Analysis and Applications of Convolution and Subordination in Geometric Function Theory

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A Dissertation Submitted in the Partial Fulfillment of the **Doctor of Philosophy in Mathematics**

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Dedicated

To My

Loving Mother and Father (late)

My Wife and My family

And respectful teachers

Whose affection is reason on every success in my life.

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Muhammad Naeem

Contents

0. Research Profile

1. Naeem M., Hussain S., Mahmood T., Khan S and Darus M. A new subclass of analytic functions defined by using Sălăgean q-differential operator. Mathematics, 2019, 7, 458-469.

2. Wang Z. G., **Naeem M.**, Hussain S., Mahmood T. and Rasheed A., A class of analytic functions related to convexity and functions with bounded turning. AIMS Mathematics, 2019, 5(3), 1926–1935.

3. Mahmood T., **Naeem M.**, Hussain S., Khan S. and Altınkaya S., A subclass of analytic functions defined by using Mittag-Leffler function. Honam Mathematical Journal, 2020 (Accepted).

4. Naeem M., Hussain S . Sakar F M., Mahmood T. and Rasheed A. Subclasses of uniformly convex and starlike functions associated with Bessel functions. Turkish Journal of Mathematics, 2019, 43, 2433 – 2443.

5. Wang Z. G., Hussain S., **Naeem M.**, Mahmood T. and Khan S., A subclass of univalent functions associated with q-analogue of Choi-Saigo-Srivastava operator Hacettepe Journal of Mathematics & Statistics, 2020, 49(4) 1471-1479.

6. Naeem M., Hussain S., Khan S., Mahmood T., Darus M. and Shareef Z., Janowski Type q-Convex and q- Close-to-Convex Functions Associated with q-Conic Domain. Mathematic, 2020, 8(3), 440-452.

0.1. Introduction

The branch of Mathematics in which we study the geometric properties of analytic functions (AF) is called geometric function theory (GFT). Cauchy started to develop the substructures around 200 years ago during 1814 – 1831. Cauchy, Riemann and Weierstrass were considered as pioneers of this field. Since that time, this theory is expanded in many directions. Riemann contributed to this field by introducing the Riemann mapping theorem in 1851. Riemann proved the existence of an AF $\mathcal{L}$ that maps from a connected domain $D_1 \neq \mathbb{C}$ in the z-plane on to a connected domain D_2 in the w-plane. This result of Riemann mapping laid down the foundation of GFT. This famous theorem and then Koebe's theory provide many fundamental results of geometric function theory. His initial work on this field can be found in [53]. Gronwall's proposed the area theorem in 1914. This idea revolves around some conformal mappings in open unit disc. A lot of applications of GFT can be found in other fields of sciences like Electronics, Medicines, Physics etc. see [91, 101, 109] and cited therein

Any function $\mathcal{L}$ is in the class S if it is univalent and normalized function. Any function $\mathcal{L} \in S$ has series of the form

$$\mathcal{L}(w) = w + \sum_{m=2}^{\infty} a_m w^m.$$

In above equation, a_2 the second coefficient, of a normalized univalent function estimated by Bieberbach [16] in 1916. Further, Bieberbach also conjectured about the mth coefficient. He has shown that $|a_2| \leq 2$ and mentioned $|a_m| \leq m$ is generally valid and is called Bieberbach conjecture. Bieberbach conjecture was proved to be true for $m = 3$ by Lowner [59] in 1923. The same result was proved for various other values of m by Janöwski [43], Goodman [38, 39], Littlewood [56], Pinchuk [83], Pommerenke [84], Rogosinski [88] and Ruscheweyh [93]. For several years this challenging and famous conjecture inspired mathematicians which results in the development of various novel procedures in complex analysis. Univalent function theory has its own significance. In 1916, Bieberbach conjecture failed to explain some situations due to which several new subclasses of S are proposed and investigated such as starlike functions by Alexander [6]. These starlike functions (SF) were further studied by Nevanlinna [72]. Alexander [6] also developed a class of convex functions (CF). In 1915, Alexander give us a connection between the members of the classes C and S^* of CF and SF respectively. Alexander theorem states that $\mathcal{L} \in S$ be in C if and only if $w\mathcal{L}' \in S^*$. Later Kaplan [51] developed the class of close to convex functions (CCF) and explained it geometrically. The geometric interpretation means that $\mathcal{L} \in K$ maps every circle arc $|w| < 1$ onto a closed path for which tangent rotates with an increase in θ for both clockwise and counterclockwise direction but can not turn back. Several approaches of geometric function theory were suggested by de-Branges [18], when he solved the Bieberbach conjecture in 1984. The class of Carathéodory functions [20] which maps unit disk onto right half plane is denoted by $\mathcal{P}$, play a pivotal role in GFT.

The operator plays an important role in Mathematics. Using convolution and subordination theories to define operators and study operators properties is newest area of present research in this field. Libera [54] introduced an operator called Libera operator and studied the classes S^* and C under this operator. Later, Bernardi [15] developed a generalized operator and discuss its various properties. In recent past several new operators are introduced by Ruscheweyh [93], Noor [73] and many others [21, 57, 58]. These researchers developed some interesting new classes of univalent and analytic functions which contain existing classes as special cases.

In 1991, Goodman [40, 41] introduced two important classes UC and US^* of analytic functions, which are natural generalizations of convex and starlike functions in conic domains. In 1999, for $k \geq 0$, Kanas [47] initiated the classes k-UC and k-US^* of k-uniformly convex and k-uniformly starlike functions respectively. The work of Kanas is generalized by many researchers (see [25, 88, 90]).

0.2. Chapter Wise Study

In this section, a short description of all chapters is discussed.

Chapter 1

In this chapter, we discuss some basics of this field. The proofs of the results are omitted but referred here. Some subclasses related with AF are also discussed. Many interesting and important subclasses of starlike and convex are presented. We also discuss convolution and subordination as main tools used in geometric function theory. Some linear and q-differential operator also discussed. At the end of this chapter some lemmas are given which helps us to prove certain important results.

Chapter 2

In this chapter, we initiate two new subclasses $\mathcal{U}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$ and $\mathcal{TU}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$. These classes are initiated by using the Sàlàgean q -differential operator with the help of Janöwski function These classes generalized numerous classes by selecting specific values of the parameters. We examined numerous sharp results and properties of these classes, like as extreme points (EP), distortion theorem (DT), coefficient estimates (CE), convexity, radii of star-likeness (RS), close-to-convexity, and integral mean inequalities. The content of this chapter are published in the journal, **Mathematics, 2019, 7, 458-469.**

Chapter 3

In this chapter, we initiate a new subclass k-$\mathcal{QMT}(\alpha)$ of AF, which generalizes the class of k-uniformly CF. The main purpose of this chapter is to establish several interesting relationships between k-$\mathcal{QMT}(\alpha)$ and the class $\mathcal{B}(\delta)$ of functions with bounded turning. We studied various interesting association of this class with already existing classes of AF. Certain important cases for some special values of the parameters have been obtained. The content of this chapter are published in the journal, **AIMS Mathematics, 2019, 5(3), 1926–1935.**

Chapter 4

In this chapter, we initiate two new subclasses $Q^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ and $TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$. These classes is initiated by using the Mittag-Leffler function with the help of Janowski functions. These classes generalized numerous classes by selecting specific values of the parameters. We examined numerous sharp results and properties of these classes, like as extreme points (EP), distortion theorem (DT), coefficient estimates (CE), convexity, radii of star-likeness (RS), close-to-convexity and integral mean inequalities (IMI). The content of this chapter are published in the journal, **Honam Mathematical Journal, 2020 (Accepted).**

Chapter 5

The main focus of this chapter is to set out few imperative characteristic properties for a few subclasses of uniformly SF and CF which are initiated here by infers of the normalized condition of the generalized Bessel functions (BF) to be univalent inside the $\widetilde{H}$. Furthermore, we as well develop up some results about of these subclasses related to a particular integral operator. The content of this chapter are published in the journal, **Turkish Journal of Mathematics, 2019, 43, 2433 – 2443.**

Chapter 6

The main focus of this chapter is to initiate a subclass $Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ of AF using subordinations along with the newly defined q-analogue of Choi-Saigo-Srivastava operator. Some results, such as integral representation (IR), coefficient estimates (CE), radius of starlikeness, linear combination (LC), weighted mean (WM) and arithmetic means (AM) for this class are derived. The content of this chapter are published in the journal, **Hacettepe Journal of Mathematics & Statistics, 2020, 49(4) 1471-1479.**

Chapter 7

In this chapter, with the help q-conic domain $\left(\Omega_{k,q}[\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}]\right)$, q-Janowski type functions and the concepts of quantum (or q-) calculus, we initiate new subclasses of q-convex and q-close-to-convex functions. These subclasses explores some vital geometric properties such as coefficient estimates (CE), sufficiency criteria and also convolution properties. Furthermore, we as well develop up some results about of these subclasses with those obtained in earlier investigations is also provided. The content of this chapter are published in the journal, **Mathematic, 2020, 8(3), 440-452.**

Chapter 1

Preliminaries

The main objective of this chapter, is to examine few fundamental ideas of GFT, which give us a foundation for the later work. Here we will inquired few fundamental definitions and classical results. The details of such results study standard texts [29, 39, 85]. At the starting, we concentrated on the classes $\mathcal{A}$ and S, which are of normalized AF and normalized analytic univalent functions respectively. Next we discussed subclasses of $\mathcal{A}$ and S At the end of this chapter, we introduced such lemmas which are helpful to understand upcoming chapters. In this chapter we presented such results which are already known and will be referred properly.

1.1 Analytic functions (AF) and Univalent functions (UF)

Firstly, we define the class $\mathcal{A}$ and S of normalized AF and normalized UF.

1.1.1 Analytic Functions [29, 39, 85]

If a function $\mathcal{L}$, whose derivative exists at w_0 and also in neighborhood of w_0 then such a function is called AF. If $\mathcal{L}(w)$ is anaytic for all $w \in D$, then $\mathcal{L}$ is AF in whole domain D.
If we take $w \in D = \mathbb{C}$, where $\mathbb{C}$ is set of complex numbers and $\mathcal{L}(w)$ is AF then $\mathcal{L}$ is called entire function. Occasionally it is tough to use arbitrary domain D, so by well-known Riemann mapping theorem, we use open unit disc $\widetilde{H} = \{w : |w| < 1\}$ instead of D.

1.1.2 The Class $\mathcal{A}$

In class $\mathcal{A}$, all AF with the normalized condition $\mathcal{L}(0) = 0$ and $\mathcal{L}'(0) = 1$ exists. Generally, normalization does not influence on Taylor's series. For $\mathcal{L} \in \mathcal{A}$ has a series form

$$\mathcal{L}(w) = w + \sum_{m=2}^{\infty} a_m w^m. \tag{1.1.1}$$

1.1.3 The Class T

In class T, all those functions of class of $\mathcal{A}$ having negative coefficients in Taylor's series. Thus $\mathcal{L} \in T$ is given by

$$\mathcal{L}(w) = w - \sum_{m=2}^{\infty} |a_m| w^m. \tag{1.1.2}$$

Initially, Silverman [100] study this class.

1.1.4 Univalent Functions [29, 39]

If in the domain D and range $\mathcal{L}(D)$ of a function $\mathcal{L}$, having 1-1 correspondence then such function is UF. Mathematically, $\mathcal{L}(w)$ is univalent if $w_m \neq w_n$ implies $\mathcal{L}(w_m) \neq \mathcal{L}(w_n)$, for all $w_m \neq w_n \in D$.

Koebe function

$$K_b(w) = w\frac{1}{(1-w)^2}, \; w \in \tilde{H}, \tag{1.1.3}$$

is one of simplest example of UF while $\mathcal{L}(w) = w^2$ is not UF.

1.1.5 The Class S

All such functions belonging to S which belong to $\mathcal{A}$ and also UF Koebe function is very well known example for the class S. UF plays pivotal role in this field.

1.2 Caratheordory functions

We have noticed that, lot of functions whose mapping cover the whole complex plane but also we have some functions which maps right half plane such complex-valued functions known as

Caratheordory functions [29, 39]. All such mappings belonging to the class $\mathcal{P}$ These functions are of incredible significance in GFT. Numerous subclasses of AF and UF are associated with $\mathcal{P}$.

1.2.1 The Class $\mathcal{P}$

Let Þ be AF with $\text{Þ}(0) = 1$. Then Þ belonging to the class $\mathcal{P}$, if $Re\text{Þ}(w) > 0$ and has the power series

$$\text{Þ}(w) = 1 + \sum_{m=1}^{\infty} a_m w^m, \quad w \in \widetilde{H}. \tag{1.2.1}$$

The Möbius function

$$M_\Theta = 1 + 2\sum_{m=1}^{\infty} w^m = \frac{1+w}{1-w}, \; w \in \widetilde{H}, \tag{1.2.2}$$

is an element of the class $\mathcal{P}$. It is not necessary a function belonging to $\mathcal{P}$ is also univalent. For example

$$\text{Þ}_0(w) = 1 + w^m \in \mathcal{P}, \; \forall\, m \geq 0. \tag{1.2.3}$$

this function for $m \geq 2$, is not univalent. The Möbius function (M_Θ) defined by (1.2.2) plays the central and extremal role in class $\mathcal{P}$.

1.2.2 Subordination

Primarily Lindelof [55] initiate the notion of subordination in 1909. Afterward Littlewood [56] and Rogosinski [88, 89] made further advancements.

Definition of subordination

Let $\mathcal{L}, l \in \mathcal{A}$ then $\mathcal{L} \prec l$ (read it as $\mathcal{L}$ is subordinate to l) if and only if we have $s \in \mathcal{A}$ with the condition $|s(w)| < 1$ and $s(0) = 0$, for $w \in \widetilde{H}$ such that $l(s(w)) = \mathcal{L}(w)$. Particularly if $\mathcal{L} \in S$ in $\widetilde{H}$ then $\mathcal{L} \prec l$ is equivalent to $\mathcal{L}(0) - l(0) = 0$ and $\mathcal{L}(\widetilde{H}) \subseteq l(\widetilde{H})$.

1.3 Some Important Subclasses of Univalent Functions

In this portion, we presented such classes which have specific geometric conditions. We briefly talk about a few essential subclasses of AF which map $\widetilde{H}$ onto the starlike domain and convex domain. We consider a few fundamental properties of these classes and the connection of these classes with the Carathĕordory class $\mathcal{P}$ will moreover be explored.

1.3.1 Starlike Function (SF) [29, 39]

Let a domain $D \subseteq \mathbb{C}$, is star shaped about a point w_0, if any other point $w \in D$ joining with w_0 by line segment $\overline{ww_0}$ lies with in D. The function $\mathcal{L}$ under which image of domain $\widetilde{H}$ is a starshaped region with respect to $w_0 \neq 0$ is known as starlike function.
Mathematically, a normalized AF function $\mathcal{L}$ is SF if

$$Re\left(\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}\right) > 0, \ w \in \widetilde{H} \tag{1.3.1}$$

or equivalently

$$\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)} \in \mathcal{P}, \ w \in \widetilde{H} \tag{1.3.2}$$

Let us denote S^* be the class normalized starlike functions. Thus

$$S^* = \left\{\mathcal{L} \in \mathcal{A} : Re\left(\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}\right) > 0, \ w \in \widetilde{H}\right\}. \tag{1.3.3}$$

With respect to origin, the Koebe function is an example of SF The class of SF was begun by Alexander [6] and examined by Nevanlinna [72].

1.3.2 Convex Function (CF) [29, 39]

Let a domain $D \subseteq \mathbb{C}$, is convex if joining two arbitrary points of D with a line segment lies completely in D. A function $\mathcal{L} \in S$ in which $\mathcal{L}\left(\widetilde{H}\right)$ is a convex, is a CF.
Mathematically, a normalized AF function $\mathcal{L}$ is CF if

$$Re\left(\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)}\right) > 0, \ w \in \widetilde{H}. \tag{1.3.4}$$

or equivalently

$$\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)} \in \mathcal{P}, \ w \in \widetilde{H}. \tag{1.3.5}$$

Let us denotes C be the class of normalized convex functions. Thus

$$C = \left\{ \mathcal{L} \in S : \ Re\left(\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)} \right) > 0, \ w \in \widetilde{H} \right\}. \tag{1.3.6}$$

The function

$$\mathcal{L}(w) = \frac{w}{1-w} \tag{1.3.7}$$

is convex. The class of CF was first examined by Alexander [6].

Note that, $C \subset S^* \subset S$.

1.3.3 Alexander Theorem

In 1952, Alexander [6] give us a theorem between the members of the classes C and S^*.

Let $\mathcal{L}$ be normalized AF in $\widetilde{H}$ Then $\mathcal{L} \in C$ if and only if $w\mathcal{L}' \in S^*$

1.3.4 Close to Convex Function (CCF) [38]

A mapping $\mathcal{L} \in \mathcal{A}$ is CCF, if it satisfy the following condition,

$$l \in C, \ Re\left(\frac{\mathcal{L}'(w}{l'(w)} \right) > 0, \ w \in \widetilde{H}. \tag{1.3.8}$$

or equivalently

$$l \in C, \ \frac{\mathcal{L}'(w)}{l'(w)} \in \mathcal{P}, \ w \in \widetilde{H} \tag{1.3.9}$$

Let us denote K be the class of normalized CCF.

$$K = \left\{ \mathcal{L} \in \mathcal{A} : \ l \in C, \ Re\left(\frac{\mathcal{L}'(w}{l'(w)} \right) > 0, \ w \in \widetilde{H} \right\}. \tag{1.3.10}$$

Kaplan [51] initiated the study of this class.

1.4 Some Generalized Subclasses of Analytic functions

In this portion, the subclasses which have specific geometric conditions of order α, where $0 \leq \alpha < 1$ are presented.

1.4.1 The Class $\mathcal{P}(\alpha)$ [39]

Let $\text{þ} \in \mathcal{P}$. Then þ belong to the class $\mathcal{P}(\alpha)$ if

$$0 \leq \alpha < 1, \ Re\text{Þ}(w) > \alpha, \ w \in \widetilde{H} \tag{1.4 1}$$

The function þ can moreover be written as

$$\text{þ}(w) = \alpha + (1-\alpha)\text{þ}_1(w), \ \text{þ}_1 \in \mathcal{P}, w \in \widetilde{H} \tag{1.4.2}$$

Let $d \in \mathbb{C} \setminus \{0\}$ and $\text{þ} \in \mathcal{A}$. Then þ belonging to $\mathcal{P}(d)$ iff there exist a mapping $\text{þ}_1 \in \mathcal{P}$ such that

$$\text{þ}(w) = (1-d) + d\text{þ}_1(w), \ w \in \widetilde{H}$$

For $d = 1-\alpha, \ 0 \leq \alpha < 1, \ \mathcal{P} \supset \mathcal{P}(\alpha) = \mathcal{P}(d)$.

1.4.2 The Class $S^*(\alpha)$

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L}$ belong to the class $S^*(\alpha)$ if

$$0 \leq \alpha < 1, \ Re\left(\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}\right) > \alpha \quad w \in \widetilde{H} \tag{1.4.3}$$

or equivalently

$$0 \leq \alpha < 1, \ \frac{w\mathcal{L}'}{\mathcal{L}} \in \mathcal{P}(\alpha), \ w \in \widetilde{H} \tag{1.4.4}$$

All normalized SF of order α is denoted by the class $S^*(\alpha)$.

$$S^*(\alpha) = \left\{\mathcal{L} \in \mathcal{A} : Re\left(\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}\right) > \alpha, \ 0 \leq \alpha < 1, \ w \in \widetilde{H}\right\}. \tag{1.4.5}$$

Roberston [87] initiate this class in 1936.

Note $S^*(0) = S^*$.

1.4.3 The Class $C(\alpha)$

Let $\mathcal{L} \in S$ Then $\mathcal{L}$ belong to the class $C(\alpha)$ if

$$0 \le \alpha < 1, \; Re\left(\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)}\right) > \alpha, \; w \in \tilde{H}. \tag{1.4.6}$$

or equivalently

$$0 \le \alpha < 1, \; \frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)} \in \mathcal{P}(\alpha), \; w \in \tilde{H} \tag{1.4.7}$$

All normalized CF of order α is denoted by the class $C(\alpha)$.

$$C(\alpha) = \left\{ f \in S \,.\, Re\left(\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)}\right) > \alpha, \; 0 \le \alpha < 1, \; w \in \tilde{H} \right\} \tag{1.4 8}$$

Roberston [87] initiate this class in 1936.

Note that. $C = C(0)$

1.4.4 The Class $K(\alpha)$ [39]

A function $\mathcal{L} \in \mathcal{A}$ is said to be CCF of order α if it satisfy the following condition

$$Re\left(\frac{\mathcal{L}'(w}{l'(w)}\right) > \alpha, \; 0 \le \alpha < 1, \; w \in \tilde{H}. \tag{1.4.9}$$

or equivalently

$$\frac{\mathcal{L}'(w}{l'(w)} \in \mathcal{P}(\alpha), \; 0 \le \alpha < 1, \; w \in \tilde{H}. \tag{1.4.10}$$

Here l is convex function of order α.

All normalized CCF of order α is denoted by the class $K(\alpha)$

Note that $K(0) = K$.

In 1991, Goodman [40, 41] initiate two important classes UC and US^* of uniformly convex function (UCF) and uniformly starlike function (USF) respectively as below

1.4.5 The Class UC

A function $\mathcal{L} \in \mathcal{A}$ defined in (1 1.1) in class UC if it satisfy the following condition

$$\mathcal{L} \in UC \text{ if and only if } \mathcal{L} \in \mathcal{A} : Re\left((w-\vartheta)\frac{\mathcal{L}''(w)}{\mathcal{L}'(w)}+1\right) > 0,\ \vartheta, w \in \widetilde{H},$$

according to Ronning [90] and Ma et al. [63]

$$\mathcal{L} \in UC \text{ if and only if } \mathcal{L} \in \mathcal{A}\quad Re\left(\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)}\right) > \left|\frac{w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right|,\ w \in \widetilde{H}. \tag{1.4.11}$$

All such functions are in class UC.

1.4.6 The Class US^*

A function $\mathcal{L} \in \mathcal{A}$ defined in (1 1 1) in class US^* if it satisfy the following condition

$$\mathcal{L} \in UC \text{ if and only if } \mathcal{L} \in \mathcal{A}\quad Re\left(\frac{(w-\vartheta)\mathcal{L}'(w)}{\mathcal{L}(w)-\mathcal{L}(\eta)}\right) > 0.\ \vartheta, w \in \widetilde{H}.$$

according to Ronning [90] and Ma et al. [63]

$$\mathcal{L} \in US^* \text{ if and only if } \mathcal{L} \in \mathcal{A} : Re\left(\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}\right) > \left|\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}-1\right|,\ w \in \widetilde{H} \tag{1 4 12}$$

All such functions are in class US^*. Geometrically, $\mathcal{L} \in \mathcal{A}$ in classes US^* or UC which maps every circular segment η which is in $\widetilde{H}$, with center $\varphi \in \widetilde{H}$ onto a starlike arc or a convex arc respectively.

For $k \geq 0$, Kanas [47] inaugurate the class k-UC and k-US^* as:

1.4.7 The Class k-UC

A function $\mathcal{L} \in S$ is belong to class k-UC if it satisfy

$$\mathcal{L} \in k-UC \text{ if and only if } \mathcal{L} \in S \cdot Re\left(\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)}\right) > k\left|\frac{w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right|,\ w \in \widetilde{H}. \tag{1.4.13}$$

1.4.8 The Class k-US^*

Let $\mathcal{L} \in S$ Then $\mathcal{L}$ belong to the class k-US^* if it satisfy

$$\mathcal{L} \in k - US^* \text{ if and only if } \mathcal{L} \in S \text{ . } Re\left(\frac{w\mathcal{L}'}{\mathcal{L}}\right) > k\left|\frac{w\mathcal{L}'}{\mathcal{L}} - 1\right|, \ w \in \widetilde{H}. \tag{1.4.14}$$

Observe that 0-$UC \equiv C$, 0-$US^* \equiv US^*$ and 1-$UC \equiv UC$, 1-$US^* \equiv US^*$.

Geometrical representation of theses classes is very interesting for $k \geq 0$. for detail see [43. 48. 49. 50. 76. 77. 62, 113].

For $(0 \leq \alpha < 1)$, in [88. 90]. Ronning initiated the two classes k-$US^*(\alpha)$ and k-$UC(\alpha)$ as:

1.4.9 The Class k-$US^*(\alpha)$

Let $\mathcal{L} \in S$. Then $\mathcal{L}$ belong to the class k-$US^*(\alpha)$ if it satisfy

$$\mathcal{L} \in k - US^*(\alpha) \text{ iff } Re\left\{\frac{w\mathcal{L}'}{\mathcal{L}} - \alpha\right\} > k\left|\frac{w\mathcal{L}'}{\mathcal{L}} - 1\right|, \ w \in \widetilde{H}. \tag{1.4.15}$$

1.4.10 The Class k-$UC(\alpha)$

Let $\mathcal{L} \in S$. Then $\mathcal{L}$ belong to the class k-$UC(\alpha)$ if it satisfy

$$\mathcal{L} \in k - UC(\alpha) \text{ iff } \mathcal{L} \in S \text{ . } Re\left(\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)} - \alpha\right) > k\left|\frac{w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right|, \ w \in \widetilde{H}. \tag{1.4.16}$$

For $(0 \leq \alpha < \beta \leq 1)$ and $k(1-\beta) < 1-\alpha$, El-Ashwah et al [25]. initiate two important subclasses k-$UC(\alpha, \beta)$ and k-$US^*(\alpha, \beta)$ of k-uniformly convex and starlike functions as

1.4.11 The Class k-$UC(\alpha, \beta)$

Let $\mathcal{L} \in S$. Then $\mathcal{L}$ belong to the class k-$UC(\alpha, \beta)$ if it satisfy

$$\mathcal{L} \in k - UC(\alpha, \beta) \text{ if and only if } Re\left\{\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)} - \alpha\right\} > k\left|\frac{(w\mathcal{L}'(w))'}{\mathcal{L}'(w)} - \beta\right|, \ w \in \widetilde{H} \tag{1.4.17}$$

1.4.12 The Class k-$US^*(\alpha, \beta)$

Let $\mathcal{L} \in S$. Then $\mathcal{L}$ belong to the class k-$US^*(\alpha, \beta)$ if it satisfy

$$\mathcal{L} \in k - US^*(\alpha, \beta) \text{ if and only if } Re\left\{\frac{w\mathcal{L}'}{\mathcal{L}} - \alpha\right\} > k\left|\frac{w\mathcal{L}'}{\mathcal{L}} - \beta\right|, \; w \in \widetilde{H} \tag{1.4.18}$$

1.4.13 The Class M_α

For $0 \leq \alpha \leq 1$, Mocanu [69] initiate the class M_α of mappings $\mathcal{L} \in A$ such that $\frac{\mathcal{L}(w)\mathcal{L}'(w)}{w} \neq 0$ for all $w \in \widetilde{H}$ and

$$\text{Re}\left\{(1-\alpha)\frac{w\mathcal{L}'}{\mathcal{L}} + \alpha\frac{(w\mathcal{L}')'}{\mathcal{L}'}\right\} > 0, \; w \in \widetilde{H}. \tag{1.4.19}$$

Geometrically, as a class of mappings that maps the circle centered at the origin on α-convex arcs and derived the condition (1.4.19). For particular values of α, we obtain a number of interesting classes of AF having nice geometry, for instance $M_0 = S^*$ and $M_1 = C$ are well known classes of SF and CF.

1.5 Introduction of q-Calculus

The mathematical study of q-calculus, especially q-transform, q-integral and q-fractional calculus has been a point of awesome intrigued for analysts due to its wide applications in numerous areas (see [35, 112]). A few of the prior work on the applications of the q-calculus was presented by Jackson [45, 46]. Afterward, q-analysis with geometrical properties was turned into distinguished through quantum groups. Owing to the numerous applications of q-analysis in mathematics and other area, many analysts [2, 35, 44, 45, 46, 105, 107, 108, 112] did a few noteworthy work on q-calculus. Recently, Srivastava [109], explore and investigated the mathematical application of fractional q- calculus, Integral q-calculus and fractional q-differential operators in GFT. Now days, rather of ordinary operators, many researchers like Aldweby and Darus [7] and Srivastava et al. [110], studied q-operators because of extensive use in q-calculus.

1.5.1 q-shifted factorial and q-gamma function [36, 37]

For $q, \delta_0 \in \mathbb{C}$ $(|q| < 1)$, and $m \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, q-shifted factorial is symbolized and defined as

$$\lceil \delta_0, q \rceil_m = \Pi_{t=0}^{m-1} \left(1 - \delta_0 q^t\right) = (1 - \delta_0)(1 - \delta_0 q)\left(1 - \delta_0 q^2\right) .. \left(1 - \delta_0 q^{m-1}\right), \tag{1.5.1}$$

with

$$\lceil \delta_0, q \rceil_0 = 1$$

In the sense of q- gamma function $(\Gamma_q w)$, we write for $m \in \mathbb{N}_0$

$$\left\lceil q^{\delta_0}, q_m \right\rceil = \frac{\Gamma_q (m + \delta_0)}{\Gamma_q \delta_0} (1 - q)^m , \tag{1.5.2}$$

where

$$\Gamma_q w = \frac{\lceil q, q \rceil_\infty}{\lceil q^w, q \rceil_\infty} (1 - q)^{1-w}, \quad (1 > |q|) . \tag{1.5.3}$$

note that

$$\lceil \delta_0, q \rceil_\infty = \Pi_{t=0}^{\infty} \left(1 - \delta_0 q^t\right) . \tag{1.5.4}$$

and

$$\Gamma_q (w + 1) = (\Gamma_q w) \lceil w \rfloor_q , \tag{1.5.5}$$

where $\lceil w \rfloor_q$ is a q-number and defined below.

1.5.2 q-number and q-factorial [114]

For $q \in]0, 1[$. The q-number $\left(\lceil \delta_0 \rfloor_q\right)$ and q-factorial$\left(\lceil n_0 \rfloor_q !\right)$ is defined by

$$\lceil \delta_0 \rfloor_q = -\frac{q^{\delta_0} - 1}{1 - q}, \quad (\delta_0 \in \mathbb{C}) . \tag{1 5.6}$$

and

$$\lceil \delta_0 \rfloor_q = \sum_{t=1}^{m-1} q^t = 1 + q + q^2 + . . + q^{m-1}. \quad (\delta_0 \in \mathbb{N}) . \tag{1.5.7}$$

Now q-factorial

$$\lceil n_0 \rfloor_q ! = \Pi_{t=0}^{n_0} \lceil t \rfloor_q , \quad (n_0 \in \mathbb{N}) . \tag{1.5.8}$$

and

$$\lceil n_0 \rfloor_q! = 1, \quad (n_0 = 1)$$

1.5.3 Remarks

For $q \to 1^-$, the basic q-calculus definitions reduce into classical definitions.

(i) $\lceil \delta_0 \rfloor_q = -\frac{q^{\delta_0 - 1}}{1-q} \to \delta_0 \quad (q \to 1^-)$.

(ii) For $q \to 1^-$, $\Gamma_q w$ (q- gamma function) reduce into Γw (classical gamma (Euler's) function), that is

$$\Gamma_q w \to \Gamma w \qquad (q \to 1^-)$$

(iii) For, $\lceil \delta_0, q \rceil_m$ (q-shifted factorial) defined in (1.5.1) reduces into $(\delta_0)_t$ (familiar Pochhammer symbol) as

$$(\delta_0)_t = \begin{cases} \delta_0 (\delta_0 + 1) \,..\, (\delta_0 + t - 1) & t \in \mathbb{N}, \\ 1 & t = 0. \end{cases} \tag{1.5.9}$$

we obtain $(\delta_0)_t$ as

$$(\delta_0)_t = \lim_{q \to 1^-} \frac{\lceil q^{\delta_0}, q \rceil_m}{(1-q)^m}. \tag{1.5.10}$$

1.6 Conic Domain

For $\kappa \geq 0$, Kanas and Wiśniowska [43, 49] initiated the study of AF on conic domain Ω_κ as:

$$\Omega_\kappa = \left\{ u + iv : u > \kappa \sqrt{(u-1)^2 + v^2} \right\}.$$

Note that, for $\kappa = 1$, $0 < \kappa < 1$ and $\kappa > 1$, the domain Ω_κ represents a parabola, right branch of a hyperbola and ellipse respectively. For these conic regions, the below functions $h_\kappa (w)$ play as the role of extremal functions.

$$h_\kappa (w) = \begin{cases} \frac{1+w}{1-w} & \kappa = 0, \\ 1 + \left(\log \frac{\sqrt{w}+1}{1-\sqrt{u}} \right)^2 \frac{2}{\pi^2} & \kappa = 1, \\ 1 + \sinh^2 \left\{ \arctan h \sqrt{w} \frac{2}{\pi} \left(\arccos \kappa \right) \right\} \frac{2}{1-\kappa^2} & 0 < \kappa < 1, \\ 1 + \frac{1}{\kappa^2 - 1} \sin \left(\frac{\pi}{2R(y)} \int_0^{\frac{u(u)}{\sqrt{y}}} \frac{dx}{\sqrt{1-x^2}\sqrt{1-y^2x^2}} \right) + \frac{1^2}{\kappa^2-1} & \kappa > 1. \end{cases} \tag{1.6.1}$$

where

$$u(w) = \frac{w - \sqrt{y}}{1 - \sqrt{y}w}, \quad w \in \widetilde{H}$$

If $\kappa = \cosh\left(\pi R'(y)/(4R(y))\right) \in (0, 1)$, where $R(y)$ and $R'(y) = R(\sqrt{1-y^2})$ represents the Legender's complete elliptic integral of first kind and complementary integral of $R'(y)$ and $R(y)$, see [5, 33, 43, 49, 52, 71, 94, 115]. If $h_\kappa(w) = 1 + \delta(\kappa)w + \delta_1(\kappa)w^2 + \cdots$ is taken from [49] for (1 6 1), then

$$\delta(\kappa) = \begin{cases} \frac{8(\arccos\kappa)^2}{\pi^2(1-\kappa^2)} & 0 \le \kappa < 1, \\ \frac{8}{\pi^2} & \kappa = 1, \\ \frac{\pi^2}{4\sqrt{y}(\kappa^2-1)R^2(y)(1+y)} & \kappa > 1, \end{cases} \tag{1 6.2}$$

$$\delta_1(\kappa) = \delta_2(\kappa)\,\delta(\kappa),$$

where

$$\delta_2(\kappa) = \begin{cases} \frac{T_1^2+2}{3} & 0 \le \kappa < 1, \\ \frac{2}{3} & \kappa = 1, \\ \frac{4R^2(y)(y^2+6y+1)-\pi^2}{24R^2(y)(1+y)\sqrt{y}} & \kappa > 1, \end{cases} \tag{1 6.3}$$

where $T_1 = \frac{2}{\pi}\arccos\kappa$, and $y \in (0, 1)$.

1.7 Some Important Functions

1.7.1 Mittag-Leffler function

Mittag-Leffler defined familiar Mittag-Leffler function [67, 68] $M_\alpha(w)$ by

$$M_\alpha(w) = \sum_{m=0}^{\infty} \frac{w^m}{\Gamma(\alpha - 1)}. \tag{1.7.1}$$

and Wiman [118] generalized this function by

$$M_{\alpha,\mu}(w) = \sum_{m=0}^{\infty} \frac{w^m}{\Gamma(\alpha m + \mu)} \quad (\alpha \ge 0). \tag{1.7.2}$$

where $\Re(\alpha) > 0$, $\Re(\mu) > 0$ and $\alpha, \mu \in \mathbb{C}$. Many researchers explain the Mittag-Leffler function and its generalizations see [9, 34, 66, 86, 103, 104, 106].

An impartant theory that has contributed significantly in geometric function theory is differential operator theory. Numerous researchers have worked intensively in this way, for recent work see [1, 26, 32, 75]. Elhaddad [31] introduced the following differential operator for $\mathcal{L} \in \mathcal{A}$

$$\partial D_{\chi}^{n'}(\alpha, \mu)\mathcal{L}(w) = w + \sum_{m=2}^{\infty} [1 + (m-1)\chi]^{n'} \frac{\Gamma(\mu)}{\Gamma(\alpha(m-1)+\mu)} a_m w^m. \tag{1.7.3}$$

and for $\mathcal{L} \in \mathcal{T}$

$$\partial D_{\chi}^{n'}(\alpha, \mu)\mathcal{L}(w) = w - \sum_{m=2}^{\infty} [1 + (m-1)\chi]^{n'} \frac{\Gamma(\mu)}{\Gamma(\alpha(m-1)+\mu)} a_m w^m \tag{1.7.4}$$

1.7.2 Bessel functions (BF)

Let us consider a second order linear homogeneous differential equation.

$$w^2 s''(w) + bws'(w) + \left[dw^2 - x^2(1-b)\right] s(w) = 0 \qquad (b, x, d \in \mathbb{C}) \tag{1.7.5}$$

A particular solution of (1.7.5) give a generalized Bessel functions of the first kind of order x, given in (1.7.6) defined by Baricz [12].

$$s(w) = s_{x,b,d} = \sum_{m=2}^{\infty} \frac{(-1)^m d^m}{(m)!\Gamma\left(x + m + \frac{b+1}{2}\right)} \tag{1.7.6}$$

The function $s_{x,b,d}$ is not univalent in $\widetilde{H}$ but the series given by (1.7.5) is convergent. Cho [22] defined the following transformation

$$u_{x,b,d}(w) = 2^x \Gamma\left(x + \frac{b+1}{2}\right) w^{-\frac{x}{2}} s_{x,b,d}\left(\sqrt{w}\right) \qquad \sqrt{1} = 1$$

Using well known Pochhammer symbol the following Gamma function can be defined

$$(a_0)_m = \frac{\Gamma(a_0+m)}{\Gamma(a_0)} = \begin{cases} 1 & m = 0 \\ a_0(a_0+1)(a_0+2)\cdots(a_0+m-1) & m \in \mathbb{N} \end{cases}$$

We can express $u_{x,b,d}(w) = u_x(w)$ and $u_x(w)$ can be written as follows.

$$u_x(w) = u_{x,b,d}(w) = \sum_{m=0}^{\infty} \frac{\left(-\frac{d}{4}\right)^m}{\left(x+\frac{b+1}{2}\right)_m (m)!} w^m, \text{ where } x + \frac{b+1}{2} \in N = \{1, 2, 3, \ldots\} \quad (1.7.7)$$

We write $b_s = x + \frac{b+1}{2}$.

In geometric function theory the study of generalized Bessel functions is an important topic. In this study, we refer to the studies done by Baricz [11, 12, 13, 14], Akgul [3, 4] , Sakar and Aydogan [94] , Cho et al [22] , Mondal [70] , Deniz [27, 28] and Choi [23]. Studies on Struve functions can be found in recent investigation by Srivastava [102].

1.8 Some Important Operators

1.8.1 Convolution Operator

The part of operator is exceptionally imperative in this field. By utilizing convolution theory, to characterize operator and study operator properties, are most current days research. Numerous differential and integral operators can be characterized by the help of convolution of certain AF.

Let $\mathcal{L}$ given by (1.1 1) and $l(w) = w + \sum_{m=2}^{\infty} b_m w^m$ Then convolution of $\mathcal{L}$ and l is symbolized by $\mathcal{L} * l$ and describe as:

$$(\mathcal{L} * l)(w) = w + \sum_{m=2}^{\infty} a_m b_m w^m. \quad (1.8.1)$$

1.8.2 q-Derivative operator or q-Difference operator [44, 45, 46]

The q-derivative (q-difference) operator of a mapping $\mathcal{L}$ of the form (1.1.1), for a given subset of $\mathbb{C}$ is symbolized by $(\partial\mathcal{D}_q\mathcal{L})(w)$ and described by

$$(\partial\mathcal{D}_q\mathcal{L})(w) = \begin{cases} \frac{\mathcal{L}(w)-\mathcal{L}(qw)}{w(1-q)} & w \neq 0 \\ \mathcal{L}'(0) & w = 0, \end{cases} \quad (1.8.2)$$

when $q \to 1$, the q-difference (q-derivative) operator, shrink into ordinary derivative, that is

$$\lim_{q\to 1} (\partial \mathcal{D}_q \mathcal{L})(w) = \mathcal{L}'(w),$$

provided that $\mathcal{L}'$ exists.

1.8.3 Sälägean q-derivative operator (or q-differential) operator

In 2017, Govindaraj and Sivasubramanian [42] defined the Sàlagean q-derivative (q-differential) operator as:

Let $\mathcal{L} \in \mathcal{A}$ set out in (1.1 1). Then the Sälagean q-derivative (or q-differential) is interpret as:

$$S_q^0\mathcal{L} = \mathcal{L},\ S_q^1\mathcal{L} = w\partial\mathcal{D}_q\mathcal{L}\ .\ S_q^i\mathcal{L} = w\partial\mathcal{D}_q\left(S_q^{i-1}\mathcal{L}\right),\quad i \in \mathbb{N}_0 = \{0\} \cup \mathbb{N},\ w \in \widetilde{H}.$$

simply implies

$$S_q^i\mathcal{L}(w) = \mathcal{L}(w) * L_{q,i}(w),\quad i \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}. \tag{1.8 3}$$

where

$$L_{q,i}(w) = w + \sum_{m=2}^{\infty} \lceil m \rfloor_q^i w^m,\ w \in \widetilde{H} \tag{1.8.4}$$

using (1 8 3) and (1.8.4), the series form of $S_q^i\mathcal{L}(w)$ for $\mathcal{L}$ is given by

$$S_q^i\mathcal{L}(w) = w + \sum_{m=2}^{\infty} \lceil m \rfloor_q^i a_m w^m. \tag{1.8.5}$$

Note that

$$\lim_{q\to 1} L_{q,i}(w) = w + \sum_{m=2}^{\infty} m^i w^m,$$

and

$$\lim_{q\to 1} S_q^i\mathcal{L}(w) = \mathcal{L}(w) * \left(w + \sum_{m=2}^{\infty} m^i w^m\right) = w + \sum_{m=2}^{\infty} m^i a_m w^m.$$

the last expression is famous Salāgėan derivative [95].

1.8.4 Choi-Saigo-Srivastava operator

Choi, Saigo and Srivastava [24] generalized the Noor integral operator [73] in 2002 and initiate the operator $I_{\lambda,\mu}$ which is name as Choi-Saigo-Srivastava operator. The construction of operator $I_{\lambda,\mu}$ is in the following way:

Let $\mathcal{L} \in \mathcal{A}$.

$$I_{\lambda,\mu}\mathcal{L}(w) = \mathcal{L}(w) * \mathcal{F}_{\lambda+1,\mu}(w), \quad w \in \widetilde{H}, \ \lambda > -1, \ \mu > 0.$$

where

$$\mathcal{F}_{\lambda+1,\mu}(w) = w + \sum_{m=2}^{\infty} \frac{\Gamma(\mu+m-1)\Gamma(1+\lambda)}{\Gamma(\mu)\Gamma(m+\lambda)} w^m = w + \sum_{m=2}^{\infty} \frac{\lceil\mu\rceil_{m-1}}{\lceil 1+\lambda\rceil_{m-1}} w^m \tag{1.8.6}$$

Thus, we see that

$$I_{\lambda,\mu}\mathcal{L}(w) = w + \sum_{m=2}^{\infty} \frac{\lceil\mu\rceil_{m-1}}{\lceil 1+\lambda\rceil_{m-1}} a_m w^m \tag{1.8.7}$$

1.9 Some Generalized Subclasses of Analytic Functions in q-Calculus

1.9.1 Definition[43]

In 1973, Janöwski [43] , introduced the class $\mathcal{P}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ as follows.

Let Þ$\in \mathcal{A}$ and Þ$(0) = 1$, belong to $\mathcal{P}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ if þ(w) is subordinate to $\frac{1-\widehat{\mathfrak{A}}(w)}{1+\widetilde{\mathfrak{B}}(w)}$, that is

$$þ(w) \prec \frac{1+\widehat{\mathfrak{A}}(w)}{1+\widetilde{\mathfrak{B}}(w)}, \quad -1 \le \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \le 1, \ w \in \widetilde{H} \tag{1.9.1}$$

Geometrically þ $\in \mathcal{P}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ iff Þ$(0) = 1$ and Þ$\left(\widetilde{H}\right)$ are inside the such open disc whose centered lies on real axis and end of the diameter points are $d_1 = \frac{1+\widehat{\mathfrak{A}}}{1+\widetilde{\mathfrak{B}}}$ and $d_2 = \frac{1-\widehat{\mathfrak{A}}}{1-\widetilde{\mathfrak{B}}}$. Note that, $\mathcal{P}[1,-1] = \mathcal{P}$ and $\mathcal{P}[1-2\alpha,-] = \mathcal{P}(\alpha)$.

Also according to Janowski [43]

Let Þ$\in \mathcal{A}$ and Þ$(0) = 1$, belong to $\mathcal{P}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ if and only if there exists a mapping Þ$\in \mathcal{P}$ such that

$$\frac{(\widehat{\mathfrak{A}}+1)\text{Þ}(w)-\left(\widehat{\mathfrak{A}}-1\right)}{(\widetilde{\mathfrak{B}}+1)\text{Þ}(w)-\left(\widetilde{\mathfrak{B}}-1\right)}\prec\frac{1+\widehat{\mathfrak{A}}w}{1+\widetilde{\mathfrak{B}}w}, \tag{1.9.2}$$

here. the class the of mappings with non negative real parts is $\mathcal{P}$.

1.9.2 Definition[107]

$\mathcal{L}\in\mathcal{A}$ is belonging to $\mathcal{S}^*\left(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ if and only if

$$\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}=\frac{\text{Þ}(w)(\widehat{\mathfrak{A}}+1)-\left(\widehat{\mathfrak{A}}-1\right)}{\text{Þ}(w)(\widetilde{\mathfrak{B}}+1)-\left(\widetilde{\mathfrak{B}}-1\right)}.\quad\left(-1\leq\widetilde{\mathfrak{B}}<\widehat{\mathfrak{A}}\leq 1\right). \tag{1.9 3}$$

where Þ$\in\mathcal{A}$ and Þ$(0)=1$, belong to $\mathcal{P}\left(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$

1.9.3 Definition[107]

A function $\mathcal{L}\in\mathcal{A}$ is also belonging to $\mathcal{C}\left(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ if and only if

$$\frac{\left(w\mathcal{L}'(w)\right)'}{(\mathcal{L}(w))'}=\frac{\text{Þ}(w)(\widehat{\mathfrak{A}}+1)-\left(\widehat{\mathfrak{A}}-1\right)}{\text{Þ}(w)(\widetilde{\mathfrak{B}}+1)-\left(\widetilde{\mathfrak{B}}-1\right)}.\quad\left(-1\leq\widetilde{\mathfrak{B}}<\widehat{\mathfrak{A}}\leq 1\right). \tag{1.9.4}$$

where Þ$\in\mathcal{A}$ and Þ$(0)=1$. belong to $\mathcal{P}\left(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$

1.9.4 Definition[108]

A function $\mathcal{L}\in\mathcal{A}$ is also belonging to $\mathcal{S}_q^*\left(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ if and only if

$$\frac{w\partial D_q\mathcal{L}(w)}{\mathcal{L}(w)}=\frac{(\widehat{\mathfrak{A}}+1)\widetilde{\text{Þ}}(w)-\left(\widehat{\mathfrak{A}}-1\right)}{\left(\widetilde{\mathfrak{B}}+1\right)\widetilde{\text{Þ}}(w)-\left(\widetilde{\mathfrak{B}}-1\right)},\quad\left(-1\leq\widetilde{\mathfrak{B}}<\widehat{\mathfrak{A}}\leq 1\right).\quad q\in(0.1). \tag{1.9 5}$$

Additionally by rule of subordination we are able to written as follows:

$$\frac{w\partial D_q\mathcal{L}(w)}{\mathcal{L}(w)}\prec\frac{(\widehat{\mathfrak{A}}+1)w+2+\left(\widehat{\mathfrak{A}}-1\right)qw}{\left(\widetilde{\mathfrak{B}}+1\right)w+2+\left(\widetilde{\mathfrak{B}}-1\right)qw}.$$

where

$$\widetilde{\mathrm{P}}(w) = \frac{1+w}{1-qw} \tag{1 9.6}$$

1.9.5 Definition[108]

A function $\mathcal{L} \in \mathcal{A}$ is also belonging to $\mathcal{C}_q\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ if and only if

$$\frac{\partial D_q\left(w \partial D_q \mathcal{L}(w)\right)}{\partial D_q \mathcal{L}(w)} = \frac{\widetilde{\mathrm{P}}(w)(\widehat{\mathfrak{A}}+1) - \left(\widehat{\mathfrak{A}}-1\right)}{\widetilde{\mathrm{P}}(w)\left(\widetilde{\mathfrak{B}}+1\right) - \left(\widetilde{\mathfrak{B}}-1\right)}, \quad \left(-1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1\right), \quad q \in (0.1) \tag{1.9 7}$$

Additionally by rule of subordination we are able to written as follows·

$$\frac{\partial D_q\left(w \partial D_q \mathcal{L}(w)\right)}{\partial D_q \mathcal{L}(w)} \prec \frac{w(\widehat{\mathfrak{A}}+1) + \left(\widehat{\mathfrak{A}}-1\right) qw + 2}{w\left(\widetilde{\mathfrak{B}}+1\right) + \left(\widetilde{\mathfrak{B}}-1\right) qw + 2}.$$

Next, Mahmood et al.[61] initated the class $k\text{-}\mathcal{P}_q\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ as:

1.9.6 Definition[61]

A function $h \in k\text{-}\mathcal{P}_q\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ if and only if

$$h(w) \prec \frac{\left(\widehat{\mathfrak{A}} O_1 + O_3\right) h_k(w) - \left(\widehat{\mathfrak{A}} O_1 - O_3\right)}{\left(\widetilde{\mathfrak{B}} O_1 + O_3\right) h_k(w) - \left(\widetilde{\mathfrak{B}} O_1\right) - O_3}, \quad k \geq 0, \ q \in (0\ 1) \tag{1.9.8}$$

where

$$O_1 = 1 + q \ \text{ and } \ O_3 = 3 - q.$$

Also $h_k(w)$ is defined in (1.6.1) Geometrically, the mapping $h \in k\text{-}\mathcal{P}_q\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ takes whole domain $\Omega_{k\,q}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$, $1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1$, $k \geq 0$ which is definable as:

$$\Omega_{k,q}(\widehat{\mathfrak{A}}\ \widetilde{\mathfrak{B}}) = \left\{r = u + iv : \mathfrak{Re}(\Psi) > k\left|\Psi - 1\right|\right\}.$$

where

$$\Psi = \frac{\left(\widetilde{\mathfrak{B}} O_1 - O_3\right) r(w) - \left(\widehat{\mathfrak{A}} O_1 - O_3\right)}{\left(\widetilde{\mathfrak{B}} O_1 + O_3\right) r(w) - \left(\widehat{\mathfrak{A}} O_1\right) + O_3}$$

For detail see [61].

Note that

(i) For $q \to 1$, the domain $\Omega_{\kappa\, q}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ shrinks to the $\Omega_{\kappa}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ (see [78]).

(ii) For $q \to 1$, the class $\kappa - \mathcal{P}_q\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ shrinks to the class $\kappa - \mathcal{P}\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ (see [78]).

(iii) For $q \to 1$ and $\kappa = 0$, then $\kappa - \mathcal{P}_q\left(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right) = \mathcal{P}(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ also $\kappa - \mathcal{P}(1, -1) = \mathcal{P}(h_\kappa)$ (see [43]).

1.9.7 Definition[61]

Let $\mathcal{L} \in \mathcal{A}$ is belonging to k-$\mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$, if and only if

$$\mathfrak{Re}\left(\frac{(\mathfrak{D}O_1 - O_3)\frac{w\partial D_q\mathcal{L}(w)}{\mathcal{L}(w)} - (\mathfrak{C}O_1 - O_3)}{(\mathfrak{D}O_1 + O_3)\frac{w\partial D_q\mathcal{L}(w)}{\mathcal{L}(w)} - (\mathfrak{C}O_1 + O_3)}\right) > k\left|\frac{(\mathfrak{D}O_1 - O_3)\frac{w\partial D_q\mathcal{L}(u)}{\mathcal{L}(w)} - (\mathfrak{C}O_1 - O_3)}{(\mathfrak{D}O_1 + O_3)\frac{w\partial D_q\mathcal{L}(w)}{\mathcal{L}(u)} - (\mathfrak{C}O_1 + O_3)} - 1\right|$$

Or equivalently

$$\frac{w\partial D_q\mathcal{L}(w)}{\mathcal{L}(w)} \in k - \mathcal{P}_q(\mathfrak{C}, \mathfrak{D}).$$

where $k \geq 0$, $-1 \leq \mathfrak{D} < \mathfrak{C} \leq 1$.

We can see that when $q \to 1$, then $\kappa - \mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$ diminishes to the renowned class which is stated in [78].

1.10 Some Important Lemmas

1.10.1 Lemma[57]

If $\mathcal{L}$ and g are AF in $\widetilde{H}$ with

$$\mathcal{L} \prec g$$

then, for $0 < p$ and $w = re^{i\theta}$, $(0 < r < 1)$,

$$\int_0^{2\pi} |\mathcal{L}(w)|^p \, d\theta \leq \int_0^{2\pi} |g(w)|^p \, d\theta. \tag{1.10 1}$$

1.10.2 Lemma[26]

Let $\hbar_0$ is AF in $\widetilde{H}$ with $\hbar(0) = 1$ and $\beta > 0$. $0 \leq \gamma < 1$ If

$$\hbar_0(w) + \frac{\beta w \hbar_0'(w)}{\hbar_0(w)} \prec \frac{1 + w(1 - 2\gamma)}{1 - w}. \tag{1.10.2}$$

then

$$\hbar_0(w) \prec \frac{1 + w(1 - 2\delta)}{1 - w}.$$

where

$$\delta = \frac{\sqrt{(2\gamma - \beta)^2 + 8\beta} + (2\gamma - \beta)}{4}. \tag{1.10 3}$$

1.10.3 Lemma [81]

Let $\hbar$ be AF in $\widetilde{H}$ of the form

$$\hbar(w) = 1 + \sum_{m=j}^{\infty} b_m w^m, \; b_j \neq 0,$$

with $\hbar(w) \neq 0$ in $\widetilde{H}$. If, $\exists$ a point w_0, $|w_0| < 1$ such that for $\{|w_0| > |w|\}$ $|\arg \hbar(w)| < \frac{\pi\rho}{2}$ and $|\arg \hbar(w_0)| = \frac{\pi\rho}{2}$ for some $\rho > 0$, then we have $\frac{w_0 \hbar'(w_0)}{\hbar(w_0)} = i\ell\rho$, where

$$\begin{cases} \ell \geq \frac{m}{2}\left(c + \frac{1}{c}\right). & \text{when } \arg \hbar(w_0) = \frac{\pi\rho}{2}. \\ \ell \leq -\frac{m}{2}\left(c + \frac{1}{c}\right). & \text{when } \arg \hbar(w_0) = -\frac{\pi\rho}{2}. \end{cases}$$

where $(\hbar(w_0))^{1/\rho} = \pm ic \quad (c > 0)$.

Above lemma is the generalized form of Nunokawa's lemma .

1.10.4 Lemma[116]

Let positive measure on $[0,1]$ is ε Let $\dot{U}$ defined on $\widetilde{H}\times[0,1]$ is a complex-valued function with $\mathring{U}(..t)$ and AF in $\widetilde{H}$ for every $t\in[0,1]$ and for all $w\in\widetilde{H}$, ε-integrable on $[0,1]$ be $\dot{U}(w,)$. Suppose that $\dot{U}(-r,t)$ is real, $0<\mathrm{Re}\,\dot{U}(w,t)$ and $1/\dot{U}(-r,t)\leq\mathrm{Re}\left\{1/\dot{U}(w,t)\right\}$ for $t\in[0,1]$ and $|w|\leq r<1$. If

$$\mathring{U}(w)=\int_0^1\dot{U}(w,t)\,d\varepsilon(t),$$

then $1/\mathcal{F}(-r)\leq\mathrm{Re}\{1/\mathcal{F}(w)\}$.

1.10.5 Lemma[65]

If $-1\leq\mathfrak{D}<\mathfrak{C}\leq1$, $\lambda>0$ and γ be a complex number satisfying $-\lambda(1-\mathfrak{C})/(1-\mathfrak{D})\leq\mathrm{Re}\{\gamma\}$. then the differential equation

$$s(w)+\frac{ws'(w)}{\lambda s(w)+\gamma}=\frac{1+\mathfrak{C}w}{1+\mathfrak{D}w},\quad w\in\widetilde{H}.$$

has a univalent solution in $\widetilde{H}$ given by

$$s(w)=\begin{cases}\dfrac{w^{\lambda+\gamma}(1+\mathfrak{D}w)^{\lambda(\mathfrak{C}-\mathfrak{D})/\mathfrak{D}}}{\lambda\int_0^w t^{\lambda+\gamma-1}(1+\mathfrak{D}t)^{\lambda(\mathfrak{C}-\mathfrak{D})/\mathfrak{D}}dt}-\frac{\gamma}{\lambda}, & \mathfrak{D}\neq0,\\[2ex] \dfrac{w^{\lambda+\gamma}e^{\lambda\mathfrak{C}w}}{\lambda\int_0^w t^{\lambda+\gamma-1}e^{\lambda\mathfrak{C}t}dt}-\frac{\gamma}{\lambda}, & \mathfrak{D}=0.\end{cases}$$

If $r(w)=1+c_1w+c_2w^2+\ldots$ is AF in $\widetilde{H}$ and satisfying

$$r(w)+\frac{wr'(w)}{\lambda r(w)+\gamma}\prec\frac{1+\mathfrak{C}w}{1+\mathfrak{D}w},\quad w\in\widetilde{H},$$

then

$$r(w)\prec s(w)\prec\frac{1+\mathfrak{C}w}{1+\mathfrak{D}w},$$

and $s(w)$ is the best dominant.

1.10.6 Lemma[117, Chapter 14]

Let n, x and $y \neq 0, -1, -2, .$ be complex numbers. Then, for $0 < \operatorname{Re} x < \operatorname{Re} y$,

$$\begin{aligned}
(i) \quad {}_2G_1(n, x, y; w) &= \frac{\Gamma(y)}{\Gamma(y-x)\Gamma(x)} \int_0^1 s^{x-1}(1-s)^{y-1-x}(1-sw)^{-n}\, ds \\
(ii) \quad {}_2G_1(n, x, y; w) &= {}_2G_1(x, n, y; w), \\
(iii) \quad {}_2G_1(n, x, y; w) &= (1-w)^{-n}\ {}_2G_1\left(n, y-x, y, \frac{w}{w-1}\right)
\end{aligned}$$

1.10.7 Lemma[73]

. Let $-1 \leq \widetilde{\mathfrak{B}}_2 \leq \widetilde{\mathfrak{B}}_1 < \widehat{\mathfrak{A}}_1 \leq \widehat{\mathfrak{A}}_2 \leq 1$. Then

$$\frac{1+\widehat{\mathfrak{A}}_1 w}{1+\widetilde{\mathfrak{B}}_1 w} \prec \frac{1+\widehat{\mathfrak{A}}_2 w}{1+\widetilde{\mathfrak{B}}_2 w}.$$

1.10.8 Lemma[89]

Suppose $1+\sum_{m=1}^{\infty} c_m w^m = d(w) \prec H(w) = 1+\sum_{m=1}^{\infty} C_m w^m$ If $H(\widetilde{H})$ is convex and $H(w) \in \mathcal{A}$, then

$$|c_m| \leq |C_1|, \quad m \geq 1.$$

1.10.9 Lemma[61]

Suppose $d(w) = 1+\sum_{m=1}^{\infty} c_m w^m \in k\text{-}\mathcal{P}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, then

$$|c_m| \leq \left|\delta\left(k, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)\right| = \frac{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})O_1)}{4}\delta(k),$$

where $\delta(k)$ is given by (1 6.2)

1.10.10 Lemma[61]

For $k \geq 0$, let $d \in k\text{-}\mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$

$$d(w) = w + \sum_{m=2}^{\infty} b_m w^m, \qquad w \in \widetilde{H}.$$

then

$$|b_m| \leq \prod_{n=0}^{m-2} \left(\frac{\left|\delta(k)(\mathfrak{C} - \mathfrak{D})\,(O_1) - 4q\,\lceil n \rfloor_{qe}\,\mathbb{D}\right|}{4q\,\lceil n+1 \rfloor_q} \right).$$

where $\delta(k)$ is given by (1.6.2).

1.10.11 Lemma[66]

Suppose $d \in \mathcal{S}^*$ and $\mathcal{L} \in \mathcal{C}$, then $G \in \mathcal{S}$, we have

$$\frac{\mathcal{L}(w) * d(w)G(w)}{\mathcal{L}(w) * d(w)} \in \overline{co}(G(\widetilde{H})), \qquad w \in \widetilde{H}.$$

here "*" represent convolution and $\overline{co}(G(\widetilde{H})$ represent the closed convex hull $G(\widetilde{H})$.

1.10.12 Lemma[61]

The function $\mathcal{L} \in \mathcal{A}$ is belonging to the class $k\text{-}\mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$. if

$$\sum_{m=2}^{\infty} \left\{ 2O_3(1+k)q\,\lceil m-1 \rfloor_q + \left|(\mathfrak{D}O_1) + O_3)\,\lceil m \rfloor_q - (\mathfrak{C}O_1) + O_3)\right| \right\} |a_m|$$
$$\leq O_1)\,|\mathfrak{D} - \mathfrak{C}|$$

Chapter 2

A New Subclass of Analytic Functions Defined by using Sälägean q-Differential Operator

In this chapter, we initiate two new subclasses $\mathcal{U}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$ and $T\mathcal{U}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$. These classes are initiated by using the Sälagean q -differential operator with the help of Janöwski function. These classes generalized numerous classes by selecting specific values of the parameters. We examined numerous sharp results and properties of these classes, like as extreme points (EP), distortion theorem (DT), coefficient estimates (CE), convexity, radii of star-likeness (RS), close-to-convexity, and integral mean inequalities.

2.1 Introduction

2.1.1 Definition

Let $\mathcal{L}\in\mathcal{A}$. Then $\mathcal{L}\in\mathcal{U}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$ if fulfill the subordination condition

$$\frac{S_q^{\imath}\mathcal{L}(w)}{S_q^{\jmath}\mathcal{L}(w)}-\beta\left|\frac{S_q^{\imath}\mathcal{L}(w)}{S_q^{\jmath}\mathcal{L}(w)}-1\right|\prec\frac{1+\widehat{\mathfrak{A}}w}{1+\widetilde{\mathfrak{B}}w}.$$

where $w\in\widetilde{H}$, $S_q^{\jmath}\mathcal{L}(w)\neq 0$. $q\in(0,1)$, $\beta\geq 0$, $-1\leq\widetilde{\mathfrak{B}}<\widehat{\mathfrak{A}}\leq 1$. for $\imath>\jmath$, $\imath\in\mathbb{N}$ and $\jmath\in\mathbb{N}_0$

By taking notable values of parameters, we get numerous critical subclasses examined by different creators.

2.1.2 Remarks

(i) For $\widehat{\mathfrak{A}} = 1-2\alpha$, $q \to 1$, and $\widetilde{\mathfrak{B}} = -1$, the class $\mathcal{U}_{i,j}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ shrinks into the class $\widetilde{N}_{i,j}(\alpha, \beta)$, $(0 \leq \alpha < 1)$ studied by Eker and Owa [30].
(ii) For $i = 1$, $q \to 1$, $\widehat{\mathfrak{A}} = 1-2\alpha$, $j = 0$ and $\widetilde{\mathfrak{B}} = -1$ the class $\mathcal{U}_{i,j}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ shrinks into the class $\mathcal{U}_0\mathcal{S}(\alpha, \beta)$, $(0 \leq \alpha < 1)$ studied by Shams et.al [97].
(iii) For $q \to 1$, $\widehat{\mathfrak{A}} = 1-2\alpha$, $\widetilde{\mathfrak{B}} = -1$, $i = 2$, and $j = 1$, the class $\mathcal{U}_{i,j}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ shrinks into the class $\mathcal{U}_0\mathcal{K}(\alpha, \beta)$, $(0 \leq \alpha < 1)$ studied by Shams et al [98].
(iv) For $\beta = 0$, $i = 1$, $q \to 1$ and $j = 0$, the class $\mathcal{U}_{i,j}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ shrinks into the class $\mathcal{S}^*(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, studied by Janowski [43].
(v) For $\beta = 0$, $i = 2$, $q \to 1$ and $j = 1$, the class $\mathcal{U}_{i,j}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ shrinks into the class $\mathcal{K}_0(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, studied by Padmanabhan and Ganesan [82].

2.1.3 Definition

As T is the subclass of $\mathcal{A}$ having negative coefficients in Maclaurin's series defined in (1.1.2). Here, we denote the class $\mathcal{TU}_{i,j}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}) = \mathcal{U}_{i,j}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}) \cap \mathcal{T}$.
For appropriate possibility of the parameters q, i, j, β, $\widehat{\mathfrak{A}}$ and $\widetilde{\mathfrak{B}}$, we are able to get assorted subclasses of $\mathcal{T}$.

2.1.4 Remarks

(i) $\mathcal{TU}_{j+1,j}(0, \beta, 1-2\alpha, -1) = \mathcal{TS}(j, \alpha, \beta)$, $(0 \leq \alpha < 1, \beta \geq 0, j \in N_0)$ (see Rosy and Murugusundaramoorthy [92] and Aouf [10]).
(ii) $\mathcal{TU}_{1,0}(0, 1, 1-2\alpha, -1) = \mathcal{S}_p\mathcal{T}(\alpha)$ and $\mathcal{TU}_{2,1}(0, 1, 1-2\alpha, -1) = \mathcal{UCT}(\alpha)$ $(0 \leq \alpha < 1)$ (see Bharati et al. [17]).
(iii) $\mathcal{TU}_{1,0}(0, 0, 1-2\alpha, -1) = \mathcal{T}^*(\alpha)$ and $\mathcal{TU}_{2,1}(0, 0, 1-2\alpha, -1) = \mathcal{C}(\alpha)$, $(0 \leq \alpha < 1)$ (see Silverman [96]).

2.2 Main Results

Coefficient estimates

2.2.1 Theorem

A function $\mathcal{L}$ defined in (1 1.1) is belong to $\mathcal{U}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$ if

$$\sum_{m=2}^{\infty}\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}a_m\leq\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}. \tag{2.2.1}$$

Proof. For the proof of Theorem 2.2 1, it is enough to show that

$$1>\left|\frac{p(w)-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}p(w)}\right|.$$

where

$$p(w)=\frac{S_q^{\imath}\mathcal{L}(w)}{S_q^{\jmath}\mathcal{L}(w)}-\beta\left|\frac{S_q^{\imath}\mathcal{L}(w)}{S_q^{\jmath}\mathcal{L}(w)}-1\right|$$

Now

$$\begin{aligned}
&\left|\frac{p(w)-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}p(w)}\right|\\
=&\left|\frac{S_q^{\imath}\mathcal{L}(w)-S_q^{\jmath}\mathcal{L}(w)-\beta e^{\imath\theta}\left|S_q^{\imath}\mathcal{L}(w)-S_q^{\jmath}\mathcal{L}(w)\right|}{\widehat{\mathfrak{A}}S_q^{\jmath}\mathcal{L}(w)-\widetilde{\mathfrak{B}}\left[S_q^{\imath}\mathcal{L}(w)-\beta e^{\imath\theta}\left|S_q^{\imath}\mathcal{L}(w)-S_q^{\jmath}\mathcal{L}(w)\right|\right]}\right|\\
=&\left|\frac{\sum_{m=2}^{\infty}\left\{\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m-\beta e^{\imath\theta}\left|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m\right|\right\}}{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)w-\left[\sum_{m=2}^{\infty}\left(\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right)a_mw^m-\widetilde{\mathfrak{B}}\beta e^{\imath\theta}\left|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m\right|\right]}\right|\\
\leq&\frac{\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)|a_m||w|^m+\beta\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)|a_m||w|^m}{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})|w|-\left[\sum_{m=2}^{\infty}\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right||a_m||w|^m+\beta\left|\widetilde{\mathfrak{B}}\right|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)|a_m||w|^m\right]}\\
\leq&\frac{\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)(1+\beta)|a_m|}{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})-\sum_{m=2}^{\infty}\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right||a_m|-\beta\left|\widetilde{\mathfrak{B}}\right|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)|a_m|}
\end{aligned}$$

This last expression is bounded above by 1 if

$$\sum_{m=2}^{\infty}\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}a_m\leq\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}},$$

and subsequently the proof is completed. ■

Theorem (2.2.2), shown that the condition (2.2 1) is also required for functions $\mathcal{L}\in\mathcal{TU}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$.

2.2.2 Theorem

A function $\mathcal{L}$ defined in (1 1.2) is belong to $\mathcal{TU}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$ if and only if

$$\sum_{m=2}^{\infty}\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right|\right\}a_m\leq\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}.$$

Proof. Since $\mathcal{U}_{\imath,\jmath}\left(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)\supset\mathcal{TU}_{\imath,\jmath}\left(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$, for $\mathcal{L}\in\mathcal{TU}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$, we have

$$1>\left|\frac{p(w)-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}p(w)}\right|,$$

where

$$p(w)=\frac{S_q^{\imath}\mathcal{L}(w)}{S_q^{\jmath}\mathcal{L}(w)}-\beta\left|\frac{S_q^{\imath}\mathcal{L}(w)}{S_q^{\jmath}\mathcal{L}(w)}-1\right|.$$

then

$$\left|\begin{array}{c}\sum_{m=2}^{\infty}\left\{\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m+\beta e^{\imath\theta}\left|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m\right|\right\}\\ \times\left\{\begin{array}{c}\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)w+\sum_{m=2}^{\infty}\left(\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right)a_mw^m\\ +\widetilde{\mathfrak{B}}\beta e^{\imath\theta}\left|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m\right|\end{array}\right\}^{-1}\end{array}\right|<1.$$

Since $\text{Re}(w)\leq|w|$, then we obtain

$$\text{Re}\left\{\begin{array}{c}\sum_{m=2}^{\infty}\left\{\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m+\beta e^{\imath\theta}\left|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m\right|\right\}\\ \times\left\{\begin{array}{c}\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)w+\sum_{m=2}^{\infty}\left(\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right)a_mw^m\\ +\widetilde{\mathfrak{B}}\beta e^{\imath\theta}\left|\sum_{m=2}^{\infty}\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)a_mw^m\right|\end{array}\right\}^{-1}\end{array}\right\}<1$$

Now choosing w to be real and letting $w \to -1$, we obtain

$$\sum_{m=2}^{\infty}\left\{\left(1+\beta\left(1-\widetilde{\mathfrak{B}}\right)\right)\left(\lceil m\rfloor_q^{i}-\lceil m\rfloor_q^{j}\right)-\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{j}\right|\right\}a_m \leq \widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}.$$

Or, equivalently

$$\sum_{m=2}^{\infty}\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{i}-\lceil m\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{j}\right|\right\}a_m \leq \widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}.$$

This completes the proof ■

2.2.3 Corollary

A function $\mathcal{L}$ defined in (1 1.2) is belong to $\mathcal{TU}_{i,j}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$, then for $m \geq 2$

$$a_m \leq \frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{i}-\lceil m\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{j}\right|\right\}} \tag{2 2.2}$$

The result is sharp, for the function

$$\mathcal{L}(w)=w-\frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{i}-\lceil m\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{j}\right|\right\}}w^2,\ m \geq 2. \tag{2.2.3}$$

That is, equality can be attained for the function defined in (2 2 3) .

Distortion Theorems

2.2.4 Theorem

Let $\mathcal{L} \in \mathcal{TU}_{i,j}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$. Then

$$|\mathcal{L}(w)| \geq |w| - \frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{i}-\lceil 2\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil 2\rfloor_q^{j}\right|\right\}}|w|^2.$$

and

$$|\mathcal{L}(w)| \leq |w| + \frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{i}-\lceil 2\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil 2\rfloor_q^{j}\right|\right\}}|w|^2.$$

this is a sharp result.

Proof. In view of Theorem (2.2.2), let

$$\Phi(m)=\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}$$

for $(m\geq 2)$. $\Phi(m)$ is increasing mapping, therefore:

$$\Phi(2)\sum_{m=2}^{\infty}|a_m|\leq\sum_{m=2}^{\infty}\Phi(m)\,|a_m|\leq\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}},$$

that is:

$$\sum_{m=2}^{\infty}|a_m|\leq\frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\Phi(2)}$$

Thus, we have

$$|\mathcal{L}(w)|\leq|w|+|w|^2\sum_{m=2}^{\infty}|a_m|,$$

$$|\mathcal{L}(w)|\leq|w|+\frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath}-\lceil 2\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}}|w|^2.$$

Similarly, we get

$$\begin{aligned}|\mathcal{L}(w)| &\geq |w|-\sum_{m=2}^{\infty}|a_m|\,|w|^m\geq|w|-|w|^2\sum_{m=2}^{\infty}|a_m|\\ &\geq |w|-\frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath}-\lceil 2\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}}|w|^2\end{aligned}$$

Finally, the equality can be attained for the function:

$$\mathcal{L}(w)=w-\frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath}-\lceil 2\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}}w^2 \tag{2.2.4}$$

At $|w|=r$ and $w=re^{\imath(2k+1)\pi}$ $(k\in\mathbb{Z})$. This completes the Theorem (2 2.4). ■

2.2.5 Theorem

Let $\mathcal{L} \in \mathcal{TU}_{\imath,\jmath}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$. Then

$$\left|\mathcal{L}'(w)\right| \geq 1 - \frac{2\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath} - \lceil 2\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}} |w|.$$

and

$$\left|\mathcal{L}'(w)\right| \leq 1 + \frac{2\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath} - \lceil 2\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}} |w|.$$

The result is sharp.

Proof. As for $(m \geq 2)$, $\frac{\Phi(m)}{m}$ is an increasing mapping, In view of Theorem (2.2.2), we have:

$$\left(\frac{\Phi(2)}{2}\right)\left(\sum_{m=2}^{\infty} m\left|a_m\right|\right) \leq \sum_{m=2}^{\infty} m\left(\frac{\Phi(m)}{m}\right)\left|a_m\right| = \sum_{m=2}^{\infty}\Phi(m)\left|a_m\right| \leq \left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right).$$

that is:

$$\sum_{m=2}^{\infty} m\left|a_m\right| \leq \frac{2\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)}{\Phi(2)}$$

Thus we have:

$$\left|\mathcal{L}'(w)\right| \leq 1 + |w|\sum_{m=2}^{\infty} m\left|a_m\right|.$$

$$\left|\mathcal{L}(w)\right| \leq 1 + \frac{2\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath} - \lceil 2\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}} |w|.$$

Similarly, we get:

$$\begin{aligned}
\left|\mathcal{L}(w)\right| &\geq 1 - |w|\sum_{m=2}^{\infty} m\left|a_m\right| \\
&\geq 1 - \frac{2\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath} - \lceil 2\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}} |w|
\end{aligned}$$

Thus we have

$$\left|\frac{w\mathcal{L}'(w)}{\mathcal{L}(w)}-1\right|\leq 1-\varphi,$$

if and only if

$$\frac{\sum_{m=2}^{\infty}(m-\varphi)a_m\left|w\right|^{m-1}}{(1-\varphi)}\leq 1. \tag{2.2.8}$$

But, by Theorem (2 2.2), (2.2 8) will be true if

$$\left(\frac{m-\varphi}{1-\varphi}\right)\left|w\right|^{m-1}\leq\frac{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{i}-\lceil m\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{j}\right|\right\}}{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})}.$$

that is, if

$$|w|\leq\left\{\left(\frac{1-\varphi}{m-\varphi}\right)\times\frac{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{i}-\lceil m\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{j}\right|\right\}}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}\right\}^{\frac{1}{m-1}}, \text{ for } m\geq 2.$$

This implies. for $m\geq 2$

$$r_1=\inf_{m\geq 2}\left\{\left(\frac{1-\varphi}{m-\varphi}\right)\times\frac{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{i}-\lceil m\rfloor_q^{j}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{i}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{j}\right|\right\}}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}\right\}^{\frac{1}{m-1}}.$$

This completes (2.2 5). To prove (2.2.6) and (2 2.7), it is sufficient to show that

$$\left|1+\frac{w\mathcal{L}''(w)}{\mathcal{L}'(w)}-1\right|\leq 1-\varphi\quad(|w|<r_2,\ 0\leq\varphi<1).$$

and

$$\left|\mathcal{L}'(w)-1\right|\leq 1-\varphi\quad(|w|<r_3,\ 0\leq\varphi<1).$$

respectively. ■

Extreme Points

2.2.7 Theorem

Let

$$\mathcal{L}_m(w) = w - \frac{(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}})}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath} - \lceil m\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}} w^m, \quad m = 2, 3.$$

and

$$\mathcal{L}_1(w) = w.$$

Then $\mathcal{L} \in \mathcal{TU}_{\imath,\jmath}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ if and only if $\mathcal{L}(w)$ take the form

$$\mathcal{L}(w) = \sum_{m=1}^{\infty} \eta_m \mathcal{L}_m(w).$$

where

$$\eta_m \geq 0, \qquad \sum_{m=1}^{\infty} \eta_m = 1.$$

Proof. Suppose that

$$\begin{aligned}
\mathcal{L}(w) &= \sum_{m=1}^{\infty} \eta_m \mathcal{L}_m(w) \\
&= w - \sum_{m=2}^{\infty} \eta_m \frac{(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}})}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath} - \lceil m\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}} w^m
\end{aligned}$$

Then, from Theorem (2.2.2), we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \left[\frac{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath} - \lceil m\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}})}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath} - \lceil m\rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath} - \widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}} \eta_m \right] \\
&= (\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}) \sum_{m=2}^{\infty} \eta_m = (\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}})(1 - \eta_1) \leq (\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}})
\end{aligned}$$

Thus, in view of Theorem (2.2.2), we find that $\mathcal{L} \in \mathcal{TU}_{\imath,\jmath}(q, \beta, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$. Conversely, let us suppose that $\mathcal{L} \in \mathcal{TU}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$, then, since

$$a_m \leq \frac{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}}$$

By setting

$$\eta_m = \frac{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}}{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})}a_m$$

$$\eta_1 = 1-\sum_{m=2}^{\infty}\eta_m$$

Thus clearly, we have

$$\mathcal{L}(w) = \sum_{m=1}^{\infty}\eta_m\mathcal{L}_m(w).$$

Theorem (2 2 7) is completed. ■

2.2.8 Corollary

For class $\mathcal{TU}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$. EP are given by

$$\mathcal{L}_1(w) = w,$$

$$\mathcal{L}_m(w) = w - \frac{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m\rfloor_q^{\imath}-\lceil m\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil m\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil m\rfloor_q^{\jmath}\right|\right\}}w^m,\ m=2,3,\ldots$$

Integral Means Inequalities

2.2.9 Theorem

Suppose that $\mathcal{L} \in \mathcal{TU}_{\imath,\jmath}(q,\beta,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$, $p>0$, $-1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1, \beta > 0, \imath \in \mathbb{N}, \jmath \in \mathbb{N}_0, \imath > \jmath$, and $\mathcal{L}_2(w)$ is defined by

$$\mathcal{L}_2(w) = w - \frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\left\{\left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2\rfloor_q^{\imath}-\lceil 2\rfloor_q^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\lceil 2\rfloor_q^{\imath}-\widehat{\mathfrak{A}}\lceil 2\rfloor_q^{\jmath}\right|\right\}}w^2,$$

then $w = re^{\imath\theta}$, $(0 < r < 1)$, we have

$$\int_0^{2\pi} |\mathcal{L}(w)|^p \, d\theta \leq \int_0^{2\pi} |\mathcal{L}_2(w)|^p \, d\theta.$$

[redacted]roof. For

$$\mathcal{L}(w) = w - \sum_{m=2}^{\infty} a_m w^m, \quad a_m \geq 0,$$

the relation (1.10.1) is eq[redacted]t to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{m=2}^{\infty} a_m w^{m-1} \right|^p d\theta \quad [redacted]$$

$$\leq \int_0^{2\pi} \left| 1 - \frac{\widehat{\mathfrak{A}} \; [redacted]}{\left\{ \left(1 + \beta \left(1 + \left|\widetilde{\mathfrak{B}}\right|\right)\right) \left(\lceil 2 \rfloor_q^{\imath} - \lceil 2 \rfloor_q \; [redacted] \; \lceil 2 \rfloor_q^{\imath} - \widehat{\mathfrak{A}} \lceil 2 \rfloor_q^{\jmath} \right| \right\}} w \right|^p d\theta.$$

By applying Lemma (1.10.1), it is suffices to show that

$$1 - \sum_{m=2}^{\infty} a_m w^{m-1} \prec 1 - \frac{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}}{\left\{ \left(1 + \beta \left(1 + \left|\widetilde{\mathfrak{B}}\right|\right)\right) \left(\lceil 2 \rfloor_q^{\imath} - \lceil 2 \rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}} \lceil 2 \rfloor_q^{\imath} - [redacted] \right| \right\}} w.$$

By setting

$$1 - \sum_{m=2}^{\infty} a_m w^{m-1} = 1 - \frac{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}}{\left\{ \left(1 + \beta \left(1 + \left|\widetilde{\mathfrak{B}}\right|\right)\right) \left(\lceil 2 \rfloor_q^{\imath} - \lceil 2 \rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}} \lceil 2 \rfloor_q^{\imath} - \widehat{\mathfrak{A}} \lceil 2 \rfloor_q^{\jmath}\right| [redacted]} [redacted]).$$

and using (2.2.1), we obtain

$$
\begin{aligned}
|\Psi(w)| &= \left| \sum_{m=2}^{\infty} \frac{\left\{ \left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2 \rfloor_q^{\imath} - \lceil 2 \rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}} \lceil 2 \rfloor_q^{\imath} - \widehat{\mathfrak{A}} \lceil 2 \rfloor_q^{\jmath}\right| \right\}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} a_m w^{m-1} \right| \\
&\leq |w| \sum_{m=2}^{\infty} \frac{\left\{ \left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil 2 \rfloor_q^{\imath} - \lceil 2 \rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}} \lceil 2 \rfloor_q^{\imath} - \widehat{\mathfrak{A}} \lceil 2 \rfloor_q^{\jmath}\right| \right\}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} a_m \\
&\leq |w| \sum_{m=2}^{\infty} \frac{\left\{ \left(1+\beta\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right)\left(\lceil m \rfloor_q^{\imath} - \lceil m \rfloor_q^{\jmath}\right) + \left|\widetilde{\mathfrak{B}} \lceil m \rfloor_q^{\imath} - \widehat{\mathfrak{A}} \lceil m \rfloor_q^{\jmath}\right| \right\}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} a_m \\
&\leq |w| < 1.
\end{aligned}
$$

■

Chapter 3

A class of analytic functions related to convexity and functions with bounded turning

In this chapter, we define a new subclass k-$\mathcal{QMT}(\alpha)$ of analytic functions, which generalizes the class of k-uniformly convex functions. The main purpose of this chapter is to establish several interesting relationships between k-$\mathcal{QMT}(\alpha)$ and the class $\mathcal{B}(\delta)$ of functions with bounded turning. We studied various interesting relationships of this class with already existing classes of analytic functions. Certain important cases for some special values of the parameters have been obtained.

3.1 Introduction

3.1.1 Bounded turning

In [80], it is proved that if $\mathrm{Re}(\mathcal{L}') > 0$ in $\widetilde{H}$, then $\mathcal{L}$ is univalent in $\widetilde{H}$. In 1972, MacGregor [64] studied the class $\mathcal{B}$ of functions with bounded turning, a function $\mathcal{L} \in \mathcal{B}$ if it satisfies the condition $\mathrm{Re}(\mathcal{L}') > 0$ for $w \in \widetilde{H}$. A natural generalization of the class $\mathcal{B}$ is $\mathcal{B}(\delta)$ $(0 \le \delta < 1)$, a function $\mathcal{L} \in \mathcal{B}(\delta)$ if it satisfies the condition

$$\operatorname{Re}\left(\mathcal{L}'\right) > \delta \quad (w \in \widetilde{H}, 0 \leq \delta < 1),$$

for details associated with the class $\mathcal{B}(\delta)$ (see [19, 79])

3.1.2 Definition

Let $\mathcal{L} \in \mathcal{A}$ and $k \geq 0$, $0 \leq \alpha \leq 1$. Then $\mathcal{L} \in k\text{-}\mathcal{QMT}(\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{\left(w\mathcal{L}'(w)\right)'}{\mathcal{L}'(w)}\right\} > k\left|(1-\alpha)\mathcal{L}'(w) + \alpha\frac{\left(w\mathcal{L}'(w)\right)'}{\mathcal{L}'(w)} - 1\right|, \quad w \in \widetilde{H} \tag{3.1.1}$$

It is worth mentioning that for special value of parameter we obtained number of already defined classes, here some of them are listed below

(i) $k\text{-}\mathcal{QMT}(1) = k\text{-}\mathcal{UCV}$

(ii) $0\text{-}\mathcal{QMT}(\delta) = \mathcal{C}$

Next, we provide an example of a function belonging to $k\text{-}\mathcal{QMT}(\alpha)$.

3.1.3 Example

Next, we provide an example of a function belonging to $k\text{-}\mathcal{QMT}(\alpha)$.

The function $\mathcal{L}(w) = \frac{w}{1-Aw}$ is in the class $k\text{-}\mathcal{QMT}(\alpha)$ for

$$k \leq \frac{1-b^2}{b\sqrt{b(1+\alpha)\left\{b(1+\alpha)+2\right\}+4}}, \tag{3.1.2}$$

where $|A| = b$

Proof. For $Aw = be^{i\theta}$, (3.1.1) become

$$\operatorname{Re}\left(\frac{1+be^{i\theta}}{1-be^{i\theta}}\right) \geq \frac{k\left|(1+\alpha)\left(be^{i\theta}\right)^2 - 2be^{i\theta}\right|}{|1-be^{i\theta}|^2}. \tag{3.1.3}$$

It can be easily be seen that

$$\operatorname{Re}\left(\frac{1+be^{i\theta}}{1-be^{i\theta}}\right) = \frac{1-b^2}{|1-be^{i\theta}|^2}. \tag{3.1.4}$$

Using (3.1 3), (3.1 4) and after some simple calculation, we obtain (3.1 2). ■

3.2 Main Results

3.2.1 Theorem

Let $0 \leq \alpha < 1$ and $k \geq \frac{1}{1-\alpha}$. If $\mathcal{L} \in k\text{-}\mathcal{QMT}(\alpha)$, then $\mathcal{L} \in B(\delta)$, where

$$\delta = \frac{(2\gamma - \beta) + \sqrt{(2\gamma - \beta)^2 + 8\beta}}{4}, \tag{3.2.1}$$

with $\beta = \frac{1+\alpha k}{k(1-\alpha)}$ and $\gamma = \frac{k-\alpha k-1}{1-\alpha}$.

Proof. Let

$$\mathcal{L}'(w) = \hbar(w),$$

where $\hbar$ is analytic in $\widetilde{H}$ with $\hbar(0) = 1$. From Inequality (3 1 1) which takes the form

$$\begin{aligned} \mathrm{Re}\left\{1 + \frac{w\hbar'(w)}{\hbar(w)}\right\} &> k\left|(1-\alpha)\hbar(w) + \alpha\left(1 + \frac{w\hbar'(w)}{\hbar(w)}\right) - 1\right| \\ &= k\left|1 - \alpha - \hbar(w) + \alpha\hbar(w) - \alpha\frac{w\hbar'(w)}{\hbar(w)}\right|, \end{aligned}$$

we find that

$$\mathrm{Re}\left\{\hbar(w) + \left(\frac{1+\alpha k}{k(1-\alpha)}\right)\frac{w\hbar'(w)}{\hbar(w)}\right\} > \frac{k-\alpha k-1}{1-\alpha},$$

which can be rewritten as

$$\mathrm{Re}\left\{\hbar(w) + \frac{w\hbar'(w)}{\frac{1}{\beta}\hbar(w)}\right\} > \gamma,$$

where $\beta = \frac{1+\alpha k}{k(1-\alpha)}$ and $\gamma = \frac{k-\alpha k-1}{1-\alpha}$. The above relation can be written in the following Briot-Bouquet differential subordination

$$\hbar(w) + \frac{w\hbar'(w)}{\frac{1}{\beta}\hbar(w)} \prec \frac{1 + (1-2\gamma)w}{1-w}.$$

Thus, by Lemma 1 10 2, we obtain

$$\hbar \prec \frac{1+(1-2\delta)\,w}{1-w}, \tag{3.2.2}$$

where

$$\delta = \frac{(2\gamma-\beta)+\sqrt{(2\gamma-\beta)^2+8\beta}}{4}.$$

with $\beta = \frac{1+\alpha k}{k(1-\alpha)}$ and $\gamma = \frac{k-\alpha k-1}{1-\alpha}$. This implies that $\mathcal{L} \in B(\delta)$. We thus complete the proof of Theorem 3.2 1. ■

Special Cases

(i) For $\alpha = 0$ and $k = 1$, and for $\mathcal{L} \in 1\text{-}\mathcal{QMT}(0)$, we have $\mathcal{L} \in B(\delta)$, where $\delta \simeq 0\,50$ that is $\mathcal{L} \in 1\text{-}\mathcal{QMT}(0)$ implies $\mathcal{L}$ is bounded turning of order 0.5

(ii) For $\alpha = \frac{1}{2}$ and $k = 1$ and for $\mathcal{L} \in 1\text{-}\mathcal{QMT}\left(\frac{1}{2}\right)$, we have $\mathcal{L} \in B(\delta)$, where $\delta \simeq 0\,50$ that is $\mathcal{L} \in 1\text{-}\mathcal{QMT}\left(\frac{1}{2}\right)$ implies $\mathcal{L}$ is bounded turning of order 0.5

(iii) For $\alpha = 0$ then

$$\delta_1 = \frac{(2k^2-2k-1)+\sqrt{4k^4-8k^3+12k+1}}{4k}.$$

In other words for $\mathcal{L} \in k\text{-}\mathcal{QMT}(0)$, we have $\mathcal{L} \in B(\delta_1)$.

3.2.2 Theorem

Let $0 < \alpha \le 1$, $0 < \beta < 1$, $c > 0$, $k \ge 1$, $m \ge j+1\ (j \in \mathbb{N})$, $|\ell| \ge \frac{m}{2}\left(c - \frac{1}{c}\right)$ and

$$\left|\alpha\beta\ell \pm (1-\alpha)\,c^{\beta}\sin\frac{\beta\pi}{2}\right| \ge 1. \tag{3.2.3}$$

If

$$\mathcal{L}(w) = w + \sum_{m=j+1}^{\infty} a_m w^m \quad (a_{j+1} \neq 0)$$

and $\mathcal{L} \in k\text{-}\mathcal{QMT}(\alpha)$, then $\mathcal{L} \in \mathcal{B}(\beta_0)$, where

$$\beta_0 = \min\{\beta : \beta \in (0,1)\}$$

such that (3 2.3) holds.

Proof. By the assumption, we have

$$\mathcal{L}'(w) = \hbar(w) = 1 + \sum_{m=j}^{\infty} c_m w^m, \ c_j \neq 0 \tag{3.2.4}$$

In view of (3 1 1) and (3 2.4), we get

$$\operatorname{Re}\left\{1 + \frac{w\hbar'(w)}{\hbar(w)}\right\} > k\left|(1-\alpha)\hbar(w) + \alpha\left(1 + \frac{w\hbar'(w)}{\hbar(w)}\right) - 1\right|$$

If there exists a point $w_0 \in \widetilde{H}$ such that

$$|\arg \hbar(w)| < \frac{\beta\pi}{2}, \quad (|w| < |w_0|, 0 < \beta < 1)$$

and

$$|\arg \hbar(w_0)| = \frac{\beta\pi}{2}, \ 0 < \beta < 1$$

then from Lemma 1 10.3, we know that

$$\frac{w_0\hbar'(w_0)}{\hbar(w_0)} = i\ell\beta,$$

where

$$(\hbar(w_0))^{1/\beta} = \pm ic \quad (c > 0)$$

and

$$\ell : \begin{cases} \ell \geq \frac{m}{2}\left(c + \frac{1}{c}\right), & \left(\arg \hbar(w_0) = \frac{\beta\pi}{2}\right), \\ \ell \leq -\frac{m}{2}\left(c + \frac{1}{c}\right), & \left(\arg \hbar(w_0) = -\frac{\beta\pi}{2}\right) \end{cases}$$

For the case

$$\arg \hbar(w_0) = \frac{\beta\pi}{2},$$

we get

$$\operatorname{Re}\left\{1 + \frac{w\hbar'(w_0)}{\hbar(w_0)}\right\} = \operatorname{Re}\{1 + i\ell\beta\} = 1. \tag{3.2 5}$$

Moreover, we find from (3.2 3) that

$$\begin{aligned}
& k\left|(1-\alpha)\hbar(w_0)+\alpha\left(1+\frac{w\hbar'(w_0)}{\hbar(w_0)}\right)-1\right| \\
= & k\left|(1-\alpha)(\hbar(w_0)-1)+\alpha\frac{w\hbar'(w_0)}{\hbar(w_0)}\right| \\
= & k\left|(1-\alpha)\left\{(\pm ic)^{\beta}-1\right\}+i\alpha\beta l\right| \\
= & k(1-\alpha)^2\left(c^{\beta}\cos\beta\frac{\pi}{2}-1\right)^2+\left\{\alpha l\beta\pm(1-\alpha)c^{\beta}\sin\beta\frac{\pi}{2}\right\}^2 \\
\geq & 1
\end{aligned} \tag{3 2.6}$$

By virtue of (3 2 5) and (3.2.6), we have

$$\operatorname{Re}\left\{1+\frac{w\hbar'(w_0)}{\hbar(w_0)}\right\}\leq k\left|(1-\alpha)\hbar(w_0)+\alpha\left(1+\frac{w\hbar'(w_0)}{\hbar(w_0)}\right)-1\right|,$$

which is contradiction to the definition of k-$\mathcal{QMT}(\alpha)$. Since $\beta_0=\min\{\beta \;\; \beta\in(0.1)\}$ such that (3.2 3) holds, we can deduce that $\mathcal{L}\in\mathcal{B}(\beta_0)$ By using the similar method as given above, we can prove the case

$$\arg\hbar(w_0)=-\frac{\beta\pi}{2}.$$

is true. The proof of Theorem 3.2.2 is thus completed. ∎

3.2.3 Theorem

Let $0\leq\alpha<1$ and $k\geq\frac{1}{1-\alpha}$ If $\mathcal{L}\in k$-$\mathcal{QMT}(\alpha)$, then

$$\mathcal{L}'(w)\prec s(w)=\frac{1}{g(w)}.$$

where

$$g(w)=\left[{}_2G_1\left(\frac{2}{\beta}.1.\frac{1}{\beta}+1.\frac{w}{w-1}\right)\right]$$

with $\beta=\frac{1+\alpha k}{k(1-\alpha)}$.

Proof. Suppose that

$$\mathcal{L}'(w)=\hbar(w).$$

From the Theorem 3 2 1, we see that

$$\hbar(w) + \frac{w\hbar'(w)}{\frac{1}{\beta}\hbar(w)} \prec \frac{1+(1-2\gamma)w}{1-w} \prec \frac{1+w}{1-w}.$$

where $\beta = \frac{1+\alpha k}{k(1-\alpha)}$ and $\gamma = \frac{k-\alpha k-1}{1-\alpha}$.

If we set $\lambda = \frac{1}{\beta}$, $\gamma = 0$, $\mathfrak{C} = 1$ and $\mathfrak{D} = -1$ Lemma 1.10 5, then

$$\hbar(w) \prec s(w) = \frac{1}{g(w)} = \frac{w^{\frac{1}{\beta}}(1-w)^{-\frac{2}{\beta}}}{1/\beta \int_0^u t^{1/\beta-1}(1-t)^{-2/\beta}\,dt}$$

By putting $t = uw$, and using Lemma 1.10.6, we obtain

$$\begin{aligned}\hbar(w) \prec s(w) = \frac{1}{g(w)} &= \frac{1}{1/\beta(1-w)^{-\frac{2}{\beta}} \int_0^1 u^{1/\beta-1}(1-uw)^{-2/\beta}\,du} \\ &= \left\{ {}_2G_1\left(\frac{2}{\beta}(1-\gamma), 1; \frac{1}{\beta}+1; \frac{w}{w-1}\right)\right\}^{-1}.\end{aligned}$$

which is the desired result of Theorem 3.2.3. ■

3.2.4 Theorem

If $0 < \beta < 1$ and $0 \le \nu < 1$. If $\mathcal{L} \in k\text{-}\mathcal{QMT}(\alpha)$, then

$$\operatorname{Re}\mathcal{L}' > \left\{ {}_2G_1\left(\frac{2}{\beta}(1-\mu), 1, \frac{1}{\beta}+1\ \frac{1}{2}\right)\right\}^{-1},$$

or equivalently $k\text{-}\mathcal{QMT}(\alpha) \subset B(\mu_0)$ where

$$\mu_0 = \left\{ {}_2G_1\left(\frac{2}{\beta}(1-\mu), 1; \frac{1}{\beta}+1\ \frac{1}{2}\right)\right\}^{-1}.$$

Proof. For $n = \frac{2}{\beta}(1-\mu)$, $x = \frac{1}{\beta}$, $y = \frac{1}{\beta}+1$, we have

$$\begin{aligned}
\mathcal{F}(w) &= (1+Dw)^n \int_0^1 t^{x-1}(1+Dtw)^{-n}\,dt \\
\mathcal{F}(w) &= \frac{\Gamma(x)}{\Gamma(y)}\,{}_2G_1\left(1,n,y;\frac{w}{w-1}\right) \qquad (3.2.7)
\end{aligned}$$

To prove $k\text{-}\mathcal{QMT}(\alpha) \subset B(\mu_0)$, it is suffices to prove

$$\inf_{|w|<1}\{\operatorname{Re} q(w)\} = q(-1).$$

we need to show that

$$\operatorname{Re}\{1/\mathcal{F}(w)\} \geq 1/\mathcal{F}(-1).$$

By using Lemma 1.10.4 and (3.2.7) it follows that

$$\mathcal{F}(w) = \int_0^1 \mathcal{F}(w,t)\,d\varepsilon(t).$$

where

$$\begin{aligned}
\mathcal{F}(w,t) &= \frac{1-w}{1-(1-t)w}, \quad (0 \leq t \leq 1). \\
d\varepsilon(t) &= \frac{\Gamma(x)}{\Gamma(n)\Gamma(y-n)} t^{n-1}(1-t)^{y-n-1}\,dt.
\end{aligned}$$

which is a positive measure on $[0,1]$

It is clear that $\operatorname{Re}\mathcal{F}(w,t) > 0$ and $\mathcal{F}(-r,t)$ is real for $0 \leq |w| \leq r < 1$ and $t \in [0,1]$ Also

$$\operatorname{Re}\left\{\frac{1}{\mathcal{F}(w,t)}\right\} = \operatorname{Re}\left\{\frac{1-(1-t)w}{1-w}\right\} \geq \frac{1+(1-t)r}{1+r} = \frac{1}{\mathcal{F}(-r\,t)}$$

for $|w| \leq r < 1$. Therefore using Lemma 1.10.4, we get

$$\operatorname{Re}\{1/\mathcal{F}(w)\} \geq 1/\mathcal{F}(-r)$$

Now letting $r \to 1^-$, it follows

$$\mathrm{Re}\{1/\mathcal{F}(w)\} \geq 1/\mathcal{F}(-1).$$

Thus, we deduce that k-$\mathcal{QMT}(\alpha) \subset B(\mu_0)$. ■

Chapter 4

A Subclass of Analytic Functions Defined by using Mittag-Leffler Function

In this chapter, we initiate two new subclasses $Q^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ and $TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$. These classes is initiated by using the Mittag-Leffler function with the help of Janòwski functions. These classes generalized numerous classes by selecting specific values of the parameters. We examined numerous sharp results and properties of these classes, like as extreme points (EP), distortion theorem (DT), coefficient estimates (CE), convexity, radii of star-likeness (RS), close-to-convexity and integral mean inequalities (IMI).

4.1 Introduction

4.1.1 Definition

Let $\mathcal{L}\in\mathcal{A}$. Then $\mathcal{L}\in Q^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ if fulfill the following subordination relationship:

$$\frac{\partial D^{\imath}_{\chi}(\alpha,\mu)\mathcal{L}(w)}{\partial D^{\jmath}_{\chi}(\alpha,\mu)\mathcal{L}(w)}-\gamma\left|\frac{\partial D^{\imath}_{\chi}(\alpha,\mu)\mathcal{L}(w)}{\partial D^{\jmath}_{\chi}(\alpha,\mu)\mathcal{L}(w)}-1\right|\prec\frac{1+\widehat{\mathfrak{A}}w}{1+\widetilde{\mathfrak{B}}w},$$

where $w \in \widetilde{H}$, $\partial D^{\jmath}_{\chi}(\alpha,\mu)\mathcal{L}(w) \neq 0$, $\alpha,\gamma,\mu,\chi \geq 0$, $-1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1$. for $\imath > \jmath$. $\imath \in \mathbb{N}$ and $\jmath \in \mathbb{N}_0$.and $\partial D^{\imath}_{\chi}(\alpha,\mu)\mathcal{L}(w)$ defined in (1.7.3).

By taking notable values of parameters, we get numerous critical subclasses examined by different creators

($\imath$) $Q^{0,1}_{1,\imath,\jmath}(\gamma, 1-2\varepsilon, -1) = E_{\imath,\jmath}(\gamma,\varepsilon)$, [30],

($\imath\imath$) $Q^{0,1}_{1,1,0}(\gamma, 1-2\varepsilon, -1) = UE(\gamma,\varepsilon)$, [97],

($\imath\imath\imath$) $Q^{0,1}_{1,2,0}(\gamma, 1-2\varepsilon, -1) = UE(\gamma,\varepsilon)$. [98],

($\imath\imath\imath$) $Q^{0,1}_{1,1,0}\left(0,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right) = S^{*}\left(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$. [43],

($\imath v$) $Q^{0,1}_{1,2,0}\left(0,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right) = K\left(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$, [82].

4.1.2 Definition

As T be the subclass of $\mathcal{A}$ having negative coefficients in Maclaurin's series defined in (1 1.2). Here, we denote the class $TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right) = Q^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right) \cap \mathcal{T}$.

For appropriate possibility of the parameters $\chi, \alpha, \mu, \widehat{\mathfrak{A}}, \imath, \widetilde{\mathfrak{B}}, \jmath, \gamma$, we are able to get assorted subclasses of $\mathcal{T}$.

($\imath$) $TQ^{0,1}_{1,\imath+1,\imath}(\gamma, 1-2\varepsilon, -1) = TS(\imath,\gamma,\varepsilon)$. [8],

($\imath\imath$) $TQ^{0,1}_{1,1,0}(1, 1-2\varepsilon, -1) = S_pT(\varepsilon)$, [17],

($\imath\imath\imath$) $TQ^{0,1}_{1,1,0}(0, 1-2\varepsilon, -1) = T^{*}(\varepsilon)$, [100].

4.2 Main Results

Coefficient estimates

4.2.1 Theorem

A function $\mathcal{L}$ defined in (1 1.1) is belong to $Q^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ if

$$\sum_{m=2}^{\infty} v\left[\{1+\gamma(1+|I|)\}(\phi^{\imath}-\phi^{\jmath}) + \left|\left(\widetilde{\mathfrak{B}}\phi^{\imath} - \widehat{\mathfrak{A}}\phi^{\jmath}\right)\right|\right]|a_m| \leq \widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}. \tag{4.2.1}$$

where $\phi = 1 + (m-1)\chi$ and $v = \frac{\Gamma(\mu)}{\Gamma(\alpha(m-1)+\mu)}$

Proof. We need to show that

$$\left|\frac{p(w)-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}p(w)}\right|<1.$$

where

$$p(w)=\frac{\partial D_{\chi}^{\imath}(\alpha,\mu)\mathcal{L}(w)}{\partial D_{\chi}^{\jmath}(\alpha,\mu)\mathcal{L}(w)}-\gamma\left|\frac{\partial D_{\chi}^{\imath}(\alpha,\mu)\mathcal{L}(w)}{\partial D_{\chi}^{\jmath}(\alpha,\mu)\mathcal{L}(w)}-1\right|.$$

Hence, we obtain

$$\begin{aligned}
&\left|\frac{p(w)-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}p(w)}\right| \\
= &\left|\frac{\partial D_{\chi}^{\imath}(\alpha,\mu)\mathcal{L}(w)-\partial D_{\chi}^{\jmath}(\alpha,\mu)\mathcal{L}(w)-\gamma e^{\imath\theta}\left|\partial D_{\chi}^{\imath}(\alpha,\mu)\mathcal{L}(w)-\partial D_{\chi}^{\jmath}(\alpha,\mu)\mathcal{L}(w)\right|}{\widehat{\mathfrak{A}}\partial D_{\chi}^{\jmath}(\alpha,\mu)\mathcal{L}(w)-\widetilde{\mathfrak{B}}\left[\partial D_{\chi}^{\imath}(\alpha,\mu)\mathcal{L}(w)-\gamma e^{\imath\theta}\left|\partial D_{\chi}^{\imath}(\alpha,\mu)\mathcal{L}(w)-\partial D_{\chi}^{\jmath}(\alpha,\mu)\mathcal{L}(w)\right|\right]}\right| \\
= &\left|\frac{\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)a_m w^m-\gamma e^{\imath\theta}\left|\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)a_m w^m\right|}{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})w-\left[\sum_{m=2}^{\infty}v\left(\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right)a_m w^m-\gamma\widetilde{\mathfrak{B}}e^{\imath\theta}\left|\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)a_m w^m\right|\right]}\right| \\
\leq &\frac{\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)|a_m||w|^m+\gamma\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)|a_m||w|^m}{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})|w|-\left[\sum_{m=2}^{\infty}v\left|\left(\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right)\right||a_m||w|^m+\gamma\left|\widetilde{\mathfrak{B}}\right|\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)|a_m||w|^m\right]} \\
\leq &\frac{\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)(1+\gamma)|a_m|}{(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})-\sum_{m=2}^{\infty}v\left|\left(\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right)\right||a_m|-\gamma\left|\widetilde{\mathfrak{B}}\right|\sum_{m=2}^{\infty}v\left(\phi^{\imath}-\phi^{\jmath}\right)|a_m|}
\end{aligned}$$

This last expression is bounded above by 1 if

$$\sum_{m=2}^{\infty}v\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left(\phi^{\imath}-\phi^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]|a_m|\leq\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}.$$

and hence the proof is completed. ■

Theorem 4.2.2, shown that the condition (4.2 1) is also required for functions $\mathcal{L}\in TQ_{\chi,\imath,\jmath}^{\alpha,\mu}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$.

4.2.2 Theorem

A function $\mathcal{L}$ defined in (1.1.2) is belonging to $TQ_{\chi,\imath,\jmath}^{\alpha,\mu}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$ if and only if

$$\sum_{m=2}^{\infty}v\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left(\phi^{\imath}-\phi^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]|a_m|\leq\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}.$$

where $v=\frac{\Gamma(\mu)}{\Gamma(\alpha(m-1)+\mu)}$ and $\phi=1+(m-1)\chi$

Proof. Since

$$Q^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right) \supseteq TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right),$$

by making use of the same method given in Theorem (4.2 1), we immediately proof of Theorem (4.2.2) ∎

4.2.3 Corollary

A function $\mathcal{L}$ defined in (1 1.2) is belonging to $TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$, then

$$a_m \leq \frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\upsilon\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath}-\phi^{\jmath})+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]} \quad (m \geq 2). \tag{4.2.2}$$

The sharpness of this result, we have the function:

$$\mathcal{L}(w) = w - \frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\upsilon\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath}-\phi^{\jmath})+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]}w^m \quad (m \geq 2). \tag{4.2.3}$$

That is, equality can be attained for the function defined in (4.2 3).
Next, we discuss, distortion result and growth result for $\mathcal{L}$ in the class $TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$:

4.2.4 Theorem

Let $\mathcal{L} \in TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$. Then

$$|\mathcal{L}(w)| \geq |w| - \frac{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left((1+\chi)^{\imath}-(1+\chi)^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}(1+\chi)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|^2$$

and

$$|\mathcal{L}(w)| \leq |w| + \frac{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left((1+\chi)^{\imath}-(1+\chi)^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}(1+\chi)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|^2$$

Proof. In view of Theorem 4.2.2, consider

$$\delta(m) = \upsilon\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath}-\phi^{\jmath})+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|.$$

where $\phi = 1 + (m-1)\chi$ and $v = \frac{\Gamma(\mu)}{\Gamma(\alpha(m-1)+\mu)}$, $\delta(m)$ is an increasing function for m $(m \geq 2)$.

This implies that:

$$\delta(2)\sum_{m=2}^{\infty}|a_m| \leq \sum_{m=2}^{\infty}\delta(m)|a_m| \leq \widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}.$$

that is:

$$\sum_{m=2}^{\infty}|a_m| \leq \frac{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}}{\delta(2)}$$

Thus, we have:

$$|\mathcal{L}(w)| \leq |w| + \sum_{m=2}^{\infty}|a_m||w|^2.$$

$$|\mathcal{L}(w)| \leq |w| + \frac{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left((1+\chi)^{\imath}-(1+\chi)^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}(1+\chi)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|^2$$

Similarly, we get:

$$\begin{aligned}|\mathcal{L}(w)| &\geq |w| - \sum_{m=2}^{\infty}|a_m||w|^m \geq |w| - \sum_{m=2}^{\infty}|a_m||w|^2 \\ &\geq |w| - \frac{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left((\chi+1)^{\imath}-(1+\chi)^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}(\chi+1)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|^2.\end{aligned}$$

Finally, the equality can be attained for the function:

$$\mathcal{L}(w) = w - \frac{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left((\chi+1)^{\imath}-(1+\chi)^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}(\chi+1)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}w^2 \tag{4.2.4}$$

at $|w| = r$ and $w = re^{\imath(2k+1)\pi}$ $(k \in \mathbb{Z})$. This completes Theorem 4.2.4 ■

4.2.5 Theorem

Let $\mathcal{L} \in TQ_{\chi,\imath,\jmath}^{\alpha,\mu}\left(\gamma, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$. Then

$$\left|\mathcal{L}'(w)\right| \geq 1 - \frac{2\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left((\chi+1)^{\imath}-(1+\chi)^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}(\chi+1)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|,$$

and

$$\left|\mathcal{L}'(w)\right| \leq 1+\frac{2\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left((1+\chi)^{\imath}-(1+\chi)^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}(1+\chi)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|.$$

This result is sharp.

Proof. In view of Theorem 4.2.2, suppose that

$$\delta(m)=\upsilon\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}\left(\phi^{\imath}-\phi^{\jmath}\right)+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|.$$

where $\phi=1+(m-1)\chi$ and $\upsilon=\frac{\Gamma(\mu)}{\Gamma(\alpha(m-1)+\mu)}$, $\frac{\Phi(m)}{m}$ is an increasing function for m $(m\geq 2)$ Similarly, we obtain:

$$\left(\sum_{m=2}^{\infty}|a_m|\,m\right)\left(\frac{\delta(2)}{2}\right)\leq\sum_{m=2}^{\infty}m\left(\frac{\delta(m)}{m}\right)|a_m|=\sum_{m=2}^{\infty}\delta(m)\,|a_m|\leq\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right),$$

that is:

$$\sum_{m=2}^{\infty}|a_m|\,m\leq\left(\frac{2}{\delta(2)}\right)\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right).$$

and consequently:

$$\left|\mathcal{L}'(w)\right|\leq 1+\sum_{m=2}^{\infty}m\,|a_m|\,|w|,$$

$$\left|\mathcal{L}'(w)\right| \leq 1+\frac{2\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left((\chi+1)^{\imath}-(1+\chi)^{\jmath}\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}+\left|\widetilde{\mathfrak{B}}(\chi+1)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|$$

Also, we get

$$\begin{aligned}\left|\mathcal{L}'(w)\right| &\geq 1-\sum_{m=2}^{\infty}m\,|a_m|\,|w|\\ \left|\mathcal{L}'(w)\right| &\geq 1-\frac{2\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left((\chi+1)^{\imath}-(1+\chi)^{\jmath}\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}+\left|\widetilde{\mathfrak{B}}(1+\chi)^{\imath}-\widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]}|w|.\end{aligned}$$

Finally, we can see that the assertions of Theorem 4.2.5, the equality can be attained for the the function defined by (4.2.4). Theorem 4.2.5 is completed. ■

Next discussion is Radii of Starlikeness (RS), Convexity and Close-to-Convexity for $TQ_{\chi\, \imath, \jmath}^{\alpha, \mu}\left(\gamma, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$

4.2.6 Theorem

Let $\mathcal{L} \in TQ_{\lambda\, \imath, \jmath}^{\alpha, \mu}\left(\gamma, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$. Then

(i) For $|w| < r_1$ and $(0 \leq \alpha < 1)$, $\mathcal{L}$ is starlike of order α, where:

$$r_1 = \inf_{m \geq 2} \left\{ \left(\frac{1-\alpha}{m-\alpha} \right) \times \frac{\upsilon \left[\left\{ 1 + \gamma \left(1 + \left| \widetilde{\mathfrak{B}} \right| \right) \right\} (\phi^{\imath} - \phi^{\jmath}) + \left| \widetilde{\mathfrak{B}} \phi^{\imath} - \widehat{\mathfrak{A}} \phi^{\jmath} \right| \right]}{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}} \right)} \right\}^{\frac{1}{m-1}}, \quad (4.2.5)$$

(ii) For $|w| < r_2$ and $(0 \leq \alpha < 1)$, $\mathcal{L}$ is convex of order α, where:

$$r_2 = \inf_{m \geq 2} \left\{ \left(\frac{1-\alpha}{m(m-\alpha)} \right) \times \frac{\upsilon \left[\left\{ 1 + \gamma \left(1 + \left| \widetilde{\mathfrak{B}} \right| \right) \right\} (\phi^{\imath} - \phi^{\jmath}) + \left| \widetilde{\mathfrak{B}} \phi^{\imath} - \widehat{\mathfrak{A}} \phi^{\jmath} \right| \right]}{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}} \right)} \right\}^{\frac{1}{m-1}}. \quad (4.2.6)$$

(iii) For $|w| < r_3$ and $(0 \leq \alpha < 1)$, $\mathcal{L}$ is close to convex of order α, where:

$$r_3 = \inf_{m \geq 2} \left\{ \left(\frac{1-\alpha}{m} \right) \times \frac{\upsilon \left[\left\{ 1 + \gamma \left(1 + \left| \widetilde{\mathfrak{B}} \right| \right) \right\} (\phi^{\imath} - \phi^{\jmath}) + \left| \widetilde{\mathfrak{B}} \phi^{\imath} - \widehat{\mathfrak{A}} \phi^{\jmath} \right| \right]}{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}} \right)} \right\}^{\frac{1}{m-1}} \quad (4.2.7)$$

All results are sharp, for the mapping $\mathcal{L}$ given in (4.2.3).

Proof. Here enough to show:

$$\left| \frac{w\mathcal{L}'(w)}{\mathcal{L}(w)} - 1 \right| \leq 1 - \alpha \quad \text{for } |w| < r_1,$$

where r_1 is given by (4.2.5). Indeed we find from (1.1.2) that is

$$\left| \frac{w\mathcal{L}'(w)}{\mathcal{L}(w)} - 1 \right| \leq \frac{\sum_{m=2}^{\infty} (m-1) a_m |w|^{m-1}}{1 - \sum_{m=2}^{\infty} a_m |w|^{m-1}}.$$

Thus, we have

$$\left| \frac{w\mathcal{L}'(w)}{\mathcal{L}(w)} - 1 \right| \leq 1 - \alpha,$$

if and only if

$$\frac{\sum_{m=2}^{\infty}(m-\alpha)a_m |w|^{m-1}}{(1-\alpha)} \leq 1 \tag{4.2.8}$$

But, by Theorem 4 2.2, (4 2 8) will be true if

$$\left(\frac{m-\alpha}{1-\alpha}\right)|w|^{m-1} \leq \frac{\upsilon\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath}-\phi^{\jmath})+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}},$$

that is, if

$$|w| \leq \left\{\left(\frac{1-\alpha}{m-\alpha}\right) \times \frac{\upsilon\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath}-\phi^{\jmath})+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}\right\}^{\frac{1}{m-1}} \quad (m \geq 2)$$

this implies

$$r_1 = \inf_{m\geq 2}\left\{\left(\frac{1-\alpha}{m-\alpha}\right) \times \frac{\upsilon\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath}-\phi^{\jmath})+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]}{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)}\right\}^{\frac{1}{m-1}} \quad (m \geq 2)$$

This completes (4 2.5).

To prove (4 2.6) and (4.2.7) it is sufficient to show that

$$\left|1+\frac{w\mathcal{L}''(w)}{\mathcal{L}'(w)}-1\right| \leq 1-\alpha \qquad (|w| < r_2,\ 0 \leq \alpha < 1),$$

and

$$\left|\mathcal{L}'(w)-1\right| \leq 1-\alpha \qquad (|w| < r_3,\ 0 \leq \alpha < 1).$$

■

Next, we discussed extreme points for $TQ^{\alpha\,\mu}_{\chi,\imath,\jmath}\left(\gamma,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}\right)$.

4.2.7 Theorem

Let

$$\mathcal{L}_m(w) = w - \frac{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}{\upsilon\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath}-\phi^{\jmath})+\left|\widetilde{\mathfrak{B}}\phi^{\imath}-\widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]}w^m \qquad (m \geq 2).$$

and

$$\mathcal{L}_1(w) = w.$$

Then $\mathcal{L} \in TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$ if and only if $\mathcal{L}(w)$ take the form

$$\mathcal{L}(w) = \sum_{m=1}^{\infty} \eta_m \mathcal{L}_m(w),$$

where

$$\eta_m \geq 0, \qquad \sum_{m=1}^{\infty} \eta_m = 1.$$

Proof. Suppose that

$$\begin{aligned}
\mathcal{L}(w) &= \sum_{m=1}^{\infty} \eta_m \mathcal{L}_m(w) \\
&= w - \sum_{m=2}^{\infty} \eta_m \frac{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}}{v\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^\imath - \phi^\jmath) + \left|\widetilde{\mathfrak{B}}\phi^\imath - \widehat{\mathfrak{A}}\phi^\jmath\right|\right]} w^m
\end{aligned}$$

Then, from Theorem 4.2.2, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty}\left[\frac{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right) v\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^\imath - \phi^\jmath) + \left|\widetilde{\mathfrak{B}}\phi^\imath - \widehat{\mathfrak{A}}\phi^\jmath\right|\right]}{v\left[\left|\widetilde{\mathfrak{B}}\phi^\imath - \widehat{\mathfrak{A}}\phi^\jmath\right| + \left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^\imath - \phi^\jmath)\right]}\eta_m\right] \\
&= (\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}) \sum_{m=2}^{\infty} \eta_m \\
&= (\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}})(1 - \eta_1) \\
&\leq (\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}).
\end{aligned}$$

Thus, in view of Theorem 4.2.2, we find that $\mathcal{L} \in TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$. Conversely, let us suppose that $\mathcal{L} \in TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}\right)$, then, since

$$a_m \leq \frac{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}}{v\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^\imath - \phi^\jmath) + \left|\widetilde{\mathfrak{B}}\phi^\imath - \widehat{\mathfrak{A}}\phi^\jmath\right|\right]}$$

by setting

$$\eta_m = \frac{\upsilon\left[\left|\widetilde{\mathfrak{B}}\phi^{\imath} - \widehat{\mathfrak{A}}\phi^{\jmath}\right| + \left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath} - \phi^{\jmath})\right]}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} a_m \qquad (m \geq 2)$$

and

$$\eta_1 = 1 - \sum_{m=2}^{\infty} \eta_m.$$

we have

$$\mathcal{L}(w) = \sum_{m=1}^{\infty} \eta_m \mathcal{L}_m(w)$$

Theorem completed. ■

4.2.8 Corollary

For class $TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma, \widehat{\mathfrak{A}}. \widetilde{\mathfrak{B}}\right)$. EP are as:

$$\mathcal{L}_1(w) = w,$$

and

$$\mathcal{L}_m(w) = w - \frac{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}}{\upsilon\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^{\imath} - \varphi^{\jmath}) + \left|\widetilde{\mathfrak{B}}\phi^{\imath} - \widehat{\mathfrak{A}}\phi^{\jmath}\right|\right]} w^m \qquad (m \geq 2).$$

Integral Means Inequalities

4.2.9 Theorem

Suppose that $\mathcal{L} \in TQ^{\alpha,\mu}_{\chi,\imath,\jmath}\left(\gamma, \widehat{\mathfrak{A}}. \widetilde{\mathfrak{B}}\right)$ and $\mathcal{L}_2(w)$ is defined by

$$\mathcal{L}_2(w) = w - \frac{\left(\widehat{\widehat{\mathfrak{A}}} - \widetilde{\mathfrak{B}}\right)\Gamma(\alpha+\mu)}{\Gamma\mu\left[\left((\chi+1)^{\imath} - (1+\chi)^{\jmath}\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\} + \left|\widetilde{\mathfrak{B}}(\chi+1)^{\imath} - \widehat{\mathfrak{A}}(1+\chi)^{\jmath}\right|\right]} w^2$$

then for $w = re^{i\theta}(0 < r < 1)$, we have

$$\int_0^{2\pi} |\mathcal{L}(w)|^p \, d\theta \le \int_0^{2\pi} |\mathcal{L}_2(w)|^p \, d\theta$$

Proof. Let $\mathcal{L}(w) = w - \sum\limits_{m=2}^{\infty} a_m w^m$ ($a_m \ge 0$) then we must show that

$$\begin{aligned}
&\int_0^{2\pi} \left| 1 - \sum_{m=2}^{\infty} a_m w^{m-1} \right|^p d\theta \\
\le \; &\int_0^{2\pi} \left| 1 - \frac{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right) \Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left((\chi+1)^i - (1+\chi)^j\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\} + \left|\widetilde{\mathfrak{B}}(\chi+1)^i - \widehat{\mathfrak{A}}(1+\chi)^j\right|\right]} w \right|^p d\theta
\end{aligned}$$

By Lemma 1.10.1, it is enough to show that

$$1 - \sum_{m=2}^{\infty} a_m w^{m-1} \prec 1 - \frac{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right) \Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left((\chi+1)^i - (1+\chi)^j\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\} + \left|\widetilde{\mathfrak{B}}(\chi+1)^i - \widehat{\mathfrak{A}}(1+\chi)^j\right|\right]} w.$$

By setting

$$1 - \sum_{m=2}^{\infty} a_m w^{m-1} = 1 - \frac{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right) \Gamma(\alpha+\mu)}{\Gamma(\mu)\left[\left((\chi+1)^i - (1+\chi)^j\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\} + \left|\widetilde{\mathfrak{B}}(\chi+1)^i - \widehat{\mathfrak{A}}(1+\chi)^j\right|\right]} \varpi(w).$$

and using (4 2.1), we get

$$\begin{aligned}
|\varpi(w)| &= \left| \sum_{m=2}^{\infty} \frac{\Gamma(\mu)\left[\left((\chi+1)^i - (1+\chi)^j\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\} + \left|\widetilde{\mathfrak{B}}(\chi+1)^i - \widehat{\mathfrak{A}}(1+\chi)^j\right|\right]}{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right) \Gamma(\alpha+\mu)} a_m w^{m-1} \right| \\
&\le |w| \sum_{m=2}^{\infty} \frac{\Gamma(\mu)\left[\left((\chi+1)^i - (1+\chi)^j\right)\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\} + \left|\widetilde{\mathfrak{B}}(\chi+1)^i - \widehat{\mathfrak{A}}(1+\chi)^j\right|\right]}{\left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\right) \Gamma(\alpha+\mu)} a_m \\
&\le |w| \sum_{m=2}^{\infty} \frac{v\left[\left\{1+\gamma\left(1+\left|\widetilde{\mathfrak{B}}\right|\right)\right\}(\phi^i - \phi^j) + \left|\widetilde{\mathfrak{B}}\phi^i - \widehat{\mathfrak{A}}\phi^j\right|\right]}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} a_m \\
&\le |w| < 1.
\end{aligned}$$

Theorem 4.2.9 is completed ■

Chapter 5

Subclasses of uniformly convex and starlike functions associated with Bessel functions

Applications of Bessel functions have been commonly utilized in UF theory. The main focus of this chapter is to set out few imperative characteristic properties for a few subclasses of uniformly SF and CF which are initiated here by infers of the normalized condition of the generalized BF to be univalent inside the $\widetilde{H}$. Furthermore, we as well develop up some results about of these subclasses related to a particular integral operator.

5.1 Introduction

Throughout this chapter. unless otherwise stated. the parameters of alpha, beta and eta are considered as $(\alpha \geq 0)$, $(0 \leq \beta < 1)$ and $(0 \leq \eta \leq 1)$.

5.1.1 Definition

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L} \in Q_x(\alpha, \beta, \eta)$ if it fulfill the successive condition:

$$Re\left\{\frac{w\mathcal{L}'(w) + \eta w^2\mathcal{L}''(w)}{\mathcal{L}(w)}\right\} \geq \alpha\left|\frac{w\mathcal{L}'(w) + \eta w^2\mathcal{L}''(w)}{\mathcal{L}(w)} - 1\right| + \beta \tag{5.1.1}$$

5.1.2 Definition

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L} \in Q_x CV(\alpha, \beta, \eta)$ if it fulfill the successive condition:

$$Re\left\{\frac{\mathcal{L}'(w)+\eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right\} \geq \alpha\left|\frac{\mathcal{L}'(w)+\eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}-1\right|+\beta \tag{5.1.2}$$

5.1.3 Definition

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L} \in PS(\alpha, \eta)$ if it fulfill the successive condition:

$$Re\left\{\frac{w\mathcal{L}'(w)+\eta w^2\mathcal{L}''(w)}{\mathcal{L}(w)}\right\}+\alpha \geq \left|\frac{w\mathcal{L}'(w)+\eta w^2\mathcal{L}''(w)}{\mathcal{L}(w)}-\alpha\right| \tag{5.1.3}$$

5.1.4 Definition

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L} \in PCV(\alpha, \eta)$ if it fulfill the successive condition:

$$Re\left\{\frac{\mathcal{L}'(w)+\eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right\}+\alpha \geq \left|\frac{\mathcal{L}'(w)+\eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}-\alpha\right|. \tag{5.1.4}$$

We note that $PST(\alpha, \eta) = PS(\alpha, \eta) \cap T$ and $PCVT(\alpha, \eta) = PCV(\alpha, \eta) \cap T$

For special value of parameter, we obtain some of the previously studied classes. Some of them are listed below.

i $\quad Q_x(\alpha, \beta, 0) = SP(\alpha, \beta)$ [22].

ii $\quad Q_x CV(\alpha, \beta, 1) = UCV(\alpha, \beta)$ [17].

iii $\quad PS(\alpha, 0) = x(\alpha)$ [22].

iv $\quad PCV(\alpha, 1) = CP(\alpha)$ [22].

In present chapter, we obtained sufficient conditions for $\mathcal{L} \in \mathcal{A}$, to be in $Q_x(\alpha, \beta, \eta)$ and $Q_x CV(\alpha, \beta, \eta)$. We also determined necessary and sufficient conditions for $\mathcal{L} \in \mathcal{A}$, to be in the $PS(\alpha, \eta)$ and $PCV(\alpha, \eta)$. Furthermore, we determined sufficient conditions for wu_x to be in $Q_x(\alpha, \beta, \eta)$ and $Q_x CV(\alpha\ \beta, \eta)$ also for $w(2-u_x)$ to be in the function classes $PS(\alpha\ \eta)$ and $PCV(\alpha, \eta)$. We consider an integral operator related to the function u_x. Also, some corollaries related to main theorems have been presented.

5.2 Main Results

In this section, some theorems and corollaries related to our main results will be given.

5.2.1 Theorem

A sufficient condition for $\mathcal{L} \in \mathcal{A}$, is belonging to $Q_x(\alpha, \beta, \eta)$, if it fulfill the successive inequality (5 2 1):

$$\sum_{m=2}^{\infty} \{m(\eta m - \eta + 1)(1+\alpha) - \alpha - \beta\} |a_m| \leq 1 - \beta \tag{5.2.1}$$

Proof. It is enough to show that

$$\alpha \left| \frac{w\mathcal{L}'(w) + \eta w^2 \mathcal{L}''(w)}{\mathcal{L}(w)} - 1 \right| - Re \left\{ \frac{w\mathcal{L}'(w) + \eta w^2 \mathcal{L}''(w)}{\mathcal{L}(w)} - 1 \right\} \leq 1 - \beta.$$

Let us consider the following inequalities

$$\begin{aligned}
&\alpha \left| \frac{w\mathcal{L}'(w) + \eta w^2 \mathcal{L}''(w)}{\mathcal{L}(w)} - 1 \right| - Re \left\{ \frac{w\mathcal{L}'(w) + \eta w^2 \mathcal{L}''(w)}{\mathcal{L}(w)} - 1 \right\} \\
&\leq (\alpha+1) \left| \frac{w\mathcal{L}'(w) + \eta w^2 \mathcal{L}''(w) - \mathcal{L}(w)}{\mathcal{L}(w)} \right| \\
&\leq \frac{(\alpha+1)\left(\sum_{m=2}^{\infty} (\eta m^2 - \eta m + m - 1) |a_m| \right)}{1 - \sum_{m=2}^{\infty} |a_m|}.
\end{aligned}$$

This preceding expression is bounded above by $1-\beta$, if the following inequality holds

$$\sum_{m=2}^{\infty} \{m(\eta m - \eta + 1)(1+\alpha) - \alpha - \beta\} |a_m| \leq 1 - \beta.$$

■

It is remarkable that a necessary and sufficient condition for $\mathcal{L} \in T$ is belonging to $Q_x(\alpha, \beta, \eta)$ is that the condition (5.2 1) holds.

5.2.2 Theorem

A sufficient condition for $\mathcal{L} \in \mathcal{A}$, is belonging to $Q_x CV(\alpha, \beta, \eta)$, if it fulfill the successive inequality (5.2 2).

$$\sum_{m=2}^{\infty} m\left\{\eta(m-1)(1+\alpha)+1-\beta\right\}|a_m| \leq 1-\beta \tag{5.2.2}$$

Proof. It is enough to show that

$$\alpha\left|\frac{\mathcal{L}'(w)+\eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}-1\right|-Re\left\{\frac{w\mathcal{L}'(w)+\eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}-1\right\} \leq 1-\beta.$$

Let's consider the following inequalities

$$\begin{aligned} &\alpha\left|\frac{\eta w\mathcal{L}''(w)+\mathcal{L}'(w)}{\mathcal{L}'(w)}-1\right|-Re\left\{\frac{\eta w\mathcal{L}''(w)+w\mathcal{L}'(w)}{\mathcal{L}'(w)}-1\right\} \\ &\leq (1+\alpha)\left|\frac{\eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right| \\ &\leq \frac{\eta(1+\alpha)\sum_{m=2}^{\infty}(m-1)m|a_m|}{1-\sum_{m=2}^{\infty}(m|a_m|)} \end{aligned}$$

This preceding expression is bounded above by $1-\beta$, if the following inequality holds

$$\sum_{m=2}^{\infty} m\left\{\eta(m-1)(1+\alpha)+1-\beta\right\}|a_m| \leq 1-\beta$$

■

It is remarkable that a necessary and sufficient condition for $\mathcal{L} \in T$ is belonging to $Q_x CV(\alpha, \beta, \eta)$ is that the condition (5.2.2) is satisfied.

5.2.3 Theorem

A necessary and sufficient condition for $\mathcal{L} \in T$ is belonging to $PS(\alpha, \eta)$ is that the following inequality (5.2.3) holds.

$$\sum_{m=2}^{\infty}\left\{\eta m(m-1)+m-\alpha\right\}|a_m| \leq 1+\alpha \tag{5.2.3}$$

Proof. Let's consider the following inequalities

$$Re\left\{\frac{\eta w^2\mathcal{L}''(w)+w\mathcal{L}'(w)}{\mathcal{L}(w)}\right\}+\alpha\geq\left|\frac{\eta w^2\mathcal{L}''(w)+w\mathcal{L}'(w)}{\mathcal{L}(w)}-\alpha\right|,$$

which leads to

$$\left|\frac{w\mathcal{L}'(w)+\eta w^2\mathcal{L}''(w)}{\mathcal{L}(w)}-\alpha\right|-Re\left\{\frac{w\mathcal{L}'(w)+\eta w^2\mathcal{L}''(w)}{\mathcal{L}(w)}\right\}\leq 2\alpha$$

Consider

$$\begin{aligned}
&\left|\frac{\eta w^2\mathcal{L}''(w)+w\mathcal{L}'(w)}{\mathcal{L}(w)}-\alpha\right|-Re\left\{\frac{\eta w^2\mathcal{L}''(w)+w\mathcal{L}'(w)}{\mathcal{L}(w)}\right\}\\
&\leq 2\left|\frac{w\mathcal{L}'(w)+\eta w^2\mathcal{L}''(w)}{\mathcal{L}(w)}\right|\\
&=2\frac{\left|z-\sum\limits_{m=2}^{\infty}\left(\eta m^2-\eta m+m\right)\left(a_m w^m\right)\right|}{\left|z-\sum\limits_{m=2}^{\infty}\left(a_m w^m\right)\right|}\\
&=2\frac{\left|1-\sum\limits_{m=2}^{\infty}\left(m+\eta m^2-\eta m\right)\left(a_m w^{m-1}\right)\right|}{\left|1-\sum\limits_{m=2}^{\infty}a_m w^{m-1}\right|}.
\end{aligned}$$

This preceding expression is bounded above by 2α if the following inequalities hold

$$\begin{aligned}
&\sum_{m=2}^{\infty}\left(\eta m^2-\eta m+m\right)|a_m|-1\leq\alpha\left(1+\sum_{m=2}^{\infty}|a_m|\right)\\
&\sum_{m=2}^{\infty}\left\{\eta m(m-1)+m-\alpha\right\}|a_m|\leq 1+\alpha.
\end{aligned}$$

■

5.2.4 Theorem

A necessary and sufficient condition for $\mathcal{L} \in T$ is belonging to $PCV\,(\alpha\ \eta)$ is that, the following inequality (5.2 4) holds.

$$\sum_{m=2}^{\infty} \left[\eta m\,(m-1) + m\,(1-\alpha)\right] |a_m| \leq 1+\alpha \tag{5.2 4}$$

Proof. Let us consider the following inequalities

$$Re\left\{\frac{\mathcal{L}'(w) + \eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right\} + \alpha \geq \left|\frac{\mathcal{L}'(w) + \eta w\mathcal{L}''(w)}{\mathcal{L}'(w)} - \alpha\right|,$$

which leads to

$$\left|\frac{\mathcal{L}'(w) + \eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right| - Re\left\{\frac{\mathcal{L}'(w) + \eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right\} \leq 2\alpha.$$

Consider

$$\begin{aligned}
&\left|\frac{\mathcal{L}'(w) + \eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right| - Re\left\{\frac{\mathcal{L}'(w) + \eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right\} \\
&\leq 2\left|\frac{\mathcal{L}'(w) + \eta w\mathcal{L}''(w)}{\mathcal{L}'(w)}\right| \\
&= 2\frac{\left|z - \sum\limits_{m=2}^{\infty} \left(\eta m^2 - \eta m + m\right) a_m w^m\right|}{\left|z - \sum\limits_{m=2}^{\infty} m a_m w^m\right|} \\
&= 2\frac{\left|1 - \sum\limits_{m=2}^{\infty} \left(m + \eta m^2 - \eta m\right)\left(a_m w^{m-1}\right)\right|}{\left|1 - \sum\limits_{m=2}^{\infty} \left(m a_m w^{m-1}\right)\right|}.
\end{aligned}$$

This preceding expression is bounded above by 2α if the following inequality holds

$$\sum_{m=2}^{\infty} \left[\eta m\,(m-1) + m\,(1-\alpha)\right] |a_m| \leq 1+\alpha$$

■

5.2.5 Theorem

If $d < 0$ and $b_s < 0$ then $wu_x \in Q_x(\alpha, \beta, \eta)$ if

$$\eta(1+\alpha)u_x''(1) + (2\eta+1)(1+\alpha)u_x'(1) + (1-\beta)[u_x(1)-1] \leq 1-\beta \tag{5.2.5}$$

Proof. Since

$$wu_x(w) = w + \sum_{m=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} w^m, \tag{5.2.6}$$

on account of Theorem 5.2.1, it is enough to show that

$$\pounds(d, b_s, \alpha, \beta, \eta) = \sum_{m=2}^{\infty} [m(1+\alpha)(\eta m - \eta + 1) - (\alpha+\beta)] \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!(b_s)_{m-1}} \leq 1-\beta.$$

Doing some simple calculations, we have:

$$
\begin{aligned}
&\pounds\left(d. b_s, \alpha. \beta, \eta\right) \\
&= \sum_{m=2}^{\infty}\left[(1+\alpha)\left(\eta m^2-\eta m+m\right)-(\alpha-\beta)\right] \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=2}^{\infty}\left[(1+\alpha)(\eta m-1)(m-1)+1-\beta\right] \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=2}^{\infty}\left\{\eta\left(m^2-m\right)(1+\alpha)+(m-1)(1+\alpha)+1-\beta\right\} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=2}^{\infty} \eta(m-1)(m-2)(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}}+\sum_{m=2}^{\infty}(2\eta+1)(m-1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&+\sum_{m=2}^{\infty}(1-\beta) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=2}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-3)!\,(b_s)_{m-1}}+\sum_{m=2}^{\infty}(2\eta+1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-2)!\,(b_s)_{m-1}} \\
&+\sum_{m=2}^{\infty}(1-\beta) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=0}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m+2}}{(b_s)_{m+2}\,(m)!}+\sum_{m=0}^{\infty}(2\eta+1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m+1}}{(b_s)_{m+1}\,(m)!}+\sum_{m=0}^{\infty}(1-\beta) \frac{\left(-\frac{d}{4}\right)^{m+1}}{(b_s)_{m+1}\,(m+1)!} \\
&= \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^2}{b_s\left(b_s+1\right)} u_{x+2}(1)+(2\eta+1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)}{b_s} u_{x+1}(1)+(1-\beta)\left\{u_x(1)-1\right\} \\
&= \eta(1+\alpha) u_x^{\prime\prime}(1)+(2\eta+1)(1+\alpha) u_x^{\prime}(1)+(1-\beta)\left\{u_x(1)-1\right\} \qquad (5.2.7)
\end{aligned}
$$

Therefore the last expression (5 2.7) is bounded above by $1-\beta$ if the condition (5 2.5) is satisfied. ■

5.2.6 Corollary

If $d<0$ and $b_s<0$ then, $w\left(2-u_x\right) \in Q_x(\alpha, \beta, \eta)$ if and only if (5.2.5) is satisfied.

Proof. Since

$$
w\left(2-u_x\right)=w-\sum_{m=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} w^m .
$$

by utilizing the procedure given within the verification of Theorem 5 2.5. we arrive instantly Corollary 5 2.6 ■

5.2.7 Theorem

If $d < 0$ and $b_s < 0$ then $wu_x \in Q_xCV(\alpha, \beta, \eta)$ if

$$\eta(1+\alpha)u_x''(1) + [2\eta(1+\alpha)+(1-\beta)]u_x'(1) + (1-\beta)[u_x(1)-1] \le 1-\beta \tag{5.2.8}$$

Proof. Since

$$wu_x(w) = w + \sum_{m=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} w^m. \tag{5.2.9}$$

on account of Theorem 5 2 2. it is enough to show that

$$f(d, b_s, \alpha, \beta, \eta) = \sum_{m=2}^{\infty} m[\eta(1+\alpha)(m-1)+(1-\beta)] \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!(b_s)_{m-1}} \le 1-\beta.$$

Doing some simple calculations, we have.

$$
\begin{aligned}
& f(d, b_s, \alpha, \beta, \eta) \\
&= \sum_{m=2}^{\infty} \eta(1+\alpha)\left(m^2-m\right) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} + \sum_{m=2}^{\infty} m(1-\beta) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=2}^{\infty} \eta(1+\alpha)\{2(m-1)+(m-1)(m-2)\} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&+ \sum_{m=2}^{\infty} (1-\beta)\{(m-1)+1\} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=2}^{\infty} \eta(m-1)(1+\alpha)(m-2) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} + \sum_{m=2}^{\infty} 2\eta(m-1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&+ \sum_{m=2}^{\infty} (m-1)(1-\beta) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} + \sum_{m=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=3}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-3)!\,(b_s)_{m-1}} + \sum_{m=2}^{\infty} 2\eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-2)!\,(b_s)_{m-1}} \\
&+ \sum_{m=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-2)!\,(b_s)_{m-1}} + \sum_{m=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}} \\
&= \sum_{m=0}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m+2}}{(b_s)_{m+2}\,(m)!} + \sum_{m=0}^{\infty} 2\eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m+1}}{(b_s)_{m+1}\,(m)!} + \sum_{m=0}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{m+1}}{(b_s)_{m+1}\,(m)!} \\
&+ \sum_{m=0}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{m+1}}{(b_s)_{m+1}\,(m+1)!} \\
&= \sum_{m=0}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{m+2}}{(b_s)_{m+2}\,(m)!} + \sum_{m=0}^{\infty} \{2\eta(1+\alpha)+(1-\beta)\} \frac{\left(-\frac{d}{4}\right)^{m+1}}{(b_s)_{m+1}\,(m)!} \\
&+ \sum_{m=0}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{m+1}}{(b_s)_{m+1}\,(m+1)!} \\
&= \eta(1+\alpha)\, u_x''(1) + \{2\eta(1+\alpha)+(1-\beta)\}\, u_x'(1) + (1-\beta)\{u_x(1)-1\}
\end{aligned}
$$

Therefore the last expression is bounded above by $1-\beta$, if the condition (5.2.8) is satisfied. ■

5.2.8 Corollary

If $d < 0$ and $b_s < 0$ then, $u(2-u_x) \in Q_x CV(\alpha, \beta, \eta)$ if and only if the inequality (5.2.8) is satisfied.

Proof. Since

$$w(2-u_x) = w - \sum_{m=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} w^m.$$

by utilizing the procedure given within the verification of Theorem 5 2 7, we arrive instantly Corollary 5.2 8 ■

5.2.9 Theorem

If $d < 0$ and $b_s < 0$ then $w(2-u_x) \in PS(\alpha, \eta)$ if

$$\eta u_x''(1) + (2\eta+1) u_x'(1) + (1-\alpha) u_x(1) \leq 2. \tag{5.2 10}$$

Proof. Since

$$w(2-u_x(w)) = w - \sum_{m=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} w^m$$

on account of Theorem 5.2.3. it is sufficient to show that

$$r(d, b_s, \alpha, \eta) = \sum_{m=2}^{\infty} [m\eta(m-1) + (m-\alpha)] \frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!(b_s)_{m-1}} \leq 1+\alpha.$$

Doing some simple calculations, we have:

$$\begin{aligned}
& r(d, b_s, \alpha, \eta) \\
&= \sum_{m=2}^{\infty} [\eta m(m-1) + m - \alpha] \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} \\
&= \sum_{m=2}^{\infty} \left[\eta\left(m^2 - m\right) + m - \alpha\right] \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} \\
&= \sum_{m=2}^{\infty} \eta\left(m^2 - m\right) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} + \sum_{m=2}^{\infty} (m-1) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} + \sum_{m=2}^{\infty} (1-\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} \\
&= \sum_{m=2}^{\infty} \eta(m-1)(m-2) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} + \sum_{m=2}^{\infty} (2\eta+1)(m-1) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} \\
&+ \sum_{m=2}^{\infty} (1-\alpha) \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} \\
&= \eta u_x''(1) + (2\eta+1) u_x'(1) + (1-\alpha)[u_x(1) - 1]
\end{aligned}$$

Therefore, the last expression is bounded above by $1+\alpha$ if the condition (5 2 10) is satisfied.

$$\eta u_x^{''}(1)+(2\eta+1)\,u_x^{'}(1)+[u_x(1)-1](1-\alpha)\leq 1+\alpha$$
$$\eta u_x^{''}(1)+(2\eta+1)\,u_x^{'}(1)+(1-\alpha)\,u_x(1)\leq 2$$

■

5.2.10 Theorem

If $d<0$ and $b_s<0$ then $w(2-u_x)\in PCV(\alpha,\eta)$ if and only if

$$\eta u_x^{''}(1)+(2\eta+1-\alpha)\,u_x^{'}(1)+(1-\alpha)\,u_x(1)\leq 2.$$

Proof. Since

$$w(2-u_x(w))=w-\sum_{m=2}^{\infty}\frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}}w^m.$$

on account of Theorem 5.2.4, it is sufficient to show that

$$h(d.b_s,\alpha,\eta)=\sum_{m=2}^{\infty}[m\eta(m-1)+(1-\alpha)\,m]\,\frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}}\leq 1+\alpha.$$

Doing some simple calculations, we have:

$$\begin{aligned}
&h(d.b_s,\alpha,\eta)\\
=\;&\sum_{m=2}^{\infty}[m\eta(m-1)+(1-\alpha)\,m]\,\frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}}\\
=\;&\sum_{m=2}^{\infty}\eta(m-1)(m-2)\,\frac{\left(-\frac{d}{4}\right)^{m-1}}{(m-1)!\,(b_s)_{m-1}}+\sum_{m=2}^{\infty}(2\eta+1-\alpha)\,\frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!}\\
&+\sum_{m=2}^{\infty}(1-\alpha)\,\frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!}\\
=\;&\eta u_x^{''}(1)+(2\eta+1-\alpha)\,u_x^{'}(1)+(1-\alpha)\,[u_x(1)-1]\,.
\end{aligned}$$

This preceding expression is bounded above by $1+\alpha$ if the following inequality holds

$$\eta u_x''(1) + (2\eta + 1 - \alpha) u_x'(1) + (1-\alpha) u_x(1) \leq 2.$$

■

Within the another two theorems given below, we get results about of comparable sorts in association with a specific integral operator $T(d, b_s, w)$ expressed by

$$T(d, b_s, w) = \int_0^w [2 - u_x(t)]\, dt. \tag{5.2.11}$$

5.2.11 Theorem

If $d < 0$ and $b_s < 0$ then $T(d, b_s, w) \in Q_x(\alpha, \beta, \eta)$ if and only if the condition (5.2 5) is satisfied.

Proof. Since

$$T(d, b_s, w) = w - \sum_{m=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} w^m,$$

on account of Theorem 5.2.1, we need only to show that

$$\sum_{m=2}^{\infty} [m(\eta m - \eta + 1)(1+\alpha) - \alpha - \beta] |a_m| \leq 1 - \beta$$

The rest portion of the proof of this theorem is the same to the proof of Theorem 5.2 5. ■

5.2.12 Theorem

If $d < 0$ and $b_s < 0$ then $T(d, b_s, w) \in PS(\alpha, \eta)$ if and only if the condition (5 2.10) is satisfied.

Proof. Since

$$T(d, b_s, w) = w - \sum_{m=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{m-1}}{(b_s)_{m-1}(m-1)!} w^m.$$

on account of Theorem 5 2 3, we need only to show that

$$\sum_{m=2}^{\infty} \eta m(m-1) + m - \alpha |a_m| \leq 1 + \alpha$$

The rest portion of the proof of this theorem is the same as the proof of Theorem 5 2.9 ■

Chapter 6

A subclass of univalent functions associated with q-analogue of Choi-Saigo-Srivastava operator

The main objective of this chapter is to initiate a subclass $Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ of AF using subordinations along with the newly defined q-analogue of Choi-Saigo-Srivastava operator. Some results, such as coefficient estimates (CE), integral representation (IR), linear combination (LC), weighted mean (WM), arithmetic means (AM) and radius of starlikeness for this class are derived.

6.1 Introduction

With the help of convolution and the definition of q-derivative, we generalized Choi-Saigo-Srivastava operator [24], into q-Choi-Saigo-Srivastava operator as:

Let $\mathcal{L} \in \mathcal{A}$,

$$I^q_{\lambda,\mu}\mathcal{L}(w) = \mathcal{L}(w) * \mathcal{F}_{q\, \lambda+1\, \mu}(w), \quad w \in \widetilde{H},\ \lambda > -1,\ \mu > 0.$$

where

$$\mathcal{F}_{q\, \lambda+1,\mu}(w) = w + \sum_{m=2}^{\infty} \frac{\Gamma_q(\mu+m-1)\Gamma_q(1+\lambda)}{\Gamma_q(\mu)\Gamma_q(m+\lambda)} w^m = w + \sum_{m=2}^{\infty} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} w^m. \qquad (6.1.1)$$

Thus, we see that

$$I^q_{\lambda,\mu}\mathcal{L}(w) = w + \sum_{m=2}^{\infty} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda\ q \rceil_{m-1}} a_m w^m. \tag{6.1.2}$$

Clearly

$$I^q_{0,2}\mathcal{L}(w) = w\partial D_q \mathcal{L}(w) \text{ and } I^q_{1,2}\mathcal{L}(w) = \mathcal{L}(w)$$

From (6.1.2), we can easily get the identity

$$\lceil \lambda + 1 \rfloor_q I^q_{\lambda,\mu}\mathcal{L}(w) = q^\lambda w \partial D_q \left(I^q_{\lambda+1,\mu}\mathcal{L}(w) \right) + \lceil \lambda \rfloor_q I^q_{\lambda+1,\mu}\mathcal{L}(w), \tag{6.1.3}$$

and

$$q^\lambda w \partial D_q \left(I^q_{\lambda,\mu}\mathcal{L}(w) \right) = \lceil \mu \rfloor_q I^q_{\lambda,\mu+1}\mathcal{L}(w) - \lceil \mu - 1 \rfloor_q I^q_{\lambda\ \mu}\mathcal{L}(w) \tag{6.1.4}$$

If $q \to 1$, the relationships (6.1.3) and (6.1.4) imply that

$$\begin{aligned} w\left(I_{\lambda+1}\mathcal{L}(w)\right)' &= (1+\lambda)\, I_{\lambda,\mu}\mathcal{L}(w) - \lambda I_{\lambda+1,\mu}\mathcal{L}(w). \\ w\left(I_{\lambda\ \mu}\mathcal{L}(w)\right)' &= \mu I_{\lambda,\mu+1}\mathcal{L}(w) - (\mu-1)\, I_{\lambda+1,\mu}\mathcal{L}(w). \end{aligned}$$

which is the well know identities of Choi-Saigo-Srivastava operator. By taking notable values of parameters, we get numerous know operator studied earlier in the literature.

Special cases

1. For $\mu = 2$. we obtain, q-analogue of Noor Integral operator studied in [105], which is define as:

$$I^q_{\lambda,2}\mathcal{L}(w) = w + \sum_{m=2}^{\infty} \frac{\lceil m \rfloor_q!}{\lceil 1+\lambda.q \rceil_{m-1}} a_m w^m.$$

2 For $\mu = 2$, $q \mapsto 1$,we obtain, differential operator studied in [74], which is define as:

$$I^m\mathcal{L}(w) = w + \sum_{m=2}^{\infty} \frac{m!}{(1+\lambda)_{m-1}} a_m w^m.$$

3. For $\mu = 2$, $\lambda = 1-\alpha$, and $q \mapsto 1$. we obtain Owa-Srivastava operator studied in [99] which is define as:

$$I_{1-\alpha,2}\mathcal{L}(w) = w + \sum_{m=2}^{\infty} \frac{\Gamma(m+1)\Gamma(2-\alpha)}{\Gamma(m+1-\alpha)} a_m w^m.$$

The aim of this chapter, to investigate the following subclass of AF associated with the operator $I^q_{\lambda,\mu}$.

6.1.1 Definition

A function $\mathcal{L} \in \mathcal{A}$ is belonging to $Q^*_q(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ if it satisfies

$$\frac{w\partial D_q\left(I^q_{\lambda,\mu}\mathcal{L}(w)\right)}{I^q_{\lambda,\mu}\mathcal{L}(w)} \prec \frac{1+\widehat{\mathfrak{A}}w}{1+\widetilde{\mathfrak{B}}w}$$

Equivalently, a function $\mathcal{L} \in \mathcal{A}$ is belonging to $Q^*_q(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, if and only if

$$\left| \frac{\frac{w\partial D_q(I^q_{\lambda,\mu}\mathcal{L}(w))}{I^q_{\lambda,\mu}\mathcal{L}(w)} - 1}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}\left(\frac{w\partial D_q(I^q_{\lambda,\mu}\mathcal{L}(w))}{I^q_{\lambda,\mu}\mathcal{L}(w)}\right)} \right| < 1 \tag{6.1.5}$$

Throughout our discussion we assume that $\lambda > -1$, $\mu > 0$, $0 < q < 1$ and $-1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1$, unless otherwise stated. We also suppose that all coefficients a_m of $\mathcal{L}$ are real positive numbers.

6.2 Main Results

6.2.1 Theorem

Let $\mathcal{L} \in \mathcal{A}$ be of the form (1.1.1). Then $\mathcal{L} \in Q^*_q(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, if and only if

$$\sum_{m=2}^{\infty} \left\{ \lceil m \rfloor_q (1-\widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}} \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} a_m < \widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}} \tag{6.2.1}$$

Proof. Assuming that (6.2.1) holds. To show that $\mathcal{L} \in Q^*_q(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, we only need to prove the inequality (6 1.5). For this, we consider

$$\left|\frac{\frac{w\partial D_q(I^q_{\lambda,\mu}\mathcal{L}(w))}{I^q_{\lambda,\mu}\mathcal{L}(w)}-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\left(\frac{w\partial D_q(I^q_{\lambda,\mu}\mathcal{L}(w))}{I^q_{\lambda,\mu}\mathcal{L}(w)}\right)}\right| = \left|\frac{\sum_{m=2}^{\infty}(\lceil m\rfloor_q-1)\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m w^m}{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)w+\sum_{m=2}^{\infty}\left\{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\lceil m\rfloor_q\right\}\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m w^m}\right|$$

$$\leq \frac{\sum_{m=2}^{\infty}(\lceil m\rfloor_q-1)\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m}{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)-\sum_{m=2}^{\infty}\left\{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\lceil m\rfloor_q\right\}\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m} < 1.$$

where we have used (1.6.2), (6.1.2), and (6.2.1) and this completes the direct part. Conversely, let $\mathcal{L}\in Q_q^*(\lambda,\mu,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$ be of the form (1.1.1), then from (6.1.5) along with (6.1.2), we have

$$\left|\frac{\frac{w\partial D_q(I^q_{\lambda,\mu}\mathcal{L}(w))}{I^q_{\lambda,\mu}\mathcal{L}(w)}-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\left(\frac{w\partial D_q(I^q_{\lambda,\mu}\mathcal{L}(u))}{I^q_{\lambda,\mu}\mathcal{L}(w)}\right)}\right| = \left|\frac{\sum_{m=2}^{\infty}(\lceil m\rfloor_q-1)\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m w^m}{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)w+\sum_{m=2}^{\infty}\left\{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\lceil m\rfloor_q\right\}\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m w^m}\right| < 1.$$

Since $|\mathrm{Re}\,w| < |w|$, we have

$$\mathrm{Re}\left(\frac{\sum_{m=2}^{\infty}(\lceil m\rfloor_q-1)\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m w^m}{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)+\sum_{m=2}^{\infty}\left\{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\lceil m\rfloor_q\right\}\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda,q\rceil_{m-1}}a_m w^m}\right) < 1. \tag{6.2.2}$$

Now, by choosing such value of w from the real axis such that $\frac{w\partial D_q(I^q_{\lambda,\mu}\mathcal{L}(w))}{I^q_{\lambda,\mu}\mathcal{L}(w)}$ is real. For real values and $w\to 1^-$, also after quiteing the denominator in (6.2.2), we obtain the required inequality (6.2.1). ■

6.2.2 Theorem

Let $\mathcal{L}\in Q_q^*(\lambda,\mu,\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}})$. Then

$$I^q_{\lambda,\mu}\mathcal{L}(w) = \exp\int_0^w \frac{1}{t}\left(\frac{1-\widehat{\mathfrak{A}}\phi(t)}{1-\widetilde{\mathfrak{B}}\varphi(t)}\right)d_q(t).$$

with $|\varphi(w)| < 1$.

Proof. Let $\mathcal{L} \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ and setting

$$\frac{w\partial D_q I_{\lambda,\mu}^q \mathcal{L}(w)}{I_{\lambda,\mu}^q \mathcal{L}(w)} = h(w).$$

with

$$h(w) \prec \frac{1+\widehat{\mathfrak{A}}w}{1+\widetilde{\mathfrak{B}}w}.$$

equivalently, we can write

$$\left|\frac{h(w)-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}h(w)}\right| < 1,$$

then, we have

$$\frac{h(w)-1}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}h(w)} = \phi(w).$$

where $|\phi(w)| < 1$ Thus we can rewrite

$$\frac{\partial D_q\left(I_{\lambda,\mu}^q \mathcal{L}(w)\right)}{I_{\lambda,\mu}^q \mathcal{L}(w)} = \frac{1}{w}\left(\frac{1-\widehat{\mathfrak{A}}\phi(t)}{1-\widetilde{\mathfrak{B}}\phi(t)}\right)$$

and assist by basic calculation of integration, we get the required result. ■

6.2.3 Theorem

Let $\mathcal{L}_j \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ and have the form

$$\mathcal{L}_j(w) = w + \sum_{k=1}^{\infty} a_{k,j} w^k. \quad \text{for } j = 1, 2, 3, \ldots, l.$$

then $F \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, where

$$F(w) = \sum_{j=1}^{l} c_j \mathcal{L}_j(w) \quad \text{with} \quad \sum_{j=1}^{l} c_j = 1.$$

Proof. By the idea of Theorem 6.2.1, we write as

$$\sum_{m=2}^{\infty}\left\{\frac{\left\{\lceil m\rfloor_q(1-\widetilde{\mathfrak{B}})-1+\widehat{\mathfrak{A}}\right\}\frac{\lceil \mu\, q\rceil_{m-1}}{\lceil 1+\lambda\, q\rceil_{m-1}}}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}\right\} a_{m,j} < 1.$$

Therefore

$$\begin{aligned} F(w) &= \sum_{j=2}^{l} c_j \left(w + \sum_{m=2}^{\infty} a_{m,j} w^m \right) = w + \sum_{j=2}^{l} \sum_{m=2}^{\infty} c_j a_{m,j} w^m \\ &= w + \sum_{m=2}^{\infty} \left(\sum_{j=2}^{l} c_j a_{m,j} \right) w^m \end{aligned}$$

However,

$$\begin{aligned} &\sum_{m=2}^{\infty} \frac{\left\{ \lceil m \rfloor_q (1 - \widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}} \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} \left(\sum_{j=2}^{l} a_{m,j} c_j \right) \\ = &\sum_{j=2}^{l} \left\{ \sum_{m=2}^{\infty} \frac{\left\{ \lceil m \rfloor_q (1 - \widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}} \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} a_{m,j} \right\} c_j \leq 1, \end{aligned}$$

then $F \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$. ■

6.2.4 Theorem

Let $\mathcal{L}, g \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ Then the weighted mean h_j, where h_j is defined by

$$h_j(w) = \frac{(1-j)\,\mathcal{L}(w) + (1+j)\,g(w)}{2}, \qquad (6.2\ 3)$$

is also in $Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$

Proof. From (6.2.3), we can write

$$h_j(w) = w + \sum_{m=2}^{\infty} \left\{ \frac{(1-j)\,a_m + (1+j)\,\widetilde{\mathfrak{B}}_m}{2} \right\} w^m.$$

To prove that $h_j(w) \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, we need to show that

$$\sum_{m=2}^{\infty} \frac{\left\{ \lceil m \rfloor_q (1 - \widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}} \right\}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} \left\{ \frac{(1-j)\,a_m + (1+j)\,\widetilde{\mathfrak{B}}_m}{2} \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} < 1.$$

For this, consider

$$\begin{aligned}
&\sum_{m=2}^{\infty} \frac{\left\{\lceil m \rfloor_q (1-\widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}}\right\}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} \left\{\frac{(1-\jmath)\, a_m + (1+\jmath)\, \widetilde{\mathfrak{B}}_m}{2}\right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} \\
= \; & \frac{(1-\jmath)}{2} \sum_{m=2}^{\infty} \frac{\left\{\lceil m \rfloor_q (1-\widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}}\right\}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} a_m \\
& + \frac{(1+\jmath)}{2} \sum_{m=2}^{\infty} \frac{\left\{\lceil m \rfloor_q (1-\widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}}\right\}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} \widetilde{\mathfrak{B}}_m \\
< \; & \frac{(1-\jmath)}{2} + \frac{(1+\jmath)}{2} = 1
\end{aligned}$$

by using inequality (6.2.1), the proof is completed. ■

6.2.5 Theorem

Let $\mathcal{L}_\jmath$ with $\jmath = 1, 2, \ldots \alpha$ is belonging to the $Q_q^*(\lambda, \mu\ \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$. Then the arithmetic mean h, where h is defined by

$$h(w) = \frac{1}{\alpha} \sum_{\jmath=1}^{\alpha} \mathcal{L}_\jmath(w), \tag{6.2.4}$$

is also in $Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}. \widetilde{\mathfrak{B}})$.

Proof. From (6.2.4), we can write

$$\begin{aligned}
h(w) &= \frac{1}{\alpha} \sum_{\jmath=1}^{\alpha} \left(w + \sum_{m=2}^{\infty} a_{m,\jmath} w^m \right) \\
&= w + \sum_{m=2}^{\infty} \left(\frac{1}{\alpha} \sum_{\jmath=1}^{\alpha} a_{m\,\jmath} \right) w^m.
\end{aligned} \tag{6.2.5}$$

Since $\mathcal{L}_j \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, for every $j = 1, 2, 3, \ldots, \alpha$, therefore using (6.2.5) and (6.2.1), we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \left\{ \lceil m \rfloor_q (1-\widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}} \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} \left(\frac{1}{\alpha} \sum_{j=1}^{\alpha} a_{m,j} \right) \\
= &\ \frac{1}{\alpha} \sum_{j=1}^{\alpha} \left(\sum_{m=2}^{\infty} \left\{ \lceil m \rfloor_q (1-\widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}} \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}} a_{m,j} \right) \\
\leq &\ \frac{1}{\alpha} \sum_{j=1}^{\alpha} \left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}} \right) \\
= &\ \widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}},
\end{aligned}$$

and this completes the proof. ■

6.2.6 Theorem

Let $\mathcal{L} \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$. Then $\mathcal{L} \in \mathcal{S}^*(\gamma)$, for $|w| < r_1$, where

$$r_1 = \left[\frac{(1-\gamma) \left\{ (1-\widetilde{\mathfrak{B}}) \lceil m \rfloor_q + \widehat{\mathfrak{A}} - 1 \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}}}{(m-\gamma) \left(\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}} \right)} \right]^{\frac{1}{m-1}}$$

Proof. Let $\mathcal{L} \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$. To prove $\mathcal{L} \in \mathcal{S}^*(\gamma)$, we only need to show

$$\left| \frac{\left(\frac{u\mathcal{L}'(u)}{\mathcal{L}(w)} \right) - 1}{\left(\frac{w\mathcal{L}'(u)}{\mathcal{L}(w)} \right) + 1 - 2\gamma} \right| < 1.$$

By using (1.1.1) along with some simple computations yield

$$\sum_{m=2}^{\infty} \left(\frac{m-\gamma}{1-\gamma} \right) |a_m| \, |w|^{m-1} < 1. \tag{6.2.6}$$

Since $\mathcal{L} \in Q_q^*(\lambda, \mu, \widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, therefore from (6.2.1), we can easily obtain

$$\sum_{m=2}^{\infty} \frac{\left\{ \lceil m \rfloor_q (1-\widetilde{\mathfrak{B}}) - 1 + \widehat{\mathfrak{A}} \right\} \frac{\lceil \mu, q \rceil_{m-1}}{\lceil 1+\lambda, q \rceil_{m-1}}}{\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}}} |a_m| < 1 \tag{6.2.7}$$

Now the inequality (6.2.6) is true, if the following inequality

$$\sum_{m=2}^{\infty}\left(\frac{m-\gamma}{1-\gamma}\right)|a_m||w|^{m-1}<\sum_{m=2}^{\infty}\frac{\left\{\lceil m\rfloor_q(1-\widetilde{\mathfrak{B}})-1+\widehat{\mathfrak{A}}\right\}\frac{\lceil\mu,q\rceil_{m-1}}{\lceil 1+\lambda\,q\rceil_{m-1}}}{\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}}|a_m|.$$

holds, which implies that

$$|w|^{m-1}<\frac{(1-\gamma)\left\{\lceil m\rfloor_q(1-\widetilde{\mathfrak{B}})-1+\widehat{\mathfrak{A}}\right\}\frac{\lceil\mu\,q\rceil_{m-1}}{\lceil 1+\lambda\,q\rceil_{m-1}}}{\left(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}}\right)(m-\gamma)},$$

and thus we get the required result. ■

6.2.7 Theorem

Let $-1\leq\widetilde{\mathfrak{B}}_2\leq\widetilde{\mathfrak{B}}_1<\widehat{\mathfrak{A}}_1\leq\widehat{\mathfrak{A}}_2\leq 1$, and $I^q_{\lambda+1,\mu}\mathcal{L}(w)\neq 0$ in $\widetilde{H}$. and this satisfy

$$\frac{\left(\lceil\lambda+1\rfloor_q\right)I^q_{\lambda,\mu}\mathcal{L}(w)}{q^{\lambda}I^q_{\lambda+1,\mu}\mathcal{L}(w)}-\frac{\lceil\lambda\rfloor_q}{q^{\lambda}}\prec\frac{1+\widehat{\mathfrak{A}}_1w}{1+\widetilde{\mathfrak{B}}_1w}.$$

Then $\mathcal{L}\in Q^*_q\left(\lambda+1,\mu,\widehat{\mathfrak{A}}_2,\widetilde{\mathfrak{B}}_2\right)$.

Proof. Since $I^q_{\lambda+1,\mu}\mathcal{L}(w)\neq 0$ in $\widetilde{H}$, we define the function $p(w)$ by

$$\frac{w\partial D_q\left(I^q_{\lambda+1,\mu}\mathcal{L}(w)\right)}{I^q_{\lambda+1,\mu}\mathcal{L}(w)}=p(w). \tag{6.2.8}$$

By virtue of (6.1.3), we obtain

$$\frac{\left(\lceil\lambda+1\rfloor_q\right)I^q_{\lambda,\mu}\mathcal{L}(w)}{q^{\lambda}I^q_{\lambda+1,\mu}\mathcal{L}(w)}-\frac{\lceil\lambda\rfloor_q}{q^{\lambda}}=p(w).$$

Therefore, using (6.2.8), we have

$$\frac{w\partial D_q\left(I^q_{\lambda+1,\mu}\mathcal{L}(w)\right)}{I^q_{\lambda+1,\mu}\mathcal{L}(w)}=p(w)\prec\frac{1+\widehat{\mathfrak{A}}_1w}{1+\widetilde{\mathfrak{B}}_1w}.$$

by Lemma (1.10.7), we deduce that $\mathcal{L}\in Q^*_q\left(\lambda+1,\mu,\widehat{\mathfrak{A}}_2,\widetilde{\mathfrak{B}}_2\right)$. ■

Chapter 7

Janowski type q-convex and q-close-to-convex functions associated with q-conic domain

In this chapter, with the help q-conic domain $\left(\Omega_{k,q}[\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}}]\right)$ q-Janowski type functions and the concepts of quantum (or q-) calculus, we initiate new subclasses of q-convex and q-close-to-convex functions. These subclasses explores some vital geometric properties such as coefficient estimates (CE), sufficiency criteria and also convolution properties. Furthermore, we as well develop up some results about of these subclasses with those obtained in earlier investigations is also provided.

7.1 Introduction

Here we introduce the following classes $\kappa-\mathcal{UCV}_q(\mathfrak{C},\mathfrak{D})$, $\kappa-\mathcal{UK}_q(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}},\mathfrak{C},\mathfrak{D})$ and $\kappa-\mathcal{UQ}_q(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}},\mathfrak{C},\mathfrak{D})$ of analytic functions.

7.1.1 Definition

Let $\mathcal{L} \in \mathcal{A}$, be in the class $k-\mathcal{UCV}_q(\mathfrak{C}.\mathfrak{D})$ if and only if

$$\mathfrak{Re}\left(\frac{(\mathfrak{D}O_1 - O_3)\frac{\partial D_q(w\partial D_q\mathcal{L}(w))}{\partial D_q\mathcal{L}(w)} - (\mathfrak{C}O_1 - O_3)}{(\mathfrak{D}O_1 + O_3)\frac{\partial D_q(w\partial D_q\mathcal{L}(w))}{\partial D_q\mathcal{L}(u)} - (\mathfrak{C}O_1 + O_3)}\right) > k\left|\frac{(\mathfrak{D}O_1 - O_3)\frac{\partial D_q(w\partial D_q\mathcal{L}(w))}{\partial D_q\mathcal{L}(w)} - (\mathfrak{C}O_1 - O_3)}{(\mathfrak{D}O_1 + O_3)\frac{\partial D_q(w\partial D_q\mathcal{L}(u))}{\partial D_q\mathcal{L}(w)} - (\mathfrak{C}O_1 + O_3)} - 1\right|.$$

Or equivalently,

$$\frac{\partial D_q\left(w\partial D_q\mathcal{L}(w)\right)}{\partial D_q\mathcal{L}(w)} \in k - \mathcal{P}_q(\mathfrak{C}.\mathfrak{D}).$$

where $k \geq 0.\ -1 \leq \mathfrak{D} < \mathfrak{C} \leq 1$

One can clearly see that

$$\mathcal{L} \in \kappa - \mathcal{UCV}_q(\mathfrak{C}.\mathfrak{D}) \Leftrightarrow w\partial D_q(w) \in \kappa - \mathcal{ST}_q(\mathfrak{C}.\mathfrak{D}). \qquad (7.1\ 1)$$

Here the class $\kappa - \mathcal{UCV}_q(\mathfrak{C}\ \mathfrak{D})$ reduces to a well known class defined in [78] when $q \to 1$.

7.1.2 Definition

Let $\mathcal{L} \in \mathcal{A}$, be in the class k-$\mathcal{UK}_q(\widehat{\mathfrak{A}}.\widetilde{\mathfrak{B}}.\mathfrak{C}.\mathfrak{D})$ if and only if there exist $l \in k$-$\mathcal{ST}_q(\mathfrak{C},\mathfrak{D})$. such that

$$\mathfrak{Re}\left(\frac{\left(\widetilde{\mathfrak{B}}O_1 - O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)} - \left(\widehat{\mathfrak{A}}O_1 - O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1 + O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)} - \left(\widehat{\mathfrak{A}}O_1) + O_3\right)}\right) > k\left|\frac{\left(\widetilde{\mathfrak{B}}O_1 - O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)} - \left(\widehat{\mathfrak{A}}O_1 - O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1 + O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)} - \left(\widehat{\mathfrak{A}}O_1) + O_3\right)} - 1\right|$$

We can write equivalently

$$\frac{w\partial D_q\mathcal{L}(w)}{l(w)} \in k - \mathcal{P}_q(\widehat{\mathfrak{A}}.\widetilde{\mathfrak{B}}).$$

where $-1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1,\ k \geq 0.\ -1 \leq \mathfrak{D} < \mathfrak{C} \leq 1.$

Here the class k-$\mathcal{UK}_q(\widehat{\mathfrak{A}}.\widetilde{\mathfrak{B}}.\mathfrak{C}.\mathfrak{D})$ shrinks into the class defined in (see [60]) when $q \to 1$

7.1.3 Definition

Let $\mathcal{L} \in \mathcal{A}$. belong to the class $k\text{-}\mathcal{UQ}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}\ \mathfrak{D})$ if and only if there exists $l \in k\text{-}\mathcal{CV}_q(\mathfrak{C}, \mathfrak{D})$. such that

$$\mathfrak{Re}\left(\frac{\left(\widetilde{\mathfrak{B}}O_1 - O_3\right)\frac{\partial D_q(w\partial D_q\mathcal{L}(w))}{\partial D_q l(w)} - \left(\widehat{\mathfrak{A}}O_1 - O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1 + O_3\right)\frac{\partial D_q(w\partial D_q\mathcal{L}(w))}{\partial D_q l(w)} - \left(\widehat{\mathfrak{A}}O_1) + O_3\right)}\right) > k\left|\frac{\left(\widetilde{\mathfrak{B}}O_1 - O_3\right)\frac{\partial D_q(w\partial D_q\mathcal{L}(w))}{\partial D_q l(w)} - \left(\widehat{\mathfrak{A}}O_1 - O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1 + O_3\right)\frac{\partial D_q(w\partial D_q\mathcal{L}(w))}{\partial D_q l(w)} - \left(\widehat{\mathfrak{A}}O_1) + O_3\right)} - 1\right|$$

Or equivalently,

$$\frac{\partial D_q\left(w\partial D_q\mathcal{L}(w)\right)}{\partial D_q l(w)} \in k - \mathcal{P}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}),$$

where for $k \geq 0$, $-1 \leq \mathfrak{D} < \mathfrak{C} \leq 1$, $-1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1$.

It is simple to verify this

$$\mathcal{L} \in \kappa - \mathcal{UQ}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D}) \Leftrightarrow w\partial D_q\mathcal{L} \in \kappa - \mathcal{UK}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D}). \tag{7.1.2}$$

Here the class $\kappa - \mathcal{UQ}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$ shrinks into the class defined in [60] when $q \to 1$,.

In this chapter, we assume that $q \in (0,1)$, $k \geq 0$, $-1 \leq \mathfrak{D} < \mathfrak{C} \leq 1$. and $-1 \leq \widetilde{\mathfrak{B}} < \widehat{\mathfrak{A}} \leq 1$ unless otherwise specified.

7.2 Main Results

7.2.1 Theorem

Let $\mathcal{L} \in \mathcal{A}$, then $\mathcal{L}$ is in the class $k\text{-}\mathcal{UCV}_q(\mathfrak{C}, \mathfrak{D})$, if the following inequality holds

$$\sum_{m=2}^{\infty} [m]_q \left\{2O_3(k+1)q\,[m-1]_q + \left|(\mathfrak{D}O_1) + O_3)\,[m]_q - (\mathfrak{C}O_1) + O_3)\right|\right\}|a_m|$$
$$\leq O_1\,|\mathfrak{D} - \mathfrak{C}|$$

Proof. By Lemma 1.10.12 and relation (7.1.1) the proof is straightforward. ∎

For $q \to 1^-$, in Theorem 7 2.1, then we obtained following corollary, proved by Malik and Noor [78].

7.2.2 Corollary

Let $\mathcal{L} \in \mathcal{A}$, then $\mathcal{L}$ belong to k-$\mathcal{UCV}(\mathfrak{C}, \mathfrak{D})$, if the following inequality holds

$$\sum_{m=2}^{\infty} m\left\{2(k+1)(m-1)+|m(\mathfrak{D}+1)-(\mathfrak{C}+1)|\right\}|a_m| \leq |\mathfrak{D}-\mathfrak{C}|.$$

7.2.3 Theorem

Let $\mathcal{L} \in \mathcal{A}$, then $\mathcal{L}$ is in the class k-$\mathcal{UK}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$, if the condition (7.2.1) holds

$$\sum_{m=2}^{\infty}\left\{2O_3(k+1)\left|b_m-\lceil m\rfloor_q a_m\right|+\left|\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\lceil m\rfloor_q a_m-\left(\widehat{\mathfrak{A}}O_1 \div O_3\right)b_m\right|\right\}$$
$$\leq O_1\left|\widetilde{\mathfrak{B}}-\widehat{\mathfrak{A}}\right| \quad (7\ 2.1)$$

Proof. Presume that (7.2.1) holds, then it is to be show enough that

$$k\left|\frac{\left(\widetilde{\mathfrak{B}}O_1-O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1-O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\frac{w\partial D_q\mathcal{L}(u)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1+O_3\right)}-1\right|$$
$$-\operatorname{Re}\left\{\frac{\left(\widetilde{\mathfrak{B}}O_1-O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1-O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\frac{u\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1\right)+O_3\right)}-1\right\}$$
$$<1$$

We have

$$
\begin{aligned}
&k\left|\frac{\left(\widetilde{\mathfrak{B}}O_1-O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1-O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\frac{w\partial D_q\mathcal{L}(u)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1\right)+O_3\right)}-1\right| \\
&-\operatorname{Re}\left\{\frac{\left(\widetilde{\mathfrak{B}}O_1-O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1-O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\frac{u\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1+O_3\right)}-1\right\} \\
&\leq(k+1)\left|\frac{\left(\widetilde{\mathfrak{B}}O_1-O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1-O_3\right)}{\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\frac{w\partial D_q\mathcal{L}(w)}{l(w)}-\left(\widehat{\mathfrak{A}}O_1+O_3\right)}-1\right| \\
&=2O_3(k+1)\left|\frac{l(w)-w\partial D_q\mathcal{L}(w)}{\left(\widetilde{\mathfrak{B}}O_1+O_3\right)w\partial D_q\mathcal{L}(w)-\left(\widehat{\mathfrak{A}}O_1)+O_3\right)l(w)}\right| \\
&=2O_3(k+1)\left|\frac{\sum_{m=2}^{\infty}\left\{b_m-\lceil m\rfloor_q a_m\right\}w^m}{O_1\left(\widetilde{\mathfrak{B}}-\widehat{\mathfrak{A}}\right)w+\sum_{m=2}^{\infty}\left\{\widetilde{\mathfrak{B}}O_1)+O_3)\lceil m\rfloor_q a_m-\left(\widehat{\mathfrak{A}}O_1+O_3\right)b_m\right\}w^m}\right| \\
&\leq\frac{2O_3(k+1)\sum_{m=2}^{\infty}\left\{\left|b_m-\lceil m\rfloor_q a_m\right|\right\}}{O_1)\left|\widetilde{\mathfrak{B}}-\widehat{\mathfrak{A}}\right|-\sum_{m=2}^{\infty}\left|\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\lceil m\rfloor_q a_m-\left(\widehat{\mathfrak{A}}O_1+O_3\right)b_m\right|}. \qquad (7.2\ 2)
\end{aligned}
$$

The expression (7.2.2) is bounded above by 1 if

$$
\begin{aligned}
&\sum_{m=2}^{\infty}\left[2O_3(k+1)\left|b_m-\lceil m\rfloor_q a_m\right|+\left|\left(\widetilde{\mathfrak{B}}O_1+O_3\right)\lceil m\rfloor_q a_m-\left(\widehat{\mathfrak{A}}O_1+O_3\right)b_m\right|\right] \\
&\leq O_1\left|\widetilde{\mathfrak{B}}-\widehat{\mathfrak{A}}\right|.
\end{aligned}
$$

■

7.2.4 Corollary[60]

Let $\mathcal{L}\in\mathcal{A}$. Then $\mathcal{L}$ is in the class $k\text{-}\mathcal{UK}_{q\to1}(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}},\mathfrak{C},\mathfrak{D})=k\text{-}\mathcal{UK}(\widehat{\mathfrak{A}},\widetilde{\mathfrak{B}},\mathfrak{C},\mathfrak{D})$, if the following condition holds:

$$
\sum_{m=2}^{\infty}\left\{2(k+1)\left|b_m-ma_m\right|+\left|(\widetilde{\mathfrak{B}}+1)ma_m-(\widehat{\mathfrak{A}}+1)b_m\right|\right\}\leq\left|\widetilde{\mathfrak{B}}-\widehat{\mathfrak{A}}\right|.
$$

7.2.5 Theorem

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L}$ is in the class k-$\mathcal{UQ}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$, if the following condition holds:

$$\sum_{m=2}^{\infty} \lceil m \rfloor_q \left[2O_3(k+1) \left| b_m - \lceil m \rfloor_q a_m \right| + \left| (\widetilde{\mathfrak{B}} O_1 + O_3) \lceil m \rfloor_q a_m - (\widehat{\mathfrak{A}} O_1 + O_3) b_m \right| \right]$$
$$\leq O_1 \left| \widetilde{\mathfrak{B}} - \widehat{\mathfrak{A}} \right|$$

Proof. By Theorem 7.2.3 and relation (7.1.2) the proof is straight forward. ■

7.2.6 Corollary[60]

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L}$ is in the class k-$\mathcal{UK}_{q \to 1}(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D}) = k$-$\mathcal{UQ}(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$, if

$$\sum_{m=2}^{\infty} m \left\{ 2(k+1) \left| b_m - m a_m \right| + \left| (\widetilde{\mathfrak{B}} + 1) m a_m - (\widehat{\mathfrak{A}} + 1) b_m \right| \right\} \leq \left| \widetilde{\mathfrak{B}} - \widehat{\mathfrak{A}} \right|.$$

7.2.7 Corollary[111]

Let $\mathcal{L} \in \mathcal{A}$. Then $\mathcal{L}$ is in the class $1 - \mathcal{UK}_{q \to 1}(1 - 2\tau, -1, 1, -1) = \mathcal{UK}(\tau)$ if

$$\sum_{m=2}^{\infty} m^2 \left| a_m \right| \leq \frac{1-\tau}{2}$$

7.2.8 Theorem

Let $\mathcal{L} \in k$-$\mathcal{UCV}_q(\mathfrak{C}, \mathfrak{D})$, is of the form (1.1.1). Then

$$|a_m| \leq \frac{1}{\lceil m \rfloor_q} \prod_{n=0}^{m-2} \left(\frac{\left| \delta(k)(\mathfrak{C} - \mathfrak{D})(O_1) - 4q \lceil n \rfloor_{qe} \mathbb{D} \right|}{4q \lceil n+1 \rfloor_q} \right).$$

where $\delta(k)$ is given by (1.6.2).

Proof. By Lemma 1.10.10 and relation (7.1.1) the proof is straightforward. ■

For $q \to 1^-$, Theorem 7.2.8 bring to the following corollary, proved by Noor [78]

7.2.9 Corollary

Let $\mathcal{L} \in k\text{-}\mathcal{UCV}(\mathfrak{C}.\mathfrak{D})$. Then

$$|a_m| \leq \frac{1}{m} \prod_{n=0}^{m-2} \left(\frac{|\delta(k)(\mathfrak{C} - \mathfrak{D}) - 2n\mathfrak{D}|}{2\,(n+1)} \right).$$

where $\delta(k)$ is given by (1.6 2).

7.2.10 Theorem

If $\mathcal{L} \in k\text{-}\mathcal{UK}_q(\widehat{\mathfrak{A}}.\widetilde{\mathfrak{B}}.\mathfrak{C}.\mathfrak{D})$ and $l(w) \in k\text{-}\mathcal{ST}_q(\mathfrak{C}.\mathfrak{D})$. Then

$$|a_m| \leq \begin{cases} \frac{1}{\lceil m \rfloor_q} \prod_{n=0}^{m-2} \frac{|\delta(k)(\mathfrak{C}-\mathfrak{D})(O_1)-4q\lceil n \rfloor_{q\epsilon}\mathbb{D}|}{4q\lceil n+1 \rfloor_q} \\ +\frac{\delta(k)(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})O_1)}{4\lceil m \rfloor_q} \sum_{j=1}^{m-1} \prod_{n=0}^{j-2} \frac{|\delta(k)(\mathfrak{C}-\mathfrak{D})(O_1)-4q\lceil j \rfloor_{q\epsilon}\mathbb{D}|}{4q\lceil j+1 \rfloor_q}, \quad m \geq 2. \end{cases}$$

where $\delta(k)$ is given in (1.6.2).

Proof. Let us take

$$\frac{w\partial D_q\mathcal{L}(w)}{l(w)} = h(w) \tag{7.2.3}$$

where

$$h(w) \in k - \mathcal{P}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}) \text{ and } l(w) \in k - \mathcal{ST}_q(\mathfrak{C}.\mathfrak{D}).$$

Now from (7.2.3), we have

$$w\partial D_q\mathcal{L}(w) = l(w)h(w),$$

which implies that

$$w + \sum_{m=2}^{\infty} \lceil m \rfloor_q a_m w^m = \left(1 + \sum_{m=1}^{\infty} c_m w^m\right)\left(w + \sum_{m=2}^{\infty} b_m w^m\right).$$

By equating w^m coefficients

$$\lceil m \rfloor_q a_m = b_m + \sum_{j=1}^{m-1} b_j c_{m-j}. \quad a = 1.\ b_1 = 1.$$

This implies that

$$\lceil m \rfloor_q |a_m| \le |b_m| + \sum_{j=1}^{m-1} |b_j| |c_{m-j}| \tag{7.2.4}$$

Since $h \in k\text{-}\mathcal{P}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$, therefore by using Lemma 1.10 9 on (7.2.4), we have

$$\lceil m \rfloor_q |a_m| \le |b_m| + \frac{\delta(k) O_1 (\widehat{\mathfrak{A}} - \widetilde{\mathfrak{B}})}{4} \sum_{j=1}^{m-1} |b_j| . \tag{7.2 5}$$

Again $l \in k\text{-}\mathcal{ST}_q(\mathfrak{C}. \mathfrak{D})$, therefore by using Lemma 1.10.10 on (7.2.5). we have

$$|a_m| \le \begin{cases} \frac{1}{\lceil m \rfloor_q} \prod_{n=0}^{m-2} \left(\frac{|\delta(k)(\mathfrak{C}-\mathfrak{D})O_1) - 4q \lceil n \rfloor_q \mathfrak{D}|}{4q \lceil n+1 \rfloor_q} \right) \\ \\ + \frac{\delta(k)(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})O_1)}{4 \lceil m \rfloor_q} \sum_{j=1}^{m-1} \prod_{n=0}^{j-2} \left(\frac{|\delta(k)(\mathfrak{C}-\mathfrak{D})(O_1) - 4q \lceil n \rfloor_{qe} \mathfrak{D}|}{4q \lceil n+1 \rfloor_q} \right) \end{cases}.$$

■

7.2.11 Corollary[60]

If $\mathcal{L} \in k\text{-}\mathcal{UK}_{q \to 1}(\widehat{\mathfrak{A}}. \widetilde{\mathfrak{B}}. \mathfrak{C}, \mathfrak{D}) = k\text{-}\mathcal{UK}(\widehat{\mathfrak{A}}. \widetilde{\mathfrak{B}}. \mathfrak{C}, \mathfrak{D})$. Then

$$|a_m| \le \begin{cases} \frac{1}{m} \prod_{n=0}^{m-2} \left(\frac{|\delta(k)(\mathfrak{C}-\mathfrak{D}) - 2n\mathfrak{D}|}{2(n+1)} \right) \\ \\ + \frac{\delta(k)(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})}{2m} \sum_{j=1}^{m-1} \prod_{n=0}^{j-2} \left(\frac{|\delta(k)(\mathfrak{C}-\mathfrak{D}) - 2n\mathfrak{D}|}{2(n+1)} \right), \quad m \ge 2 \end{cases}$$

where $\delta(k)$ is defined by (1.6.2).

7.2.12 Corollary[76]

If $\mathcal{L}(w) \in k\text{-}\mathcal{UK}_{q \to 1}(1. -1, 1. -1) = k\text{-}\mathcal{UK}$. Then

$$|a_m| \le \frac{(\delta(k))_{m-1}}{m!} + \frac{\delta(k)}{m} \sum_{j=0}^{m-1} \frac{(\delta(k))_{j-1}}{(j-1)!}. \quad m \ge 2.$$

7.2.13 Corollary[51]

If $\mathcal{L} \in 0 - \mathcal{UK}_{q\to 1}(0, 1, -1, 1, -1) = \mathcal{K}$. Then

$$|a_m| \leq m, \quad m \geq 2.$$

7.2.14 Theorem

If $\mathcal{L} \in k\text{-}\mathcal{UQ}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$. Then

$$|a_m| \leq \left\{ \begin{array}{c} \frac{1}{\left(\lceil m \rfloor_q\right)^2} \prod_{n=0}^{m-2} \frac{\left|\delta(k)(\mathfrak{C}-\mathfrak{D})(O_1)-4q\lceil n \rfloor_{q\epsilon} \mathbb{D}\right|}{4q\lceil n+1 \rfloor_q} \\ +\frac{\delta(k)(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})O_1)}{4\left(\lceil m \rfloor_q\right)^2} \sum_{j=1}^{m-1} \prod_{n=0}^{j-2} \frac{\left|\delta(k)(\mathfrak{C}-\mathfrak{D})(O_1)-4q\lceil j \rfloor_{q\epsilon} \mathbb{D}\right|}{4q\lceil j+1 \rfloor_q} \end{array} \right. , \quad m \geq 2.$$

where $\delta(k)$ is defined by (1.6.2).

Proof. By Theorem 7.2.10 and relation (7.1.2) the proof is straight forward. ■

7.2.15 Corollary[60]

If $\mathcal{L} \in k\text{-}\mathcal{UQ}_{q\to 1}(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D}) = \mathcal{UQ}(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$ and is of the form (1.1.1). Then

$$|a_m| \leq \left\{ \begin{array}{c} \frac{1}{m^2} \prod_{n=0}^{m-2} \left(\frac{|\delta(k)(\mathfrak{C}-\mathfrak{D})-2n\mathfrak{D}|}{2(n+1)}\right) \\ +\frac{\delta(k)(\widehat{\mathfrak{A}}-\widetilde{\mathfrak{B}})}{2m^2} \sum_{j=1}^{m-1} \prod_{n=0}^{j-2} \left(\frac{|\delta(k)(\mathfrak{C}-\mathfrak{D})-2n\mathfrak{D}|}{2(n+1)}\right) \end{array} \right. , \quad m \geq 2.$$

7.2.16 Theorem

If $\mathcal{L} \in k\text{-}\mathcal{P}(\mathfrak{C}, \mathfrak{D})$ and $\widehat{\chi} \in \mathcal{C}$, then $\mathcal{L} * \widehat{\chi} \in k\text{-}\mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$

Proof. Here we need to prove that

$$\frac{w \partial D_q \left(\widehat{\chi}(w) * \mathcal{L}(w)\right)}{\left(\widehat{\chi}(w) * \mathcal{L}(w)\right)} \in k - \mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$$

Consider

$$\frac{\widehat{\chi}(w) * \mathcal{L}(w) \left(\frac{w \partial D_q \mathcal{L}(w)}{\mathcal{L}(w)}\right)}{\widehat{\chi}(w) * \mathcal{L}(w)}$$
$$\frac{\widehat{\chi}(w) * \mathcal{L}(w) \Psi(w)}{\widehat{\chi}(w) * \mathcal{L}(w)}.$$

where $\frac{w\partial D_q\mathcal{L}(w)}{\mathcal{L}(w)} = \Psi(w) \in \mathcal{P}_q(\mathfrak{C}, \mathfrak{D})$. By using Lemma 1.10 11, we will obtain the required result. ∎

7.2.17 Theorem

If $\mathcal{L} \in k\text{-}\mathcal{UK}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$ and $\widehat{\chi} \in \mathcal{C}$, then $\mathcal{L} * \widehat{\chi} \in k\text{-}\mathcal{UK}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$.

Proof. Since $\mathcal{L} \in k\text{-}\mathcal{UK}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}}, \mathfrak{C}, \mathfrak{D})$, there exist $l \in k\text{-}\mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$. such that $\frac{w\partial D_q\mathcal{L}(w)}{l(w)} \in k\text{-}\mathcal{P}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$.

It follows from Lemma 1.10 11 that $\widehat{\chi} * l \in k\text{-}\mathcal{ST}_q(\mathfrak{C}, \mathfrak{D})$.

Consider

$$\frac{w\partial D_q\left(\widehat{\chi}(w) * \mathcal{L}(w)\right)}{\left(\widehat{\chi}(w) * l(w)\right)} = \frac{\widehat{\chi}(w) * \left(w\partial D_q\mathcal{L}(w)\right)}{\left(\widehat{\chi}(w) * l(w)\right)}.$$

$$\frac{\widehat{\chi}(w) * \left(\frac{w\partial D_q\mathcal{L}(w)}{l(w)}\right) l(w)}{\widehat{\chi}(w) * \mathcal{L}(w)}$$
$$\frac{\widehat{\chi}(w) * F(w)l(w)}{\widehat{\chi}(w) * l(w)}.$$

where $F \in k\text{-}\mathcal{ST}_q(\widehat{\mathfrak{A}}, \widetilde{\mathfrak{B}})$ By using Lemma 1.10.11, we will obtain the required result. ∎

Conclusions

This thesis is mainly in the field of GFT of single complex variable. We studied some new subclasses of AF within the open unit disk $\widetilde{H}$ by utilizing latest techniques such as subordination and convolution.

It is worth mentioning that operator plays a central role in almost all branches of analysis. In this thesis we discuss Sälägean q-differential operator. We profoundly examined the analytic and geometric properties of these recently defined classes. For these classes certain important and interesting results like coefficients bounds growth results, distortion theorems extreme points, radius of convex, close-to-convex and starlikeness have obtained We also effectively introduce q-analogue of Choi-Saigo-Srivastava operator and initiate a subclass of AF. We utilized the subordination methods to drive our key findings such as coefficient estimates (CE) integral representation (IR), linear combination (LC), weighted mean (WM), arithmetic means (AM) and radius of starlikeness for this class. Our initiated classes generalized numerous classes by choosing specific values of the parameters.

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