Fixed Points of Generalized Contraction Mappings in Abstract Spaces

By

Shaif Saleh Ahmed Alshoraify

Reg. No. 61-FBAS/PHDMA/S16

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
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Pakistan
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Student Name: Shaif Saleh Ahmed Alshoraify **Signature:** ______________

Examination Committee:

a) External Examiner 1:
Prof. Dr. Mujahid Abbas/ Professor,
Government College Lahor **Signature:** ______________

b) External Examiner 2:
Prof. Dr. Akbar Azam/ Professor,
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c) Internal Examiner:
Dr. Tahir Mahmood/ Assistant Professor,
Dept of Math & Stat, IIU **Signature:** _______________

Supervisor Name:
Prof. Dr. Muhammad Arshad

Signature: _______________

Name of Dean/HOD:
Prof. Dr. Muhammed Irfan Khan **Signature:** ______________

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To those who are very tired for my sake and have been looking forward to the hope of the harvest moments, may Allah protect them, prolong their lives and provide them with health and wellness; my parents.

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Contents

Preface

For a mapping $J : L \to L$, an element $a \in L$ is said to be fixed point of J if $Ja = a$. Choosing a member a_0 of L and using the pattern $a_{n+1} = Ja_n$, generate a sequence $\{a_n\}$ from a_0 in L, where n being from $\mathbb{N}$, such a sequence is described to be a Picard sequence in the literature. Iteration of such a sequence plays a key role in the iteration of a fixed point of some given mapping with special contractive impositions. The study of fixed point theory has been a wide field of research since the existence of non-linear analysis and researcher opted to locate a solution point of different mathematical and engineering problems via reshaping the problems in the frame of fixed point numerical and then iterating its fixed point, indeed the solution of the problems.

Banach (1992) tendered debate of the metric fixed point theory by establishing the first theorem termed as Banach contraction principle, directing researchers to a new field in mathematics fixed point theorem, that is composed of topology, geometry and analysis. For a metric space, a self-mapping J on L is said to be Banach contraction if J satisfies the inequality $d(Ja, Jy) \leq kd(a, y)$ for each $a, y \in L$ provided that $0 < k < 1$. The statement of Banach contraction principle states that for a complete metric space L, and a mapping J, J has a unique fixed point in L. Despite of the brief statement and proof, the theorem accommodate a comprehensive knowledge of finding solution to complex problems in various fields of studies, such as game theory, functional analysis, dynamical system, PDEs and even biosciences. As the iteration process is involved in the theorem, thus it can be easily applied on a computer system. It was proved that Banach contraction is uniformly continuous on the set L, however, to extend this idea with a weaker condition, Kannan [55] was the first who gave the idea of non-continuous operator satisfying a new contractive condition. He proved the persistence of a fixed point even if the operator is non-continuous. He described a weaker form of the Banach contraction as $d(Ja, Jy) \leq k\,[d(a, Ja) + d(y, Jy)]$ for all $a, y \in L$ where $0 \leq k < \frac{1}{2}$. This was further exploited by Chatterjee [37] in somehow closer form but quite different in nature. He replaced the Kannan inequality by $d(Ja, Jy) \leq k\,[d(a, Jy) + d(y, Ja)]$ for all $a, y \in L$ where $0 \leq k < \frac{1}{2}$. One can observe that these three principles are independent of each other and had provided base to the research of fixed point theory.

The results of fixed point in the entire field had played an important role in different research activities, for instance (see [12, 20, 23, 24, 28, 38, 39, 47, 48, 58, 82, 103]). Fixed point theory has applications in various fields of mathematics and other sciences, for example [10, 52, 65, 69]. Fréchet [45] tendered debate on the topic of metric which now a days contribute a central part in both applied and pure mathematics. The metric space and its generalized forms are essential as well as the inevitable part of many branches of mathematics. After exploring its application in various fields of mathematics and other sciences, mathematicians were compelled to come up with the more general and extended versions of metric spaces like Matthews [68] put forwarded the concept of partial metric version of Banach contraction theorem [27]. Several authors, then presented interesting results on partial metric spaces and its topological properties [3, 46, 89]. Ran and Reurings [78] explaining the notion of partial order and gave applications. In 2008, Jachymski [51] had the excellent idea to use the metric space endowed with a graph and the contraction condition of Banach will be satisfied only for the edges of the graph.

In parallel to this, Bakhtin [26] and Czerwik [41] introduced the definition of b-metric space. Several authors interesting the notion of b-metric space to establish the persistence of fixed point for contraction maps for instance, see [7, 20, 22, 42, 86, 104]. Also, among all these extensions of metric space is the quasi metric space that was introduced by Wilson [101]. The second condition i.e., commutativity, does not hold in general in quasi metric spaces. Using this generalization, various authors investigated fixed point for different mappings [25, 35, 60, 67, 83, 102]. This discussion was followed by the topic of dislocated metric space [6, 15] and dislocated quasi metric space [16, 30, 92, 96, 103]. Dislocated quasi b-metric space was investigated by Klin-eam and Suanoom [64], for fixed point and fixed point theorems in complete dislocated quasi b-metric space was explored [1, 77]. Moreover, Kamran *et al.* [56] elongated the concept of b-metric space to extended b-metric space while replacing the parameter $s \geq 1$ in the triangle inequality by the control function $\theta : L \times L \to [1, +\infty)$ see also [87]. The same notion was also generalized by Mlaiki *et al.* [71] in a different way with inserting a control function instead of the constant 's' and hence named his newly defined metric as controlled metric type spaces. The arguments continued and Abdeljawad *et al.* [2] added another notion labeled as double controlled metric type spaces.

Besides all the extensions of metric space mentioned above, a big portion of the literature is composed of the multi-valued maps instead of single-valued maps. It was Nadler in 1969 [72] who commenced discussion on maps that map on a set of points rather than those mapped to single points only. He took Banach contraction as a special case of the Hausdorff Pompeiu metric and establish fixed point theorem for multi-valued mappings. Later, this approach was furthered by many renown researchers [22, 29, 92, 95, 98, 99] and likewise results were established in a variety of metric spaces. To study fixed point in the area, the definition of fixed point was rephrased as, for a multi-valued mapping J, a point a in the set L is named as fixed point of J if $a \in Ja$. Applications in engineering, economics, Nash equilibria and game theory in fixed point results of multivalued mappings have introduced see [11, 18, 36, 66].

This thesis is a composition of theorems based on fixed point for both single-valued and multi-valued mappings having certain contractive imposition. Fixed point concept is exploited for iterating common fixed point of such maps. The dissertation is a mixture of diverse knowledge of metric space and its extensions having different methodology of iterating a solution or fixed point of distinct maps. This dissertation is split into four chapters. Following is a chapter wise content distribution of our thesis.

Chapter 1, focuses fundamental notions and definitions that are primarily involved in the main results or helps in understanding the results. Similarly, some important pre-existing fixed point theorems are given in the chapter.

Chapter 2, explains in detail, notion of complete left (right) K-sequentially quasi metric space. Also, we have presented the concept of α-Alt mapping and using that fixed point results which established for $F-\mu_s-\rho_s^*$-contraction in the setting of quasi b-metric space. The chapter further throw light on the applicability of the main theorems with concrete arguments.

Chapter 3, is related to common fixed point of coincidence as well as common fixed point of four maps carrying specific impositions. Working in partial metric space, theorems has been proved for fixed point. The multi-valued maps of fixed point is iterated for F-contractions in abstract spaces with application.

Chapter 4, this chapter demonstrate fixed point of distinct contractions in left double controlled quasi and dislocated quasi metric type spaces. The main results are explained using concrete examples. Throughout this thesis $\mathbb{N}$, $\mathbb{R}^+$and $\mathbb{R}$ denote the set of all natural numbers, the set of all non-negative real numbers and the set of all real numbers, respectively.

Chapter 1

Preliminaries

This chapter is based on discussion about the evolution of fixed point theory. It throws light on metric fixed point theory and its gradual upbringing in the field of non-linear analysis. The fundamental notions and founding fixed point theorems are described in this chapter, these notions/theorems enable us to understand our main work and also help us in proving these important results. This chapter portrays the said results as generalizations of the renown metric space. Section 1.1, tender discussion on the topics quasi metric, quasi b-metric, dislocated quasi metric and dislocated quasi b-metric spaces. Section 1.2, define double controlled metric type spaces while Section 1.3, all others basic that empower our research.

1.1 Quasi and Dislocated Quasi b-Metric Spaces

1.1.1 Definition [64]

Let $L \neq \{\}$ and $s \geq 1$ a real number. A mapping $d_{qb} : L \times L \to [0, +\infty)$ is called a dislocated quasi b-metric , if the below conditions satisfied:

(a) if $d_{qb}(a, y) = d_{qb}(y, a) = 0$, then $a = y$;

(b) $d_{qb}(a, y) \leq s\,[d_{qb}(a, e) + d_{qb}(e, y)]$, for any $a, y, e \in L$.

The pair (L, d_{qb}) is called a dislocated quasi b-metric space. The following remarks can be observed:

(a) if $s = 1$, then a dislocated quasi b-metric space becomes a dislocated quasi metric space [92];

(b) if $s = 1$ and $a = y$ implies $d_{qb}(a, y) = d_{qb}(y, a) = 0$, then (L, d_{qb}) becomes a quasi metric space [101, 102];

(c) if $d_{qb}(a, y) = d_{qb}(y, a)$ and $a = y$ implies $d_{qb}(a, y) = 0$, therefor (L, d_{qb}) becomes a b-metric space [104]. For $a \in L$ and $\varepsilon > 0$, $B_{d_{qb}}(a, \varepsilon) = \{y \in L : \max\{d_{qb}(a, y), d_{qb}(y, a)\} < \varepsilon\}$ and $\overline{B_{d_{qb}}(a, \varepsilon)} = \{y \in L : \max\{d_{qb}(a, y), d_{qb}(y, a)\} \leq \varepsilon\}$ are open and closed ball in (L, d_{qb}) respectively.

1.1.2 Example [88]

Let $L = \{1, 2, 3\}$. Define the function q_s on $L \times L$ as $q_s(a, y) = \frac{1}{a^2}$ for every $a > y$, $q_s(a, y) = 1$ for $a < y$, and $q_s(a, y) = 0$, for $a = y$, with $q_s(a, y) \neq (1, 2)$ and $q_s(1, 2) = \frac{16}{9}$. Then (L, q_s) is a quasi b-metric space with coefficient $s = 2$. It is neither a b-metric space since $q_s(1, 2) = \frac{16}{9} \neq q_s(2, 1) = \frac{1}{4}$, nor a quasi metric space since $q_s(1, 2) = \frac{16}{9} > \frac{10}{9} = q_s(1, 3) + q_s(3, 2)$.

Reilly *et al.* [83] established the concept of left as well as right K-Cauchy sequence and left (right) K-sequentially complete quasi metric space.

1.1.3 Definition [83]

Consider (L, q) is a quasi metric space. Then a sequence $\{a_n\}$ in (L, q) is called:

(a) Left (right) K-Cauchy, if for every $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such as $q(a_m, a_n) < \varepsilon$ (respectively $q(a_n, a_m) < \varepsilon$) for all $m > n \geq n_0$.

(b) A converges to a, if $\lim_{n \to +\infty} q(a_n, a) = \lim_{n \to +\infty} q(a, a) = 0$ and the point a in this case is called a limit of the sequence $\{a_n\}$.

(c) (L, q) is called left (right) K-sequentially complete if each left (right) K-Cauchy sequence in q convergent such as $q(a, a) = 0$.

1.1.4 Example [77]

Let $L = R^+$ and $p > 1$. Define $d_{qb} : L \times L \to R^+$ by $d_{qb}(a, y) = |a - y|^p + |a|^p$ for all $a, y \in L$. Then (L, d_{qb}) is a dislocated quasi b-metric space with $s = 2^p > 1$. But it is not a quasi b-metric space. Also is not a dislocated b-metric space. It is obvious that (L, d_{qb}) is neither b-metric space nor dislocated quasi metric space.

1.1.5 Definition [30, 83]

Let (L, d_{qb}) be a dislocated quasi b-metric space. Let $\{a_n\}$ is a sequence in (L, d_{qb}), then

(a) $\{a_n\}$ is called left (right) Cauchy if for each $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such as $d_{qb}(a_m, a_n) < \varepsilon$ (respectively $d_{qb}(a_n, a_m) < \varepsilon$), for all $m > n \geq n_0$.

(b) $\{a_n\}$ dislocated quasi b-converges to $a \in L$, if $\lim\limits_{n\to+\infty} d_{qb}(a_n, a) = \lim\limits_{n\to+\infty} d_{qb}(a, a_n) = 0$ or for any $\varepsilon > 0$, $\exists\ n_0 \in \mathbb{N}$, such as $\forall\ n > n_0$, $d_{qb}(a, a_n) < \varepsilon$ and $d_{qb}(a_n, a) < \varepsilon$ and the point a in this case is called a d_{qb}-limit of $\{a_n\}$.

(c) (L, d_{qb}) is a complete if and only if every Cauchy sequence in L is convergent.

1.1.6 Definition [30]

Let (L, q_s) is a quasi b-metric space. Then a sequence $\{a_n\}$ in (L, q_s) is called:

(a) Left (right) K-Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such as $q_s(a_m, a_n) < \varepsilon$ (respectively $q_s(a_n, a_m) < \varepsilon$) for all $m > n \geq n_0$.

(b) Converges to a, if $\lim_{n\to+\infty} q_s(a_n, a) = \lim_{n\to+\infty} q_s(a, a_n) = 0$. In this case, the point a is called a limit of the sequence $\{a_n\}$.

(c) (L, q_s) is called left (right) K-sequentially complete if every left (right) K-Cauchy sequence in it is q_s-convergent.

1.1.7 Definition [32, 92]

Let (L, d_{qb}) is a dislocated b-quasi metric space. Let A be a nonempty subset of L and $a \in L$. We say, an element $y_0 \in A$ is a left (right) best approximation in A if,

$$\begin{aligned} d_{qb}(a, A) &= d_{qb}(a, y_0), \text{ where } d_{qb}(a, A) = \inf_{y\in A} d_{qb}(a, y) \\ \text{and } d_{qb}(A, a) &= d_{qb}(y_0, a), \text{ where } d_{qb}(A, a) = \inf_{y\in A} d_{qb}(y, a). \end{aligned}$$

If every $a \in L$ has a best approximation in A, then A is called a proximinal set.

It is clear that if $d_{qb}(a, A) = d_{qb}(A, a) = 0$, then $a \in A$. But, if $a \in A$, then $d_{qb}(a, A)$ or $d_{qb}(A, a)$ may not equal to zero. We denote $P(L)$ the set of all proximinal subsets of L.

1.1.8 Definition [92]

The function $H_{d_{qb}} : P(L) \times P(L) \to [0, +\infty)$, which defined as

$$H_{d_{qb}}(C, F) = \max\{\sup_{l \in C} d_{qb}(l, F), \ \sup_{b \in F} d_{qb}(C, b)\}$$

is called Hausdorff dislocated quasi b-metric on $P(L)$. Also $(P(L), H_{d_{qb}})$ is known as Hausdorff dislocated quasi b-metric space.

1.1.9 Lemma [95]

Every closed set A in a left (right) K-seqnentially complete quasi metric space L is a left (right) K-sequentially complete.

1.2 Double Controlled Metric Type Spaces

1.2.1 Definition [56]

Let $L \neq \{\}$ and $\theta : L \times L \to [1, +\infty)$. The function $q : L \times L \to [0, +\infty)$ is called an extended b-metric, if for each $a, y, e \in L$ the below satisfied:

(q_1) $q(a, y) = 0 \Leftrightarrow a = y$;

(q_2) $q(a, y) = q(y, a)$;

(q_3) $q(a, y) \leq \theta(a, y)[q(a, e) + q(e, y)]$. The pair (L, q) is called an extended b-metric space.

1.2.2 Definition [71]

Given $\beta : L \times L \to [1, +\infty)$, where L is nonempty. Let $q : L \times L \to [0, +\infty)$. Suppose that for all $a, y, e \in L$ the below conditions satisfied:

(i) $q(a, y) = 0 \Leftrightarrow a = y$,

(ii) $q(a, y) = q(y, a)$,

(iii) $q(a, y) \leq \beta(a, e)q(a, e) + \beta(e, y)q(e, y)$. Then, q is called a controlled metric type and (L, q) is called a controlled metric type space.

1.2.3 Definition [2]

Given thye functions $\alpha, \mu : L \times L \to [1, +\infty)$. If $q : L \times L \to [0, +\infty)$ satisfies the follows: for all $a, y, e \in L$,

(i) $q(a, y) = 0 \Leftrightarrow a = y$,

(ii) $q(a, y) = q(y, a)$,

(iii) $q(a, y) \leq \alpha(a, e)q(a, e) + \mu(e, y)q(e, y)$.

Then, q is called double controlled metric type with the functions α, μ and the pair (L, q) is called double controlled metric type space with the functions α, μ.

1.2.4 Example [2]

Let $L = [0, +\infty)$. Define q by

$$q(a, y) = \begin{cases} 0, \Leftrightarrow a = y, \\ \frac{1}{a}, \text{ if } a \geq 1 \text{ and } y \in [0, 1), \\ \frac{1}{y}, \text{ if } y \geq 1 \text{ and } a \in [0, 1), \\ 1, \text{ if not.} \end{cases}$$

Consider $\alpha, \mu : L \times L \to [1, +\infty)$ as

$$\alpha(a, y) = \begin{cases} a, \text{ if } a, y \geq 1, \\ 1, \text{ if not.} \end{cases} \quad \mu(a, y) = \begin{cases} 1, \text{ if } a, y < 1, \\ \max\{a, y\}, \text{ if not.} \end{cases}$$

Then, q is a double controlled metric type but, q is not an extended b-metric when considering the same function $\mu = \alpha$. Indeed,

$$q(0, \frac{1}{2}) = 1 > \frac{2}{3} = \frac{1}{3} + \frac{1}{3} = \alpha(0, 3)\, q(0, 3) + \alpha\left(3, \frac{1}{2}\right) q\left(3, \frac{1}{2}\right).$$

1.2.5 Definition [2]

Let (L, q) is a double controlled metric type space with two functions, the sequence $\{a_n\}$ is called:

(i) A convergent to some a in L, if for each $\varepsilon > 0$, there is some integer N_ε such as $q(a_n, a) < \varepsilon$ for each $n > N_\varepsilon$. It is written as $\lim_{n\to+\infty} a_n = a$.

(ii) A Cauchy if for every $\varepsilon > 0$, $q(a_m, a_n) < \varepsilon$ for each $m > n > N_\varepsilon$, where N_ε is some integer.

(iii) (L, q) is called complete if each Cauchy sequence is convergent.

1.2.6 Theorem [2]

Let (L, q) is a complete double controlled metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and let $J : L \to L$ is a given map. Assume that:

$$q(Ja, Jy) \leq kq(a, y), \text{ for each } a, y \in L \text{ provided that } k \in (0, 1).$$

For $r_0 \in L$, choose $r_n = J^n r_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\alpha(r_{i+1}, r_{i+2})}{\alpha(r_i, r_{i+1})} \mu(r_{i+1}, r_m) < \frac{1}{k}.$$

In addition, for every $r \in L$, we have

$$\lim_{n \to +\infty} \alpha(r, r_n) \text{ and } \lim_{n \to +\infty} \mu(r_n, r) \text{ exist and are finite.}$$

Then, J has a unique fixed point $r^* \in L$.

1.3 Some Basic Concepts

1.3.1 Definition [9]

Consider $\mu \in \Psi$ and Ψ denotes the set of functions $\mu : [0, +\infty) \to [0, +\infty)$ satisfied the follows:

(Ψ_1) μ is non decreasing.

(Ψ_2) For all $t > 0$, such as $\sum_{k=0}^{+\infty} \mu^k(t) < +\infty$, where μ^k is the k^{th} iterate of μ. The function $\mu \in \Psi$ is called comparison function.

1.3.2 Lemma [9]

Let $\mu \in \Psi$. Then, we have

(i) $\mu(t) < t$, for all $t > 0$,

(ii) $\mu(0) = 0$.

1.3.3 Definition [53]

Consider S and Q are two self-mappings on L. If $Sa = Qa$, for some $a \in L$, then a is called coincidence point of S and Q.

1.3.4 Definition [53]

A pair (S, Q) of self-maps defined on L is called a weakly compatible if they commute at there coincidence points. (i.e. if $Sa = Qa$, for some $a \in L$, then $SQa = QSa$).

1.3.5 Definition [84]

Consider (L, d) be a metric space, $J : L \to L$ be a given map and $\alpha : L \times L \to [0, +\infty)$. The map J is called an α-admissible if, for each $a, y \in L$,

$$\alpha(a, y) \geq 1 \Rightarrow \alpha(Ja, Jy) \geq 1.$$

1.3.6 Definition [73]

Let $J, \Im, S, Q : L \to L$ are maps of a non-empty set L and $\alpha : J(L) \cup \Im(L) \times J(L) \cup \Im(L) \to [0, +\infty)$ is a mapping. A pair (S, Q) is called an α-admissible with respect to J and $\Im$, if for all $a, y \in L$, $\alpha(Ja, \Im y) \geq 1$ or $\alpha(\Im a, Jy) \geq 1$, implies

$$\alpha\,(Sa, Qy) \geq 1 \text{ and } \alpha\,(Qa, Sy) \geq 1.$$

1.3.7 Definition [68]

Consider $L \neq \{\}$ and $p : L \times L \to [0, +\infty)$ such as, for each $a, y, e \in L$, if the below conditions satisfied:

(i) $a = y \Leftrightarrow p(a, a) = p(a, y) = p(y, y)$;

(ii) $p(a, a) \leq p(a, y)$;

(iii) $p(a, y) = p(y, a)$;

(iv) $p(a, y) \leq p(a, e) + p(e, y) - p(e, e)$. Then the pair (L, p) is called a partial metric space and p is called a partial metric on L. If p is a partial on L, then the function $p^s : L \times\ L \to [0, +\infty)$

defined by $p^s(a,y) = 2p(a,y) - p(a,a) - p(y,y)$ satisfies the conditions of a metric space on L and hence it is a usual metric on L.

1.3.8 Lemma [3]

A sequence $\{a_n\}$ in a partial metric space (L,p) is:

(i) A Cauchy sequence if and only if it is a Cauchy sequence in a metric space (L,p^s).

(ii) A complete if and only if a metric space (L,p^s) is complete. Moreover,

$$\lim_{n\to+\infty} p^s(a,a_n) = 0 \Leftrightarrow \lim_{n\to+\infty} p(a,a_n) = \lim_{n,m\to+\infty} p(a_n,a_m) = p(a,a).$$

1.3.9 Lemma [3]

Consider (L,p) be a partial metric space and $J : L \to L$ is a given map. J is called a continuous at $a_0 \in L$, if it is sequentially continuous at a_0, that is, if and only if

$$\forall\{a_n\} \subset L, \quad \lim_{n\to+\infty} a_n = a_0 \text{ implies } \lim_{n\to+\infty} Ja_n = Ja_0.$$

1.3.10 Lemma [3]

Assume that $a_n \to e$ as $n \to +\infty$ in a partial metric space (L,p) whenever $p(e,e) = 0$. Then

$$\lim_{n\to+\infty} p(a_n,y) = p\left(e,y\right), \text{ for all } y \in L.$$

1.3.11 Definition [78]

Let $(L,\preceq)$ be a partially ordered set and $J : L \to L$ is a given map. We called J is non decreasing with respect to $\preceq$ if $a,y \in L,\ a \preceq y \Rightarrow Ja \preceq Jy$.

1.3.12 Definition [78]

Let d is a metric on L and $(L,\preceq)$ be a partially ordered set. We called $(L,\preceq,d)$ is regular if for all nondecreasing sequence $\{a_n\} \subset L$ such as $a_n \to a \in L$ as $n \to +\infty$, there exists a subsequence $\left\{a_{n(\hat{\jmath})}\right\}$ of $\{a_n\}$ such as $a_{n(\hat{\jmath})} \preceq a$ for all $\hat{\jmath}$.

1.3.13 Definition [100]

Consider $\mathcal{F}$ is the set of all functions $F : [0, +\infty) \to \mathbb{R}$ such as:

(F1) F is strictly increasing, that is for each $a, y \in [0, +\infty)$ whenever $a < y$ implies $F(a) < F(y)$;

(F2) for all sequence $\{\vartheta_n\}_{n=1}^{+\infty}$ of positive numbers, $\lim_{n\to+\infty} \vartheta_n = 0$ if and only if $\lim_{n\to+\infty} F(\vartheta_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such as $\lim_{\vartheta\to 0^+} \vartheta^k F(\vartheta) = 0$.

1.3.14 Definition [100]

Consider (L, d) be a metric space. A mapping $J : L \times L$ is called F-contraction if there exists $\tau > 0$, we have

$\forall \;\; a,\, y \in L,\, d(Ja, Jy) > 0 \Rightarrow \tau + F(d(Ja, Jy)) \leq F(d(a, y))$.

Ali *et al.* [7] see also [40] extended the set of mapping $\mathcal{F}$ defined by [100] to the set $\mathcal{F}_S$ of each functions $F : [0, +\infty) \to \mathbb{R}$ such as

(F4) For every sequence $\{\vartheta_n\}$ of positive real numbers such as $\tau + F(s\vartheta_n) \leq F(\vartheta_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$, we have $\tau + F(s^n\vartheta_n) \leq F(s^{n-1}\vartheta_{n-1})$, for each $n \in \mathbb{N}$.

Chapter 2

Fixed Point Results for Locally Contractive Multivalued Mappings in Quasi and Quasi b-Metric Spaces

2.1 Introduction

After, many authors extended the Banach contraction theorem by using many different type of contractions in many spaces. The concept of a quasi metric space was first introduced in 1930 [101]. Many authors have provided several extensions of this result by considering various types of contractions and extended this concept to quasi b-metric space see [21, 43, 88]. Nadler [72] gave a new direction to the field when he mapped the contraction maps on a set of points instead of a single point. Various authors investigated fixed point theory for such mappings in many directions. On the other hand, Reilly *et al.* [83] presented the concept of left (right) K-Cauchy sequence and complete left (right) K-sequentially in complete quasi metric space. Recently, Altun *et al.* [9] proved a significant result concerning the existence of common fixed point satisfying a contraction with new restriction of order in a complete ordered metric space. Arshad *et al.* [16] observed that there were mappings which had fixed points but fixed point for results were not established for such maps and introduced a contraction on closed ball satisfied common fixed points for such maps.

In Section 2.2, we extended the result given by Altun *et al.* [9] in five ways: a pair of multi-valued maps, open ball with new generalized contraction, left K-sequentially complete quasi metric space and generalized a function $\alpha : L \times L \to [0, +\infty)$. We applied our results to obtain the contractions along with a graph and a partial ordered spaces.

Wardowski [100] introduced the concept of F-contraction and investigated fixed point results for different mappings (see also [4, 5, 7, 17, 49, 50, 58, 62, 76]). In Section 2.3, we discuss a recent generalization of quasi b-metric space and introduce $F - \mu_s - \rho_s^*$ contraction which is an extension of many announced contractions. Fixed point results for some of such contractions have been obtained. We achieve results endowed with a graph and in ordered left K-sequentially complete quasi b-metric space. An application is presented to ensure the existence of unique common solution point for integral equations and lastly we give an application to obtain the unique solution of functional equations that rises in dynamic programming.

2.1.1 Lemma [95]

Let (L, q_s) be a quasi b-metric space. Let $(P(L), H_{q_s})$ be a Hausdorff quasi b-metric space on $P(L)$. Then, for each $C, F \in P(L)$ and for all $l \in C$, there exists $b_l \in F$ such as $H_{q_s}(C, F) \geq q_s(l, b_l)$ and $H_{q_s}(F, C) \geq q_s(b_l, l)$, where $q_s(l, F) = q_s(l, b_l)$ and $q_s(F, l) = q_s(b_l, l)$.

2.1.2 Definition

Let $L \neq \{\}$ and $\alpha : L \times L \to [0, +\infty)$ is a map whenever $\alpha(a, y) \geq 1$ and $\alpha(y, a) \geq 1$ implies $a = y$. Let $M \subseteq L$, define $\alpha^*(a, M) = \inf\{\alpha(a, l),\ l \in M\}$ and $\alpha^*(M, y) = \inf\{\alpha(b, y),\ b \in M\}$.

2.1.3 Definition

Let $L \neq \{\}$ and $\rho_s : L \times L \to [0, +\infty)$ is a map. Let $M \subseteq L$, define $\rho_s^*(a, M) = \inf\{\rho_s(a, l),\ l \in M\}$ and $\rho_s^*(M, y) = \inf\{\rho_s(b, y),\ b \in M\}$.

2.1.4 Definition [76]

Let $\mathcal{F}$ is the set of each strictly increasing functions $F : [0, +\infty) \to \mathbb{R}$, i.e for each $a, y \in [0, +\infty)$, if $a < y$, then $F(a) < F(y)$.

2.1.5 Definition [31]

Let $s \geq 1$ and let a function $\mu_s : [0, +\infty) \to [0, +\infty)$ satisfies:

(Ψ_{s1}) μ_s is non-decreasing;

(Ψ_{s2}) for each $t > 0$, such as $\sum_{k=0}^{\infty} s^k \mu_s^k (t) < +\infty$, where μ_s^k is the k^{th} iterate of μ_s. Then the function μ_s is called b-comparison function. Let $s \geq 1$, the function $\mu_s (t) = bt$, $t \in [0, +\infty)$ with $0 < b < \frac{1}{s}$ is a b-comparison function. For each value of 's' in the given example, we can obtain infinitely many b-comparison functions by taking different values of 'b'. Denote the set of all b-comparison functions by Ψ_s. If we take $s = 1$, then μ_s is called (c)-comparison function. If $\mu (t) = \frac{t}{1+t}$, then μ is a (c)-comparison function. Denoted the set of all (c)-comparison functions by Ψ.

2.1.6 Lemma [31]

Let $\mu_s \in \Psi_s$. Then, we have

(i) $s\mu_s (t) < t$, for each $t > 0$,

(ii) $\mu_s (0) = 0$.

Clearly $s\mu_s (t) < t$ for each $t > 0$ implies $s^{n+1}\mu_s^{n+1}(t) < s^n \mu_s^n (t)$.

2.2 Fixed Point Results for A pair of Multivalued Mappings in Quasi Metric Spaces via New Approach

Results given in this section have been published in [90]

Let (L, q) be a quasi metric space, $a_0 \in L$ and $J : L \to P(L)$ be a multi-valued map on L. Since Ja_0 is a proximinal set, so there exists $a_1 \in Ja_0$ such as $q(a_0, Ja_0) = q(a_0, a_1)$ and $q(Ja_0, a_0) = q(a_1, a_0)$. Now, for $a_1 \in L$, there exist $a_2 \in Ja_1$ such as $q(a_1, Ja_1) = q(a_1, a_2)$ and $q(Ja_1, a_1) = q(a_2, a_1)$. Continuing this way, we generate a sequence a_n of points in L such as $a_{n+1} \in Ja_n$, $q(a_n, Ja_n) = q(a_n, a_{n+1})$ and $q(Ja_n, a_n) = q(a_{n+1}, a_n)$. We denote this iterative sequence $\{LJ(a_n)\}$ and say that $\{LJ(a_n)\}$ is a sequence in L generated by a_0.

2.2.1 Theorem

Consider (L, q) be a left K-sequentially complete quasi metric space, $S, J : L \to P(L)$ be the multivalued mappings, $\mu \in \Psi$, $a_0 \in L$, $r > 0$ and $\alpha : L \times L \to [0, +\infty)$. Suppose that:

(i) For every $a, y \in B_q(a_0, r) \cap \{LJ(a_n)\}$ with $\alpha^*(Sa, a) \geq 1$, $\alpha^*(y, Sy) \geq 1$, we have

$$\max\{H_q(Ja, Jy), H_q(Jy, Ja)\} \leq \mu(P_q(a, y)), \tag{2.1}$$

where

$$P_q(a, y) = \max\left\{q(a, y), q(a, Ja), \frac{q(a, Ja)\, q(a, Jy) + q(y, Jy)\, q(y, Ja)}{q(a, Jy) + q(y, Ja)}\right\}.$$

(ii) $$\sum_{p=0}^{j} \max\{\mu^p(q(a_1, a_0)), \mu^p(q(a_0, a_1))\} < r, \text{ for each } j \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

(iii) If $a \in B_q(a_0, r)$, $q(a, Ja) = q(a, y)$ and $q(Ja, a) = q(y, a)$, then

$$\text{(a) } \alpha^*(a, Sa) \geq 1, \text{ implies } \alpha^*(Sy, y) \geq 1, \text{ (b) } \alpha^*(Sa, a) \geq 1, \text{ implies } \alpha^*(y, Sy) \geq 1.$$

(iv) The set $G(S) = \{a : \alpha^*(a, Sa) \geq 1 \text{ and } a \in B_q(a_0, r)\}$ contains a_0 and closed.

Then, the subsequence $\{a_{2n}\}$ of $\{LJ(a_n)\}$ is a sequence in $G(S)$ and a sequence $\{a_{2n}\} \to a^* \in G(S)$. Also, if inequality (2.1) satisfied for $a, y \in \{a^*\}$. Then a^* is a common fixed point of J and S in $B_q(a_0, r)$.

Proof. As a_0 is any element of $G(S)$, from condition (iv) $\alpha^*(a_0, Sa_0) \geq 1$. Consider the sequence $\{LJ(a_n)\}$. Then there exists $a_1 \in Ja_0$ such as

$$q(a_0, Ja_0) = q(a_0, a_1) \text{ and } q(Ja_0, a_0) = q(a_1, a_0).$$

From condition (iii) $\alpha^*(Sa_1, a_1) \geq 1$. In particular, (2.2) holds for $j = 0$, so

$$\max\{q(a_1, a_0), q(a_0, a_1)\} < r.$$

Therefore, $q(a_1, a_0) < r$ and $q(a_0, a_1) < r$. Hence $a_1 \in B_q(a_0, r)$. Let $a_2, ...a_j \in B_q(a_0, r) \cap$

$\{LJ(a_n)\}$, $\alpha^*(a_j, Sa_j) \geq 1$ and $\alpha^*(Sa_{j+1}, a_{j+1}) \geq 1$, for some $j \in \mathbb{N}$, where $j = 2p$, $p = 2, 3, ..., \frac{j}{2}$. By using Lemma 2.1.1, we have

$$\begin{aligned} q(a_{2p}, a_{2p+1}) &\leq H_q(Ja_{2p-1}, Ja_{2p}) \\ &\leq \max\{H_q(Ja_{2p-1}, Ja_{2p}), H_q(Ja_{2p}, Ja_{2p-1})\}. \end{aligned}$$

As $a_{2p-1}, a_{2p} \in B_q(a_0, r) \cap \{LJ(a_n)\}$, $\alpha^*(a_{2p}, Sa_{2p}) \geq 1$ and $\alpha^*(Sa_{2p-1}, a_{2p-1}) \geq 1$, by (2.1), we have

$$\begin{aligned} q(a_{2p}, a_{2p+1}) &\leq \mu\Big(\max\Big\{q(a_{2p-1}, a_{2p}), q(a_{2p-1}, a_{2p}), \\ &\quad \frac{q(a_{2p-1}, a_{2p})\, q(a_{2p-1}, Ja_{2p}) + q(a_{2p}, a_{2p+1})\, q(a_{2p}, Ja_{2p-1})}{q(a_{2p-1}, Ja_{2p}) + q(a_{2p}, Ja_{2p-1})}\Big\}\Big). \end{aligned}$$

$$q(a_{2p}, a_{2p+1}) \leq \mu(q(a_{2p-1}, a_{2p})). \tag{2.3}$$

It implies that,

$$q(a_{2p}, a_{2p+1}) \leq \max\{\mu(q(a_{2p-1}, a_{2p})), \mu(q(a_{2p}, a_{2p-1}))\}. \tag{2.4}$$

Again by Lemma 2.1.1, we have

$$\begin{aligned} q(a_{2p-1}, a_{2p}) &\leq H_q(Ja_{2p-2}, Ja_{2p-1}) \\ &\leq \max\{H_q(Ja_{2p-2}, Ja_{2p-1}), H_q(Ja_{2p-1}, Ja_{2p-2})\}. \end{aligned}$$

As $a_{2p-1}, a_{2p-2} \in B_q(a_0, r) \cap \{LJa_n\}$, $\alpha^*(Sa_{2p-1}, a_{2p-1}) \geq 1$ and $\alpha^*(a_{2p-2}, Sa_{2p-2}) \geq 1$, by (2.1), we have

$$\begin{aligned} q(a_{2p-1}, a_{2p}) &\leq \mu(\max\{q(a_{2p-1}, a_{2p-2}), q(a_{2p-1}, a_{2p}), q(a_{2p-2}, a_{2p-1})\}) \\ &= \mu(\max\{q(a_{2p-1}, a_{2p-2}), q(a_{2p-2}, a_{2p-1})\}). \end{aligned}$$

As μ is non decreasing function, so

$$\mu(q(a_{2p-1}, a_{2p})) \leq \max\left\{\mu^2(q(a_{2p-1}, a_{2p-2})), \mu^2(q(a_{2p-2}, a_{2p-1}))\right\}. \tag{2.5}$$

Using (2.5) in (2.3), we have

$$q\left(a_{2p}, a_{2p+1}\right) \leq \max\left\{\mu^2\left(q\left(a_{2p-1}, a_{2p-2}\right)\right), \mu^2\left(q\left(a_{2p-2}, a_{2p-1}\right)\right)\right\}. \tag{2.6}$$

Now, by Lemma 2.1.1

$$q\left(a_{2p-2}, a_{2p-1}\right) \leq H_q\left(Ja_{2p-3}, Ja_{2p-2}\right).$$

As $a_{2p-3}, a_{2p-2} \in B_q(a_0, r) \cap \{LJa_n\}$, $\alpha^*\left(a_{2p-2}, Sa_{2p-2}\right) \geq 1$ and $\alpha^*\left(Sa_{2p-3}, a_{2p-3}\right) \geq 1$, by (2.1), we have

$$q\left(a_{2p-2}, a_{2p-1}\right) \leq \mu\left(q\left(a_{2p-3}, a_{2p-2}\right)\right). \tag{2.7}$$

It implies that,

$$\mu^2\left(q\left(a_{2p-2}, a_{2p-1}\right)\right) \leq \mu^2\left(\mu\left(\max\left\{q\left(a_{2p-3}, a_{2p-2}\right), q\left(a_{2p-2}, a_{2p-3}\right)\right\}\right)\right). \tag{2.8}$$

Now, by Lemma 2.1.1

$$q\left(a_{2p-1}, a_{2p-2}\right) \leq H_q\left(Ja_{2p-2}, Ja_{2p-3}\right).$$

As $a_{2p-3}, a_{2p-2} \in B_q(a_0, r) \cap \{LJa_n\}$, $\alpha^*\left(Sa_{2p-3}, a_{2p-3}\right) \geq 1$ and $\alpha^*\left(a_{2p-2}, Sa_{2p-2}\right) \geq 1$, by (2.1), we have

$$q\left(a_{2p-1}, a_{2p-2}\right) \leq \mu\left(\max\left\{q\left(a_{2p-2}, a_{2p-3}\right), q\left(a_{2p-3}, a_{2p-2}\right)\right\}\right).$$

As μ is non decreasing function, so

$$\mu^2\left(q\left(a_{2p-1}, a_{2p-2}\right)\right) \leq \mu^2\left(\mu\left(\max\left\{q\left(a_{2p-2}, a_{2p-3}\right), q\left(a_{2p-3}, a_{2p-2}\right)\right\}\right)\right). \tag{2.9}$$

Combining inequalities (2.6), (2.8) and (2.9), we have

$$q\left(a_{2p}, a_{2p+1}\right) \leq \max\left\{\mu^3 q\left(a_{2p-3}, a_{2p-2}\right), \mu^3 q\left(a_{2p-2}, a_{2p-3}\right)\right\}. \tag{2.10}$$

Following the patterns of inequalities (2.4), (2.6) and (2.10), we have

$$q\left(a_{2p}, a_{2p+1}\right) \leq \max\left\{\mu^{2p}\left(q\left(a_0, a_1\right), \mu^{2p}\left(q\left(a_1, a_0\right)\right)\right)\right\}. \tag{2.11}$$

Also, by Lemma 2.1.1, we have

$$q(a_{2p+1}, a_{2p}) \leq H_q(Ja_{2p}, Ja_{2p-1}).$$

As $a_{2p-1}, a_{2p} \in B_q(a_0, r) \cap \{LJ(a_n)\}$, $\alpha^*(Sa_{2p-1}, a_{2p-1}) \geq 1$ and $\alpha^*(a_{2p}, Sa_{2p}) \geq 1$, by (2.1), we have

$$q(a_{2p+1}, a_{2p}) \leq \mu(q(a_{2p-1}, a_{2p})), \tag{2.12}$$

which implies that,

$$q(a_{2p+1}, a_{2p}) \leq \max\{\mu(q(a_{2p-1}, a_{2p})), \mu(q(a_{2p}, a_{2p-1}))\}. \tag{2.13}$$

Using (2.5) in (2.12), we have

$$q(a_{2p+1}, a_{2p}) \leq \max\left\{\mu^2(q(a_{2p-1}, a_{2p-2})), \mu^2(q(a_{2p-2}, a_{2p-1}))\right\}. \tag{2.14}$$

Combining the inequalities (2.8), (2.9) and (2.14), we have

$$q(a_{2p+1}, a_{2p}) \leq \max\left\{\mu^3 q(a_{2p-3}, a_{2p-2}), \mu^3 q(a_{2p-2}, a_{2p-3})\right\}. \tag{2.15}$$

Following the patterns of inequalities (2.13), (2.14) and (2.15), we have

$$q(a_{2p+1}, a_{2p}) \leq \max\left\{\mu^{2p}(q(a_1, a_0)), \mu^{2p}(q(a_0, a_1))\right\}. \tag{2.16}$$

Now, by using the inequalities (2.11), (2.2) and the triangle inequality, we have

$$q(a_0, a_{2p+1}) \leq \sum_{j=0}^{2p} \max\left\{\mu^j q(a_1, a_0), \mu^j q(a_0, a_1)\right\} < r. \tag{2.17}$$

Similarly, by using inequalities (2.16), (2.2) and the triangle inequality, we have

$$q(a_{2p+1}, a_0) \leq \sum_{j=0}^{2p} \max\left\{\mu^j(q(a_1, a_0)), \mu^j(q(a_0, a_1))\right\} < r. \tag{2.18}$$

By inequality (2.17) and (2.18) , we have $a_{2p+1} \in B_q(a_0, r)$. Also $q\left(a_{2p+1}, Ja_{2p+1}\right) = q\left(a_{2p+1}, a_{2p+2}\right)$ and $q\left(Ja_{2p+1}, a_{2p+1}\right) = q\left(a_{2p+2}, a_{2p+1}\right)$. As $\alpha^*\left(Sa_{2p+1}, a_{2p+1}\right) \geq 1$, so from condition (iii), we have $\alpha^*\left(a_{2p+2}, Sa_{2p+2}\right) \geq 1$. Similarly, we have

$$q\left(a_{2p+1}, a_{2p+2}\right) \leq \max\left\{\mu^{2p+1}\left(q\left(a_1, a_0\right)\right), \mu^{2p+1}\left(q\left(a_0, a_1\right)\right)\right\} \tag{2.19}$$

and

$$q\left(a_{2p+2}, a_{2p+1}\right) \leq \max\left\{\mu^{2p+1}\left(q\left(a_1, a_0\right)\right), \mu^{2p+1}\left(q\left(a_0, a_1\right)\right)\right\}. \tag{2.20}$$

Also,

$$q\left(a_0, a_{2p+2}\right) \leq r \text{ and } q\left(a_{2p+2}, a_0\right) \leq r.$$

It following that $a_{2p+2} \in B_q(a_0, r)$. Also

$$q\left(a_{2p+2}, Ja_{2p+2}\right) = q\left(a_{2p+2}, a_{2p+3}\right) \text{ and } q\left(Ja_{2p+2}, a_{2p+2}\right) = q\left(a_{2p+3}, a_{2p+2}\right).$$

As $\alpha^*\left(a_{2p+2}, Sa_{2p+2}\right) \geq 1$, so from condition (iii) we have $\alpha^*\left(Sa_{2p+3}, a_{2p+3}\right) \geq 1$. Hence by mathematical induction $a_n \in B_q(a_0, r)$, $\alpha^*\left(a_{2n}, Sa_{2n}\right) \geq 1$ and $\alpha^*\left(Sa_{2n+1}, a_{2n+1}\right) \geq 1$, for each $n \in \mathbb{N}$. Also, $a_{2n} \in G\left(S\right)$. The inequalities (2.11), (2.16), (2.19) and (2.20) can be written as

$$q\left(a_n, a_{n+1}\right) \leq \max\left\{\mu^n\left(q\left(a_1, a_0\right)\right), \mu^n\left(q\left(a_0, a_1\right)\right)\right\}, \tag{2.21}$$

$$q\left(a_{n+1}, a_n\right) \leq \max\left\{\mu^n\left(q\left(a_1, a_0\right)\right), \mu^n\left(q\left(a_0, a_1\right)\right)\right\}, \tag{2.22}$$

for each $n \in \mathbb{N}$. Let $k_1\left(\varepsilon\right) \in \mathbb{N}$ and fix $\varepsilon > 0$ such as

$$\sum_{k \geq k_1(\varepsilon)} \max\left\{\mu^k\left(q\left(a_1, a_0\right)\right), \mu^k\left(q\left(a_0, a_1\right)\right)\right\} < \varepsilon.$$

Let $n, m \in \mathbb{N}$ with $m > n > k_1(\varepsilon)$, then

$$\begin{aligned} q(a_n, a_m) &\leq \sum_{k=n}^{m-1} q(a_k, a_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \max\left\{\mu^k(q(a_1, a_0)), \mu^k(q(a_0, a_1))\right\} \\ q(a_n, a_m) &< \sum_{k \geq k_1(\varepsilon)} \max\{\mu^n q(a_1, a_0), \mu^n q(a_0, a_1)\} < \varepsilon. \end{aligned}$$

This proved that $\{LJ(a_n)\}$ is a left K-Cauchy sequence in (L, q) and since (L, q) is a left K sequentially complete, so $\{LJ(a_n)\} \to a^* \in L$ and

$$\lim_{n \to +\infty} q(a_{2n}, a^*) = \lim_{n \to +\infty} q(a^*, a_{2n}) = 0. \tag{2.23}$$

As $\{a_{2n}\}$ is a subsequence of $\{LJ(a_n)\}$, so $a_{2n} \to a^*$. Also, $\{a_{2n}\}$ is a sequence in $G(S)$ and $G(S)$ is closed, so $a^* \in G(S)$ and therefore

$$\alpha^*(a^*, Sa^*) \geq 1. \tag{2.24}$$

Now,

$$q(a^*, a^*) \leq q(a^*, a_{2n}) + q(a_{2n}, a^*).$$

It implies that $q(a^*, a^*) = 0$. Now, by Lemma 2.1.1, we have

$$q(a^*, Ja^*) \leq q(a^*, a_{2n+2}) + H_q(Ja_{2n+1}, Ja^*).$$

By assumption, inequality (2.1) holds for a^*. Also $\alpha^*(Sa_{2n+1}, a_{2n+1}) \geq 1$ and $\alpha^*(a^*, Sa^*) \geq 1$, so

$$\begin{aligned} q(a^*, Ja^*) \leq q(a^*, a_{2n+2}) + \mu\Big(\max\Big\{&q(a_{2n+1}, a^*), q(a_{2n+1}, a_{2n+2}), \\ &\frac{q(a_{2n+1}, a_{2n+2})\, q(a_{2n+1}, Ja^*) + q(a^*, Ja^*)\, q(a^*, Ja_{2n+1})}{q(a_{2n+1}, Ja^*) + q(a^*, Ja_{2n+1})}\Big\}\Big). \end{aligned}$$

Since, $q(a^*, Ja_{2n+1}) \leq q(a^*, a_{2n+2})$.

Putting limit as n tends to infinity of above inequality, we get

$$\lim_{n\to+\infty} q\left(a^*, Ja_{2n+1}\right) = 0. \tag{2.25}$$

Putting limit as n tends to infinity and using inequality (2.23) and (2.25), we get

$$q\left(a^*, Ja^*\right) = 0. \tag{2.26}$$

Now,

$$q\left(Ja^*, a^*\right) \leq H_q\left(Ja^*, Ja_{2n+1}\right) + q\left(a_{2n+2}, Ja^*\right).$$

As inequality (2.1) hold for a^*, $\alpha^*\left(a^*, Sa^*\right) \geq 1$ and $\alpha^*\left(Sa_{2n+1}, a_{2n+1}\right) \geq 1$, then

$$q\left(Ja^*, a^*\right) \leq \left(\max\left\{q\left(a_{2n+1}, a^*\right), q\left(a_{2n+1}, a_{2n+2}\right),\right.\right.$$

$$\left.\left.\frac{q\left(a_{2n+1}, a_{2n+2}\right) q\left(a_{2n+1}, Ja^*\right) + q\left(a^*, Ja^*\right) q\left(a^*, Ja_{2n+1}\right)}{q\left(a_{2n+1}, Ja^*\right) + q\left(a^*, Ja_{2n+1}\right)}\right\}\right) + q\left(a_{2n+2}, a^*\right).$$

Putting limit as n tends to infinity and using (2.23) and (2.26), we get

$$q\left(Ja^*, a^*\right) = 0. \tag{2.27}$$

From inequalities (2.26) and (2.27), we have $a^* \in Ja^*$. As $\alpha\left(a^*, Sa^*\right) \geq 1$ and $q\left(a^*, Ja^*\right) = q\left(Ja^*, a^*\right) = q\left(0, 0\right),$ then from (iii)

$$\alpha^*\left(Sa^*, a^*\right) \geq 1. \tag{2.28}$$

From (2.24) and (2.28), we have $\alpha^*\left(a^*, Sa^*\right) \geq 1$, $\alpha^*\left(Sa^*, a^*\right) \geq 1$. This implies $\alpha\left(a^*, y\right) \geq 1$, $\alpha\left(y, a^*\right) \geq 1$, for all $y \in Sa^*$. Thus, by Definition 2.1.2, $a^* = y$. Hence, a^* is a common fixed point of S and J. ■

2.2.2 Example

Let $L=[0,+\infty)$ and $q(a,y)=\begin{cases} a+2y & \text{if } a\neq y \\ 0 & \text{if } a=y \end{cases}$, for $(a,y)\in L\times L$, then (L,q) is left (right) K-sequentially complete quasi metric space. Consider μ is a function on $[0,+\infty)$ define by $\mu(t)=\frac{3t}{4}$. Let $\mathcal{R}$ is a binary relation on L defined as

$$\begin{aligned} \mathcal{R} &= \left\{(a,\frac{a}{4}) : a\in\left\{0,1,\frac{1}{16},\frac{1}{256},\frac{1}{4096},...\right\}\right\} \\ &\quad \cup\left\{(\frac{a}{4},a) : a\in\left\{\frac{1}{4},\frac{1}{64},\frac{1}{1024},...\right\}\right\}. \end{aligned}$$

Define the pair of multivalued mappings $J,S:L\to P(L)$ by

$$Ja=\begin{cases} \left[\frac{a}{4},\frac{a}{2}\right], \text{ if } a\in[0,1], \\ [a+1,a+2], \text{ if } a\in(1,+\infty). \end{cases}, \quad Sa=\begin{cases} \{\frac{a}{4}\}, \text{ if } a\in[0,1], \\ \{2a\}, \text{ if } a\in(1,+\infty). \end{cases}$$

Define $\alpha : L\times L\to[0,+\infty)$ as follows:

$$\alpha(a,y)=\begin{cases} 1, \text{ if } (a,y)\in\mathcal{R}, \\ \frac{1}{2}, \text{ if } a,y\in[0,10)\wedge(a,y)\notin\mathcal{R}, \\ 3, \text{ otherwise.} \end{cases}$$

$$\begin{aligned} A &= \{a:\alpha^*(a,Sa)\geq 1\}=\left\{0,1,\frac{1}{16},\frac{1}{256},\frac{1}{4096},...\right\}. \\ B &= \{y:\alpha^*(Sy,y)\geq 1\}=\left\{0,\frac{1}{4},\frac{1}{64},\frac{1}{1024},...\right\}. \end{aligned}$$

Let $a_0=1$ and $r=21$, $B_q(a_0,r)=[0,10)$. Then,

$$\begin{aligned} G(S) &= \{a:\alpha^*(a,Sa)\geq 1 \text{ and } a\in B_q(a_{0,}r)\} \\ &= \left\{0,1,\frac{1}{16},\frac{1}{256},...\right\}. \end{aligned}$$

Clearly $G(S)$ is closed and contains a_0, so condition (iv) is satisfied. Now, as $\frac{1}{4^{n-1}} \in B_q(a_0, r)$, for each $n \in \mathbb{N}$

$$q(\frac{1}{4^{n-1}}, J\frac{1}{4^{n-1}}) = q(\frac{1}{4^{n-1}}, \frac{1}{4 \times 4^{n-1}}).$$

and

$$q(J\frac{1}{4^{n-1}}, \frac{1}{4^{n-1}}) = q(\frac{1}{4 \times 4^{n-1}}, \frac{1}{4^{n-1}}).$$

As $\alpha^*\left(\frac{1}{4^{n-1}}, S\frac{1}{4^{n-1}}\right) \geq 1$ implies $\alpha^*\left(S\frac{1}{4\times 4^{n-1}}, \frac{1}{4\times 4^{n-1}}\right) \geq 1$, if n is odd. Also, $\alpha^*\left(S\frac{1}{4^{n-1}}, \frac{1}{4^{n-1}}\right) \geq 1$ implies $\alpha^*\left(\frac{1}{4\times 4^{n-1}}, S\frac{1}{4\times 4^{n-1}}\right) \geq 1$, if n is even. Also, $0 \in B_q(a_0, r)$, $q(0, J0) = q(0, 0)$, $q(J0, 0) = q(0, 0)$. As $\alpha^*(0, S0) \geq 1$ if and only if $\alpha^*(S0, 0) \geq 1$. Hence, condition (iii) is satisfied. Now, $2, 3 \in B_q(a_0, r)$ with $\alpha^*(S3, 3) \ngeq 1, \alpha^*(2, S2) \ngeq 1$,

$$\max\{H_q(J2, J3), H_q(J3, J2)\} = \max\{11, 13\} = 13 \geq P_q(2, 3),$$

this explain that a contractive condition is not satisfied on whole $B_q(a_0, r)$. Now, when we take $11, 12 \in L$ with $\alpha^*(S11, 11) \geq 1$, $\alpha^*(12, S12) \geq 1$, we get

$$\max\{H_q(J11, J12), H_q(J12, J11)\} = \max\{40, 38\} = 40 \geq P_q(a, y).$$

Therefore, the contractive condition is not satisfied on L and $B_q(a_0, r)$.

Now, if we take $a,\ y \in B_q(a_0, r) \cap \{LJa_n\}$ with $\alpha^*(Sa, a) \geq 1$, $\alpha^*(y, Sy) \geq 1$, then in general $a = \frac{1}{4^{n-1}},\ y = \frac{1}{4^{m-1}}$, where n is even, m is odd.

Case i: For $n \leq m$, we have

$$\begin{aligned} H(Ja, Jy) &= H\left(\left[\frac{1}{4 \times 4^{n-1}}, \frac{1}{2 \times 4^{n-1}}\right], \left[\frac{1}{4 \times 4^{m-1}}, \frac{1}{2 \times 4^{m-1}}\right]\right) \\ &= \max\left\{q\left(\frac{1}{2 \times 4^{n-1}}, \frac{1}{4 \times 4^{m-1}}\right), q\left(\frac{1}{4 \times 4^{n-1}}, \frac{1}{2 \times 4^{m-1}}\right)\right\} \\ &= \max\left\{\frac{1}{2 \times 4^{n-1}} + \frac{1}{2 \times 4^{m-1}}, \frac{1}{4 \times 4^{n-1}} + \frac{1}{4^{m-1}}\right\} \\ &= \max\left\{\frac{4^{m-n}+1}{2 \times 4^{m-1}}, \frac{4^{m-n}+4}{4 \times 4^{m-1}}\right\} = \frac{4^{m-n}+1}{2 \times 4^{m-1}}. \end{aligned}$$

$$H(Jy, Ja) = \max\left\{\frac{1+4^{m-n}}{2 \times 4^{m-1}}, \frac{1+4 \times 4^{m-n}}{4 \times 4^{m-1}}\right\}.$$

Now, we have

$$\frac{1+4\times 4^{m-n}}{4\times 4^{m-1}} < \frac{3}{4}\left(\frac{\left(\frac{3}{2\times 4^{n-1}}\right)\left(\frac{4\times 4^{m-n}+2}{4\times 4^{m-1}}\right)+\left(\frac{3}{2\times 4^{m-1}}\right)\left(\frac{4+2\times 4^{m-n}}{4\times 4^{m-1}}\right)}{\left(\frac{4\times 4^{m-n}+2}{4\times 4^{m-1}}\right)+\left(\frac{4+2\times 4^{m-n}}{4\times 4^{m-1}}\right)}\right),$$

$$\text{or } \max\left\{H_q\left(Ja,Jy\right),H_q\left(Jy,Ja\right)\right\} \leq \mu\left(P_q(a,y)\right).$$

Case ii: Similarly, for $n > m$, we have

$$\begin{aligned}\max\left\{H_q\left(Ja,Jy\right),H_q\left(Jy,Ja\right)\right\} &= \frac{1+4\times 4^{n-m}}{4\times 4^{n-1}}\\ &< \frac{3}{4}\left(\frac{1+2\times 4^{n-m}}{4^{n-1}}\right)=\mu\left(P_q(a,y)\right).\end{aligned}$$

Case iii: If $a = 0$, $y = \frac{1}{4^{m-1}}$, we get

$$\begin{aligned}\max\left\{H_q\left(Ja,Jy\right),H_q\left(Jy,Ja\right)\right\} &= \max\left\{\frac{1}{4^{m-1}},\frac{1}{2\times 4^{m-1}}\right\}=\frac{1}{4^{m-1}}\\ &< \frac{2}{4^{m-1}}=\mu\left(P_q(a,y)\right).\end{aligned}$$

Case iv: If $a = \frac{1}{4^{n-1}}$, $y = 0$, we get

$$\max\left\{H_q\left(Ja,Jy\right),H_q\left(Jy,Ja\right)\right\}=\frac{1}{4^{n-1}} \leq \mu\left(P_q(a,y)\right).$$

Case v: Inequality (2.1) is trivially satisfied when we take $a = 0$ and $y = 0$. Also,

$$\sum_{p=0}^{j}\max\left\{\mu^p\left(q\left(a_1,a_0\right)\right),\mu^p\left(q\left(a_0,a_1\right)\right)\right\}=9<21=r.$$

Thus, all the hypothesis of Theorem 2.2.1 hold. Moreover, 0 is a common fixed point of J and S.

By dropping a left K-sequentially complete quasi metric space and taking complete metric space, we have the below result.

2.2.3 Theorem

Consider (L, d) be a complete metric space, $r > 0$, $a_0 \in L$, $S, J : L \to P(L)$ be the multivalued mappings on $B(a_0, r)$, $\mu \in \Psi$ and $\alpha : L \times L \to [0, +\infty)$. Assume that the below assumptions satisfied:

(i) for each $a, y \in B_d(a_0, r) \cap \{LJ(a_n)\}$ with $\alpha^*(Sa, a) \geq 1, \alpha^*(y, Sy) \geq 1$, we have

$$H_d(Ja, Jy) \leq \mu(P_d(a, y)),$$

(ii) $\sum_{p=0}^{j} \mu^p(d(a_1, a_0)) < r$, for each $j \in \mathbb{N} \cup \{0\}$.

(iii) if $a \in B(a_0, r)$, $d(a, Ja) = d(a, y)$, then

(a) $\alpha^*(a, Sa) \geq 1$, implies $\alpha^*(Sy, y) \geq 1$, (b) $\alpha^*(Sa, a) \geq 1$, implies $\alpha^*(y, Sy) \geq 1$,

(iv) $G(S) = \{a : \alpha(a, Sa) \geq 1 \text{ and } a \in B(a_0, r)\}$ is closed and contains a_0. Then, the subsequence $\{a_{2n}\}$ of $\{LJ(a_n)\}$ is a sequence in $G(S)$ also, a sequence $\{a_{2n}\} \to a^* \in G(S)$ and $q(a^*, a^*) = 0$. If inequality (i) satisfied for a^*. Then J and S have a common fixed point a^* in $B(a_0, r)$.

2.2.4 Theorem

Consider (L, q) be a complete left K-sequentially quasi metric space, $\alpha : L \times L \to [0, +\infty)$, $\mu \in \Psi$, $a_0 \in L$ and $S, J : L \to P(L)$. Suppose that:

(i) for all $a, y \in L \cap \{LJ(a_n)\}$ with $\alpha^*(Sa, a) \geq 1$, $\alpha^*(y, Sy) \geq 1$, we have

$$\max\{H_q(Ja, Jy), H_q(Jy, Ja)\} \leq \mu(P_q(a, y)),$$

where

$$P_q(a, y) = \max\left\{q(a, y), q(a, Ja), \frac{q(a, Ja)\, q(a, Jy) + q(y, Jy)\, q(y, Ja)}{q(a, Jy) + q(y, Ja)}\right\}.$$

(ii) if $q(a, Ja) = q(a, y)$ and $q(Ja, a) = q(y, a)$, then

(a) $\alpha^*(a, Sa) \geq 1$, implies $\alpha^*(Sy, y) \geq 1$, (b) $\alpha^*(Sa, a) \geq 1$, implies $\alpha^*(y, Sy) \geq 1$,

(iii) the set $G(S) = \{a : \alpha(a, Sa) \geq 1\}$ contains a_0 and is closed.

Then, the subsequence $\{a_{2n}\}$ of $\{LJ(a_n)\}$ is a sequence in $G(S)$ also, a sequence $\{a_{2n}\} \to a^* \in$

$G(S)$ and $q(a^*, a^*) = 0$. If inequality (i) is satisfied for a^*. Then J and S have a common fixed point a^* in L.

By taking self mappings, we obtain the below result.

2.2.5 Theorem

Consider (L, q) be a complete left K-sequentially quasi metric space, $\alpha : L \times L \to [0, +\infty)$ is a function, $r > 0$, $\mu \in \Psi$, $S, J : L \to L$, $a_0 \in L$ and $a_n = Ja_{n-1}$ is a Picard sequence. Assume that the below assumptions satisfied:

(i) for all $a, y \in B_q(a_0, r) \cap \{a_n\}$ with $\alpha(Sa, a) \geq 1$ and $\alpha(y, Sy) \geq 1$, we have

$$\max\{q(Ja, Jy), q(Jy, Ja)\} \leq \mu(P_q(a, y)),$$

where

$$P_q(a, y) = \max\left\{q(a, y), q(a, Ja), \frac{q(a, Ja)\, q(a, Jy) + q(y, Jy)\, q(y, Ja)}{q(a, Jy) + q(y, Ja)}\right\}.$$

(ii) $\sum\limits_{p=0}^{j} \max\{\mu^p(q(a_1, a_0)), \mu^p(q(a_0, a_1))\} < r$, for each $j \in \mathbb{N} \cup \{0\}$.

(iii) if $a \in B_q(a_0, r)$, then

(a) $\alpha(a, y) \geq 1$ implies $\alpha(SJa, Ja) \geq 1$,

(b) $\alpha^*(y, a) \geq 1$ implies $\alpha(Ja, SJa) \geq 1$,

(vi) $G(S) = \{a : \alpha(a, y) \geq 1$ and $a \in B_q(a_0, r)\}$ contains a_0 and is closed. Then, the subsequence $\{a_{2n}\}$ of $\{a_n\}$ is a sequence in $G(S)$ also, a sequence $\{a_{2n}\} \to a^* \in G(S)$ and $q(a^*, a^*) = 0$. If inequality (i) satisfied for a^*. Then J and S have a common fixed point a^* in $B_q(a_0, r)$.

Now, we apply our results to obtain the contractions endowed with a graph and a partial ordered spaces as follows;

2.2.6 Definition

Let (L, q) be a quasi metric space along with a graph G and $S, J : L \to P(L)$ be multivalued maps. Consider that for $r > 0$, $a_0 \in B_q(a_0, r)$ and $\mu \in \Psi$, the following conditions hold:

$$\max\{H_q(Ja, Jy), H_q(Jy, Ja)\} \leq \mu(P_q(a, y)), \tag{2.29}$$

for all $a, y \in B_q(a_0, r) \cap \{LJ(a_n)\}$ with $\{(y, v) \in E(G) : v \in Sy\}$ and $\{(u, a) \in E(G), u \in Sa\}$, where

$$P_q(a, y) = \max\left\{q(a, y), q(a, Ja), \frac{q(a, Ja)\, q(a, Jy) + q(y, Jy)\, q(y, Ja)}{q(a, Jy) + q(y, Ja)}\right\}.$$

Then, the pair (S, J) is called a μ-graphic contractive multivalued maps on open ball.

2.2.7 Theorem

Consider (L, q) be a complete left K-sequentially quasi metric space with graph G. Let $a_0 \in B_q(a_0, r)$, $r > 0$ and S, $J : L \to P(L)$ is μ-graphic contractive multivalued mappings on $B_q(a_0, r)$. Suppose that the below assumptions are satisfied:

(i) $\sum_{p=0}^{j} \max\{\mu^p q(a_1, a_0), \mu^p q(a_0, a_1)\} < r$, for each $j \in \mathbb{N} \cup \{0\}$;

(ii) if $a \in B_q(a_0, r)$, $q(a, Ja) = q(a, y)$, $q(Ja, a) = q(y, a)$, then

(a) $(a, u) \in E(G)$, for each $u \in Sa$ implies $(v, y) \in E(G)$, for each $v \in Sy$,

(b) $(u, a) \in E(G)$, for each $u \in Sa$ implies $(y, v) \in E(G)$, for each $v \in Sy$.

(iii) The set $G(S) = \{a : (a, y) \in E(G) \text{ for each } y \in Sa \text{ and } a \in B_q(a_0, r)\}$ is closed and contains a_0. Then, the subsequence $\{a_{2n}\}$ of $\{LJ(a_n)\}$ is a sequence in $G(S)$ and a sequence $\{a_{2n}\} \to a^* \in G(S)$. Also, if inequality (2.29) satisfied for a^*. Then J and S have a common fixed point a^* in $B_q(a_0, r)$.

Proof. Define $\alpha : L \times L \to \mathbb{R}_+$, by $\alpha(y, v) = 1$, for all $v \in Sy$, if and only if $y \in B_q(a_0, r) \cap \{LJ(a_n)\}$ with $\{(y, v) \in E(G) : v \in Sy\}$. Also $\alpha(u, a) = 1$, for all $u \in Sa$, if and only if $a \in B_q(a_0, r) \cap \{LJ(a_n)\}$ with $\{(u, a) \in E(G), u \in Sa\}$. Moreover $\alpha(a, y) = 0$, otherwise. Now, as (S, J) is a μ-graphic contractive multivalued mappings on open ball, so inequality (2.29), implies inequality (2.1). Assumption (i) of Theorem 2.2.7 implies assumption (ii) of

Theorem 2.2.1. Assumption (ii) of Theorem 2.2.7 implies assumption (iii) of Theorem 2.2.1. Assumption (iii) of Theorem 2.2.7. implies assumption (iv) of Theorem 2.2.1. So, all hypothesis of Theorem 2.2.1 hold. Thus, the subsequence $\{a_{2n}\}$ of $\{LJ(a_n)\} \in G(S)$, for each $n \in \mathbb{N} \cup \{0\}$ and a sequence $\{a_{2n}\} \to a^* \in G(s)$. Also if inequality (2.29) is satisfied for a^*, then inequality (2.1) is satisfied for a^*. Then J and S have a common fixed point a^* in $B_q(a_0, r)$. ■

2.2.8 Theorem

Consider $(L, \preceq, q)$ is an ordered complete left K sequentially quasi metric space, $r > 0$, $a_0 \in L$ and S, $J : L \to P(L)$ be a non decreasing mappings on $B_q(a_0, r)$, with respect to $\preceq$ and there is some $\mu \in \Psi$. Suppose that:

(i) for all $(a, y) \in B_q(a_0, r) \cap \{LJ(a_n)\}$ with $Sa \preceq a$ and $y \preceq Sy$, we have

$$\max\{H_q(Ja, Jy), H_q(Jy, Ja)\} \leq \mu(P_q(a, y)), \tag{2.30}$$

where

$$P_q(a, y) = \max\left\{q(a, y), q(a, Ja), \frac{q(a, Ja)\, q(a, Jy) + q(y, Jy)\, q(y, Ja)}{q(a, Jy) + q(y, Ja)}\right\}.$$

(ii) $\sum_{p=0}^{j} \max\{\mu^p q(a_1, a_0), \ \mu^p q(a_0, a_1)\} < r$, for each $j \in \mathbb{N} \cup \{0\}$.

(iii) if $a \in B_q(a_0, r)$, $q(a, Ja) = q(a, y)$ and $q(Ja, a) = q(y, a)$, then

(a) $a \preceq Sa$, implies $Sy \preceq y$, (b) $Sa \preceq a$, implies $y \preceq Sy$.

(iv) $G(S) = \{a : a \preceq Sa \text{ and } a \in B_q(a_0, r)\}$ contains a_0 and is closed.

Then, the subsequence $\{a_{2n}\}$ of $\{LJ(a_n)\}$ is a sequence in $G(S)$, for each $n \in \mathbb{N} \cup \{0\}$. Also $\{a_{2n}\} \to a^* \in G(S)$. If inequality (2.30) satisfied for a^*. Then there is a single common fixed point a^* of J and S in $B_q(a_0, r)$.

Proof. Define $\alpha : L \times L \to \mathbb{R}_+$, by $\alpha(y, v) = 1$, for all $v \in Sy$, if and only if $y \in B_q(a_0, r) \cap \{LJ(a_n)\}$ with $y \preceq v$, $v \in Sa$. Also $\alpha(u, a) = 1$, for all $u \in Sa$, if and only if $a \in B_q(a_0, r) \cap \{LJ(a_n)\}$ with $a \succeq u$, $u \in Sa$. Moreover $\alpha(a, y) = 0$, otherwise. Then, clearly Assumption (i)-(iv) of Theorem 2.2.8 implies assumption (i)-(iv) of Theorem 2.2.1. Hence, the subsequence $\{a_{2n}\}$ of $\{LJ(a_n)\} \in G(S)$, for each $n \in \mathbb{N} \cup \{0\}$, a sequence $\{a_{2n}\} \to a^* \in G(s)$

and if inequality (2.30) holds for a^*, then inequality (2.1) is satisfied for a^*. Thus, J and S have a common fixed point a^* in $B_q(a_0, r)$. ∎

2.2.9 Remarks

(i) By taking six proper subsets of $P_q(a, y)$ instead of $P_q(a, y)$, we can obtain six new corollaries for each of theorems; Theorem 2.2.1, Theorem 2.2.3, Theorem 2.2.4, Theorem 2.2.5, Theorem 2.2.7 and Theorem 2.2.8.

(ii) Fixed point result in right K-sequentially quasi metric space can be obtained in a similar way.

2.3 Fixed Point Results for $F - \mu_s - \rho_s^*$ Contraction in Quasi b-Metric Spaces with Some Applications

Results given in this section will appear in [94]

2.3.1 Definition

Let (L, q_s, s) be a left K-sequentially complete quasi b-metric space, $\rho_s : L \times L \to [0, +\infty)$ and $S, J : L \to P(L)$ are the multivalued maps. The pair (S, J) is called $F - \mu_s - \rho_s^*$ contraction on the intersection of a sequence and open ball, if $\mu_s \in \Psi$, $F \in \mathcal{F}$, $a_0 \in L$, $r, \tau > 0$, $a, y \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$, $\rho_s^*(Sy, y) \geq s$, $\rho_s^*(a, Sa) \geq s$, $q_s(a, Jy) + q_s(y, Sa) \neq 0$ and $\max\{H_{q_s}(Sa, Jy), H_{q_s}(Jy, Sa), Q_s(a, y), Q_s(y, a)\} > 0$, then

$$\tau + \max\{F(H_{q_s}(Sa, Jy)), F(H_{q_s}(Jy, Sa))\} \leq F(\mu_s(Q_s(a, y))), \tag{2.31}$$

where

$$Q_s(a, y) = \max\left\{q_s(a, y), q_s(a, Sa), \frac{q_s(a, Sa)\, q_s(a, Jy) + q_s(y, Jy)\, q_s(y, Sa)}{q_s(a, Jy) + q_s(y, Sa)}\right\}.$$

Also, if $q_s(a, Jy) + q_s(y, Sa) = 0$, then $\max\{H_{q_s}(Sa, Jy), H_{q_s}(Jy, Sa), Q_s(a, y), Q_s(y, a)\} = 0$. Moreover,

$$\sum_{p=0}^{j} s^{p+1} \left[\max\left\{\mu_s^p\left(q_s\left(a_1, a_0\right)\right), \mu_s^p\left(q_s\left(a_0, a_1\right)\right)\right\}\right] < r, \text{ for each } j \in \mathbb{N} \cup \{0\}. \tag{2.32}$$

2.3.2 Theorem

Let (L, q_s, s) be a left K-sequentially quasi b-metric space, $\rho_s : L \times L \to [0, +\infty)$, $S, J : L \to P(L)$ and (S, J) is $F - \mu_s - \rho_s^*$ contraction on open ball. Suppose that:

(i) If $a \in B_{q_s}(a_0, r)$,

(a) $\rho_s^*(a, Sa) \geq s$, $q_s(a, Sa) = q_s(a, y)$ and $q_s(Sa, a) = q_s(y, a)$ implies $\rho_s^*(Sy, y) \geq s$,

(b) $\rho_s^*(Sa, a) \geq s$, $q_s(a, Ja) = q_s(a, y)$ and $q_s(Ja, a) = q_s(y, a)$ implies $\rho_s^*(y, Sy) \geq s$.

(ii) The set $G(S) = \{a : \rho_s^*(a, Sa) \geq s \text{ and } a \in B_{q_s}(a_0, r)\}$ contains a_0 and is closed.

Then, the subsequence $\{a_{2n}\}$ of $\{JS(a_n)\}$ is a sequence in $G(S)$, $\{a_{2n}\} \to a^* \in G(S)$ and $q_s(a^*, a^*) = 0$. Also, if inequality (2.31) is satisfied for a^*. Then J and S have a common fixed point a^* in $B_{q_s}(a_0, r)$.

Proof. Consider the sequence $\{JS(a_n)\}$ generated by a_0. As a_0 is any element of $G(S)$, from assumption (ii) $\rho_s^*(a_0, Sa_0) \geq s$ and $a_0 \in B_{q_s}(a_0, r)$. Then there exists $a_1 \in Sa_0$ such as $q_s(a_0, Sa_0) = q_s(a_0, a_1)$ and $q_s(Sa_0, a_0) = q_s(a_1, a_0)$. From condition (i) $\rho_s^*(Sa_1, a_1) \geq s$. By (2.32), we have

$$\max\left\{q_s\left(a_1, a_0\right), q_s\left(a_0, a_1\right)\right\} \leq \sum_{p=0}^{j} s^{p+1} \left[\max\left\{\mu_s^p\left(q_s\left(a_1, a_0\right)\right), \mu_s^p\left(q_s\left(a_0, a_1\right)\right)\right\}\right] < r.$$

That is $q_s(a_1, a_0) < r$ and $q_s(a_0, a_1) < r$. Hence, $a_1 \in B_{q_s}(a_0, r)$. Also

$$q_s\left(a_1, Ja_1\right) = q_s\left(a_1, a_2\right) \text{ and } q_s\left(Ja_1, a_1\right) = q_s\left(a_2, a_1\right).$$

As $\rho_s^*(Sa_1, a_1) \geq s$, so from assumption (i), we have $\rho_s^*(a_2, Sa_2) \geq s$. Now, by Lemma 2.1.1, we have

$$q_s(a_{2p}, a_{2p+1}) \leq H_{q_s}(Ja_{2p-1}, Sa_{2p}),\ q_s(a_{2p+1}, a_{2p}) \leq H_{q_s}(Sa_{2p}, Ja_{2p-1}) \tag{2.33}$$

and

$$q_s(a_{2p+1}, a_{2p+2}) \leq H_{q_s}(Sa_{2p}, Ja_{2p+1}),\ q_s(a_{2p+2}, a_{2p+1}) \leq H_{q_s}(Ja_{2p+1}, Sa_{2p}). \tag{2.34}$$

By the triangle inequality, we have

$$q_s(a_0, a_2) \leq s\left[q_s(a_0, a_1) + q_s(a_1, a_2)\right]. \tag{2.35}$$

By using (2.34), we have

$$\tau + F\left(q_s(a_1, a_2)\right) \leq \tau + F\left(H_{q_s}(Sa_0, Ja_1)\right),$$

$$\tau + F\left(q_s(a_1, a_2)\right) \leq \tau + \max\left\{F\left(H_{q_s}(Sa_0, Ja_1)\right), F\left(H_{q_s}(Ja_1, Sa_0)\right)\right\}. \tag{2.36}$$

Now, let $a_{2p'}, a_{2p'+1}$ is two consecutive elements of the sequence $\{JS(a_n)\}$. Clearly, if

$$\max\{H_{q_s}(Sa_{2p'}, Ja_{2p'+1}),\ H_{q_s}(Ja_{2p'+1}, Sa_{2p'}),\ Q_s(a_{2p'}, a_{2p'+1}),\ Q_s(a_{2p'+1}, a_{2p'})\} \not> 0,$$

for some $p' \in \mathbb{N} \cup \{0\}$, or if $q_s(a_{2p'}, Ja_{2p'+1}) + q_s(a_{2p'+1}, Sa_{2p'}) = 0$, then

$$H_{q_s}(Sa_{2p'}, Ja_{2p'+1}) = H_{q_s}(Ja_{2p'+1}, Sa_{2p'}) = Q_s(a_{2p'}, a_{2p'+1}) = Q_s(a_{2p'+1}, a_{2p'}) = 0.$$

If $Q_s(a_{2p'}, a_{2p'+1}) = 0$, then $q_s(a_{2p'}, a_{2p'+1}) = 0$. Also, if $Q_s(a_{2p'+1}, a_{2p'}) = 0$, then $q_s(a_{2p'+1}, a_{2p'}) = 0$. So, $a_{2p'+1} = a_{2p'}$ and $a_{2p'} \in Sa_{2p'}$. Now, $H_{q_s}(Sa_{2p'}, Ja_{2p'+1}) = 0$ implies $q_s(a_{2p'+1}, Ja_{2p'+1}) = 0$ and $H_{q_s}(Ja_{2p'+1}, Sa_{2p'}) = 0$ implies $q_s(Ja_{2p'+1}, a_{2p'+1}) = 0$. So, $a_{2p'+1} \in Ja_{2p'+1}$ and hence $a_{2p'}$ is a common fixed point of S and J. Therefore, the proof is done. Now, suppose

$$\max\{H_{q_s}(Sa_{2p}, Ja_{2p+1}),\ H_{q_s}(Ja_{2p+1}, Sa_{2p}),\ Q_s(a_{2p}, a_{2p+1}),\ Q_s(a_{2p+1}, a_{2p})\} > 0,$$

and $q_s(a_{2p}, Ja_{2p+1}) + q_s(a_{2p+1}, Sa_{2p}) \neq 0$ for all $p \in \{0\} \cup \mathbb{N}$. As $a_0, a_1 \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$, $\rho_s^*(Sa_1, a_1) \geq s$, and $\rho_s^*(a_0, Sa_0) \geq s$, by using (2.31) in (2.36), we have

$$\tau + F(q_s(a_1, a_2)) \leq F(\mu_s(Q_s(a_0, a_1))) = F(\mu_s(q_s(a_0, a_1))).$$

Since F is strictly increasing and $\tau > 0$, so $q_s(a_1, a_2) < \mu_s(q_s(a_0, a_1))$. Now, inequality (2.35) implies

$$q_s(a_0, a_2) \leq \sum_{p=0}^{1} s^{p+1} [\max\{\mu_s^p(q_s(a_1, a_0)), \mu_s^p(q_s(a_0, a_1))\}] < r.$$

Now, by using (2.34), we have

$$\begin{aligned} \tau + F(q_s(a_2, a_1)) &\leq \tau + F(H_{q_s}(Ja_1, Sa_0)) \\ &\leq \tau + \max\{F(H_{q_s}(Ja_1, Sa_0)), F(H_{q_s}(Sa_0, Ja_1))\}. \end{aligned}$$

As $a_1, a_0 \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$, $\rho_s^*(a_0, Sa_0) \geq s$ and $\rho_s^*(Sa_1, a_1) \geq s$, then by (2.31), we have

$$\tau + F(q_s(a_2, a_1)) \leq F(\mu_s(Q_s(a_0, a_1))) \leq F(\mu_s(q_s(a_0, a_1))).$$

Since F is strictly increasing and $\tau > 0$, so

$$q_s(a_2, a_1) < \mu_s(\max\{q_s(a_1, a_0), q_s(a_0, a_1)\}).$$

Now, by the triangle inequality

$$d(a_2, a_0) \leq \sum_{p=0}^{j} s^{p+1} [\max\{\mu_s^p(q_s(a_1, a_0)), \mu_s^p(q_s(a_0, a_1))\}] < r.$$

That is, $q_s(a_0, a_2) < r$ and $q_s(a_2, a_0) < r$. So $a_2 \in B_{q_s}(a_0, r)$. Also

$$q_s(a_2, Sa_2) = q_s(a_2, a_3) \text{ and } q_s(Sa_2, a_2) = q_s(a_3, a_2).$$

As $\rho_s^*(a_2, Sa_2) \geq s$, so from assumption (i), we have $\rho_s^*(Sa_3, a_3) \geq s$. Let $a_3, ...a_j \in B_{q_s}(a_0, r)$, and $\rho_s^*(a_0, Sa_0) \geq s$, $\rho_s^*(Sa_1, a_1) \geq s$, $\rho_s^*(a_2, Sa_2) \geq s$, $\rho_s^*(Sa_3, a_3) \geq s, \cdots$, $\rho_s^*(Sa_{j+1}, a_{j+1}) \geq$

s, for some $j \in \mathbb{N}$, where $j = 2p$, $p = 1, 2, 3, ..., \frac{j}{2}$. Now by using (2.33), we have

$$\begin{aligned} \tau + F\left(q_s\left(a_{2p}, a_{2p+1}\right)\right) &\leq \tau + F\left(H_{q_s}\left(Ja_{2p-1}, Sa_{2p}\right)\right) \\ &\leq \tau + \max\left\{F\left(H_{q_s}\left(Ja_{2p-1}, Sa_{2p}\right)\right), F\left(H_{q_s}\left(Sa_{2p}, Ja_{2p-1}\right)\right)\right\}. \end{aligned}$$

As $a_{2p-1}, a_{2p} \in B_{q_s}(a_0, r) \cap \{JS\left(a_n\right)\}$, $\rho_s^*\left(a_{2p}, Sa_{2p}\right) \geq s$, $\rho_s^*\left(Sa_{2p-1}, a_{2p-1}\right) \geq s$ and $\max\{H_{q_s}(Ja_{2p-1}, Sa_{2p}), H_{q_s}(Sa_{2p}, Ja_{2p-1}), Q_s(a_{2p-1}, a_{2p}), Q_s(a_{2p}, a_{2p-1})\} > 0$, by (2.31), we have

$$\begin{aligned} \tau + F\left(q_s\left(a_{2p}, a_{2p+1}\right)\right) &\leq F(\mu_s(Q_s(a_{2p}, a_{2p-1}))) \\ &= F\left(\mu_s\left(\max\{q_s\left(a_{2p}, a_{2p-1}\right), q_s\left(a_{2p}, a_{2p+1}\right), q_s\left(a_{2p-1}, a_{2p}\right)\}\right)\right). \end{aligned}$$

If $\max\{q_s\left(a_{2p}, a_{2p-1}\right), q_s\left(a_{2p}, a_{2p+1}\right), q_s\left(a_{2p-1}, a_{2p}\right)\} = q_s\left(a_{2p}, a_{2p+1}\right)$, then

$$\tau + F\left(q_s\left(a_{2p}, a_{2p+1}\right)\right) \leq F\left(\mu_s\left(q_s\left(a_{2p}, a_{2p+1}\right)\right)\right),$$

implies $q_s\left(a_{2p}, a_{2p+1}\right) < \mu_s\left(q_s\left(a_{2p}, a_{2p+1}\right)\right) < s\mu_s\left(q_s\left(a_{2p}, a_{2p+1}\right)\right)$. A contradiction, so

$$\tau + F\left(q_s\left(a_{2p}, a_{2p+1}\right)\right) \leq F(\mu_s\left(\max\left\{q_s\left(a_{2p-1}, a_{2p}\right), q_s\left(a_{2p}, a_{2p-1}\right)\right\}\right)),$$

Since F is strictly increasing and $\tau > 0$, so

$$q_s\left(a_{2p}, a_{2p+1}\right) < \max\left\{\mu_s\left(q_s\left(a_{2p-1}, a_{2p}\right)\right), \mu_s\left(q_s\left(a_{2p}, a_{2p-1}\right)\right)\right\}. \tag{2.37}$$

Now, by (2.34), we have

$$\tau + F\left(q_s\left(a_{2p-1}, a_{2p}\right)\right) \leq \tau + F\left(H_{q_s}\left(Sa_{2p-2}, Ja_{2p-1}\right)\right)$$

$$\leq \tau + F\left(\max\left\{H_{q_s}\left(Sa_{2p-2}, Ja_{2p-1}\right), H_{q_s}\left(Ja_{2p-1}, Sa_{2p-2}\right)\right\}\right).$$

As $a_{2p-1}, a_{2p-2} \in B_{q_s}(a_0, r) \cap \{JSa_n\}$, $\rho_s^*\left(Sa_{2p-1}, a_{2p-1}\right) \geq s$, $\rho_s^*\left(a_{2p-2}, Sa_{2p-2}\right) \geq s$ and $\max\{H_{q_s}(Sa_{2p-2}, Ja_{2p-1}),\ H_{q_s}(Ja_{2p-1}, Sa_{2p-2}),\ Q_s(a_{2p-2}, a_{2p-1}),\ Q_s(a_{2p-1}, a_{2p-2})\} > 0$, by

(2.31), we have

$$\tau + F\left(q_s\left(a_{2p-1}, a_{2p}\right)\right) \leq F(\mu_s\left(Q_s(a_{2p-2}, a_{2p-1})\right)) = F\left(\mu_s\left(q_s\left(a_{2p-2}, a_{2p-1}\right)\right)\right).$$

Since F is strictly increasing and $\tau > 0$, so

$$q_s\left(a_{2p-1}, a_{2p}\right) < \mu_s\left(q_s\left(a_{2p-2}, a_{2p-1}\right)\right),$$

$$q_s\left(a_{2p-1}, a_{2p}\right) < \mu_s\left(\max\left\{q_s\left(a_{2p-1}, a_{2p-2}\right), q_s\left(a_{2p-2}, a_{2p-1}\right)\right\}\right).$$

As μ_s is non decreasing function, so

$$\mu_s\left(q_s\left(a_{2p-1}, a_{2p}\right)\right) < \max\left\{\mu_s^2\left(q_s\left(a_{2p-1}, a_{2p-2}\right)\right), \mu_s^2\left(q_s\left(a_{2p-2}, a_{2p-1}\right)\right)\right\}. \tag{2.38}$$

Now, by (2.34), we have

$$\tau + F\left(q_s\left(a_{2p}, a_{2p-1}\right)\right) \leq \tau + F\left(H_{q_s}\left(Ja_{2p-1}, Sa_{2p-2}\right)\right)$$

$$\leq \tau + F\left(\max\left\{H_{q_s}\left(Sa_{2p-2}, Ja_{2p-1}\right), H_{q_s}\left(Ja_{2p-1}, Sa_{2p-2}\right)\right\}\right).$$

By (2.31), we have

$$\tau + F\left(q_s\left(a_{2p}, a_{2p-1}\right)\right) \leq F(\mu_s\left(Q_s(a_{2p-2}, a_{2p-1})\right)) = F\left(\mu_s\left(q_s\left(a_{2p-2}, a_{2p-1}\right)\right)\right).$$

Since F is strictly increasing and $\tau > 0$, so

$$q_s\left(a_{2p}, a_{2p-1}\right) < \mu_s\left(q_s\left(a_{2p-2}, a_{2p-1}\right)\right) < \mu_s\left(\max\left\{q_s\left(a_{2p-1}, a_{2p-2}\right), q_s\left(a_{2p-2}, a_{2p-1}\right)\right\}\right).$$

As μ_s is non decreasing function, so

$$\mu_s\left(q_s\left(a_{2p}, a_{2p-1}\right)\right) < \max\left\{\mu_s^2\left(q_s\left(a_{2p-1}, a_{2p-2}\right)\right), \mu_s^2\left(q_s\left(a_{2p-2}, a_{2p-1}\right)\right)\right\}. \tag{2.39}$$

Now, by merging (2.38) and (2.39), we have

$$\max\left\{\mu_s\left(q_s\left(a_{2p-1},a_{2p}\right)\right),\ \mu_s\left(q_s\left(a_{2p},a_{2p-1}\right)\right)\right\}$$

$$<\max\left\{\mu_s^2\left(q_s\left(a_{2p-1},a_{2p-2}\right)\right),\ \mu_s^2\left(q_s\left(a_{2p-2},a_{2p-1}\right)\right)\right\}. \tag{2.40}$$

Using (2.40) in (2.37), we have

$$q_s\left(a_{2p},a_{2p+1}\right)<\max\left\{\mu_s^2\left(q_s\left(a_{2p-1},a_{2p-2}\right)\right),\ \mu_s^2\left(q_s\left(a_{2p-2},a_{2p-1}\right)\right)\right\}. \tag{2.41}$$

Now, by using (2.33), we have

$$\tau+F\left(q_s(a_{2p-2},a_{2p-1})\right)\leq\tau+F\left(H_{q_s}(Ja_{2p-3},Sa_{2p-2})\right)$$

$$\leq\tau+\max\left\{F\left(H_{q_s}(Ja_{2p-3},Sa_{2p-2})\right),F\left(H_{q_s}(Sa_{2p-2},Ja_{2p-3})\right)\right\}.$$

As $a_{2p-3},a_{2p-2}\in B_{q_s}(a_0,r)\cap\{JSa_n\}$, $\rho_s^*\left(Sa_{2p-3},a_{2p-3}\right)\geq s$, $\rho_s^*\left(a_{2p-2},Sa_{2p-2}\right)\geq s$ and $\max\{H_{q_s}(Sa_{2p-2},Ja_{2p-1}),\ H_{q_s}(Ja_{2p-1},Sa_{2p-2}),\ Q_s(a_{2p-2},a_{2p-1}),\ Q_s(a_{2p-1},a_{2p-2})\}>0$, by (2.31), we have

$$\begin{aligned}\tau+F\left(q_s(a_{2p-2},a_{2p-1})\right) &\leq F(\mu_s\left(Q_s(a_{2p-2},a_{2p-3})\right))\\ &= F\left(\mu_s\left(\max\left\{q_s\left(a_{2p-2},a_{2p-3}\right),q_s\left(a_{2p-2},a_{2p-1}\right),q_s\left(a_{2p-3},a_{2p-2}\right)\right\}\right)\right)\\ &= F\left(\mu_s\left(\max\left\{q_s\left(a_{2p-2},a_{2p-3}\right),q_s\left(a_{2p-3},a_{2p-2}\right)\right\}\right)\right),\end{aligned}$$

which implies that,

$$q_s(a_{2p-2},a_{2p-1})<\mu_s(\max\left\{q_s\left(a_{2p-2},a_{2p-3}\right),q_s(a_{2p-3},a_{2p-2})\right\}),$$

$$\mu_s^2q_s(a_{2p-2},a_{2p-1})<\mu_s^3(\max\left\{q_s(a_{2p-3},a_{2p-2}),q_s(a_{2p-2},a_{2p-3})\right\}). \tag{2.42}$$

Now, by using (2.34), we have

$$\tau+F\left(q_s(a_{2p-1},a_{2p-2})\right)\leq\tau+F\left(H_{q_s}(Sa_{2p-2},Ja_{2p-3})\right)$$

$$\leq \tau + \max\left\{F\left(H_{q_s}(Ja_{2p-3}, Sa_{2p-2})\right), F\left(H_{q_s}(Sa_{2p-2}, Ja_{2p-3})\right)\right\}.$$

As $a_{2p-3}, a_{2p-2} \in B_{q_s}(a_0, r) \cap \{JSa_n\}$, $\rho_s^*(Sa_{2p-3}, a_{2p-3}) \geq s$, $\rho_s^*(a_{2p-2}, Sa_{2p-2}) \geq s$ and $\max\{H_{q_s}(Sa_{2p-2}, Ja_{2p-1}),\ H_{q_s}(Ja_{2p-1}, Sa_{2p-2}),\ Q_s(a_{2p-2}, a_{2p-1}),\ Q_s(a_{2p-1}, a_{2p-2})\} > 0$, by (2.31), we have

$$\tau + F\left(q_s(a_{2p-1}, a_{2p-2})\right) \leq F(\mu_s\left(Q_s(a_{2p-2}, a_{2p-3})\right))$$

$$= F\left(\mu_s\left(\max\left\{q_s\left(a_{2p-2}, a_{2p-3}\right), q_s\left(a_{2p-2}, a_{2p-1}\right), q_s\left(a_{2p-3}, a_{2p-2}\right)\right\}\right)\right).$$

As,

$$\begin{aligned} q_s(a_{2p-2}, a_{2p-1}) &< \mu_s(\max\left\{q_s\left(a_{2p-2}, a_{2p-3}\right), q_s(a_{2p-3}, a_{2p-2})\right\}) \\ &\leq s\mu_s(\max\left\{q_s\left(a_{2p-2}, a_{2p-3}\right), q_s(a_{2p-3}, a_{2p-2})\right\}) \\ &< \max\left\{q_s\left(a_{2p-2}, a_{2p-3}\right), q_s(a_{2p-3}, a_{2p-2})\right\}. \end{aligned}$$

So,

$$\tau + F\left(q_s(a_{2p-1}, a_{2p-2})\right) < F\left(\mu_s\left(\max\left\{q_s\left(a_{2p-2}, a_{2p-3}\right), q_s\left(a_{2p-3}, a_{2p-2}\right)\right\}\right)\right),$$

which implies that,

$$q_s(a_{2p-1}, a_{2p-2}) < \mu_s(\max\left\{q_s\left(a_{2p-2}, a_{2p-3}\right), q_s(a_{2p-3}, a_{2p-2})\right\}),$$

$$\mu_s^2 q_s(a_{2p-1}, a_{2p-2}) < \mu_s^3(\max\left\{q_s(a_{2p-3}, a_{2p-2}), q_s(a_{2p-2}, a_{2p-3})\right\}). \tag{2.43}$$

Now, by (2.42) and (2.43), we have

$$\max\left\{\mu_s^2 q_s(a_{2p-2}, a_{2p-1}), \mu_s^2 q_s(a_{2p-1}, a_{2p-2})\right\}$$

$$< \mu_s^3(\max\left\{q_s(a_{2p-3}, a_{2p-2}), q_s(a_{2p-2}, a_{2p-3})\right\}). \tag{2.44}$$

Using (2.44) in (2.41), we have

$$q_s(a_{2p}, a_{2p+1}) \leq \max\left\{\mu_s^3(q_s(a_{2p-3}, a_{2p-2})), \mu_s^3(q_s(a_{2p-2}, a_{2p-3}))\right\}. \tag{2.45}$$

Following the patterns of inequalities (2.37), (2.41) and (2.45), we have

$$q_s(a_{2p}, a_{2p+1}) \le \max\left\{\mu_s^{2p}(q_s(a_0, a_1)), \mu_s^{2p}(q_s(a_1, a_0))\right\}.$$

As $j = 2p$, so

$$q_s(a_j, a_{j+1}) \le \max\left\{\mu_s^{j}(q_s(a_0, a_1)), \mu_s^{j}(q_s(a_1, a_0))\right\}. \tag{2.46}$$

Now, by using (2.33) , we have

$$\tau + F\left(q_s(a_{2p+1}, a_{2p})\right) \le \tau + F\left(H_{q_s}(Sa_{2p}, Ja_{2p-1})\right)$$

$$\le \tau + \max\left\{F\left(H_{q_s}(Ja_{2p-1}, Sa_{2p})\right), F\left(H_{q_s}(Sa_{2p}, Ja_{2p-1})\right)\right\}.$$

As $a_{2p-1}, a_{2p} \in B_{q_s}(a_0, r) \cap \{JS\,(a_n)\}$, $\rho_s^*\,(a_{2p}, Sa_{2p}) \ge s$, $\rho_s^*\,(Sa_{2p-1}, a_{2p-1}) \ge s$ and max $\{H_{q_s}(Ja_{2p-1}, Sa_{2p}),\ H_{q_s}(Sa_{2p}, Ja_{2p-1}),\ Q_s(a_{2p-1}, a_{2p}),\ Q_s(a_{2p}, a_{2p-1})\} > 0$, by (2.31), we have

$$\begin{aligned} \tau + F\left(q_s(a_{2p+1}, a_{2p})\right) &\le F(\mu_s\left(Q_s(a_{2p}, a_{2p-1})\right)) \\ &= F\left(\mu_s\left(\max\{q_s\left(a_{2p}, a_{2p-1}\right), q_s\left(a_{2p}, a_{2p+1}\right), q_s\left(a_{2p-1}, a_{2p}\right)\}\right)\right). \end{aligned}$$

By inequality (2.37), we have

$$\tau + F\left(q_s(a_{2p+1}, a_{2p})\right) < F\left(\mu_s\left(\max\left\{q_s\left(a_{2p-1}, a_{2p}\right), q_s\left(a_{2p}, a_{2p-1}\right)\right\}\right)\right).$$

Now,

$$q_s(a_{2p+1}, a_{2p}) < \max\left\{\mu_s\left(q_s(a_{2p-1}, a_{2p})\right), \mu_s(q_s(a_{2p}, a_{2p-1})\right\}. \tag{2.47}$$

Now, using (2.40) and (2.47), we have

$$q_s(a_{2p+1}, a_{2p}) \le \max\left\{\mu_s^2(q_s(a_{2p-1}, a_{2p-2})), \mu_s^2(q_s(a_{2p-2}, a_{2p-1}))\right\}. \tag{2.48}$$

Now, using (2.44) and (2.48), we have

$$q_s(a_{2p+1}, a_{2p}) \le \max\left\{\mu_s^3(q_s(a_{2p-3}, a_{2p-2})), \mu_s^3(q_s(a_{2p-2}, a_{2p-3}))\right\}. \tag{2.49}$$

Following the patterns of inequalities (2.47), (2.48) and (2.49), we have

$$q_s(a_{2p+1}, a_{2p}) \leq \max\left\{\mu_s^{2p}(q_s(a_0, a_1)), \mu_s^{2p}(q_s(a_1, a_0))\right\}.$$

As $j = 2p$, so

$$q_s(a_{j+1}, a_j) \leq \max\left\{\mu_s^{j}(q_s(a_0, a_1)), \mu_s^{j}(q_s(a_1, a_0))\right\}. \tag{2.50}$$

Now, if $j = 2p - 1$, then inequalities (2.46) and (2.50) can be obtained by using similar arguments. Now, by using the triangle inequality, (2.46) and (2.32), we have

$$\begin{aligned}
q_s(a_0, a_{j+1}) &\leq sq_s(a_0, a_1) + s^2 q_s(a_1, a_2) + ... + s^j q_s(a_{j-1}, a_j) + s^j q_s(a_j, a_{j+1}) \\
&\leq sq_s(a_0, a_1) + ... + s^j q_s(a_{j-1}, a_j) + s^{j+1} q_s(a_j, a_{j+1}) \\
&< sq_s(a_0, a_1) + s^2 \mu_s q_s(a_0, a_1) + ... + s^{j+1} \mu_s^j q_s(a_0, a_1) \\
&< \sum_{p=0}^{j} s^{p+1} \left[\max\left\{\mu_s^p(q_s(a_1, a_0)), \mu_s^p(q_s(a_0, a_1))\right\}\right] < r.
\end{aligned}$$

Similarly, by using the triangle inequality, (2.50) and (2.32), we have

$$q_s(a_{j+1}, a_0) < \sum_{p=0}^{j} s^{p+1} \left[\max\left\{\mu_s^p(q_s(a_1, a_0)), \mu_s^p(q_s(a_0, a_1))\right\}\right] < r,$$

$$q_s(a_0, a_{j+1}) < r \text{ and } q_s(a_{j+1}, a_0) < r.$$

It following that $a_{j+1} \in B_{q_s}(a_0, r)$. Also $\rho_s^*(Sa_{j+1}, a_{j+1}) \geq s$, $q_s(a_{j+1}, Ja_{j+1}) = q_s(a_{j+1}, a_{j+2})$ and $q_s(Ja_{j+1}, a_{j+1}) = q_s(a_{j+2}, a_{j+1})$, so from assumption (i), we have $\rho_s^*(a_{j+2}, Sa_{j+2}) \geq s$. Now, if $a_3, ...a_l \in B_{q_s}(a_0, r)$ and $\rho_s^*(a_0, Sa_0) \geq s$, $\rho_s^*(Sa_1, a_1) \geq s$, $\rho_s^*(Sa_3, a_3) \geq s$, $\cdots$, $\rho_s^*(a_{l+1}, Sa_{l+1}) \geq s$, for some $l \in \mathbb{N}$, where $l = 2p + 1$, $p = 1, 2, 3, ..., \frac{l-1}{2}$, then similarly we obtain $a_{l+1} \in B_{q_s}(a_0, r)$ and $\rho_s^*(Sa_{l+2}, a_{l+2}) \geq s$. Therefore, by mathematical induction $a_n \in B_{q_s}(a_0, r)$, $\rho_s^*(a_{2n}, Sa_{2n}) \geq s$ and $\rho_s^*(Sa_{2n+1}, a_{2n+1}) \geq s$, for each $n \in \mathbb{N} \cup \{0\}$. Also, $a_{2n} \in G(S)$. Now inequalities (2.46) and (2.50) can be written as

$$q_s(a_n, a_{n+1}) < \max\left\{\mu_s^n(q_s(a_1, a_0)), \mu_s^n(q_s(a_0, a_1))\right\}, \tag{2.51}$$

$$q_s(a_{n+1}, a_n) < \max\{\mu_s^n(q_s(a_1, a_0)), \mu_s^n(q_s(a_0, a_1))\}, \tag{2.52}$$

for all $n \in \mathbb{N}$. As $\sum_{w=1}^{+\infty} s^w \mu_s^w(t) < +\infty$, then the series

$$\sum_{w=1}^{+\infty} s^w \mu_s^w(\max\{\mu_s^{e-1}(q_s(a_1, a_0)), \mu_s^{e-1}(q_s(a_0, a_1))\})$$

converges for each $e \in \mathbb{N}$. As $s\mu_s(t) < t$, so

$$\begin{aligned} & s^{w+1}\mu_s^{w+1}(\max\{\mu_s^{e-1}(q_s(a_1, a_0)), \mu_s^{e-1}(q_s(a_0, a_1))\}) \\ < \; & s^w \mu_s^w(\max\{\mu_s^{e-1}(q_s(a_1, a_0)), \mu_s^{e-1}(q_s(a_0, a_1))\}), \text{ for all } w \in \mathbb{N}. \end{aligned}$$

So, for fix $\varepsilon > 0$, there exists $k_1(\varepsilon) \in \mathbb{N}$ such as

$$\sum_{j=1}^{+\infty} s^j \mu_s^j \left(\max\left\{\mu_s^{k_1(\varepsilon)-1}(q_s(a_1, a_0)), \mu_s^{k_1(\varepsilon)-1}(q_s(a_0, a_1))\right\}\right) < \varepsilon.$$

Let $m, k, h \in \mathbb{N}$ with $m > k > k_1(\varepsilon)$, then

$$\begin{aligned} q_s(a_k, a_m) &= q_s(a_k, a_{k+h}) \leq s q_s(a_k, a_{k+1}) + s^2 q_s(a_{k+1}, a_{k+2}) + ... + s^h q_s(a_{k+h-1}, a_{k+h}) \\ &< s\mu_s^k(\max\{q_s(a_1, a_0), q_s(a_0, a_1)\}) + s^2\mu_s^{k+1}(\max\{q_s(a_1, a_0), q_s(a_0, a_1)\}) \\ &\quad + ... + s^h \mu_s^{k+h-1}(\max\{q_s(a_1, a_0), q_s(a_0, a_1)\}) \\ &= s\mu_s \max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\} + \\ &\quad s^2\mu_s^2 \max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\} \\ &\quad + ... + s^h \mu_s^h \max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\} \\ &< \sum_{j=1}^{+\infty} s^j \mu_s^j \left(\max\left\{\mu_s^{k-1}(q_s(a_1, a_0)), \mu_s^{k-1}(q_s(a_0, a_1))\right\}\right) \\ &< \sum_{j=1}^{+\infty} s^j \mu_s^j \left(\max\left\{\mu_s^{k_1(\varepsilon)-1}(q_s(a_1, a_0)), \mu_s^{k_1(\varepsilon)-1}(q_s(a_0, a_1))\right\}\right) < \varepsilon. \end{aligned}$$

Thus, $\{JS(a_n)\}$ is a left K-Cauchy sequence in (L, q_s). As (L, q_s) is a left K-sequentially complete, so $\{JS(a_n)\} \to a^* \in L$ and

$$\lim_{n\to+\infty} q_s(a_n, a^*) = \lim_{n\to+\infty} q_s(a^*, a_n) = 0. \tag{2.53}$$

As $\{a_{2n}\}$ is a subsequence of $\{JS(a_n)\}$, so $a_{2n} \to a^*$. Also, $\{a_{2n}\} \in G(S)$ and $G(S)$ is closed, so $a^* \in G(S)$ and therefore

$$\rho_s^*(a^*, Sa^*) \geq s. \tag{2.54}$$

Now, we show that a^* is a fixed point for S. We claim that $q_s(a^*, Sa^*) = q_s(Sa^*, a^*) = 0$. On contrary, we assume that $q_s(a^*, Sa^*) > 0$. Now

$$q_s(a^*, Sa^*) \leq s(q_s(a^*, a_{2n+2}) + q_s(a_{2n+2}, Sa^*)). \tag{2.55}$$

Then, there exists $n_0 \in \mathbb{N}$ such as $q_s(a_{2n+2}, Sa^*) > 0$ for each $n \geq n_0$. By Lemma 2.1.1, $0 < q_s(a_{2n+2}, Sa^*) \leq H_{q_s}(Ja_{2n+1}, Sa^*)$, so

$$\max\{H_{q_s}(Ja_{2n+1}, Sa^*),\ H_{q_s}(Sa^*, Ja_{2n+1}),\ Q_s(a_{2n+1}, a^*),\ Q_s(a^*, a_{2n+1})\} > 0,$$

for all $n \geq n_0$. By Lemma 2.1.1, we have

$$\begin{aligned} & \tau + F(q_s(a_{2n+2}, Sa^*)) \leq \tau + F(H_{q_s}(Ja_{2n+1}, Sa^*)) \\ \leq\ & \tau + \max\{F(H_{q_s}(Sa^*, Ja_{2n+1})), F(H_{q_s}(Ja_{2n+1}, Sa^*))\}. \end{aligned}$$

By assumption, inequality (2.31) holds for a^*. Also $\rho_s^*(a^*, Sa^*) \geq s$ and $\rho_s^*(Sa_{2n+1}, a_{2n+1}) \geq s$, by (2.31), we have

$$\tau + F(q_s(a_{2n+2}, Sa^*)) \leq F(\mu_s(Q_s(a^*, a_{2n+1}))).$$

Since F is strictly increasing, we have

$$q_s(a_{2n+2}, Sa^*) < \mu_s(Q_s(a^*, a_{2n+1})).$$

Putting $\lim_{n\to+\infty}$ im the above inequality, we get

$$\lim_{n\to+\infty} q_s\left(a_{2n+2}, Sa^*\right) < \lim_{n\to+\infty} \mu_s\left(Q_s(a^*, a_{2n+1})\right). \tag{2.56}$$

Now,

$$\begin{aligned} Q_s(a^*, a_{2n+1}) &= \max\left\{q_s\left(a^*, a_{2n+1}\right), q_s\left(a^*, Sa^*\right), \right.\\ &\left.\frac{q_s\left(a^*, Sa^*\right) q_s\left(a^*, Ja_{2n+1}\right) + q_s\left(a_{2n+1}, Ja_{2n+1}\right) q_s\left(a_{2n+1}, Sa^*\right)}{q_s\left(a^*, Ja_{2n+1}\right) + q_s\left(a_{2n+1}, Sa^*\right)}\right\} \end{aligned}$$

$$\begin{aligned} \leq\ & \max\left\{q_s\left(a^*, a_{2n+1}\right), q_s\left(a^*, Sa^*\right), \right.\\ &\left.\frac{q_s\left(a^*, Sa^*\right) q_s\left(a^*, a_{2n+2}\right) + q_s\left(a_{2n+1}, a_{2n+2}\right) q_s\left(a_{2n+1}, Sa^*\right)}{q_s\left(a^*, Ja_{2n+1}\right) + q_s\left(a_{2n+1}, Sa^*\right)}\right\}. \end{aligned}$$

Putting $\lim_{n\to+\infty}$ in the above inequality, we get

$$\lim_{n\to+\infty} \left(Q_s(a^*, a_{2n+1})\right) \leq q_s\left(a^*, Sa^*\right).$$

Now, inequality (2.56) implies

$$\lim_{n\to+\infty} q_s\left(a_{2n+2}, Sa^*\right) < \mu_s\left(q_s\left(a^*, Sa^*\right)\right).$$

Putting limit as n tends to infinity on inequality (2.55) and using the above inequality, we have

$$q_s\left(a^*, Sa^*\right) < \mu_s\left(q_s\left(a^*, Sa^*\right)\right) < s\mu_s\left(q_s\left(a^*, Sa^*\right)\right).$$

As $s\mu_s(t) < t$, so our assumption is wrong and $q_s\left(a^*, Sa^*\right) = 0$. Now, assume that $q_s(Sa^*, a^*) > 0$, then there is some $n_1 \in \mathbb{N}$ such as $q_s(Sa^*, a_{2n+2}) > 0$ for all $n \geq n_1$. By Lemma 2.1.1, $0 < q_s\left(Sa^*, a_{2n+2}\right) \leq H_{q_s}(Sa^*, Ja_{2n+1})$, so

$$\max\left\{H_{q_s}\left(Ja_{2n+1}, Sa^*\right),\ H_{q_s}(Sa^*, Ja_{2n+1}),\ Q_s(a_{2n+1}, a^*),\ Q_s(a^*, a_{2n+1})\right\} > 0,$$

for all $n \geq n_1$. As inequality (2.31) hold for a^*, $\rho_s^*\left(a^*, Sa^*\right) \geq s$ and $\rho_s^*\left(Sa_{2n+1}, a_{2n+1}\right) \geq s$, then by Lemma 2.1.1 and (2.31), we have

$$\tau + F\left(q_s\left(Sa^*, a_{2n+2}\right)\right) \leq F(\mu_s\left(Q_s(a^*, a_{2n+1})\right)).$$

Since F is strictly increasing, we have

$$q_s\left(Sa^*, a_{2n+2}\right) < \mu_s\left(Q_s(a^*, a_{2n+1})\right).$$

Letting $\lim\limits_{n\to+\infty}$ in the above inequality, we get

$$\lim_{n\to+\infty} q_s\left(Sa^*, a_{2n+2}\right) < \lim_{n\to+\infty} \mu_s\left(Q_s(a^*, a_{2n+1})\right) < q_s\left(a^*, Sa^*\right) = 0.$$

Now,

$$q_s(Sa^*, a^*) \leq sq_s(Sa^*, a_{2n+2}) + sq_s(a_{2n+2}, a^*).$$

Letting $\lim\limits_{n\to+\infty}$ in the above inequality, we get

$$q_s(Sa^*, a^*) < 0.$$

Which is a contradiction, so $q_s(Sa^*, a^*) = 0$. Hence $a^* \in Sa^*$. As $\rho_s^*\left(a^*, Sa^*\right) \geq s$ and $q_s\left(a^*, Sa^*\right) = q_s\left(Sa^*, a^*\right) = q_s\left(a^*, a^*\right)$, then assumption (i) implies that $\rho_s^*\left(Sa^*, a^*\right) \geq s$. Now, following similar lines as above we obtain that a^* is a fixed point for J. Hence, $a^* \in Ja^*$. Hence, the pair (S, J) has a common fixed point a^* in $B_{q_s}(a_0, r)$. ■

2.3.3 Example

Let $L = [0, +\infty)$. Define $q_s : L \times L \to [0, +\infty)$ by $q_s\left(a, y\right) = \left(a + 2y\right)^2$, if $a \neq y$ and $q_s\left(a, y\right) = 0$, if $a = y$. Then (L, q_s) is a left (right) K-sequentially complete quasi b-metric with $s = 2$. Consider $\mathcal{R}$ is a binary relation on L defined by

$$\begin{aligned}\mathcal{R} \;&=\; \left\{(a, \frac{a}{5}) : a \in \{0, 1, \frac{1}{25}, \frac{1}{625}, ...\}\right\} \\ &\quad \cup \left\{(\frac{a}{5}, a) : a \in \{\frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\}\right\}.\end{aligned}$$

Consider μ_s a function on $[0,+\infty)$ defined by $\mu_s(t) = \frac{3t}{8}$. Define the pair of multivalued mappings $J, S : L \to P(L)$ by

$$Ja = \begin{cases} \left[\frac{a}{5}, \frac{a}{4}\right], \text{ if } a \in [0,1], \\ \left[4a^3, a^6+5\right], \text{ if } a \in (1,+\infty). \end{cases} \quad Sa = \begin{cases} \{\frac{a}{5}\}, \text{ if } a \in [0,1], \\ \left[a^4, a^7\right], \text{ if } a \in (1,+\infty). \end{cases}$$

Define $\rho : L \times L \to [0,+\infty)$ as follows:

$$\rho(a,y) = \begin{cases} 2, \text{ if } (a,y) \in \mathcal{R} \\ \frac{1}{6}, \text{ otherwise.} \end{cases}$$

$$\begin{aligned} A &= \{a : \rho_2^*(a, Sa) \geq 2\} = \left\{0, 1, \frac{1}{25}, \frac{1}{625}, ...\right\}. \\ B &= \{y : \rho_2^*(Sy, y) \geq 2\} = \left\{0, \frac{1}{5}, \frac{1}{125}, \frac{1}{3125}, ...\right\}. \end{aligned}$$

Let $a_0 = 1$ and $r = 49$, then $B_{q_s}(a_0, r) = [0, 3)$. Now,

$$\begin{aligned} G(S) &= \{a : \rho_2^*(a, Sa) \geq 2 \text{ and } a \in B_{q_s}(a_{0,}r)\} \\ &= \left\{0, 1, \frac{1}{25}, \frac{1}{625}, ...\right\}. \end{aligned}$$

Clearly $G(S)$ is closed and contains a_0. Therefore, the condition (ii) of Theorem 2.3.2 holds. As, $\frac{1}{5^{n-1}} \in B_{q_s}(a_0, r)$, for each $n \in \mathbb{N}$, we have

$$q_s(\frac{1}{5^{n-1}}, J\frac{1}{5^{n-1}}) = q_s(\frac{1}{5^{n-1}}, \frac{1}{5 \times 5^{n-1}}) \text{ and } q_s(J\frac{1}{5^{n-1}}, \frac{1}{5^{n-1}}) = q_s([\frac{1}{5 \times 5^{n-1}}, \frac{1}{5^{n-1}}).$$

Obvious, $\rho^*\left(\frac{1}{5^{n-1}}, S\frac{1}{5^{n-1}}\right) \geq 2$, for all $n \in \{1,3,5,...\}$ implies $\rho^*\left(S\frac{1}{5\times 5^{n-1}}, \frac{1}{5\times 5^{n-1}}\right) \geq 2$, for all $n \in \{1,3,5,...\}$. Also, $\rho^*\left(S\frac{1}{5^{n-1}}, \frac{1}{5^{n-1}}\right) \geq 2$, for all $n \in \{2,4,6,...\}$ implies $\rho^*\left(\frac{1}{5\times 5^{n-1}}, S\frac{1}{5\times 5^{n-1}}\right) \geq 2$, for all $n \in \{2,4,6,...\}$. Also, $0 \in B_{q_s}(a_0, r)$, $q_s(0, J0) = q_s(0,0)$, $q_s(J0, 0) = q_s(0,0)$ and $\rho^*(0, S0) \geq 2$ if and only if $\rho^*(S0, 0) \geq 2$. Therefore, the condition (i) of Theorem 2.3.2 hold. Now, for each $a, y \in B_{q_s}(a_0, r) \cap \{LJa_n\}$ with $\rho_2^*(Sy, y) \geq 2$, $\rho_2^*(a, Sa) \geq 2$. In general for n is odd, m is even, $n, m \in \mathbb{N}$

$$a = \frac{1}{5^{n-1}}, \ y = \frac{1}{5^{m-1}}.$$

Defined $F : [0, +\infty) \to R$ by the formula $F(a) = \ln(a)$ and $\tau \in (0, \frac{1}{58})$. After some calculation, it can easily be proved that (S, J) is a $F - \mu_s - \rho_s^*$ contraction on open ball. Hence, all the hypothesis of Theorem 2.3.2 hold. Hence, the pair (S, J) has a common fixed point 0.

2.3.4 Theorem

Consider (L, d) be a metric space and $S, J : L \to L$. Assume that the below hypothesis satisfy:

(i) the set $G = \{a \in L : \rho(a, Sa) \geq 1\}$ is closed and non-empty,

(ii) there exists a function $\mu \in \Psi$ such as for every $(a, y) \in L \times L$, $\rho(a, Sa) \geq 1$, $\rho(Sy, y) \geq 1$ implies $d(Sa, Jy) \leq \mu(d(a, y))$,

(iii) for every $a \in L$, we have $\rho(a, Sa) \geq 1$ implies $\rho(Ja, SJa) \geq 1$, and $\rho(Sa, a) \geq 1$ implies $\rho(SJa, Ja) \geq 1$. Then, for any $a_0 \in G$, the Picard sequence $\{J^n a_0\}$ converges to some $a^* \in L$ and a^* is a common fixed point of J and S.

2.3.5 Remarks

(i) By taking non-empty proper subsets of $Q_s(a, y)$ instead of $Q_s(a, y)$ in Theorem 2.3.2, we can obtain six different new results.

(ii) By taking non-empty proper subsets of $Q_s(a, y)$ instead of $Q_s(a, y)$ in Theorem 2.3.4, we can obtain six different new results.

Now, we achieve fixed point results for graphic F-μ_s-ρ_s^* contractions in quasi b-metric space.

2.3.6 Definition

Let (L, q_s) be a quasi b-metric space along with a graph G and $S, J : L \to P(L)$ are multivalued mappings. The pair (S, J) is called $F - \mu_s$-graphic contraction on the intersection of an open ball and a sequence, if $\mu_s \in \Psi$, $F \in \mathcal{F}$, $a_0 \in L$, $r, \tau > 0$, $a, y \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$, $\{(a, v) \in E(G) : v \in Sa\}$ and $\{(u, y) \in E(G) : u \in Sy\}$, $q_s(a, Jy) + q_s(y, Sa) \neq 0$ and $\max\{H_{q_s}(Sa, Jy), H_{q_s}(Jy, Sa), Q_s(a, y), Q_s(y, a)\} > 0$, then

$$\text{(i)} \quad \tau + \max\{F(H_{q_s}(Sa, Jy)), F(H_{q_s}(Jy, Sa))\} \leq F(\mu_s(Q_s(a, y))) \tag{2.57}$$

and if $q_s(a, Jy) + q_s(y, Sa) = 0$, then $\max\{H_{q_s}(Sa, Jy), H_{q_s}(Jy, Sa), Q_s(a, y), Q_s(y, a)\} = 0$.

$$\text{(ii)} \quad \sum_{p=0}^{j} s^{p+1}\left[\max\left\{\mu_s^p\left(q_s\left(a_1, a_0\right)\right), \mu_s^p\left(q_s\left(a_0, a_1\right)\right)\right\}\right] < r, \text{ for each } j \in \mathbb{N} \cup \{0\}. \tag{2.58}$$

2.3.7 Theorem

Consider (L, q_s) be a left K-sequentially complete quasi b-metric space along with graph G. Let $a_0 \in B_{q_s}(a_0, r)$, $r > 0$ and (S, J) is a $F - \mu_s$-graphic contraction on the intersection of a sequence and open ball. Suppose that the below assumptions are satisfied:

(i) if $a \in B_{q_s}(a_0, r)$, (a) $\{(a, v) \in E(G) : v \in Sa\}$, $q_s(a, Sa) = q_s(a, y)$ and $q_s(Sa, a) = q_s(y, a)$, then $\{(u, y) \in E(G) : u \in Sy\}$.

(b) $\{(v, a) \in E(G) : v \in Sa\}$, $q_s(a, Ja) = q_s(a, y)$

and $q_s(Ja, a) = q_s(y, a)$, then $\{(y, u) \in E(G) : u \in Sy\}$;

(ii) the set $A(S) = \{a : (a, v) \in E(G) \text{ for all } v \in Sa \text{ and } a \in B_{q_s}(a_0, r)\}$ is closed and contains a_0. Then, the subsequence $\{a_{2n}\}$ of $\{JS(a_n)\}$ is a sequence in $G(S)$ and $\{a_{2n}\} \to a^* \in G(S)$. Also, if inequality (2.57) satisfied for a^*. Then, J and S have a common fixed point a^* in $B_{q_s}(a_0, r)$.

Proof. Define $\rho : L \times L \to [0, +\infty)$ by $\rho(a, v) = s$, for all $v \in Sa$, and $a \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$ with $\{(a, v) \in E(G) : v \in Sa\}$. Also, $\rho(u, y) = s$, for each $u \in Sy$ and $y \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$ with $\{(u, y) \in E(G) : u \in Sy\}$. Moreover, $\rho(a, y) = 0$, for all other element of L. Now, as (S, J) is a $F - \mu_s$-graphic contraction. So inequality (2.57) implies inequality (2.31). Inequality (2.58) implies inequality (2.32). Assumption (i) of Theorem 2.3.7 implies assumption (i) of Theorem 2.3.2 and assumption (ii) of Theorem 2.3.7 implies assumption (ii) of Theorem 2.3.2. So, all assumptions of Theorem 2.3.2 hold. Hence the subsequence $\{a_{2n}\}$ of $\{JS(a_n)\}$ is a sequence in $A(S)$, for each $n \in \mathbb{N} \cup \{0\}$ and $\{a_{2n}\} \to a^* \in A(s)$. Also, if inequality (2.57) holds for a^*, then inequality (2.31) holds for a^*. Thus, J and S have a common fixed point a^* in $B_{q_s}(a_0, r)$. ∎

2.3.8 Theorem

Consider (L, d) be a complete metric space along with graph G and S, $J : L \to L$ are the self maps. Assume that the below assumptions satisfied:

(i) there exists a function $\mu \in \Psi$ such as for every $(a, y) \in L \times L$, $(a, Sa) \in E(G)$, $(Sy, y) \in E(G) \Rightarrow d(Sa, Jy) \leq \mu(d(a, y))$;

(ii) if $(a, Sa) \in E(G)$, then $(Ja, SJa) \in E(G)$ and if $(Sa, a) \in E(G)$, then $(SJa, Ja) \in E(G)$;

(iii) $G(S) = \{a : (a, Sa) \in E(G)\}$ is closed and non-empty.

Then, J and S have a common fixed point a^* in L.

Now, we will apply the various fixed point results on a complete left (right) K-sequentially quasi b-metric space endowed with a partial order as following.

2.3.9 Theorem

Consider $(L, \preceq)$ be a partial order set and (L, q_s) be a complete left (right) K-seqnentially quasi b-metric space. Let $a_0 \in L$, $r, \tau > 0$ and S, $J : L \to P(L)$ are the mappings on $B_{q_s}(a_0, r)$. Assume that there is some function $\mu_s \in \Psi$, F is strictly increasing mapping and suppose that:

(i) for each $(a, y) \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$ with $Sy \preceq y$ and $a \preceq Sa$, if $\max\{H_{q_s}(Sa, Jy), H_{q_s}(Jy, Sa), Q_s(a, y), Q_s(y, a)\} > 0$ and $q_s(a, Jy) + q_s(y, Sa) \neq 0$, we have

$$\tau + \max\{F(H_{q_s}(Sa, Jy)), F(H_{q_s}(Jy, Sa))\} < F(\mu_s(Q_s(a, y)))$$

where,

$$Q_s(a, y) = \max\left\{q_s(a, y), q_s(a, Sa), \frac{q_s(a, Sa)\, q_s(a, Jy) + q_s(y, Jy)\, q_s(y, Sa)}{q_s(a, Jy) + q_s(y, Sa)}\right\},$$

if $q_s(a, Jy) + q_s(y, Sa) = 0$, then

$$\max\{H_{q_s}(Sa, Jy),\ H_{q_s}(Jy, Sa),\ Q_s(a, y),\ Q_s(y, a)\} = 0.$$

(ii) $\sum\limits_{p=0}^{j} s^{p+1}\left[\max\{\mu^p q_s(a_1, a_0), \mu^p q_s(a_0, a_1)\}\right] < r$, for each $j \in \mathbb{N} \cup \{0\}$;

(iii) if $a \in B_{q_s}(a_0, r)$,

(a) $a \preceq Sa$, $q_s(a, Sa) = q_s(a, y)$ and $q_s(Sa, a) = q_s(y, a)$ implies $Sy \preceq y$;

(b) $Sa \preceq a$, $q_s(a, Ja) = q_s(a, y)$ and $q_s(Ja, a) = q_s(y, a)$ implies $y \preceq Sy$;

(iv) $G(S) = \{a : a \preceq Sa \text{ and } a \in B_{q_s}(a_0, r)\}$ contains a_0 and is closed. Then, the subsequence $\{a_{2n}\}$ of $\{JS(a_n)\} \in G(S)$ and $\{a_{2n}\} \to a^* \in G(S)$. Also, if assumption (i) is satisfied for a^*. Then J and S have a common fixed point a^* in $B_{q_s}(a_0, r)$.

Proof. Define $\rho : L \times L \to [0, +\infty)$, by $\rho(a, v) = s$, for all $v \in Sa$, where $a \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$ with $a \preceq Sa$. Also $\rho(u, y) = s$, for each $u \in Sy$, where $y \in B_{q_s}(a_0, r) \cap \{JS(a_n)\}$ with $y \succeq Sy$. Moreover $\rho(a, y) = 0$, for all other element of L. It is easy to see that assumptions (i), (ii), (iii) and (iv) of Theorem 2.3.9 implies inequality (2.31), inequality (2.32), assumption (i) and assumption (ii) of Theorem 2.3.2 respectively. So, all assumptions of Theorem 2.3.2 are satisfied. Thus, the subsequence $\{a_{2n}\}$ of $\{JS(a_n)\}$ is a sequence in $G(S)$, for each $n \in \mathbb{N} \cup \{0\}$ and a sequence $\{a_{2n}\} \to a^* \in G(s)$. Also, if assumption (i) holds for a^*, then inequality (2.31) satisfied for a^*. Hence, J and S have a common fixed point a^* in $B_{q_s}(a_0, r)$. ∎

2.3.10 Theorem

Consider $(L, \preceq, d)$ be an ordered metric space and $S, J : L \to L$ are the self mappings, suppose that:

(i) the set $G = \{a \in L : a \preceq Sa\}$ is closed and non-empty,

(ii) there exists $\mu \in \Psi$ such as for every $(a, y) \in L \times L$, $a \preceq Sa$, $y \succeq Sy \Rightarrow d(Sa, Jy) \leq \mu(d(a, y))$,

(iii) for every $a \in L$, we have $a \preceq Sa \Rightarrow Ja \succeq SJa$, $a \succeq Sa \Rightarrow Ja \preceq SJa$.

Then, for any $a_0 \in G$, the Picard sequence $\{J^n a_0\}$ converges to some $a \in L$ and a^* is a common fixed point.

Application to system of integral equations

Let S, $J : L \to L$ are two self maps and $a_0 \in L$. Let $a_1 = Sa_0$, $a_2 = Ja_1$, $a_3 = Sa_2$ and so on. In this way, we generate a sequence a_n in L such as

$$a_{2p+1} = Sa_{2p} \text{ and } a_{2p+2} = Ja_{2p+1}, \text{ (where } p = 0, 1, 2, \dots).$$

We say that a sequence $\{JS(a_n)\} \in L$ generated by a_0.

2.3.11 Definition

Let (L, q_s) be a left (right) K-sequentially complete quasi b-metric space and $S, J : L \to L$. S and J is called an $F - \mu_s$ contraction, if there is some $F \in \mathcal{F}$, $\tau > 0$, $a, y \in L$, $\max\{q_s(Sa, Jy), q_s(Jy, Sa), Q_s(a, y), Q_s(y, a)\} > 0$ and $q_s(a, Jy) + q_s(y, Sa) \neq 0$, then

$$\tau + \max\{F(q_s(Sa, Jy)), F(q_s(Jy, Sa))\} \leq F(\mu_s(Q_s(a, y))), \tag{2.59}$$

if $q_s(a, Jy) + q_s(y, Sa) = 0$, then

$$\max\{q_s(Sa, Jy),\ q_s(Jy, Sa),\ Q_s(a, y),\ Q_s(y, a)\} = 0$$

where,

$$Q_s(a, y) = \max\left\{q_s(a, y), q_s(a, Sa), \frac{q_s(a, Sa)\, q_s(a, Jy) + q_s(y, Jy)\, q_s(y, Sa)}{q_s(a, Jy) + q_s(y, Sa)}\right\}. \tag{2.60}$$

2.3.12 Theorem

Let (L, q_s) be a left (right) K-sequentially complete quasi b-metric space with a parameter $s \geq 1$ and (S, J) is an $F - \mu_s$ contraction. Then, $\{JS(a_n)\} \to a^* \in L$. Also, if a^* satisfies (2.59), then S and J have a unique common fixed point a^* in L.

Proof. We have only to prove a uniqueness. Let u is another common fixed point of S and J. If $\max\{q_s(Su, Ja^*), q_s(Ja^*, Su), Q_s(a^*, u), Q_s(u, a^*)\} \ngtr 0$, or if $q_s(a^*, Ju) + q_s(u, Sa^*) = 0$, then $q_s(Su, Ja^*) = 0$ and $q_s(Ja^*, Su) = 0$, which further implies $q_s(u, a^*) = q_s(a^*, u) = 0$ and hence $u = a^*$. Now, suppose $q_s(a^*, u) > 0$, then $\max\{q_s(Su, Ja^*), q_s(Ja^*, Su), Q_s(a^*, u), Q_s(u, a^*)\} > 0$ and $q_s(a^*, Ju) + q_s(u, Sa^*) \neq 0$. Then, we have

$$\begin{aligned} \tau + F(q_s(Su, Ja^*)) &\leq \tau + \max\{F(q_s(Su, Ja^*)), F(q_s(Ja^*, Su))\} \\ &\leq F(\mu_s(Q_s(a^*, u))). \end{aligned}$$

This implies that,

$$q_s(u, a^*) < \mu_s(q_s(u, a^*)) < s\mu_s(q_s(u, a^*))$$

which is contradiction. Then, we get $q_s(u, a^*) = 0$. Similarly we obtain $q_s(a^*, u) = 0$. Hence $a^* = u$. ■

Now, as an application, we discuss Theorem 2.3.12 to find solution of the system of Volterra type integral equations. Consider the following integral equations:

$$u(t) = \int_0^t K_1(t, s, u(s))ds, \tag{2.61}$$

$$v(t) = \int_0^t K_2(t, s, v(s))ds \tag{2.62}$$

for each $t \in [0, 1]$. We find a solution of (2.61) and (2.62). Let $L = C([0, 1], [0, +\infty[)$ is the set of all continuous functions on $[0, 1]$, endowed with the complete a left (right) K-sequentially quasi b-metric. For $u \in C([0, 1], [0, +\infty[)$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,1]} \{(u(t)) e^{-\tau t}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$q_\tau(u, v) = \left[\sup_{t\in[0,1]} \{(u(t) + 2v(t))e^{-\tau t}\}\right]^2 = \|u + 2v\|_\tau^2,$$

for each $u, v \in C([0, 1], [0, +\infty))$, with these settings, $(C([0, 1], [0, +\infty)), q_\tau)$ becomes a quasi b-metric space.

2.3.13 Theorem

Assume that the below conditions are hold:

(i) $K_1,\ K_2 : [I] \times [I] \times [0, +\infty[\to [0, +\infty[$;

(ii) Define

$$\begin{aligned} Ju(t) &= \int_0^t K(t, s, u(s))ds, \\ Jv(t) &= \int_0^t K(t, s, v(s))ds. \end{aligned}$$

Let, $\tau > 1$, such as

$$\max\{K(t,s,u) + 2K(t,s,v),\ K(t,s,v) + 2K(t,s,u)\} < \frac{\Im(u,v)}{\tau\Im(u,v)+1}, \tag{2.63}$$

for each $t, s \in [0,1]$ and $u, v \in C([0,1],[0,+\infty[)$, where

$$\Im(u,v) = \mu_s\left(\max\left\{\begin{array}{c} ||u+2v||^2, \|u+2Su\|^2, \\ \frac{\|u+2Su\|^2\|u+2Jv\|^2+\|v+2Jv\|^2\|v+2Su\|^2}{\|u+2Jv\|^2+\|v+2Su\|^2} \end{array}\right\}\right).$$

Then, the integral equation (2.61) and (2.62) have a unique common solution.

Proof. By condition (ii)

$$\begin{aligned} & |\max\{Su + 2Jv,\ Jv + 2Su\}| \\ = \ & \max\left\{\int_0^t (K(t,s,u) + 2K(t,s,v))ds,\ \int_0^t (K(t,s,v) + 2K(t,s,u))ds\right\} \\ < \ & \int_0^t \frac{\Im(u,v)}{\tau\Im(u,v)+1} e^{\tau s} ds < \frac{\Im(u,v)}{\tau\Im(u,v)+1}\int_0^t e^{\tau s} ds, \end{aligned}$$

$$\begin{aligned} |\max\{Su + 2Jv, Jv + 2Su\}| &< \frac{\Im(u,v)}{\tau\Im(u,v)+1} e^{\tau s}, \\ |\max\{Su + 2Jv, Jv + 2Su\}|\, e^{-\tau s} &< \frac{\Im(u,v)}{\tau\Im(u,v)+1}, \\ ||\max\{Su + 2Jv, Jv + 2Su\}||_\tau &< \frac{\Im(u,v)}{\tau\Im(u,v)+1}. \end{aligned}$$

This implies

$$\frac{\tau\Im(u,v)+1}{\Im(u,v)} < \frac{1}{||\max\{Ju + 2Jv, Jv + 2Ju\}||_\tau}.$$

That is,

$$\tau + \frac{1}{\Im(u,v)} < \frac{1}{||\max\{Su + 2Jv, Jv + 2Su\}||_\tau},$$

which further implies

$$\begin{aligned} \tau - \frac{1}{||\max\{Su + 2Jv, Jv + 2Su\}||_\tau} &< \frac{-1}{\Im(u,v)} \\ \tau + \max\{\frac{-1}{||Su + 2Jv||_\tau}, \frac{-1}{||Jv + 2Su||_\tau}\} &< \frac{-1}{\Im(u,v)}. \end{aligned}$$

Therefore, all assumptions of Theorem 2.3.13 are satisfied for $F(v) = \frac{-1}{\sqrt{v}}$, $v > 0$ and $q_\tau(u,v) = \|u + 2v\|_\tau^2$. Thus, integral equations given in (2.61) and (2.62) have a unique common solution. ■

Application to functional equations

We derive an application for the solution of a functional equation arising in dynamic programming. Consider U and V two Banach spaces, $P \subseteq U$, $Q \subseteq V$ and

$$\begin{aligned} f &: P \times Q \to P \\ g, u &: P \times Q \to \mathbb{R} \\ M, N &: P \times Q \times \mathbb{R} \to \mathbb{R}. \end{aligned}$$

For further results on dynamic programming, we refer to [33, 34, 74]. Assume that P and Q are the state and decision spaces, respectively. The problem related to dynamic programming is reduced to solve the following functional equations:

$$p(\gamma) = \sup_{\alpha \in Q}\{g(\gamma,\alpha) + M(\gamma,\alpha,p(f(\gamma,\alpha)))\}, \tag{2.64}$$

$$q(\gamma) = \sup_{\alpha \in Q}\{u(\gamma,\alpha) + N(\gamma,\alpha,q(f(\gamma,\alpha)))\}, \tag{2.65}$$

for $\gamma \in P$. We aim to give the existence and uniqueness of a common and bounded solution of equations (2.64) and (2.65). Suppose $B(P)$ is the set of all bounded real valued functions on P. Consider,

$$d_s(h,k) = \|h - k\|_{+\infty}^2 = \sup_{\gamma \in P} |h(\gamma) - k(\gamma)|^2, \tag{2.66}$$

for all $h, k \in B(P)$. Then $(B(P), d_s)$ is a quasi b-metric space. Assume that

$(C1)$: $M, N, g,$ and u are bounded.

$(C2)$: For $\gamma \in P$, $h \in B(P)$, $S, J : B(P) \to B(P)$, take

$$Sh(\gamma) = \sup_{\alpha \in Q}\{g(\gamma, \alpha) + M(\gamma, \alpha, h(f(\gamma, \alpha)))\}, \tag{2.67}$$

$$Jh(\gamma) = \sup_{\alpha \in Q}\{u(\gamma, \alpha) + N(\gamma, \alpha, h(f(\gamma, \alpha)))\}. \tag{2.68}$$

Moreover, for every $(\gamma, \alpha) \in P \times Q$, $h, k \in B(P)$, $t \in P$ and $\tau > 0$, implies

$$|M(\gamma, \alpha, h(t)) - N(\gamma, \alpha, k(t)| \leq D(h, k)e^{-\tau} \tag{2.69}$$

where,

$$D(h, k) = \mu_s \left(\max \left\{ \begin{array}{c} |h(t) - k(t)|^2, |h(t) - Sh(t)|^2, \\ \frac{|h(t)-Sh(t)|^2|h(t)-Tk(t)|^2+|k(t)-Tk(t)|^2|k(t)-Sh(t)|^2}{|h(t)-Tk(t)|^2+|k(t)-Sh(t)|^2} \end{array} \right\} \right).$$

2.3.14 Theorem

Assume that the conditions $(C1)$, $(C2)$ and (2.69) hold. Then equation (2.64) and (2.65) have a unique common and bounded solution in $B(P)$.

Proof. Take any $\lambda > 0$. By using definition of supremum in equation (2.67) and (2.68), there exist $h_1, h_2 \in B(P)$ and $\alpha_1, \alpha_2 \in Q$ such as

$$(Sh_1) < g(\gamma, \alpha_1) + M(\gamma, \alpha_1, h_1(f(\gamma, \alpha_1))) + \lambda, \tag{2.70}$$

$$(Jh_2) < g(\gamma, \alpha_2) + N(\gamma, \alpha_2, h_2(f(\gamma, \alpha_2))) + \lambda. \tag{2.71}$$

Again using definition of supremum, we have

$$(Sh_1) \geq g(\gamma, \alpha_2) + M(\gamma, \alpha_2, h_1(f(\gamma, \alpha_2))), \tag{2.72}$$

$$(Jh_2) \geq g(\gamma, \alpha_1) + N(\gamma, \alpha_1, h_2(f(\gamma, \alpha_1))). \tag{2.73}$$

Then equation (2.70) and (2.73) together with equation (2.69) implies

$$\begin{aligned}(Sh_1)(\gamma) - (Jh_2)(\gamma) &\leq M(\gamma, \alpha_1, h_1(f(\gamma, \alpha_1))) - N(\gamma, \alpha_1, h_2(f(\gamma, \alpha_1))) + \lambda \\ &\leq |M(\gamma, \alpha_1, h_1(f(\gamma, \alpha_1))) - N(\gamma, \alpha_1, h_2(f(\gamma, \alpha_1)))| + \lambda \\ &\leq D(h, k)e^{-\tau} + \lambda.\end{aligned}$$

Since, $\lambda > 0$ is arbitrary, we get

$$\begin{aligned}|Sh_1(\gamma) - Jh_2(\gamma)| &\leq D(h, k)e^{-\tau} \\ e^{\tau} |Sh_1(\gamma) - Jh_2(\gamma)| &\leq D(h, k).\end{aligned}$$

Which further implies that,

$$\tau + \ln |Sh_1(\gamma) - Jh_2(\gamma)| \leq \ln(D(h, k).$$

Therefore, all requirements of Theorem 2.3.14 hold for $F(g) = \ln g$, $g > 0$ and $d_\tau(h, k) = \|h - k\|_\tau^2$. Thus, there exists a common fixed point $h^* \in B(W)$ of J and S, that is, $h^*(\gamma)$ is a unique common solution of equations (2.64) and (2.65). ■

Chapter 3

Fixed Points and Common Fixed Points in Abstract Spaces

3.1 Introduction

Since Banach contraction principle has been exploited by number of researchers in different ways (see [44, 53, 61]), Meir and Keeler [70] are among the mathematicians whose idea got attention by many scholars and academicians. They demonstrated their contraction as, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such as $\varepsilon \leq d(a, y) < \varepsilon + \delta(\varepsilon)$ implies $d(Ja, Jy) < \varepsilon$ for any $a, y \in L$. Various authors [19, 54, 73, 80, 85] worked on the topic and found interesting extensions of their works. Rhodes and Jungck [53] set in motion the use of weakly compatibility and made a comparative analysis of weakly compatible and compatible maps. They claimed that the first one implies the later one but the converse does not hold. Moreover, since, the debate of common fixed point of two or more maps is under discussion since long, Patel *et al.* [73] investigated common fixed point theorems of α-admissible maps whenever there were four maps under consideration. In addition to the said ideas, Karapinar [59] used contractions that involve rational expressions and discussed the existence of a fixed point in metric space. There are also modified results of multi-valued F-contraction by Rasham *et al.* [79] that comprise of a pair of maps.

Section 3.2, following the pattern of Patel *et al.* [73] and examine common fixed point theorems for single-maps based on the merged $(\alpha - \psi)$-Meir-Keeler-Khan type contractive in complete metric space via α-admissible mappings. These results also reflect the idea presented by Redjel *et al.* [80]. In section 3.3, focuses on partial metric space and using $\alpha - \psi - K$-contractive maps, iterated fixed point which extend the results of Karapinar [59]. The same results are also deduced in ordered partial metric space. In section 3.4, searches for fixed point of a pair of multivalued maps accommodating certain impositions on the proximinal subsets of L. Modified versions of F-contraction on a sequence in dislocated b-quasi metric space and $F\rho_s^*$ Khan type contraction are stated while considering b-quasi metric space. The section imply application of the proved theorems to the solution of integral equations. Following are the results we intend to use in our research. Fisher [44] revised the Khan [61] idea with an improved version given below.

3.1.1 Theorem [44]

Consider J be a self mapping on a complete metric space (L, d) satisfied:

$$d(Ja, Jy) \leq \mu \frac{d\,(a, Ja)\,d\,(a, Jy) + d\,(y, Jy)\,d\,(y, Ja)}{d(a, Jy) + d(y, Ja)},\ \mu \in \,]0, 1[$$

if $d(a, Jy) + d(y, Ja) \neq 0$ and $d(Ja, Jy) = 0$, if $d(a, Jy) + d(y, Ja) = 0$. Then J has a unique fixed point $a^* \in L$. Moreover, for each $a_0 \in L$, the sequence $\{J^n a_0\}$ converges to a^*.

3.1.2 Lemma [7]

Let (L, d_b, s) be a b-metric space and $\{a_n\}$ be any sequence in L, there is some $\tau > 0$ and $F \in \mathcal{F}_S$ such as $\tau + F(sd_{qb}(a_n, a_{n+1})) \leq F(d_{qb}(a_{n-1}, a_n))$, $n \in \mathbb{N}$. Then $\{a_n\}$ is a Cauchy sequence in L.

3.2 Common Fixed Points for Generalized $(\alpha - \psi)$-Meir-Keeler-Khan Mappings in Metric Spaces

Results given in this section have been published in [13]

We introduced the class of common fixed point results for two pairs of weakly compatible self maps in complete metric space satisfies $(\alpha - \psi)$-Meir-Keeler-Khan type contractive via

α-admissible mappings.

3.2.1 Definition

Consider (L, d) be a complete metric space. The self-mappings $J, \Im, S, Q : L \to L$ are called $(\alpha - \psi)$-Meir-Keeler-Khan type, if there exists $\psi \in \Psi$ and $\alpha : J(L) \cup \Im(L) \times J(L) \cup \Im(L) \to [0, +\infty)$ satisfied the below condition:

For each $\varepsilon > 0$, there is some $\delta(\varepsilon) > 0$ such as,

$$\varepsilon \leq \psi\left(\frac{d(Ja, Sa)\, d(Ja, Qy) + d(\Im y, Qy)\, d(\Im y, Sa)}{d(Ja, Qy) + d(\Im y, Sa)}\right) < \varepsilon + \delta(\varepsilon)$$

implies

$$\alpha(Ja, \Im y)\, d(Sa, Qy) < \varepsilon. \tag{3.1}$$

3.2.2 Remark

It is easy to see that if $J, \Im, S, Q : L \to L$ be $(\alpha - \psi)$-Meir-Keeler-Khan type mappings, then

$$\alpha(Ja, \Im y)\, d(Sa, Qy) \leq \psi\left(\frac{d(Ja, Sa)\, d(Ja, Qy) + d(\Im y, Qy)\, d(\Im y, Sa)}{d(Ja, Qy) + d(\Im y, Sa)}\right), \text{ for all } a, y \in L. \tag{3.2}$$

3.2.3 Theorem

Consider (L, d) be a complete metric space and $J, \Im, S, Q : L \to L$ be an $(\alpha - \psi)$-Meir-Keeler-Khan type mappings such as $Q(L) \subseteq J(L)$ and $S(L) \subseteq \Im(L)$. Assume that:

(i) the pair (S, Q) is α-admissible with respect to J and $\Im$ (shortly $\alpha_{J,\Im}$-admissible);

(ii) there exists $a_0 \in L$ such as $\alpha(Ja_0, Sa_0) \geq 1$;

(iii) one of $J, \Im, S$ and Q is continuous.

(iv) (S, J) and $(Q, \Im)$ are weakly compatible pairs of self-mappings.

Then $J, \Im, S$ and Q have a common fixed point $u^* \in L$.

Proof. By hypothesis (ii), there is some $a_0 \in L$ such as $\alpha(Ja_0, Sa_0) \geq 1$. Define the sequences $\{a_n\}$ and $\{y_n\}$ in L such as

$$y_{2p} = Sa_{2p} = \Im a_{2p+1} \text{ and } y_{2p+1} = Qa_{2p+1} = Ja_{2p+2}. \tag{3.3}$$

This can be done, since $Q(L) \subseteq J(L)$ and $S(L) \subseteq \Im(L)$. Since (S,Q) is $\alpha_{J,\Im}$-admissible, we have

$$\alpha\left(Ja_0, Sa_0\right) = \alpha\left(Ja_0, \Im a_1\right) \geq 1 \text{ implies } \alpha\left(Sa_0, Qa_1\right) \geq 1 \text{ and } \alpha\left(Qa_0, Sa_1\right) \geq 1,$$

which gives,

$$\alpha\left(\Im a_1, Ja_2\right) \geq 1 = \alpha\left(y_0, y_1\right) \geq 1.$$

Again by (i), we have

$$\alpha\left(\Im a_1, Qa_1\right) = \alpha\left(\Im a_1, Ja_2\right) \geq 1 \text{ implies } \alpha\left(Qa_1, Sa_2\right) \geq 1 \text{ and } \alpha\left(Sa_1, Qa_2\right) \geq 1,$$

which gives,

$$\alpha\left(Ja_2, \Im a_3\right) = \alpha\left(y_1, y_2\right) \geq 1.$$

Inductively, we obtain

$$\alpha\left(y_{2p}, y_{2p+1}\right) \geq 1, \ p = 0, 1, 2, \ldots. \tag{3.4}$$

That is $\alpha\left(Ja_{2p}, \Im a_{2p+1}\right) \geq 1$ and $\alpha\left(\Im a_{2p+1}, Ja_{2p+2}\right) \geq 1$. By (3.2) and (3.4), we get

$$\begin{aligned}
d\left(y_{2p}, y_{2p+1}\right) &= d\left(Sa_{2p}, Qa_{2p+1}\right) \leq \alpha\left(Ja_{2p}, \Im a_{2p+1}\right) d\left(Sa_{2p}, Qa_{2p+1}\right) \\
&\leq \psi\left(\frac{d\left(Ja_{2p}, Sa_{2p}\right) d\left(Ja_{2p}, Qa_{2p+1}\right) + d\left(\Im a_{2p+1}, Qa_{2p+1}\right) d\left(\Im a_{2p+1}, Sa_{2p}\right)}{d\left(Ja_{2p}, Qa_{2p+1}\right) + d\left(\Im a_{2p+1}, Sa_{2p}\right)}\right) \\
&\leq \psi\left(\frac{d\left(Qa_{2p-1}, Sa_{2p}\right) d\left(Qa_{2p-1}, Qa_{2p+1}\right) + d\left(Sa_{2p}, Qa_{2p+1}\right) d\left(Sa_{2p}, Sa_{2p}\right)}{d\left(Qa_{2p-1}, Qa_{2p+1}\right) + d\left(Sa_{2p}, Sa_{2p}\right)}\right) \\
&\leq \psi\left(\frac{d\left(Qa_{2p-1}, Sa_{2p}\right) d\left(Qa_{2p-1}, Qa_{2p+1}\right)}{d\left(Qa_{2p-1}, Qa_{2p+1}\right)}\right) \leq \psi d\left(Qa_{2p-1}, Sa_{2p}\right) \\
&\leq \psi d\left(y_{2p-1}, y_{2p}\right), \text{ for all } p \in \mathbb{N}.
\end{aligned}$$

Now,

$$\begin{aligned} d\left(y_{2p-1}, y_{2p}\right) &= d\left(Qa_{2p-1}, Sa_{2p}\right) \leq \alpha\left(\Im a_{2p-1}, Ja_{2p}\right) d\left(Qa_{2p-1}, Sa_{2p}\right) \\ &\leq \psi\left(\frac{d\left(\Im a_{2p-1}, Qa_{2p-1}\right) d\left(\Im a_{2p-1}, Sa_{2p}\right) + d\left(Ja_{2p}, Sa_{2p}\right) d\left(Ja_{2p}, Qa_{2p-1}\right)}{d\left(\Im a_{2p-1}, Sa_{2p}\right) + d\left(Ja_{2p}, Qa_{2p-1}\right)}\right) \\ &\leq \psi\left(\frac{d\left(Sa_{2p-2}, Qa_{2p-1}\right) d\left(Sa_{2p-2}, Sa_{2p}\right) + d\left(Qa_{2p-1}, Sa_{2p}\right) d\left(Qa_{2p-1}, Qa_{2p-1}\right)}{d\left(Sa_{2p-2}, Sa_{2p}\right) + d\left(Qa_{2p-1}, Qa_{2p-1}\right)}\right) \\ &\leq \psi d\left(Sa_{2p-2}, Qa_{2p-1}\right) \leq \psi\left(y_{2p-2}, y_{2p-1}\right). \end{aligned}$$

That is,

$$d\left(y_{2p}, y_{2p+1}\right) \leq \psi d\left(y_{2p-1}, y_{2p}\right) \leq \psi^2 d\left(y_{2p-2}, y_{2p-1}\right).$$

Continuing in this manner, we obtain

$$d\left(y_{2p}, y_{2p+1}\right) \leq \psi^{2p} d\left(y_0, y_1\right).$$

We can write a above inequality as,

$$d\left(y_n, y_{n+1}\right) \leq \psi^n d\left(y_0, y_1\right).$$

Now, we show that $\{y_n\}$ is a Cauchy sequence. For each $\varepsilon > 0$ there is some $n(\varepsilon) \in \mathbb{N}$ with $\sum_{n \geq n(\epsilon)} \psi^n \left(d\left(y_0, y_1\right)\right) < \varepsilon$. Let $n, m \in \mathbb{N}$ such as $n > m > n(\varepsilon)$, by the triangle inequality, we get

$$d\left(y_m, y_n\right) \leq \sum_{k=m}^{n-1} d\left(y_k, y_{k+1}\right) \leq \sum_{k=m} \psi^k \left(d\left(y_0, y_1\right)\right) \leq \sum_{k=n(\epsilon)} \psi^k \left(d\left(y_0, y_1\right)\right) < \varepsilon.$$

Which shows that $\{y_n\}$ is a Cauchy sequence in a complete metric space (L, d). $\exists\, u^* \in L$ such as $\lim_{n\to+\infty} y_n = u^*$ and sequentially, Sa_{2n}, $\Im a_{2n+1}$, Qa_{2n+1}, $Ja_{2n+2} \to u^*$, as $n \to +\infty$. By assumption (iii)

$$\lim_{n\to+\infty} Sa_{2n} = \lim_{n\to+\infty} \Im a_{2n+1} = \lim_{n\to+\infty} Qa_{2n+1} = \lim_{n\to+\infty} Ja_{2n+2} = u^*.$$

As, $Q(L) \subseteq J(L)$, there exists $u \in L$ such as $u^* = Ju$. By using (3.2) and (3.4), we have

$$\begin{aligned}
d(Su,u^*) &\leq d(Su,Qa_{2n+1}) + d(Qa_{2n+1},u^*) \\
&\leq \alpha(Ju,\Im a_{2n+1})\, d(Su,Qa_{2n+1}) + d(Qa_{2n+1},u^*) \\
&\leq \psi\left(\frac{d(Ju,Su)\,d(Ju,Qa_{2n+1}) + d(\Im a_{2n+1},Qa_{2n+1})\,d(\Im a_{2n+1},Su)}{d(Ju,Qa_{2n+1}) + d(\Im a_{2n+1},Su)}\right) + d(Qa_{2n+1},u^*) \\
&\leq \psi\left(\frac{d(u^*,Su)\,d(u^*,Qa_{2n+1}) + d(Sa_{2n},Qa_{2n+1})\,d(Sa_{2n},Su)}{d(u^*,Qa_{2n+1}) + d(Sa_{2n},Su)}\right) + d(Qa_{2n+1},u^*).
\end{aligned}$$

Letting $\lim_{n\to+\infty}$ of above inequality, we get $d(Su,u^*) \leq 0$. Thus, $Su = u^*$. So, $Ju = Su = u^*$. Therefore, u is a coincidence point of J and S. Since the pair of mappings S and J are weakly compatible, we obtain

$$\begin{aligned}
SJu &= JSu, \\
Su^* &= Ju^*.
\end{aligned}$$

Since $S(L) \subseteq \Im(L)$, there exists a point $v \in L$ such as $u^* = \Im v$. By (3.2) and (3.4), we have

$$\begin{aligned}
d(u^*,Qv) &= d(Su,Qv) \leq \alpha(Ju,\Im v)\, d(Su,Qv) \\
&\leq \psi\left(\frac{d(u^*,u^*)\,d(u^*,Qv) + d(u^*,Qv)\,d(u^*,u^*)}{d(u^*,Qv) + d(u^*,u^*)}\right) \leq \psi(0).
\end{aligned}$$

That is $d(u^*,Qv) = 0$. Thus, $u^* = Qv$. Therefore, $Qv = \Im v = u^*$. So v is coincident point of $\Im$ and Q. As, the pair of maps $\Im$ and Q are weakly compatible

$$\begin{aligned}
\Im Qv &= Q\Im v, \\
\Im u^* &= Qu^*.
\end{aligned}$$

Now, we show that u^* is a fixed point of S. By (3.2) and (3.4), we have

$$\begin{aligned}
d(Su^*,u^*) &= d(Su^*,Qv) \leq \alpha(Ju^*,\Im v)\, d(Su^*,Qv) \\
&\leq \psi\left(\frac{d(Ju^*,Su^*)\,d(Ju^*,Qv) + d(\Im v,Qv)\,d(\Im v,Su^*)}{d(Ju^*,Qv) + d(\Im v,Su^*)}\right) = 0.
\end{aligned}$$

So, $d(Su^*,u^*) = 0$. Thus, $Su^* = u^*$. Therefore, $Su^* = Ju^* = u^*$. Now, we show that u^* is fixed point of Q. By using (3.2) and (3.4), we get

$$d(u^*, Qu^*) = d(Su^*, Qu^*) \leq \alpha(Ju^*, \Im u^*) d(Su^*, Qu^*) \leq \psi(0) = 0.$$

Thus, $d(u^*, Qu^*) = 0$. Therefore, $u^* = Qu^*$. So, $Qu^* = \Im u^* = u^*$. Thus, $Su^* = Ju^* = Qu^* = \Im u^* = u^*$.

Hence, u^* is a common fixed point of $J, \Im, S$ and Q. ■

3.2.4 Theorem

Let (L, d) be a complete metric space and $J, \Im, S, Q : L \to L$ are an $(\alpha - \psi)$-Meir-Keeler-Khan type mappings such as $Q(L) \subseteq J(L)$ and $S(L) \subseteq \Im(L)$. Assume that:

(i) the pair (S, Q) is $\alpha_{J,\Im}$-admissible;

(ii) there exists $a_0 \in L$ such as $\alpha(Ja_0, Sa_0) \geq 1$;

(iii) if $\{y_n\}$ is a sequence in L such as $\alpha(y_n, y_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $y_n \to u^* \in L$ as n tends to infinity, then $\alpha(y_n, u^*) \geq 1$, for each $n \in \mathbb{N}$.

Then, $J, \Im, S$ and Q have a common fixed point $u^* \in L$ provided (S, J) and $(Q, \Im)$ are weakly compatible pairs of self-mappings.

Proof. Similar lines of Theorem 3.2.3, we obtain a sequence $\{y_n\}$ in L defined by:

$$y_{2n} = Sa_{2n} = \Im a_{2n+1} \text{ and } y_{2n+1} = Qa_{2n+1} = Ja_{2n+2},$$

for all $n \geq 0$, which converges to some $u^* \in L$. Sequentially,

$$Sa_{2n}, \Im a_{2n+1},\ Qa_{2n+1},\ Ja_{2n+2} \to u^*,\ \text{as } n \to +\infty.$$

Since $Q(L) \subseteq J(L)$, there exists $u \in L$ such as $u^* = Ju$. By (iii) and (2.4), we get

$$\begin{aligned} d(Su, u^*) &= d(Su, Qa_{2n+1}) \leq \alpha(Ju, \Im a_{2n+1}) d(Su, Qa_{2n+1}) \\ &\leq \psi\left(\frac{d(Ju, Su) d(Ju, Qa_{2n+1}) + d(\Im a_{2n+1}, Qa_{2n+1}) d(\Im a_{2n+1}, Su)}{d(Ju, Qa_{2n+1}) + d(\Im a_{2n+1}, Su)}\right) \\ &\leq \psi\left(\frac{d(u^*, Su) d(u^*, Qa_{2n+1}) + d(Sa_{2n}, Qa_{2n+1}) d(Sa_{2n}, Su)}{d(u^*, Qa_{2n+1}) + d(Sa_{2n}, Su)}\right). \end{aligned}$$

Letting $\lim_{n\to+\infty}$, in above inequality, we have

$$d(Su, u^*) \leq 0$$

Thus, $Su = u^*$, so, $Ju = Su = u^*$, Therefore u is a coincidence point of J and S. As, the pair of mappings S and J are weakly compatible, we have

$$\begin{aligned} SJu &= JSu, \\ Su^* &= Ju^*. \end{aligned}$$

Similarly, as $S(L) \subseteq \Im(L)$, we get $d(u^*, Qv) = 0$. That, $u^* = Qv$. Therefore $Qv = \Im v = u^*$. So v is coincident point of $\Im$ and Q. As, the pair of maps $(\Im, Q)$ are weakly compatible so,

$$\begin{aligned} \Im Qv &= Q\Im v, \\ \Im u^* &= Qu^*. \end{aligned}$$

We can easily show that u^* is fixed point of S and Q and the proof is completed. ■

For a uniqueness of the fixed point of our results we considered the below condition:
(H) For all common fixed points a and y of $J, \Im, S$ and Q, there is some $v \in L$ such as $\alpha(a, v) \geq 1$ and $\alpha(y, v) \geq 1$.

3.2.5 Theorem

Adding the condition (H) to the statement of Theorem 3.2.3 or 3.2.4, we get the uniqueness of common fixed point of S, J, Q and $\Im$.

Proof. The existence of a fixed point has proved in Theorem 3.2.3 (respectively Theorem 3.2.4). Now, assume that $\hat{w}$ is another common fixed point of $J, \Im, S$ and Q such as $u^* \neq \hat{w}$. By using condition (H), there is some $v \in L$ such as $\alpha(Ju^*, v) \geq 1$ and $\alpha(\Im\hat{w}, v) \geq 1$. Define a sequence $\{v_p\}$ in L by

$$\begin{aligned} v_0 &= Sv_0 = \Im v_1\ ,\ v_{2p} = Sv_{2p} = \Im v_{2p+1} \text{ and} \\ v_1 &= Qv_1 = Jv_2\ , v_{2p+1} = Qv_{2p+1} = Jv_{2p+2}\ , \end{aligned}$$

for all $p \geq 0$. Since the pair (S, Q) is $\alpha_{J,\Im}$- admissible, we obtain

$$\alpha\left(u^*, v_{2p}\right) \geq 1 \text{ and } \alpha\left(\hat{w}, v_{2p}\right) \geq 1, \text{ for all } p.$$

Now, by using Remark 3.2.2, we have

$$\begin{aligned}
d\left(u^*, v_{2p+1}\right) &= d\left(Su^*, Qv_{2p+1}\right) \leq \alpha\left(Ju^*, \Im v_{2p+1}\right) d\left(Su^*, Qv_{2p+1}\right) \\
&\leq \psi\left(\frac{d\left(Ju^*, Su^*\right) d\left(Ju^*, Qv_{2p+1}\right) + d\left(\Im v_{2p+1}, Qv_{2p+1}\right) d\left(\Im v_{2p+1}, Su^*\right)}{d\left(Ju^*, Qv_{2p+1}\right) + d\left(\Im v_{2p+1}, Su^*\right)}\right) \\
&\leq \psi\left(\frac{d\left(Su^*, Su^*\right) d\left(Su^*, Qv_{2p+1}\right) + d\left(\Im v_{2p+1}, Qv_{2p+1}\right) d\left(\Im v_{2p+1}, Su^*\right)}{d\left(Su^*, Qv_{2p+1}\right) + d\left(\Im v_{2p+1}, Su^*\right)}\right).
\end{aligned}$$

By the triangle inequality

$$\begin{aligned}
d\left(\Im v_{2p+1}, Qv_{2p+1}\right) &\leq d\left(Su^*, Qv_{2p+1}\right) + d\left(\Im v_{2p+1}, Su^*\right) \\
&\leq \psi\left(\frac{d\left(\Im v_{2p+1}, Qv_{2p+1}\right) d\left(\Im v_{2p+1}, Su^*\right)}{d\left(Su^*, Qv_{2p+1}\right) + d\left(\Im v_{2p+1}, Su^*\right)}\right) \\
&\leq \psi d\left(\Im v_{2p+1}, Su^*\right) \leq \psi d\left(u^*, v_{2p}\right).
\end{aligned}$$

Iteratively, this inequality implies

$$d\left(u^*, v_{2p+1}\right) \leq \psi^{2p+1}\left(d\left(u^*, v_0\right)\right), \text{ for all } p.$$

Putting $p \to +\infty$, in above inequality, we get

$$\lim_{p \to +\infty} d\left(v_{2p}, u^*\right) = 0. \tag{3.5}$$

$$\lim_{p \to +\infty} d\left(v_{2p}, \hat{w}\right) = 0. \tag{3.6}$$

By the uniqueness of the limit (3.5), (3.6) we get $u^* = \hat{w}$. ■

The below example is to illustrate Theorem 3.2.3.

3.2.6 Example

Consider $L = [2, 20]$ and (L, d) be usual metric space. Define J, $\Im$, S and Q as follows:

$$S(a) = 2, \text{ for all } a. \ Q(a) = \begin{cases} 2, & \text{if } a \in [2,5) \cup [6,20], \\ a+1, & \text{if } a \in [5,6). \end{cases}$$

$$J(a) = \begin{cases} a, & \text{if } a \in [2,7], \\ 7, & \text{if } a \in (7,20]. \end{cases} \quad \Im(a) = \begin{cases} 2, & \text{if } a = 2 \\ 3, & \text{if } a \in (2,5) \cup [6,20], \\ a+3, & \text{if } a \in [5,6). \end{cases}$$

Indeed, $Q(L) \subseteq J(L)$ and $S(L) \subseteq \Im(L)$, we note $Sa = Ja$ for which $a = 2$ implies $SJa = JSa$ and $Qa = \Im a$ implies $Q\Im a = \Im Qa$. Thus the pairs$\{S, J\}$ and $\{Q, \Im\}$ are weakly compatible. Consider $\varepsilon = \frac{3}{4}$ and suppose that $\psi(t) = \frac{3t}{4}$, then $J, \Im, S$ and Q satisfy the $(\alpha - \psi)$-Meir-Keeler-Khan type contractive condition with the mapping $\alpha : J(L) \cup \Im(L) \times J(L) \cup \Im(L) \to \mathbb{R}^+$ which defined by

$$\alpha(a, y) = \begin{cases} 1, & \text{if } a, y \in [2,5) \cup [9,20], \\ \frac{1}{10}, & \text{otherwise.} \end{cases}$$

Clearly $a = 2$ is our unique common fixed point. Indeed, hypothesis (ii) is satisfied with $a_0 = 2 \in L$ with $\alpha(2, 2) \geq 1$. Thus, all the hypothesis of Theorem 3.2.3 are hold.

3.2.7 Corollary

Consider (L, d) be a complete metric space and $Q : L \to L$ is an $(\alpha - \psi)$-Meir-Keeler-Khan mapping. Assume that:

(i) Q is an α-admissibl and continuous;

(ii) there exists $a_0 \in L$ such as $\alpha(a_0, Q(a_0)) \geq 1$. Then, Q has a fixed point in L.

Proof. Immediately by taking $S = Q = \Im = J$ in the Theorem 3.2.3. ■

3.2.8 Corollary

Consider (L, d) be a complete metric space and let $Q : L \to L$ is an (α, ψ)-Meir-Keeler-Khan map. Assume that:

(i) there exists $a_0 \in L$ such as $\alpha(a_0, Q(a_0)) \geq 1$;

(ii) Q is an α-admissible;

(iii) if $\{a_n\}$ is a sequence in L such as $\alpha(a_n, a_{n+1}) \geq 1$ for each $n \in \mathbb{N}$ and $a_n \to a \in L$ as n tends to infinity, then $\alpha(a_n, a) \geq 1$, for each $n \in \mathbb{N}$. Then, Q has a fixed point in L.

In the Theorem 3.2.4, if we take $\psi(t) = \lambda t$, where $\lambda \in (0,1)$ and $\alpha(Ja, \Im y) = 1$, for each $a, y \in L$, we obtain the below results.

3.2.9 Corollary

Consider (L, d) be a complete metric space. Let J, $\Im$, S, $Q : L \to L$ be the mappings satisfies the below condition:

For $\varepsilon > 0$, there exist $\acute{\delta} > 0$ such as,

$$\varepsilon \leq \lambda \left(\frac{d(Ja, Sa)\, d(Ja, Qy) + d(\Im y, Qy)\, d(\Im y, Sa)}{d(Ja, Qy) + d(\Im y, Sa)} \right) < \varepsilon + \acute{\delta} \tag{3.7}$$

$$\Rightarrow d(Sa, Qy) < \varepsilon.$$

Then, $J, \Im, Q$ and S have a unique common fixed point $u^* \in L$. Moreover, for each a_0 the sequence $\{Q^n a_0\}$ converge to u^*.

Proof. Let $\mu \in]0,1[$ and choose $\lambda_0 \in]0,1[$ with $\lambda_0 > \mu$. Fix $\varepsilon > 0$. If we take $\acute{\delta} = \varepsilon \left(\frac{1}{\mu} - \frac{1}{\lambda_0} \right)$. Assume that

$$\frac{1}{\lambda_0}\varepsilon \leq \frac{d(Ja, Sa)\, d(Ja, Qy) + d(\Im y, Qy)\, d(\Im y, Sa)}{d(Ja, Qy) + d(\Im y, Sa)} < \frac{1}{\lambda_0}\varepsilon + \acute{\delta},$$

it following that

$$d(Sa, Qy) < \mu \frac{d(Ja, Sa)\, d(Ja, Qy) + d(\Im y, Qy)\, d(\Im y, Sa)}{d(Ja, Qy) + d(\Im y, Sa)}$$

$$< \mu \left(\frac{1}{\lambda_0}\varepsilon + \acute{\delta} \right)$$

$$= \mu \left(\frac{1}{\lambda_0}\varepsilon + \varepsilon \left(\frac{1}{\mu} - \frac{1}{\lambda_0} \right) \right) = \varepsilon.$$

Hence (3.7) is satisfied which makes Theorem 3.1.1 an immediate consequence of Corollary 3.2.9. ■

Following the idea of Samet [85], according of Corollary 3.2.9 we obtain an integral version for Fisher's result. Now, we start by the below result.

3.2.10 Theorem

Consider (L,d) be a complete metric space and $J,\Im,S,Q : L \to L$, $\lambda \in]0,1[$. Suppose that there exists $\rho : [0,+\infty[\to [0,+\infty[$ satisfied the below assumptions;

(i) ρ is nondecreasing and right continuous;

(ii) $\rho(0)=0$ and $\rho(t)>0$ for all $t>0$;

(iii) for each $\varepsilon>0$, there exist $\acute{\delta}>0$ such as

$$\frac{1}{\lambda}\varepsilon < \rho\left(\frac{d(Ja,Sa)\,d(Ja,Qy)+d(\Im y,Qy)\,d(\Im y,Sa)}{d(Ja,Qy)+d(\Im y,Sa)}\right) < \frac{1}{\lambda}\varepsilon+\acute{\delta}$$

implies $\rho\left(\frac{1}{\lambda}d(Sa,Qy)\right) < \frac{1}{\lambda}\varepsilon$, for all $a,y\in L$. Then inequality (3.7) hold.

Proof. Fix $\varepsilon>0$, since $\rho\left(\frac{1}{\lambda}\varepsilon\right)>0$, by assumption (iii) there exists $\beta>0$ such as

$$\rho\left(\frac{1}{\lambda}\varepsilon\right) < \rho\left(\frac{d(Ja,Sa)\,d(Ja,Qy)+d(\Im y,Qy)\,d(\Im y,Sa)}{d(Ja,Qy)+d(\Im y,Sa)}\right) < \rho\left(\frac{1}{\lambda}\varepsilon\right)+\beta$$

$$\text{implies } \rho\left(\frac{1}{\lambda}d(Sa,Qy)\right) < \rho\left(\frac{1}{\lambda}\varepsilon\right). \tag{3.8}$$

Since ρ is right continuous, there exists $\acute{\delta}>0$ such as;

$$\rho\left(\frac{1}{\lambda}\varepsilon+\acute{\delta}\right) < \rho\left(\frac{1}{\lambda}\varepsilon\right)+\beta.$$

For all $a,y\in L$, such as

$$\frac{1}{\lambda}\varepsilon < \frac{d(Ja,Sa)\,d(Ja,Qy)+d(\Im y,Qy)\,d(\Im y,Sa)}{d(Ja,Qy)+d(\Im y,Sa)} < \frac{1}{\lambda}\varepsilon+\acute{\delta}.$$

Since ρ is nondecreasing, we have

$$\begin{aligned}\rho\left(\frac{1}{\lambda}\varepsilon\right) &< \rho\left(\frac{d(Ja,Sa)\,d(Ja,Qy)+d(\Im y,Qy)\,d(\Im y,Sa)}{d(Ja,Qy)+d(\Im y,Sa)}\right) \\ &< \rho\left(\frac{1}{\lambda}\varepsilon+\acute{\delta}\right) < \rho\left(\frac{1}{\lambda}\varepsilon\right)+\beta.\end{aligned}$$

Then by (3.8), we have

$$\rho\left(\frac{1}{\lambda}d(Sa,Qy)\right) < \rho\left(\frac{1}{\lambda}\varepsilon\right),$$

which implies that $d(Sa,Qy) < \varepsilon$. Then (3.7) is satisfied. ■

Now, Ξ denote the set of all map $g : [0,+\infty[\to [0,+\infty[$ satisfied the below assumptions:

(i) $g(0) = 0$ and $g(t) > 0$ for all $t > 0$;

(ii) g continuous and nondecreasing.

3.2.11 Corollary

Let (L,d) be a complete metric space and $J,\Im,S,Q : L \to L$ are mappings, let $g \in \Xi$ such as for $\varepsilon > 0$ there exist $\acute{\delta} > 0$, with

$$\frac{1}{\lambda}\varepsilon < g\left(\frac{d(Ja,Sa)\,d(Ja,Qy)+d(\Im y,Qy)\,d(\Im y,Sa)}{d(Ja,Qy)+d(\Im y,Sa)}\right) < \frac{1}{\lambda}\varepsilon+\acute{\delta}$$

$$\text{implies } g\left(\frac{1}{\lambda}d(Sa,Qy)\right) < \frac{1}{\lambda}\varepsilon.$$

Then, (3.7) is satisfied.

Proof. Since every continuos function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is right continuous, the proof follows immediately from Theorem 3.2.10. ■

3.2.12 Corollary

Consider (L,d) be a complete metric space and J, $\Im$, S and Q are four maps from L into itself. Let $\varphi : [0,+\infty[\to [0,+\infty[$ is a locally integrable function such as

$$\int_0^t \varphi(u)\,du > 0, \text{ for all } t > 0.$$

Assume that $\varepsilon > 0$, there exist $\acute{\delta} > 0$ such as

$$\frac{1}{\lambda}\varepsilon \leq \int_0^{\frac{d(Ja,Sa)d(Ja,Qy)+d(\Im y,Qy)d(\Im y,Sa)}{d(Ja,Qy)+d(\Im y,Sa)}} \varphi(t)\,dt < \frac{1}{\lambda}\varepsilon + \acute{\delta}$$

$$\text{implies } \int_0^{\frac{1}{\lambda}d(Sa,Qy)} \varphi(t)\,dt < \frac{1}{\lambda}\varepsilon. \tag{3.9}$$

Thus, (3.7) is satisfied. Now, we obtain an integral version of Khan result as follows.

3.2.13 Corollary

Consider (L,d) be a complete metric space and $J, \Im, S, Q : L \to L$ be self mappings. Let $\varphi : [0,+\infty[\to [0,+\infty[$ be a locally integrable function such as $\int_0^t \varphi(u)\,du > 0$, for each $t > 0$ and let $\lambda \in]0,1[$. Assume that $J, \Im, S$ and Q satisfied the below condition. For all $a, y \in L$,

$$\int_0^{\frac{1}{\lambda}d(Sa,Qa)} \varphi(t)\,dt \leq \acute{\mu} \int_0^{\frac{d(Ja,Sa)d(Ja,Qy)+d(\Im y,Qy)d(\Im y,Sa)}{d(Ja,Qy)+d(\Im y,Sa)}} \varphi(t)\,dt, \tag{3.10}$$

where, $\acute{\mu} \in (0,1)$. Then $J, \Im, S$ and Q have an unique common fixed point. Moreover, for any $a \in L$, the sequence $\{y^n(a)\}$ converges to a^*.

Proof. Let $\varepsilon > 0$. it is easy to observe that (3.9) is satisfied. Take $\acute{\delta} = \frac{\varepsilon}{\lambda}\left(\frac{1}{\acute{\mu}} - 1\right)$, then (3.7) is satisfied. ■

3.3 Some Fixed Points of $\alpha - \psi - K$-Contractive Mapping in Partial Metric Spaces

Results given in this section have been published in [14]

We are starting to introducing the generalized of $\alpha - \psi - K$-contractive in the context of partial metric space as follows:

3.3.1 Definition

Let (L,p) be a partial metric space and $J : L \to L$ is a map. J is called $\alpha - \psi - K$-contractive if there exists functions $\alpha : L \times L \to [0,+\infty)$ and $\psi \in \Psi$ such as for each $a, y \in L$, $a \neq y$, we have

$$\alpha(a,y)p(J(a),J(y)) \leq \psi(K(a,y)) \tag{3.11}$$

whenever,

$$K(a,y) = \max\left\{p(a,y), \frac{p(a,Ja)+p(y,Jy)}{2}, \frac{p(a,Jy)+p(y,Ja)}{2}, \right.$$
$$\left. \frac{p(a,Ja)\,p(y,Jy)}{p(a,y)}, \frac{p(y,Jy)\,[1+p(a,Ja)]}{[1+p(a,y)]}\right\}.$$

3.3.2 Theorem

Let (L,p) be a complete partial metric space and $J : L \to L$ be $\alpha - \psi - K$-contractive map satisfies the below hypothesis:

(i) J is α-admissible;

(ii) there exists $a_0 \in L$ such as $\alpha(a_0, Ja_0) \geq 1$.

(iii) J is a continuous. Then $u \in L$ such as $J(u) = u$.

Proof. From hypothesis (ii), there exists $a_0 \in L$ such as $\alpha(a_0, Ja_0) \geq 1$. We generate a sequence $\{a_n\}$ in L as follows: $a_{n+1} = Ja_n$ for each $n \in \mathbb{N} \cup \{0\}$. If $a_{n0} = a_{n0+1}$ for some $n_0 \in \mathbb{N}_0$, then $u = a_{n_0}$ is a fixed point of J. Suppose that $a_n \neq a_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. By (i), we have

$$\begin{aligned}
\alpha(a_0,a_1) &= \alpha(a_0,Ja_0) \geq 1 \Rightarrow \alpha(Ja_0,Ja_1) = \alpha(a_1,a_2) \geq 1, \\
\alpha(a_1,a_2) &= \alpha(a_1,Ja_1) \geq 1 \Rightarrow \alpha(Ja_1,Ja_2) = \alpha(a_2,a_3) \geq 1.
\end{aligned}$$

By induction, we get

$$\alpha(a_n, a_{n+1}) \geq 1. \tag{3.12}$$

From (3.11) and (3.12), we have

$$p(a_{n+1},a_{n+2}) = p(Ja_n,Ja_{n+1}) \leq \alpha(a_n,a_{n+1})\,p(Ja_n,Ja_{n+1}) \leq \psi(K(a_n,a_{n+1}))$$

$$\leq \psi \max \left\{ p\left(a_n, a_{n+1}\right), \frac{p\left(a_n, Ja_n\right) + p\left(a_{n+1}, Ja_{n+1}\right)}{2}, \frac{p\left(a_n, Ja_{n+1}\right) + p\left(a_{n+1}, Ja_n\right)}{2}, \right.$$

$$\left. \frac{p\left(a_n, Ja_n\right) p\left(a_{n+1}, Ja_{n+1}\right)}{p\left(a_n, a_{n+1}\right)}, \frac{p\left(a_{n+1}, Ja_{n+1}\right)\left[1 + p\left(a_n, Ja_n\right)\right]}{\left[1 + p\left(a_n, a_{n+1}\right)\right]} \right\}$$

$$= \psi \max \left\{ p\left(a_n, a_{n+1}\right), \frac{p\left(a_n, a_{n+1}\right) + p\left(a_{n+1}, a_{n+2}\right)}{2}, \frac{p\left(a_n, a_{n+2}\right) + p\left(a_{n+1}, a_{n+1}\right)}{2}, \right.$$

$$\left. \frac{p\left(a_n, a_{n+1}\right) p\left(a_{n+1}, a_{n+2}\right)}{p\left(a_n, a_{n+1}\right)}, \frac{p\left(a_{n+1}, a_{n+2}\right)\left[1 + p\left(a_n, a_{n+1}\right)\right]}{\left[1 + p\left(a_n, a_{n+1}\right)\right]} \right\}$$

$$\leq \psi \max \left\{ p\left(a_n, a_{n+1}\right), p\left(a_{n+1}, a_{n+2}\right) \right\},$$

$$p\left(a_{n+1}, a_{n+2}\right) \leq \psi\left(\max \left\{ p\left(a_n, a_{n+1}\right), p\left(a_{n+1}, a_{n+2}\right) \right\}\right), \text{ for all } n.$$

If $\max \left\{ p\left(a_n, a_{n+1}\right), p\left(a_{n+1}, a_{n+2}\right) \right\} = p\left(a_{n+1}, a_{n+2}\right)$, then

$$p\left(a_{n+1}, a_{n+2}\right) \leq \psi\left(p\left(a_{n+1}, a_{n+2}\right)\right) < p\left(a_{n+1}, a_{n+2}\right).$$

Which is contradiction. Thus, $\max \left\{ p\left(a_n, a_{n+1}\right), p\left(a_{n+1}, a_{n+2}\right) \right\} = p\left(a_n, a_{n+1}\right)$, for all $n \in \mathbb{N}$. Hence,

$$p\left(a_{n+1}, a_{n+2}\right) \leq \psi\left(p\left(a_n, a_{n+1}\right)\right). \tag{3.13}$$

Continuing in this process inductively, we obtain

$$p\left(a_n, a_{n+1}\right) \leq \psi^n\left(p\left(a_0, a_1\right)\right), \text{ for each } n \in \mathbb{N}. \tag{3.14}$$

Using the definition of partial metric, we have

$$\max \left\{ p\left(a_n, a_n\right), p\left(a_{n+1}, a_{n+1}\right) \right\} \leq p\left(a_n, a_{n+1}\right). \tag{3.15}$$

Using inequality (3.14), we have

$$\max \left\{ p\left(a_n, a_n\right), p\left(a_{n+1}, a_{n+1}\right) \right\} \leq \psi^n\left(a_0, a_1\right). \tag{3.16}$$

By (3.15) and (3.16), we have

$$\begin{aligned}p^s(a_n, a_{n+1}) &= 2p(a_n, a_{n+1}) - p(a_n, a_n) - p(a_{n+1}, a_{n+1}) \\ &\leq 2p(a_n, a_{n+1}) + p(a_n, a_n) + p(a_{n+1}, a_{n+1}) \leq 4\psi^n(a_0, a_1). \qquad (3.17)\end{aligned}$$

Using inequality (3.17), we get

$$\begin{aligned}p^s(a_{n+k}, a_n) &\leq p^s(a_{n+k}, a_{n+k-1}) + ... + p^s(a_{n+1}, a_n) \\ &\leq 4\psi^{n+k-1}p(a_0, a_1) + ... + 4\psi^n p(a_0, a_1) \leq 4\sum_{i=n}^{n+k-1}\psi^i p(a_0, a_1).\end{aligned}$$

As $\sum_{i=0}^{+\infty}\psi^i p(a_0, a_1)$ is a convergent. We obtain that $\{a_n\}$ is a Cauchy sequence in a metric space (L, p^s). Now, by using Lemma 1.3.8 and the completeness of (L, p), we conclude the completeness of (L, p^s). Therefore the sequence $\{a_n\}$ is a convergent in (L, p^s), say $\lim_{n\to+\infty} p^s(a_n, u) = 0$. By Lemma 1.3.8, we have

$$p(u, u) = \lim_{n\to+\infty} p(a_n, u) = \lim_{m,n\to+\infty} p(a_m, a_n). \qquad (3.18)$$

Now, since $\{a_n\}$ is a Cauchy sequence in (L, p^s), we get

$$\lim_{m,n\to+\infty} p^s(a_m, a_n) = 0. \qquad (3.19)$$

View of inequality (3.16), we have

$$\lim_{n\to+\infty} p(a_n, a_n) = 0. \qquad (3.20)$$

From (3.19), (3.20) and definition of p^s, we conclude that

$$\lim_{m,n\to+\infty} p(a_m, a_n) = 0.$$

On using (3.18), we have

$$p(u, u) = \lim_{n\to+\infty} p(a_n, u) = \lim_{m,n\to+\infty} p(a_m, a_n) = 0. \qquad (3.21)$$

Now, we show that $Ju = u$. By Lemma 1.3.9, we get

$$p(Ju, Ju) = \lim_{n\to+\infty} p(Ja_n, Ju) = \lim_{m,n\to+\infty} p(Ja_m, Ja_n). \tag{3.22}$$

That is,

$$p(Ju, Ju) = \lim_{m,n\to+\infty} p(a_{m+1}, a_{n+1}) = 0. \tag{3.23}$$

Using inequality (3.21) and (3.23), we get

$$p(u, u) = p(Ju, Ju) = 0. \tag{3.24}$$

By Lemma 1.3.10, we have

$$\lim_{n\to+\infty} p(a_n, Ju) = p(u, Ju). \tag{3.25}$$

Therefore, using (3.22) , (3.24) and (3.25), we have

$$p(Ju, Ju) = p(u, u) = p(u, Ju) = 0.$$

Thus $u = Ju$. Hence u is a fixed point of J. ■

3.3.3 Theorem

Consider (L, p) is a complete partial metric space and $J : L \to L$ is an $\alpha - \psi - K$-contractive map. Suppose that:

(i) there exists $a_0 \in L$ such as $\alpha(a_0, J(a_0)) \geq 1$;

(ii) J is α-admissible;

(iii) if $\{a_n\}$ is a sequence in L such as $\alpha(a_n, a_{n+1}) \geq 1$, for each $n \in \mathbb{N}$ and $a_n \to a \in L$ as n tends to infinity, then there exists a subsequence $\left\{a_{n(\hat{\jmath})}\right\}$ such as $\alpha\left(a_{n(\hat{\jmath})}, a\right) \geq 1$, for each $\hat{\jmath} \in \mathbb{N}$. Then, J has a fixed point in L.

Proof. Following the similar lines of the Theorem 3.3.2, we obtain that a sequence $\{a_n\}$ given by $a_{n+1} = Ja_n$, where $n =$ 0,1,2,... converges to some $u \in L$. From (3.12) and given hypotheses, there exists a subsequence $\left\{a_{n(\hat{\jmath})}\right\}$ of $\{a_n\}$ such as

$$\alpha\left(a_{n(\hat{\jmath})}, a\right) \geq 1, \text{ for all } \hat{\jmath}. \tag{3.26}$$

Now, we proceed to prove that u is a fixed point of J. Suppose the contrary $p(u, Ju) > 0$. Therefore, by using (3.11) and (3.26), we have

$$\begin{aligned} p(u, Ju) &\leq p(u, a_{n(\hat{\jmath})+1}) + p(a_{n(\hat{\jmath})+1}, Ju) - p(a_{n(\hat{\jmath})+1}, a_{n(\hat{\jmath})+1}) \\ &\leq p(u, a_{n(\hat{\jmath})+1}) + p(a_{n(\hat{\jmath})+1}, Ju) = p(u, a_{n(\hat{\jmath})+1}) + p(Ja_{n(\hat{\jmath})}, Ju) \\ &\leq p(u, a_{n(\hat{\jmath})+1}) + \alpha(a_{n(\hat{\jmath})}, u)p(Ja_{n(\hat{\jmath})}, Ju) \leq p(u, a_{n(\hat{\jmath})+1}) + \psi\left(K(a_{n(\hat{\jmath})}, u)\right). \end{aligned}$$

$$p(u, Ju) \leq p(u, a_{n(\hat{\jmath})+1}) + \psi \max\left\{p\left(a_{n(\hat{\jmath})}, u\right), \frac{p\left(a_{n(\hat{\jmath})}, a_{n(\hat{\jmath})+1}\right) + p(u, Ju)}{2},\right.$$

$$\left.\frac{p\left(a_{n(\hat{\jmath})}, Ju\right) + p\left(u, a_{n(\hat{\jmath})+1}\right)}{2}, \frac{p\left(a_{n(\hat{\jmath})}, a_{n(\hat{\jmath})+1}\right) p\left(u, Ju\right)}{p\left(a_{n(\hat{\jmath})}, u\right)}, \frac{p\left(u, Ju\right)\left[1 + p\left(a_{n(\hat{\jmath})}, a_{n(\hat{\jmath})+1}\right)\right]}{\left[1 + p\left(a_{n(\hat{\jmath})}, u\right)\right]}\right\}. \tag{3.27}$$

Letting limit $\hat{\jmath}$ tends to infinity in (3.27), we get

$$p(u, Ju) \leq \psi\left(p(u, Ju)\right) < p(u, Ju).$$

Which is a contradiction. Therefore, $p(u, Ju) = 0$. Hence $Ju = u$. ■

3.3.4 Corollary [59]

Consider (L, d) be a complete metric space and $J : L \to L$ is an $\alpha - \psi - K$-contractive map. Assume that:

(i) J is an α-admissible and continuous;

(ii) there exists $a_0 \in L$ such as $\alpha\left(a_0, J\left(a_0\right)\right) \geq 1$. Then, J has a fixed point in L.

3.3.5 Corollary [89]

Consider (L, p) be a complete partial metric space, $\alpha : L \times L \to [0, +\infty)$ be an a function, $\alpha \in \Psi$ and J is generalized $\alpha - \psi$ contractive type mapping on L. Suppose that:

(i) J is an α-admissible and continuous;

(ii) there exists $a_0 \in L$ such as $\alpha(a_0, J(a_0)) \geq 1$. Then, J has a fixed point in L.

3.3.6 Corollary [57]

Consider (L, d) be a complete metric space. Let $J : L \to L$ is a mapping. Suppose that there exists $\psi \in \Psi$ such as

$$d(J(a), J(y)) \leq \psi(M(a, y)),$$

for each $(a, y) \in L \times L$, where

$$M(a, y) = \max\left\{d(a, y), \frac{d(a, Ja) + d(y, Jy)}{2}, \frac{d(a, Jy) + d(y, Ja)}{2}\right\}.$$

Then, J has a unique fixed point.

3.3.7 Example

Let $L = \mathbb{R}^+$ where, (L, p) is a complete partial metric space with partial metric defined by $p(a, y) = \max\{a, y\}$. Defined $J : L \to L$ by

$$J(a) = \begin{cases} \frac{a}{6}, \text{ if } a \in [0, 1], \\ \frac{a}{3} - \frac{1}{6}, \text{ if } a \in (1, 2], \\ a - \frac{3}{2}, \text{ if } a > 2. \end{cases} \quad \text{Let } \alpha(a, y) = \begin{cases} 2, \text{ if } a, y \in [0, 1], \\ \frac{4}{3}, \text{ if } a, y \in (1, 2], \\ 0, \text{ otherwise.} \end{cases}$$

Clearly J is a continuous for all $a \in L$. To show that J is α-admissible: Let $a, y \in L$ such as $\alpha(a, y) \geq 1$, by definition of α we have $a, y \in [0, 1]$ implies $\alpha(Ja, Jy) = \alpha(a, y) \geq 1$. Similarity in the case $a, y \in (1, 2]$. Now, the cases arises:

(i) In the case $a, y \in [0, 1]$, we have

$$\begin{aligned} \alpha(a, y) p(Ja, Jy) &= \alpha(a, y) p\left(\frac{a}{6}, \frac{y}{6}\right) = 2\max\{\frac{a}{6}, \frac{y}{6}\} \leq \frac{1}{3} p(a, y) \\ &\leq \psi(K(a, y)). \end{aligned}$$

(ii) Also, holds in the case $a, y \in (1, 2]$.

(iii) In the case $a \in [0, 1]$, $y \in (1, 2]$, we have

$$\alpha(a, y) p(Ja, Jy) = \alpha(a, y) p\left(\frac{a}{6}, \frac{y}{3} - \frac{1}{6}\right) = 0.$$

(iv) For $y \in [0, 1]$, $a \in (1, 2]$, we have

$$\alpha(a, y) p(Ja, Jy) = \alpha(a, y) p\left(\frac{a}{3} - \frac{1}{6}, \frac{y}{6}\right) = 0.$$

(v) If a or y is not in $[0, 2]$, then $\alpha(a, y) = 0$, that is

$$\alpha(a, y) p(Ja, Jy) \leq \psi(K(a, y)).$$

Thus, J is $\alpha - \psi - K$-contractive with $\psi(t) = \frac{t}{3}$, for each $t \geq 0$. Indeed, for $a_0 = 1$, we have

$$\alpha(a_0, J(a_0)) = \alpha(1, J1) = \alpha\left(1, \frac{1}{6}\right) = 2.$$

Now, all hypothesis of Theorem 3.3.2 are satisfied. In fact, 0 is a fixed point.

We can not apply Corollary 3.3.4 because, $\frac{2}{9} \not\leq \frac{1}{6}$. Indeed, if $a = \frac{3}{2}$, $y = 2$, then
$\alpha\left(\frac{3}{2}, 2\right) p^s\left(\frac{1}{2} - \frac{1}{6}, \frac{2}{3} - \frac{1}{6}\right) = \frac{4}{3} p^s\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{4}{3}\left[2p\left(\frac{1}{3}, \frac{1}{2}\right) - p\left(\frac{1}{3}, \frac{1}{3}\right) - p\left(\frac{1}{2}, \frac{1}{2}\right)\right] = \frac{2}{9}$ and
$p^s\left(\frac{3}{2}, 2\right) = 2p\left(\frac{3}{2}, 2\right) - p\left(\frac{3}{2}, \frac{3}{2}\right) - p(2, 2) = \frac{1}{2}$, $\psi\left(\frac{1}{2}\right) = \frac{1}{6}$.

3.3.8 Example

Consider $L = \{1, 2, 3, 4\}$ and the function $p : L \times L \to [0, +\infty)$ defined as
$p(3, 4) = p(1, 2) = 3$, $p(1, 3) = p(2, 4) = 5$, $p(1, 4) = p(2, 3) = 4$, $p(a, y) = p(y, a)$,
$p(1, 1) = 0$, $p(4, 4) = p(3, 3) = p(2, 2) = 2$. Define a self mapping J as: $J(1) = 4$, $J(3) = 3$, $J(2) = 4$, $J(4) = 3$, for all $a \in L$. Obviously p is a partial metric on L, but not metric (since $p(a, a) \neq 0$, for $a \in \{2, 3, 4\}$). Clearly J is an $\alpha - \psi - K$-contractive map with $\psi(t) = \frac{4}{5}t$, $t \geq 0$. In fact, for each $a, y \in L$, we have

$$\alpha(a, y) p(J(a), J(y)) \leq \psi(K(a, y)),\ \alpha(a, y) = \begin{cases} 1, & \text{if } (a, y) \neq (3, 3), \\ \frac{1}{4}, & \text{if } (a, y) = (3, 3). \end{cases}$$

Moreover, there exists $a_0 \in L$ such as $\alpha(a_0, Ja_0) = 1$. In fact, for $a_0 = 1$, we have

$$\alpha(1, J1) = \alpha(1, 4) = 1.$$

Let $\{a_n\}$ is a sequence in L such as $\alpha(a_n, a_{n+1}) \geq 1$, for each $n \in \mathbb{N}$ and $a_n \to a \in L$, as n tends to infinity, for some $a \in L$. From the definition of α, for all n, we have $a_n \neq 0$. Thus, $a \neq 0$ and we have $\alpha(a_n, a) \geq 1$ for all n. Also, J is α-admissible. For this, we have

$$\alpha(a, y) \geq 1 \Rightarrow a \neq 3,\ y \neq 3 \Rightarrow Ja \neq 3,\ Jy \neq 3 \Rightarrow \alpha(Ja, Jy) \geq 1.$$

Consequently, J has a fixed point. In this case, 3 is a fixed point.

We denote by fix(J) the set of fixed points of J, for the uniqueness, we need the below additional condition:

(H) For each $a, y \in$ fix (J), there exists $e \in L$ such as $\alpha(e, a) \geq 1$ and $\alpha(e, y) \geq 1$.

3.3.9 Theorem

Adding the condition (H) to the hypotheses of Theorem 3.3.3, we get that u is a unique of fixed point of J.

Proof. Assume that v is another fixed point of J. From (H), there exists $e \in L$ such as

$$\alpha(e, u) \geq 1, \quad \alpha(e, v) \geq 1. \tag{3.28}$$

Since J is α-admissible, from (3.28), we have

$$\alpha(J^n(e), u) \geq 1, \quad \alpha(J^n(e), v) \geq 1, \tag{3.29}$$

for all $n \geq 0$. Define the sequence $\{e_n\}$ in L by $e_{n+1} = J(e_n)$ for each $n = 0, 1, 2, ...$ and $e_0 = e$. Suppose that $d(e_n, u) > 0$. By (3.29) and using the technique given in Theorem 3.2.5, it can easily be proved that $u = v$. ■

We apply our results to obtain fixed points partial metric spaces endowed with a partial ordered.

3.3.10 Corollary

Let $(L, \preceq)$ be a partially ordered set and p is a partial metric on L such as (L, p) is complete. Let $J : L \to L$ is a nondecreasing mapping with respect to $\preceq$ and satisfies the following inequality:

$$p(J(a), J(y)) \leq \psi(K(a, y)), \text{ for all } a, y \in L \text{ with } a \succeq y,$$

where

$$K(a, y) = \max\left\{p(a, y), \frac{p(a, Ja) + p(y, Jy)}{2}, \frac{p(a, Jy) + p(y, Ja)}{2}, \frac{p(a, Ja)\,p(y, Jy)}{p(a, y)}, \frac{p(y, Jy)\,[1 + p(a, Ja)]}{[1 + p(a, y)]}\right\}.$$

Assume that the below conditions satisfied:

(i) $(L, \preceq, p)$ is regular or J is continuous;

(ii) there exists $a_0 \in L$ such as $a_0 \preceq Ja_0$. Then J has a fixed point. Moreover, if for $a, y \in L$, there exists $e \in L$ such as $a \preceq e$ and $y \preceq e$. Then, J has unique fixed point.

Proof. The proof comes easily from the follows. Define a map $\alpha : L \times L \to [0, +\infty)$ by $\alpha(a, y) = \begin{cases} 1, \text{ if } a \preceq y \text{ or } a \succeq y, \\ 0, \text{ otherwise.} \end{cases}$ ■

3.4 New Types of F-Contraction for Multivalued Mappings and Related Fixed Point Results in Abstract Spaces

Results given in this section have been published in [8]

3.4.1 Lemma

Let (L, d_{qb}, s) be a dislocated b-quasi metric space. Let $(P(L), H_{d_{qb}})$ be the Hausdorff dislocated b-quasi metric space on $P(L)$. Then, for each $C, F \in P(L)$ and for each $l \in C$, there exists $b_l \in F$, such as $H_{d_{qb}}(C, F) \geq d_{qb}(l, b_l)$ and $H_{d_{qb}}(F, C) \geq d_{qb}(b_l, l)$, where $d_{qb}(l, F) = d_{qb}(l, b_l)$ and $d_{qb}(F, l) = d_{qb}(b_l, l)$.

3.4.2 Lemma

Let (L, d_{qb}, s) is a dislocated b-quasi metric space. Let $\{a_n\}$ is any sequence in L there is some $\tau > 0$ and $F \in \mathcal{F}_S$ such as

$$\tau + F(s \max\{d_{qb}(a_n, a_{n+1}), d_{qb}(a_{n+1}, a_n)\}) \leq F(\max\{d_{qb}(a_{n-1}, a_n), d_{qb}(a_n, a_{n-1})\}) \quad (3.30)$$

for each $n \in \mathbb{N}$. Then $\{a_n\}$ is a Cauchy sequence in L.

Proof. Let $\vartheta_n = \max\{d_{qb}(a_n, a_{n+1}), d_{qb}(a_{n+1}, a_n)\}$, for all $n \in \mathbb{N}$. Therefore, by (3.30) and property (F4), we have

$$\tau + F(s^n \vartheta_n) \leq F(s^{n-1} \vartheta_{n-1}), \; n \in \mathbb{N}.$$

Similar the technique as given in [7], we obtain $\{a_n\}$ is a Cauchy sequence in L. ■

Let (L, d_{qb}) be a dislocated b-quasi metric space, $a_0 \in L$ and $S, J : L \to P(L)$ be multifunctions on L. Let $a_1 \in Sa_0$ is an element such as $d_{qb}(a_0, Sa_0) = d_{qb}(a_0, a_1)$, $d_{qb}(Sa_0, a_0) = d_{qb}(a_1, a_0)$. Let $a_2 \in Ja_1$ be such as $d_{qb}(a_1, Ja_1) = d_{qb}(a_1, a_2)$, $d_{qb}(Ja_1, a_1) = d_{qb}(a_2, a_1)$. Let $a_3 \in Sa_2$ be such as $d_{qb}(a_2, Sa_2) = d_{qb}(a_2, a_3)$ and so on. Thus, we generate a sequence a_n of points in L such as $a_{2n+1} \in Sa_{2n}$ and $a_{2n+2} \in Ja_{2n+1}$, with $d_{qb}(a_{2n}, Sa_{2n}) = d_{qb}(a_{2n}, a_{2n+1})$, $d_{qb}(Sa_{2n}, a_{2n}) = d_{qb}(a_{2n+1}, a_{2n})$, and $d_{qb}(a_{2n+1}, Ja_{2n+1}) = d_{qb}(a_{2n+1}, a_{2n+2})$, $d_{qb}(Ja_{2n+1}, a_{2n+1}) = d_{qb}(a_{2n+2}, a_{2n+1})$, where $n = 0, 1, 2, \cdots$. We denote this iterative sequence by $\{JS(a_n)\}$. We say that $\{JS(a_n)\}$ is a sequence in L generated by a_0. If $J = S$, then we say that $\{LJ(a_n)\}$ is a sequence in L generated by a_0.

3.4.3 Definition

Let (L, d_{qb}, s) be a dislocated b-quasi metric space and $S, J : L \to P(L)$ are two multivalued mappings. The pair (S, J) is called a DQF-contraction, if there exists $F \in \mathcal{F}_S$ and $\tau, c > 0$ whenever for any two consecutive points a, y belonging to the range of an iterative sequence $\{JS(a_n)\}$ with $\max\{H_{d_{qb}}(Sa, Jy), H_{d_{qb}}(Jy, Sa), D_{qb}(a, y), D_{qb}(y, a)\} > 0$, we have

$$\tau + \max\{F(sH_{d_{qb}}(Sa, Jy)), F(sH_{d_{qb}}(Jy, Sa))\} \leq \min\{F(D_{qb}(a, y)), F(D_{qb}(y, a))\} \quad (3.31)$$

where,

$$D_{qb}(a,y) = \max\left\{d_{qb}(a,y), \frac{d_{qb}(a,Sa).d_{qb}(y,Jy)}{c+\max\{d_{qb}(a,y),d_{qb}(y,a)\}}, d_{qb}(a,Sa), d_{qb}(y,Jy)\right\}. \tag{3.32}$$

3.4.4 Theorem

Consider (L, d_{qb}, s) be a complete dislocated b-quasi metric with $s \geq 1$ and (S, J) be a DQF-contraction. Then $\{JS(a_n)\} \to u \in L$. Also, if (3.31) holds for each $a, y \in \{u\}$, then S and J have a common fixed point u in L and $d_{qb}(u, u) = 0$.

Proof. Let $\{JS(a_n)\}$ is the iterative sequence in L generated by a point $a_0 \in L$. If

$$\max\{H_{d_{qb}}(Sa_{2p'}, Ja_{2p'+1}), H_{d_{qb}}(Ja_{2p'+1}, Sa_{2p'}), D_{qb}(a_{2p'}, a_{2p'+1}), D_{qb}(a_{2p'+1}, a_{2p'})\} \not> 0$$

for some $p' \in \mathbb{N} \cup \{0\}$, then

$$H_{d_{qb}}(Sa_{2p'}, Ja_{2p'+1}) = H_{d_{qb}}(Ja_{2p'+1}, Sa_{2p'}) = D_{qb}(a_{2p'}, a_{2p'+1}) = D_{qb}(a_{2p'+1}, a_{2p'}) = 0$$

Clearly, if $D_{qb}(a_{2p'}, a_{2p'+1}) = 0$, then $d_{qb}(a_{2p'}, a_{2p'+1}) = 0$. Also $D_{qb}(a_{2p'+1}, a_{2p'}) = 0$ implies $d_{qb}(a_{2p'+1}, a_{2p'}) = 0$. So, $a_{2p'} = a_{2p'+1}$ and $a_{2p'} \in Sa_{2p'}$. Now, $H_{d_{qb}}(Sa_{2p'}, Ja_{2p'+1}) = 0$ implies $d_{qb}(a_{2p'+1}, Ja_{2p'+1}) = 0$ and $H_{d_{qb}}(Ja_{2p'+1}, Sa_{2p'}) = 0$ implies $d_{qb}(Ja_{2p'+1}, a_{2p'+1}) = 0$. So, $a_{2p'+1} \in Ja_{2p'+1}$ and $a_{2p'}$ is a common fixed point of S and J. So the proof is completed in this case. Now, let

$$\max\{H_{d_{qb}}(Sa_{2p}, Ja_{2p+1}), H_{d_{qb}}(Ja_{2p+1}, Sa_{2p}), D_{qb}(a_{2p}, a_{2p+1}), D_{qb}(a_{2p+1}, a_{2p})\} > 0,$$

for all $p \in \mathbb{N} \cup \{0\}$. By Lemma 3.4.1, we have

$$d_{qb}(a_{2p}, a_{2p+1}) \leq H_{d_{qb}}(Ja_{2p-1}, Sa_{2p}),\ d_{qb}(a_{2p+1}, a_{2p}) \leq H_{d_{qb}}(Sa_{2p}, Ja_{2p-1}), \tag{3.33}$$

and

$$d_{qb}(a_{2p+1}, a_{2p+2}) \leq H_{d_{qb}}(Sa_{2p}, Ja_{2p+1}),\ d_{qb}(a_{2p+2}, a_{2p+1}) \leq H_{d_{qb}}(Ja_{2p+1}, Sa_{2p}). \tag{3.34}$$

From (3.34), (F1) and using the condition (3.31), we get

$$\begin{aligned} F(sd_{qb}(a_{2p+1},a_{2p+2})) &\leq F(sH_{d_{qb}}(Sa_{2p},Ja_{2p+1})) \\ &\leq \max\{F(sH_{d_{qb}}(Sa_{2p},Ja_{2p+1})),F(sH_{d_{qb}}(Ja_{2p+1},Sa_{2p}))\} \\ &\leq \min\{F(D_{qb}(a_{2p},a_{2p+1})),F(D_{qb}(a_{2p+1},a_{2p}))\}-\tau \\ &\leq F(D_{qb}(a_{2p},a_{2p+1}))-\tau. \end{aligned}$$

From (3.32), we have

$$\begin{aligned} D_{qb}(a_{2p},a_{2p+1}) &= \max\{d_{qb}(a_{2p},a_{2p+1}),\frac{d_{qb}\left(a_{2p},Sa_{2p}\right).d_{qb}\left(a_{2p+1},Ja_{2p+1}\right)}{c+\max\{d_{qb}\left(a_{2p},a_{2p+1}\right),d_{qb}\left(a_{2p+1},a_{2p}\right)\}}, \\ &\quad d_{qb}(a_{2p},Sa_{2p}),d_{qb}(a_{2p+1},Ja_{2p+1})\} \\ &= \max\{d_{qb}(a_{2p},a_{2p+1}),\frac{d_{qb}\left(a_{2p},a_{2p+1}\right).d_{qb}\left(a_{2p+1},a_{2p+2}\right)}{c+\max\{d_{qb}\left(a_{2p},a_{2p+1}\right),d_{qb}\left(a_{2p+1},a_{2p}\right)\}}, \\ &\quad d_{qb}(a_{2p},a_{2p+1}),d_{qb}(a_{2p+1},a_{2p+2})\} \\ &\leq \max\{d_{qb}(a_{2p},a_{2p+1}),d_{qb}(a_{2p+1},a_{2p+2})\}. \end{aligned}$$

If, $\max\{d_{qb}(a_{2p},a_{2p+1}),d_{qb}(a_{2p+1},a_{2p+2})\}=d_{qb}(a_{2p+1},a_{2p+2})$, then

$$F(sd_{qb}(a_{2p+1},a_{2p+2}))\leq F(d_{qb}(a_{2p+1},a_{2p+2}))-\tau.$$

It is a contradiction due to (F1) and $s\geq 1$. Thus,

$$F(sd_{qb}(a_{2p+1},a_{2p+2}))\leq F(d_{qb}(a_{2p},a_{2p+1}))-\tau, \tag{3.35}$$

$$F(sd_{qb}(a_{2p+1},a_{2p+2}))\leq F(\max\{d_{qb}(a_{2p},a_{2p+1}),d_{qb}(a_{2p+1},a_{2p})\})-\tau. \tag{3.36}$$

From (3.34), (F1) and using the condition (3.31), we get

$$\begin{aligned}
F(sd_{qb}(a_{2p+2}, a_{2p+1})) &\leq F(sH_{d_{qb}}(Ja_{2p+1}, Sa_{2p})) \\
&\leq \max\{F(sH_{d_{qb}}(Sa_{2p}, Ja_{2p+1})), F(sH_{d_{qb}}(Ja_{2p+1}, Sa_{2p}))\} \\
&\leq \min\{F(D_{qb}(a_{2p}, a_{2p+1})), F(D_{qb}(a_{2p+1}, a_{2p}))\} - \tau \\
&\leq F(D_{qb}(a_{2p}, a_{2p+1})) - \tau \\
&= F(\max\{d_{qb}(a_{2p}, a_{2p+1}), d_{qb}(a_{2p+1}, a_{2p+2})\}) - \tau.
\end{aligned}$$

By using (3.35) and (F1), we get

$$\begin{aligned}
F(sd_{qb}(a_{2p+2}, a_{2p+1})) &\leq F(\max\{d_{qb}(a_{2p}, a_{2p+1}), \frac{1}{s}d_{qb}(a_{2p}, a_{2p+1})\}) - \tau \\
&= F(d_{qb}(a_{2p}, a_{2p+1})) - \tau \\
&\leq F(\max\{d_{qb}(a_{2p}, a_{2p+1}), d_{qb}(a_{2p+1}, a_{2p})\}) - \tau.
\end{aligned}$$

$$F(sd_{qb}(a_{2p+2}, a_{2p+1})) \leq F(\max\{d_{qb}(a_{2p}, a_{2p+1}), d_{qb}(a_{2p+1}, a_{2p})\}) - \tau. \tag{3.37}$$

Combining (3.36) and (3.37), we get

$$\max\{F(sd_{qb}(a_{2p+2}, a_{2p+1})), F(sd_{qb}(a_{2p+1}, a_{2p+2}))\} \leq F(\max\{d_{qb}(a_{2p}, a_{2p+1}), d_{qb}(a_{2p+1}, a_{2p})\}) - \tau. \tag{3.38}$$

By using (3.33) and (3.31), we have

$$\begin{aligned}
F(sd_{qb}(a_{2p}, a_{2p+1})) &\leq F(sH_{d_{qb}}(Ja_{2p-1}, Sa_{2p})) \\
&\leq \max\{F(sH_{d_{qb}}(Sa_{2p}, Ja_{2p-1})), F(sH_{d_{qb}}(Ja_{2p-1}, Sa_{2p}))\} \\
&\leq \min\{F(D_{qb}(a_{2p-1}, a_{2p})), F(D_{qb}(a_{2p}, a_{2p-1}))\} - \tau \\
&\leq F(D_{qb}(a_{2p}, a_{2p-1})) - \tau.
\end{aligned}$$

From (3.32), we have

$$\begin{aligned}D_{qb}(a_{2p},a_{2p-1}) &= \max\{d_{qb}(a_{2p},a_{2p-1}),\frac{d_{qb}\left(a_{2p},a_{2p+1}\right).d_{qb}\left(a_{2p-1},a_{2p}\right)}{c+\max\left\{d_{qb}\left(a_{2p},a_{2p-1}\right),d_{qb}\left(a_{2p-1},a_{2p}\right)\right\}},\\ &\quad d_{qb}(a_{2p},a_{2p+1}),d_{qb}(a_{2p-1},a_{2p})\}\\ &\leq \max\{d_{qb}(a_{2p},a_{2p-1}),d_{qb}(a_{2p-1},a_{2p}),d_{qb}(a_{2p},a_{2p+1})\}.\end{aligned}$$

If $\max\{d_{qb}(a_{2p},a_{2p-1}),d_{qb}(a_{2p-1},a_{2p}),d_{qb}(a_{2p},a_{2p+1})\}=d_{qb}(a_{2p},a_{2p+1})$, then

$$F(sd_{qb}(a_{2p},a_{2p+1}))\leq F(d_{qb}(a_{2p},a_{2p+1}))-\tau.$$

A contradiction due to (F1). Thus,

$$F(sd_{qb}(a_{2p},a_{2p+1}))\leq F(\max\{d_{qb}(a_{2p-1},a_{2p}),d_{qb}(a_{2p},a_{2p-1})\})-\tau. \tag{3.39}$$

By using (3.33) and (3.32), we have

$$\begin{aligned}F(sd_{qb}(a_{2p+1},a_{2p})) &\leq F(sH_{d_{qb}}(Sa_{2p},Ja_{2p-1}))\leq F(D_{qb}(a_{2p},a_{2p-1}))-\tau\\ &\leq F(\max\{d_{qb}(a_{2p},a_{2p-1}),d_{qb}(a_{2p-1},a_{2p}),d_{qb}(a_{2p},a_{2p+1})\})-\tau.\end{aligned}$$

From (3.39), $d_{qb}(a_{2p},a_{2p+1})<\max\{d_{qb}(a_{2p-1},a_{2p}),d_{qb}(a_{2p},a_{2p-1})\}$, so

$$F(sd_{qb}(a_{2p+1},a_{2p}))\leq F(\max\{d_{qb}(a_{2p},a_{2p-1}),d_{qb}(a_{2p-1},a_{2p})\})-\tau. \tag{3.40}$$

Combining (3.39) and (3.40), we get

$$\max\{F(sd_{qb}(a_{2p},a_{2p+1})),F(sd_{qb}(a_{2p+1},a_{2p}))\}\leq \max\{d_{qb}(a_{2p},a_{2p-1}),d_{qb}(a_{2p-1},a_{2p})\}-\tau. \tag{3.41}$$

Combining (3.38) and (3.41), we get

$$\tau+F(s\max\{d_{qb}(a_n,a_{n+1}),d_{qb}(a_{n+1},a_n)\})\leq F(\max\{d_{qb}(a_{n-1},a_n),d_{qb}(a_n,a_{n-1})\}). \tag{3.42}$$

By Lemma 3.4.2, $\{JS(a_n)\}$ is a Cauchy sequence in (L,d_{qb}). As, (L,d_{qb}) is a complete dislocated b-quasi metric space, so there exists $u \in L$ such as $\{JS(a_n)\} \to u$

$$\lim_{n\to+\infty} d_{qb}(a_n, u) = \lim_{n\to+\infty} d_{qb}(u, a_n) = 0. \tag{3.43}$$

Now, suppose $d_{qb}(u, Ju) > 0$, then $D_{qb}(a_{2n}, u) > 0$, so

$$\max\{H_{d_{qb}}(Sa_{2n}, Ju), H_{d_{qb}}(Ju, Sa_{2n}), D_{qb}(a_{2n}, u), D_{qb}(u, a_{2n})\} > 0.$$

By using Lemma 3.4.1 and (3.31), we have

$$\begin{aligned} \tau + F(sd_{qb}(a_{2n+1}, Ju)) &\leq \tau + \max\{F(sH_{d_{qb}}(Sa_{2n}, Ju)), F(sH_{d_{qb}}(Ju, Sa_{2n}))\} \\ &\leq \min\{F(D_{qb}(a_{2n}, u)), F(D_{qb}(u, a_{2n}))\} \leq F(D_{qb}(a_{2n}, u)). \end{aligned}$$

As, F is strictly increasing, so

$$sd_{qb}(a_{2n+1}, Ju) < D_{qb}(a_{2n}, u).$$

Taking $\lim\limits_{n\to+\infty}$ in above inequality, we get

$$\lim_{n\to+\infty} sd_{qb}(a_{2n+1}, Ju) < \lim_{n\to+\infty} D_{qb}(a_{2n}, u) \tag{3.44}$$

From (3.32), we have

$$D_{qb}(a_{2n}, u) = \max\{d_{qb}(a_{2n}, u), \frac{d_{qb}(a_{2n}, a_{2n+1}).d_{qb}(u, Ju)}{c + \max\{d_{qb}(a_{2n}, u), d_{qb}(u, a_{2n})\}}, d_{qb}(a_{2n}, a_{2n+1}), d_{qb}(u, Ju)\}.$$

Taking limit as n tends to infinity in above inequality and by using (3.43), we get

$$\lim_{n\to+\infty} D_{qb}(a_{2n}, u) = d_{qb}(u, Ju). \tag{3.45}$$

Using inequality (3.45) in (3.44), we get

$$\lim_{n\to+\infty} sd_{qb}(a_{2n+1}, Ju) < d_{qb}(u, Ju). \tag{3.46}$$

Now,

$$d_{qb}(u, Ju) \le sd_{qb}(u, a_{2n+1}) + sd_{qb}(a_{2n+1}, Ju).$$

Taking $\lim_{n\to+\infty}$ on above inequality, we get

$$d_{qb}(u, Ju) \le s \lim_{n\to+\infty} d_{qb}(u, a_{2n+1}) + \lim_{n\to+\infty} sd_{qb}(a_{2n+1}, Ju). \tag{3.47}$$

Using inequality (3.43) and (3.46) in (3.47), we get

$$d_{qb}(u, Ju) < d_{qb}(u, Ju).$$

Which is a contradiction, so $d_{qb}(u, Ju) = 0$. Now, suppose $d_{qb}(Ju, u) > 0$, then there exists $n_0 \in \mathbb{N}$ such as $d_{qb}(Ju, a_{2n+1}) > 0$ for all $n \ge n_0$. By Lemma 3.4.1 $d_{qb}(Ju, a_{2n+1}) \le H_{d_{qb}}(Ju, Sa_{2n})$, so

$$\max\{H_{d_{qb}}(Sa_{2n}, Ju), H_{d_{qb}}(Ju, Sa_{2n}), D_{qb}(a_{2n}, u), D_{qb}(u, a_{2n})\} > 0,$$

for all $n \ge n_0$. Following similar arguments as above, we get

$$\lim_{n\to+\infty} sd_{qb}(Ju, a_{2n+1}) < d_{qb}(u, Ju) = 0. \tag{3.48}$$

Now,

$$d_{qb}(Ju, u) \le sd_{qb}(Ju, a_{2n+1}) + sd_{qb}(a_{2n+1}, u).$$

Taking $\lim_{n\to+\infty}$ on above inequality and using inequality (3.43) and (3.48), we get

$$d_{qb}(Ju, u) \le 0.$$

Which is a contradiction, so $d_{qb}(Ju, u) = 0$. Thus $u \in Ju$. Similar lines as above we obtain that $d_{qb}(Su, u) = 0$ and $d_{qb}(u, Su) = 0$. Hence, the pair (S, J) has a common fixed point u in (L, d_{qb}). Now,

$$d_{qb}(u, u) \le d_{qb}(u, Ju) + d_{qb}(Ju, u) \le 0.$$

Therefore $d_{qb}(u, u) = 0$ and the proof is completed. ■

3.4.5 Example

Let $L = \{0\} \cup \mathbb{Q}^+$ and $d_{qb}(a, y) = (a + 2y)^2$ if $a \neq y$, and $d_{qb}(a, y) = 0$, if $a = y$. Then (L, d_{qb}) is a dislocated b-quasi metric space with $s = 2$. Define the mappings $S, J : L \to P(L)$ as follows:

$$S(a) = \begin{cases} \left[\frac{1}{4}a, \frac{2}{5}a\right] \cap \mathbb{Q}^+, & \text{for all } a \in \{0, 7, \frac{7}{4}, \frac{7}{12}, \frac{7}{48}, \cdots\}, \\ [a+1, a+4] \cap \mathbb{Q}^+, & \text{otherwise.} \end{cases}$$

$$J(y) = \begin{cases} \left[\frac{1}{3}y, \frac{3}{8}y\right] \cap \mathbb{Q}^+, & \text{for all } y \in \{0, 7, \frac{7}{4}, \frac{7}{12}, \frac{7}{48}, \cdots\}, \\ [y+3, y+6] \cap \mathbb{Q}^+, & \text{otherwise.} \end{cases}$$

Case 1: If, $\tau + \max\{F(sH_{d_{qb}}(Sa, Jy)), F(sH_{d_{qb}}(Ja, Sy))\} = \tau + F(sH_{d_{qb}}(Sa, Jy)) \leq \min\{F(D_{qb}(a, y)), F(D_{qb}(y, a))\}$ satisfied. Define the function $F : [0, +\infty) \to R$ as $F(a) = \ln(a)$ for all $a \in [0, +\infty)$ and $\tau > 0$. Since $a, y \in L$, $\tau = \ln(1.2)$ and let $a_0 = 7$, so the sequence defined as $\{JS(a_n)\} = \{7, \frac{7}{4}, \frac{7}{12}, \frac{7}{48}, \cdots\} \in L$ and generated by $a_0 = 7$. Also, $\{JS(a_n)\} \to 0$. Now, if $a, y \in \{JS(a_n)\} \cup \{0\}$, we have

$$\begin{aligned} sH_{d_{qb}}(Sa, Jy) &= 2H_{d_{qb}}\left(\left[\frac{1}{4}a, \frac{2}{5}a\right], \left[\frac{1}{3}y, \frac{3}{8}y\right]\right) \\ &= 2\max\left[\left\{\sup_{l \in Sa} d_{qb}\left(l, \left[\frac{1}{3}y, \frac{3}{8}y\right]\right), \sup_{b \in Jy} d_{qb}\left(\left[\frac{1}{4}a, \frac{2}{5}a\right], b\right)\right\}\right] \\ &= 2\max\left\{d_{qb}\left(\frac{2a}{5}, \frac{y}{3}\right), d_{qb}\left(\frac{a}{4}, \frac{3}{8}y\right)\right\} \\ &= 2\max\left\{\left(\frac{2a}{5} + \frac{2y}{3}\right)^2, \left(\frac{a}{4} + \frac{3}{4}y\right)^2\right\}. \end{aligned}$$

Also,

$$\begin{aligned} D_{q_b}(a, y) = \max\Bigg\{ & d_{qb}(a, y), \frac{d_{qb}\left(a, \left[\frac{a}{4}, \frac{2a}{5}\right]\right).d_{qb}\left(y, \left[\frac{y}{3}, \frac{3y}{8}\right]\right)}{1 + \max\{d_{qb}(a, y), d_{qb}(y, a)\}}, \\ & d_{qb}\left(a, \left[\frac{a}{4}, \frac{2a}{5}\right]\right), d_{qb}\left(y, \left[\frac{y}{3}, \frac{3y}{8}\right]\right)\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{d_{qb}(a,y), \frac{d_{qb}(a,\frac{a}{4}).d_{qb}(y,\frac{y}{3})}{1+\max\{d_{qb}(a,y),d_{qb}(y,a)\}}, d_{qb}(a,\frac{a}{4}), d_{qb}(y,\frac{y}{3})\right\} \\
&= \max\left\{(a+2y)^2, \frac{(5ay)^2}{4(1+(a+2y)^2)}, \left(\frac{3a}{2}\right)^2, \left(\frac{5y}{3}\right)^2\right\} = (a+2y)^2.
\end{aligned}
$$

Case (i). If $\max\left\{\left(\frac{2a}{5}+\frac{2y}{3}\right)^2, \left(\frac{a}{4}+\frac{3}{4}y\right)^2\right\} = \left(\frac{a}{4}+\frac{3}{4}y\right)^2$, and $\tau = \ln(1.2)$, then we have

$$
\begin{aligned}
3\,(a+3y)^2 &\leq 20\,(a+2y)^2 \\
\frac{6}{5}\left(\frac{a}{4}+\frac{3}{4}y\right)^2 &\leq (a+2y)^2 \\
\ln(1.2)+\ln\left(\frac{a}{4}+\frac{3}{4}y\right)^2 &\leq \ln(a+2y)^2.
\end{aligned}
$$

Which implies that, $\tau + F(sH_{d_{qb}}(Sa,Jy) \leq F(D_{q_b}(a,y))$.
Case (ii). Similarly, if $\max\left\{\left(\frac{2a}{5}+\frac{2y}{3}\right)^2, \left(\frac{a}{4}+\frac{3}{4}y\right)^2\right\} = \left(\frac{2a}{5}+\frac{2y}{3}\right)^2$ and $\tau = \ln(1.2)$, then we have

$$
\begin{aligned}
48\,(3a+5y)^2 &\leq 1125\,(a+2y)^2 \\
\frac{6}{5}\left(\frac{2a}{5}+\frac{2y}{3}\right)^2 &\leq (a+2y)^2 \\
\ln(1.2)+\ln\left(\frac{2a}{5}+\frac{2y}{3}\right)^2 &\leq \ln(a+2y)^2.
\end{aligned}
$$

Hence,

$$
\tau + F(sH_{d_{qb}}(Sa,Jy) \leq F(D_{q_b}(a,y)).
$$

Case 2: If $\max\{\tau + F(sH_{d_{qb}}(Sa,Jy)), \tau + F(sH_{d_{qb}}(Ja,Sy))\} = \tau + F(sH_{d_{qb}}(Ja,Sy))$ holds.

$$
\begin{aligned}
sH_{d_{qb}}(Ja,Sy) &= 2\max\left[\left\{\sup_{b\in Ja} d_{qb}(b,Sy), \sup_{l\in Sy} d_{qb}(Ja,l)\right\}\right] \\
&= 2\max\left[\left\{\sup_{b\in Ja} d_{qb}\left(b,\left[\frac{1}{4}y,\frac{2}{5}y\right]\right), \sup_{l\in Sy} d_{qb}\left(\left[\frac{1}{3}a,\frac{3}{8}a\right],l\right)\right\}\right] \\
&= 2\max\left\{d_{qb}\left(\frac{3a}{8},\frac{y}{4}\right), d_{qb}\left(\frac{a}{3},\frac{2y}{5}\right)\right\} \\
&= 2\max\left\{\left(\frac{3a}{8}+\frac{2y}{4}\right)^2, \left(\frac{a}{3}+\frac{4y}{5}\right)^2\right\},
\end{aligned}
$$

where

$$\begin{aligned} D_{q_b}(y,a) &= \max\left\{ d_{qb}(y,a), \frac{d_{qb}\left(a,\left[\frac{a}{4},\frac{2a}{5}\right]\right).d_{qb}\left(y,\left[\frac{y}{3},\frac{3y}{8}\right]\right)}{1+\max\{d_{qb}(a,y),d_{qb}(y,a)\}},\right. \\ &\left. d_{qb}\left(a,\left[\frac{a}{4},\frac{2a}{5}\right]\right), d_{qb}\left(y,\left[\frac{y}{3},\frac{3y}{8}\right]\right)\right\} \end{aligned}$$

$$= \max\left\{ d_{qb}(y,a), \frac{d_{qb}(a,\frac{a}{4}).d_{qb}(y,\frac{y}{3})}{1+\max\{d_{qb}(a,y),d_{qb}(y,a)\}}, d_{qb}(a,\frac{a}{4}), d_{qb}(y,\frac{y}{3})\right\}$$

$$D_{q_b}(y,a) = \max\left\{(y+2a)^2, \frac{(5ay)^2}{4(1+(y+2a)^2)}, \left(\frac{3a}{2}\right)^2, \left(\frac{5y}{3}\right)^2\right\} = (y+2a)^2.$$

Case (i). If, $\max\left\{\left(\frac{3a}{8}+\frac{2y}{4}\right)^2, \left(\frac{a}{3}+\frac{4y}{5}\right)^2\right\} = \left(\frac{a}{3}+\frac{4y}{5}\right)^2$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned} 12\,(5a+12y)^2 &\leq 1125\,(y+2a)^2 \\ \frac{6}{5}\left(\frac{a}{3}+\frac{4y}{5}\right)^2 &\leq (y+2a)^2 \\ \ln(1.2)+\ln\left(\frac{a}{3}+\frac{4y}{5}\right)^2 &\leq \ln(y+2a)^2, \text{ so} \\ \tau + F(sH_{d_{qb}}(Ja,Sy) &\leq F(D_{q_b}(y,a)). \end{aligned}$$

Case (ii). If $\max\left\{\left(\frac{3a}{8}+\frac{2y}{4}\right)^2, \left(\frac{a}{3}+\frac{4y}{5}\right)^2\right\} = \left(\frac{3a}{8}+\frac{2y}{4}\right)^2$, and $\tau = \ln(1.2)$, then we have

$$\begin{aligned} 12\,(3a+4y)^2 &\leq 320\,(y+2a)^2 \\ \frac{6}{5}\left(\frac{3a}{8}+\frac{2y}{4}\right)^2 &\leq (y+2a)^2 \\ \ln(1.2)+\ln\left(\frac{3a}{8}+\frac{2y}{4}\right)^2 &\leq \ln(y+2a)^2. \end{aligned}$$

Hence, $\tau + F(sH_{d_{qb}}(Ja,Sy) \leq F(D_{q_b}(y,a))$. Now, note that if $a, y \notin \{JS(a_n)\}$, the contraction is not satisfied. Thus, all assumptions of Theorem 3.4.4 hold. In fact 0 is a common fixed point.

Drop J in Theorem 3.4.4, we obtain the below Theorem.

3.4.6 Theorem

Consider (L, d_{qb}) be a complete dislocated b-quasi metric space with $s \geq 1$ and $S : L \rightarrow P(L)$ be a multivalued mapping such as for every two consecutive points a, y belonging to the range of an iterative sequence $\{S(a_n)\}$ with $D_{q_b}(a, y) > 0$, $F \in \mathcal{F}_S$, $\tau, c > 0$

$$\tau + F(sH_{q_b}(Sa, Sy)) \leq F(D_{q_b}(a, y)), \tag{3.49}$$

where

$$D_{q_b}(a, y) = \max\left\{d_{qb}(a, y), \frac{d_{qb}(a, Sa) \, .d_{qb}(y, Sy)}{c + d_{qb}(a, y)}, d_{qb}(a, Sa), d_{qb}(y, Sy)\right\}. \tag{3.50}$$

Then $\{S(a_n)\} \rightarrow u \in L$. Moreover, if (3.49) is satisfied for $\{u\}$ and $d_{qb}(u, u) = 0$, then S has a fixed point.

3.4.7 Remark

Taking the different values of $D_{qb}(a, y)$ in (3.32), different results on F-contractions can be obtained as corollaries of the Theorem 3.4.4.

$F\rho_s^*$-Khan type contraction in quasi b-metric spaces

Piri *et al.* [75] restated the notion of Khan [61] and Fisher [44] however, included rational expressions in discussion. Piri *et al.* [76] also revised the results of F_k-Khan-type self-mapping into a new form inside a complete metric space. We also demonstrated a new kind of rational contraction investigated its fixed point using b- quasi metric space. This furthers Khan fixed point theorem. Finally, we stated multi-valued $F\rho_s^*$-Khan-type multivalued for more than one maps in b- quasi metric space.

3.4.8 Definition

Let $L \neq \{\}$, $s \geq 1$ and $\rho_s : L \times L \rightarrow [0, +\infty)$ is a map such as $\rho_s(a, y) \geq s$ and $\rho_s(y, a) \geq s$, implies $a = y$. Let $M \subseteq L$, define $\rho_s^*(a, M)$=inf $\{\rho_s(a, l), l \in M\}$ and $\rho_s^*(M, y) = \inf\{\rho_s(b, y), b \in M\}$. Let $S, J : L \rightarrow P(L)$ be the multivalued mappings, then the pair (S, J) is called ρ_s^*-Alt multivalued mapping, if $a \in L$

$$\text{(a) } \rho_s^*(a, Sa) \geq s, \; q_b(a, Sa) = q_b(a, y) \text{ and } q_b(Sa, a) = q_b(y, a) \text{ implies } \rho_s^*(Sy, y) \geq s,$$
$$\text{(b) } \rho_s^*(Sa, a) \geq s, \; q_b(a, Ja) = q_b(a, y) \text{ and } q_b(Ja, a) = q_b(y, a) \text{ implies } \rho_s^*(y, Sy) \geq s.$$

3.4.9 Definition

Let (L, q_b, s) be a b-quasi metric space and (S, J) is a pair of ρ_s^* multivalued mapping. Then (S, J) is called $F\rho_s^*$ Khan type contraction, if $\tau > 0$ and there exists $F \in \mathcal{F}_S$ whenever for any two consecutive points a, y belonging to the range of an iterative sequence $\{JS(a_n)\}$ with $\rho_s^*(Sy, y) \geq s$, $\rho_s^*(a, Sa) \geq s$ and $\max\{H_{q_b}(Sa, Jy), H_{q_b}(Jy, Sa), q_b(a, y), q_b(y, a)\} > 0$, we have

$$\tau + \max\{F(sH_{q_b}(Sa, Jy)), F(sH_{q_b}(Jy, Sa) \leq \min\{F(Q_b(a, y)), F(Q_b(y, a))\}, \tag{3.51}$$

where,

$$Q_b(a, y) = \frac{q_b(a, Sa)\, q_b(a, Jy) + q_b(y, Jy)\, q_b(y, Sa)}{\max\{q_b(a, Jy), q_b(y, Sa)\}}. \tag{3.52}$$

3.4.10 Theorem

Consider (L, q_b, s) be a complete b- quasi metric space with $s \geq 1$. Let $\rho_s : L \times L \to [0, +\infty)$ and (S, J) is a pair of $F\rho_s^*$ Khan type contraction and the set $G(S) = \{a : \rho_s^*(a, Sa) \geq s\}$ is closed and contains a_0. Then $\{JS(a_n)\} \to u \in L$. If (3.51) is satisfied for each $a, y \in \{u\}$. Then, there is a single common fixed point of S and J in L and $q_b(u, u) = 0$.

As a_0 is any element of $G(S)$, from condition of the theorem $\rho_s^*(a_0, Sa_0) \geq s$. Let $\{JS(a_n)\} \in L$ is a sequence generated by a point $a_0 \in L$. Let $a_{2p'}, a_{2p'+1}$ are elements of this sequence. Clearly, if

$$\max\{H_{q_b}(Sa_{2p'}, Ja_{2p'+1}), H_{q_b}(Ja_{2p'+1}, Sa_{2p'}), q_b(a_{2p'}, a_{2p'+1}), q_b(a_{2p'+1}, a_{2p'})\} \not> 0,$$

for some $p' \in \mathbb{N} \cup \{0\}$, then

$$H_{q_b}(Sa_{2p'}, Ja_{2p'+1}) = H_{q_b}(Ja_{2p'+1}, Sa_{2p'}) = q_b(a_{2p'}, a_{2p'+1}) = q_b(a_{2p'+1}, a_{2p'}) = 0.$$

As $q_b(a_{2p'}, a_{2p'+1}) = q_b(a_{2p'+1}, a_{2p'}) = 0$, so $a_{2p'} = a_{2p'+1}$ and $a_{2p'} \in Sa_{2p'}$.
Now, $H_{q_b}(Sa_{2p'}, Ja_{2p'+1}) = 0$ implies $q_b(a_{2p'+1}, Ja_{2p'+1}) = 0$ and $H_{q_b}(Ja_{2p'+1}, Sa_{2p'}) = 0$ implies $q_b(Ja_{2p'+1}, a_{2p'+1}) = 0$. So, $a_{2p'+1} \in Ja_{2p'+1}$ and $a_{2p'}$ is a common fixed point of S and J. So the proof is done. In order to find common fixed point of S and J, when

$$\max\{H_{q_b}(Sa_{2p}, Ja_{2p+1}), H_{q_b}(Ja_{2p+1}, Sa_{2p}), q_b(a_{2p}, a_{2p+1}), q_b(a_{2p+1}, a_{2p})\} > 0,$$

for all $p \in \{0\} \cup \mathbb{N}$. Since $\rho_s^*(a_0, Sa_0) \geq s$, $q_b(a_0, Sa_0) = q_b(a_0, a_1)$ and $q_b(Sa_0, a_0) = q_b(a_1, a_0)$. As (S, J) is ρ_s^* multivalued mapping, so $\rho_s^*(Sa_1, a_1) \geq s$. Now, $\rho_s^*(Sa_1, a_1) \geq s$, $q_b(a_1, Ja_1) = q_b(a_1, a_2)$ and $q_b(Ja_1, a_1) = q_b(a_2, a_1)$ implies that $\rho_s^*(a_2, Sa_2) \geq s$. By induction we deduce that $\rho_s^*(a_{2p}, Sa_{2p}) \geq s$ and $\rho_s^*(Sa_{2p+1}, a_{2p+1}) \geq s$, for all $p = 0, 1, 2, \cdots$. Now, by Lemma 2.1.1, we have

$$q_b(a_{2p}, a_{2p+1}) \leq H_{q_b}(Ja_{2p-1}, Sa_{2p}),\ q_b(a_{2p+1}, a_{2p}) \leq H_{q_b}(Sa_{2p}, Ja_{2p-1}) \tag{3.53}$$

and

$$q_b(a_{2p+1}, a_{2p+2}) \leq H_{q_b}(Sa_{2p}, Ja_{2p+1}),\ q_b(a_{2p+2}, a_{2p+1}) \leq H_{q_b}(Ja_{2p+1}, Sa_{2p}). \tag{3.54}$$

As $s \geq 1$, then (3.54) implies

$$\begin{aligned} F(sq_b(a_{2p+1}, a_{2p+2})) &\leq F(sH_{q_b}(Sa_{2p}, Ja_{2p+1})) \\ &\leq \max\{F(sH_{q_b}(Sa_{2p}, Ja_{2p+1})), F(sH_{q_b}(Ja_{2p+1}, Sa_{2p}))\}. \end{aligned}$$

As $a_{2p}, a_{2p+1} \in \{JS(a_n)\}$, $\rho_s^*(a_{2p}, Sa_{2p}) \geq s$ and $\rho_s^*(Sa_{2p+1}, a_{2p+1}) \geq s$, by using the condition (3.51), we get

$$\begin{aligned} F(sq_b(a_{2p+1}, a_{2p+2})) &\leq \min\{F(Q_b(a_{2p}, a_{2p+1})), F(Q_b(a_{2p+1}, a_{2p}))\} - \tau \\ &\leq F(Q_b(a_{2p}, a_{2p+1})) - \tau. \end{aligned}$$

From (3.52), we get

$$\begin{aligned} Q_b(a_{2p}, a_{2p+1}) &= \frac{q_b(a_{2p}, Sa_{2p})q_b(a_{2p}, Ja_{2p+1}) + q_b(a_{2p+1}, Ja_{2p+1})q_b(a_{2p+1}, Sa_{2p})}{\max\{q_b(a_{2p}, Ja_{2p+1}), q_b\,(Sa_{2p}, a_{2p+1})\}} \\ &= \frac{q_b(a_{2p}, a_{2p+1}).q_b(a_{2p}, Ja_{2p+1}) + q_b(a_{2p+1}, a_{2p+2}) \times 0}{\max\{q_b(a_{2p}, Ja_{2p+1}), 0\}} \\ &= q_b(a_{2p}, a_{2p+1}). \end{aligned}$$

Therefore,

$$F(sq_b(a_{2p+1}, a_{2p+2}) \leq F(q_b(a_{2p}, a_{2p+1})) - \tau. \tag{3.55}$$

This implies

$$F(sq_b(a_{2p+1}, a_{2p+2})) \leq F\left(\max\{q_b(a_{2p}, a_{2p+1}), q_b(a_{2p+1}, a_{2p})\}\right) - \tau. \tag{3.56}$$

As $s \geq 1$, then (3.54) implies

$$\begin{aligned} F(sq_b(a_{2p+2}, a_{2p+1})) &\leq F(sH_{q_b}(Ja_{2p+1}, Sa_{2p})) \\ &\leq \max\{F(sH_{q_b}(Ja_{2p+1}, Sa_{2p})), F(sH_{q_b}(Sa_{2p}, Ja_{2p+1}))\} \end{aligned}$$

As $a_{2p+1}, a_{2p} \in \{JS(a_n)\}$, $\rho_s^*\,(Sa_{2p+1,}, a_{2p+1}) \geq s$ and $\rho_s^*\,(a_{2p}, Sa_{2p}) \geq s$, then using the condition (3.51), we get

$$\begin{aligned} F(sq_b(a_{2p+2}, a_{2p+1})) &\leq \min\left\{F\left(Q_b(a_{2p}, a_{2p+1})\right), F\left(Q_b(a_{2p+1}, a_{2p})\right)\right\} - \tau \\ &\leq F\left(Q_b(a_{2p}, a_{2p+1})\right) - \tau = F\left(q_b(a_{2p}, a_{2p+1})\right) - \tau. \end{aligned}$$

Therefore,

$$F(sq_b(a_{2p+2}, a_{2p+1})) \leq F\left(\max\{q_b(a_{2p}, a_{2p+1}), q_b(a_{2p+1}, a_{2p})\}\right) - \tau. \tag{3.57}$$

Combining (3.56) and (3.57), we get

$$\max\left\{F(sq_b(a_{2p+1}, a_{2p+2})), F(sq_b(a_{2p+2}, a_{2p+1}))\right\} \leq F\left(\max\{q_b(a_{2p}, a_{2p+1}), q_b(a_{2p+1}, a_{2p})\}\right) - \tau. \tag{3.58}$$

As $s \geq 1$, then (3.53) implies

$$\begin{aligned} F\left(sq_b(a_{2p}, a_{2p+1})\right) &\leq (sH_{q_b}(Ja_{2p-1}, Sa_{2p})) \\ &\leq \max\{F(sH_{q_b}(Sa_{2p}, Ja_{2p-1})), F(sH_{q_b}(Ja_{2p-1}, Sa_{2p}))\}. \end{aligned}$$

As $a_{2p}, a_{2p-1} \in \{JS(a_n)\}$, $\rho_s^*(a_{2p,}, Sa_{2p}) \geq s$ and $\rho_s^*(Sa_{2p-1}, a_{2p-1}) \geq s$, then by using the condition (3.51), we get

$$\begin{aligned} F\left(sq_b(a_{2p}, a_{2p+1})\right) &\leq \min\left\{F\left(Q_b(a_{2p}, a_{2p-1})\right), F\left(Q_b(a_{2p-1}, a_{2p})\right)\right\} - \tau \\ &\leq F\left(Q_b(a_{2p}, a_{2p-1})\right) - \tau. \end{aligned}$$

$$\begin{aligned} F\left(sq_b(a_{2p}, a_{2p+1})\right) &\leq F\left(\frac{q_b(a_{2p}, Sa_{2p})q_b(a_{2p}, Ja_{2p-1}) + q_b(a_{2p-1}, Ja_{2p-1})q_b(a_{2p-1}, Sa_{2p})}{\max\{q_b(a_{2p}, Ja_{2p-1}), q_b(a_{2p-1}, Sa_{2p})\}}\right) - \tau \\ &\leq F(q_b(a_{2p-1}, a_{2p})) - \tau. \end{aligned}$$

Therefore,

$$F\left(sq_b(a_{2p}, a_{2p+1})\right) \leq F\left(\max\{q_b(a_{2p-1}, a_{2p}), q_b(a_{2p}, a_{2p-1})\}\right) - \tau. \quad (3.59)$$

Similarly, by using (3.51), (3.52) and (3.53), we get

$$F\left(sq_b(a_{2p+1}, a_{2p})\right) \leq F\left(\max\{q_b(a_{2p-1}, a_{2p}), q_b(a_{2p}, a_{2p-1})\}\right) - \tau. \quad (3.60)$$

Combining (3.59) and (3.60), we get

$$\tau + F(s\max\{q_b(a_{2p}, a_{2p+1})), F(q_b(a_{2p+1}, a_{2p})) \leq F\left(\max\{q_b(a_{2p-1}, a_{2p}), q_b(a_{2p}, a_{2p-1})\}\right). \quad (3.61)$$

Combining (3.58) and (3.61), we get

$$\tau + F\left(s\max\left\{q_b(a_n, a_{n+1}), q_b(a_{n+1}, a_n)\right\}\right) \leq F\left(\max\{q_b(a_{n-1}, a_n), q_b(a_n, a_{n-1})\}\right). \quad (3.62)$$

By Lemma 3.4.2, $\{JS(a_n)\}$ is a Cauchy sequence in (L, q_b). As $\rho_s^*(a_{2n,}, Sa_n) \geq s$ for all $n \in \mathbb{N}$, so $\{a_{2n}\}$ is a subsequence of $\{JS(a_n)\}$ contained in $G(S)$. As $G(S)$ is closed, so there exists

$u \in G(S)$ such as $\{a_{2n}\} \to u$,

$$\lim_{n\to+\infty} q_b(a_n, u) = \lim_{n\to+\infty} q_b(u, a_n) = 0. \tag{3.63}$$

Also, $\rho_s^*(u, Su) \geq s$. To complete the proof similar arguments as Theorem 3.4.4, we obtain

Proof. $q_b(u, Su) = 0$ and $q_b(Su, u) = 0$. Hence $u \in Su$. As $\rho_s^*(u, Su) \geq s$ and $q_b(u, Su) = q_b(Su, u) = q_b(0, 0)$, then Definition 3.4.8 implies

$$\rho_s^*(Su, u) \geq s.$$

Similar arguments as above, we get

$q_b(u, Ju) = 0$ and $q_b(Ju, u) = 0$. Hence $u \in Ju$. Hence, the pair (S, J) has a common fixed point u in (L, q_b).

■

Single valued result with application to system of integral equations

Let $S, J : L \to L$ are the self maps and $a_0 \in L$. Let $a_1 = Sa_0$, $a_2 = Ja_1$, $a_3 = Sa_2$ and so on. We generate a sequence a_n in L such as

$$a_{2n+1} = Sa_{2n} \text{ and } a_{2n+2} = Ja_{2n+1}, \text{ (where } n = 0, 1, 2, \dots).$$

We say that $\{JS(a_n)\}$ is a sequence in L generated by a_0.

The following result is obtained by replacing the multivalued mappings with the single valued mappings in Theorem 3.4.4. Our results generalizes Theorem 24 in [79]. Also, we prove uniqueness of common fixed point in our results.

3.4.11 Theorem

Consider (L, d_{q_b}) be a complete dislocated b-quasi metric space, $s \geq 1$ and $S, J : L \to L$ are two self maps. If there exists $F \in \mathcal{F}_S$ and $\tau, c > 0$ whenever for any two consecutive points a, y belonging to the range of an iterative sequence $\{JS(a_n)\}$ with $\max\{d_{qb}(Sa, Jy)$, $d_{qb}(Jy, Sa)$,

$D_{qb}(a,y),\ D_{qb}(y,a)\} > 0$, we have

$$\tau + \max\{F(sd_{qb}(Sa,Jy)), F(sd_{qb}(Jy,Sa))\} \leq \min\{F(D_{q_b}(a,y)), F(D_{q_b}(y,a))\}, \tag{3.64}$$

where

$$D_{q_b}(a,y) = \max\left\{d_{qb}(a,y), \frac{d_{qb}\left(a,Sa\right).d_{qb}\left(y,Jy\right)}{c+\max\{d_{qb}\left(a,y\right), d_{qb}\left(y,a\right)\}}, d_{qb}(a,Sa), d_{qb}(y,Jy)\right\}. \tag{3.65}$$

Then, $\{JS(a_n)\} \to u \in L$. Also, if u satisfies (3.64), the pair (S,J) has a common fixed point in L and $d_{q_b}(u,u) = 0$.

Proof. Consider that a^* is another common fixed point of S and J and $d_{qb}(Su, Ja^*) > 0$. Then, we get

$$\begin{aligned} & \tau + F\left(sd_{qb}(Su, Ja^*)\right) \\ \leq\ & F\left(\max\left\{d_{qb}(u,a^*), \frac{d_{qb}\left(u,Su\right).d_{qb}\left(a^*,Ja^*\right)}{1+\max\{d_{qb}\left(u,a^*\right), d_{qb}\left(a^*,u\right)\}}, d_{qb}(u,Su), d_{qb}(a^*,Ja^*)\right\}\right), \end{aligned}$$

this implies that

$$sd_{qb}(u,a^*) < d_{qb}(u,a^*).$$

Which is contradiction. Then $d_{qb}(Su, Ja^*) = 0$. Also

$$\begin{aligned} & \tau + F\left(sd_{qb}(Sa^*, Ju)\right) \\ \leq\ & F\left(\max\left\{d_{qb}(a^*,u), \frac{d_{qb}\left(a^*,Sa^*\right).d_{qb}\left(u,Ju\right)}{1+\max\{d_{qb}\left(a^*,u\right), d_{qb}\left(u,a^*\right)\}}, d_{qb}(a^*,Sa^*), d_{qb}(u,Ju)\right\}\right), \end{aligned}$$

Then, we get $d_{qb}(Sa^*, Ju) = 0$. So, $a^* = u$. ■

3.4.12 Corollary

Consider (L, d_{q_b}) be a complete dislocated b-quasi metric space, $s \geq 1$ and $S, J : L \to L$ are two self mappings. If there exists $F \in \mathcal{F}_S$ and $\tau, c > 0$ such as for any two consecutive points a, y belonging to the range of an iterative sequence $\{JS(a_n)\}$ with $\max\{d_{qb}(Sa,Jy), D_{qb}(a,y)\} > 0$, we have

$$\tau + F(sd_{qb}(Sa,Jy)) \leq F(D_{q_b}(a,y)), \tag{3.66}$$

where

$$D_{q_b}(a,y) = \max\left\{d_{qb}(a,y), \frac{d_{qb}(a,Sa).d_{qb}(y,Jy)}{c+d_{qb}(a,y)}, d_{qb}(a,Sa), d_{qb}(y,Jy)\right\}.$$

Then $\{JS(a_n)\} \to u \in L$. Also, if u satisfies (3.66), then S and J have a unique common fixed point in L and $d_{q_b}(u,u) = 0$.

Let $\mathcal{F}$ is the set of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ defined by [100].

3.4.13 Corollary

Let (L, d_{q_b}) be a complete dislocated quasi metric space and $S, J : L \to L$ are two self mappings. If there exists $F \in \mathcal{F}$ and $\tau, c > 0$ such as for any two consecutive points a, y belonging to the range of an iterative sequence $\{JS(a_n)\}$ with $\max\{d_{qb}(Sa,Jy), d_{qb}(Jy,Sa), D_{qb}(a,y), D_{qb}(y,a)\} > 0$, we have

$$\tau + \max\{F(d_{qb}(Sa,Jy)), F(d_{qb}(Jy,Sa))\} \leq \min\{F(D_{q_b}(a,y)), F(D_{q_b}(y,a))\}, \tag{3.67}$$

where

$$D_{q_b}(a,y) = \max\left\{d_{qb}(a,y), \frac{d_{qb}(a,Sa).d_{qb}(y,Jy)}{c+\max\{d_{qb}(a,y), d_{qb}(y,a)\}}, d_{qb}(a,Sa), d_{qb}(y,Jy)\right\}.$$

Then $\{JS(a_n)\} \to u \in L$. If u satisfies (3.67), then the pair (S, J) has a unique common fixed point u in L and $d_{q_b}(u,u) = 0$.

Now, as an application, we discuss the application of Theorem 3.4.11 to find solution of the system of Volterra type integral equations. Consider the following integral equations:

$$u(t) = \int_0^t K_1(t,s,u(s))ds, \tag{3.68}$$

$$v(t) = \int_0^t K_2(t,s,v(s))ds \tag{3.69}$$

for each $t \in [I]$. We find the solution of (3.68) and (3.69). Let $L = C([I],[0,+\infty[)$ is the set of each continuous functions on $[I]$, endowed with the complete dislocated b-quasi metric. For

$u \in C([I],[0,+\infty[)$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,1]} \{u(t)e^{-\tau t}\}$; $\tau > 0$ is arbitrary. Then define

$$d_\tau(u,v) = \left[\sup_{t\in[0,1]} \{(u(t)+2v(t))e^{-\tau t}\}\right]^2 = \|u+2v\|_\tau^2$$

for each $u, v \in C([I],[0,+\infty[)$, with these settings, $(C([I],[0,+\infty[),d_\tau)$ becomes a dislocated b-quasi metric space.

In the below theorem we prove the existence of solution of integral equations.

3.4.14 Theorem

Suppose that the below assumptions hold:

(i) $K_1, K_2 : [0,1]\times[0,1]\times[0,+\infty) \to [0,+\infty)$ and $f, g : [I] \to [0,+\infty)$ are continuous;

(ii) Define

$$\begin{aligned} Su(t) &= \int_0^t K_1(t,s,u(s))ds, \\ Jv(t) &= \int_0^t K_2(t,s,v(s))ds. \end{aligned}$$

Also, assume there exist $\tau > 1$

$$\max\{K_1(t,s,u)+2K_2(t,s,v), K_2(t,s,v)+2K_1(t,s,u)\} \le \sqrt{\tau e^{2\tau s-\tau}\min\{M(u,v),M(v,u)\}}, \tag{3.70}$$

for any $t, s \in [I]$ and $u, v \in C([I],[0,+\infty[)$, where

$$M(u,v) = \max\left\{ \begin{array}{c} ||u+2v||^2, \frac{||u+2Su||^2||v+2Jv||^2}{c+\max\{||u+2v||^2,||v+2u||^2\}}, \\ ||u+2Su||^2, ||v+2Jv||^2 \end{array} \right\}.$$

Then, integral equations (3.68) and (3.69) has a unique solution.

Proof. By assumption (ii) and (3.70), we have

$$\begin{aligned}\max\{Su+2Jv, Jv+2Su\} &= \max\left\{\int_0^t (K_1(t,s,u)+2K_2(t,s,v))ds, \int_0^t (K_2(t,s,v)+2K_1(t,s,u))ds\right\} \\ &\leq \int_0^t \sqrt{\tau e^{2\tau s-\tau}\min\{M(u,v),M(v,u)\}}ds.\end{aligned}$$

$$\begin{aligned}(\max\{Su+2Jv, Jv+2Su\})^2 &\leq \tau e^{-\tau}\min\{M(u,v),M(v,u)\}\int_0^t e^{2\tau s}ds \\ &\leq \frac{1}{2}e^{-\tau}\min\{M(u,v),M(v,u)\}e^{2\tau t}.\end{aligned}$$

This implies,

$$\left(\max\{Su+2Jv, Jv+2Su\}e^{-\tau t}\right)^2 \leq \frac{1}{2}e^{-\tau}\min\{M(u,v),M(v,u)\}.$$

That is,

$$2\|\max\{Su+2Jv, Jv+2Su\}\|_\tau^2 \leq e^{-\tau}\min\{M(u,v),M(v,u)\},$$

which further implies,

$$\begin{aligned}\tau+2\ln\|\max\{Su+2Jv, Jv+2Su\}\|_\tau^2 &\leq \ln\min\{M(u,v),M(v,u)\}, \\ \tau+\max\{s\ln\|Su+2Jv\|_\tau^2, s\ln\|Jv+2Su\}\|_\tau^2\} &\leq \ln\min\{M(u,v),M(v,u)\}.\end{aligned}$$

Thus, all the assumptions of Theorem 3.4.11 are hold for $F(a)=\ln a$, $d_\tau(u,v)=\|u+2v\|_\tau^2$, $s=2$. Thus, the system of integral equations given in (3.68) and (3.69) has a common unique solution. ■

Chapter 4

Double Controlled Quasi and Dislocated Quasi Metric Type Spaces

Theory and definitions given in this section have been published in [91, 93, 97].

4.1 Introduction

Abdeljawad *et al.* [2] generalized the idea of controlled metric type spaces and introduced double controlled metric type spaces. They replaced the control function $\theta(a, y)$ in the triangle inequality by two control functions $\alpha(a, y)$ and $\mu(a, y)$. In this chapter we introduce double controlled quasi, dislocated quasi metric type spaces and obtain fixed points, common fixed points of single-valued and multivalued mappings satisfying different type contractions.

Section 4.1, contains some basic definitions of the concept of double controlled quasi and dislocated quasi metric type spaces. In section 4.2, we introduce and prove some unique fixed point results involving new types of contraction single-valued maps in double controlled quasi metric type spaces. In section 4.3, we introduce some fixed points of multivalued mappings satisfying rational type contractions, common fixed points of Reich type and Kannan type contractions in double controlled quasi metric type spaces. In section 4.4, we obtain fixed point results for a pair of multi-valued maps satisfying Kannan type double controlled contraction in a left K-sequentially complete double controlled dislocated quasi metric type space.

4.1.1 Definition

Given $\alpha, \mu : L \times L \to [1, +\infty)$. If $\rho_q : L \times L \to [0, +\infty)$ satisfies the below conditions:
(ρ_q1) $\rho_q(a, y) = \rho_q(y, a) = 0$, then $a = y$;
(ρ_q2) $\rho_q(a, y) \leq \alpha(a, e)\rho_q(a, e) + \mu(e, y)\rho_q(e, y)$, for all $a, y, e \in L$.

Then, ρ_q is called a double controlled dislocated quasi metric type with the functions α, μ and (L, ρ_q) is called a double controlled dislocated quasi metric type space. If $\mu(e, y) = \alpha(e, y)$ then (L, ρ_q) is called a controlled dislocated quasi metric type space. If $a = y$, then (L, ρ_q) is called a double controlled quasi metric type space. For $a \in L$ and $\varepsilon > 0$, $B_{\rho_q}(a, \varepsilon) = \{y \in L : \rho_q(a, y) < \varepsilon$ and $\rho_q(y, a) < \varepsilon\}$ and $\overline{B_{\rho_q}(a, \varepsilon)} = \{y \in L : \rho_q(a, y) \leq \varepsilon$ and $\rho_q(y, a) \leq \varepsilon\}$ are open and closed ball in (L, ρ_q) respectively.

4.1.2 Remark

Any quasi metric space, double controlled metric type space and controlled quasi metric type space are double controlled quasi metric type space but, the converse is not true in general (see examples 4.1.3, 4.2.2, 4.2.5, 4.3.4, 4.3.12 and 4.3.15).

4.1.3 Example

Let $L = \{0, 1, 2\}$. Defined $q : L \times L \to [0,+\infty)$ by $q(0, 1) = 4$, $q(0, 2) = 1$, $q(1, 0) = 3 = q(1, 2)$, $q(2, 0) = 0$, $q(2, 1) = 2$, $q(0, 0) = q(1, 1) = q(2, 2) = 0$.
Defined $\alpha, \mu : L \times L \to [1,+\infty)$ as $\alpha(0, 1) = \alpha(1, 0) = \alpha(1, 2) = 1$, $\alpha(0, 2) = \frac{5}{4}$, $\alpha(2, 0) = \frac{10}{9}$, $\alpha(2, 1) = \frac{20}{19}$, $\alpha(0, 0) = \alpha(1, 1) = \alpha(2, 2) = 1$, $\mu(0, 1) = \mu(1, 0) = \mu(0, 2) = \mu(1, 2) = 1$, $\mu(2, 0) = \frac{3}{2}$, $\mu(2, 1) = \frac{11}{8}$, $\mu(0, 0) = \mu(1, 1) = \mu(2, 2) = 1$.
Note that the usual triangle inequality in quasi metric is not satisfied. Let $a = 0$, $e = 2$, $y = 1$, we have

$$q(0, 1) = 4 > 3 = q(0, 2) + q(2, 1).$$

Clearly q is double controlled quasi metric type for all $a, y, e \in L$, but it is not a controlled quasi metric type. In fact,

$$q(0,1) = 4 > \frac{255}{76} = \alpha\,(0,2)\,q(0,2) + \alpha(2,1)q(2,1).$$

Also, it is not double controlled metric type space because, we have

$$q(0,1) = 4 = \alpha\,(0,2)\,q(0,2) + \mu(2,1)q(2,1) \neq q(1,0).$$

4.1.4 Definition

Let (L, ρ_q) be a double controlled dislocated quasi metric type space with two functions. A sequence $\{a_c\}$ is convergent to some a in L if and only if $\lim_{c\to+\infty} \rho_q(a_c, a) = \lim_{c\to+\infty} \rho_q(a, a_c) = 0$.

4.1.5 Definition

Let (L, ρ_q) be a double controlled dislocated quasi metric type space with two functions.
(i) A sequence $\{a_c\}$ is a left Cauchy if and only if for every $\varepsilon > 0$ such as $\rho_q(a_m, a_c) < \varepsilon$, for all $c > m > c_\varepsilon$, where c_ε is some integer or $\lim_{c,m\to+\infty} \rho_q(a_m, a_c) = 0$.
(ii) A sequence $\{a_c\}$ is a right Cauchy if and only if for every $\varepsilon > 0$ such as $\rho_q(a_m, a_c) < \varepsilon$, for all $m > c > c_\varepsilon$, where c_ε is some integer.
(iii) The sequence $\{a_c\}$ is a dual Cauchy if and only if it is left Cauchy as well as right Cauchy.

4.1.6 Definition

Let (L, ρ_q) be double controlled dislocated quasi metric type space, then
(i) Ever left-Cauchy sequence in L is convergent $\Leftrightarrow$ it is left complete.
(ii) Each right-Cauchy sequence in L is convergent $\Leftrightarrow$ it is right-complete .
(iii) Every left-Cauchy as well as right-Cauchy sequence in L is convergent $\Leftrightarrow$ it is dual complete.

4.1.7 Definition

Consider (L, ρ_q) be a double controlled dislocated quasi metric type space. Let A be a nonempty subset of L and $l \in L$. A point $y_0 \in A$ is called a best approximation in A if

$$\begin{aligned} \rho_q(l, A) &= \rho_q(l, y_0), \text{ where } \rho_q(l, A) = \inf_{y \in A} \rho_q(l, y) \\ \text{and } \rho_q(A, l) &= \rho_q(y_0, l), \text{ where } \rho_q(A, l) = \inf_{y \in A} \rho_q(y, l). \end{aligned}$$

If each $l \in L$ has a best approximation in A, then A is known as proximinal set. $P(L)$ is equal to the set of all proximinal subsets of L.

4.1.8 Definition

The function $H_{\rho_q} : P(L) \times P(L) \to [0, +\infty)$ defined by

$$H_{\rho_q}(C, F) = \max \left\{ \sup_{a \in C} \rho_q(a, F), \ \sup_{b \in F} \rho_q(C, b) \right\}$$

is called Hausdorff double controlled dislocated quasi metric type on $P(L)$. Also $(P(L), H_{\rho_q})$ is known as Hausdorff double controlled dislocated quasi metric type space. Following similar arguments of Lemma 1.7 given by Shoaib [92], we obtain the below Lemma.

4.1.9 Lemma

Let (L, ρ_q) be a double controlled dislocated quasi metric type space. Let $(P(L), H_{\rho_q})$ is a Hausdorff double controlled dislocated quasi metric type space on $P(L)$. Then, for all $C, F \in P(L)$ and for each $l \in C$, there exists $b_l \in F$, such as $H_{\rho_q}(C, F) \geq \rho_q(l, b_l)$ and $H_{\rho_q}(F, C) \geq \rho_q(b_l, l)$.

4.1.10 Remark

Any dislocated quasi metric space or double controlled metric type space is double controlled dislocated quasi metric type space but, the converse is not true in general. Also, a controlled dislocated quasi metric type space is also double controlled dislocated quasi metric type space, but the converse in general is not true (see examples 4.1.11 and 4.4.4).

4.1.11 Example

Let $L = \{0,1,2,3\}$. Define $\rho_q : L \times L \to [0,+\infty)$ by: $\rho_q(0,1) = 0$, $\rho_q(0,2) = 1$, $\rho_q(0,3) = \frac{1}{4}$, $\rho_q(1,0) = \frac{1}{2}$, $\rho_q(1,2) = 2$, $\rho_q(1,3) = \frac{1}{3}$, $\rho_q(2,0) = \frac{1}{2}$, $\rho_q(2,1) = 1$, $\rho_q(2,3) = \frac{1}{3}$, $\rho_q(3,0) = \frac{3}{2}$, $\rho_q(3,1) = 2$, $\rho_q(3,2) = \frac{1}{4}$, $\rho_q(0,0) = \frac{1}{2}$, $\rho_q(1,1) = 0$, $\rho_q(2,2) = 2$, $\rho_q(3,3) = 0$. Define $\alpha, \mu : L \times L \to [1,+\infty)$ as: $\alpha(0,1) = \alpha(1,2) = \alpha(0,2) = 1$, $\alpha(1,0) = \frac{4}{3}$, $\alpha(2,0) = 2$, $\alpha(3,1) = \frac{4}{3}$, $\alpha(2,3) = 3$, $\alpha(0,3) = \frac{4}{3}$, $\alpha(1,3) = 3$, $\alpha(2,1) = 1$, $\alpha(3,0) = \frac{4}{3}$, $\alpha(3,2) = 2$, $\alpha(0,0) = \alpha(1,1) = \alpha(2,2) = \alpha(3,3) = 1$,

$\mu(1,2) = \mu(2,1) = \frac{3}{2}$, $\mu(2,0) = 2$, $\mu(3,0) = \mu(0,3) = \mu(1,0) = \mu(0,1) = \mu(1,3) = \mu(3,1) = 1$, $\mu(3,2) = 4$, $\mu(2,3) = 1$, $\mu(0,2) = \frac{4}{3}$, $\mu(0,0) = \mu(1,1) = \mu(2,2) = \mu(3,3) = 1$. It is obvious that ρ_q is double controlled dislocated quasi metric type for all $a, y, e \in L$. It is clear that ρ_q is not double controlled metric type space. Also, it is not controlled dislocated quasi metric type. Indeed,

$$\rho_q(1,2) = 2 > \frac{3}{2} = \alpha(1,3)\,\rho_q(1,3) + \alpha(3,2)\rho_q(3,2).$$

4.2 Double Controlled Quasi Metric Type Spaces and Some Results

Results given in this section have been published in [97]

In this section, we generalize the definition of fixed point for double controlled quasi metric type spaces with two incomparable functions α and μ which are follows:

4.2.1 Theorem

Consider (L, q) be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and let $J : L \to L$ is a given map. Assume that the below restrictions are satisfied:

There exists $k \in (0,1)$ such as

$$q(Jv, Jy) \leq kq(v, y), \text{ for all } v, y \in L. \tag{4.1}$$

For $a_0 \in L$, choose $a_c = J^c a_0$. Assume that

$$\lim_{i,m\to+\infty} \frac{\alpha\left(a_{i+1},a_{i+2}\right)}{\alpha\left(a_i,a_{i+1}\right)}\mu\left(a_{i+1},a_m\right) < \frac{1}{k}. \tag{4.2}$$

In addition, for every $a \in L$, we have

$$\lim_{c\to+\infty} \alpha\left(a,a_c\right) \text{ and } \lim_{c\to+\infty} \mu\left(a_c,a\right) \text{ exist and are finite.} \tag{4.3}$$

Then, J has a unique fixed point $a^* \in L$.

Proof. Let $a_0 \in L$ is any element. $\{a_c\}$ is the sequence defined as above, if $a_0 = Ja_0$, then a_0 is a fixed point of J. By (4.1), we have

$$q(a_c, a_{c+1}) \leq k^c q(a_0, a_1), \;\; c \in \mathbb{N}. \tag{4.4}$$

For each natural numbers $c < m$, we have

$$\begin{aligned}
q(a_c, a_m) &\leq \alpha(a_c, a_{c+1})q(a_c, a_{c+1}) + \mu(a_{c+1}, a_m)q(a_{c+1}, a_m) \\
&\leq \alpha(a_c, a_{c+1})q(a_c, a_{c+1}) + \mu(a_{c+1}, a_m)\alpha(a_{c+1}, a_{c+2})q(a_{c+1}, a_{c+2}) \\
&\quad +\mu(a_{c+1}, a_m)\mu(a_{c+2}, a_m)q(a_{c+2}, a_m) \\
&\leq \alpha(a_c, a_{c+1})q(a_c, a_{c+1}) + \mu(a_{c+1}, a_m)\alpha(a_{c+1}, a_{c+2})q(a_{c+1}, a_{c+2}) \\
&\quad +\mu(a_{c+1}, a_m)\mu(a_{c+2}, a_m)\alpha(a_{c+2}, a_{c+3})q(a_{c+2}, a_{c+3}) \\
&\quad +\mu(a_{c+1}, a_m)\mu(a_{c+2}, a_m)\mu(a_{c+3}, a_m)q(a_{c+3}, a_m) \leq ...
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha(a_c,a_{c+1})q(a_c,a_{c+1}) + \sum_{i=c+1}^{m-2}\left(\prod_{j=c+1}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)q(a_i,a_{i+1}) \\
&\quad + \prod_{k=c+1}^{m-1}\mu\left(a_k,a_m\right)q(a_{m-1},a_m) \\
&\leq \alpha(a_c,a_{c+1})q(a_c,a_{c+1}) + \sum_{i=c+1}^{m-2}\left(\prod_{j=c+1}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)k^i q(a_0,a_1) \\
&\quad + \prod_{i=c+1}^{m-1}\mu\left(a_i,a_m\right)k^{m-1}q(a_0,a_1) \\
&\leq \alpha(a_c,a_{c+1})k^c q(a_0,a_1) + \sum_{i=c+1}^{m-2}\left(\prod_{j=c+1}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)k^i q(a_0,a_1) \\
&\quad + \prod_{i=c+1}^{m-1}\mu\left(a_i,a_m\right)k^{m-1}\alpha\left(a_{m-1},a_m\right)q(a_0,a_1) \\
&= \alpha(a_c,a_{c+1})k^c q(a_0,a_1) + \sum_{i=c+1}^{m-1}\left(\prod_{j=c+1}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)k^i q(a_0,a_1) \\
&\leq \alpha(a_c,a_{c+1})k^c q(a_0,a_1) + \sum_{i=c+1}^{m-1}\left(\prod_{j=0}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)k^i q(a_0,a_1).
\end{aligned}
$$

Let, $S_c = \sum_{i=0}^{c}\left(\prod_{j=0}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)k^i$.
Hence, we have

$$
q(a_c,a_m) \leq q(a_0,a_1)\left[k^c\alpha(a_c,a_{c+1}) + (S_{m-1}-S_c)\right]. \tag{4.5}
$$

Let, $r_i = \left(\prod_{j=0}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)k^i$. By using (4.2), we have $\lim_{i\to+\infty}\frac{r_{i+1}}{r_i} < 1$. By ratio test the infinite series $\sum_{i=1}^{+\infty}\left(\prod_{j=0}^{i}\mu\left(a_j,a_m\right)\right)\alpha\left(a_i,a_{i+1}\right)k^i$ is convergent and let c,m tending to infinity in (4.5), it implies that

$$
\lim_{c,m\to+\infty} q(a_c,a_m) = 0. \tag{4.6}
$$

Since (L,q) is a left complete double controlled quasi metric type space, there exists some $a^* \in L$ such as

$$
\lim_{c\to+\infty} q(a_c,a^*) = \lim_{c\to+\infty} q(a^*,a_c) = 0. \tag{4.7}
$$

By (4.1) and using the triangle inequality, we have

$$q(a^*, Ja^*) \leq \alpha(a^*, a_{c+1})q(a^*, a_{c+1}) + \mu(a_{c+1}, Ja^*)q(a_{c+1}, Ja^*)$$

$$\leq \alpha(a^*, a_{c+1})q(a^*, a_{c+1}) + k\mu(a_{c+1}, Ja^*)q(a_c, a^*).$$

By taking limit c tends to infinity together with (4.3) and (4.7), we get $q(a^*, Ja^*) = 0$, that is $Ja^* = a^*$. Now, we have to show that the fixed point of J is unique for this, let $\xi \in L$ is another fixed point of J such as $J\xi = \xi$ and $a^* \neq \xi$, we have

$$q(a^*, \xi) = q(Ja^*, J\xi) \leq kq(a^*, \xi).$$

So $a^* = \xi$. Hence, a^* is the unique fixed point of J. ■

4.2.2 Example

Let $L = \{0, 1, 2\}$. Define $q : L \times L \to [0,+\infty)$ and $\alpha, \mu : L \times L \to [1, +\infty)$ by

$q(v,y)$	0	1	2
0	0	$\frac{3}{4}$	$\frac{1}{8}$
1	$\frac{2}{5}$	0	$\frac{4}{5}$
2	$\frac{1}{5}$	$\frac{1}{4}$	0

,

$\alpha(v,y)$	0	1	2
0	1	$\frac{21}{20}$	2
1	$\frac{3}{2}$	1	1
2	1	1	1

and

$\mu(v,y)$	0	1	2
0	1	$\frac{11}{10}$	$\frac{5}{3}$
1	1	1	$\frac{10}{9}$
2	1	2	1

.

It is easy to see that (L, q) is a double controlled quasi-metric type and the given function q is not a controlled metric type for the function α. Indeed,

$$q(1, 2) = \frac{4}{5} > \frac{63}{80} = \alpha(1, 0)\, q(0, 2) + \alpha(0, 1)q(0, 1).$$

Take $J0 = J2 = 0$, $J1 = 2$ and $k = \frac{1}{2}$. We observe the below cases:

(i) If $v = 0$, $y = 1$, we have

$$q(Jv, Jy) = \frac{1}{8} < \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}.$$

Also, hold if $v = 1$, $y = 0$.

(ii) It is clear, in the case $v = 0$, $y = 2$. Also, if $v = 2$, $y = 0$,

(iii) If $v = 1$, $y = 2$, we get

$$\frac{1}{5} \leq \frac{1}{2} \times \frac{4}{5} = \frac{2}{5}.$$

Similarly, in case $v = 2$, $y = 1$. So (4.1) holds. Now let $a_0 = 2$, we have $a_1 = Ja_0 = J2 = 0$, $a_2 = 0$, $a_3 = 0, \cdots$

$$\lim_{i,m \to +\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) = 1 < 2 = \frac{1}{k}.$$

That is, (4.2) holds. In addition, for each $a \in L$, we have

$$\lim_{c \to +\infty} \alpha(a, a_c) = \alpha(a, 0) < +\infty \text{ and } \lim_{c \to +\infty} \mu(a_c, a) = \mu(0, a) < +\infty.$$

That is, (4.3) holds. Hence all hypothesis of Theorem 4.2.1 hold and $a = 0$ is a unique fixed point.

Now, we introduce the concept of $\alpha - \mu - k$ double controlled contraction and prove related fixed point results with some examples.

4.2.3 Definition

Let $L \neq \{\}$, (L, q) be a left complete double controlled quasi metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and $J : L \to L$ is a given mapping. Assume that there exists a real number k such as

$$\begin{aligned} h_1 &= \sup\{k\alpha(v, y),\ v, y \in L\} < \frac{1}{2}. \\ h_2 &= \sup\{k\mu(v, y),\ v, y \in L\} < \frac{1}{2}. \end{aligned}$$

Suppose that:

$$q(Jv, Jy) \leq k[q(v, Jy) + q(y, Jv)], \text{ for all } v, y \in L. \tag{4.8}$$

For $a_0 \in L$ and $a_c = J^c a_0$, we have

$$\lim_{i,m \to +\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) < \frac{1-h}{h}, \tag{4.9}$$

where $\max\{h_1, h_2\} = h$. Then, J is called $\alpha - \mu - k$ double controlled contraction.

4.2.4 Theorem

Let (L, q) be a left complete double controlled quasi metric type space with the functions α, μ $: L \times L \to [1, +\infty)$ and let $J : L \to L$ is $\alpha - \mu - k$ double controlled contraction. Suppose that, for each $a \in L$, we have

$$\lim_{c\to+\infty} \alpha(a, a_c) < +\infty \text{ and } \lim_{c\to+\infty} \mu(a_c, a) < +\infty. \tag{4.10}$$

Then, J has a unique fixed point $a^* \in L$.

Proof. Consider $a_0 \in L$ is any element. $\{a_c\}$ is the sequence defined as above, if $a_0 = Ja_0$, then a_0 is a fixed point of J. By (4.8), we have

$$\begin{aligned}
q(a_c, a_{c+1}) &= q(Ja_{c-1}, Ja_c) \leq k\left[q(a_{c-1}, Ja_c) + q(a_c, Ja_{c-1})\right] \\
&\leq k\left[q(a_{c-1}, a_{c+1}) + q(a_c, a_c)\right] \\
&\leq k\alpha(a_{c-1}, a_c)q(a_{c-1}, a_c) + k\mu(a_c, a_{c+1})q(a_c, a_{c+1}) \\
&\leq h_1 q(a_{c-1}, a_c) + h_2 q(a_c, a_{c+1}), \text{ (by Definition 4.2.3)} \\
&\leq hq(a_{c-1}, a_c) + hq(a_c, a_{c+1}) \\
q(a_c, a_{c+1}) &\leq \frac{h}{1-h} q(a_{c-1}, a_c). \qquad (4.11)
\end{aligned}$$

Now,

$$\begin{aligned}
q(a_{c-1}, a_c) &= q(Ja_{c-2}, Ja_{c-1}) \leq k\left[q(a_{c-2}, a_c) + q(a_{c-1}, a_{c-1})\right] \\
&\leq k\alpha(a_{c-2}, a_{c-1})q(a_{c-2}, a_{c-1}) + k\mu(a_{c-1}, a_c)q(a_{c-1}, a_c) \\
&\leq h_1 q(a_{c-2}, a_{c-1}) + h_2 q(a_{c-1}, a_c) \text{ (by Definition 4.2.3)}, \\
q(a_{c-1}, a_c) &\leq \frac{h}{1-h} q(a_{c-2}, a_{c-1}). \qquad (4.12)
\end{aligned}$$

Combining (4.11) and (4.12), we get

$$q(a_c, a_{c+1}) \leq \left(\frac{h}{1-h}\right)^2 q(a_{c-2}, a_{c-1}). \tag{4.13}$$

Continuing in this way, we obtain

$$q(a_c, a_{c+1}) \leq \left(\frac{h}{1-h}\right)^c q(a_0, a_1). \tag{4.14}$$

Now, to prove that $\{a_c\}$ is a Cauchy sequence, for each natural numbers $c < m$, by (4.14) and using the technique given in Theorem 4.2.1, we have

$$S_c = \sum_{i=0}^{c} \left(\prod_{j=0}^{i} \mu\left(a_j, a_m\right)\right) \alpha\left(a_i, a_{i+1}\right) \left(\frac{h}{1-h}\right)^i.$$

Hence, we have

$$q(a_c, a_m) \leq q(a_0, a_1)\left[\left(\frac{h}{1-h}\right)^c \alpha(a_c, a_{c+1}) + (S_{m-1} - S_c)\right]. \tag{4.15}$$

The i^{th} term of the sequence $\{S_c\}$ is $r_i = \left(\prod\limits_{j=0}^{i} \mu\left(a_j, a_m\right)\right) \alpha\left(a_i, a_{i+1}\right) \left(\frac{h}{1-h}\right)^i$. By (4.9), we have $\lim\limits_{i\to+\infty} \frac{r_{i+1}}{r_i} < 1$. By ratio test $\{S_c\}$ converges and so $\{S_c\}$ is Cauchy. Letting c, m tend to $+\infty$ in (4.15) yields

$$\lim_{c,m\to+\infty} q(a_c, a_m) = 0. \tag{4.16}$$

So the sequence $\{a_c\}$ is a left Cauchy. As (L, q) is a left complete double controlled quasi metric type space, there is some $a^* \in L$ such as

$$\lim_{c\to+\infty} q(a_c, a^*) = 0 = \lim_{c\to+\infty} q(a^*, a_c). \tag{4.17}$$

We claim that $Ja^* = a^*$. By (4.8), we have

$$q(a^*, Ja^*) \leq \alpha(a^*, a_{c+1})q(a^*, a_{c+1}) + \mu(a_{c+1}, Ja^*)q(a_{c+1}, Ja^*)$$

$$
\begin{aligned}
&\leq \alpha(a^*, a_{c+1})q(a^*, a_{c+1}) + \mu(a_{c+1}, Ja^*)k\left[q(a_c, Ja^*) + q(a^*, a_{c+1})\right] \\
&\leq \alpha(a^*, a_{c+1})q(a^*, a_{c+1}) + \mu(a_{c+1}, Ja^*)k^2\left[q(a_{c-1}, Ja^*) + q(a^*, a_c)\right] \\
&\quad +\mu(a_{c+1}, Ja^*)kq(a^*, a_{c+1}) \\
&\leq \alpha(a^*, a_{c+1})q(a^*, a_{c+1}) + \mu(a_{c+1}, Ja^*)k^2[\alpha(a_{c-1}, a^*)q(a_{c-1}, a^*) \\
&\quad +\mu(a^*, Ja^*)q(a^*, Ja^*) + q(a^*, a_c)] + \mu(a_{c+1}, Ja^*)kq(a^*, a_{c+1}) \\
&\leq \alpha(a^*, a_{c+1})q(a^*, a_{c+1}) + \mu(a_{c+1}, Ja^*)k^2\alpha(a_{c-1}, a^*)q(a_{c-1}, a^*) \\
&\quad + (h_2)^2\, q(a^*, Ja^*) + \mu(a_{c+1}, Ja^*)k^2q(a^*, a_c) + \mu(a_{c+1}, Ja^*)kq(a^*, a_{c+1}).
\end{aligned}
$$

By taking limit as c tends to infinity together with (4.17), we get

$$\left(1 - (h_2)^2\right) q(a^*, Ja^*) \leq 0.$$

Hence, $a^* = Ja^*$, which is a contradiction. Let a^{**} in L is another fixed point of J such as $Ja^{**} = a^{**}$ and $a^* \neq a^{**}$, we have

$$
\begin{aligned}
q(a^*, a^{**}) &= q(Ja^*, Ja^{**}) \leq k\left[q(a^*, Ja^{**}) + q(a^{**}, Ja^*)\right] \\
&\leq k\left[q(a^*, a^{**}) + q(a^{**}, a^*)\right] \\
&\leq \left(\frac{k}{1-k}\right) q(a^{**}, a^*) \\
&\leq \left(\frac{k}{1-k}\right)^2 q(a^*, a^{**}) \\
&\vdots \\
&\leq \left(\frac{k}{1-k}\right)^{2n} q(a^*, a^{**}).
\end{aligned}
$$

By taking limit as n tends to infinity, we have $a^* = a^{**}$. Hence, a^* is a unique fixed point of J.

■

4.2.5 Example

Take $L = \{0, 1, 2\}$. Define $q : L \times L \to [0,+\infty)$ and $\alpha, \mu : L \times L \to [1, +\infty)$ by

$q(v,y)$	0	1	2
0	0	$\frac{2}{5}$	$\frac{1}{4}$
1	$\frac{9}{20}$	0	$\frac{4}{5}$
2	$\frac{1}{5}$	$\frac{7}{10}$	0

,

$\alpha(a,y)$	0	1	2
0	1	$\frac{102}{100}$	1
1	$\frac{6}{5}$	1	1
2	$\frac{11}{10}$	1	1

and

$\mu(a,y)$	0	1	2
0	1	$\frac{6}{5}$	$\frac{11}{10}$
1	1	1	1
2	1	1	1

.

It is easy that (L,q) is a double controlled quasi-metric type. The given q is not a controlled metric space for the function α. Indeed,

$$q(2,1)=\frac{7}{10}>\frac{307}{500}=\alpha\left(2,0\right)q(2,0)+\alpha(0,1)q(0,1).$$

Take $J0=J2=2$, $J1=0$ and $k=\frac{2}{5}$, we observe the below cases:

(i) If $v=0$, $y=1$, we have

$$\begin{aligned} q\left(Jv,Jy\right) &= \frac{1}{5}\leq\frac{8}{25}=\frac{2}{5}\left[0+\frac{4}{5}\right] \\ &= k\left[q\left(v,Jy\right)+q\left(y,Jv\right)\right]. \end{aligned}$$

If $v=1$, $y=0$, we get

$$q\left(Jv,Jy\right)=\frac{1}{4}\leq\frac{8}{25}=k\left[q\left(v,Jy\right)+q\left(y,Jv\right)\right].$$

(ii) It is straightforward, in the case $v=0$, $y=2$.

(iii) If $v=1$, $y=2$, we get

$$\frac{1}{4}\leq\frac{2}{5}=\frac{2}{5}\left[\frac{4}{5}+\frac{1}{5}\right]=k\left[q\left(v,Jy\right)+q\left(y,Jv\right)\right].$$

Similarly, in the case when we take $v=2$, $y=1$, that is the inequality (4.8) holds. Now, we have

$$h_1=\sup\left\{k\alpha\left(v,y\right),\ v,y\in L\right\}<\frac{1}{2}\text{ and }h_2=\sup\left\{k\mu\left(v,y\right),\ v,y\in L\right\}<\frac{1}{2}.$$

$h=\max\{h_1,h_2\}=\frac{12}{25}$. Now, let $a_0=2$, we have $a_1=Ja_0=J2=2$, $a_2=2$, $a_3=2,\cdots$

$$\lim_{i,m\to+\infty}\frac{\alpha\left(a_{i+1},a_{i+2}\right)}{\alpha\left(a_i,a_{i+1}\right)}\mu\left(a_{i+1},a_m\right)=1<\frac{1-h}{h},$$

which shows that (4.9) holds. Now, for each $a \in L$, we have

$$\lim_{c \to +\infty} \alpha (a, a_c) < +\infty \text{ and } \lim_{c \to +\infty} \mu (a_c, a) < +\infty.$$

That is, (4.10) holds. All hypothesis of Theorem 4.2.4 are satisfied and $a = 2$ is unique fixed point.

4.2.6 Definition

Let (L, q) be a complete quasi b-metric space. $J : L \to L$ is called Chatterjee type b-contraction if the following conditions are satisfied:

$$q (Jv, Jy) \leq k [q (v, Jy) + q (y, Jv)],$$

for all $v, y \in L$, $k \in (0, \frac{1}{2})$ and

$$b < \frac{1 - kb}{kb}. \tag{4.18}$$

4.2.7 Theorem

Consider (L, q) be a complete quasi b-metric space and $J : L \to L$ is Chatterjee type b-contraction. Then, J has a unique fixed point.

4.2.8 Remark

In the example 4.2.5, q is quasi b-metric with $b = \frac{16}{13}$, but we can not apply Theorem 4.2.7 because J is not Chatterjee type b-contraction. Indeed, $b \not< \frac{1-kb}{kb}$, for all $b \geq \frac{16}{13}$.

4.3 Fixed Point Results in Double Controlled Quasi Metric Type Spaces

Results given in this section have been published in [93]

Let (L, q) be a double controlled quasi metric type space, $a_0 \in L$ and $J : L \to P(L)$ be multifunctions on L. Now, we consider the arguments of a sequence $\{LJ(a_c)\}$ as appearing in the beginning of section 2.2 and we say that $\{LJ(a_c)\}$ is a sequence in L generated by a_0 under double controlled quasi metric type q. We can define $\{LJ(a_c)\}$ in other metrics in a similar way.

4.3.1 Definition

Let (L, q) be a complete double controlled quasi-metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$. A multivalued mapping $J : L \to P(L)$ is called a double controlled rational contraction if the following conditions are satisfied:

$$H_q(Jv, Jy) \leq k(Q(v, y)), \tag{4.19}$$

for all $v, y \in L$, $0 < k < 1$ and

$$Q(v, y) = \max\left\{q(v, y), q(v, Jv), \frac{q(v, Jv)\, q(v, Jy) + q(y, Jy)\, q(y, Jv)}{q(v, Jy) + q(y, Jv)}\right\}.$$

Also, for $a_0 \in L$, take $a_c \in \{LJ(a_c)\}$ such as

$$\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) < \frac{1}{k}. \tag{4.20}$$

Now, we prove that an operator J satisfying certain rational contraction has a fixed point in double controlled quasi metric type space.

4.3.2 Theorem

Consider (L, q) be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and $J : L \to P(L)$ is a double controlled rational contraction. Suppose that, for every $a \in L$

$$\lim_{c \to +\infty} \alpha(a, a_c), \quad \lim_{c \to +\infty} \mu(a_c, a) \text{ exist and are finite.} \tag{4.21}$$

Then, J has a fixed point $a^* \in L$.

Proof. By Lemma 4.1.9 and using inequality (4.19), we have

$$q(a_c, a_{c+1}) \leq H_q(Ja_{c-1}, Ja_c) \leq k(Q(a_{c-1}, a_c)).$$

$$\begin{aligned} Q(a_{c-1},a_c) &\leq \max\left\{q\left(a_{c-1},a_c\right),q\left(a_{c-1},a_c\right),\right. \\ &\left.\frac{q\left(a_{c-1},a_c\right)q\left(a_{c-1},Ja_c\right)+q\left(a_c,a_{c+1}\right)q\left(a_c,Ja_{c-1}\right)}{q\left(a_{c-1},Ja_c\right)+q\left(a_c,Ja_{c-1}\right)}\right\} \\ &= q(a_{c-1},a_c). \end{aligned}$$

Therefore,

$$q\left(a_c,a_{c+1}\right) \leq kq(a_{c-1},a_c). \tag{4.22}$$

Now,

$$q\left(a_{c-1},a_c\right) \leq H_q\left(Ja_{c-2},Ja_{c-1}\right) \leq k\left(Q(a_{c-2},a_{c-1})\right).$$

$$\begin{aligned} Q(a_{c-2},a_{c-1}) &= \max\left\{q\left(a_{c-2},a_{c-1}\right),q\left(a_{c-2},a_{c-1}\right),\right. \\ &\left.\frac{q\left(a_{c-2},a_{c-1}\right)q\left(a_{c-2},Ja_{c-1}\right)+q\left(a_{c-1},a_c\right)q\left(a_{c-1},Ja_{c-2}\right)}{q\left(a_{c-2},Ja_{c-1}\right)+q\left(a_{c-1},Ja_{c-2}\right)}\right\}. \end{aligned}$$

Therefore,

$$q\left(a_{c-1},a_c\right) \leq kq(a_{c-2},a_{c-1}). \tag{4.23}$$

Using (4.23) in (4.22), we have

$$q\left(a_c,a_{c+1}\right) \leq k^2q(a_{c-2},a_{c-1}).$$

Continuing in this way, we obtain

$$q\left(a_c,a_{c+1}\right) \leq k^cq(a_0,a_1). \tag{4.24}$$

Now, by (4.24) and using the technique given in Theorem 4.2.1, it can easily be proved that $\{a_c\}$ is a left Cauchy sequence. So, for each natural numbers with $c < m$, we have

$$\lim_{c,m\to+\infty} q(a_c,a_m) = 0. \tag{4.25}$$

Since (L,q) is a left complete double controlled quasi metric type space, there is some $a^* \in L$ such as

$$\lim_{c\to+\infty} q(a_c,a^*) = \lim_{c\to+\infty} q(a^*,a_c) = 0. \tag{4.26}$$

By using the triangle inequality and then (4.19), we have

$$\begin{aligned} q(a^*,Ja^*) &\leq \alpha(a^*,a_{c+1})q(a^*,a_{c+1})+\mu(a_{c+1},Ja^*)q(a_{c+1},Ja^*) \\ &\leq \alpha(a^*,a_{c+1})q(a^*,a_{c+1})+\mu(a_{c+1},Ja^*)\max\left\{q\left(a_c,a^*\right),q\left(a_c,a_{c+1}\right),\right. \\ &\left.\frac{q\left(a_c,a_{c+1}\right)q\left(a_c,Ja^*\right)+q\left(a^*,Ja^*\right)q\left(a^*,a_{c+1}\right)}{q\left(a_c,Ja^*\right)+q\left(a^*,a_{c+1}\right)}\right\}. \end{aligned}$$

Using (4.21), (4.25) and (4.26), we get $q(a^*,Ja^*)\leq 0$. That is, $a^*\in Ja^*$. Thus, a^* is a fixed point of J. ■

4.3.3 Theorem

Consider (L,q) be a left complete double controlled quasi metric type space with the functions $\alpha,\mu:L\times L\to[1,+\infty)$ and $J:L\to L$ is a map such as:

$$q\left(Jv,Jy\right)\leq k\left(Q\left(v,y\right)\right),$$

for all $v,y\in L$, $0\leq k\leq 1$ and

$$Q\left(v,y\right)=\max\left\{q\left(v,y\right),q\left(v,Jv\right),\frac{q\left(v,Jv\right)q\left(v,Jy\right)+q\left(y,Jy\right)q\left(y,Jv\right)}{q(v,Jy)+q(y,Jv)}\right\}.$$

Suppose that, for every $a\in L$ and for the Picard sequence $\{a_c\}$

$$\lim_{c\to+\infty}\alpha\left(a,a_c\right),\quad \lim_{c\to+\infty}\mu\left(a_c,a\right)\ \text{are finite and}\ \sup_{m\geq1}\lim_{i\to+\infty}\frac{\alpha\left(a_{i+1},a_{i+2}\right)}{\alpha\left(a_i,a_{i+1}\right)}\mu\left(a_{i+1},a_m\right)<\frac{1}{k}.$$

Then J has a fixed point $a^*\in L$. We present the below example to illustrate Theorem 4.3.3.

4.3.4 Example

Let $L=\{0,1,2,3\}$. Define $q:L\times L\to[0,+\infty)$ by $q(0,1)=1$, $q(0,2)=4$, $q(0,3)=5$, $q(1,0)=0$, $q(1,2)=10$, $q(1,3)=1$, $q(2,0)=7$, $q(2,1)=3$, $q(2,3)=5$, $q(3,0)=3$, $q(3,1)=6$, $q(3,2)=2$, $q(0,0)=q(1,1)=q(2,2)=q(3,3)=0$. Define $\alpha,\mu:L\times L\to[1,+\infty)$ as $\alpha(0,1)=2$, $\alpha(1,3)=2$, $\alpha(2,1)=\frac{7}{3}$, $\alpha(3,0)=\frac{4}{3}$, $\alpha(3,2)=\frac{3}{2}$ and $\alpha(v,y)=1$, if otherwise, $\mu(1,2)=\mu(2,1)=\mu(2,0)=\mu(3,0)=\mu(0,3)=\mu(2,3)=\mu(3,1)=1$, $\mu(1,0)=\frac{3}{2}$, $\mu(0,1)=$

2, $\mu(1,3)=3$, $\mu(3,2)=4$, $\mu(0,2)=\frac{5}{2}$, $\mu(3,3)=\mu(2,2)=\mu(1,1)=\mu(0,0)=1$. Clearly (L,q) is a double controlled quasi metric type space, but it is not a controlled quasi metric type space. Indeed,

$$q(1,2)=10>4=\alpha\left(1,0\right)q(1,0)+\alpha(0,2)q(0,2).$$

Also, it is not double controlled metric type space. Take $J0=J1=0$, $J2=J3=1$ and $k=\frac{1}{3}$. We observe that

$$q\left(Jv,Jy\right)\leq k\left(Q\left(v,y\right)\right),\ \text{for all } v,y\in L.$$

Let $a_0=2$, we have $a_1=Ja_0=J2=1$, $a_2=Ja_1=0$, $a_3=Ja_2=0,\cdots$.

$$\sup_{m\geq 1}\lim_{i\to+\infty}\frac{\alpha\left(a_{i+1},a_{i+2}\right)}{\alpha\left(a_i,a_{i+1}\right)}\mu\left(a_{i+1},a_m\right)=2<3=\frac{1}{k}.$$

Also, for every $a\in L$, we have

$$\lim_{c\to+\infty}\alpha\left(a,a_c\right)<+\infty \text{ and } \lim_{c\to+\infty}\mu\left(a_c,a\right)<+\infty.$$

All assumption of Theorem 4.3.3 hold and $a^*=0$ is a fixed point.

4.3.5 Theorem

Let (L,q) be a left complete quasi b-metric space and $J:L\to L$ is a map. Assume that there is some $k\in(0,1)$ such as

$$q\left(Jv,Jy\right)\leq k\left(Q\left(v,y\right)\right)$$

whenever,

$$Q\left(v,y\right)=\max\left\{q\left(v,y\right),q\left(v,Jv\right),\frac{q\left(v,Jv\right)q\left(v,Jy\right)+q\left(y,Jy\right)q\left(y,Jv\right)}{q(v,Jy)+q(y,Jv)}\right\},$$

for all $v,y\in L$. Assume that $0<kb<1$. Then, J has a fixed point $a^*\in L$.

4.3.6 Remark

In the example 4.3.4, note that q is a quasi b metric with $b=\frac{10}{3}$, but we can not apply Theorem 4.3.5 for any $b=\frac{10}{3}$ and $k=\frac{1}{3}$, because $kb\nless 1$.

4.3.7 Theorem

Let (L, q) be a left complete quasi metric space and $J : L \to P(L)$ is a multivalued map. Suppose that there is some $0 < k < 1$ such as for any $v, y \in L$

$$H_q(Jv, Jy) \le k\left(\max\left\{q(v,y), q(v,Jv), \frac{q(v,Jv)\,q(v,Jy) + q(y,Jy)\,q(y,Jv)}{q(v,Jy)+q(y,Jv)}\right\}\right).$$

Then, J has a fixed point $a^* \in L$.

Now, we consider a sequence $\{JS(a_n)\}$ as in the beginning section 4.4. We introduce double controlled Reich type contraction.

4.3.8 Definition

Let $L \neq \{\}$, (L, q) be a left complete double controlled quasi-metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and $S, J : L \to P(L)$ be a multivalued mappings. Assume that:

$$H_q(Sv, Jy) \le t(q(v,y)) + k(q(v,Sv) + q(y,Jy))$$

and

$$H_q(Jv, Sy) \le t(q(v,y)) + k(q(v,Jv) + q(y,Sy)), \tag{4.27}$$

for each $v, y \in L$, $0 < t + 2k < 1$. For $a_0 \in L$, choose $a_c \in \{JS(a_c)\}$, we have

$$\sup_{m\ge1}\lim_{i\to+\infty}\frac{\alpha(a_{i+1},a_{i+2})}{\alpha(a_i,a_{i+1})}\mu(a_{i+1},a_m) < \frac{1-k}{t+k}. \tag{4.28}$$

Then the pair (S, J) is called a double controlled Reich type contraction.

The following results extend the results of Reich [81].

4.3.9 Theorem

Let $S, J : L \to P(L)$ be the multivalued mappings, (L, q) be a left complete double controlled quasi metric type space and (S, J) is a pair of double controlled Reich type contraction. Suppose that, for all $a \in L$

$$\lim_{c\to+\infty}\alpha(a,a_c)\ \text{ is finite and }\ \lim_{c\to+\infty}\mu(a_c,a) < \frac{1}{k}. \tag{4.29}$$

Then, S and J have a common fixed point $\dot{z}$ in L.

Proof. Consider the sequence $\{JS(a_c)\}$. Now, by Lemma 4.1.9, we have

$$q(a_{2c}, a_{2c+1}) \leq H_q(Ja_{2c-1}, Sa_{2c}) \tag{4.30}$$

By using the condition (4.27), we get

$$q(a_{2c}, a_{2c+1}) \leq t(q(a_{2c-1}, a_{2c}) + k(q(a_{2c-1}, Ja_{2c-1}) + q(a_{2c}, Sa_{2c})$$

$$\begin{aligned} &\leq t(q(a_{2c-1}, a_{2c}) + k\left(q(a_{2c-1}, a_{2c}) + q(a_{2c}, a_{2c+1})\right) \\ q(a_{2c}, a_{2c+1}) &\leq \frac{t+k}{1-k}\left(q(a_{2c-1}, a_{2c})\right) \\ &= \eta\left(q(a_{2c-1}, a_{2c})\right). \end{aligned} \tag{4.31}$$

Now, by Lemma 1.11, we have

$$q(a_{2c-1}, a_{2c}) \leq H_q(Sa_{2c-2}, Ja_{2c-1}).$$

So,by using the condition (4.27), we have

$$q(a_{2c-1}, a_{2c}) \leq tq(a_{2c-2}, a_{2c-1}) + k\left(q(a_{2c-2}, Sa_{2c-2}) + q(a_{2c-1}, Ja_{2c-1})\right)$$

$$\begin{aligned} &\leq tq(a_{2c-2}, a_{2c-1}) + k\left(q(a_{2c-2}, a_{2c-1}) + q(a_{2c-1}, a_{2c})\right) \\ &\leq \frac{t+k}{1-k}\left(q(a_{2c-2}, a_{2c-1})\right) = \eta\left(q(a_{2c-2}, a_{2c-1})\right), \end{aligned} \tag{4.32}$$

where $\eta = \frac{t+k}{1-k}$. Using (4.31) in (4.32), we have

$$q(a_{2c}, a_{2c+1}) \leq \eta^2 q(a_{2c-2}, a_{2c-1}). \tag{4.33}$$

Now, by Lemma 4.1.9 we have

$$q(a_{2c-2}, a_{2c-1}) \leq H_q(Ja_{2c-3}, Sa_{2c-2}).$$

Using the condition (4.27), we have

$$\begin{aligned} q(a_{2c-2}, a_{2c-1}) &\leq tq(a_{2c-3}, a_{2c-2}) + k\left(q(a_{2c-3}, a_{2c-2}) + q(a_{2c-2}, a_{2c-1})\right) \\ &\leq \eta^3\left(q(a_{2c-3}, a_{2c-2})\right). \end{aligned} \tag{4.34}$$

From (4.33) and (4.34), we have

$$\eta^2(q(a_{2c-2}, a_{2c-1})) \leq \eta^3(q(a_{2c-3}, a_{2c-2})). \tag{4.35}$$

Using (4.35) in (4.31), we have

$$q(a_{2c}, a_{2c+1}) \leq \eta^3(q(a_{2c-3}, a_{2c-2})).$$

Continuing in this way, we get

$$q(a_{2c}, a_{2c+1}) \leq \eta^{2c}(q(a_0, a_1)). \tag{4.36}$$

Similarly, by Lemma 4.1.9, we have

$$q(a_{2c-1}, a_{2c}) \leq \eta^{2c-1}(q(a_0, a_1)). \tag{4.37}$$

Combating inequality (4.36) and (2.37), we have

$$q(a_c, a_{c+1}) \leq \eta^c(q(a_0, a_1)). \tag{4.38}$$

Now, by using (4.38) and by using the technique given in [2], it can easily be proved that $\{a_c\}$ is a left Cauchy sequence. So, for all natural numbers with $c < m$, we have

$$\lim_{c,m \to +\infty} q(a_c, a_m) = 0. \tag{4.39}$$

Since (L, q) is a left complete double controlled quasi metric type space. So $\{a_c\} \to \dot{z} \in L$, that is

$$\lim_{c\to+\infty} q(a_c, \dot{z}) = \lim_{c\to+\infty} q(\dot{z}, a_c) = 0. \tag{4.40}$$

To show that $\dot{z}$ is a common fixed point. We claim that $q(\dot{z}, J\dot{z}) = 0$. On contrary suppose $q(\dot{z},J\dot{z}) > 0$. Now by Lemma 4.1.9, we have

$$q(a_{2c+1}, J\dot{z}) \leq H_q(Sa_{2c}, J\dot{z}).$$

$$q(a_{2c+1}, J\dot{z}) \leq t(q(a_{2c}, \dot{z})) + k\left[q(a_{2c}, a_{2c+1}) + q(\dot{z}, J\dot{z}))\right]. \tag{4.41}$$

Taking $\lim_{c\to+\infty}$ of inequality (4.41), we get

$$\lim_{c\to+\infty} q(a_{2c+1}, J\dot{z}) \leq t \lim_{c\to+\infty} q(a_{2c}, \dot{z}) + k \lim_{c\to+\infty} \left[q(a_{2c}, a_{2c+1}) + q(\dot{z}, J\dot{z}))\right].$$

By using inequalities (4.39) and (4.40), we get

$$\lim_{c\to+\infty} q(a_{2c+1}, J\dot{z}) \leq k(q(\dot{z}, J\dot{z})). \tag{4.42}$$

Now,

$$q(\dot{z}, J\dot{z}) \leq \alpha(\dot{z}, a_{2c+1})q(\dot{z}, a_{2c+1}) + \mu(a_{2c+1}, J\dot{z})q(a_{2c+1}, J\dot{z}).$$

Taking $\lim_{c\to+\infty}$ and by using inequalities (4.29), (4.40) and (4.42), we get

$$q(\dot{z}, J\dot{z}) < q(\dot{z}, J\dot{z}).$$

It is a contradiction, therefore $q(\dot{z}, J\dot{z}) = 0$. Thus, $\dot{z} \in J\dot{z}$. Now, suppose $q(\dot{z}, S\dot{z}) > 0$. By Lemma 4.1.9, we have

$$q(a_{2c}, S\dot{z}) \leq H_q(Ja_{2c-1}, S\dot{z}).$$

By inequality (4.27), we get

$$q(a_{2c}, S\dot{z}) \leq t(q(a_{2c-1}, \dot{z})) + k\left[q(a_{2c-1}, a_{2c}) + q(\dot{z}, S\dot{z})\right].$$

Taking $\lim\limits_{c\to+\infty}$ of above inequality, we get

$$\lim_{c\to+\infty} q(a_{2c}, S\dot{z}) \leq k(q(\dot{z}, S\dot{z})). \tag{4.43}$$

Now,

$$q(\dot{z}, S\dot{z}) \leq \alpha(\dot{z}, a_{2c})q(\dot{z}, a_{2c}) + \mu(a_{2c}, S\dot{z})q(a_{2c}, S\dot{z}).$$

Taking $\lim\limits_{c\to+\infty}$ and by using inequality (4.29), (4.40) and (4.43), we get

$$q(\dot{z}, S\dot{z}) < q(\dot{z}, S\dot{z}).$$

A contradiction. Thus, $\dot{z} \in S\dot{z}$. Hence, $\dot{z}$ is a common fixed point for S and J. ■

4.3.10 Theorem

Let (L, q) be a left complete double controlled quasi metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and $S, J : L \to L$ be the mappings such as:

$$q(Sv, Jy) \leq t(q(v, y)) + k(q(v, Sv) + q(y, Jy))$$

and

$$q(Jv, Sy) \leq t(q(v, y)) + k(q(v, Jv) + q(y, Sy)),$$

for each $v, y \in L$, $0 < t + 2k < 1$. Suppose that, for every $a \in L$ and for the Picard sequence $\{a_c\}$

$$\begin{aligned} \lim_{c\to+\infty} \alpha(a, a_c) \text{ is finite, } \lim_{c\to+\infty} \mu(a_c, a) &< \frac{1}{k} \text{ and} \\ \sup_{m\geq 1} \lim_{i\to+\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) &< \frac{1-k}{t+k}. \end{aligned}$$

Then S and J have a common fixed point $a^* \in L$.

4.3.11 Theorem

Consider (L,q) be a left complete quasi b-metric type space with the functions $\alpha,\mu : L\times L \to [1,+\infty)$ and $S,J : L \to L$ be the mappings such as:

$$q(Sv,Jy) \le t(q(v,y)) + k(q(v,Sv)+q(y,Jy))$$

and

$$q(Jv,Sy) \le t(q(v,y)) + k(q(v,Jv)+q(y,Sy)),$$

for each $v,y \in L$, $0 < t+2k < 1$ and $b < \frac{1-k}{t+k}$. Then S and J have a common fixed point $a^* \in L$.

4.3.12 Example

Let $L = \left\{0,\frac{1}{2},\frac{1}{4},1\right\}$. Define $q : L\times L \to [0,+\infty)$ by $q(0,\frac{1}{2}) = 1$, $q(0,\frac{1}{4}) = \frac{1}{3}$, $q(\frac{1}{4},0) = \frac{1}{5}$, $q(\frac{1}{2},0) = 1$, $q\left(\frac{1}{4},\frac{1}{2}\right) = 3$, $q\left(\frac{1}{4},1\right) = \frac{1}{2}$, $q\left(1,\frac{1}{4}\right) = \frac{1}{3}$ and $q(v,y) = |v-y|$, if otherwise. Define $\alpha,\mu : L\times L \to [1,+\infty)$ as follows $\alpha\left(\frac{1}{2},\frac{1}{4}\right) = \frac{16}{5}$, $\alpha(0,\frac{1}{4}) = \frac{3}{2}$, $\alpha(\frac{1}{4},1) = 3$, $\alpha(1,\frac{1}{4}) = \frac{12}{5}$ and $\alpha(v,y) = 1$, if otherwise. $\mu(0,\frac{1}{2}) = \frac{14}{5}$, $\mu(1,\frac{1}{2}) = 3$ and $\mu(v,y) = 1$, if otherwise. Clearly q is double controlled quasi metric type for all $v,y,e \in L$. Let, $J0 = \{0\}$, $J\frac{1}{2} = \left\{\frac{1}{4}\right\}$, $J\frac{1}{4} = \{0\}$, $J1 = \left\{\frac{1}{4}\right\}$, $S0 = S\frac{1}{4} = \{0\}$, $S\frac{1}{2} = \left\{\frac{1}{4}\right\}$, $S1 = \{0\}$ and $t = \frac{2}{5}$, $k = \frac{1}{4}$. Now, if we take the case $v = \frac{1}{2}$, $y = \frac{1}{4}$, we have

$H_q(S\frac{1}{2}, J\frac{1}{4}) = H_q\left(\left\{\frac{1}{4}\right\},\{0\}\right) = q\left(\frac{1}{4},0\right) = \frac{1}{5} \le \frac{17}{80} = t(q(v,y)) + k(q(v,Sv)+q(y,Jy))$. Also, satisfied for all $v,y \in L$. That is inequality (4.27) satisfied. Now, let $a_0 = 1$, we have $a_1 = Sa_0 = 0$, $a_2 = Ja_1 = 0$, $a_3 = Sa_2 = 0$

$$\sup_{m\ge 1}\lim_{i\to+\infty}\frac{\alpha\left(a_{i+1},a_{i+2}\right)}{\alpha\left(a_i,a_{i+1}\right)}\mu\left(a_{i+1},a_m\right) = 1 < \frac{15}{13} = \frac{1-k}{t+k},$$

which shows that inequality (4.27) and (4.28) holds. Thus the pair (S,J) is double controlled Reich type contraction. Finally, for every $a \in L$, we obtain

$$\lim_{c\to+\infty}\alpha\left(a,a_c\right) \text{ is finite, } \lim_{c\to+\infty}\mu\left(a_c,a\right) < \frac{1}{k}.$$

All assumption of Theorem 4.3.9 are hold and $\dot{z} = 0$ is a common fixed point.

Note that q is a quasi b-metric with $b = 3$, but Theorem 4.3.11 can not be applied because $b \nless \frac{15}{13} = \frac{1-k}{t+k}$. Therefore, this example shows that our generalization from quasi b-metric space to double controlled quasi metric type spaces is genuine. Also, It is not controlled quasi metric type space. Indeed,

$$q(\frac{1}{4}, \frac{1}{2}) = 3 > \frac{6}{5} = \alpha\left(\frac{1}{4}, 0\right) q\left(\frac{1}{4}, 0\right) + \alpha(0, \frac{1}{2}) q(0, \frac{1}{2}).$$

Taking $t = 0$ in Theorem 4.3.9, we obtain the below theorem of Kannan-type.

4.3.13 Theorem

Consider (L, q) be a left complete double controlled quasi metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and $S, J : L \to P(L)$ are the multivalued mappings such as:

$$H_q(Sv, Jy) \leq k(q(v, Sv) + q(y, Jy)) \text{ and } H_q(Jv, Sy) \leq k(q(v, Jv) + q(y, Sy)),$$

for each $v, y \in L$, $0 < 2k < 1$. Suppose that, for every $a \in L$ and for the sequence $\{JS(a_c)\}$, we have

$$\begin{aligned} \lim_{c \to +\infty} \alpha(a, a_c) \text{ is finite, } \lim_{c \to +\infty} \mu(a_c, a) &< \frac{1}{k} \text{ and} \\ \sup_{m \geq 1} \lim_{i \to +\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) &< \frac{1-k}{k}. \end{aligned}$$

Then, there is a single common fixed point for S and J in L.

Taking $S = J$, we get the below result.

4.3.14 Theorem

Consider (L, q) be a left complete double controlled quasi metric type space with the functions $\alpha, \mu : L \times L \to [1, +\infty)$ and $J : L \to P(L)$ is a multivalued map such as:

$$H_q(Jv, Jy) \leq k(q(v, Jv) + q(y, Jy))$$

for each $v, y \in L$, $0 < 2k < 1$. Suppose that, for every $a \in L$ and for the sequence $\{J(a_c)\}$, we have

$$\begin{aligned} \lim_{c \to +\infty} \alpha(a, a_c) \text{ is finite}, \lim_{c \to +\infty} \mu(a_c, a) &< \frac{1}{k} \text{ and} \\ \sup_{m \geq 1} \lim_{i \to +\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) &< \frac{1-k}{k}. \end{aligned}$$

Then, J has a fixed point.

4.3.15 Example

Let $L = [0, 3)$. Defined $q : L \times L \to [0,+\infty)$ as

$$q(v, y) = \begin{cases} 0, \text{ if } v = y, \\ (v - y)^2 + v, \text{ otherwise.} \end{cases}$$

with

$$\alpha(v, y) = \begin{cases} 2, \text{ if } v, y \geq 1, \\ \frac{v+2}{2}, \text{ otherwise.} \end{cases}, \mu(v, y) = \begin{cases} 1 \quad \text{if } v, y \geq 1, \\ \frac{y+2}{2}, \text{ otherwise.} \end{cases}$$

Clearly (L, q) is double controlled quasi metric type space. Choose $Jv = \left\{\frac{v}{4}\right\}$ and $k = \frac{2}{5}$. It is clear that J is Kannan type double controlled contraction. Also, for each $a \in L$, we have

$$\lim_{c \to +\infty} \alpha(a, a_c) < +\infty, \lim_{c \to +\infty} \mu(a_c, a) < \frac{1}{k}.$$

Thus, all hypotheses of Theorem 4.3.14 hold and $\dot{z} = 0$ is a fixed point.

4.3.16 Theorem

Let (L, q) be a left complete quasi b-metric space and $J : L \to P(L)$ be a mapping such as:

$$H_q(Jv, Jy) \leq k[q(v, Jv) + q(y, Jy)],$$

for each $v, y \in L$, $k \in [0, \frac{1}{2})$ and $b < \frac{1-k}{k}$. Then J has a fixed point $a^* \in L$.

4.3.17 Remark

In the example 4.3.15, $q(v,y) = (v-y)^2 + v$ is a quasi b metric with $b \geq 2$, but we can not apply Theorem 4.3.16 because J is not Kannan type b-contraction. Indeed $b \nless \frac{3}{2} = \frac{1-k}{k}$.

4.4 Double Controlled Dislocated Quasi Metric Type Spaces and Some Results

Results given in this section have been published in [91]

Let (L, ρ_q) be a double controlled complete dislocated quasi metric type space, $a_0 \in L$ and $S, J : L \to P(L)$ are multifunctions on L. Let $a_1 \in Sa_0$ is an element such as $\rho_q(a_0, Sa_0) = \rho_q(a_0, a_1)$, $\rho_q(Sa_0, a_0) = \rho_q(a_1, a_0)$. Let $a_2 \in Ja_1$ be such as $\rho_q(a_1, Ja_1) = \rho_q(a_1, a_2)$, $\rho_q(Ja_1, a_1) = \rho_q(a_2, a_1)$. Let $a_3 \in Sa_2$ be such as $\rho_q(a_2, Sa_2) = \rho_q(a_2, a_3)$ and so on. Thus, we generate a sequence a_n of members in L such as $a_{2n+1} \in Sa_{2n}$ and $a_{2n+2} \in Ja_{2n+1}$, with $\rho_q(a_{2n}, Sa_{2n}) = \rho_q(a_{2n}, a_{2n+1})$, $\rho_q(Sa_{2n}, a_{2n}) = \rho_q(a_{2n+1}, a_{2n})$, and $\rho_q(a_{2n+1}, Ja_{2n+1}) = \rho_q(a_{2n+1}, a_{2n+2})$, $\rho_q(Ja_{2n+1}, a_{2n+1}) = \rho_q(a_{2n+2}, a_{2n+1})$, where $n = 0, 1, 2, \cdots$. We denote this iterative sequence by $\{JS(a_n)\}$. We say that $\{JS(a_n)\}$ is a sequence in L generated by a_0. If $J = S$, then we say that $\{J(a_n)\}$ is a sequence in L generated by a_0. Let $M \subseteq L$, define $\beta^*(a, M) = \inf\{\beta(a, l),\ l \in M\}$ and $\beta^*(M, y) = \inf\{\beta(b, y),\ b \in M\}$.

4.4.1 Definition

Let $L \neq \{\}$ and $\beta : L \times L \to [0, +\infty)$ be a map such as $\beta(a, y) \geq 1$ and $\beta(y, a) \geq 1$, implies $a = y$. Let $S, J : L \to P(L)$ are the multi-valued maps, then the pair (S, J) is said to be β^*-Alt multivalued mapping, if

(a) $\beta^*(a, Sa) \geq 1,\ \rho_q(a, Sa) = \rho_q(a, y)$ and $\rho_q(Sa, a) = \rho_q(y, a)$ implies $\beta^*(Sy, y) \geq 1$.

(b) $\beta^*(Sa, a) \geq 1,\ \rho_q(a, Ja) = \rho_q(a, y)$ and $\rho_q(Ja, a) = \rho_q(y, a)$ implies $\beta^*(y, Sy) \geq 1$.

4.4.2 Definition

Let (L, ρ_q) be a complete double controlled dislocated quasi metric type space and (S, J) be a pair of β^* multivalued mapping. Then (S, J) is called β^* Kannan type double controlled contraction, if for any two consecutive points a, y belonging to the range of a sequence $\{JS(a_n)\}$ with $\beta^*(Sy, y) \geq 1$, $\beta^*(v, Sv) \geq 1$ and $\rho_q(v, y) > 0$, we have

$$\max\left\{H_{\rho_q}(Sv, Jy),\ H_{\rho_q}(Jy, Sv)\right\} \leq k(\rho_q(y, Jy) + \rho_q(v, Sv)), \tag{4.44}$$

where, $k \in [0, \frac{1}{2})$. Also, the terms of the sequence $\{JS(a_n)\}$ satisfy the following

$$\sup_{m\geq 1} \lim_{i\to+\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) < \frac{1-k}{k}. \tag{4.45}$$

4.4.3 Theorem

Let (L, ρ_q) be a left K-sequentially complete double controlled dislocated quasi metric type space. (S, J) is the pair of β^* Kannan type double controlled contraction. Assume that:
(i) The set $G(S) = \{a : \beta^*(a, Sa) \geq 1\}$ is closed and contains a_0.
(ii) For every $a \in L$, we have

$$\lim_{n\to+\infty} \alpha(a, a_n) \text{ and } \lim_{n\to+\infty} \mu(a_n, a) < \frac{1}{k}. \tag{4.46}$$

Then $\{JS(a_n)\} \to a \in L$. Also, if (4.44) holds for each $a, y \in \{a^*\}$ then, there is a single common fixed point for S and J in L and $\rho_q(a^*, a^*) = 0$.

Proof. As a_0 is any element of $G(S)$, from condition (i) $\beta^*(a_0, Sa_0) \geq 1$. Let $\{JS(a_n)\}$ is the iterative sequence in L generated by a point $a_0 \in L$.
Since $\beta^*(a_0, Sa_0) \geq 1$, $\rho_q(a_0, Sa_0) = \rho_q(a_0, a_1)$ and $\rho_q(Sa_0, a_0) = \rho_q(a_1, a_0)$. As (S, J) is β^* multivalued mapping, so $\beta^*(Sa_1, a_1) \geq 1$. Now, $\beta^*(Sa_1, a_1) \geq 1$, $\rho_q(a_1, Ja_1) = \rho_q(a_1, a_2)$ and $\rho_q(Ja_1, a_1) = \rho_q(a_2, a_1)$ implies that $\beta^*(a_2, Sa_2) \geq 1$. By induction we deduce that $\beta^*(a_{2c}, Sa_{2c}) \geq 1$ and $\beta^*(Sa_{2c+1}, a_{2c+1}) \geq 1$, for all $c = 0, 1, 2, \cdots$. Now, by Lemma 4.1.9, we have

$$\rho_q(a_{2c}, a_{2c+1}) \leq H_{\rho_q}(Ja_{2c-1}, Sa_{2c}) \tag{4.47}$$

and

$$\rho_q(a_{2c+1}, a_{2c+2}) \leq H_{\rho_q}(Sa_{2c}, Ja_{2c+1}). \tag{4.48}$$

Using (4.48), implies

$$\rho_q(a_{2c+1}, a_{2c+2}) \leq H_{\rho_q}(Sa_{2c}, Ja_{2c+1}).$$

As $a_{2c}, a_{2c+1} \in \{JS(a_n)\}$, $\beta^*(a_{2c}, Sa_{2c}) \geq 1$ and $\beta^*(Sa_{2c+1}, a_{2c+1}) \geq 1$, by using the condition (4.44) in inequality (4.48), we have

$$\begin{aligned} \rho_q(a_{2c+1}, a_{2c+2}) &\leq k\left[\rho_q(a_{2c}, Sa_{2c}) + \rho_q(a_{2c+1}, Ja_{2c+1})\right] \\ &\leq k\left(\rho_q(a_{2c}, a_{2c+1}) + \rho_q(a_{2c+1}, a_{2c+2})\right) \end{aligned}$$

$$\leq \frac{k}{1-k}\left(\rho_q(a_{2c}, a_{2c+1})\right) \leq \mu\left(\rho_q(a_{2c}, a_{2c+1})\right), \text{ where } \mu = \frac{k}{1-k}. \tag{4.49}$$

Now, by (4.47), we have

$$\rho_q(a_{2c}, a_{2c+1}) \leq H_{\rho_q}(Ja_{2c-1}, Sa_{2c}).$$

As $a_{2c}, a_{2c-1} \in \{JS(a_n)\}$, $\beta^*(a_{2c}, Sa_{2c}) \geq 1$ and $\beta^*(Sa_{2c-1}, a_{2c-1}) \geq 1$, by using the condition (4.44) in inequality (4.47), we get

$$\begin{aligned} \rho_q(a_{2c}, a_{2c+1}) &\leq k(\rho_q(a_{2c-1}, Ja_{2c-1}) + \rho_q(a_{2c}, Sa_{2c}) \leq \frac{k}{1-k}\left(\rho_q(a_{2c-1}, a_{2c})\right) \\ &\leq \mu\left(\rho_q(a_{2c-1}, a_{2c})\right), \text{ where } \mu = \frac{k}{1-k}. \end{aligned} \tag{4.50}$$

As $a_{2c-2}, a_{2c-1} \in \{JS(a_n)\}$, $\beta^*(a_{2c-2}, Sa_{2c-2}) \geq 1$ and $\beta^*(Sa_{2c-1}, a_{2c-1}) \geq 1$, by using the condition (4.44), we get

$$\rho_q(a_{2c-1}, a_{2c}) \leq \mu\left(\rho_q(a_{2c-2}, a_{2c-1})\right). \tag{4.51}$$

Using (4.51) in (4.50), we have

$$\rho_q(a_{2c}, a_{2c+1}) \leq \mu^2 \rho_q(a_{2c-2}, a_{2c-1}). \tag{4.52}$$

As $a_{2c-2}, a_{2c-1} \in \{JS(a_n)\}$, $\beta^*(Sa_{2c-1}, a_{2c-1}) \geq 1$ and $\beta^*(a_{2c-2}, Sa_{2c-2}) \geq 1$, by using the condition (4.44), we get

$$\rho_q(a_{2c-2}, a_{2c-1}) \leq \mu^3\left(\rho_q(a_{2c-3}, a_{2c-2})\right). \tag{4.53}$$

From (4.52) and (4.53), we have

$$\mu^2(\rho_q(a_{2c-2}, a_{2c-1})) \leq \mu^3(\rho_q(a_{2c-3}, a_{2c-2})). \tag{4.54}$$

Using (4.54) in (4.50), we have

$$\rho_q(a_{2c}, a_{2c+1})\} \leq \mu^3(\rho_q(a_{2c-3}, a_{2c-2})).$$

Continuing in this way, we get

$$\rho_q(a_{2c}, a_{2c+1})\} \leq \mu^{2c}(\rho_q(a_0, a_1)). \tag{4.55}$$

Now, by using (4.49), (4.50), (4.51) and continuing in this way, we get

$$\rho_q(a_{2c+1}, a_{2c+2}) \leq \mu^{2c+1}(\rho_q(a_0, a_1)). \tag{4.56}$$

Combining the inequalities (4.55) and (4.56), we have

$$\rho_q(a_n, a_{n+1}) \leq \mu^n(\rho_q(a_0, a_1)). \tag{4.57}$$

Now, by (4.57) and using the technique given in 4.2.1, it can easily be proved that $\{a_n\}$ is a left Cauchy sequence. So, for all natural numbers with $n < m$, we have

$$\lim_{n,m\to+\infty} \rho_q(a_n, a_m) = 0. \tag{4.58}$$

So, the sequence $\{JS(a_n)\}$ is a left Cauchy sequence. Since (L, ρ_q) is a left K-sequentially double controlled complete dislocated quasi metric type space, so there exists $a^* \in L$ such as $\{JS(a_n)\} \to a^* \in L$, that is

$$\lim_{n\to+\infty} \rho_q(a_n, a^*) = \lim_{n\to+\infty} \rho_q(a^*, a_n) = 0. \tag{4.59}$$

Since (L, ρ_q) is a left K-sequentially complete and $G(S)$ is closed subset of L, so $\big(G(S), \rho_q\big)$ is a left K-sequentially complete. As $\beta^*(a_{2c,}, Sa_{2c}) \geq 1$ for all $c \in \mathbb{N}$. So $\{a_{2n}\}$ is a sequence of

$\{JS(a_n)\}$ contained in $G(S)$. By completeness of $\big(G(S), \rho_q\big)$ and uniqueness of limit $\{a_{2n}\} \to a^*$, that is

$$\beta^* (a^*, Sa^*) \geq 1.$$

Now, we suppose that $\rho_q (a^*, Sa^*) \neq 0$. By Lemma 4.1.9, we have

$$\rho_q(a_{2n}, Sa^*) \leq H_{\rho_q}(Ja_{2n-1}, Sa^*).$$

As $\beta^* (a^*, Sa^*) \geq 1$ and $\beta^* (Sa_{2n-1}, a_{2n-1}) \geq 1$, by using (4.44), we have

$$\rho_q(a_{2n}, Sa^*) \leq k \left[\rho_q(a_{2n-1}, a_{2n}) + \rho_q(a^*, Sa^*)\right]. \tag{4.60}$$

Taking $\lim\limits_{n\to+\infty}$ of inequality (4.60), we get

$$\lim_{n\to+\infty} \rho_q(a_{2n}, Sa^*) \leq \lim_{n\to+\infty} k \left[\rho_q(a_{2n-1}, a_{2n}) + \rho_q(a^*, Sa^*))\right],$$

$$\lim_{n\to+\infty} \rho_q(a_{2n}, Sa^*) \leq k(\rho_q(a^*, Sa^*)). \tag{4.61}$$

Now,

$$\rho_q(a^*, Sa^*) \leq \alpha(a^*, a_{2n-1})\rho_q(a^*, a_{2n-1}) + \mu(a_{2n}, Sa^*)\rho_q(a_{2n}, Sa^*). \tag{4.62}$$

Taking $\lim\limits_{n\to+\infty}$ of inequality (4.62) and using inequality (4.46) and (4.59), we get

$$\rho_q(a^*, Sa^*) \leq \lim_{n\to+\infty} \mu(a_{2n}, Sa^*)\rho_q(a_{2n}, Sa^*). \tag{4.63}$$

Using inequality (4.46) and (4.61) in inequality (4.63), we get that

$$\rho_q(a^*, Sa^*) < \rho_q(a^*, Sa^*).$$

It is a contradiction, therefore

$$\rho_q(a^*, Sa^*) = 0. \tag{4.64}$$

Now, suppose that $\rho_q\left(Sa^*, a^*\right) = 0$. By Lemma 4.1.9 and (4.44), we have

$$\rho_q(Sa^*, a_{2n}) \leq H_{\rho_q}(Sa^*, Ja_{2n-1}).$$

Now, $\beta^*\left(a^*, Sa^*\right) \geq 1$ and $\beta^*\left(Sa_{2n-1}, a_{2n-1}\right) \geq 1$, so by (4.44), we have

$$\rho_q(Sa^*, a_{2n}) \leq k\left[\rho_q(a^*, Sa^*) + \rho_q(a_{2n-1}, a_{2n})\right]. \tag{4.65}$$

Taking $\lim\limits_{n\to+\infty}$ of inequality (4.65), we get

$$\lim_{n\to+\infty}\rho_q(Sa^*, a_{2n}) \leq \lim_{n\to+\infty} k\left[\rho_q(a_{2n-1}, a_{2n}) + \rho_q(a^*, Sa^*))\right],$$

$$\lim_{n\to+\infty}\rho_q(Sa, a_{2n}) \leq k(\rho_q(a^*, Sa^*)). \tag{4.66}$$

Now,

$$\rho_q(Sa^*, a^*) \leq \alpha(Sa^*, a_{2n})\rho_q(Sa^*, a_{2n}) + \mu(a_{2n}, a^*)\rho_q(a_{2n}, a^*). \tag{4.67}$$

Taking $\lim\limits_{n\to+\infty}$ of inequality (4.67) and using inequality (4.46) and (4.59), we get

$$\rho_q(Sa^*, a^*) \leq \frac{1}{k}\rho_q(Sa^*, a_{2n}). \tag{4.68}$$

By using inequality (4.46) and (4.66) in inequality (4.68), we obtain

$$\rho_q(Sa^*, a^*) < \rho_q(a^*, Sa^*) = 0, \text{ by (4.64)}.$$

It is a contradiction. Hence, $a^* \in Sa^*$. Now,

$$\rho_q(a^*, a^*) \leq \alpha(a^*, a_{2n})\rho_q(a^*, a_{2n}) + \mu\left(a_{2n}, a^*\right)\rho_q(a_{2n}, a^*).$$

This implies $\rho_q(a^*, a^*) = 0$ as $n \to +\infty$. As $\beta^*\left(a^*, Sa^*\right) \geq 1$ and $\rho_q\left(a^*, Sa^*\right) = \rho_q\left(Sa^*, a^*\right) = 0$. So, Definition 4.4.1 implies

$$\beta^*\left(Sa^*, a^*\right) \geq 1.$$

Now, by Lemma 4.1.9, we have

$$\rho_q(a_{2n+1}, Ja^*) \leq H_{\rho_q}(Sa_{2n}, Ja^*).$$

As, $\beta^*(a_{2n}, Sa_{2n}) \geq 1$ and $\beta^*(Sa^*, a^*) \geq 1$, so by (4.44), we get

$$\rho_q(a_{2n+1}, Ja^*) \leq k\left[\rho_q(a_{2n}, a_{2n+1}) + \rho_q(a^*, Ja^*))\right]. \tag{4.69}$$

Taking $\lim_{n\to+\infty}$ of inequality (4.69), we have

$$\lim_{n\to+\infty} \rho_q(a_{2n+1}, Ja^*) \leq \lim_{n\to+\infty} k\left[\rho_q(a_{2n}, a_{2n+1}) + \rho_q(a^*, Ja^*))\right].$$

Taking $\lim_{n\to+\infty}$ and using inequality (4.58), we have

$$\lim_{n\to+\infty} \rho_q(a_{2n+1}, Ja^*) \leq k(\rho_q(a^*, Ja^*)). \tag{4.70}$$

Since,

$$\rho_q(a^*, Ja^*) \leq \alpha(a^*, a_{2n})\rho_q(a^*, a_{2n}) + \mu(a_{2n+1}, Ja^*)\rho_q(a_{2n+1}, Ja^*).$$

Taking $\lim_{n\to+\infty}$ for above inequality and using inequality (4.46) and (4.59), we get

$$\rho_q(a^*, Ja^*) \leq \frac{1}{k}\left(\rho_q(a_{2n+1}, Ja^*)\right).$$

By using inequality (4.46) and (4.70), we get that

$$\rho_q(a^*, Ja^*) < \rho_q(a^*, Ja^*).$$

It is a contradiction. Thus $\rho_q(a^*, Ja^*) = 0$. Similar arguments as above, we get

$$\rho_q(Ja^*, a^*) = 0.$$

Hence $a^* \in Ja^*$. Thus, a^* is a common fixed point of S and J. ■

4.4.4 Example

Let $L=[0,+\infty)\cap\mathbb{Q}^{+}$ and defined $\rho_q(a,y)=(a+2y)^2$ if $a\neq y$, and $\rho_q(a,y)=0$, if $a=y$. Then (L,ρ_q) is a complete double controlled dislocated quasi metric type space with

$$\alpha(a,y)=\begin{cases} 2, \text{ if } a,\ y\geq 1, \\ \frac{a+2}{2}, \text{ otherwise.}\end{cases} \qquad \mu(a,y)=\begin{cases} 1, \text{ if } a,\ y\geq 1, \\ \frac{y+2}{2}, \text{ otherwise.}\end{cases}$$

Let,

$$\beta(a,y)=\begin{cases} 1, \text{ if } a\in\mathcal{A} \text{ and } y\in\mathcal{B} \\ \frac{1}{4}, \text{ otherwise.}\end{cases}$$

Now,

$$\begin{aligned} A &= \{a:\beta^{*}(a,Sa)\geq 1\}=\left\{0,1,\frac{1}{64},\frac{1}{4096},...\right\}. \\ B &= \{y:\beta^{*}(Sy,y)\geq 1\}=\left\{0,\frac{1}{8},\frac{1}{512},...\right\}. \end{aligned}$$

Define the maps $S,J:L\to P(L)$ as:

$$J(y)=\begin{cases} \left[\frac{y}{8},\frac{y}{4}\right]\cap\mathbb{Q}^{+}, \text{ for all } y\in\{0,1,\frac{1}{8},\frac{1}{64},\frac{1}{512},\frac{1}{4096},\cdots\}, \\ [y+2,\ 2(y+1)]\cap\mathbb{Q}^{+}, \text{ if otherwise.}\end{cases}$$

$$S(y)=\begin{cases} \{\frac{1}{8}y\}\cap\mathbb{Q}^{+}, \text{ for all } y\in\{0,1,\frac{1}{8},\frac{1}{64},\frac{1}{512},\frac{1}{4096},\cdots\}, \\ [y+1,y+3]\cap\mathbb{Q}^{+}, \text{ if otherwise.}\end{cases}$$

The given ρ_q is not a controlled dislocated quasi metric type space for the function α. Indeed,

$$\rho_q(1,3)=49>37.5=\alpha(1,0)\,\rho_q(1,0)+\alpha(0,3)\rho_q(0,3).$$

Now, $\rho_q(a_0,Sa_0)=\rho_q(1,S1)=\rho_q\left(1,\frac{1}{8}\right)=\left(1+\frac{2}{8}\right)^2=\left(\frac{5}{4}\right)^2$, we define the sequence $\{JS(a_n)\}=\{1,\frac{1}{8},\frac{1}{64},\frac{1}{512},\frac{1}{4096},\cdots\}$ in L generated by $y_0=1$.
Note that $\beta^{*}(a,Sa)\geq 1$, $\rho_q(a,Sa)=\rho_q(a,y)$ and $\rho_q(Sa,a)=\rho_q(y,a)$ implies $\beta^{*}(Sy,y)\geq 1$. Also, $\beta^{*}(Sa,a)\geq 1$, $\rho_q(a,Ja)=\rho_q(a,y)$ and $\rho_q(Ja,a)=\rho_q(y,a)$ implies $\beta^{*}(y,Sy)\geq 1$. So the pair (S,J) is β^{*}-Alt multivalued mapping on $\{JS(a_n)\}$.

Now, for all $a, y \in L \cap \{JS(a_n)\}$ with $\beta^*(Sy, y) \geq 1$, $\beta^*(a, Sa) \geq 1$ and $k = \frac{2}{5}$, we observe that all cases are satisfied that is

$$\max\left\{H_{\rho_q}(Sa, Jy), H_{\rho_q}(Jy, Sa)\right\} \leq k(\rho_q(a, Sa) + \rho_q(y, Jy)).$$

Now, let $a_0 = 1$, we have $a_1 = Sa_0 = \frac{1}{8}$, $a_2 = Ja_1 = \frac{1}{64}$, $a_3 = Sa_2 = \frac{1}{512}, \ldots$.

$$\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\alpha(a_{i+1}, a_{i+2})}{\alpha(a_i, a_{i+1})} \mu(a_{i+1}, a_m) = 0.71 < \frac{1-k}{k} = \frac{3}{2}.$$

That is, the pair (S, J) is β^* Kannan type double controlled contraction. Let $a_0 = 1$, we have

$$\begin{aligned} G(S) &= \{a : \beta^*(a, Sa) \geq 1 \text{ and } a \in \{JS(a_n)\}\} \\ &= \left\{0, 1, \frac{1}{64}, \frac{1}{4096}, \ldots\right\}. \end{aligned}$$

That (i) is hold. Finally, for every $a \in \{JS(a_n)\}$, we have

$$\lim_{n \to +\infty} \alpha(a, a_n) < \frac{5}{2} \text{ and } \lim_{n \to +\infty} \mu(a_n, a) < \frac{5}{2}.$$

Thus, each hypothesis of Theorem 4.4.3 hold. In fact 0 is a a single common fixed point of S and J.

4.4.5 Definition

Let (L, ρ_q) be a complete dislocated quasi b-metric type space and S, J be a β^*-Alt multivalued map. Then the pair (S, J) is called β^* Kannan type b-contraction, for every two consecutive points a, y belonging to the range of a sequence $\{JS(a_n)\}$ with $\beta^*(Sy, y) \geq 1$, $\beta^*(a, Sa) \geq 1$ and $\rho_q(a, y) > 0$, we have

$$\max\left\{H_{\rho_q}(Sy, Ja), H_{\rho_q}(Ja, Sy)\right\} \leq k(\rho_q(a, Ja) + \rho_q(y, Sy)), \qquad 4.71$$

whenever, $k \in [0, \frac{1}{2})$ and $b < \frac{1-k}{k}$.

4.4.6 Theorem

Consider (L, ρ_q) be a left K-sequentially complete dislocated quasi b-metric space and a pair (S, J) is a β^* Kannan type b-contraction. Assume that: the set $G(S) = \{a : \beta^* (a, Sa) \geq 1\}$ is contains a_0 and closed. Then $\{JS(a_n)\}\} \to a^* \in L$. Also, if (4.71) holds for each $a, y \in \{a^*\}$, then S and J have a common fixed point a^* in L and $\rho_q(a^*, a^*) = 0$.

4.4.7 Remark

In the Example 4.4.4, $\rho_q(a, y) = (a + 2y)^2$ is a dislocated quasi b-metric with $b = 2$, but we can not apply Theorem 4.4.6 because the pair (S, J) is not β^* Kannan type b-contraction. Indeed $b \nless \frac{1-k}{k} = \frac{3}{2}$.

4.4.8 Corollary

Let (L, ρ) be a left K-sequentially complete b-metric space. Let the pair (S, J) be a β^* Kannan type contraction. Assume that the set $G(S) = \{a : \beta^* (a, Sa) \geq 1\}$ is closed and contains a_0. Then, S and J have a common fixed point a^* in L.

Conclusion: The main aim of this chapter is to introduce double controlled quasi and dislocated quasi metric type spaces and related definitions as a generalization of double controlled metric-type spaces. We have removed one and a half restriction out of three restrictions of double controlled metric-type spaces. These new spaces is a generalization of metric space, quasi metric space, dislocated metric space, dislocated quasi metric space, partial metric space, quasi partial metric space, b-metric space, quasi b-metric space, dislocated b-metric space, dislocated quasi b-metric space, extended b-metric space, dislocated extended b-metric space, quasi-extended b-metric space, dislocated quasi-extended b-metric space and double controlled metric space. We establish new generalized contractions and obtain fixed point results for single-valued as well as a pair of multivalued maps in complete double controlled quasi metric type spaces and in left K-sequentially complete double controlled dislocated quasi metric type spaces. New results in ordered spaces and new results for graphic contractions can be obtained as corollaries of our results. Some examples have been built to demonstrate the novelty of results.

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