\hat{\mathrm{S}}$ denoting the membership grade of $e \in \underset{.}{\mathrm{V}} \times \underset{.}{\mathrm{V}}$. The closed subinterval $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^U]$ satisfying the conditions:

   $\hat{\mathrm{S}}^L(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \leq \min\left(\hat{\mathrm{S}}^L(\underset{.}{\mathrm{t}}_i), \hat{\mathrm{S}}^L(\underset{.}{\mathrm{t}}_j)\right)$ and

   $\hat{\mathrm{S}}^U(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \leq \min\left(\hat{\mathrm{S}}^U(\underset{.}{\mathrm{t}}_i), \hat{\mathrm{S}}^U(\underset{.}{\mathrm{t}}_j)\right)$ such that $\hat{\mathrm{S}}^U(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \geq \min\left(\hat{\mathrm{S}}^L(\underset{.}{\mathrm{t}}_i), \hat{\mathrm{S}}^L(\underset{.}{\mathrm{t}}_j)\right)$.

This definition is supported with the help of following example in which we show that none of the intervals is undefined and that Definition 6.1.3 works perfectly.

6.1.4 Example

The following IVFG depicted in Figure 64 is in accordance with Definition 6.1.3.

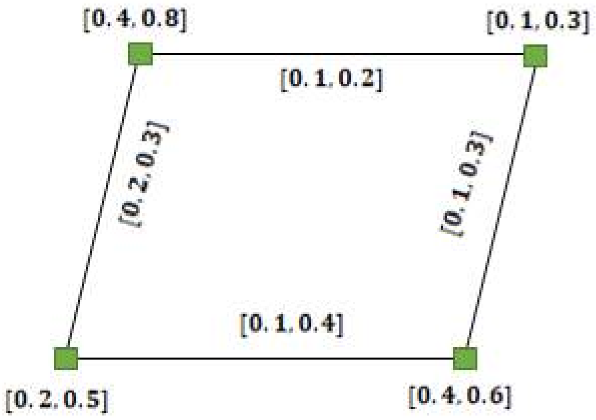


Figure 64 (Interval valued fuzzy graph compatible with Definition 6.1.3)

Next we provide a critical study of IVIFGs with a definition which is based on IVIFG which yield undefined closed sub-intervals. First, we review the existing definition and the existing shortcomings with the help of examples. Then the new definition is developed and supported by an example.

6.1.5 Definition [85]

An IVIFG is a duplet $\dot{\mathrm{G}} = (\dot{\mathrm{V}}, \dot{\mathrm{E}})$ where the set of nodes is denoted by $\dot{\mathrm{V}}$ and $\dot{\mathrm{E}}$ is the collection of edges between these nodes such that

1) Every $ṭ \in \dot{\mathrm{V}}$ is characterized by two functions $\hat{\mathrm{S}}$ and $Đ$ denoting the membership and non-membership grades of $ṭ \in \dot{\mathrm{V}}$. Basically, $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^U]$ and $Đ = [Đ^L, Đ^U]$ are closed subintervals of the unit interval [0, 1] with a condition that $0 \leq \hat{\mathrm{S}}^U + Đ^U \leq 1$. Moreover, the term $R = [R^L, R^U]$ denote the hesitancy level of $ṭ \in \dot{\mathrm{V}}$ such that $R^U = 1 - \hat{\mathrm{S}}^L - Đ^L$ and $R^L = 1 - \hat{\mathrm{S}}^U - Đ^U$.
2) Every $e \in \dot{\mathrm{E}}$ is characterized by two functions $\hat{\mathrm{S}}$ and $Đ$ denoting the membership and non-membership grades of $e \in \dot{\mathrm{V}} \times \dot{\mathrm{V}}$. Basically $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^U]$ and $Đ = [Đ^L, Đ^U]$ satisfying the conditions:

$$\hat{\mathrm{S}}^L(ṭ_i, ṭ_j) \leq \min\left(\hat{\mathrm{S}}^L(ṭ_i), \hat{\mathrm{S}}^L(ṭ_j)\right), \hat{\mathrm{S}}^U(ṭ_i, ṭ_j) \leq \min\left(\hat{\mathrm{S}}^U(ṭ_i), \hat{\mathrm{S}}^U(ṭ_j)\right)$$

$$Đ^L(ṭ_i, ṭ_j) \geq \max\left(Đ^L(ṭ_i), Đ^L(ṭ_j)\right), Đ^U(ṭ_i, ṭ_j) \geq \max\left(Đ^U(ṭ_i), Đ^U(ṭ_j)\right)$$

provided that $0 \leq \hat{\mathrm{S}}^U + Đ^U \leq 1$. Moreover, the term $\check{\mathrm{R}} = [\check{\mathrm{R}}^L, \check{\mathrm{R}}^U]$ denote the hesitancy level of $e \in \dot{\mathrm{E}}$ such that $\check{\mathrm{R}}^U = 1 - \hat{\mathrm{S}}^L - Đ^L$ and $\check{\mathrm{R}}^L = 1 - \hat{\mathrm{S}}^U - Đ^U$.

This Definition 6.1.5 seems to be weak in two ways. The first reason is that IVIFG is a generalization of IFG where the non-membership grade of edge is defined as $Đ(ṭ_i, ṭ_j) \leq \max\left(Đ(ṭ_i), Đ(ṭ_j)\right)$ which is not followed in defining IVIFG raising a question on its perfection. Another shortcoming in Definition 6.1.5 is we may sometime have obtained undefined closed intervals which should be avoided. For further illustration, consider the example as follows.

6.1.6 Example

Consider the following IVIFG in Figure 65 where it is observed that we get some undefined intervals by applying Definition 6.1.5. The undefined closed intervals are highlighted.

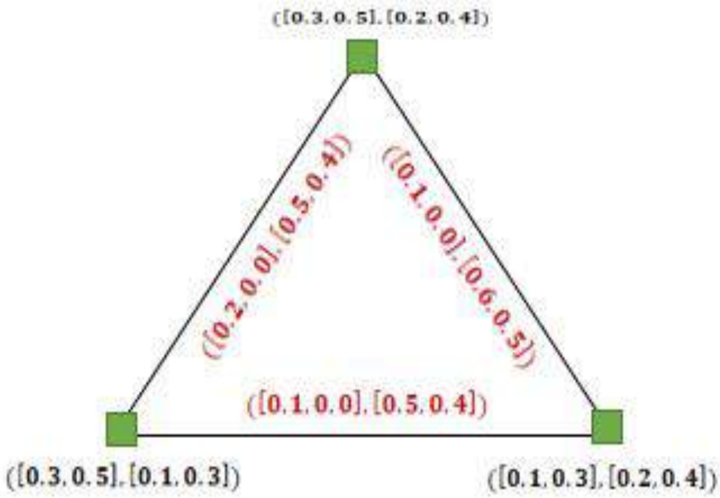


Figure 65 (Interval valued intuitionistic fuzzy graph obtained using Definition 6.1.5 having undefined intervals)

Keeping in mind the weakness of Definition 6.1.5 as demonstrated by Example 6.1.6 , we propose a new definition of IVIFG which is well defined and truly a generalization of IFG.

6.1.7 Definition

An IVIFG is a pair $\dot{G} = (\underset{.}{V}, \dot{E})$ where the set of nodes is denoted by $\underset{.}{V}$ and $\dot{E}$ is the collection of edges between these nodes such that

1) Every $\underset{.}{t} \in \underset{.}{V}$ is characterized by two functions $\hat{S}$ and Đ denoting the membership and non-membership grades of $\underset{.}{t} \in \underset{.}{V}$. Basically, $\hat{S} = [\hat{S}^L, \hat{S}^U]$ and $Đ = [Đ^L, Đ^U]$ are closed subintervals of the unit interval [0, 1] with a condition that $0 \leq \hat{S}^U + Đ^U \leq 1$. Moreover,

the term $R = [R^L, R^U]$ denotes the hesitancy level of ţ ∈ Ṿ such that $R^U = 1 - \hat{S}^L - Đ^L$ and $R^L = 1 - \hat{S}^U - Đ^U$.

2) Every e ∈ Ẻ is characterized by two functions Ŝ and Đ denoting the membership and non-membership grades of e ∈ Ṿ × Ṿ. Basically $\hat{S} = [\hat{S}^L, \hat{S}^U]$ and $Đ = [Đ^L, Đ^U]$ are such that:

$\hat{S}^L(ţ_i, ţ_j) \leq \min\left(\hat{S}^L(ţ_i), \hat{S}^L(ţ_j)\right)$.

$\hat{S}^U(ţ_i, ţ_j) \leq \min\left(\hat{S}^U(ţ_i), \hat{S}^U(ţ_j)\right)$ such that $\hat{S}^U(ţ_i, ţ_j) \geq \min\left(\hat{S}^L(ţ_i), \hat{S}^L(ţ_j)\right)$.

$Đ^L(ţ_i, ţ_j) \leq \max\left(Đ^L(ţ_i), Đ^L(ţ_j)\right)$.

$Đ^U(ţ_i, ţ_j) \leq \max\left(Đ^U(ţ_i), Đ^U(ţ_j)\right)$ such that $Đ^U(ţ_i, ţ_j) \geq \max\left(Đ^L(ţ_i), Đ^L(ţ_j)\right)$.

provided that $0 \leq \hat{S}^U + Đ^U \leq 1$. Moreover, the term $\check{R} = [\check{R}^L, \check{R}^U]$ denote the hesitancy level of e ∈ Ẻ such that $\check{R}^U = 1 - \hat{S}^L - Đ^L$ and $\check{R}^L = 1 - \hat{S}^U - Đ^U$.

The following example shows that Definition 6.1.7 improves the concept of IVIFG and there is no chance of getting undefined intervals.

6.1.8 Example

Consider the graph depicted in Figure 66 demonstrating the new definition of IVIFG.

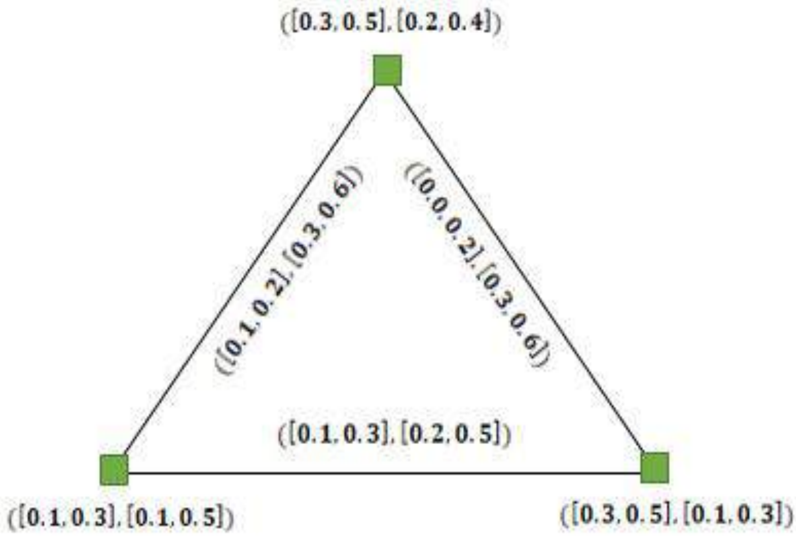


Figure 66 (Interval valued intuitionistic fuzzy graph)

The next section is based on shortcomings of existing definitions of the complement of an IVFG and of an IVIFG together with the development of new definitions. The existing and new definitions are demonstrated with examples.

6.2 New Definitions for Complement of Interval Valued Fuzzy Graph and Interval Valued Intuitionistic Fuzzy Graph

It is observed that the definition of complement of IVFG did not provide justifiable results on some occasions. The results obtained using the current definition of IVFG leads us to obtain some undefined intervals, and therefore in this section we propose a new definition for complement of IVFG.

For the first time, the complement of IVFG was defined in [84]. First, we review the existing definition and with the help of an example we pointed out its shortcomings.

6.2.1 Definition [84]

The complement of an IVFG $\dot{G} = (\underset{.}{V}, \dot{E})$ is defined by $\dot{G}^c = (\underset{.}{V}^c, \dot{E}^c)$ where $\underset{.}{V}^c = \underset{.}{V}$ and the membership and non-membership grades of $\dot{E}$ satisfying the conditions:

$(\hat{S}^L)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right) - \hat{S}^L\left((\underset{.}{t}_i, \underset{.}{t}_j)\right)$ $\quad (\hat{S}^U)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^U(\underset{.}{t}_i), \hat{S}^U(\underset{.}{t}_j)\right) - \hat{S}^U(\underset{.}{t}_i, \underset{.}{t}_j)$.

6.2.2 Example

In the following two IVFGs, Figure 68 represents the complement of IVFG depicted in Figure 67 based on based on existing definition which leads us to certain undefined intervals as highlighted.

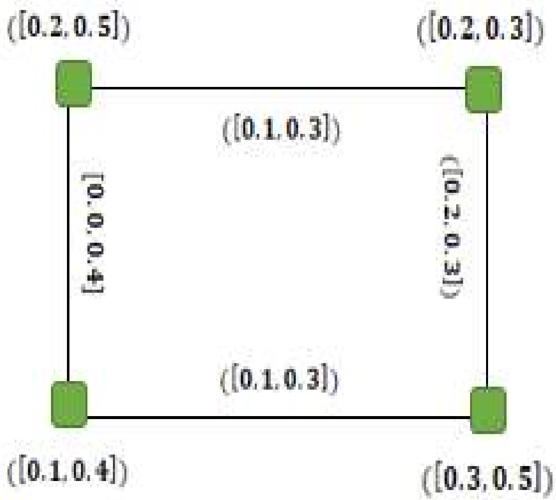


Figure 67 (Interval valued fuzzy graph)

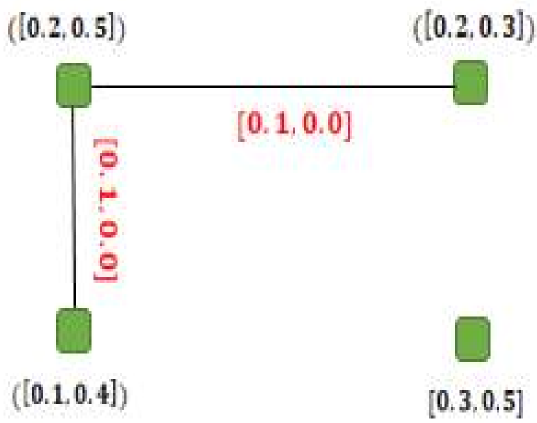


Figure 68 (Complement of graph depicted in Figure 67 based on Definition (6.2.1) giving us undefined intervals)

Another definition of complement of IVFG is defined in [79] which is described as:

6.2.3 Definition [79]

The complement of an IVFG $\dot{G} = (\underset{.}{V}, \dot{E})$ is defined by $\dot{G}^c = (\underset{.}{V}^c, \dot{E}^c)$ where $\underset{.}{V}^c = \underset{.}{V}$ and the membership functions of $\dot{E}^c$ satisfies the following:

$$(\hat{S}^L)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \begin{cases} 0 & if \quad \hat{S}^L(\underset{.}{t}_i, \underset{.}{t}_j) \geq 0 \\ \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right) & if \quad \hat{S}^L(\underset{.}{t}_i, \underset{.}{t}_j) = 0 \end{cases}$$

$$(\hat{S}^U)^c(ṭ_i, ṭ_j) = \begin{cases} 0 & if \quad \hat{S}^U(ṭ_i, ṭ_j) \geq 0 \\ \min\left(\hat{S}^U(ṭ_i), \hat{S}^U(ṭ_j)\right) & if \quad \hat{S}^U(ṭ_i, ṭ_j) = 0 \end{cases}$$

This definition is not well defined as it is not valid for all types of IVFGs. The problem in this definition of complement is that it does not possesses the property $(\dot{G}^c)^c = \dot{G}$. This is demonstrated in the following example.

6.2.4 Example

The complement of IVFG as exhibited in Figure 67 is depicted in Figure 69 below and it is observed that all the edges have disappeared surprisingly which seems insignificant. Now we determine the complement of Figure 69 again and put it on display in Figure 70 which clearly indicates that the basic result of complement i.e. $(\dot{G}^c)^c = \dot{G}$ does not hold true for Definition 6.2.3.

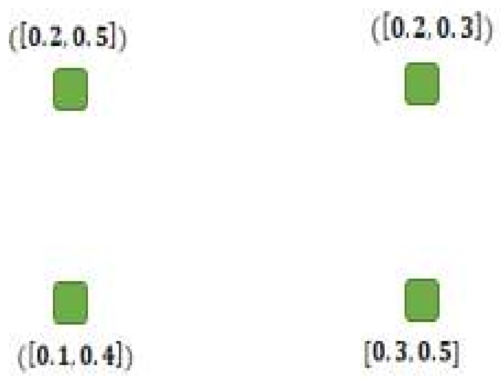


Figure 69 (Complement of Figure (67))

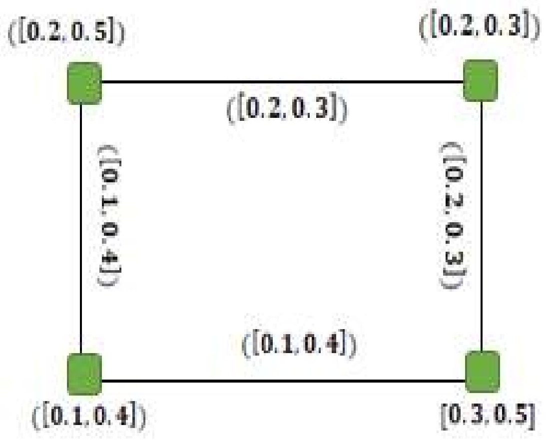


Figure 70 (Complement of Figure (69) based on Definition (6.2.3))

Moreover, Definition 6.2.3 is used to find the complement of a complete IVFG in [79] by Akram and Dudek where this definition possesses the property of a complement i.e. $(\dot{G}^c)^c = \dot{G}$. But we need a definition of complement which can be used for entire range of IVFG and not only complete IVFG therefore we develop a new well-defined definition for the complement of IVFGs as follows.

6.2.5 Definition

The complement of IVFG $\dot{G} = (\underset{.}{V}, \dot{E})$ is defined by $\dot{G}^c = (\underset{.}{V}^c, \dot{E}^c)$ where $\underset{.}{V}^c = \underset{.}{V}$ and the membership grades of $\dot{E}$ satisfies the following properties:

$$(\hat{S}^L)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right) - \hat{S}^L\left((\underset{.}{t}_i, \underset{.}{t}_j)\right).$$

$$(\hat{S}^U)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^U(\underset{.}{t}_i), \hat{S}^U(\underset{.}{t}_j)\right) - \hat{S}^U(\underset{.}{t}_i, \underset{.}{t}_j) + \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right).$$

In the following examples, we not only demonstrate the new definition but also verify that the basic property of the notion of complement as presented in definition 6.2.5 is available, i.e. $(\dot{G}^c)^c = \dot{G}$.

6.2.6 Example

In what follows, Figure 71 represents the complement of Figure 67 and Figure 72 represents the complement of Figure 71. Hence, as we claimed the equality $(\dot{G}^c)^c = \dot{G}$ holds true for our proposed new definition.

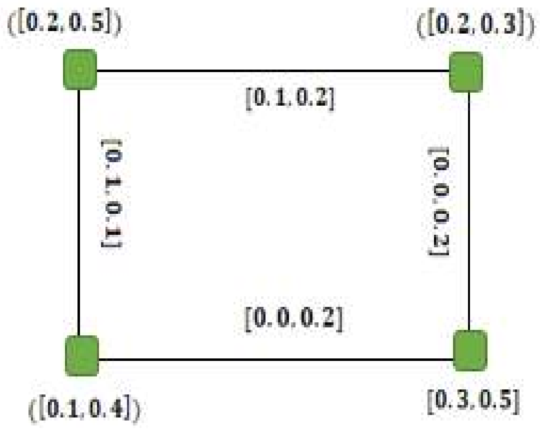


Figure 71 (Complement of graph depicted in Figure (67))

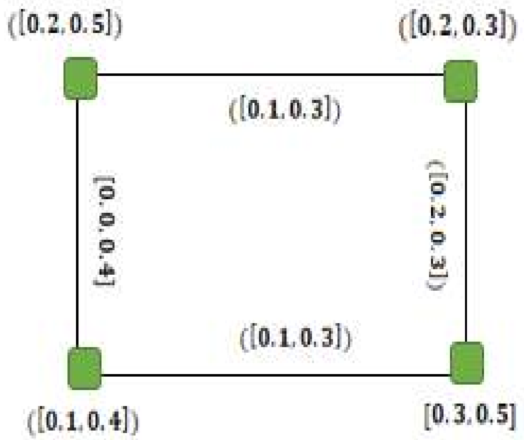


Figure 72 (The complement of graph depicted in Figure (71))

Now we develop the definition of complement for IVIFGs because the existing definition does not make any sense in some cases as we described for IVFGs.

6.2.7 Definition

The complement of IVIFG $\dot{G} = (\underset{\cdot}{V}, \dot{E})$ is defined by $\dot{G}^c = (\underset{\cdot}{V}^c, \dot{E}^c)$ where $\underset{\cdot}{V}^c = \underset{\cdot}{V}$ and the membership and non-membership grades of $\dot{E}$ satisfied the conditions:

$(\hat{S}^L)^c(ṭ_i, ṭ_j) = \min\left(\hat{S}^L(ṭ_i), \hat{S}^L(ṭ_j)\right) - \hat{S}^L\left((ṭ_i, ṭ_j)\right).$

$(\hat{S}^U)^c(ṭ_i, ṭ_j) = \min\left(\hat{S}^U(ṭ_i), \hat{S}^U(ṭ_j)\right) - \hat{S}^U(ṭ_i, ṭ_j) + \min\left(\hat{S}^L(ṭ_i), \hat{S}^L(ṭ_j)\right).$

$(Đ^L)^c(ṭ_i, ṭ_j) = \max\left(Đ^L(ṭ_i), Đ^L(ṭ_j)\right) - Đ^L\left((ṭ_i, ṭ_j)\right).$

$(Đ^U)^c(ṭ_i, ṭ_j) = \max\left(Đ^U(ṭ_i), Đ^U(ṭ_j)\right) - Đ^U(ṭ_i, ṭ_j) + \max\left(Đ^L(ṭ_i), Đ^L(ṭ_j)\right).$

6.2.8 Example

Consider the following IVIFGs where the graph depicted in Figure 74 represents the complement of Figure 73. Moreover, through some easy calculations, one can easily verify that $(\dot{G}^c)^c = \dot{G}$.

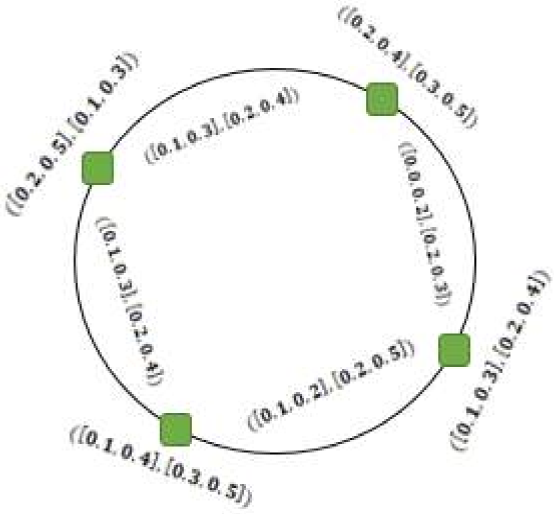


Figure 73 (Interval valued intuitionistic fuzzy graph)

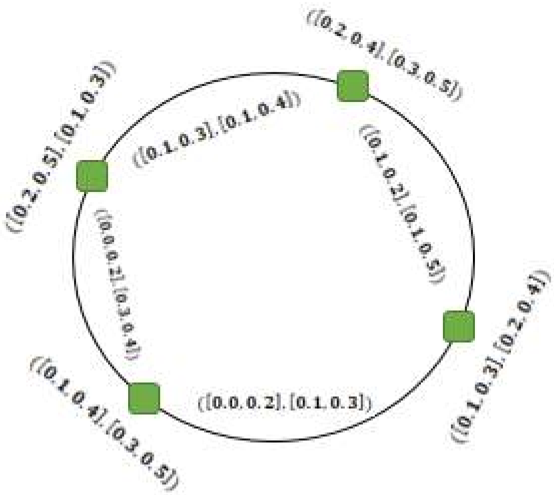


Figure 74 (Complement of interval valued intuitionistic fuzzy graph depicted in Figure (73))

In the next section, we discuss the limitations of SVNGS, IVNGs, and their complements. In addition, we develop some new valid definitions.

6.3 Improvements in Single Valued Neutrosophic Graphs and Interval Valued Neutrosophic Graphs

In this section, we study the SVNGs proposed in [89] and IVNGs developed in [94] and we describe their shortcomings with some examples. Then a new definition is proposed for both SVNGs and IVNGs and their complements are discussed. In support of these new definitions, we present some examples.

6.3.1 Definition [89]

A SVNG is a pair $\dot{G} = (\underset{\cdot}{V}, \dot{E})$ where $\underset{\cdot}{V}$ is the set of nodes and $\dot{E}$ is the collection of edges between these nodes such that

1) Every $\underset{\cdot}{t} \in \underset{\cdot}{V}$ is characterized by three functions $\hat{S}: \underset{\cdot}{V} \to [0, 1]$, $\hat{I}: \underset{\cdot}{V} \to [0, 1]$ and $Đ: \underset{\cdot}{V} \to [0, 1]$ denoting the membership, neutral and non-membership grades of $\underset{\cdot}{t} \in \underset{\cdot}{V}$ which satisfy $0 \leq \hat{S} + \hat{I} + Đ \leq 3$.

2) Every $e \in \dot{\mathrm{E}}$ is characterized by three functions $\hat{\mathrm{S}}: \underset{.}{\mathrm{V}} \times \underset{.}{\mathrm{V}} \rightarrow [0,1], \hat{\mathrm{I}}: \underset{.}{\mathrm{V}} \times \underset{.}{\mathrm{V}} \rightarrow [0,1]$ and $\mathrm{Đ}: \underset{.}{\mathrm{V}} \times \underset{.}{\mathrm{V}} \rightarrow [0,1]$ denoting the membership, neutral and non-membership grades of $e \in \underset{.}{\mathrm{V}} \times \underset{.}{\mathrm{V}}$ satisfies:

$$\hat{\mathrm{S}}(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \leq \min\left(\hat{\mathrm{S}}(\underset{.}{\mathrm{t}}_i), \hat{\mathrm{S}}(\underset{.}{\mathrm{t}}_j)\right)$$

$$\hat{\mathrm{I}}(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \geq \max\left(\hat{\mathrm{I}}(\underset{.}{\mathrm{t}}_i), \hat{\mathrm{I}}(\underset{.}{\mathrm{t}}_j)\right)$$

$$\mathrm{Đ}(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \geq \max\left(\mathrm{Đ}(\underset{.}{\mathrm{t}}_i), \mathrm{Đ}(\underset{.}{\mathrm{t}}_j)\right)$$

Which satisfy the inequality $0 \leq \hat{\mathrm{S}} + \hat{\mathrm{I}} + \mathrm{Đ} \leq 3$.

It is observed that this definition is not suitable and needed to be modified based on two reasons. The first reason lies in the fact that as SVNG is a generalization of IFG. So, by exempting the indeterminacy value Definition 6.3.1 should reduce to IFG but in actually it does not happen as the non-membership degree of an edge in IFG and in SVNG are defined in a different way i.e. the non-membership degree for IFG is defined as $\mathrm{Đ}(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \leq \max\left(\mathrm{Đ}(\underset{.}{\mathrm{t}}_i), \mathrm{Đ}(\underset{.}{\mathrm{t}}_j)\right)$ while in case of SVNGs it is defined as $\mathrm{Đ}(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) \geq \max\left(\mathrm{Đ}(\underset{.}{\mathrm{t}}_i), \mathrm{Đ}(\underset{.}{\mathrm{t}}_j)\right)$.

The second reason is based on definition of complement of SVNGs. As the complement of an SVNG needed to be an SVNG but it did not happen in some occasions. We present the current definition of complement of SVNG and with the help of an example describe that the complement of an SVNG is not well defined.

6.3.2 Definition [89]

The complement of SVNG $\dot{\mathrm{G}} = \left(\underset{.}{\mathrm{V}}, \dot{\mathrm{E}}\right)$ is defined by $\dot{\mathrm{G}}^c = \left(\underset{.}{\mathrm{V}}^c, \dot{\mathrm{E}}^c\right)$ where $\underset{.}{\mathrm{V}}^c = \underset{.}{\mathrm{V}}$ and the membership, neutral value and non-membership grades of $\dot{\mathrm{E}}$ satisfies the conditions:

$(\hat{\mathrm{S}})^c(\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j) = \min\left(\hat{\mathrm{S}}(\underset{.}{\mathrm{t}}_i), \hat{\mathrm{S}}(\underset{.}{\mathrm{t}}_j)\right) - \hat{\mathrm{S}}\left((\underset{.}{\mathrm{t}}_i, \underset{.}{\mathrm{t}}_j)\right)$.

$(\hat{\mathrm{I}})^{c}(ṭ_i, ṭ_j) = \max\left(\hat{\mathrm{I}}(ṭ_i), \hat{\mathrm{I}}(ṭ_j)\right) - \hat{\mathrm{I}}\left((ṭ_i, ṭ_j)\right).$

$(\mathrm{Đ})^{c}(ṭ_i, ṭ_j) = \max\left(\mathrm{Đ}(ṭ_i), \mathrm{Đ}(ṭ_j)\right) - \mathrm{Đ}\left((ṭ_i, ṭ_j)\right).$

Keeping in mind Definition 6.3.2 and consider the following example, where talking the complement of an SVNG does not remain an SVNG. All this leads to some modification in the basic definition of SVNG which not only generalizes an IFG but also compatible to satisfy the basic properties of taking complement.

6.3.3 Example

In the following figures, Figure 75 represents an SVNGs and Figure 76 represents the complement of SVNGs depicted in Figure 75. It is clear from Figure 76 that the complement of SVNG is not an SVNG.

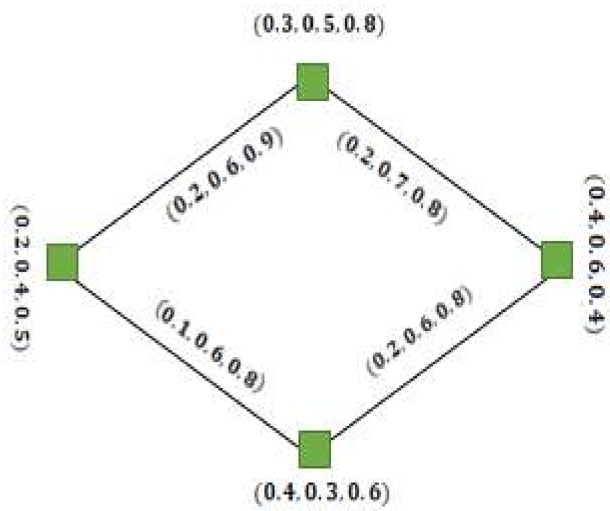


Figure 75 (Single valued neutrosophic graph)

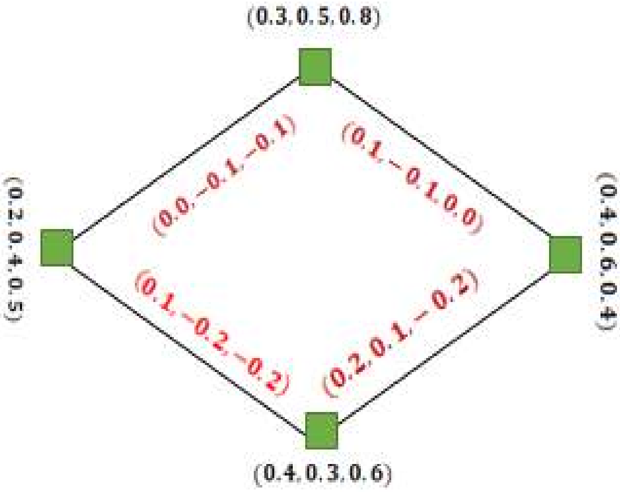


Figure 76 (Complement of single valued neutrosophic graphic depicted in Figure (75))

Now we present a new definition for SVNG as a generalization of IFG which is well-defined.

6.3.4 Definition

An SVNG is a pair $\dot{\mathrm{G}} = (\text{Ṿ}, \text{Ẻ})$ where Ṿ is the set of nodes and Ẻ is the collection of edges between these nodes such that

1. Every $\text{ṭ} \in \text{Ṿ}$ is characterized by three functions $\hat{\mathrm{S}}: \text{Ṿ} \to [0, 1], \hat{\mathrm{I}}: \text{Ṿ} \to [0, 1]$ and $\text{Đ}: \text{Ṿ} \to [0, 1]$ denoting the membership, neutral and non-membership grades of $\text{ṭ} \in \text{Ṿ}$ which satisfy the inequality $0 \leq \hat{\mathrm{S}} + \hat{\mathrm{I}} + \text{Đ} \leq 3$.
2. Every $e \in \text{Ẻ}$ is characterized by three functions $\hat{\mathrm{S}}: \text{Ṿ} \times \text{Ṿ} \to [0, 1], \hat{\mathrm{I}}: \text{Ṿ} \times \text{Ṿ} \to [0, 1]$ and $\text{Đ}: \text{Ṿ} \times \text{Ṿ} \to [0, 1]$ denoting the membership, neutral and non-membership grades of $e \in \text{Ṿ} \times \text{Ṿ}$ satisfying

$$\hat{\mathrm{S}}(\text{ṭ}_i, \text{ṭ}_j) \leq \min\left(\hat{\mathrm{S}}(\text{ṭ}_i), \hat{\mathrm{S}}(\text{ṭ}_j)\right)$$

$$\hat{\mathrm{I}}(\text{ṭ}_i, \text{ṭ}_j) \leq \max\left(\hat{\mathrm{I}}(\text{ṭ}_i), \hat{\mathrm{I}}(\text{ṭ}_j)\right)$$

$$\text{Đ}(\text{ṭ}_i, \text{ṭ}_j) \leq \max\left(\text{Đ}(\text{ṭ}_i), \text{Đ}(\text{ṭ}_j)\right)$$

provided that $0 \leq \hat{\mathrm{S}} + \hat{\mathrm{I}} + \text{Đ} \leq 3$.

6.3.5 Example

In the following figures, Figure 77 represents an SVNG compatible with the new definition and Figure 78 represents the complement of the SVNG depicted in Figure 77.

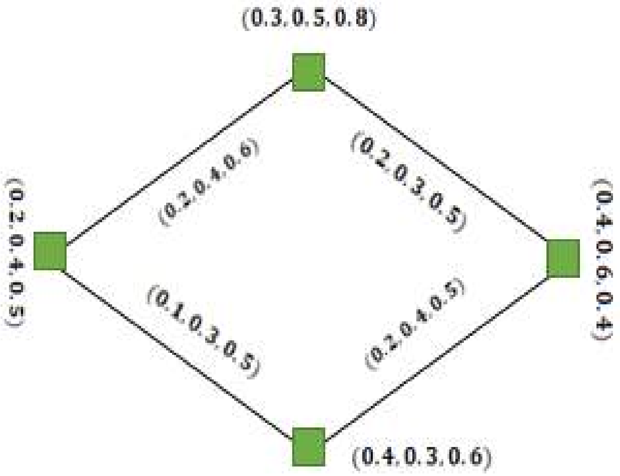


Figure 77 (Single valued neutrosophic graph)

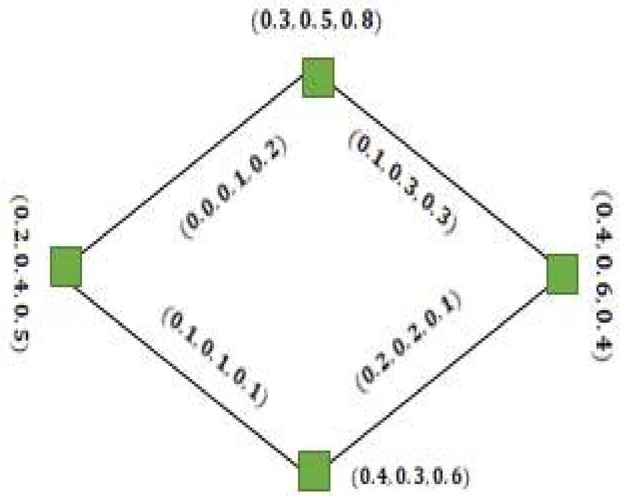


Figure 78 (Complement of graph depicted in Figure (77))

Likewise, in SVNGs the same kind of flaws exists in the definition of IVNGs. First, we present the existing definition of IVNG and point out its limitation with the help of an example. Then we develop the new definition for IVNG and support it with an example. After developing the new definition for IVNGs, we discuss the concept of complement for IVNGs by considering the defects exist in forming complements of IVFG and IVIFG.

6.3.6 Definition [94]

An IVNG is a pair $\dot{\mathrm{G}} = (\mathrm{\d{V}}, \dot{\mathrm{E}})$ where $\mathrm{\d{V}}$ is the set of nodes and $\dot{\mathrm{E}}$ is the collection of edges between these nodes such that

1. Every $\d{t} \in \mathrm{\d{V}}$ is characterized by three functions $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ denoting the membership, neutral and non-membership grades of $\d{t} \in \mathrm{\d{V}}$. Basically, $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^U], \hat{\mathrm{I}} = [\hat{\mathrm{I}}^L, \hat{\mathrm{I}}^U]$ and $\mathrm{Đ} = [\mathrm{Đ}^L, \mathrm{Đ}^U]$ are closed subintervals of the unit interval [0, 1] which satisfy the inequality $0 \le \hat{\mathrm{S}}^U + \hat{\mathrm{I}}^U + \mathrm{Đ}^U \le 3$.
2. Every $e \in \dot{\mathrm{E}}$ is characterized by three functions $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ denoting the membership, neutral and non-membership grades of $e \in \mathrm{\d{V}} \times \mathrm{\d{V}}$. Basically $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^L], \hat{\mathrm{I}} = [\hat{\mathrm{I}}^L, \hat{\mathrm{I}}^U]$ and $\mathrm{Đ} = [\mathrm{Đ}^L, \mathrm{Đ}^L]$ are such that:

$$\hat{\mathrm{S}}^L(\d{t}_i, \d{t}_j) \le \min\left(\hat{\mathrm{S}}^L(\d{t}_i), \hat{\mathrm{S}}^L(\d{t}_j)\right) \qquad \hat{\mathrm{S}}^U(\d{t}_i, \d{t}_j) \le \min\left(\hat{\mathrm{S}}^U(\d{t}_i), \hat{\mathrm{S}}^U(\d{t}_j)\right)$$

$$\hat{\mathrm{I}}^L(\d{t}_i, \d{t}_j) \ge \max\left(\hat{\mathrm{I}}^L(\d{t}_i), \hat{\mathrm{I}}^L(\d{t}_j)\right) \qquad \hat{\mathrm{I}}^U(\d{t}_i, \d{t}_j) \ge \max\left(\hat{\mathrm{I}}^U(\d{t}_i), \hat{\mathrm{I}}^U(\d{t}_j)\right)$$

$$\mathrm{Đ}^L(\d{t}_i, \d{t}_j) \ge \max\left(\mathrm{Đ}^L(\d{t}_i), \mathrm{Đ}^L(\d{t}_j)\right) \qquad \mathrm{Đ}^U(\d{t}_i, \d{t}_j) \ge \max\left(\mathrm{Đ}^U(\d{t}_i), \mathrm{Đ}^U(\d{t}_j)\right)$$

which satisfy the inequality $0 \le \hat{\mathrm{S}}^U + \hat{\mathrm{I}}^U + \mathrm{Đ}^U \le 3$.

Likewise, IVFG and IVIFG are defined, Definition 6.3.6 sometime leads to undefined closed sub-intervals as shown in following example.

6.3.7 Example

Consider the following example of an IVNG as exhibited in Figure 79 based on Definition 6.3.6 leading us to some undefined intervals.

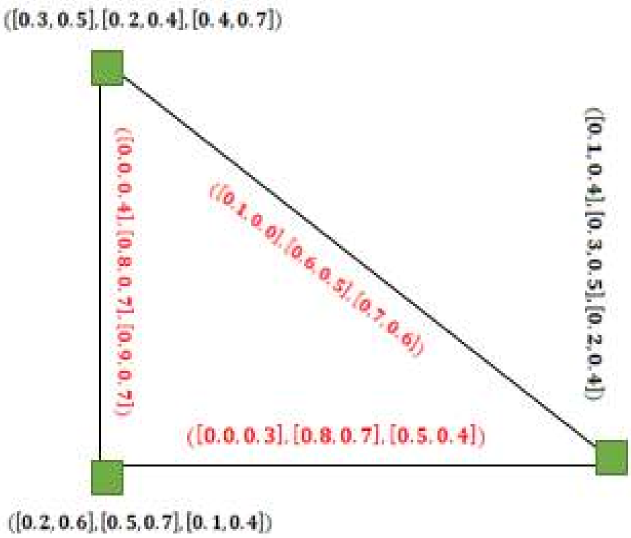


Figure 79 (Interval valued neutrosophic graph with undefined intervals)

Now we present a new definition of IVNG as follows.

6.3.8 Definition

An IVNG is a pair $\dot{\mathrm{G}} = (\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}})$ where $\underset{\cdot}{\mathrm{V}}$ is the set of nodes and $\dot{\mathrm{E}}$ is the collection of edges between these nodes such that

1) Every $\underset{\cdot}{\mathrm{t}} \in \underset{\cdot}{\mathrm{V}}$ is characterized by three functions $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ denoting the membership, neutral and non-membership grades of $\underset{\cdot}{\mathrm{t}} \in \underset{\cdot}{\mathrm{V}}$. Basically, $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^U], \hat{\mathrm{I}} = [\hat{\mathrm{I}}^L, \hat{\mathrm{I}}^U]$ and $\mathrm{Đ} = [\mathrm{Đ}^L, \mathrm{Đ}^U]$ are closed subintervals of the unit interval [0, 1] with a condition that $0 \leq \hat{\mathrm{S}}^U + \hat{\mathrm{I}}^U + \mathrm{Đ}^U \leq 3$.

2) Every $e \in \dot{\mathrm{E}}$ is characterized by three functions $\hat{\mathrm{S}}, \hat{\mathrm{I}}$ and $\mathrm{Đ}$ denoting the membership, neutral and non-membership grades of $e \in \underset{\cdot}{\mathrm{V}} \times \underset{\cdot}{\mathrm{V}}$. Basically $\hat{\mathrm{S}} = [\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^L], \hat{\mathrm{I}} = [\hat{\mathrm{I}}^L, \hat{\mathrm{I}}^U]$ and $\mathrm{Đ} = [\mathrm{Đ}^L, \mathrm{Đ}^L]$ are such that:

$$\hat{\mathrm{S}}^L(\underset{\cdot}{\mathrm{t}}_i, \underset{\cdot}{\mathrm{t}}_j) \leq \min\left(\hat{\mathrm{S}}^L(\underset{\cdot}{\mathrm{t}}_i), \hat{\mathrm{S}}^L(\underset{\cdot}{\mathrm{t}}_j)\right) \qquad \hat{\mathrm{S}}^U(\underset{\cdot}{\mathrm{t}}_i, \underset{\cdot}{\mathrm{t}}_j) \leq \min\left(\hat{\mathrm{S}}^U(\underset{\cdot}{\mathrm{t}}_i), \hat{\mathrm{S}}^U(\underset{\cdot}{\mathrm{t}}_j)\right)$$

$$\hat{\text{I}}^L(\text{ṭ}_i, \text{ṭ}_j) \leq \max\left(\hat{\text{I}}^L(\text{ṭ}_i), \hat{\text{I}}^L(\text{ṭ}_j)\right) \qquad \hat{\text{I}}^U(\text{ṭ}_i, \text{ṭ}_j) \leq \max\left(\hat{\text{I}}^U(\text{ṭ}_i), \hat{\text{I}}^U(\text{ṭ}_j)\right)$$

$$\text{Ð}^L(\text{ṭ}_i, \text{ṭ}_j) \leq \max\left(\text{Ð}^L(\text{ṭ}_i), \text{Ð}^L(\text{ṭ}_j)\right) \quad \text{Ð}^U(\text{ṭ}_i, \text{ṭ}_j) \leq \max\left(\text{Ð}^U(\text{ṭ}_i), \text{Ð}^U(\text{ṭ}_j)\right)$$

provided that $\hat{\text{S}}^U(\text{ṭ}_i, \text{ṭ}_j) \leq \hat{\text{S}}^L(\text{ṭ}_i, \text{ṭ}_j)$, $\hat{\text{I}}^L(\text{ṭ}_i, \text{ṭ}_j) \leq \hat{\text{I}}^U(\text{ṭ}_i, \text{ṭ}_j)$, $\text{Ð}^L(\text{ṭ}_i, \text{ṭ}_j) \leq \text{Ð}^U(\text{ṭ}_i, \text{ṭ}_j)$ and $0 \leq \hat{\text{S}}^U + \hat{\text{I}}^U + \text{Ð}^U \leq 3$.

6.3.9 Example

Consider the graph depicted in Figure 80 demonstrating the new definition of IVNGs.

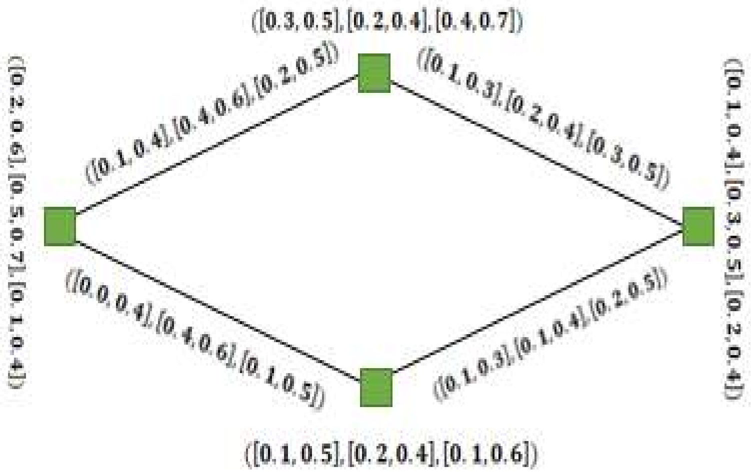


Figure 80 (Interval valued neutrosophic graph)

Next, we discuss the existing definition for taking the complement of IVNG and show its limitations with the help of examples. Then we propose a new definition for the complement of IVNG as the existing definition leads us to some undefined intervals.

6.3.10 Definition [89]

The complement of IVNG $\dot{\text{G}} = (\text{Ṿ}, \dot{\text{E}})$ is defined by $\dot{\text{G}}^c = (\text{Ṿ}^c, \dot{\text{E}}^c)$ where $\text{Ṿ}^c = \text{Ṿ}$ and the membership, abstinence and non-membership grades of $\dot{\text{E}}$ are satisfying:

$$\left(\hat{\text{S}}^L\right)^c(\text{ṭ}_i, \text{ṭ}_j) = \min\left(\hat{\text{S}}^L(\text{ṭ}_i), \hat{\text{S}}^L(\text{ṭ}_j)\right) - \hat{\text{S}}^L\left((\text{ṭ}_i, \text{ṭ}_j)\right).$$

$$(\hat{S}^U)^c(t_i, t_j) = \min\left(\hat{S}^U(t_i), \hat{S}^U(t_j)\right) - \hat{S}^U(t_i, t_j).$$

$$(\hat{I}^L)^c(t_i, t_j) = \max\left(\hat{I}^L(t_i), \hat{I}^L(t_j)\right) - \hat{I}^L\left((t_i, t_j)\right).$$

$$(\hat{I}^U)^c(t_i, t_j) = \max\left(\hat{I}^U(t_i), \hat{I}^U(t_j)\right) - \hat{I}^U(t_i, t_j).$$

$$(Đ^L)^c(t_i, t_j) = \max\left(Đ^L(t_i), Đ^L(t_j)\right) - Đ^L\left((t_i, t_j)\right).$$

$$(Đ^U)^c(t_i, t_j) = \max\left(Đ^U(t_i), Đ^U(t_j)\right) - Đ^U(t_i, t_j).$$

6.3.11 Example

The following IVNG depicted in Figure 81 shows that the Definition 6.3.10 for complement of IVNG leads us to some undefined intervals.

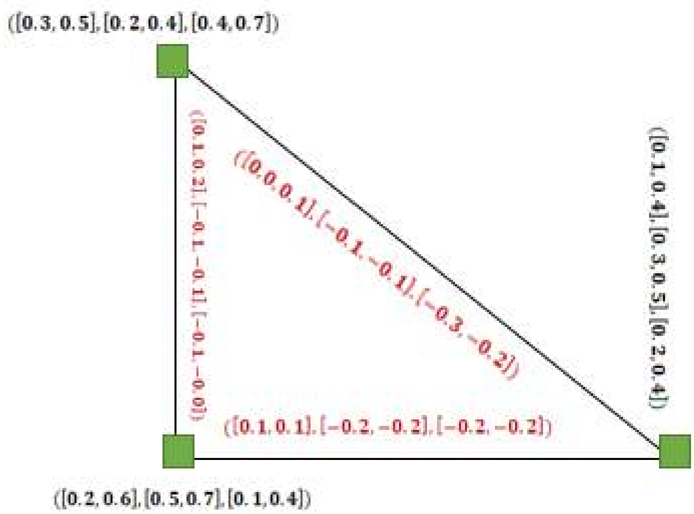


Figure 81(Complement of Figure (79))

As IVNG is a generalization of IVFG and IVIFG so its complement defined in [89] is not well-defined therefore we propose a new definition for complement of IVNG as follows.

6.3.12 Definition

The complement of IVNG $\dot{G} = (\underset{.}{V}, \dot{E})$ is defined by $\dot{G}^c = (\underset{.}{V}^c, \dot{E}^c)$ where $\underset{.}{V}^c = \underset{.}{V}$ and the membership, abstinence and non-membership grades of $\dot{E}$ are satisfying the conditions:

$(\hat{S}^L)^c(ṭ_i, ṭ_j) = \min\left(\hat{S}^L(ṭ_i), \hat{S}^L(ṭ_j)\right) - \hat{S}^L\left((ṭ_i, ṭ_j)\right).$

$(\hat{S}^U)^c(ṭ_i, ṭ_j) = \min\left(\hat{S}^U(ṭ_i), \hat{S}^U(ṭ_j)\right) - \hat{S}^U(ṭ_i, ṭ_j) + \min\left(\hat{S}^L(ṭ_i), \hat{S}^L(ṭ_j)\right).$

$(\hat{I}^L)^c(ṭ_i, ṭ_j) = \max\left(\hat{I}^L(ṭ_i), \hat{I}^L(ṭ_j)\right) - \hat{I}^L\left((ṭ_i, ṭ_j)\right).$

$(\hat{I}^U)^c(ṭ_i, ṭ_j) = \max\left(\hat{I}^U(ṭ_i), \hat{I}^U(ṭ_j)\right) - \hat{I}^U(ṭ_i, ṭ_j) + \max\left(\hat{I}^L(ṭ_i), \hat{I}^L(ṭ_j)\right).$

$(Đ^L)^c(ṭ_i, ṭ_j) = \max\left(Đ^L(ṭ_i), Đ^L(ṭ_j)\right) - Đ^L\left((ṭ_i, ṭ_j)\right).$

6.3.13 Example

Consider the following IVNGs where the graph depicted in Figure 83 is the complement of the graph depicted in Figure 82. Moreover, through some easy calculation, one can easily verify that $(\dot{G}^c)^c = \dot{G}$.

$(Đ^U)^c(ṭ_i, ṭ_j) = \max\left(Đ^U(ṭ_i), Đ^U(ṭ_j)\right) - Đ^U(ṭ_i, ṭ_j) + \max\left(Đ^L(ṭ_i), Đ^L(ṭ_j)\right).$

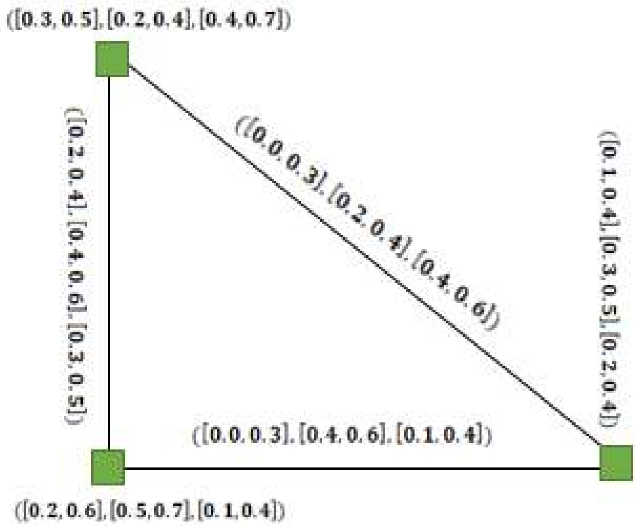


Figure 82 (Interval valued neutrosophic graph)

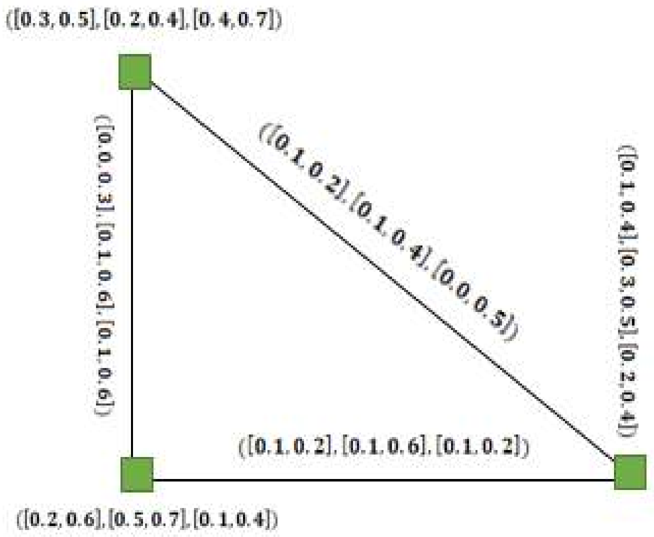


Figure 83 (Complement of graph depicted in Figure (82))

So far in our study, we have developed the theory of IVFGs and its generalizations providing examples that explained the limitation in existing definitions and the fitness of new definitions.

6.3.14 Remark

The work done proposed is applicable whether the graphs are directed or undirected.

Now a real-life application is presented to show the feasibility of proposed work. The process of ranking universities is explained based on SVNGs.

6.4 Ranking of Universities by Higher Education Commission

The higher education commission (HEC) of Pakistan needed to evaluate the Pakistani universities in research perspectives. HEC Pakistan plan to give away some financial support to those universities whose research is comparatively better. The university with good research score will earn greater financial support. Here we shall discuss the case of federal capital Islamabad. Four universities z_1, z_2, z_3 and z_4 from Islamabad are chosen representing Quaid e Azam university, National University of Science and Technology, Comsats university and International Islamic University Islamabad. The criterion for ranking of universities is

‘’research productivity’’. The members of quality assurance cell (QEC) of HEC provide their information in the form of a single valued neutrosophic preference relations (SVNPRs) as given in Figure 84. The weight vector in this case is given by $w = (0.2, 0.1, 0.3, 0.2)^T$. The detailed steps of algorithm for this process are explained followed by stepwise calculations of this problem.

6.4.1 Steps of Algorithm

Step 1: Establishing SVNPRs: This step involves the evaluation of objects by decision makers in the form of SVNGs.

Step 2: The SVNPRs forms a relational matrix known as single valued neutrosophic relational matrix (SVNRM) denoted by $\dot{\mathrm{R}} = (\dot{\mathrm{r}}_{ij})_{n\times n}$.

Step 3: This step involves the aggregation of information provided in the relational matrix in Step 2.

Step 4: This step involves the ranking of aggregated data based on score function.

Now we solve the problem stepwise demonstrating all calculations.

Step 1: this step involves the information from QEC of HEC Pakistan about universities in the form of SVNGs as exhibited in Figure 84.

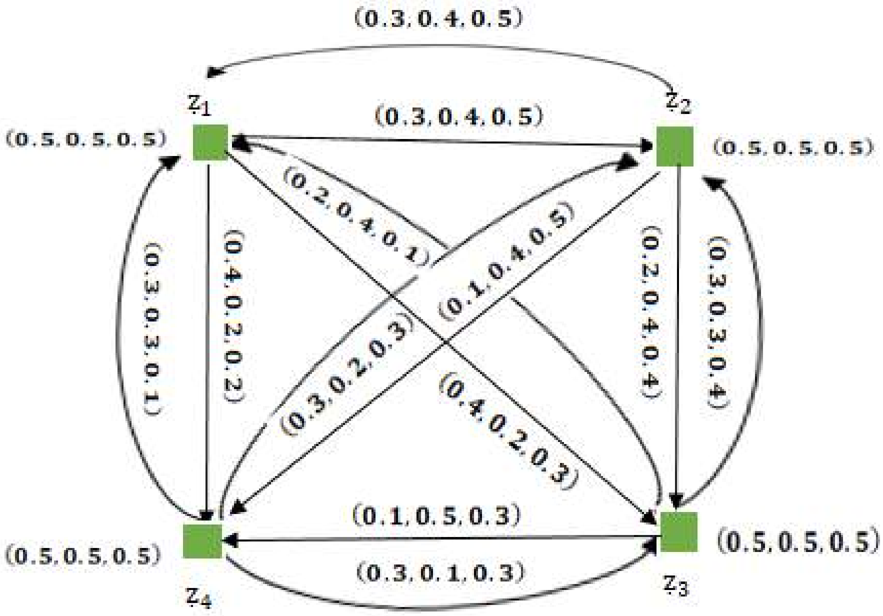


Figure 84 (Directed Network of the Single valued neutrosophic information)

Here, the importance of new definition of SVNG is clear. In decision making problems, we usually have two types of attributes known as attributes of cost type and of benefit types. In case the attributes are of cost types, the data is normalized using the definition of complement of SVNGs. Therefore, if we utilize the existing definition of SVNGs and its complement as we have done in Example 11. We obtained some undefined results. Similar is the case with other structures. All this shows the importance of new proposed definitions of several fuzzy graphs and their complements.

Step 2: From SVNPRs provided in Figure 83, we have the following relational matrix.

$$\dot{R} = (\dot{r}_{ij})_{4\times 4} = \begin{pmatrix} (0.5,0.5,0.5) & (0.3,0.4,0.5) & (0.4,0.2,0.3) & (0.4,0.2,0.2) \\ (0.3,0.4,0.5) & (0.5,0.5,0.5) & (0.2,0.4,0.4) & (0.1,0.4,0.5) \\ (0.2,0.4,0.1) & (0.3,0.3,0.4) & (0.5,0.5,0.5) & (0.1,0.5,0.3) \\ (0.3,0.3,0.1) & (0.3,0.2,0.3) & (0.3,0.1,0.3) & (0.5,0.5,0.5) \end{pmatrix}$$

Step 3: Using single-valued neutrosophic weighted averaging operator on relational matrix to aggregate the data.

$$\dot{r}_i = \hat{I}\dot{V}PFWA(\dot{r}_{i1}, \dot{r}_{i2}, \dots, \dot{r}_{in}) = \left(1 - (\prod_{j=1}^{n}(1 - \hat{S}_{ij})^{\ddot{w}_j}, \left(\prod_{j=1}^{n} \hat{I}_{ij}\right)^{\ddot{w}_j}, \left(\prod_{j=1}^{n} Đ_{ij}\right)^{\ddot{w}_j}\right)$$

The aggregation results are given as:

$$\dot{r}_1 = (0.349302063, 0.3552344, 0.4102356)$$

$$\dot{r}_2 = (0.204404888, 0.4912912, 0.5371592)$$

$$\dot{r}_3 = (0.266048598, 0.5219262, 0.3675541)$$

$$\dot{r}_4 = (0.297165899, 0.2919586, 0.3393458)$$

Step 4: The aggregated data obtained in Step 3 is ranked based on the following score function and the ranking results are given below.

$$\dot{s}(\dot{r}_i) = \frac{(\hat{S} + 1 - \hat{I} + 1 - Đ)}{3}$$

$\dot{s}(\dot{r}_1) = 0.527944021$, $\dot{s}(\dot{r}_2) = 0.391984829$, $\dot{s}(\dot{r}_3) = 0.458856099$, $\dot{s}(\dot{r}_4) = 0.555287166$

Finally, we have the following ranking

$$z_4 > z_1 > z_3 > z_2$$

The ranking results indicate that International Islamic University Islamabad stands at number 1 in research productivity among four universities of federal capital of Islamabad in the evaluation of quality assurance cell of HEC. If we use geometric aggregation operators instead of averaging aggregation operators, the results would be same.

6.5 Conclusion

In this chapter, we successfully pointed out the shortcomings existing in the current definitions of IVFG, IVIFG with the help of examples and developed a new improved definition for these concepts. It is also discussed that the current definitions of complement for IVFG and IVIFG leads us to some undefined results, therefore some new definitions for complement of IVFG and IVIFG are proposed and supported by examples. Further it is observed that the definition of SVNG has some serious flaws, described by examples. So, a new modified definition for SVNG has been developed and supported by some examples. The concept of IVNG is also modified and the validity of the modified definition is tested with examples. Throughout this chapter, we defined all those terms which were not previously defined well. In near future, we shall investigate some other results of IVFGs, IVIFG, IVNG, SVNGs for any possible shortcomings if existed and then try to improve them.

Chapter 7

An Approach Towards Decision Making and Shortest Path Problems Using the Concepts of Interval-Valued Pythagorean Fuzzy Information

7.1 Interval Valued Pythagorean Fuzzy Graphs

In this section the concept of IVPyFG is defined and some other useful graph related idea are explored. To make these ideas more understandable, each concept is supported with examples.

7.1.1 Definition [32]

An IVPyFS (over a universal set X) is of the form $A = \left\{x, \left(\hat{\mathrm{S}}(x), \mathrm{Đ}(x)\right)\right\}$ where $\hat{\mathrm{S}}$ and Đ are mappings from X to some subinterval of $[0, 1]$ i.e. $\hat{\mathrm{S}}(x) = \left[\hat{\mathrm{S}}^L(x), \hat{\mathrm{S}}^U(x)\right]$ and $\mathrm{Đ}(x) = [\mathrm{Đ}^L(x), \mathrm{Đ}^U(x)]$ with a condition $0 \leq \left(\hat{\mathrm{S}}^U(x)\right)^2 + \left(\mathrm{Đ}^U(x)\right)^2 \leq 1$.

Now we defined a score function for ranking purpose of IVPyFNs.

7.1.2 Definition

The score function for an IVPyFN $A = \left(\left[\hat{\mathrm{S}}^L, \hat{\mathrm{S}}^U\right], [\mathrm{Đ}^L, \mathrm{Đ}^U]\right)$ is defined as:

$$\hat{\mathrm{S}}C(A) = \frac{\left(\hat{\mathrm{S}}^L\right)^2 (1 - (\mathrm{Đ}^L)^2) + (\hat{\mathrm{S}}^U)^2\left((1 - (\mathrm{Đ}^U)^2)\right)}{2} \quad \hat{\mathrm{S}}C(A) \in [0, 1]$$

7.1.3 Definition

An IVPyFG is a pair $\dot{\mathrm{G}} = \left(\underset{\cdot}{\mathrm{V}}, \dot{\mathrm{E}}\right)$ where $\underset{\cdot}{\mathrm{V}}$ is the set of nodes and $\dot{\mathrm{E}}$ is the collection of edges between these nodes and

1) Every ṭ ∈ Ṿ is characterized by two functions Ŝ and Đ denote the degree of membership and non-membership of ṭ ∈ Ṿ. Basically, $\hat{S} = [\hat{S}^L, \hat{S}^L]$ and $Đ = [Đ^L, Đ^L]$ are subintervals of the unit interval [0, 1] with a condition that $0 \leq (\hat{S}^U)^2 + (Đ^U)^2 \leq 1$. Moreover, the term $R = [R^U, R^L]$ denote the refusal degree of ṭ ∈ Ṿ such that $R^U = \sqrt[n]{1 - (\hat{S}^L)^2 - (Đ^L)^2}$ and $R^L = \sqrt[n]{1 - (\hat{S}^U)^2 - (Đ^U)^2}$.

2) Every $e \in \dot{\check{E}}$ is characterized by two functions Ŝ and Đ denote the degree of membership and non-membership of $e = (\hat{S}, Đ) \in$ Ṿ × Ṿ. Basically $\hat{S} = [\hat{S}^L, \hat{S}^L]$ and $Đ = [Đ^L, Đ^L]$ are defined as:

$$\hat{S}^L(ṭ_i, ṭ_j) \leq \min\left(\hat{S}^L(ṭ_i), \hat{S}^L(ṭ_j)\right) \text{ and}$$

$$\hat{S}^U(ṭ_i, ṭ_j) \leq \min\left(\hat{S}^U(ṭ_i), \hat{S}^U(ṭ_j)\right) \text{ s.t } \hat{S}^U \geq \min\left(\hat{S}^L(ṭ_i), \hat{S}^L(ṭ_j)\right).$$

$$Đ^L(ṭ_i, ṭ_j) \leq \max\left(Đ^L(ṭ_i), Đ^L(ṭ_j)\right) \text{ and}$$

$$Đ^U(ṭ_i, ṭ_j) \leq \max\left(Đ^U(ṭ_i), Đ^U(ṭ_j)\right) \text{ s.t } Đ^U \geq \max\left(Đ^L(ṭ_i), Đ^L(ṭ_j)\right).$$

with a condition that $0 \leq (\hat{S}^U)^2 + (Đ^U(x))^2 \leq 1$. Moreover, the term $\check{R} = [\check{R}^U, \check{R}^L]$ denote the refusal degree of $e \in \dot{\check{E}}$ such that $\check{R}^U = \sqrt[n]{1 - (\hat{S}^L)^2 - (Đ^L)^2}$ and $\check{R}^L = \sqrt[n]{1 - (\hat{S}^U)^2 - (Đ^U)^2}$.

7.1.4 Theorem

IVPyFG is a generalization of IVFG and IVIFG.

Proof: We prove this result as follows:

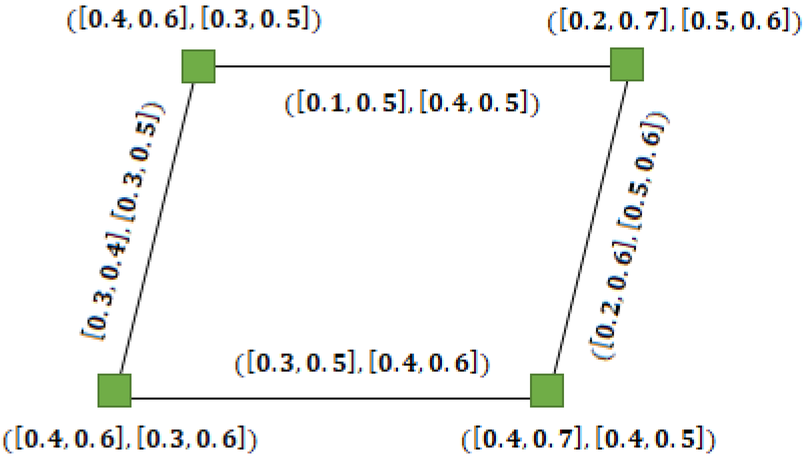
1) If we take $n = 1$. The definition of IVPyFG reduces to IVIFG.

2) If we take $n = 1$, $Đ^L = Đ^U = 0$. The definition of IVPFG reduces to IVFG.

This result shows the significance of the new concept as it the generalization of all the existing structures and can deal with the situations where the existing structures fails due to the limitations in their structures.

7.1.5 Example

The following Figure 85 is an example of IVPyFG.



.

Figure 85 (Interval valued Pythagorean fuzzy graph)

7.1.6 Definition

An IVPyFG $\dot{G}' = (\underset{.}{V}', \dot{E}')$ is said to be interval valued Pythagorean fuzzy subgraph (IVPFSG) of the graph $\dot{G} = (\underset{.}{V}, \dot{E})$ iff $\underset{.}{V}' \subseteq \underset{.}{V}$ and $\dot{E}' \subseteq \dot{E}$.

7.1.7 Example

The following Figure 86 provide is an example of IVPyFSG in Figure 85 of IVPyFG.

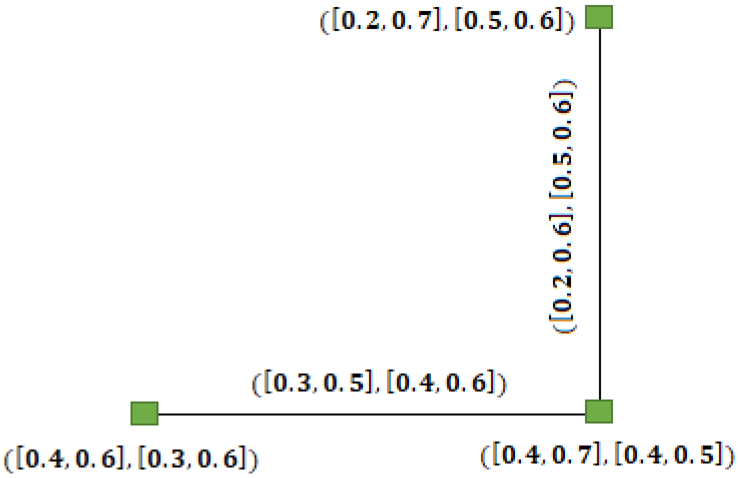


Figure 86 (Interval valued Pythagorean fuzzy subgraph)

One of the important concept in graph theory is complement of a graph which has been discussed widely in the frameworks of FGs and IFGs and other fuzzy algebraic structures. Here we defined the complement of IVPyFG which is a generalization of the complement of IVIFG and IVFG under some restrictions. The defined concept is then supported with the help of some examples and results.

7.1.8 Definition

The degree of a vertex in an IVPyFG $\dot{G} = (\underset{.}{V}, \dot{E})$ is denoted and defined by

$d(u) = \left(ɖ\hat{S}_{1L}(u), ɖ\hat{S}_{1U}(u), ɖÐ_{1L}(u), ɖÐ_{1U}(u)\right)$Where $ɖ\hat{S}_{1L}(u) = \sum_{\substack{u \neq y \\ u \in \underset{.}{V}}} \hat{S}_{2L}(uy)$, $ɖ\hat{S}_{1U}(u) = \sum_{\substack{u \neq y \\ u \in \underset{.}{V}}} \hat{S}_{2U}(uy)$ and $ɖÐ_{1L}(u) = \sum_{\substack{u \neq y \\ u,y \in \underset{.}{V}}} Ð_{2L}(uy)$, $ɖÐ_{1U}(u) = \sum_{\substack{u \neq y \\ u,y \in \underset{.}{V}}} Ð_{2U}(uy)$.

Here $(ɖ\hat{S}_{1L}(u), ɖ\hat{S}_{1U}(u))$ represents the lower and upper degrees of membership function of the vertex and $(ɖÐ_{1L}(u), ɖÐ_{1U}(u))$ represents the lower and upper degrees of non-membership function of the vertex.

7.1.9 Definition

The complement of IVPyFG $\dot{G} = (\underset{.}{V}, \dot{E})$ is defined by $\dot{G}^c = (\underset{.}{V}^c, \dot{E}^c)$ where $\underset{.}{V}^c = \underset{.}{V}$ and the membership grades of $\dot{E}$ are defined by:

$$(\hat{S}^L)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right) - \hat{S}^L\left((\underset{.}{t}_i, \underset{.}{t}_j)\right).$$

$$(\hat{S}^U)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^U(\underset{.}{t}_i), \hat{S}^U(\underset{.}{t}_j)\right) - \hat{S}^U(\underset{.}{t}_i, \underset{.}{t}_j) + \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right).$$

$$(Đ^L)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \max\left(Đ^L(\underset{.}{t}_i), Đ^L(\underset{.}{t}_j)\right) - Đ^L(\underset{.}{t}_i, \underset{.}{t}_j).$$

$$(Đ^U)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \max\left(Đ^U(\underset{.}{t}_i), Đ^U(\underset{.}{t}_j)\right) - Đ^U(\underset{.}{t}_i, \underset{.}{t}_j) + \max\left(Đ^L(\underset{.}{t}_i), Đ^L(\underset{.}{t}_j)\right).$$

7.1.10 Theorem

For IVPyFG $\dot{G} = (\underset{.}{V}, \dot{E})$, $(\dot{G}^c)^c = \dot{G}$.

Proof: Let $\dot{G} = (\underset{.}{V}, \dot{E})$ be an IVPyFG. Then by definition of complement, we prove this result as:

$$(\hat{S}^L)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right) - \hat{S}^L\left((\underset{.}{t}_i, \underset{.}{t}_j)\right).$$

$$\left((\hat{S}^L)^c\right)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right) - \left(\min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right) - \hat{S}^L\left((\underset{.}{t}_i, \underset{.}{t}_j)\right)\right) =$$
$$\hat{S}^L\left((\underset{.}{t}_i, \underset{.}{t}_j)\right).$$

$$(\hat{S}^U)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^U(\underset{.}{t}_i), \hat{S}^U(\underset{.}{t}_j)\right) - \hat{S}^U(\underset{.}{t}_i, \underset{.}{t}_j) + \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right).$$

$$\left((\hat{S}^U)^c\right)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \min\left(\hat{S}^U(\underset{.}{t}_i), \hat{S}^U(\underset{.}{t}_j)\right) - \left(\min\left(\hat{S}^U(\underset{.}{t}_i), \hat{S}^U(\underset{.}{t}_j)\right) - \hat{S}^U(\underset{.}{t}_i, \underset{.}{t}_j) +\right.$$
$$\left.\min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right)\right) + \min\left(\hat{S}^L(\underset{.}{t}_i), \hat{S}^L(\underset{.}{t}_j)\right).$$

$$\left((\hat{S}^U)^c\right)^c(\underset{.}{t}_i, \underset{.}{t}_j) = \hat{S}^U(\underset{.}{t}_i, \underset{.}{t}_j).$$

Similarly, we can prove that

$$((\text{Đ}^L)^c)^c(\d{t}_i, \d{t}_j) = \text{Đ}^L(\d{t}_i, \d{t}_j).$$

$$((\text{Đ}^U)^c)^c(\d{t}_i, \d{t}_j) = \text{Đ}^U(\d{t}_i, \d{t}_j).$$

Hence, $(\dot{G}^c)^c = \dot{G}$.

7.1.11 Example

Consider the following IVPyFGs where the graph depicted in Figure 88 is the complement of the graph depicted in Figure 87. Moreover, through some easy calculation, one can easily verify that $(\dot{G}^c)^c = \dot{G}$.

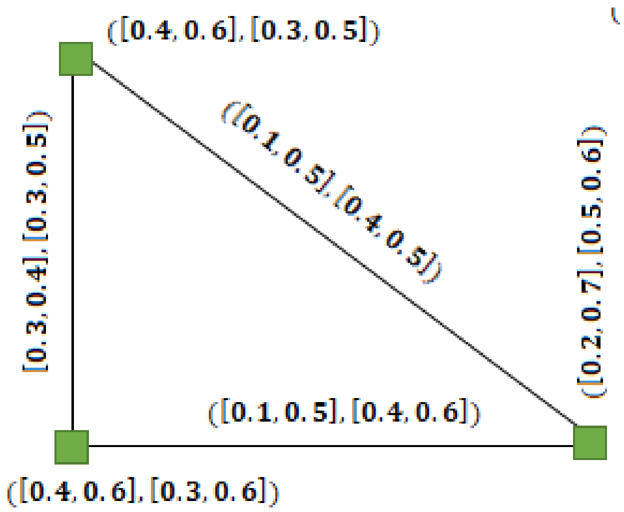


Figure 87 (Interval valued PyFG)

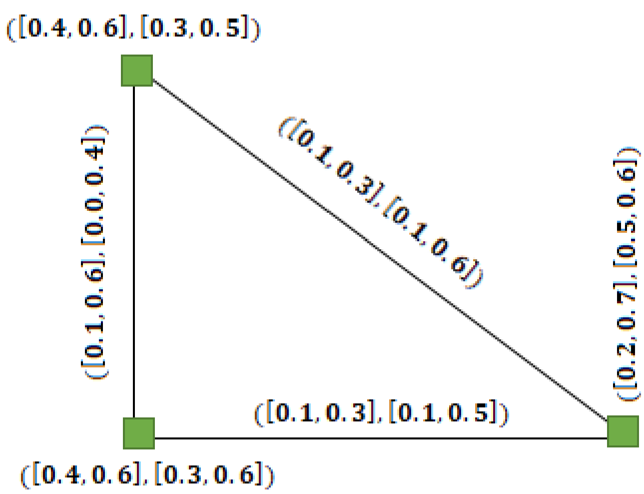


Figure 88 (Complement of IVPyFG in Figure (87))

7.1.12 Definition

An IVPyFG is known as:

5) ***Semi* Ŝ *strong*:** if $\hat{S}^L(ṭ_i, ṭ_j) = \min\left(\hat{S}^L(ṭ_i), \hat{S}^L(ṭ_j)\right)$ and $\hat{S}^U(ṭ_i, ṭ_j) = \min\left(\hat{S}^U(ṭ_i), \hat{S}^U(ṭ_j)\right)$

6) ***Semi* Ð *strong*:** if $Ð^L(ṭ_i, ṭ_j) = \max\left(Ð^L(ṭ_i), Ð^L(ṭ_j)\right)$ and $Ð^U(ṭ_i, ṭ_j) = \max\left(Ð^U(ṭ_i), Ð^U(ṭ_j)\right)$

7) ***Strong*:** if (1) and (2) holds true.

7.1.13 Example

The graph in Figure 89 is strong IVPyFG.

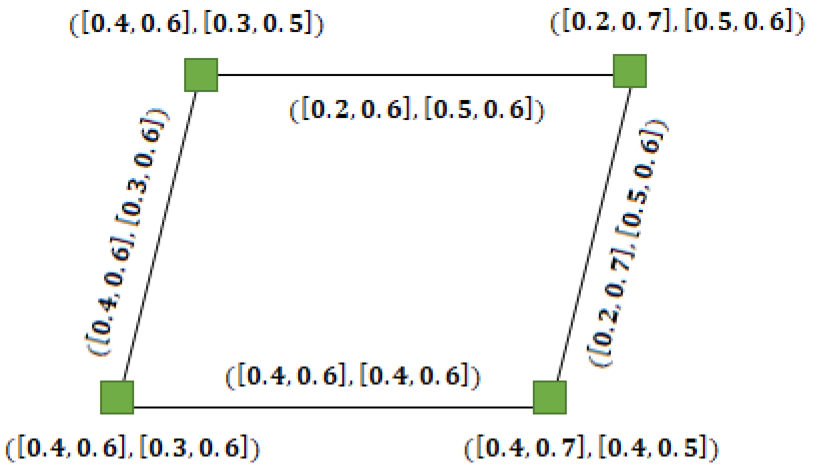


Figure 89 (Strong IVPyFG)

7.1.14 Definition

In an IVPyFG, a set of distinct nodes $ṭ_i$ $(i = 1, 2, ,3 \dots m)$ is considered as a path if there exist an edge between every two vertices $ṭ_i$ and $ṭ_j$ for $i, j = 1, 2, ,3 \dots m$. Or a set of distinct nodes $ṭ_i$ $(i = 1, 2, ,3 \dots m)$ is considered as a path if at least one of the following holds true.

1) $\hat{S}(ṭ_i, ṭ_j)$ is a non-zero subinterval of $[0, 1]$.

2) $Đ(ṭ_i, ṭ_j)$ is a non-zero subinterval of $[0, 1]$.

Consequences of Definition 7.1.14.

1) If a path consists of m vertices. Then its length is $m - 1$.

2) A path is known as cycle if its first and last vertex coincides.

3) Two vertices are said to be connected if they are joined by a path.

7.1.15 Example

The following Figure 90 is an example of Path in IVPyFG.

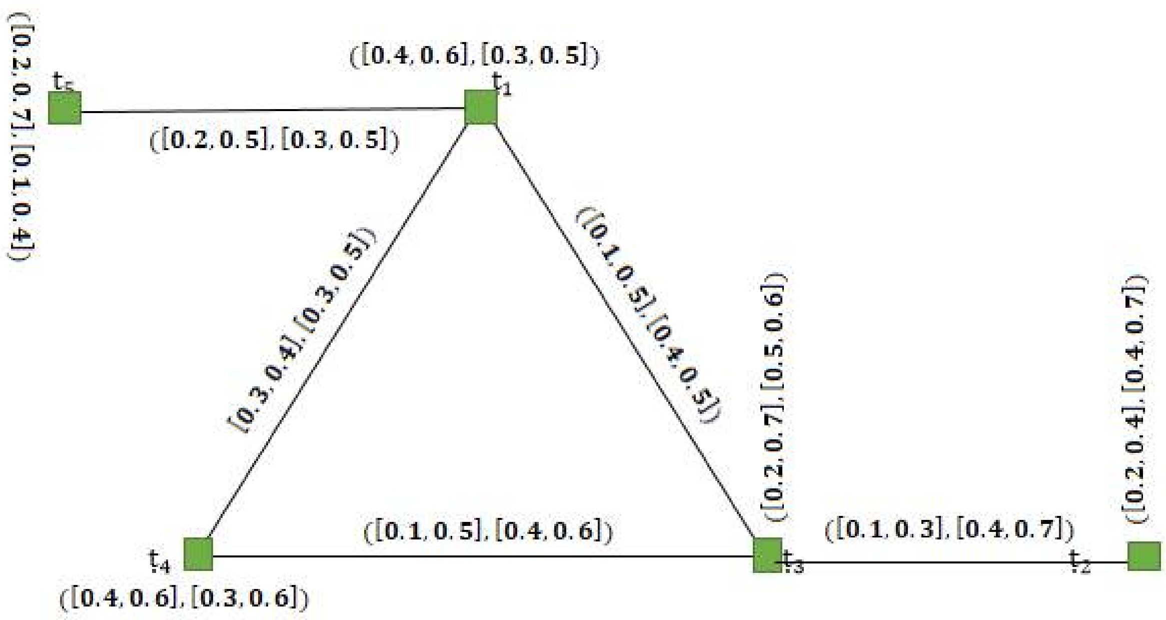


Figure 90 (A path in an Interval valued Pythagorean) fuzzy graph)

In the above Figure 90, $ṭ_1ṭ_3ṭ_4$ is path.

7.2 Applications in Decision Making

In this section, the developed approach of IVPyFGs is utilized in two decision making problems. The first problem is about the selection of best university among some universities while the second one is the famous problem of supply chain management.

7.2.1 Decision Making for the Evaluation of Best University:

In this problem, a network of universities is evaluated in a capital to obtain the best university based on performance. For this we consider a network of four universities denoted by $z_i(i = 1, 2, 3, 4)$. These four universities are being monitored and must be evaluated under the attribute "efficiency". A group of decision makers anonymously evaluated these universities and gave their opinions in the form IVPyFGs displayed using IVPyFGs.

The process of decision making is based on the following steps:

1. Obtaining information from decision makers.
2. Forming a relational matrix based on the information provided in Step 1.
3. Using interval valued Pythagorean fuzzy weighted averaging (IVPFWA) or interval valued Pythagorean fuzzy weighted geometric (IVPFWG) operator to aggregate the information of relational matrix.
4. Use ranking function to obtained optimum results.

A detailed numerical example is discussed as follows

Step 1: information from decision makers about universities is obtained in Figure 91

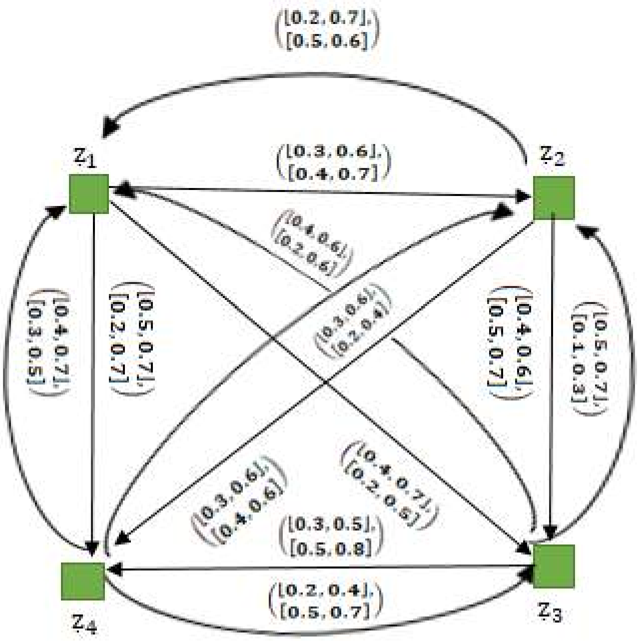


Figure 91 (Directed Network of the Interval valued Pythagorean fuzzy relation IVPyFR)

Step 2: forming relational matrix.

$$\dot{\mathrm{R}} = (\dot{\mathrm{r}}_{ij})_{4\times 4} = \begin{pmatrix} \begin{pmatrix}[0.4, 0.5],\\ [0.4, 0.5]\end{pmatrix}, & \begin{pmatrix}[0.3, 0.6],\\ [0.4, 0.7]\end{pmatrix}, & \begin{pmatrix}[0.4, 0.7],\\ [0.2, 0.5]\end{pmatrix}, & \begin{pmatrix}[0.5, 0.7],\\ [0.2, 0.7]\end{pmatrix} \\ \begin{pmatrix}[0.2, 0.7],\\ [0.5, 0.6]\end{pmatrix}, & \begin{pmatrix}[0.4, 0.5],\\ [0.4, 0.5]\end{pmatrix}, & \begin{pmatrix}[0.4, 0.6],\\ [0.5, 0.7]\end{pmatrix}, & \begin{pmatrix}[0.3, 0.6],\\ [0.4, 0.6]\end{pmatrix} \\ \begin{pmatrix}[0.4, 0.6],\\ [0.2, 0.6]\end{pmatrix}, & \begin{pmatrix}[0.5, 0.7],\\ [0.1, 0.3]\end{pmatrix}, & \begin{pmatrix}[0.4, 0.5],\\ [0.4, 0.5]\end{pmatrix}, & \begin{pmatrix}[0.3, 0.5],\\ [0.5, 0.8]\end{pmatrix} \\ \begin{pmatrix}[0.4, 0.7],\\ [0.3, 0.5]\end{pmatrix}, & \begin{pmatrix}[0.3, 0.6],\\ [0.2, 0.4]\end{pmatrix}, & \begin{pmatrix}[0.2, 0.4],\\ [0.5, 0.7]\end{pmatrix}, & \begin{pmatrix}[0.4, 0.5],\\ [0.4, 0.5]\end{pmatrix} \end{pmatrix}$$

Step 3: Using IVPFWA operator on relational matrix to aggregation of data.

$$\dot{\mathrm{r}}_i = I\dot{V}PFWA(\dot{\mathrm{r}}_{i1}, \dot{\mathrm{r}}_{i2}, \dots, \dot{\mathrm{r}}_{in})$$

$$= \left(\begin{matrix} \left[\sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\hat{\mathrm{S}}^{L}{}_{ij}\right)^2\right)^{\frac{1}{n}}}, \sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\hat{\mathrm{S}}^{U}{}_{ij}\right)^2\right)^{\frac{1}{n}}} \right], \\ \left[\left(\prod_{j=1}^{n} \mathrm{Đ}^{L}{}_{ij}\right)^{\frac{1}{n}}, \left(\prod_{j=1}^{n} \mathrm{Đ}^{U}{}_{ij}\right)^{\frac{1}{n}} \right] \end{matrix} \right)$$

$i = 1,2, \dots, n,$

We aggregate $\dot{r}_{ij}$, $j = 1,2, \dots ,4$ corresponding to the university z_i, and then get the complex IVPFN $\dot{r}_i$ of the university z_i, over all other universities.

$$\dot{r}_1 = ([0.408609, 0.636849], [0.0016, 0.030625])$$

$$\dot{r}_2 = ([0.337536, 0.608982], [0.0.01, 0.315])$$

$$\dot{r}_3 = ([0.408609, 0.587717], [0.001, 0.018])$$

$$\dot{r}_4 = ([0.337536, 0.57149], [0.003, 0.0175])$$

Step 4: Now we find the score values of $\dot{r}_i$ by using the score function:

$$ṣ(\dot{r}_i) = \frac{\left((\hat{S}^L{}_i)^2(1-(Đ^L{}_i)^2) + (\hat{S}^U{}_i)^2(1-(Đ^U{}_i)^2)\right)}{2}$$

$$ṣ(\dot{r}_1) = 0.00028, ṣ(\dot{r}_2) = 0.000356, ṣ(\dot{r}_3) = 0.000106, ṣ(\dot{r}_4) = 0.000107$$

According to $ṣ(\dot{r}_i)$ $i = 1,2, \dots ,4$ we get ranking of the universities z_i, $i = 1,2, \dots ,4$ as:

$$z_2 > z_1 > z_4 > z_3$$

Therefore, the best university is z_2.

Now the problem is solved again using IVPFWG operators and we repeat from **Step 3**.

Step 3: Using IVPFWG operator on relational matrix to aggregation of data.

$\dot{r}_i = IVPFW\dot{G}(\dot{r}_{i1}, \dot{r}_{i2}, \dots, \dot{r}_{in})$

$$= \left(\frac{\left(\prod_{j=1}^{n} \hat{S}^L{}_{ij}\right)^{\frac{1}{n}}, \left(\prod_{j=1}^{n} \hat{S}^U{}_{ij}\right)^{\frac{1}{n}},}{\sqrt{1-\left(\prod_{j=1}^{n}\left(1-\left(Đ^L{}_{ij}\right)^2\right)^{\frac{1}{n}}\right.}, \sqrt{1-\left(\prod_{j=1}^{n}\left(1-\left(Đ^U{}_{ij}\right)^2\right)^{\frac{1}{n}}\right.}} \right)$$

We aggregate $\dot{r}_{ij}$, $j = 1,2,\ldots,4$ corresponding to the university $\d{z}_i$, and then get the complex IVPFN $\dot{r}_i$ of the university $\d{z}_i$, over all other universities.

$$\dot{r}_1 = ([0.006, 0.03675], [0.319378, 0.617585])$$

$$\dot{r}_2 = ([0.0024, 0.0315], [0.454175, 0.608982])$$

$$\dot{r}_3 = ([0.006, 0.02625], [0.34691, 0.608508])$$

$$\dot{r}_4 = ([0.0024, 0.021], [0.372401, 0.547141]$$

Step 4: Now we find the score values of $\dot{r}_i$ by using the score function:

$$\d{s}(\dot{r}_1) = 0.000434, \d{s}(\dot{r}_2) = 0.000314, \d{s}(\dot{r}_3) = 0.000233, \d{s}(\dot{r}_4) = 0.000157$$

Therefore according $\d{s}(\dot{r}_1)$ we get ranking of the universities $\d{z}_i$, $i = 1,2,\ldots,4$ as:

$$\d{z}_1 > \d{z}_2 > \d{z}_3 > \d{z}_4$$

Therefore, the best university is $\d{z}_1$.

Analysis shows that the results obtained using different aggregation techniques are different. So, the choice of using aggregation operators is up to decision makers.

7.2.2 Application in Supply Chain Management

In supply chain management the partners of a company assessed based on their performances towards a supply chain. The coordination of companies depends on many aspects. In this problem, our aim is to find out most influential aspect in a supply chain. Let us consider we have four aspects i.e. service level, cost and price, quality and response time denoted by $z_i(i = 1, 2, 3, 4)$. To rank these factors, three decision makers are asked to give their preferences in the form of IVPyFNs in three matrices below.

$$R_1 = \begin{bmatrix} \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.7,0.7],\\ [0.2,0.5]\end{pmatrix} & \begin{pmatrix}[0.1,0.7],\\ [0.1,0.5]\end{pmatrix} & \begin{pmatrix}[0.5,0.7],\\ [0.6,0.7]\end{pmatrix} \\ \begin{pmatrix}[0.3,0.7],\\ [0.4,0.7]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.2,0.5],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.7],\\ [0.4,0.6]\end{pmatrix} \\ \begin{pmatrix}[0.2,0.6],\\ [0.4,0.6]\end{pmatrix} & \begin{pmatrix}[0.2,0.4],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.6],\\ [0.5,0.7]\end{pmatrix} \\ \begin{pmatrix}[0.3,0.5],\\ [0.4,0.7]\end{pmatrix} & \begin{pmatrix}[0.2,0.7],\\ [0.3,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.4],\\ [0.4,0.5]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.7],\\ [0.1,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.5],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.2,0.6],\\ [0.3,0.4]\end{pmatrix} \\ \begin{pmatrix}[0.2,0.7],\\ [0.5,0.7]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.6],\\ [0.4,0.7]\end{pmatrix} & \begin{pmatrix}[0.5,0.7],\\ [0.1,0.6]\end{pmatrix} \\ \begin{pmatrix}[0.1,0.6],\\ [0.3,0.7]\end{pmatrix} & \begin{pmatrix}[0.5,0.5],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.2,0.6],\\ [0.4,0.5]\end{pmatrix} \\ \begin{pmatrix}[0.1,0.6],\\ [0.2,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.4],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.55],\\ [0.4,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} \end{bmatrix}$$

$$R_3 = \begin{bmatrix} \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.7],\\ [0.3,0.5]\end{pmatrix} & \begin{pmatrix}[0.1,0.5],\\ [0.3,0.7]\end{pmatrix} & \begin{pmatrix}[0.2,0.6],\\ [0.5,0.6]\end{pmatrix} \\ \begin{pmatrix}[0.4,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.6],\\ [0.3,0.7]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.5]\end{pmatrix} \\ \begin{pmatrix}[0.5,0.7],\\ [0.6,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.1,0.7]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.2,0.3],\\ [0.3,0.4]\end{pmatrix} \\ \begin{pmatrix}[0.53,0.72],\\ [0.1,0.2]\end{pmatrix} & \begin{pmatrix}[0.53,0.63],\\ [0.12,0.42]\end{pmatrix} & \begin{pmatrix}[0.4,0.5],\\ [0.3,0.5]\end{pmatrix} & \begin{pmatrix}[0.5,0.6],\\ [0.5,0.6]\end{pmatrix} \end{bmatrix}$$

The information of three different reviewers are regrouped using IVPyFWA operators as:

$$I\dot{V}PFWA(\dot{r}_{i1}, \dot{r}_{i2}, \dots, \dot{r}_{in}) = \left(\begin{array}{c} \sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\hat{S}^{L}{}_{ij}\right)^2\right) \right)^{\frac{1}{n}}}, \sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\hat{S}^{U}{}_{ij}\right)^2\right) \right)^{\frac{1}{n}}}, \\ \left(\prod_{j=1}^{n} Đ^{L}{}_{ij} \right)^{\frac{1}{n}}, \left(\prod_{j=1}^{n} Đ^{U}{}_{ij} \right)^{\frac{1}{n}} \end{array} \right)$$

$$R = \begin{bmatrix} \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} & \begin{pmatrix} [0.44,0.48], \\ [0.002,0.05] \end{pmatrix} & \begin{pmatrix} [0.35,0.45], \\ [0.005,0.07] \end{pmatrix} & \begin{pmatrix} [0.38,0.46], \\ [0.03,0.056] \end{pmatrix} \\ \begin{pmatrix} [0.38,0.47], \\ [0.03,0.098] \end{pmatrix} & \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} & \begin{pmatrix} [0.4,0.45], \\ [0.02,0.098] \end{pmatrix} & \begin{pmatrix} [0.43,0.47], \\ [0.007,0.06] \end{pmatrix} \\ \begin{pmatrix} [0.38,0.47], \\ [0.024,0.084] \end{pmatrix} & \begin{pmatrix} [0.41,0.43], \\ [0.008,0.084] \end{pmatrix} & \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} & \begin{pmatrix} [0.36,0.44], \\ [0.02,0.05] \end{pmatrix} \\ \begin{pmatrix} [0.39,0.46], \\ [0.003,0.028] \end{pmatrix} & \begin{pmatrix} [0.39,0.46], \\ [0.006,0.05] \end{pmatrix} & \begin{pmatrix} [0.38,0.43], \\ [0.016,0.05] \end{pmatrix} & \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} \end{bmatrix}$$

The regrouped data is transformed into a directed IVPyFG depicted in Figure 92.

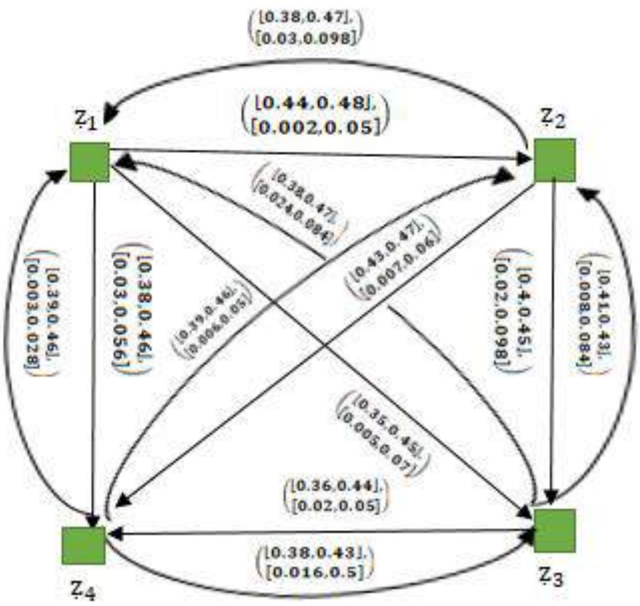


Figure 92 (Directed Network OF IVPyFR)

Using the condition $\hat{S}^{2L} \geq 0.46$, The IVPyFG in Figure 92 reduces to another IVPyFG which is depicted in Figure 93.

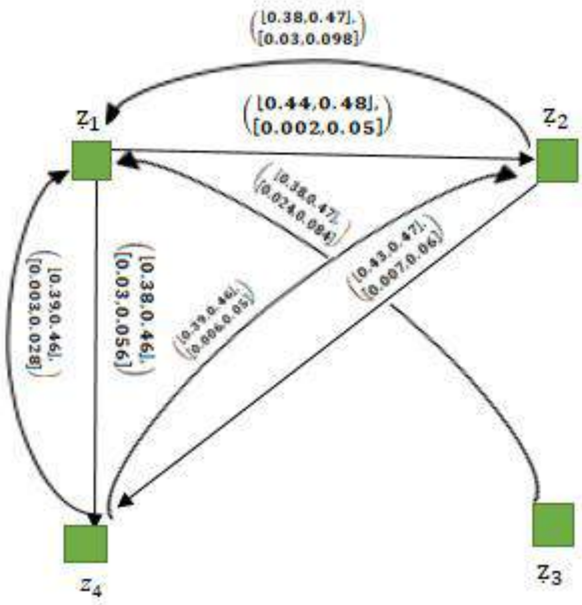


Figure 93 (Partial Directed Network OF IVPyFR)

Now we used Definition 7.1.8 to compute the out degrees of every vertex z_i as follows:

$$Out - d(z_1) = ([0.82,0.94], [0.032,0.061])$$

$$Out - d(z_2) = ([0.81,0.94], [0.05,0.196])$$

$$Out - d(z_3) = ([0.38,0.47], [0.024,0.084])$$

$$Out - d(z_4) = ([0.78,0.92], [0.009,0.078])$$

By observing the degree of membership, where the score function is available in step 4, it is clear that

$$Out - d(z_1) \geq Out - d(z_2) \geq Out - d(z_4) \geq Out - d(z_3)$$

Hence $z_1 \geq z_2 \geq z_4 \geq z_3$ which shows that cost and price is the required factor that one must keep in mind.

Now again all the steps are repeated and instead of IVPyFWA operators, we used IVPyFWG operators to regroup the data of first three matrices.

First, the information of three different reviewers are regrouped using IVPyFWA operators as:

$$I\dot{V}PFW\dot{G}(\dot{r}_{i1},\dot{r}_{i2},\ldots,\dot{r}_{in}) = \left(\frac{\left(\prod_{j=1}^{n}\hat{S}^{L}{}_{ij}\right)^{\frac{1}{n}},\left(\prod_{j=1}^{n}\hat{S}^{U}{}_{ij}\right)^{\frac{1}{n}},}{\sqrt{1-\left(\prod_{j=1}^{n}\left(1-\left(Đ^{L}{}_{ij}\right)^{2}\right)\right)^{\frac{1}{n}}},\sqrt{1-\left(\prod_{j=1}^{n}\left(1-\left(Đ^{U}{}_{ij}\right)^{2}\right)\right)^{\frac{1}{n}}}}\right)$$

$$R = \begin{bmatrix} \begin{pmatrix}[0.042,0.072],\\ [0.43,0.46]\end{pmatrix} & \begin{pmatrix}[0.035,0.011],\\ [0.36,0.44]\end{pmatrix} & \begin{pmatrix}[0.001,0.058],\\ [0.39,0.46]\end{pmatrix} & \begin{pmatrix}[0.007,0.084],\\ [0.43,0.45]\end{pmatrix} \\ \begin{pmatrix}[0.008,0.098],\\ [0.42,0.47]\end{pmatrix} & \begin{pmatrix}[0.042,0.072],\\ [0.43,0.46]\end{pmatrix} & \begin{pmatrix}[0.015,0.072],\\ [0.41,0.47]\end{pmatrix} & \begin{pmatrix}[0.042,0.098],\\ [0.40,0.45]\end{pmatrix} \\ \begin{pmatrix}[0.003,0.084],\\ [0.42,0.47]\end{pmatrix} & \begin{pmatrix}[0.02,0.04],\\ [0.41,0.47]\end{pmatrix} & \begin{pmatrix}[0.042,0.072],\\ [0.43,0.46]\end{pmatrix} & \begin{pmatrix}[0.004,0.036],\\ [0.40,0.45]\end{pmatrix} \\ \begin{pmatrix}[0.005,0.072],\\ [0.37,0.45]\end{pmatrix} & \begin{pmatrix}[0.01,0.06],\\ [0.39,0.44]\end{pmatrix} & \begin{pmatrix}[0.012,0.037],\\ [0.39,0.44]\end{pmatrix} & \begin{pmatrix}[0.042,0.072],\\ [0.43,0.46]\end{pmatrix} \end{bmatrix}$$

The regrouped data is transformed into a directed IVPyFG depicted in Figure 94.

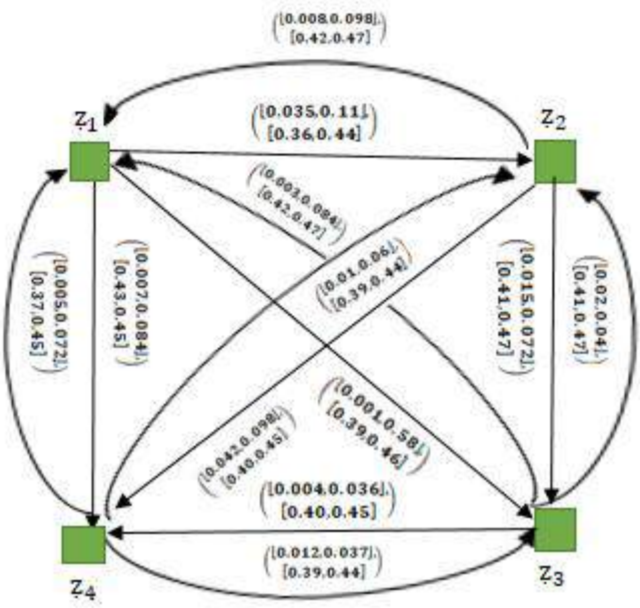


Figure 94 (Directed Network OF IVPyFR)

Using the condition $\hat{S}^{2L} \geq 0.058$, the IVPyFG in Figure 94 reduces to another IVPyFG which is depicted in Figure 95.

The regrouped data is transformed into a directed IVPyFG depicted in Figure 93.

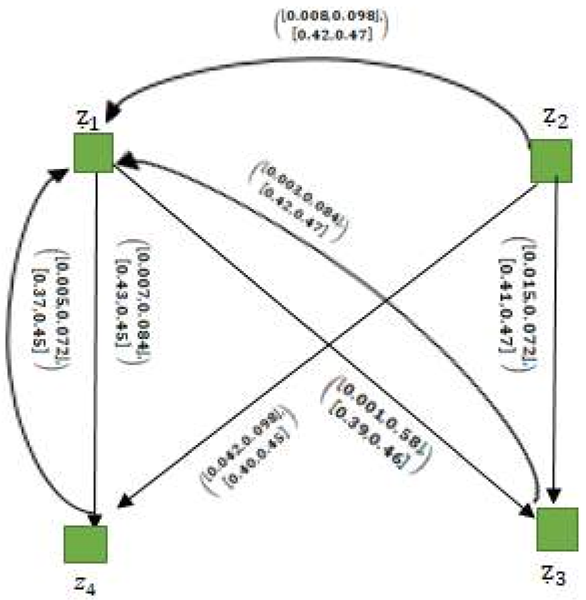


Figure 95 (Partial Directed Network OF IVPyFR)

Now we used Definition 7.1.8 to compute the out degrees of every vertex z_i as follows:

$$Out - d(z_1) = ([0.008,0.142], [0.82,0.91])$$

$$Out - d(z_2) = ([0.065,0.268], [1.21,1.39])$$

$$Out - d(z_3) = ([0.003,0.084], [0.42,0.47])$$

$$Out - d(z_4) = ([0.005,0.072], [0.37,0.45])$$

By observing the degree of membership, where the score function is available in step 4, it is clear that

$$Out - d(z_2) \geq Out - d(z_1) \geq Out - d(z_3) \geq Out - d(z_4)$$

Hence $z_2 \geq z_1 \geq z_3 \geq z_4$ which shows that cost and price is the required factor that one must keep in mind.

Hence by using even different operators, we get same results i.e. cost and price is the factor we must looked for.

7.2.2.1 Comparative Study and Advantages

The significance of the proposed new approach lies in the fact that the new approach can solve the problems that occur in the environment of PyFSs as well as IVIFSs. Here we discuss two examples having information in the form of PyFNs or IVIFNs.

Consider the decision matrix where information is in the form of PyFNs as follows:

$$\dot{\mathrm{R}} = (\dot{\mathrm{r}}_{ij})_{4\times 4} = \begin{pmatrix} (\mathbf{0.6}, \mathbf{0.6}), (\mathbf{0.3}, \mathbf{0.8}), (\mathbf{0.7}, \mathbf{0.6}), (\mathbf{0.5}, \mathbf{0.6}) \\ (\mathbf{0.8}, \mathbf{0.3}), (\mathbf{0.6}, \mathbf{0.6}), (\mathbf{0.5}, \mathbf{0.7}), (\mathbf{0.6}, \mathbf{0.5}) \\ (\mathbf{0.7}, \mathbf{0.4}), (\mathbf{0.2}, \mathbf{0.8}), (\mathbf{0.6}, \mathbf{0.6}), (\mathbf{0.7}, \mathbf{0.6}) \\ (\mathbf{0.3}, \mathbf{0.7}), (\mathbf{0.5}, \mathbf{0.6}), (\mathbf{0.4}, \mathbf{0.7}), (\mathbf{0.6}, \mathbf{0.6}) \end{pmatrix}$$

Now this type of information can easily be processed using the IVPFWA and IVPWG operators by assuming $\hat{\mathrm{S}}^L = \mathrm{Đ}^L = 0$.

Further, if we have information in the form of IVIFNs as in decision matrix below:

$$\dot{\mathrm{R}} = (\dot{\mathrm{r}}_{ij})_{4\times 4} = \begin{pmatrix} \begin{pmatrix}[\mathbf{0.3}, \mathbf{0.5}], \\ [\mathbf{0.2}, \mathbf{0.5}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3}, \mathbf{0.4}], \\ [\mathbf{0.4}, \mathbf{0.5}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.4}, \mathbf{0.5}], \\ [\mathbf{0.2}, \mathbf{0.3}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.5}, \mathbf{0.6}], \\ [\mathbf{0.2}, \mathbf{0.3}]\end{pmatrix} \\ \begin{pmatrix}[\mathbf{0.2}, \mathbf{0.4}], \\ [\mathbf{0.2}, \mathbf{0.5}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3}, \mathbf{0.5}], \\ [\mathbf{0.2}, \mathbf{0.5}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.1}, \mathbf{0.4}], \\ [\mathbf{0.2}, \mathbf{0.6}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.2}, \mathbf{0.5}], \\ [\mathbf{0.4}, \mathbf{0.5}]\end{pmatrix} \\ \begin{pmatrix}[\mathbf{0.1}, \mathbf{0.6}], \\ [\mathbf{0.3}, \mathbf{0.4}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3}, \mathbf{0.7}], \\ [\mathbf{0.1}, \mathbf{0.3}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3}, \mathbf{0.5}], \\ [\mathbf{0.2}, \mathbf{0.5}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.1}, \mathbf{0.3}], \\ [\mathbf{0.5}, \mathbf{0.6}]\end{pmatrix} \\ \begin{pmatrix}[\mathbf{0.4}, \mathbf{0.5}], \\ [\mathbf{0.3}, \mathbf{0.5}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3}, \mathbf{0.4}], \\ [\mathbf{0.2}, \mathbf{0.4}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.2}, \mathbf{0.4}], \\ [\mathbf{0.5}, \mathbf{0.6}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3}, \mathbf{0.5}], \\ [\mathbf{0.2}, \mathbf{0.5}]\end{pmatrix} \end{pmatrix}$$

Then the proposed approach can still be used by utilizing averaging and geometric aggregation operators of IVIFSs as defined in [30]. On the other hand, the aggregation tools of PyFSs or IVIFSs cannot be applied to the information of IVPyFSs due to their limited structures. All this shows the worth of new developed approach.

7.2.4 Application in Finding Shortest Path in a Networks:

Shortest path problem is one of the renowned problem of graph theory. In fuzzy graph theory, shortest path problem has been greatly studied in almost every fuzzy structure [104-107]. Here the algorithm proposed by [104] which is relatively simpler and get us optimum results. The detailed steps of algorithm along with a flowchart are next in this section followed by a numerical example and some comparative study.

7.2.4.1 Algorithm for Finding Shortest Path:

The steps of algorithm are:

8. Take the first node as $([0,0],[1,1])$ as distance of every node to itself is zero.
9. Take $i=1$.
10. Find j for $P_{1j}=P_{11}+\wedge_{j\in NP(1)}P_{1j}$ and determine P_{1j}.
11. Put $i=j$.
12. Find k for $P_{jk}=P_{1j}+\wedge_{k\in NP(1)}P_{jk}$ and determine P_{jk}.
13. This process should be continued until destination node is obtained.
14. When destination node is reached. Then algorithm is stopped.

To further demonstrate the algorithm, a flowchart is depicted in Figure 96.

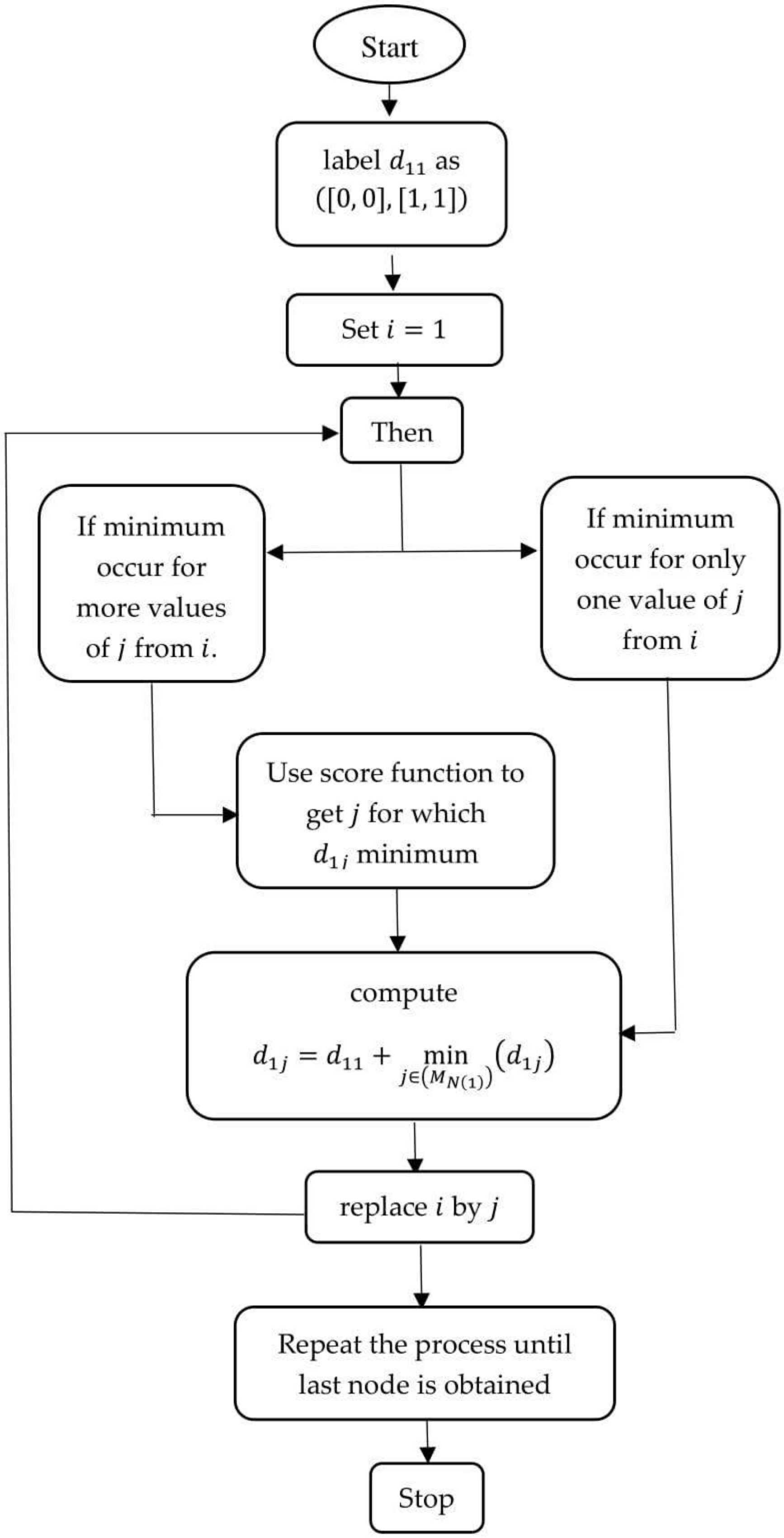


Figure 96 (Flowchart of proposed algorithm)

7.2.4.2 Example

Now consider a network of 6 nodes in Figure 97 and the distance of every two, connected nodes, is provided in the form of IVPyFNs. We proceed with the algorithm as follows

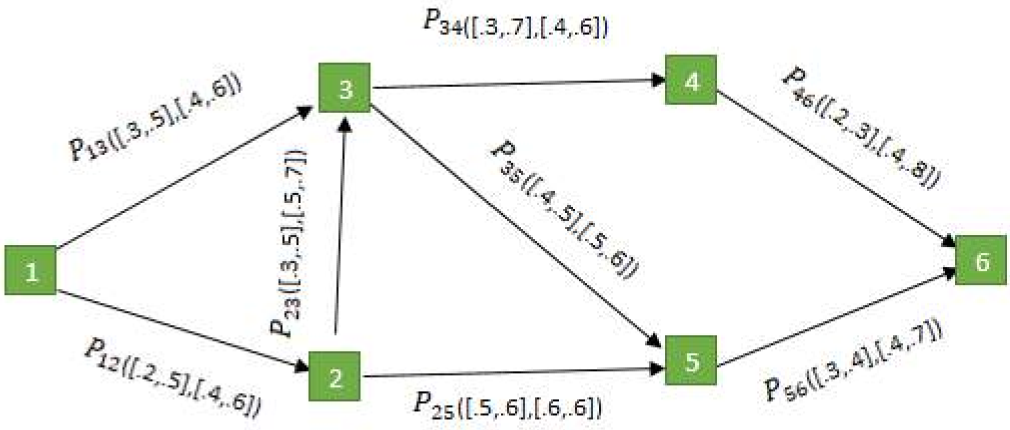


Figure 97 (Interval value Pythagorean fuzzy graph)

In In the above graph shows the source node is node 1 and the destination node is node 6, so for $n = 6$ and for the first step $P_{11} = ([0,0],[1,1])$. At first $i = 1$, so it needs to find j by the equation.

$$P_{1j} = P_{11} + \wedge_{j\in NP(1)} P_{1j} = P_{11} + \wedge_{j\in\{2,3\}} P_{1j} = P_{11} + (P_{12} \wedge P_{13})$$

$$= ([0,0],[1,1]) + \big(([.2,.5],[.4,.6]) \wedge ([.3,.5],[.4,.6])\big)$$

$$= ([0,0],[1,1]) + (0.0968 \wedge 0.1178)$$

$$= ([0,0],[1,1]) + ([.2,.5],[.4,.6])$$

$$= ([.2,.5],[.4,.6])$$

Hence for $j = 2$, $P_{12} = ([.2,.5],[.4,.6])$.

Now set $i = 2$, we need to find j by the equation

$$P_{1j} = P_{12} + \wedge_{j \in NP(2)} P_{2j} = P_{12} + \wedge_{j \in \{3,5\}} P_{2j} = P_{12} + (P_{23} \wedge P_{25})$$

$$= ([0.2, 0.5], [.4, .6]) + \big(([.3, .5], [.5, .7]) \wedge ([.5, .6], [.6, .6])\big)$$

$$= ([0.2, 0.5], [.4, .6]) + (0., 0975 \wedge 0.1952)$$

$$= ([.36, .66], [.2, .42])$$

Hence for $j = 2$, $P_{13} = ([.36, .66], [.2, .42])$.

Now set $i = 3$, we need to find j by the equation

$$P_{1j} = P_{13} + \wedge_{j \in NP(3)} P_{3j} = P_{13} + \wedge_{j \in \{4,5\}} P_{1j} = P_{13} + (P_{34} \wedge P_{35})$$

$$= ([0.36, 0.66], [0.2, 0.42]) + \big(([.3, .7], [.4, .6]) \wedge ([.4, .5], [.5, .6])\big)$$

$$= ([0.36, 0.66], [0.2, 0.42]) + (0.1946 \wedge 0.14)$$

$$= ([.52, .76], [.1, .25])$$

Hence for $j = 2$, $P_{15} = ([.52, .76], [.1, .25])$.

Now set $i = 5$, we need to find by the equation

$$P_{1j} = P_{15} + \wedge_{j \in NP(5)} P_{1j} = P_{15} + \wedge_{j \in \{6\}} P_{1j} = P_{15} + P_{56}$$

$$= ([.52, .76], [.1, .25]) + ([0.3, 0.4], [0.4, 0.7])$$

$$= ([.58, .80], [.04, .175])$$

Hence for $j = 2$, $P_{16} = ([.52, .76], [.1, .25])$.

So, the shortest path in the above graph is $P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow P_5 \longrightarrow P_6$.

If we observe Figure 97, one can say that shortest path may be $P_1 \longrightarrow P_3 \longrightarrow P_4 \longrightarrow P_6$ or $P_1 \longrightarrow P_2 \longrightarrow P_5 \longrightarrow P_6$ or $P_1 \longrightarrow P_3 \longrightarrow P_5 \longrightarrow P_6$ but in actual the shortest path neither of these and it is $P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow P_5 \longrightarrow P_6$. This shows how accurately the imprecise information is processed and shortest path is found using IVPyFGs.

7.2.4.3 Comparative Study and Advantages of Proposed Algorithm:

In this section, it is described that the proposed algorithm is applicable in the environments of IVIFSs as well as Pythagorean fuzzy networks. If we take a network in the environment of IVIFSs as depicted in Figure 98.

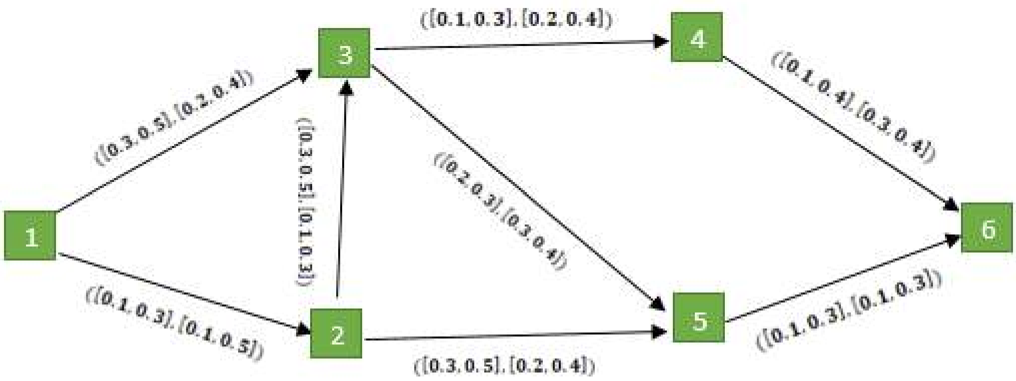


Figure 98 (Interval valued intuitionistic fuzzy network)

The shortest path in this case can similarly be found with the help of proposed algorithm using the proposed operations of addition with some conditions.

Further if we consider this kind of problem in an environment of PyFSs e.g. consider a network of nodes in Pythagorean fuzzy environment in Figure 99. The shortest path algorithm can be carried out using proposed operation with some conditions over it.

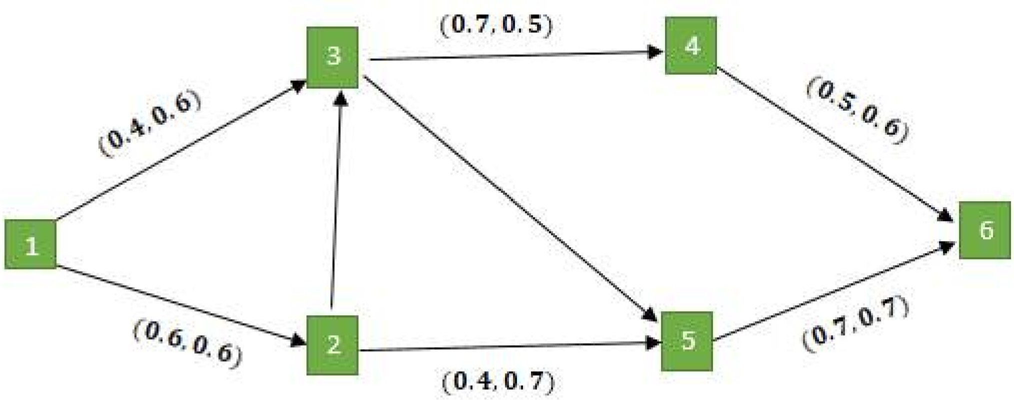


Figure 99 (fuzzy network)

Some advantages of the algorithm are:

1. The algorithm we used is very easy to be used and the results can be obtained much quicker than the existing algorithms.
2. The algorithm can be used for any kind of fuzzy structure.
3. The results obtained in current situation would be better than the results in the environment of FSs, IFSs, IVIFSs. PyFSs as IVPyFS generalizes all these structures.

7.3 Conclusion

This chapter contributed towards the theory of IVPyFGs as the concept of IVPyFG is proposed and some graph theoretic ideas were explored. Each graph related concept is supported by an example. Then the described concept of IVPyFGs is used in two decision making problems and a shortest path problem. Using the weighted averaging and weighted geometric aggregation operators, first decision making problem is solved involving the selection of best university among some universities. In second decision-making problem, a supply chain management problem is solved using the same aggregation operators and degree of an IVPyFG. The next problem was to find a shortest path in a network where path length was in the form of IVPyFNs. This problem is solved using a novel approach and the results ae discussed. We discussed the advantages of the proposed work and a comparative analysis have been established. In near future, we shall try to discuss some other shortest path algorithms and then make a comparative study.

Chapter 8

Analysis of Social Networks, Communication Networks and Shortest Path Problems in the Environment of Interval Valued q-Rung Orthopair Fuzzy Information

8.1 Interval valued q-rung orthopair fuzzy graphs

The aim of this section is to briefly introduce the framework of IVq-ROPFG and to study its consequences. Some related graphical concepts are also demonstrated with the help of examples. These basic notions are used in applications discussed in section 8.2 , section 8.3 and section 8.4.

8.1.1 Definition

A graph $\dot{G} =< \underset{.}{V}, \dot{E}, \underset{.}{A}, \underset{.}{B} >$ is known as IVq-ROPFG if

1. $\underset{.}{V} = \{\underset{.}{t}_1, \underset{.}{t}_2, \underset{.}{t}_3, \dots, \underset{.}{t}_q\}$ such that $\hat{S}_{\underset{.}{A}L}: \underset{.}{V} \to [0,1]$, $\hat{S}_{\underset{.}{A}U}: \underset{.}{V} \to [0,1]$ represent the lower and upper limits of membership degrees and $Đ_{\underset{.}{A}L}: \underset{.}{V} \to [0,1]$, $Đ_{\underset{.}{A}U}: \underset{.}{V} \to [0,1]$ represents the lower and upper limits of non-membership degrees of $\underset{.}{t}_i \in \underset{.}{V}$ respectively provided that $0 \leq (\hat{S}_{\underset{.}{A}U})^q + (Đ_{\underset{.}{A}U})^q \leq 1$ for $q \in \mathbb{Z}^+$ for all $\underset{.}{t}_i \in \underset{.}{V}, (i = 1, 2, 3, \dots m)$
2. $\dot{E} \subseteq \underset{.}{V} \times \underset{.}{V}$ where $\hat{S}_{\underset{.}{B}L}, Đ_{\underset{.}{B}L} : \underset{.}{V} \times \underset{.}{V} \to [0,1]$ and $\hat{S}_{\underset{.}{B}U}, Đ_{\underset{.}{B}U} : \underset{.}{V} \times \underset{.}{V} \to [0,1]$ such as

$\hat{S}_{\underset{.}{B}L}(\underset{.}{t}_i, \underset{.}{t}_j) \leq \min[\hat{S}_{\underset{.}{A}L}(\underset{.}{t}_i), \hat{S}_{\underset{.}{A}L}(\underset{.}{t}_j)]$, $\hat{S}_{\underset{.}{B}U}(\underset{.}{t}_i, \underset{.}{t}_j) \leq \min[\hat{S}_{\underset{.}{A}U}(\underset{.}{t}_i), \hat{S}_{\underset{.}{A}U}(\underset{.}{t}_j)]$ such that $\hat{S}_{\underset{.}{B}U}(\underset{.}{t}_i, \underset{.}{t}_j) \geq \min[\hat{S}_{\underset{.}{A}L}(\underset{.}{t}_i), \hat{S}_{\underset{.}{A}L}(\underset{.}{t}_j)]$ and $Đ_{\underset{.}{B}L}(\underset{.}{t}_i, \underset{.}{t}_j) \leq \max[Đ_{\underset{.}{A}L}(\underset{.}{t}_i), Đ_{\underset{.}{A}L}(\underset{.}{t}_j)]$, $Đ_{\underset{.}{B}U}(\underset{.}{t}_i, \underset{.}{t}_j) \leq \max[Đ_{\underset{.}{A}U}(\underset{.}{t}_i), Đ_{\underset{.}{A}U}(\underset{.}{t}_j)]$ such that $Đ_{\underset{.}{B}U}(\underset{.}{t}_i, \underset{.}{t}_j) \geq \max[Đ_{\underset{.}{A}L}(\underset{.}{t}_i), Đ_{\underset{.}{A}L}(\underset{.}{t}_j)]$ with a condition $0 \leq \left(\hat{S}_{\underset{.}{B}U}(\underset{.}{t}_i, \underset{.}{t}_j)\right)^q + \left(Đ_{\underset{.}{B}U}(\underset{.}{t}_i, \underset{.}{t}_j)\right)^q \leq 1$ for $q \in \mathbb{Z}^+$ for all $(\underset{.}{t}_i, \underset{.}{t}_j) \in \dot{E}$.

Moreover, the lower and upper limits of membership and non-membership values of a vertex $ṭ_i$ are denoted as $(ṭ_i, \hat{S}_{ẠL}, Đ_{ẠL})$ and $(ṭ_i, \hat{S}_{ẠU}, Đ_{ẠU})$ and that of an edge relation $ę_{ij} = (ṭ_i, ṭ_j)$ is denoted by $(ę_{ij}, \hat{S}_{ḄL}, Đ_{ḄL})$ and $(ę_{ij}, \hat{S}_{ḄU}, Đ_{ḄU})$. Edge between two vertices $ṭ_i, ṭ_j$ exist unless $\hat{S}_{2ij} = 0 = Đ_{2ij}$. Further, the notion Λ used for maximum and the notion $\vee$ used for minimum.

Now we defined a score function for ranking purpose of IVq-ROPFG.

8.1.2 Definition

The score function for IVq-ROPFN $A = ([\hat{S}^L, \hat{S}^U], [Đ^L, Đ^U])$ is defined as:

$$SC(A) = \frac{(\hat{S}^L)^q(1 - (Đ^L)^q) + (\hat{S}^U)^q((1 - (Đ^U)^q))}{2} \quad SC(A) \in [0, 1]$$

8.1.3 Theorem

IVq-ROPFG is a generalization of IVPyFG, IVIFG and IVFG.

Proof: We prove this result as follows:

3) If we take $q = 2$. The definition of IVq-ROPFG reduces to IVPyFG.
4) If we take $q = 1$. The definition of IVq-ROPFG reduces to IVIFG.
5) If we take $q = 1$, and $Đ_{ẠL} = Đ_{ẠU} = 0$. The definition of q-ROPFG reduces to IVFG.

This result shows the significance of the new concept as it is the generalization of all the existing structures and can deal with the situations where the existing structures failed due to the limitations in their structures.

8.1.4 Example

Let $Ṿ = \{ṭ_1, ṭ_2, ṭ_3, ṭ_4\}$ and Ė̉ be the collection of vertices and edges, respectively. Then, we have the following IVq-ROPFG for q =3.

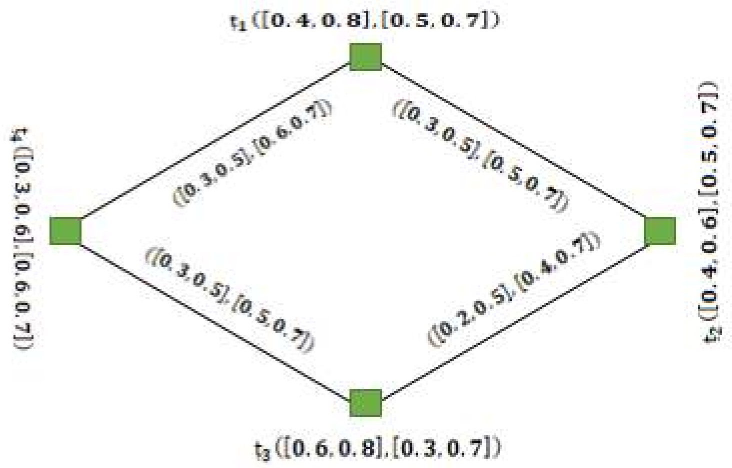


Figure 100 (Interval valued q-rung ortho pair fuzzy graph)

8.1.5 Definition

A graph $Ḥ =< V̩', \dot{E}' >$ is considered as IVq-ROF subgraph (IVq-ROFSG) of an IVq-ROFG $\dot{G} =< V̩, \dot{E} >$ i.e. $Ḥ \leq_G \dot{G}$ if $V̩'$ and $\dot{E}'$ are subsets of $V̩$ and $\dot{E}$ respectively. or $Ḥ =< V̩', \dot{E}' >$ is considered as IVq-ROFSG of IVq-ROFG $\dot{G} =< V̩, \dot{E} >$ if $\hat{S}'_{ẠLi} \leq \hat{S}_{ẠLi}, \hat{S}'_{ẠUi} \leq \hat{S}_{ẠUi}, Đ'_{ẠLi} \geq Đ_{ẠLi}, Đ'_{ẠUi} \geq Đ_{ẠUi}$ and $\hat{S}'_{ḄLi} \leq \hat{S}_{ḄLi}, \hat{S}'_{ḄUi} \leq \hat{S}_{ḄUi}, Đ'_{ḄLi} \geq Đ_{ḄLi}, Đ'_{ḄUi} \geq Đ_{ḄUi}$ for all $i, j = 1, 2, 3, 4, \ldots m$.

8.1.6 Example

An IVq-ROFSG of IVq-ROFG mentioned in previous example 8.1.4 is given below

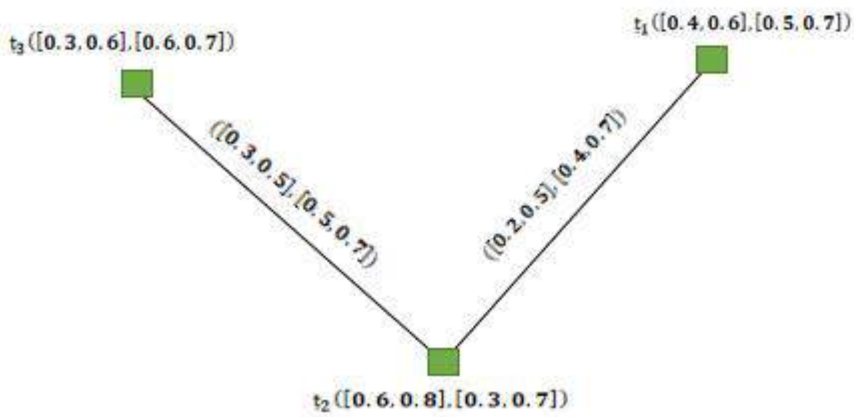


Figure 101 (IVQROPFSG of Figure (100))

8.1.7 Definition

An IVq-ROPFG $\dot{G} =< \d{V}, \dot{E} >$ is considered as

1. semi $-\hat{S}$ $strong$ if $\hat{S}_{\d{B}L} = \Lambda(\hat{S}_{\d{A}Li}, \hat{S}_{\d{A}Lj})$, $\hat{S}_{\d{B}U} = \Lambda(\hat{S}_{\d{A}Ui}, \hat{S}_{\d{A}Uj})$ for all i, j.
2. semi $-$ Đ $strong$ if $Đ_{\d{B}L} = \vee(Đ_{\d{A}Li}, Đ_{\d{A}Lj})$, $Đ_{\d{B}U} = \vee(Đ_{\d{A}Ui}, Đ_{\d{A}Uj})$for all i, j.
3. strong if $\dot{G}$ is semi $-\hat{S}$ $strong$ and $semi - Đ\ strong$ ***or*** $\dot{G}$ is $strong$ if $\hat{S}_{\d{B}L} = \Lambda(\hat{S}_{\d{A}Li}, \hat{S}_{\d{A}Lj})$, $\hat{S}_{\d{B}U} = \Lambda(\hat{S}_{\d{A}Ui}, \hat{S}_{\d{A}Uj})$ and $Đ_{\d{B}L} = \vee(Đ_{\d{A}Li}, Đ_{\d{A}Lj})$, $Đ_{\d{B}U} = \vee(Đ_{\d{A}Ui}, Đ_{\d{A}Uj})$ for all $(\d{t}_i, \d{t}_j) \in \dot{E}$.

8.1.8 Definition

An IVq-ROPFG $\dot{G} =< \d{V}, \dot{E} >$ is considered as

1. Complete$-\hat{S}$ $strong$ if $\hat{S}_{\d{B}L} = \Lambda(\hat{S}_{\d{A}Li}, \hat{S}_{\d{A}Lj})$, $\hat{S}_{\d{B}U} = \Lambda(\hat{S}_{\d{A}Ui}, \hat{S}_{\d{A}Uj})$ and $Đ_{\d{B}L} < \vee(Đ_{\d{A}Li}, Đ_{\d{A}Lj})$, $Đ_{\d{B}U} < \vee(Đ_{\d{A}Ui}, Đ_{\d{A}Uj})$for all i, j for all i, j.
2. Complete$-$Đ $strong$ if if $\hat{S}_{\d{B}L} < \Lambda(\hat{S}_{\d{A}Li}, \hat{S}_{\d{A}Lj})$, $\hat{S}_{\d{B}U} < \Lambda(\hat{S}_{\d{A}Ui}, \hat{S}_{\d{A}Uj})$ and $Đ_{\d{B}L} = \vee(Đ_{\d{A}Li}, Đ_{\d{A}Lj})$, $Đ_{\d{B}U} = \vee(Đ_{\d{A}Ui}, Đ_{\d{A}Uj})$for all i, j for all i, j.
3. Complete if $\hat{S}_{\d{B}L} = \Lambda(\hat{S}_{\d{A}Li}, \hat{S}_{\d{A}Lj})$, $\hat{S}_{\d{B}U} = \Lambda(\hat{S}_{\d{A}Ui}, \hat{S}_{\d{A}Uj})$ and $Đ_{\d{B}L} = \vee(Đ_{\d{A}Li}, Đ_{\d{A}Lj})$, $Đ_{\d{B}U} = \vee(Đ_{\d{A}Ui}, Đ_{\d{A}Uj})$ for all $(\d{t}_i, \d{t}_j) \in \dot{E}$.

8.1.9 Definition

The complement of an IVq-ROPFG $\dot{G} =< \d{V}, \dot{E} >$ is denoted and defined as $\dot{G}^c =< \d{V}^c, \dot{E}^c >$ where $\d{V}^c = \d{V}$ and the grades of $\dot{E}$ are defined by:

1. $(\hat{S}_{\d{B}L})^c(\d{t}_i, \d{t}_j) = \wedge\left(\hat{S}_{\d{A}L}(\d{t}_i), \hat{S}_{\d{A}L}(\d{t}_j)\right) - \hat{S}_{\d{B}L}\left((\d{t}_i, \d{t}_j)\right)$
2. $(\hat{S}_{\d{B}U})^c(\d{t}_i, \d{t}_j) = \wedge\left(\hat{S}_{\d{A}U}(\d{t}_i), \hat{S}_{\d{A}U}(\d{t}_j)\right) - \hat{S}_{\d{B}U}\left((\d{t}_i, \d{t}_j)\right) + \wedge\left(\hat{S}_{\d{A}L}(\d{t}_i), \hat{S}_{\d{A}L}(\d{t}_j)\right)$

3. $\left(Đ_{BL}\right)^{c}(ṭ_i, ṭ_j) = \vee\left(Đ_{AL}(ṭ_i), Đ_{AL}(ṭ_j)\right) - Đ_{BL}\left((ṭ_i, ṭ_j)\right)$.

4. $\left(Đ_{BU}\right)^{c}(ṭ_i, ṭ_j) = \vee\left(Đ_{AU}(ṭ_i), Đ_{AU}(ṭ_j)\right) - Đ_{BU}\left((ṭ_i, ṭ_j)\right) + \vee\left(Đ_{AL}(ṭ_i), Đ_{AL}(ṭ_j)\right)$

8.1.10 Example

Let $\tilde{V} = \{ṭ_1, ṭ_2, ṭ_3, ṭ_4\}$ be the set of vertices and $\dot{E}$ be the set of edges. Then the complement of an IVq-ROPFG of example 8.1.4 is given below

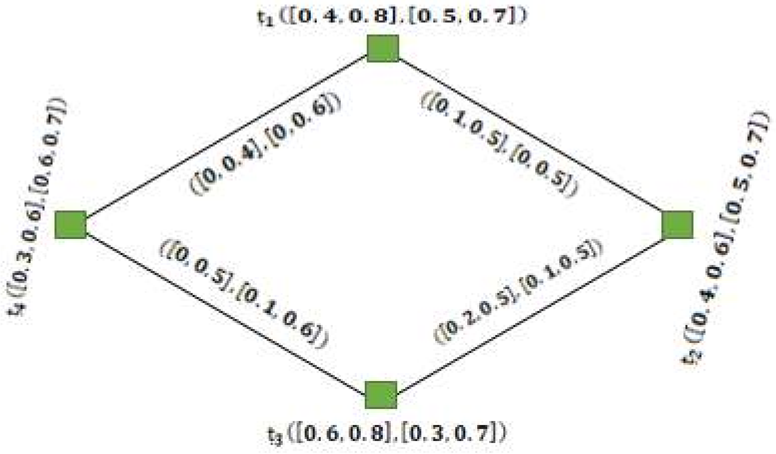


Figure 102 (Complement of IVQROPFG of example 8.1.4)

8.1.12 Remark

Likewise, in crisp sets, $((\dot{G}^c)^c) = \dot{G}$.

8.1.13 Definition

The degree of a vertex in an IVq-ROPFG $\dot{G} =< V, \dot{E} >$ is denoted and defined by

$ḍ(ṭ) = \left(ḍ\hat{S}_{1L}(ṭ), ḍ\hat{S}_{1U}(ṭ), ḍĐ_{1L}(ṭ), ḍĐ_{1U}(ṭ)\right)$ Where $ḍ\hat{S}_{1L}(ṭ) = \sum_{\substack{ṭ\neq y \\ ṭ,y\in V}} \hat{S}_{2L}(ṭy)$, $ḍ\hat{S}_{1U}(ṭ) = \sum_{\substack{ṭ\neq y \\ ṭ,y\in V}} \hat{S}_{2U}(ṭy)$ and $ḍĐ_{1L}(ṭ) = \sum_{\substack{ṭ\neq y \\ ṭ,y\in V}} Đ_{2L}(ṭy)$, $ḍĐ_{1U}(ṭ) = \sum_{\substack{ṭ\neq y \\ ṭ,y\in V}} Đ_{2U}(ṭy)$.

Here $(ḍ\hat{S}_{1L}(ṭ), ḍ\hat{S}_{1U}(ṭ))$ represents the lower and upper degrees of membership function of the vertex and $(ḍĐ_{1L}(ṭ), ḍĐ_{1U}(ṭ))$ represents the lower and upper degrees of non-membership function of the vertex.

8.1.14 Example

Let $\underset{\cdot}{V} = \{\underset{\cdot}{t}_1, \underset{\cdot}{t}_2, \underset{\cdot}{t}_3, \underset{\cdot}{t}_4\}$ and $\dot{E}$ be the collection of vertices and edges, respectively. Then, we have the following IVq-ROPFG for q =3.

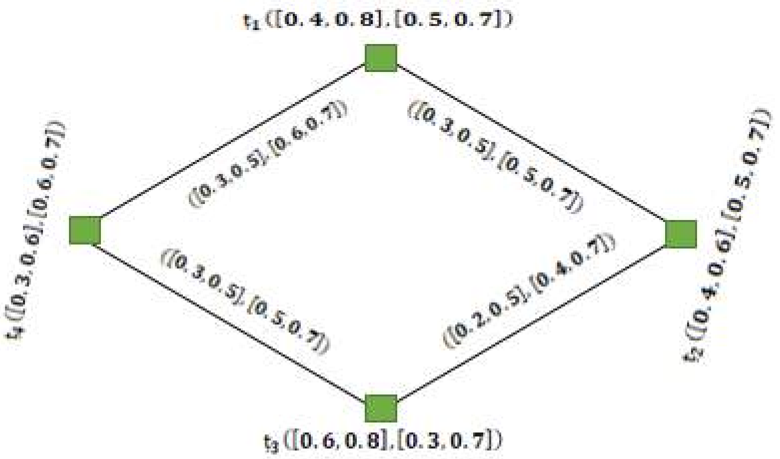


Figure 103 (Interval valued Q-rung ortho pair fuzzy graph)

Degree of vertices are:

$\mathrm{d}\underset{\cdot}{t}_1 = ([0.6, 1], [1.1,1.4], \mathrm{d}\underset{\cdot}{t}_2 = ([0.5, 1], [0.9,1.4], \mathrm{d}\underset{\cdot}{t}_3 = ([0.5, 1], [0.9,1.4], \mathrm{d}\underset{\cdot}{t}_4 = ([0.6, 1], [1.1,1.4],$

8.1.15 Proposition

Let $\dot{G} =< \underset{\cdot}{V}, \dot{E} >$ be a complete$-\hat{S}$ strong or complete$-Đ$ strong q-ROPFG. Then its complement is complete IVq-ROPFG.

Proof: Let $\dot{G}$ be a complete$-\hat{S}$ strong q-ROPFG $\Rightarrow \hat{S}_{\underset{\cdot}{B}L} = \Lambda(\hat{S}_{\underset{\cdot}{A}Li}, \hat{S}_{\underset{\cdot}{A}Lj}), \hat{S}_{\underset{\cdot}{B}U} = \Lambda(\hat{S}_{\underset{\cdot}{A}Ui}, \hat{S}_{\underset{\cdot}{A}Uj})$ and $Đ_{\underset{\cdot}{B}L} < \vee(Đ_{\underset{\cdot}{A}Li}, Đ_{\underset{\cdot}{A}Lj}), Đ_{\underset{\cdot}{B}U} < \vee(Đ_{\underset{\cdot}{A}Ui}, Đ_{\underset{\cdot}{A}Uj})$ for all i, j. To prove $\dot{G}^c$ is complete we have to prove either 1 or 2.

1. $\hat{S}_{\underset{\cdot}{B}L}{}^c > 0, \hat{S}_{\underset{\cdot}{B}U}{}^c > 0 \ or \ Đ_{\underset{\cdot}{B}L}{}^c > 0, Đ_{\underset{\cdot}{B}U}{}^c > 0$
2. $\hat{S}_{\underset{\cdot}{B}L}{}^c = 0, \hat{S}_{\underset{\cdot}{B}U}{}^c = 0 \ or \ Đ_{\underset{\cdot}{B}L}{}^c > 0, Đ_{\underset{\cdot}{B}U}{}^c > 0$

Now as $\hat{Ṣ}_{ḄL}{}^{c} = \Lambda\left(\hat{Ṣ}_{ẠLi}, \hat{Ṣ}_{ẠLj}\right) - \hat{Ṣ}_{ḄL} = \begin{cases} 0\ if\ \hat{Ṣ}_{ḄL} > 0 \\ \hat{Ṣ}_{ẠLi}\ if\ \hat{Ṣ}_{ḄL} = 0 \end{cases}$, $\hat{Ṣ}_{ḄU}{}^{c} = \Lambda\left(\hat{Ṣ}_{ẠUi}, \hat{Ṣ}_{ẠUj}\right) - \hat{Ṣ}_{ḄU} =$

$$\begin{cases} 0\ if\ \hat{Ṣ}_{ḄU} > 0 \\ \hat{Ṣ}_{ẠUi}\ if\ \hat{Ṣ}_{ḄU} = 0 \end{cases}$$

And $Đ_{ḄL}{}^{c} = \vee\left(Đ_{ẠLi}, Đ_{ẠLj}\right) - Đ_{ḄL} > 0$, $Đ_{ḄU}{}^{c} = \vee\left(Đ_{ẠUi}, Đ_{ẠUj}\right) - Đ_{ḄU} > 0$for all $i, j = 1, 2, 3, \dots q$ as $\dot{G} =< Ṿ, \dot{\acute{E}} >$ be a Complętę−Ŝ $strong$. Hence $\overline{\dot{G}} =< \overline{Ṿ}, \overline{\dot{\acute{E}}} >$ is complete. The second part can be proved analogously.

8.1.16 Definition

In an IVq-ROPFG a path is a sequence of some distinct vertices $ṭ_i (i = 1, 2, 3, \dots, q)$ if either one of the following holds for some $(i, j = 1, 2, 3, \dots, m)$

- $\hat{Ṣ}_{ḄLij} > 0$, $\hat{Ṣ}_{ḄUij} > 0$ and $Đ_{ḄLij} > 0$, $Đ_{ḄUij} > 0$
- $\hat{Ṣ}_{ḄLij} = 0$, $\hat{Ṣ}_{ḄUij} = 0$ and $Đ_{ḄLij} > 0$, $Đ_{ḄUij} > 0$
- $\hat{Ṣ}_{ḄLij} > 0$, $\hat{Ṣ}_{ḄUij} > 0$ and $Đ_{ḄLij} = 0$, $Đ_{ḄUij} = 0$

8.1.17 Definition

Let $P = ṭ_1, ṭ_2, ṭ_3, \dots, ṭ_{m+1}$ $(m > 0)$ be a path. Then its length is m. This path is known as a cycle if $ṭ_1 = ṭ_{m+1}$ for $(m \geq 3)$ while two vertices combined by a path are known as connected.

8.1.18 Example

Let $Ṿ = \{ṭ_1, ṭ_2, ṭ_3, ṭ_4\}$ and $\dot{\acute{E}}$ be the set of edges. Then, refer Figure 103.

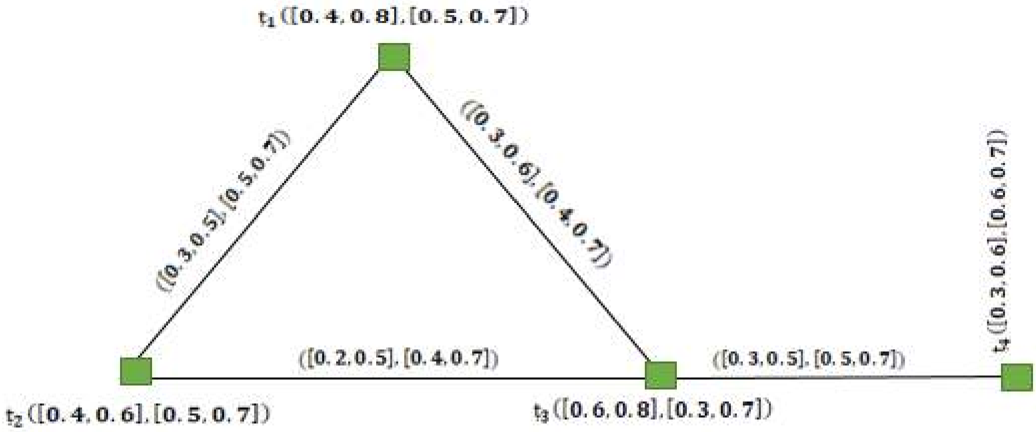


Figure 104 (Interval valued Q-rung ortho pair fuzzy graph)

Here $ṭ_1ṭ_2\ ṭ_3\ ṭ_4$ is a path and hence $ṭ_1, ṭ_2, ṭ_3, ṭ_4$ are connected vertices and the length of this path is **3.** Moreover $ṭ_1ṭ_2\ ṭ_3ṭ_1$ is a cycle.

8.1.19 Definition

Let $t \in [0,1]$. Then the triple $< Ṿ_t, [\,\hat{S}_{ẠLt},\ \hat{S}_{ẠUt}], [\,Đ_{ẠLt},\ Đ_{ẠUt}] > \subseteq Ṿ$ and $<$ $\dot{E}_t, [\,\hat{S}_{ḄLt},\ \hat{S}_{ḄUt}], [\,Đ_{ḄLt},\ Đ_{ḄUt}] > \ \subseteq \dot{E}$ for some $i = 1,2,3,\dots m$ where

$\hat{S}_{ẠLt} = \{ṭ_i \in Ṿ:\ \hat{S}_{ẠLi} \geq t\}$, $\hat{S}_{ẠUt} = \{ṭ_i \in Ṿ:\ \hat{S}_{ẠUi} \geq t\}$ and $Đ_{ẠLt} = \{ṭ_i \in Ṿ:\ Đ_{ẠLi} \leq t\}$, $Đ_{ẠUt} = \{ṭ_i \in Ṿ:\ Đ_{ẠUi} \leq t\}$, $\hat{S}_{ḄLt} = \{(ṭ_i, ṭ_i) \in Ṿ \times Ṿ:\ \hat{S}_{ḄLij} \geq t\}$, $\hat{S}_{ḄUt} = \{(ṭ_i, ṭ_i) \in Ṿ \times Ṿ:\ \hat{S}_{ḄUij} \geq t\}$and $Đ_{ḄLt} = \{(ṭ_i, ṭ_i) \in Ṿ \times Ṿ:\ Đ_{ḄLij} \leq t\}$, $Đ_{ḄUt} = \{(ṭ_i, ṭ_i) \in Ṿ \times Ṿ:\ Đ_{ḄUij} \leq t\}$.

8.1.20 Theorem

For $x, y \in [0,1]$ such as $x \leq y$, let $Ḥ =< Ṿ_x, \dot{E}_x >$ and $\dot{G} =< Ṿ_y, \dot{E}_y >$. Then $Ḥ \leq_G \dot{G}$.

Proof: To prove $Ḥ \leq_G \dot{G}$. We show that $Ṿ_x$and $\dot{E}_x$ are subsets of $Ṿ_y$ and $\dot{E}_y$ respectively. Let $ṭ_i \in Ṿ_x$ so $Đ_{ẠLi} \leq x\ \leq y$ as $x \leq y\ \Rightarrow\ Đ_{ẠLi} \leq y$ and $ṭ_i \in Ṿ_y\ \Rightarrow Ṿ_x \subseteq Ṿ_y$, $Đ_{ẠUi} \leq x\ \leq y$ as $x \leq y\ \Rightarrow\ Đ_{ẠUi} \leq y$ and $ṭ_i \in Ṿ_y\ \Rightarrow Ṿ_x \subseteq Ṿ_y$

Let $(\d{t}_i, \d{t}_i) \in \dot{E}_x$ which implies that $Đ_{\d{B}Lij} \leq x$

$$\leq y \text{ as } x \leq y$$

$$\Rightarrow Đ_{\d{B}Lij} \leq y \text{ and } (\d{t}_i, \d{t}_i) \in \dot{E}_y$$

$$\Rightarrow \dot{E}_x \subseteq \dot{E}_y.$$

Let $(\d{t}_i, \d{t}_i) \in \dot{E}_x$ which implies that $Đ_{\d{B}Uij} \leq x$

$$\leq y \text{ as } x \leq y$$

$$\Rightarrow Đ_{\d{B}Uij} \leq y \text{ and } (\d{t}_i, \d{t}_i) \in \dot{E}_y$$

$$\Rightarrow \dot{E}_x \subseteq \dot{E}_y.$$

Hence $Ḥ \leq_G \dot{G}$.

8.1.21 Theorem

Let $Ḥ =< \d{V}', \dot{E}' >$ and $\dot{G} =< \d{V}, \dot{E} >$ such that $Ḥ \leq_G \dot{G}$. Then for $x \in [0,1]$ $< \d{V}'_x, \dot{E}'_x >$ $\leq_G < \d{V}_x, \dot{E}_x >$.

Proof: Let $\d{t}_i \in \d{V}'_x \Rightarrow \hat{S}'_{\d{A}Li} \geq x \Rightarrow \hat{S}_{\d{A}Li} \geq x$ as $\hat{S}'_{\d{A}Li} \leq \hat{S}_{\d{A}Li} \Rightarrow \d{t}_i \in \d{V}_x \Rightarrow \d{V}'_x \subseteq \d{V}_x \Rightarrow \hat{S}'_{\d{A}Ui} \geq x \Rightarrow \hat{S}_{\d{A}Ui} \geq x$ as $\hat{S}'_{\d{A}\hat{S}i} \leq \hat{S}_{\d{A}Ui} \Rightarrow \d{t}_i \in \d{V}_x \Rightarrow \d{V}'_x \subseteq \d{V}_x$. Let $(\d{t}_i, \d{t}_j) \in \dot{E}'_x \Rightarrow \hat{S}'_{\d{B}Lij} \geq x \Rightarrow \hat{S}_{\d{B}Lij} \geq x$ as $\hat{S}'_{\d{B}L} \leq \hat{S}_{\d{B}L} \Rightarrow (\d{t}_i, \d{t}_j) \in \dot{E}$

$\Rightarrow \dot{E}'_x \subseteq \dot{E} \Rightarrow \hat{S}'_{\d{B}Uij} \geq x \Rightarrow \hat{S}_{\d{B}Uij} \geq x$ as $\hat{S}'_{\d{B}U} \leq \hat{S}_{\d{B}U} \Rightarrow (\d{t}_i, \d{t}_j) \in \dot{E} \Rightarrow \dot{E}'_x \subseteq \dot{E}$

Hence $< \d{V}'_x, \dot{E}'_x > \leq_G < \d{V}_x, \dot{E}_x >$.

8.2 Applications

In this section, we aim to apply the proposed idea of IVq-ROPFS to some real-life engineering problems including, shortest path problem which is a famous problem utilized in civil engineering to compute the shortest route among some possible routes. Second, we present an application of proposed ideas to discuss two engineering decision making problems. Further, IVq-ROPFGs are used in analysing a social network where the social network of co-authors of different countries is demonstrated. At last, an application of IVq-ROPFGs is studied in observing a telephone network where the incoming and outgoing calls are observed followed by a dynamic algorithm.

8.2.1 Shortest Path Problem

In advanced transportation problems of traffic engineering, the computation of shortest route is often a challenging task. Several algorithms exist in literature in order to compute the shortest route among some possible routes such as Dijkstra algorithm and Floyd's algorithm. Whenever the information about the path has lack of precision that is the information is fuzzy in fact, then fuzzy Dijkstra algorithm is utilized to compute shortest path among all possible paths. Some comprehensive work on fuzzy Dijkstra algorithm and its application in finding shortest path problem has been done in [104-107]. We followed the algorithm used in [104]. The comprehensive steps of algorithm next to a flow chart are in this section using two methods to find the shortest path subsequently pursued by two numerical examples and several related studies.

8.2.1.1 Dijkstra Algorithm for Finding Shortest Path Problem.

Dijkstra algorithm is the mainly sound or most suitable way to locate shortest path in a network. The Dijkstra algorithm's comprehensive steps for IVq-ROPFGs are acknowledged as follows:

1. Record the source node as permanent node (P) and allot it the label $([0,0],[1,1])$ because in shortest path this node is integrated by defect and the covered distance at this stage is 0.

2. Compute the label $[v_i \oplus d_{ij}, i]$ for each node j whose pathway is from node I, condition is when j is not a permanent node, in addition if j is labeled as $[v_j, k]$ by means of some additional node then substitute $[v_j, k]$ by $[v_i \oplus d_{ij}, i]$ only if $SC(v_i \oplus d_{ij})$ is less than $SC(v_j)$.
3. If all the nodes are enduringly labeled, the algorithm eliminates. Moreover, choose $[v_r, s]$ having shortest distance v_r and do again step 2 by setting $i = r$.
4. Discover the shortest path from SN to DN, by means of information of the label.

8.2.1.2 Example

Consider a traffic engineering problem of several stopes/nodes connected by roads of a certain city. The aim is to compute the shortest route from node 1 to node 6. We apply the Dijkstra algorithm to this problem in order to compute the shortest route. The information about suitability of roads from one node to another node are given in terms of IVq-ROPFNs where the first interval shows the suitability of a road while the second interval shows the non-suitability of a road. A brief demonstration of computing shortest route using Dijkstra algorithm described in section 8.2.1.1 is demonstrated as follows.

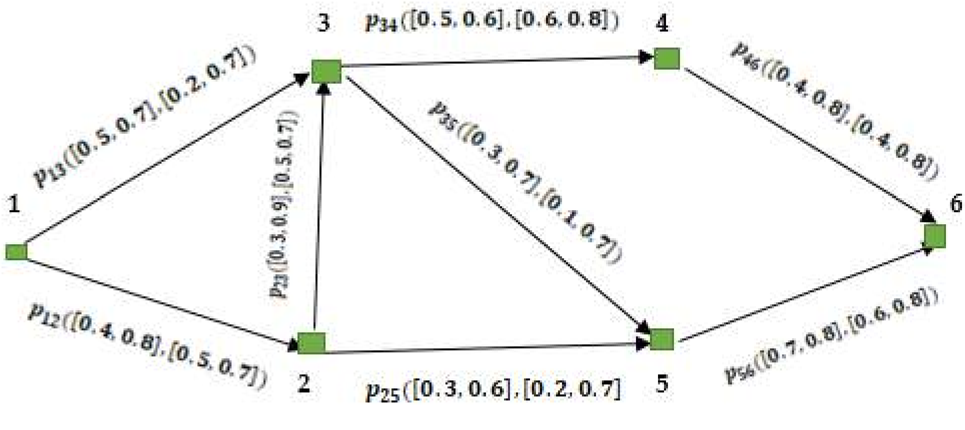


Figure 105 (Interval valued Q-rung ortho pair fuzzy graph)

The edges involved in this network are listed in Table 25.

Edges	Interval valued Q-rung ortho pair Fuzzy Distances
(p_1, p_2)	$([0.4,0.8],[0.5,0.7])$
(p_1, p_3)	$([0.5,0.7],[0.2,0.7])$
(p_2, p_3)	$([0.3,0.9],[0.5,0.7])$
(p_2, p_5)	$([0.3,0.6],[0.2,0.7])$
(p_3, p_4)	$([0.5,0.6],[0.6,0.8])$
(p_3, p_5)	$([0.3,0.7],[0.1,0.7])$
(p_4, p_6)	$([0.4,0.8],[0.4,0.8])$
(p_5, p_6)	$([0.7,0.8],[0.6,0.8])$

Table 25 (Weights of edges)

Now we apply the modified Dijkstra algorithm and the step by step computations are as follows:

Step 1: Mark node 1 as permanent as this node is in the shortest path by default.

Step 2: There are two ways directed from node 1 i.e. we may either move to node 2 or to node 3. Therefore, we have the following list of nodes in Table 26.

Nodes	Label	Status
p_1	$([0,0],[1,1])$	Permanent
p_2	$(([0.4,0.8],[0.5,0.7]), p_1)$	Temporary
p_3	$(([0.5,0.7],[0.2,0.7]), p_1)$	Temporary

Table 26 (List of nodes)

Now we to find the score of $([0.4,0.8],[0.5,0.7])$ and $([0.5,0.7],[0.2,0.7])$.

$SC([0.4,0.8],[0.5,0.7]) = 0.196192$

$SC(([0.5,0.7],[0.2,0.7]) = 0.24495$

Therefore, the score of $([0.4,0.8],[0.5,0.7])$ is less than the score of $(([0.5,0.7],[0.2,0.7])$, so we mark node 2 as permanent and labeled it by $(([0.4,0.8],[0.5,0.7]), t_1)$.

Step 3: There are two ways directed from node 3 i.e. we may either move to node 4 or to node 5. Therefore, we have the following list of nodes in Table 27.

Nodes	Label	Status
p_1	$([0,0],[1,1])$	Permanent
p_2	$(([0.4,0.8],[0.5,0.7]),p_1)$	Permanent
p_3	$(([0.5,0.7],[0.2,0.7]),p_1)$	Temporary
p_3	$(([0.016746,0.398481],[0.25,0.49]),p_2)$	Temporary
p_5	$(([0.016746,0.243058],[0.1,0.49]),p_2)$	Temporary

Table 27 (List of nodes)

Now we find the score of $([0.016746,0.398481],[0.25,0.49])$ and $([0.016746,0.243058],[0.1,0.49])$ as follows:

$SC([0.016746,0.398481],[0.25,0.49]) = 0.107892$

$SC([0.016746,0.243058],[0.1,0.49]) = 0.069515$

Here we note that the score of $([0.016746,0.243058],[0.1,0.49])$ is less than the score of $([0.016746,0.398481],[0.25,0.49])$, so therefore we mark node 5 as $(([0.016746,0.243058],[0.1,0.49]),ṭ_2)$ is label is permanent.

Step 4: There is only one way directed from node 5 i.e. we can only move to node 6. Therefore, we have the following list of nodes in Table 28.

Nodes	Label	Status
p_1	$([0,0],[1,1])$	Permanent
p_2	$(([0.4,0.8],[0.5,0.7]),p_1)$	Permanent
p_3	$(([0.5,0.7],[0.2,0.7]),p_1)$	Temporary
p_3	$(([0.016746,0.398481],[0.25,0.49]),p_2)$	Temporary
p_5	$(([0.016746,0.243058],[0.1,0.49]),p_2)$	Permanent
N_6	$(([0.12005,0.20583],[0.06,0.392]),p_5)$	Permanent

Table 28 (List of nodes)

As there is only one way from node 5 to 6. Therefore, we mark node 6 as $([0.12005,0.20583],[0.06,0.392])$ and labeled it permanent.

Step 5: Nodes 2 and 4 are the remaining temporary nodes, so their status is changed to permanent and we have the following list of nodes in Table 29.

Nodes	Label	Status
p_1	$([0,0],[1,1])$	Permanent
p_2	$(([0.4,0.8],[0.5,0.7]),p_1)$	Permanent
p_3	$(([0.5,0.7],[0.2,0.7]),p_1)$	Permanent
p_3	$(([0.016746,0.398481],[0.25,0.49]),p_2)$	Permanent
p_5	$(([0.016746,0.243058],[0.1,0.49]),p_2)$	Permanent
N_6	$(([0.12005,0.20583],[0.06,0.392]),p_5)$	Permanent

Table 29 (List of nodes)

Step 6: From Table 30, we have the following sequence of shortest path form SN to DN i.e. from node 1 to 6.

N_6	$(([\mathbf{0.12005},\mathbf{0.20583}],[\mathbf{0.06},\mathbf{0.392}]),p_5)$
p_5	$(([0.016746,0.243058],[0.1,0.49]),p_2)$
p_2	$(([0.4,0.8],[0.5,0.7]),p_1)$

Table 30 (List of nodes)

Hence the shortest path according to modified Dijkstra algorithm is $t_1 \rightarrow t_2 \rightarrow t_5 \rightarrow t_6$. Despite there are more ways from node 1 to node 6 but considering the information of the decision makers about the paths we found that the shortest path is $t_1 \rightarrow t_2 \rightarrow t_5 \rightarrow t_6$. Now we consider the same problem using a new algorithm which is demonstrated in section 8.2.1.1.

8.2.1.3 A New Algorithm for Finding Shortest Path

In this subsection, the aim is to compute the shortest path from node 1 to node 6 of Example 8.2.1.2. This new algorithm is proposed by [106] and the reason to follow this algorithm is the easy way of its computations. The programming of this method is also easy compared to the previously discussed algorithm.

8.2.1.4 Algorithm for Finding Shortest Path

The steps of algorithm are:

15. Take the first node as $([0,0],[1,1])$ as distance of every node to itself is zero.

16. Take $i = 1$.

17. Find j for $P_{1j} = P_{11} + \wedge_{j \in NP(1)} P_{1j}$ and determine P_{1j}.

18. Put $i = j$.

19. Find k for $P_{jk} = P_{1j} + \wedge_{k \in NP(1)} P_{jk}$ and determine P_{jk}.

20. This process should be continued until destination node is obtained.

21. When destination node is reached. Then algorithm is stopped.

To further demonstrate the algorithm, a flowchart is depicted in Figure 106.

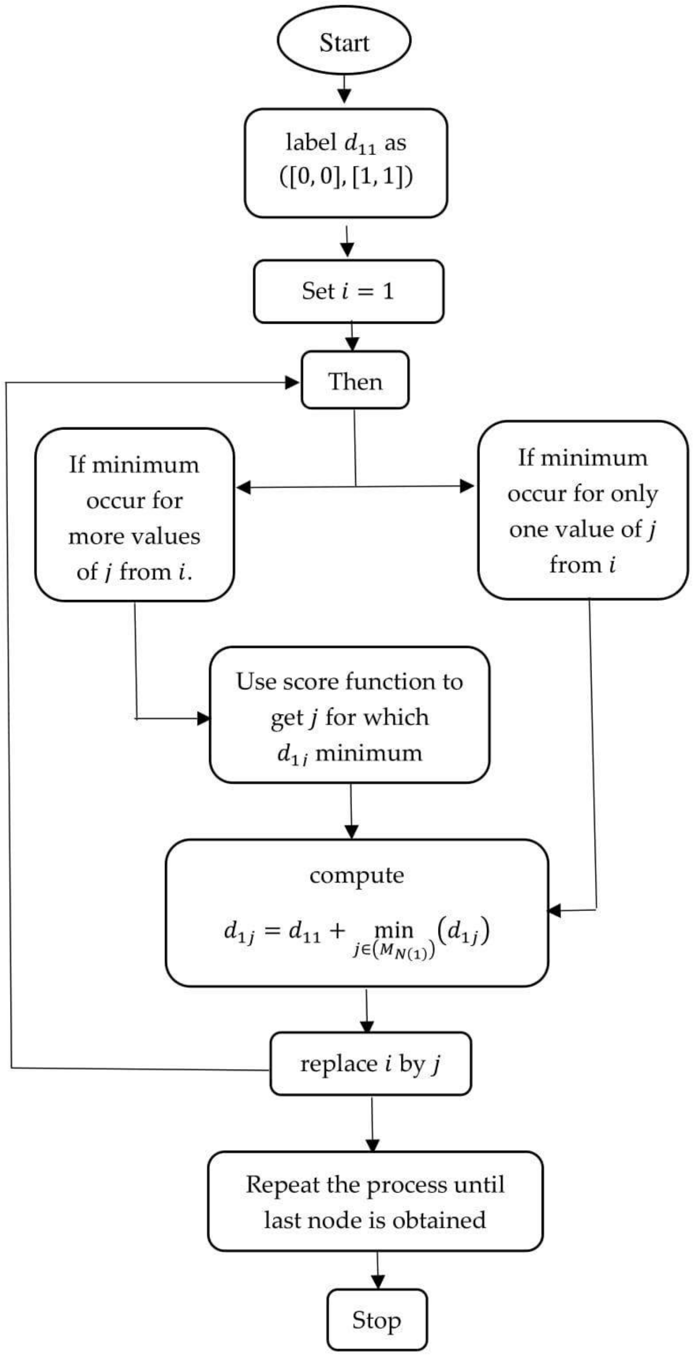


Figure 106 (Flow chart of the new algorithm)

8.2.1.5 Example 8

Here we consider the information of Example 8.2.1.2 to compute the shortest path using proposed new algorithm. The detailed steps are demonstrated as follows.

In the above Figure 105 graph shows the source node is node 1 and the destination node is node 6, so for $m = 6$ and for the first step $P_{11} = ([0,0],[1,1])$. At first $i = 1$, so it needs to find *j* by the equation.

$$P_{1j} = P_{11} + \wedge_{j \in NP(1)} P_{1j} = P_{11} + \wedge_{j \in \{2,3\}} P_{1j} = P_{11} + (P_{12} \wedge P_{13})$$

$$= ([0,0],[1,1]) + \big(([.4,.8],[.5,.7]) \wedge ([.5,.7],[.2,.7])\big)$$

$$= ([0,0],[1,1]) + (0.196192 \wedge 0.24495) = ([0,0],[1,1] + (0.196192)$$

$$= ([0,0],[1,1]) + ([.4,.8],[.5,.7]) = ([.4,.8],[.5,.7])$$

Hence for $j = 2$, $P_{12} = ([.4,.8],[.5,.7])$.

Now set $i = 2$, we need to find *j* by the equation

$$P_{1j} = P_{12} + \wedge_{j \in NP(2)} P_{2j} = P_{12} + \wedge_{j \in \{3,5\}} P_{2j} = P_{12} + (P_{23} \wedge P_{25})$$

$$= ([.4,.8],[.5,.7]) + \big(([.3,.9],[.5,.7]) \wedge ([.3,.6],[.2,.7])\big)$$

$$= ([.4,.8],[.5,.7]) + (0.,253082 \wedge 0.135) = ([.4,.8],[.5,.7]) + (0.135)$$

$$= ([.4,.8],[.5,.7]) + ([0.3,0.6],[0.2,0.7]) = ([0.044636, 0.308704],[.1,.49])$$

Hence for $j = 2$, $P_{15} = ([0.044636, 0.308704],[.1,.49])$.

Now set $i = 5$, there is only one way from node 5 which is node 6. we need to find by the equation

$$P_{1j} = P_{15} + \wedge_{j \in NP(5)} P_{1j} = P_{15} + \wedge_{j \in \{6\}} P_{1j} = P_{15} + P_{56}$$

$$= ([0.044636, 0.308704],[.1,.49]) + ([0.7,0.8],[0.6,0.8])$$

$$= ([.120052, 0.207448],[.06,.392])$$

Hence for $j = 2$, $P_{16} = ([.120052, 0.207448],[.06,.392])$

So, the shortest path in the above graph is $P_1 \longrightarrow P_2 \longrightarrow P_5 \longrightarrow P_6$. This is exactly the same path that is obtained using Dijkstra algorithm, but this method has an advantage over the Dijkstra algorithm as its computation is easy and takes less time comparatively.

8.2.1.6 Comparative Study and Advantages

As it is already discussed that IVq-ROPFS generalizes FS, IVFS, IFS, IVIFS, PyFSs, IVPyFS and q-ROPFS. Here we assume the same problem in the environment of fuzzy graphs i.e. the information of the edges is now taken in the form of fuzzy numbers as depicted in Figure 107. We apply the proposed Dijkstra algorithm as well as the new algorithm for finding the shortest path to problem with fuzzy information and analyze the results in Table 28.

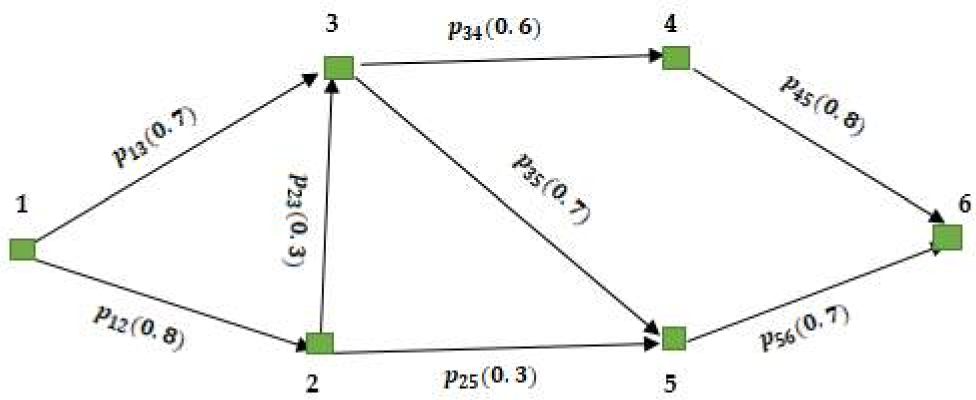


Figure 107 (Network of roads in fuzzy environment)

The shortest paths using the two algorithms in above traffic problem with fuzzy information are given in Table 31.

Algorithm	Shortest Path
Dijkstra Algorithm	$P_1 \longrightarrow P_3 \longrightarrow P_4 \longrightarrow P_6$
New Algorithm	$P_1 \longrightarrow P_3 \longrightarrow P_4 \longrightarrow P_6$

Table 31 (shortest path obtained using the proposed algorithms in fuzzy environment)

These results are clearly different than the results obtained in the environment of IVq-ROPFGs. This is because the information in the form of IVq-ROPFNs not only describe the membership degree, but it also describes the non-membership degree and hence provide better results. In case of fuzzy membership grading, the non-membership degree about the suitability of a path is not available thus increasing the vagueness. The results obtained using both algorithms in two different environments are demonstrated in Figure 108 and Figure 109 respectively where the shortest path is represented by 1→ 2 → 5 → 6. The following Figure 107 shows the shortest path in the environment of IVq-ROPFGs.

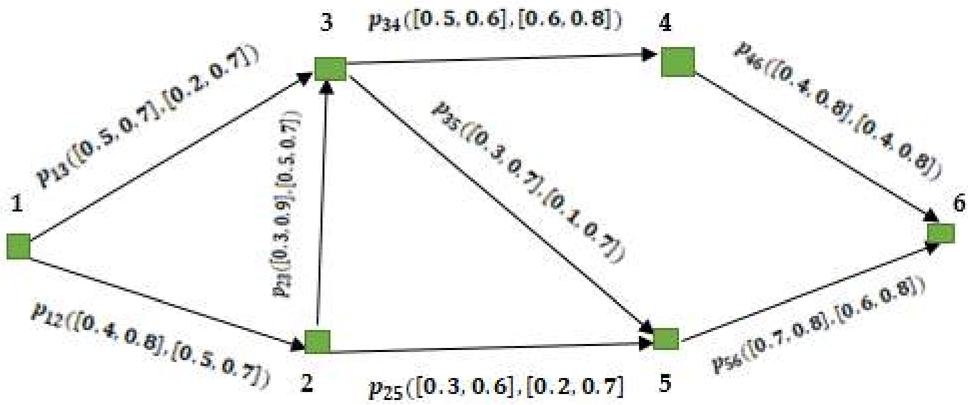


Figure 108(shortest path in Interval valued Q-rung ortho pair fuzzy environment)

The Figure 109 shows the shortest path using both algorithms in the environment of fuzzy graphs where the 1→ 3 → 4 → 6 path shows the shortest path.

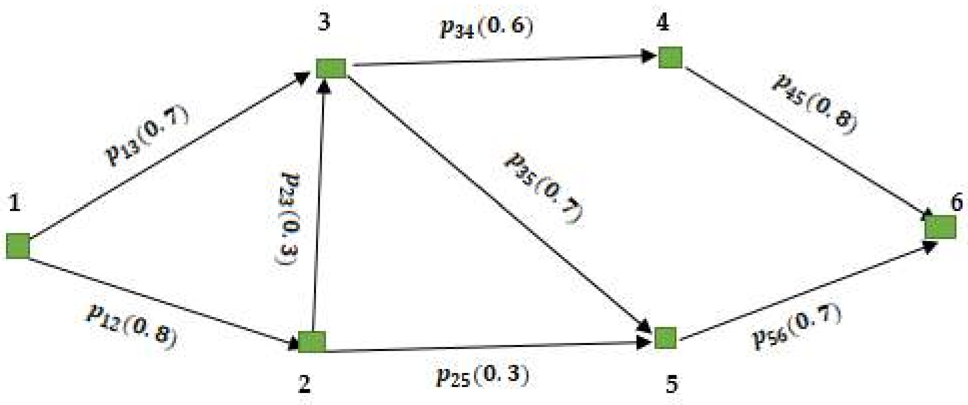


Figure 109 (shortest path in fuzzy environment)

8.3 Application of social networks

A social network is considered as the network of peoples/ groups/ countries which are socially connected. The mathematical modelling of such social networks is very essential to study the behaviour of people/groups/countries involved. Graph theory provide such platform as a graph can be considered as a network of socially connected persons or objects.

In research community, several co-authors from different countries are usually connected. The aim of this section is to apply the concept of IVQROPFGs to a social network of several co-authors of different countries. In such networks, the nodes represent the countries which the authors belong to and the edges denote the strength of their co-authorship relation i.e. to how much extent the authors involve with each other. Such networks can be very large because of the large number of scientists from enormous countries. In our case, we present a social network of research collaborators from few countries and analyse it using the concept of IVQROPFSs.

Co-authorship required due to following reasons

1. Field of expertise: In interdisciplinary research, authors from different fields of expertise join together to work on a problem which involves more that more than one field for its solution.
2. Language barriers: Sometimes to maintain the quality of language in a research project, authors from different countries required professionals from the same field but with high skills of language as well.
3. Supervision: At many occasions, authors did collaboration with high ranked researchers in order to give the research a better look by supervising the research project.
4. Funding: Funding a research project is also considered as an essential part as in some cases authors approach other researches from the same fields but with some funding sources.

In Figure 110 below, we have shown the relation of several co-authors from five countries with other countries. The nodes represent the country of the researcher while the edges show their mutual relation keeping in mind the 4 points discussed above. We analyse the degree of each node which shows the strength of relation of the authors from that country with the researchers of other countries.

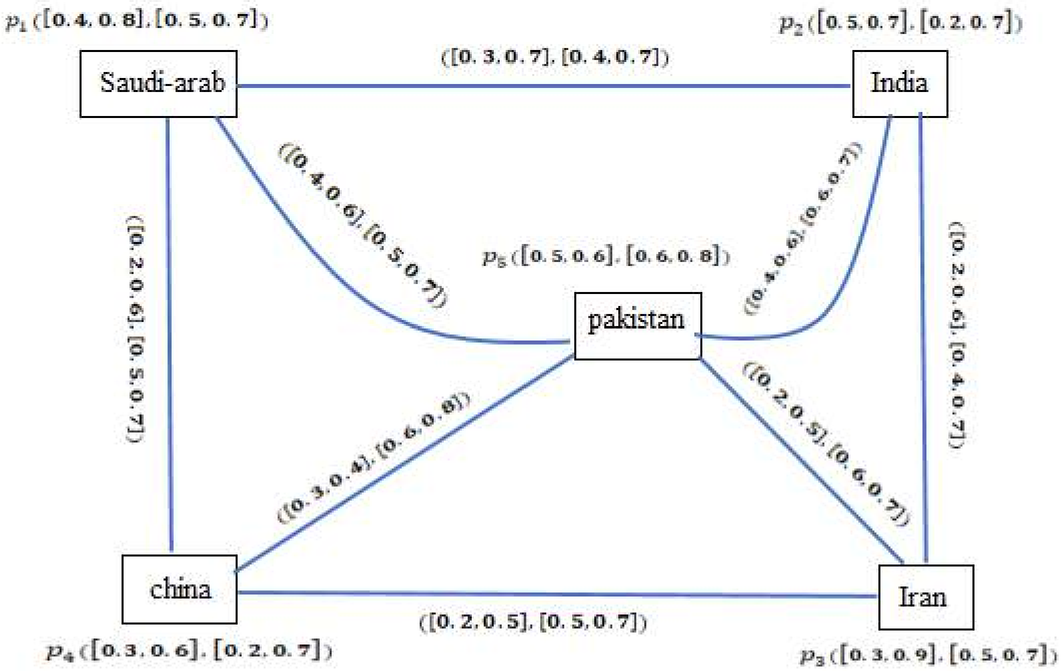


Figure 110 (Social network)

In above Figure 110, the relationship between the co-authors of every two countries is characterized by the edge in the form of IVQROPFN. In Table 29, the degree of each node is given in the form of IVQROPFN showing the strength of the relationship of the researchers of that country with the researchers of other countries. The membership grade of each value of degree shows the intensity of strength and the non-membership value shows the weakness of the relation among the co-authors. So, if the non-membership degree is smaller in contrast to membership degree then the relationship is measured as strong or else weak. Each apex degree will present us the relationship of the researchers of a country with other countries.

Country	Degree
Iran	([0.6,1.6], [1.5,2.1])
Pakistan	([1.3,2.1], [2.3,2.9])
Saudi-Arab	([0.9,1.9], [1.4,2.1])
India	([0.9,1.9], [1.4,2.1])
China	([1.2,1.5], [1.6,2.2])

Table 32 (Edge relations of Figure 110)

At the present, on basis of Definition 8.1.13 the relation's degree of each country is quantified in terms a score value. The strength of Relationship of the researchers of one country with that of other countries is fine if there is a high degree of membership otherwise the relation will be neutral or poor. The quantified terms of degrees are given in Table 33, by using the Def. (8.1.4).

Country	Score value
India	-0.35
Saudi-Arab	-0.35
Pakistan	-0.9
China	-0.55
Iran	-0.7

Table 33 (Scores of degrees of each vertex)

The computed score values of the degree of each country is given in Table 33 which clearly indicated that the score of India and Saudi-Arab is less negative, so the researchers of these countries have a better rate of co-authorship with that of other countries. Pakistan stands at number 3 in this Table. The above information is represented by a bar graph in Figure 111 for a better understanding of score values. Such bar graph analysis of data is very useful when we have large type of data that happened in real life.

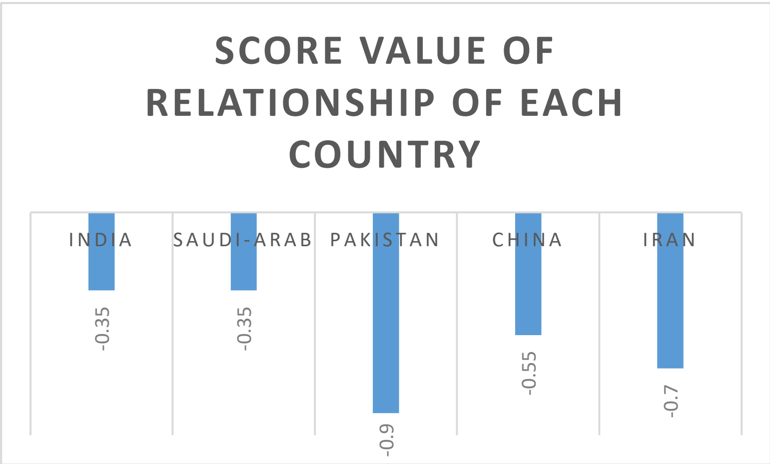


Figure 111 (Analysis of score values of Table 30)

8.3.1 Comparative Study and Advantages:

Now we establish a comparative study of the analysis of social networks in the environments of FGs and IVQROPFGs. We observe the network of co-authorship by representing the edges and vertices of the social network discussed in Figure 110 using fuzzy numbers. Consider a collaboration network of co-authors in the environment FGs in Figure 112.

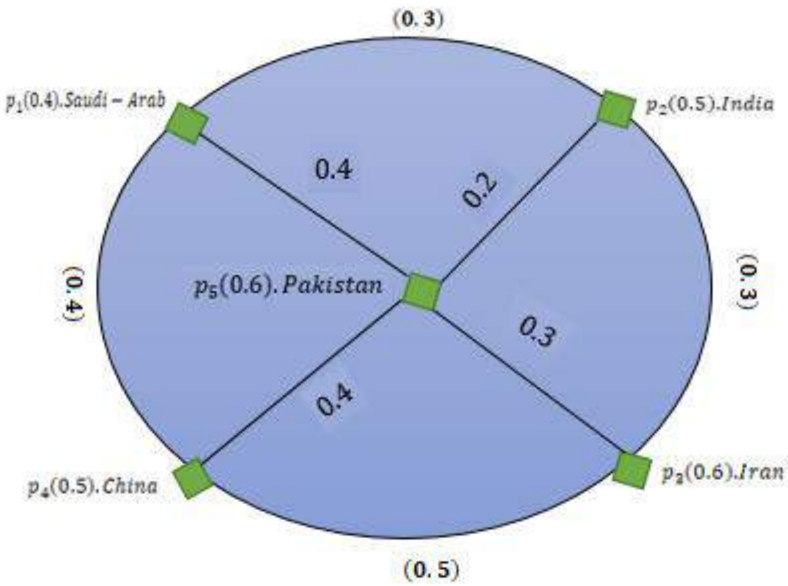


Figure 112 (Co-authorship network using fuzzy numbers)

The strength of relationship between the co-authors of different countries using fuzzy numbers are given in Table 31.

Country	Country	Relationship
Pakistan	India	(0.2)
Iran	India	(0.3)
China	Saudi-Arab	(0.4)
Pakistan	China	(0.4)
India	Saudi-Arab	(0.3)
Pakistan	Iran	(0.3)
Pakistan	Saudi-Arab	(0.4)
China	Iran	(0.5)

Table 31 (Relationship of co-authors in terms of fuzzy numbers as depicted in Fig 112)

Note that the information provided in Figure 112 are taken from Figure 110 but in the form of fuzzy numbers. The degree of each county using the definition of degree of fuzzy numbers in order to rank the countries is given in Table 32.

Country	Degree
China	1.3
Pakistan	1.3
Saudi-Arab	1.1
Iran	1.1
India	0.9

Table 32 (Degree of each node of Fig 112)

The information of Table 32 clearly indicates that China and Pakistan stand at number 1 in co-authorship network while Saudi-Arab and Iran stands at number 2 with degree 1.1. India in this regard at number 3. However if we refer to Table 30 then we came to know that India and Saudi-Arab was on top in collaboration network. The bar graph of degrees of each node of Figure 112 is depicted in Figure 113.

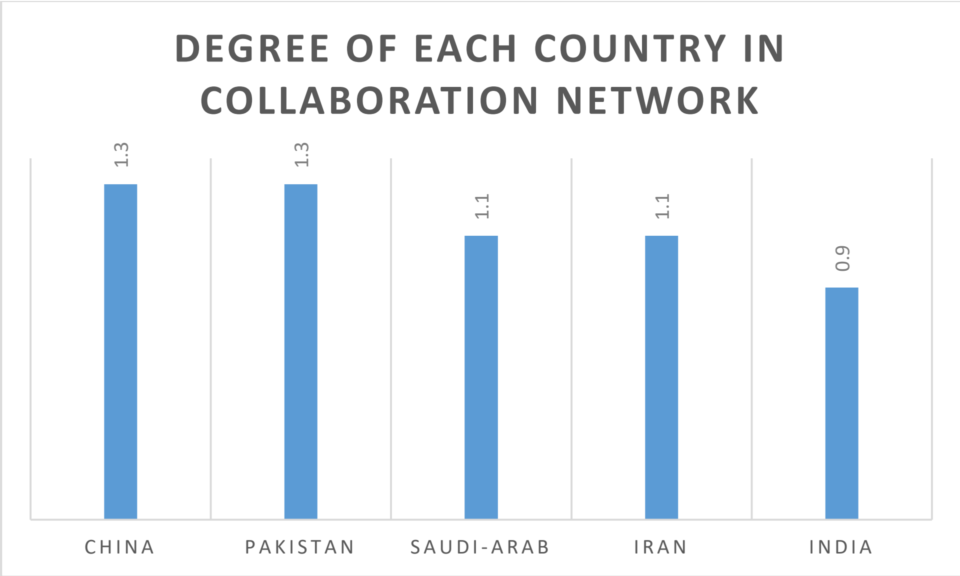


Figure 113 (Analysis of degrees of Table 32)

Note that the analysis of social network using IVq-ROPFG is more effective than using FGs because an IVq-ROPFG describes the non-membership degree as well along with membership degree while traditional fuzzy graph can only describe the membership grade of the relation of co-authors of the two countries.

8.4 Applications of IVQROPFGs in Telephonic Networks:

Telephone network is another area where graph theory is applicable as telephones connect peoples to each other. In a telephone network, people act as nodes while the edges denote their connectivity sources. There are three major types of telephone networks known as:

- Public Switched Telephone Networks (PSTNs): These networks are usually termed as public telephone networks e.g. Landline networks connected public of the world.
- Wireless Networks: These are commonly used mobile networks that connects individuals.
- Private Networks: These types of networks are used within an organization connecting the employee of that specific organization.

With a rapid increment in the usage of mobile networks, the incoming and outgoing calls gained considerable importance. In fuzzy graph theory one cannot find significant amount of work dealing with the incoming and outgoing calls which is a dynamic problem. Therefore, in this subsection, we develop an algorithm for storing the number of incoming or outgoing calls using the approach of IVQROPFGs.

In a wireless mobile network, whenever a call is made, there are three kinds of circumstances that a person can face, that are:

- Call is received
- Call is not received or rejected
- Call gets hanged or a network error accrued.

The above discussed three kinds of circumstances can be easily modeled using the membership grade, non-membership grade and the hesitancy grade of an IVQROPFN. We present a dynamic algorithm to handle the incoming and outgoing calls in a wireless mobile network.

8.4.1 Algorithm:

Let us assume a network of some peoples that are connected to each other using mobile phones. An incoming or outgoing call could face a situation of 1. call is received, 2. call is not received and 3. call is left unattended or call gets disrupted i.e. a network error accrued. The first and second situation can be considered as the membership and non-membership grade while the third case can be considered as the hesitancy degree of an IVQROPFN. The algorithm for such IVQROPF telephone network can be described in the steps given as:

1. Setup the set of vertices $\underset{.}{\mathrm{V}} = \{\underset{.}{\mathrm{t}}_i, i = 1, 2, 3, \ldots, q\ \}$. In our case each $\underset{.}{\mathrm{t}}_i = \left(\left[\hat{\mathrm{S}}^L{}_{\underset{.}{\mathrm{A}}}, \hat{\mathrm{S}}^U{}_{\underset{.}{\mathrm{A}}}\right], \left[\mathrm{Đ}^L{}_{\underset{.}{\mathrm{A}}}, \mathrm{Đ}^U{}_{\underset{.}{\mathrm{A}}}\right]\right)$ represents a telephone number where $\left[\hat{\mathrm{S}}^L{}_{\underset{.}{\mathrm{A}}}, \hat{\mathrm{S}}^U{}_{\underset{.}{\mathrm{A}}}\right]$ is the tendency of calling while $\left[\mathrm{Đ}^L{}_{\underset{.}{\mathrm{A}}}, \mathrm{Đ}^U{}_{\underset{.}{\mathrm{A}}}\right]$ is the tendency of not calling to other numbers.

2. Establish the set of edges $\dot{\mathrm{E}} = \{([\hat{S}^L{}_{\text{Ḅ}i\,j}, \hat{S}^U{}_{\text{Ḅ}ij}], [Đ^L{}_{\text{Ḅ}i\,j}, Đ^U{}_{\text{Ḅ}i\,j}])\ for\ all\ i, j = 1, 2, 3, 4, \dots, q.\}$ and calculate the degree of membership and non-membership for all $([\hat{S}^L{}_{\text{Ḅ}i\,j}, \hat{S}^U{}_{\text{Ḅ}ij}], [Đ^L{}_{\text{Ḅ}i\,j}, Đ^U{}_{\text{Ḅ}i\,j}])$ using the relation

$$\hat{S}_{\text{Ḅ}L}(ṭ_i, ṭ_j) \leq \wedge[\hat{S}_{\text{Ạ}L}(ṭ_i), \hat{S}_{\text{Ạ}L}(ṭ_j)]\ ,\ \hat{S}_{\text{Ḅ}U}(ṭ_i, ṭ_j) \leq \wedge[\hat{S}_{\text{Ạ}U}(ṭ_i), \hat{S}_{\text{Ạ}U}(ṭ_j)]$$

and

$$Đ_{\text{Ḅ}L}(ṭ_i, ṭ_j) \leq \vee[Đ_{\text{Ạ}L}(ṭ_i), Đ_{\text{Ạ}L}(ṭ_j)], Đ_{\text{Ḅ}U}(ṭ_i, ṭ_j) \leq \vee[Đ_{\text{Ạ}U}(ṭ_i), Đ_{\text{Ạ}U}(ṭ_j)]$$

3. Obtain an IVq-ROPFG $\dot{G} = < \text{Ṿ}, \dot{\mathrm{E}} >$

8.4.1 Example

Consider $\text{Ṿ} = \{ṭ_1, ṭ_2, ṭ_3, ṭ_4,\ ṭ_5\}$, the set of 5 telephone numbers making an IVq-ROPFG network. Let $\dot{\mathrm{E}} = \{(ṭ_1, ṭ_2), (ṭ_1, ṭ_3), (ṭ_2, ṭ_3), (ṭ_2, ṭ_5),\ (ṭ_4, ṭ_3), (ṭ_4, ṭ_5), (ṭ_5, ṭ_1), (ṭ_5, ṭ_3)\}$ be the set of edges whose values are given in Table 33 below in the form of IVq-ROPFN for q=2.

Edges	IVq-ROFNs
(p_1, p_2)	$([0.3,0.6], [0.4,0.7])$
(p_1, p_3)	$([0.2,0.7], [0.5,0.7])$
(p_2, p_3)	$([0.1,0.7], [0.1,0.2])$
(p_2, p_5)	$([0.4,0.5], [0.3,0.8])$
(p_4, p_3)	$([0.2,0.6], [0.3,0.7])$
(p_4, p_5)	$([0.2,0.4], [0.5,0.8])$
(p_5, p_1)	$([0.3,0.5], [0.5,0.8])$
(p_5, p_3)	$([0.2,0.4], [0.5,0.7])$

Table 33 (The number of calls between different phone numbers in the form of IVq-ROPFNs)

The telephone network discussed in Example having edges in the form of IVq-ROPFG given in Table 33 is portrayed in Figure 114.

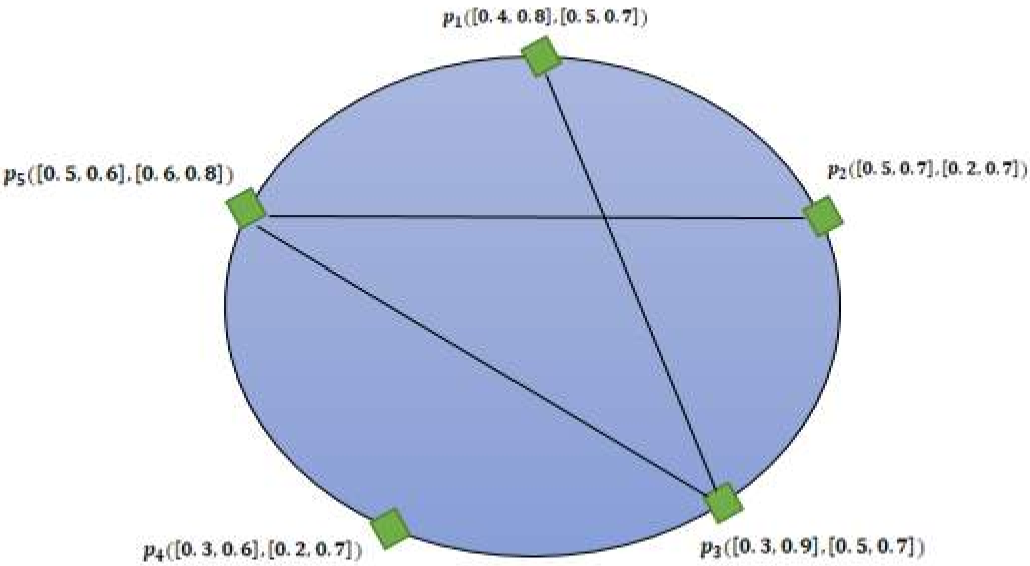


Figure 114 (Telephone network in the environment of IVq-ROPFGs)

We use Definition 8.1.13 to find the degree of each vertex telephone network depicted in Figure 114. All the degrees are given in Table 34.

Vertices	Degree
p_1	([0.5,1.3], [0.9,1.4])
p_2	([0.5,1.2], [0.6,1.5])
p_3	([0.3,0.9], [0.5,0.7])
p_4	([0.4,1], [0.8,1.5])
p_5	([0.5,0.9], [1,1.5])

Table 34 (Degree of vertices of Figure 114)

To analyse the degrees more explicitly, we use the score function of IVq-ROPFN to compute the score of degrees provided in Table 34. The score values are given in Table 35 below.

Vertices	Score
p_1	(-0.25)
p_2	(-0.2)
p_3	(0)
p_4	(-0.45)
p_5	(-0.65)

Table 35 (Score of vertices of Figure 114)

Analysing Table 35, we came to know that P_3 has zero degree which is better than the degree of other vertices which are negative showing that the effective calling ratio of ţ$_3$ is better than the calling ratios of other callers. ţ$_2$ is at number 2 in this effective calling list. For a better view, the bar graph of the data available in Table 35 is given in in Figure 115.

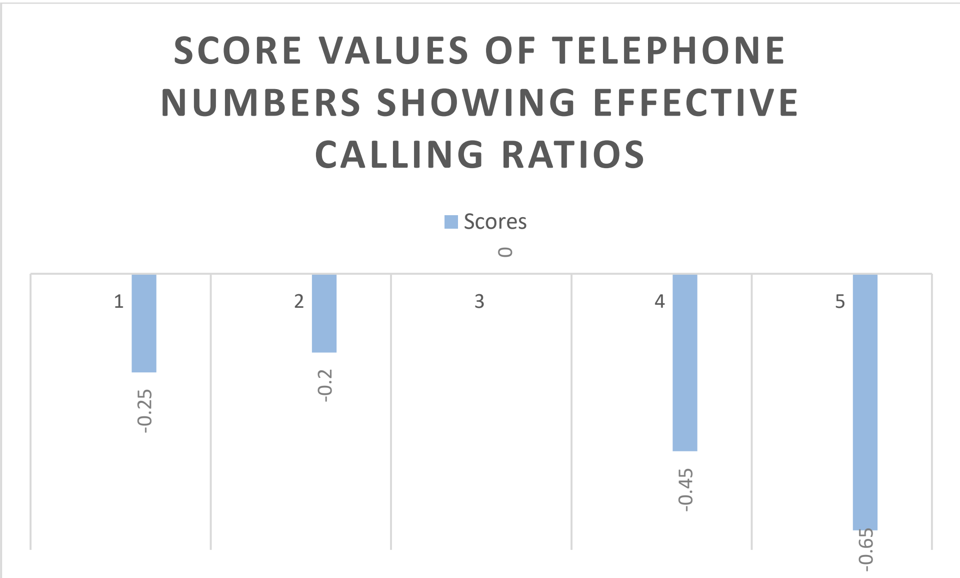


Figure 115 (Score values of degrees of effective calling ratios of telephone network discussed in Figure 114)

8.4.2 Comparative Study and Advantages

Here, we present a comparison of telephone network based on IVQROPFGs with a telephone network based on FGs. The information about the incoming or outgoing calls are shown using fuzzy numbers and a fuzzy telephone network is depicted in Figure 116 where the values of edges between several telephone numbers are given in Table 36.

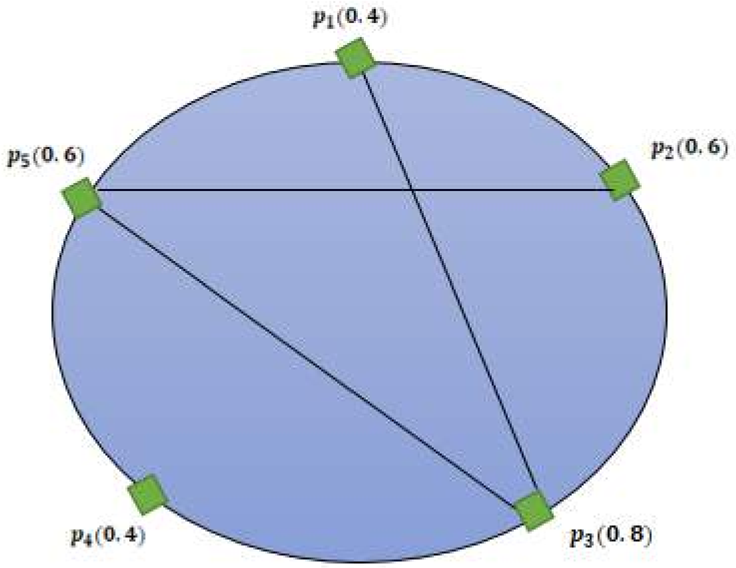


Figure 116 (Telephone network based on fuzzy graphs showing the effective number of incoming or outgoing calls)

The incoming calls received are demonstrated using fuzzy numbers while there is no information about the number of calls rejected or the calls that go interrupted by network errors. That is the main reason an FG cannot be able to demonstrate a telephone network in a better way as an IVq-ROPFG does. The information of incoming or outgoing calls in terms of fuzzy numbers is given in Table 36.

Edges	FNs
(p_1, p_2)	(0.3)
(p_1, p_3)	(0.4)
(p_2, p_3)	(0.5)
(p_2, p_5)	(0.2)
(p_4, p_3)	(0.2)
(p_4, p_5)	(0.6)
(p_5, p_1)	(0.3)
(p_5, p_3)	(0.1)

Table 36 (information about the incoming our outgoing calls in terms of fuzzy numbers) numbers)

To get a clear image of the telephonic network demonstrated in Figure 116 using FGs, we utilize the concept of degree of FGs. The degrees of each node of the FG are given in Table 37 followed by a bar graph showing the effective ratio of incoming or outgoing calls.

Vertices	Degree
p_1	1.1
p_2	1
p_3	0.8
p_4	0.5
p_5	0.9

Table 37 (Degree of vertices of telephonic network depicted in Figure 116)

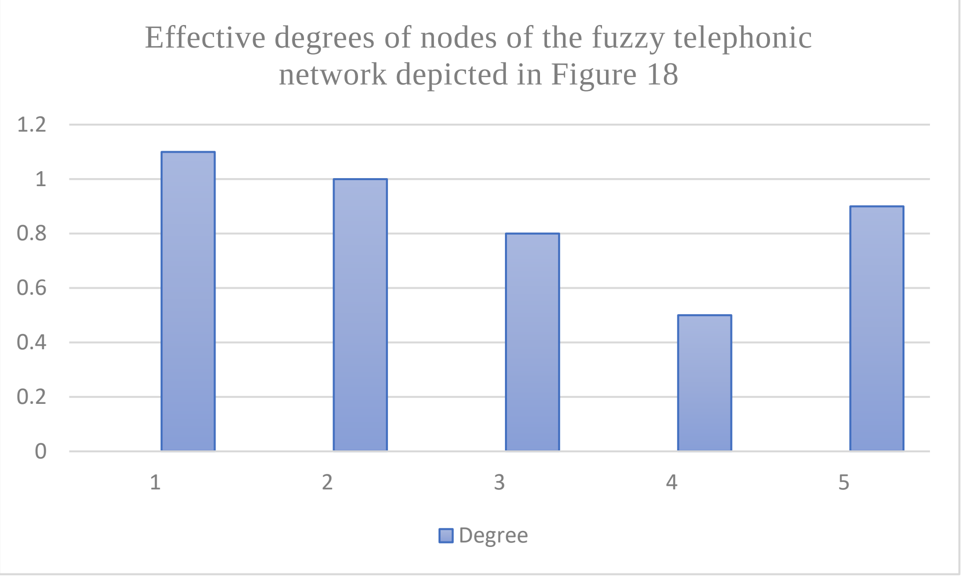


Figure 117 (Bar graph of degrees of fuzzy telephonic network)

Analyzing Table 37 and Figure 117, it is quite clear that $ṭ_1$ has the highest number of incoming or outgoing calls. $ṭ_2$ is at number second in the list. However, if we investigate the results obtained in the case of telephonic network of IVq-ROPFG in Figure 114, we came to know that $ṭ_3$ has better rate of incoming calls.

8.5 Applications in Engineering Decision Making

Decision making is one of the important tools that is highly used in several scientific fields including, engineering, economics, management sciences and other fields of machine learning and soft computing. The aim of this section is to utilize the concept of IVQROPFG in solving engineering decision making problems.

8.5.1 Algorithm

In decision making problems using graphs, we considered the family of alternatives, which are connected to each other using the edges of the graph. The aim is to evaluate all the alternatives on equal weights and to determine the best among them using some aggregation operators.

Consider a list of n alternatives needed to be evaluated under a common attribute and connected to each other by edges whose information is provided in the form of IVQROPFNs. This IVQROPFG is obtained by the information of decision makers. The steps of decision-making problem are given by:

Step 1: Formation of IVQROPFGs by decision makers in which information about the alternatives are given in the form of IVQROPFNs.

Step 2: Using the information of the IVQROPFG provided in Step 1 to construct a relational matrix.

Step 3: Using the weighted averaging and weighted geometric aggregation operators of IVQROPFSs to aggregate the information given in the relational matrix obtained in Step 2.

Step 4: Use score values to rank the alternatives for finding the best alternative among the given list of alternatives.

8.5.2 Decision Making to Evaluate the Best Engineering Project

Consider four civil engineering projects needed to be completed by the government of Pakistan in the financial year 2019-20. These four projects are denoted by $z_i(i = 1, 2, 3, 4)$ including "building a hospital", "building a road connecting two cities", "building a school in the nearby urban area" and "reconstruction of government oil refinery". The government of Pakistan needs to complete all these projects but in order of importance and need. The recognized building authority evaluate the feasibility of all these projects with the help of experts to prioritize the projects. The decision-making panel consisting of engineers, economist and other government officials. The detailed steps of this decision-making problem using the proposed algorithm are demonstrated as follows:

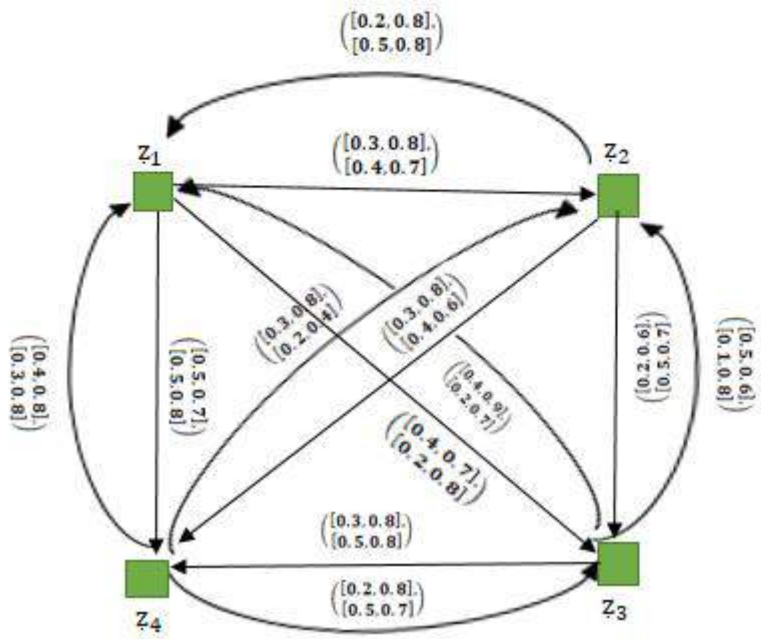


Figure 118 (Directed Network of the Interval valued q-rung ortho pair fuzzy relation IVq-ROFR)

Step 1: Formation of IVQROPFG connecting the four projects describing their suitability in terms of IVQROPFNs.

Step 2: Formation of matrix relating the four projects.

$$\dot{R} = (\dot{r}_{ij})_{4\times4} = \begin{pmatrix} \begin{pmatrix}[\mathbf{0.4, 0.6}], \\ [\mathbf{0.4, 0.6}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3, 0.8}], \\ [\mathbf{0.4, 0.7}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.4, 0.7}], \\ [\mathbf{0.2, 0.8}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.5, 0.7}], \\ [\mathbf{0.5, 0.8}]\end{pmatrix} \\ \begin{pmatrix}[\mathbf{0.2, 0.8}], \\ [\mathbf{0.5, 0.8}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.4, 0.6}], \\ [\mathbf{0.4, 0.6}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.2, 0.6}], \\ [\mathbf{0.5, 0.7}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3, 0.8}], \\ [\mathbf{0.4, 0.6}]\end{pmatrix} \\ \begin{pmatrix}[\mathbf{0.4, 0.9}], \\ [\mathbf{0.2, 0.7}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.5, 0.6}], \\ [\mathbf{0.1, 0.8}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.4, 0.6}], \\ [\mathbf{0.4, 0.6}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3, 0.8}], \\ [\mathbf{0.5, 0.8}]\end{pmatrix} \\ \begin{pmatrix}[\mathbf{0.4, 0.8}], \\ [\mathbf{0.3, 0.8}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.3, 0.8}], \\ [\mathbf{0.2, 0.4}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.2, 0.8}], \\ [\mathbf{0.5, 0.7}]\end{pmatrix}, \begin{pmatrix}[\mathbf{0.4, 0.6}], \\ [\mathbf{0.4, 0.6}]\end{pmatrix} \end{pmatrix}$$

Step 3: Using IVQROPFWA aggregation operator given below on the matrix obtained in Step 2.

$$\dot{r}_i = IVQROPFWA(\dot{r}_{i1}, \dot{r}_{i2}, \dots, \dot{r}_{in})$$

$$= \begin{pmatrix} \left[\sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\hat{S}^{L}{}_{ij}\right)^{q}\right)^{\frac{1}{n}}}, \sqrt{1 - \prod_{j=1}^{n} \left(1 - \left(\hat{S}^{U}{}_{ij}\right)^{q}\right)^{\frac{1}{n}}} \right], \\ \left[\left(\prod_{j=1}^{n} Đ^{L}{}_{ij}\right)^{\frac{1}{n}}, \left(\prod_{j=1}^{n} Đ^{U}{}_{ij}\right)^{\frac{1}{n}} \right] \end{pmatrix}$$

$i = 1,2, \dots, n,$

Now all $\dot{r}_{ij}, j = 1,2, \dots ,4$ are aggregated and the results are given below.

$\dot{r}_1 = ([0.4184, 0.7154], [0.3557, 0.7200])$

$\dot{r}_2 = ([0.3103,0.7295], [0.4472, 0.6701])$

$\dot{r}_3 = ([0.4184, 0.7819], [0.2515, 0.6260])$

$\dot{r}_4 = ([0.3516, 0.7444], [0.3310, 0.5856])$

Step 4: Using the score function defined for IVQROPFSs to rank the aggregated data obtained in Step 3. The score values are given below followed by the ranking results.

$$ṣ(ṙ_i) = \frac{\left((\hat{S}^L{}_i)^q(1-(Đ^L{}_i)^q) + (\hat{S}^U{}_i)^q(1-(Đ^U{}_i)^q)\right)}{2}$$

$ṣ(ṙ_1) = 0.1847, ṣ(ṙ_2) = 0.1629, ṣ(ṙ_3) = 0.2525, ṣ(ṙ_4) = 0.2067$

$ẓ_3 > ẓ_4 > ẓ_1 > ẓ_2$

Hence, the projects are ranked as:

- Building a school in the nearby urban area is at priority ONE.
- Reconstruction of government oil refinery is second in priority list.
- Building a hospital is at number 3 in priority list.
- Building a road connecting two cities is at last in priority list.

Now we solve the same problem using IVQROPFWG operators and to do so, we start the process from Step 3.

Step 3: Using IVQROPFWG given below to aggregate the information given in matrix obtained in Step 2.

$ṙ_i = IVQROPFWG(ṙ_{i1}, ṙ_{i2}, \dots, ṙ_{in})$

$$= \left(\begin{array}{c} \left[\left(\prod_{j=1}^{n} \hat{S}^L{}_{ij}\right)^{\frac{1}{n}}, \left(\prod_{j=1}^{n} \hat{S}^U{}_{ij}\right)^{\frac{1}{n}}\right], \\ \left[\sqrt{1-(\prod_{j=1}^{n}\left(1-\left(Đ^L{}_{ij}\right)^q\right)^{\frac{1}{n}}}, \sqrt{1-(\prod_{j=1}^{n}\left(1-\left(Đ^U{}_{ij}\right)^q\right)^{\frac{1}{n}}}\right] \end{array} \right)$$

$ṙ_1 = ([0.3936, 0.6964], [0.4129, 0.7444])$

$ṙ_2 = ([0.2632, 0.6928], [0.4586, 0.6973])$

$ṙ_3 = ([0.3936, 0.7135], [0.3885, 0.7159])$

$\dot{\mathrm{r}}_4 = ([0.3139, 0.7200], [0.3966, 0.6358]$

Step 4: Using the score function defined for IVQROPFSs to rank the aggregated data obtained in Step 3. The score values are given below followed by the ranking results.

$\underset{\cdot}{\mathrm{s}}(\dot{\mathrm{r}}_1) = 0.1559, \underset{\cdot}{\mathrm{s}}(\dot{\mathrm{r}}_2) = 0.1264, \underset{\cdot}{\mathrm{s}}(\dot{\mathrm{r}}_3) = 0.1724, \underset{\cdot}{\mathrm{s}}(\dot{\mathrm{r}}_4) = 0.1674$

$\underset{\cdot}{z}_3 > \underset{\cdot}{z}_4 > \underset{\cdot}{z}_1 > \underset{\cdot}{z}_2$

Hence, we got the same ranking as we did by using the IVQROPFWA operators:

- Building a school in the nearby urban area is at priority ONE.
- Reconstruction of government oil refinery is second in priority list.
- Building a hospital is at number 3 in priority list.
- Building a road connecting two cities is at last in priority list.

Note that, it is not necessary that the results obtained using weighted averaging and weighted geometric aggregation operators will always be the same. However, the use of weighted averaging or weighted geometric aggregation operators are up to the decision makers.

8.6 Decision Making in Supply Chain Management

Supply chain management involves the assessment of companies based on their performance in a supply chain. There are several aspects which plays vital role in a supply chain and our aim is to find out the best among those aspects.

Consider a supply chain of some engineering companies and assume that there are four factors i.e. service level, cost and price, quality and response time denoted by $z_i (i = 1, 2, 3, 4)$ playing a vital role in this chain. To compute the most influential factor among those four, we utilized the following four simple steps to evaluate the most influential factor.

1. Obtain information about given number of factors by decision makers in three or more decision matrices.
2. Regroup the information obtained in Step 1 using IVQROPFA or IVQROPFG aggregation operators into a relational matrix which results in the formation of the IVQROPFG as well.
3. Use the Definition 8.1.13 to compute the out-degrees of each factor.
4. Rank the values of out-degrees to get the most influential factor.

The information about the four factors towards their importance in the supply chain are given in three matrices below followed by the stepwise demonstration of supply chain management problem.

$$R_1 = \begin{bmatrix} \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.7,0.7], \\ [0.2,0.5] \end{pmatrix} & \begin{pmatrix} [0.1,0.7], \\ [0.1,0.5] \end{pmatrix} & \begin{pmatrix} [0.5,0.7], \\ [0.6,0.7] \end{pmatrix} \\ \begin{pmatrix} [0.3,0.7], \\ [0.4,0.7] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.2,0.5], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.5,0.7], \\ [0.4,0.6] \end{pmatrix} \\ \begin{pmatrix} [0.2,0.6], \\ [0.4,0.6] \end{pmatrix} & \begin{pmatrix} [0.2,0.4], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.6], \\ [0.5,0.7] \end{pmatrix} \\ \begin{pmatrix} [0.3,0.5], \\ [0.4,0.7] \end{pmatrix} & \begin{pmatrix} [0.2,0.7], \\ [0.3,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.4], \\ [0.4,0.5] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} \end{bmatrix}$$

$$R_2 = \begin{bmatrix} \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.5,0.7], \\ [0.1,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.5], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.2,0.6], \\ [0.3,0.4] \end{pmatrix} \\ \begin{pmatrix} [0.2,0.7], \\ [0.5,0.7] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.6], \\ [0.4,0.7] \end{pmatrix} & \begin{pmatrix} [0.5,0.7], \\ [0.1,0.6] \end{pmatrix} \\ \begin{pmatrix} [0.1,0.6], \\ [0.3,0.7] \end{pmatrix} & \begin{pmatrix} [0.5,0.5], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.2,0.6], \\ [0.4,0.5] \end{pmatrix} \\ \begin{pmatrix} [0.1,0.6], \\ [0.2,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.4], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.55], \\ [0.4,0.6] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} \end{bmatrix}$$

$$R_3 = \begin{bmatrix} \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.7], \\ [0.3,0.5] \end{pmatrix} & \begin{pmatrix} [0.1,0.5], \\ [0.3,0.7] \end{pmatrix} & \begin{pmatrix} [0.2,0.6], \\ [0.5,0.6] \end{pmatrix} \\ \begin{pmatrix} [0.4,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.3,0.6], \\ [0.3,0.7] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.5] \end{pmatrix} \\ \begin{pmatrix} [0.5,0.7], \\ [0.6,0.6] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.1,0.7] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} & \begin{pmatrix} [0.2,0.3], \\ [0.3,0.4] \end{pmatrix} \\ \begin{pmatrix} [0.53,0.72], \\ [0.1,0.2] \end{pmatrix} & \begin{pmatrix} [0.53,0.63], \\ [0.12,0.42] \end{pmatrix} & \begin{pmatrix} [0.4,0.5], \\ [0.3,0.5] \end{pmatrix} & \begin{pmatrix} [0.5,0.6], \\ [0.5,0.6] \end{pmatrix} \end{bmatrix}$$

2. Now we use the IVQROPFA operators to regroup the information given in above three decision matrices. Using the following IVQROPFA operator, the relational matrix R is formed given below.

$$IVQROPFA(\dot{r}_{i1}, \dot{r}_{i2}, \ldots, \dot{r}_{in})$$

$$= \left(\begin{array}{c} \sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\hat{S}^{L}{}_{ij}\right)^{q}\right) \right)^{\frac{1}{n}}}, \sqrt{1 - \left(\prod_{j=1}^{n} \left(1 - \left(\hat{S}^{U}{}_{ij}\right)^{q}\right) \right)^{\frac{1}{n}}}, \\ \left(\prod_{j=1}^{n} Đ^{L}{}_{ij} \right)^{\frac{1}{n}}, \left(\prod_{j=1}^{n} Đ^{U}{}_{ij} \right)^{\frac{1}{n}} \end{array} \right)$$

$$R = \begin{bmatrix} \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} & \begin{pmatrix} [0.44,0.48], \\ [0.002,0.05] \end{pmatrix} & \begin{pmatrix} [0.35,0.45], \\ [0.005,0.07] \end{pmatrix} & \begin{pmatrix} [0.38,0.46], \\ [0.03,0.056] \end{pmatrix} \\ \begin{pmatrix} [0.38,0.47], \\ [0.03,0.098] \end{pmatrix} & \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} & \begin{pmatrix} [0.4,0.45], \\ [0.02,0.098] \end{pmatrix} & \begin{pmatrix} [0.43,0.47], \\ [0.007,0.06] \end{pmatrix} \\ \begin{pmatrix} [0.38,0.47], \\ [0.024,0.084] \end{pmatrix} & \begin{pmatrix} [0.41,0.43], \\ [0.008,0.084] \end{pmatrix} & \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} & \begin{pmatrix} [0.36,0.44], \\ [0.02,0.05] \end{pmatrix} \\ \begin{pmatrix} [0.39,0.46], \\ [0.003,0.028] \end{pmatrix} & \begin{pmatrix} [0.39,0.46], \\ [0.006,0.05] \end{pmatrix} & \begin{pmatrix} [0.38,0.43], \\ [0.016,0.05] \end{pmatrix} & \begin{pmatrix} [0.43,0.46], \\ [0.042,0.072] \end{pmatrix} \end{bmatrix}$$

This relational matrix is clearly an IVQROPFG depicted in Figure 119.

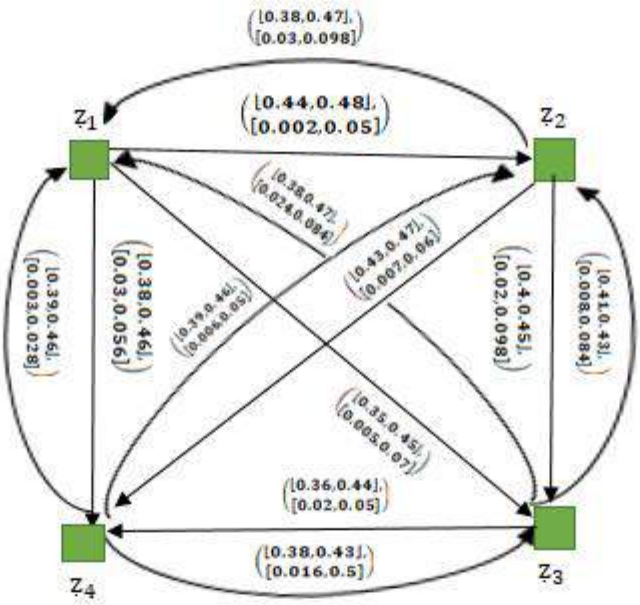


Figure 119 (Directed Network of IVQROPFG sowing the relation of four factors in the supply chain)

The Figure 119 reduces a partial IVq-ROPFG by using the condition $\hat{S}^{2L} \geq 0.46$ which is depicted in Figure 120.

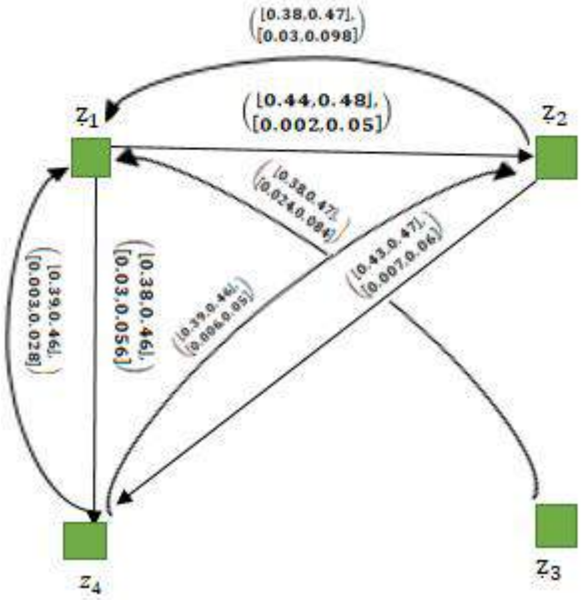


Figure 120 (Partial Directed Network of supply chain factors)

3.Utilizing the formula of out-degree to compute the out-degree values of each factor given as:

$$O\hat{S}t - d(z_1) = ([0.82,0.94], [0.032,0.061])$$

$$O\hat{S}t - d(z_2) = ([0.81,0.94], [0.05,0.196])$$

$$O\hat{S}t - d(z_3) = ([0.38,0.47], [0.024,0.084])$$

$$O\hat{S}t - d(z_4) = ([0.78,0.92], [0.009,0.078])$$

By observing the degree of membership, it is clear that

$$O\hat{S}t - d(z_1) \geq O\hat{S}t - d(z_2) \geq O\hat{S}t - d(z_4) \geq O\hat{S}t - d(z_3)$$

Hence $z_1 \geq z_2 \geq z_4 \geq z_3$ which shows that cost and price is the required factor that one must keep in mind. The results would be the same if we use IVQROPFG aggregation operators instead of IVQROPFA operators

8.6 Conclusion

In this chapter, a novel idea of IVq-ROPFG is introduced as a generalization of FG, IVFGS, IFG, IVIFG and PyFG. With the help of some remarks, the generalization of proposed work is proved. Several graphical concepts are developed for IVq-ROPFG such as the concept of subgraph, complement, degrees, out-degrees and paths etc. Each concept is demonstrated with examples. In the environment of IVq-ROPFG, a shortest path computation problem is analysed and demonstrated with the help of two different techniques. A co-authorship network of the authors of several countries and their relation is also analysed using the concepts of IVq-ROPFGs. A telephone network model is presented where the extent of calls between two or more persons is analyzed using the approach of IVq-ROPFGs. Two engineering decision making problems are discussed using the approach of IVq-ROPFGs. For every application that is assessed, a comparative study is established and the advantages of working in the area of IVq-ROPFG are demonstrated with the help of examples. In future, we plan to study few more decision-making problems using some different aggregation tools of IVq-ROPFGs.

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