Some Generalizations of Picture Fuzzy Sets and Their Applications

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A Dissertation
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
Doctor of Philosophy
In
MATHEMATICS

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DECLARATION

I hereby, declare, that this thesis neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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<u>Certificate</u>

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***DOCTOR OF PHILOSOPHY in MATHEMATICS***

We accept this dissertation as conforming to the required standard.

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2020

Dedicated

To

My

Loving parents

My family

And Respectful Supervisor

Acknowledgement

All praises to almighty **“ALLAH”** the creator of the universe, who blessed me with the knowledge and enabled me to complete the dissertation. All respects to **Holy Prophet MUHAMMAD (S.A.W),** who is the last messenger, whose life is a perfect model for the whole humanity.

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0. Research Profile

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0.1. Introduction

To deal with imprecise and uncertain events has always been a challenging task as imprecision and vagueness lie in almost every field of science. To serve the goal, Zadeh [1] proposed the notion of fuzzy set (FS) where he described the uncertainty of an object/event by a membership grade m that has a value from the interval $[0, 1]$. The value "0" means that the element does not belong to the set and the value "1" means that the element completely belongs to that set. The value between "0" and "1" tells the grade that how much an element belongs to that set. So the membership grade represents the uncertainty of human opinion. This uncertainty can be used to solve many real-life problems e.g. decision making [2], pattern recognition [3], etc.

In FS theory, information is given in only one direction but most of the human opinions are not unidirectional e.g. if someone is giving his opinion about some mobile he will discuss his advantages as well as disadvantages. So, FS does not fully expressing human opinion. To overcome this issue, Atanassov [4] proposed the notion of the intuitionistic fuzzy set (IFS) based on two grades m and n representing the membership and non-membership degree of an object with the condition that the sum of m and n must be less than or equal to 1. The IFS has proven to be useful in many areas like decision making [5, 6], pattern recognition [7], medical diagnosis [8], etc.

With the help of IFS, any uncertain event can be modeled by using two grades m and n but in real life situation, the sum of m and n may exceeds 1 because the opinion-makers evaluate these values separately. To deal with that type of information in which value of m and n exceeds 1, Yager [9] proposed the idea of the Pythagorean fuzzy set (PyFS) based on two grades m and n with the condition that the sum of squares of m and n must be less than or equal to 1. PyFS provides a considerably larger range for the values of m and n to be chosen but still, it has limited space. To obtain a space of membership and non-membership grades with no limitation, Yager [10] proposed the framework of q-rung orthopair fuzzy set (q-ROPFS) with the condition that the sum of the qth power of m and n must be less than or equal to 1, for a positive integer q. PyFSs and q-ROPFSs are used to solve many real-life problems such as decision making [11, 12], pattern recognition [13], and medical diagnosis [14], etc. The constraints of these mentioned fuzzy frameworks are discussed in Table 1.

Table 1 (fuzzy frameworks with their limitations)

Fuzzy Structures	Functions	Limitations on Functions
FS	m	$0 \leq m \leq 1$
IFS	(m, n)	$0 \leq m + n \leq 1$
PyFS	(m, n)	$0 \leq m^2 + n^2 \leq 1$
q-ROPFS	(m, n)	$0 \leq m^q + n^q \leq 1, q \in \mathbb{Z}^+$

All fuzzy models described in [1, 4, 9, 10] either use one membership grade to model an event or two but all real-life events cannot always be modeled by using these types of fuzzy frameworks. As in circumstances of voting where opinion cannot be restricted to yes or no but some refusal degree and abstinence are also involved. To model such an event, Cuong [15] used four grades membership "m", abstinence "i", non-membership "n", and refusal grade "r" and developed the concept of picture fuzzy set (PFS). Cuong's structure of PFS is of diverse nature but, likewise in IFSs, there is the restriction in PFS too that the sum of all three membership grades must not exceed 1. PFS cannot handle the information if their sum exceeds from 1. To overcome this Mahmood et al. [16] developed an important concept of spherical fuzzy set (SFS) and consequently T-spherical fuzzy set (T-SFS). SFS has the condition that square sum of m, i and n must not exceeds 1. A T-SFS allows the decision makers to choose any value from the closed unit interval regardless of any restriction. PFS, SFS, and T-SFS are used to solve many real-life problems such as decision making [17-19], etc. A description of the constraints of PFS, SFS, and T-SFS is provided in Table 2.

Table 2 (Comparison of the restrictions of PFS, SFS, and TSFS)

Fuzzy Structures	Functions	Limitations on Functions
PFS	(m, i, n)	$0 \leq m + i + n \leq 1$
SFS	(m, i, n)	$0 \leq m^2 + i^2 + n^2 \leq 1$
T-SFS	(m, i, n)	$0 \leq m^t + i^t + n^t \leq 1, t \in \mathbb{Z}^+$

A geometrical comparison among the ranges of PFSs, SFSs and T-SFSs is depicted in Figure 1 which is based on the constraints discussed in Table 2. All the numbers within and on the space of PFSs represent picture fuzzy numbers; all the numbers on and within the space of SFSs represent spherical fuzzy numbers; and all the numbers on and within space of T-SFSs represent T-spherical fuzzy numbers for $t = 20$.

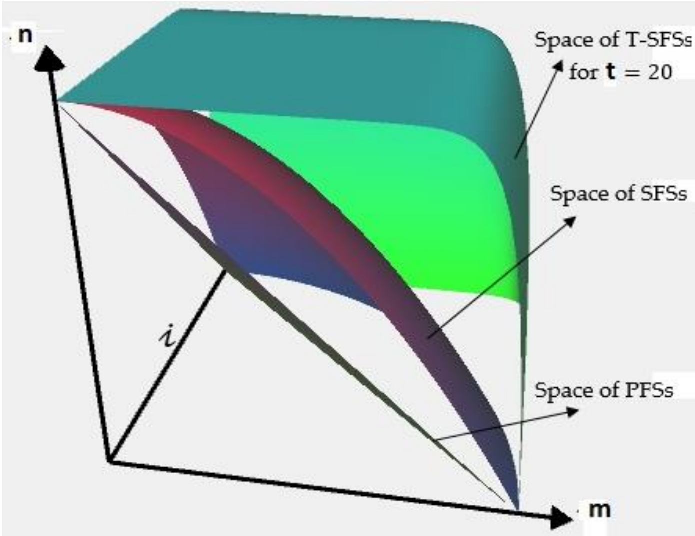

Figure 1 Comparison between PFSs, SFSs and T-SFS

From figure 1, it is easy to observe that T-SFS is much more generalized and diverse than PFS and SFS. The space for T-SFS increases with any increment in the value of t. This enables the experts to have much more values to assign to each membership, abstinence and non-membership grades.

In FS theory, the value of m is a crisp number but in some circumstances a human opinion may not be described by a single number. For this, a concept of interval-valued fuzzy set (IvFS) was proposed by Zadeh [20]. In IvFS, the membership degree is expressed by an interval which is closed sub-interval of [0, 1]. Like FSs, IvFSs have many applications in the field of decision making, pattern recognition, etc. Similarly in IFS, the value of "m" and "n" are expressed in the form of crisp number in which information may be lost. So, Atanassov and Gargov [21] proposed the concept of interval-valued IFS (IvIFS). In IvIFS, the degrees of membership and non-membership are expressed by intervals which are closed sub-intervals of [0, 1] and they keep the condition that the sum of the supremum of these sub-intervals must belong to [0, 1]. As the value of membership and non-membership in terms of the interval has more significance than the values in crisp number, therefore, the notions of PyFS and q-ROPFS are also extended to interval-valued PyFS (IvPyFS) by Peng and Yang [22] with the condition that the square sum of the supremums of these sub-intervals must belongs to [0, 1] and interval-valued q-ROPFS (Ivq-ROPFS) by Joshi et al. [23] with the condition that the sum of q^{th} power of supremums of these sub-intervals must belongs to [0, 1]. The concept of interval-valued is also applied to PFS as interval-valued PFS (IvPFS) by Coung [15] with the condition that the sum of supremums of these sub-intervals must belongs to [0, 1] and T-SFS as interval-valued T-spherical fuzzy set (IvT-SFS) by Ullah et al. [24] with the condition that the sum of t^{th} power of supremums of these sub-intervals must belongs to [0, 1].

The Similarity measure (SM) is a significant content in FS theory. SMs are widely used in the field of pattern recognition, decision making, medical diagnosis, and clustering. Many authors developed SMs for different tools of uncertainty like IFSs, PyFSs, q-ROPFSs, PFSs, SFSs and T-SFSs. Ye [25] developed some cosine SMs for IFSs. Hung and Yang [26] developed some SMs based on Hausdorff distance for IFS. Chen and Chang [27] investigated the pattern recognition problem by using SMs between IFSs based on transformation techniques. Garg and Kumar [28] proposed SMs of IFSs and studied their applications in decision making. Some SMs for IFSs were discussed in [29-32]. Nguyen et al. [33] developed SMs for PyFSs using an exponential function. Wei and Wei [34] proposed some SMs based on cosine function for PyFS and studied their applications in pattern recognition problems and medical diagnosis problems. Some SMs for PyFSs were discussed in [35-38]. Wang et al. [39] proposed SMs of q-ROPFS based on cosine function and studied their applications in scheme selection and pattern recognition. Liu et al. [40] proposed distance measure and SM between q-ROFSs. Peng and Dai [41] did a study on classroom teaching quality assessment with q-ROPFSs based on multiparametric SM. Peng and Liu [42] proposed information measures for q-ROPFSs. Some decision making problems for q-ROFSs are solved in [43-45].

Wei [46] proposed some SMs for PFSs and studied their applications in mineral field recognition and building material recognition. Wei [47] investigated strategic decision making problem by using cosine SM for PFSs. Son [48] proposed generalized PF distance measures. Wei and Gao [49] proposed picture fuzzy generalized dice SMs and used proposed SMs to solve building material recognition problem. Rafiq et al. [50] investigated decision making problem using cosine SMs for SFSs. Ullah et al. [51] proposed some SMs for T-SFSs. Some SMs and decision making problems were discussed in [52-54].

Multi-attribute decision making (MADM) is one of the most discussed problems in FS theory due to its influence in engineering, economics and management sciences. The study of MADM started in 1970 [55] to use the concept of FS in a decision-making problem. Later, the concept of IFS and its aggregation tools have been greatly used in decision making problems. Xu [56] proposed some averaging aggregation operators for IFSs. Wei [57] investigated group-decision making problem using some induced geometric aggregation operators for IFSs. Liu [58] proposed some Hamacher operators for interval-valued IFSs and studied their application in decision making problem. Liu and Chen [59] solved group decision making problem with the help of Heronian aggregation operators for IFSs. Liu and Li [60] introduced interval-valued IF power Bonferroni operators and investigated their usefulness in decision making problem. Garg [61] solved the MADM problem using intuitionistic fuzzy averaging operators. Some methods for IFS to solve the MADM problem are discussed in [62-66]. Wei and Lu [67] investigated the MADM problem with the help of Pythagorean fuzzy power aggregation operators. Garg [68] proposed PyF operators based on confidence levels and studied their application in decision making. Peng and Yuan [69] discussed some fundamental properties of PyF operators and investigated the MADM problem. Peng and Yang [70] discussed some fundamental properties of interval-valued PyF operators and investigated the MADM problem. Wei [71] introduced some interactive operators for PyFSs and investigated their usefulness in the MADM problem. Joshi [72] proposed group-generalized averaging operators and investigated their usefulness in MADM problems. Using PyFS some MADM problems are solved in [73-76]. Peng et al. [77] investigated the MADM problem by using exponential aggregation operators for q-ROPFS. Liu and Wang [78] used q-ROPF Archimedean Bonferroni operators to solve the MADM problem. Some MADM problems are studied using q-ROPFSs in [79-81].

Garg [82] proposed some averaging aggregation operators for PFS and solved MADM problem using proposed operators. Jana et al. [83] proposed picture fuzzy Dombi operator and studied their application in decision making. Wei [84] introduced some aggregation operators for PFS and investigated their usefulness in MADM problem. Wei [85] proposed picture fuzzy Hamacher operators and studied their application in MADM problem. Khan et al. [86] investigated the MADM problem using some logarithmic operators for PFS. Some aggregation operators for PFS are discussed in [87, 88]. Liu et al. [89] proposed T-spherical fuzzy power Muirhead mean operators and studied their application to MADM problem. Ullah et al. [90] proposed some T-spherical fuzzy operators and investigated their application to MADM problem.Some methods for solving MADM problem were discussed in [91-93].

In the theory of aggregation, weighted geometric and averaging operators are the widely used operators and these are based on some t-norms and t-conorms. Literature survey witnessed some other types of t-norms and t-conorms, respectively, among them Einstein t-norm and t-conorms have got some serious attention. Based on Einstein t-conorms and t-norms, several aggregation tools have been proposed for various fuzzy algebraic structures. The Einstein weighted averaging (EWA) and Einstein weighted geometric (EWG) operators of IFSs and interval valued IFSs have been investigated in [94,95]. For PyFSs, EWA and Einstein interactive aggregation operators are developed by [96,97], respectively. For further interesting work on Einstein aggregation operators and their applications in MADM, one is referred to [98–100].

0.2. Chapter Wise Study

In this section, a short description of all chapters is discussed.

Chapter 1

In this chapter, some basic notions like FS, IvFS, IFS, IvIFS, PyFS, IvPyFS, q-ROPFS, Ivq-ROPFS, PFS, IvPFS, SFS, and T-SFS are defined. Some basic operations on them like union, intersection, sum, product, scalar multiplication, power etc. are also defined on them. Further interactive operations are also defined on them. Based on Einstein t-norm and t-conorm, Einstein sum, Einstein product, Einstein scalar multiplication, and Einstein power is also defined. To rank different numbers of these frameworks score and accuracy function are also defined.

Chapter 2

In this chapter, some SMs are developed for IvPFSs due to the significance of describing the membership grades of PFS in terms of intervals. Several types cosine SMs, cotangent SMs, set-theoretic and grey SMs, four types of dice SMs and generalized dice SMs are developed. The properties of proposed SMs are also demonstrated. Using the proposed SMs, two well-known problems mineral field recognition and MADM are solved. The superiority of proposed SMs over existing SMs is demonstrated through a comparative analysis.

Chapter 3

In this chapter, some new improved operational laws are developed and their properties are studied. Based on newly developed operational laws, some series of geometric interactive improved aggregation operators namely, T-spherical fuzzy weighted geometric interactive averaging operator, T-spherical fuzzy ordered weighted geometric interactive averaging operator and T-spherical fuzzy hybrid geometric interactive averaging operator are proposed. The properties of proposed operators are also studied. Then, an algorithm for solving MADM problem using proposed operator is developed. The validity of proposed algorithm and operators is checked through numerical example. Finally, the superiority of the proposed approach is explained with a counter example to show the advantages of the proposed work.

Chapter 4

In this chapter, a series of averaging aggregation operators and interactive averaging aggregation operators under the features that each element is represented with T-SF numbers are proposed. Various properties of these operators are also studied. To rank T-SF numbers, a new score function is also proposed and some properties of newly developed score function are also studied. An algorithm for solving MADM problem using proposed operator is also developed. Solar energy is one of the best renewable sources of energy and also an environment-friendly source so the selection of solar cells is typically a multi-attribute decision-making problem. So the applicability of the developed algorithm is demonstrated with a numerical example in the selection of the solar cells and comparison of their performance with the several existing approaches.

Chapter 5

In this chapter, a series of geometric aggregation operators and interactive geometric aggregation operator is developed for T-SFS. Some properties like boundedness, monotonicity and idempotency are also studied. To rank T-SF numbers, a new score function is proposed in it. A comparison between geometric aggregation operators and interactive geometric aggregation operator is also developed with the help of a numerical example. An algorithm for solving MADM problem using proposed operator is also developed. The validity of proposed operator is checked with the help of an example. The advantages of proposed operators and a comparative analysis between proposed and existing work is also studied.

Chapter 6

In this chapter, some Einstein averaging and geometric aggregation operators for T-SFS namely T-SF EWA operator, T-SF Einstein ordered weighted averaging operator, and T-SF Einstein hybrid averaging operators, T-SF EWG operator, T-SF Einstein ordered weighted geometric operator, and T-SF Einstein hybrid geometric operators are proposed. Some properties of these operators are also studied. The MADM method is described in the environment of T-SFSs and is supported by a comprehensive numerical example using the proposed Einstein aggregation tools. The advantages of proposed operators are discussed in which some conditions are explained under which the proposed operator can reduce to other fuzzy frameworks. The comparison between existing and proposed work is also developed with the help of an example.

Chapter 7

In this chapter, some Einstein interactive operational laws are proposed. Based on these operational laws, a series of T-SF Einstein interactive averaging operators and a series of T-SF Einstein interactive geometric operators are proposed. Then, an algorithm is proposed to solve MADM problem. The algorithm is also validated by solving a numerical problem. The advantages of proposed aggregation operators are also discussed. The superiority of proposed operators over existing work is checked with the help of an example.

Chapter 8

In this chapter, some SMs based on cosine function are proposed and also some SMs based on exponential function are proposed. Some basic properties of these SMs are also studied. Then by using proposed SMs, two well-known problems namely pattern recognition and strategy decision making problems are solved. Some conditions are discussed under which the proposed SMs can reduce to other fuzzy frameworks like IFS, PyFS, q-ROPFS, SFS, and T-SFS. The superiority of proposed operators is also validated with the help of an example.

Chapter 1

Preliminaries

All the basic definitions of FS, IvFS, IFS, PyFS, q-ROPFS, PFS, SFS and T-SFS are defined here. Along with these definitions some operations for these fuzzy frameworks are also defined. Score and accuracy functions are usually used to rank the given numbers in any fuzzy frameworks. So, score and accuracy functions for all defined fuzzy environments are also defined. Some other definitions like fuzzy measure and probability function are also defined in this section.

1.1. Fuzzy Set

To deal with imprecise and uncertain events has always been a challenging task as imprecision and vagueness lie in almost every field of life. To serve the goal, Zadeh [1] proposed the notion of FS where he described the uncertainty of an object/event by a membership grade m that has a value from the unit interval $[0, 1]$.

1.1.1. Definition [1]

A FS on a non-empty set X, is defined as

$$\mathcal{F} = \{(x, m(x)) \mid x \in X\},$$

where $m: X \rightarrow [0, 1]$ is a membership function and a number m is called fuzzy number (FN).

1.1.2. Definition [1]

Some operations on fuzzy numbers $\mathcal{F}_1 = (x, m_1(x))$ and $\mathcal{F}_2 = (x, m_2(x))$ are defined as

1. $\mathcal{F}_1 \subseteq \mathcal{F}_2$ iff $m_1 \leq m_2$
2. $\mathcal{F}_1 \cup \mathcal{F}_2 = \max\{m_1, m_2\}$
3. $\mathcal{F}_1 \cap \mathcal{F}_2 = \min\{m_1, m_2\}$
4. $\mathcal{F}_1^c = 1 - m_1$

1.1.3. Definition [20]

An IvFS on a non-empty set X, is defined as

$$F = \{(x, [m_L(x), m_U(x)]) \mid x \in X\},$$

In IvFS, membership value is given in term of interval where lower limit m_L and upper limit m_U are mappings such that $m_L, m_U: X \to [0, 1]$ with the condition that $m_L \leq m_U$.

1.2. Intuitionistic Fuzzy Set

Atanassov [4] proposed the notion of the IFS based on two grades membership "m" and non-membership "n" of an object. IFSs have the condition that the sum of both m and n must belongs to closed unit interval. Some operations like union, intersection, sum, product, scalar multiplication, power etc. are also defined on IFSs.

1.2.1. Definition [4]

An IFS on X consists of membership and non-membership functions defined as $I = \{(x, m(x), n(x)) \mid x \in X\}$ such that $m, n: X \to [0,1]$ with the condition $0 \leq m(x) + n(x) \leq 1 \; \forall \; x \in X$. Further, refusal grade of x in I is $r(x) = 1 - (m(x) + n(x))$ and the pair (m, n) stands for intuitionistic fuzzy number (IFN).

1.2.2. Definition [4, 56, 63]

Some operations on IFNs $I_1 = (m_1, n_1)$ and $I_2 = (m_2, n_2)$ are defined as

1. $I_1 \subseteq I_2$ iff $m_1 \leq m_2, n_1 \geq n_2$
2. $I_1 \cup I_2 = (\max\{m_1, m_2\}, \min\{n_1, n_2\})$
3. $I_1 \cap I_2 = (\min\{m_1, m_2\}, \max\{n_1, n_2\})$
4. $I_1^c = (n_1, m_1)$
5. $I_1 \oplus I_2 = (m_1 + m_2 - m_1 m_2, n_1 n_2)$
6. $I_1 \otimes I_2 = (m_1 m_2, n_1 + n_2 - n_1 n_2)$
7. $\tau I_1 = (1 - (1 - m_1)^\tau, n_1^\tau), \quad \tau > 0$
8. $I_1^\tau = (m_1^\tau, 1 - (1 - n_1)^\tau), \quad \tau > 0$

1.2.3. Definition [61, 64]

Some interaction operations on IFNs $I_1 = (m_1, n_1)$ and $I_2 = (m_2, n_2)$ are defined as

1. $I_1 \oplus_i I_2 = \begin{pmatrix} m_1 + m_2 - m_1 m_2, \\ (1 - m_1)(1 - m_2) - (1 - m_1 - n_1)(1 - m_2 - n_2) \end{pmatrix}$
2. $I_1 \otimes_i I_2 = \big((1 - n_1)(1 - n_2) - (1 - m_1 - n_1)(1 - m_2 - n_2), n_1 + n_2 - n_1 n_2\big)$

3. $\tau I_1 = (1-(1-m_1)^\tau, (1-m_1)^\tau - (1-m_1-n_1)^\tau), \quad \tau > 0$
4. $I_1^\tau = ((1-n_1)^\tau - (1-m_1-n_1)^\tau, 1-(1-n_1)^\tau), \quad \tau > 0$

1.2.4. Definition [94]

Some Einstein operations on IFNs $I_1 = (m_1, n_1)$ and $I_2 = (m_2, n_2)$ are defined as

1. $I_1 \oplus_E I_2 = \left(\frac{m_1+m_2}{1+m_1 m_2}, \frac{n_1 n_2}{1+(1-n_1)(1-n_2)}\right)$
2. $I_1 \otimes_E I_2 = \left(\frac{m_1 m_2}{1+(1-m_1)(1-m_2)}, \frac{n_1+n_2}{1+n_1 n_2}\right)$
3. $\tau I_1 = \left(\frac{(1+m_1)^\tau-(1-m_1)^\tau}{(1+m_1)^\tau+(1-m_1)^\tau}, \frac{2n_1^\tau}{(2-n_1)^\tau+n_1^\tau}\right), \quad \tau > 0$
4. $I_1^\tau = \left(\frac{2m_1^\tau}{(2-m_1)^\tau+m_1^\tau}, \frac{(1+n_1)^\tau-(1-n_1)^\tau}{(1+n_1)^\tau+(1-n_1)^\tau}\right), \quad \tau > 0$

1.2.5. Definition [56]

For any IFN $I = (m, n)$, the score function is defined as,

$$SC(I) = m - n$$

and the accuracy function is defined as

$$AC(I) = m + n$$

The IFN which have greater score value will be superior to other. If the score of two IFNs is equal, then we rank them using accuracy value and a number is called superior if it has greater accuracy. If again accuracy values of two IFNs become equal, then both numbers are considered as similar.

1.2.6. Definition [21]

An IvIFS on X consists of membership and non-membership functions defined as $\mathcal{I} = \{\langle x, [m_L(x), m_U(x)], [n_L(x), n_U(x)]\rangle \mid x \in X\}$ such that $m_L, m_U, n_L, n_U : X \to [0,1]$ with the condition $0 \leq m_U(x) + n_U(x) \leq 1 \; \forall \; x \in X$. Further, refusal grade of x in $\mathcal{I}$ is $r(x) = [r_L(x), r_U(x)] = \left[1-\left(m_U(x)+n_U(x)\right), 1-\left(m_L(x)+n_L(x)\right)\right]$ and the pair $([m_L, m_U], [n_L, n_U])$ stands for interval-valued IFN.

1.3. Pythagorean Fuzzy Set

Yager [9] proposed the notion of the PyFS based on two grades m and n of an object with the condition that the square sum of m and n must belongs to closed unit interval. Some operations are also defined for PyFSs. To rank PyFN, score and accuracy functions are defined.

1.3.1. Definition [9]

A PyFS on X consists of membership and non-membership functions defined as $\aleph = \{(x, m(x), n(x)) \mid x \in X\}$ such that $m, n: X \to [0,1]$ with the condition $0 \leq m^2(x) + n^2(x) \leq 1\ \forall\ x \in X$. Further, refusal grade of x in $\aleph$ is $r(x) = \sqrt{1 - \left(m^2(x) + n^2(x)\right)}$ and the pair (m, n) stands for Pythagorean fuzzy number (PyFN).

1.3.2. Definition [9, 69]

Some operations on PyFNs $\aleph_1 = (m_1, n_1)$ and $\aleph_2 = (m_2, n_2)$ are defined as

1. $\aleph_1 \subseteq \aleph_2$ iff $m_1 \leq m_2, n_1 \geq n_2$
2. $\aleph_1 \cup \aleph_2 = (\max\{m_1, m_2\}, \min\{n_1, n_2\})$
3. $\aleph_1 \cap \aleph_2 = (\min\{m_1, m_2\}, \max\{n_1, n_2\})$
4. $\aleph_1^c = (n_1, m_1)$
5. $\aleph_1 \oplus \aleph_2 = \left(\sqrt{m_1^2 + m_2^2 - m_1^2 m_2^2}, n_1 n_2\right)$
6. $\aleph_1 \otimes \aleph_2 = \left(m_1 m_2, \sqrt{n_1^2 + n_2^2 - n_1^2 n_2^2}\right)$
7. $\tau\aleph_1 = \left(\sqrt{1 - (1 - m_1^2)^\tau}, n_1^\tau\right), \quad \tau > 0$
8. $\aleph_1^\tau = \left(m_1^\tau, \sqrt{1 - (1 - n_1^2)^\tau}\right), \quad \tau > 0$

1.3.3. Definition [71]

Some interaction operations on PyFNs $\aleph_1 = (m_1, n_1)$ and $\aleph_2 = (m_2, n_2)$ are defined as

1. $\aleph_1 \oplus_i \aleph_2 = \left(\begin{array}{c}\sqrt{m_1^2 + m_2^2 - m_1^2 m_2^2}, \\ \sqrt{(1 - m_1^2)(1 - m_2^2) - (1 - m_1^2 - n_1^2)(1 - m_2^2 - n_2^2)}\end{array}\right)$
2. $\aleph_1 \otimes_i \aleph_2 =$ $\left(\sqrt{(1 - n_1^2)(1 - n_2^2) - (1 - m_1^2 - n_1^2)(1 - m_2^2 - n_2^2)}, \sqrt{n_1^2 + n_2^2 - n_1^2 n_2^2}\right)$
3. $\tau\aleph_1 = \left(\sqrt{1 - (1 - m_1^2)^\tau}, \sqrt{(1 - m_1^2)^\tau - (1 - m_1^2 - n_1^2)^\tau}\right), \quad \tau > 0$
4. $\aleph_1^\tau = \left(\sqrt{(1 - n_1^2)^\tau - (1 - m_1^2 - n_1^2)^\tau}, \sqrt{1 - (1 - n_1^2)^\tau}\right), \quad \tau > 0$

1.3.4. Definition [96]

Some Einstein operations on PyFNs $\aleph_1 = (m_1, n_1)$ and $\aleph_2 = (m_2, n_2)$ are defined as

1. $\aleph_1 \oplus_E \aleph_2 = \left(\sqrt{\frac{m_1^2 + m_2^2}{1 + m_1^2 m_2^2}}, \sqrt{\frac{n_1^2 n_2^2}{1 + (1 - n_1^2)(1 - n_2^2)}} \right)$
2. $\aleph_1 \otimes_E \aleph_2 = \left(\sqrt{\frac{m_1^2 m_2^2}{1 + (1 - m_1^2)(1 - m_2^2)}}, \sqrt{\frac{n_1^2 + n_2^2}{1 + n_1^2 n_2^2}} \right)$
3. $\tau \aleph_1 = \left(\sqrt{\frac{(1 + m_1^2)^\tau - (1 - m_1^2)^\tau}{(1 + m_1^2)^\tau + (1 - m_1^2)^\tau}}, \sqrt{\frac{2(n_1^2)^\tau}{(2 - n_1^2)^\tau + (n_1^2)^\tau}} \right), \quad \tau > 0$
4. $\aleph_1^\tau = \left(\sqrt{\frac{2(m_1^2)^\tau}{(2 - m_1^2)^\tau + (m_1^2)^\tau}}, \sqrt{\frac{(1 + n_1^2)^\tau - (1 - n_1^2)^\tau}{(1 + n_1^2)^\tau + (1 - n_1^2)^\tau}} \right), \quad \tau > 0$

1.3.5. Definition [69]

For any PyFN $\aleph = (m, n)$, the score function is defined as,

$$SC(\aleph) = m^2 - n^2$$

and the accuracy function is defined as

$$AC(\aleph) = m^2 + n^2$$

The PyFN which have greater score value will be superior to other. If the score of two PyFNs is equal, then we rank them using accuracy value and a number is called superior if it has greater accuracy. If again accuracy values of two PyFNs become equal, then both numbers are considered as similar.

1.3.6. Definition [22]

An IvPyFS on X consists of membership and non-membership functions defined as $N = \{\langle x, [m_L(x), m_U(x)], [n_L(x), n_U(x)] \rangle \mid x \in X\}$ such that $m_L, m_U, n_L, n_U : X \to [0,1]$ with the condition $0 \leq (m_U(x))^2 + (n_U(x))^2 \leq 1 \; \forall \; x \in X$. Further, refusal grade of x in N is $r(x) = [r_L(x), r_U(x)] = \left[\sqrt{1 - \left((m_U(x))^2 + (n_U(x))^2 \right)}, \sqrt{1 - \left((m_L(x))^2 + (n_L(x))^2 \right)} \right]$ and the pair $([m_L, m_U], [n_L, n_U])$ stands for interval-valued PyFN.

1.4. q-Rung Orthopair Fuzzy Set

Yager [10] proposed the notion of the q-ROPFS based on two grades m and n of an object with the condition that the sum of q^{th} power of m and n degree must belongs to closed unit interval. Some operations are also defined for q-ROPFSs. To rank q-ROPFN, score and accuracy functions are defined.

1.4.1. Definition [10]

A q-ROPFS on X consists of membership and non-membership functions defined as $Q = \{(x, m(x), n(x)) \mid x \in X\}$ such that $m, n: X \to [0,1]$ with the condition $0 \leq m^q(x) + n^q(x) \leq 1 \ \forall \ x \in X$ and $q \in Z^+$. Further, refusal grade of x in Q is $r(x) = \sqrt[q]{1 - \left(m^q(x) + n^q(x)\right)}$ and the pair (m, n) stands for q-rung othopair fuzzy number (q-ROPFN).

1.4.2. Definition [10, 79]

Some operations on q-ROPFNs $Q_1 = (m_1, n_1)$ and $Q_2 = (m_2, n_2)$ are defined as

1. $Q_1 \subseteq Q_2$ iff $m_1 \leq m_2, n_1 \geq n_2$
2. $Q_1 \cup Q_2 = (\max\{m_1, m_2\}, \min\{n_1, n_2\})$
3. $Q_1 \cap Q_2 = (\min\{m_1, m_2\}, \max\{n_1, n_2\})$
4. $Q_1^c = (n_1, m_1)$
5. $Q_1 \oplus Q_2 = \left(\sqrt[q]{m_1^q + m_2^q - m_1^q m_2^q}, n_1 n_2\right)$
6. $Q_1 \otimes Q_2 = \left(m_1 m_2, \sqrt[q]{n_1^q + n_2^q - n_1^q n_2^q}\right)$
7. $\tau Q_1 = \left(\sqrt[q]{1 - \left(1 - m_1^q\right)^\tau}, n_1^\tau\right), \quad \tau > 0$
8. $Q_1^\tau = \left(m_1^\tau, \sqrt[q]{1 - \left(1 - n_1^q\right)^\tau}\right), \quad \tau > 0$

1.4.3. Definition [79]

For any q-ROPFN $Q = (m, n)$, the score function is defined as,

$$SC(Q) = m^q - n^q$$

and the accuracy function is defined as

$$AC(Q) = m^q + n^q$$

The q-ROPFN which have greater score value will be superior to other. If the score of two q-ROPFNs is equal, then we rank them using accuracy value and a number is called superior if it has greater accuracy. If again accuracy values of two q-ROPFNs become equal, then both numbers are considered as similar.

1.4.4. Definition [23]

An Ivq-ROPFS on X consists of membership and non-membership functions defined as $Q = \{\langle x, [m_L(x), m_U(x)], [n_L(x), n_U(x)]\rangle \mid x \in X\}$ such that $m_L, m_U, n_L, n_U: X \to [0,1]$ with the condition $0 \leq \left(m_U(x)\right)^q + \left(n_U(x)\right)^q \leq 1 \; \forall \; x \in X$. Further, refusal grade of x in Q is $r(x) = [r_L(x), r_U(x)] = \left[\sqrt[q]{1 - \left(\left(m_U(x)\right)^q + \left(n_U(x)\right)^q\right)}, \sqrt[q]{1 - \left(\left(m_L(x)\right)^q + \left(n_L(x)\right)^q\right)}\right]$ and the pair $([m_L, m_U], [n_L, n_U])$ stands for interval-valued q-ROPFN.

1.5. Picture Fuzzy Set

Coung [15] proposed the notion of the PFS based on three grades membership "m", abstinence "i" and non-membership "n" of an object. PFSs have the condition that the sum of m, i and n must belongs to closed unit interval. Some operations like union, intersection, sum, product, scalar multiplication, power etc. are also defined on PFSs.

1.5.1. Definition [15]

An PFS on X consists of membership, abstinence and non-membership functions defined as $P = \{(x, m(x), i(x), n(x)) \mid x \in X\}$ such that $m, i, n: X \to [0,1]$ with the condition $0 \leq m(x) + i(x) + n(x) \leq 1 \; \forall \; x \in X$. Further, refusal grade of x in P is $r(x) = 1 - (m(x) + i(x) + n(x))$ and the triplet (m, i, n) stands for picture fuzzy number (PFN).

1.5.2. Definition [15, 82, 84]

Some operations on PFNs $P_1 = (m_1, i_1, n_1)$ and $P_2 = (m_2, i_2, n_2)$ are defined as

1. $P_1 \subseteq P_2$ iff $m_1 \leq m_2, i_1 \leq i_2, n_1 \geq n_2$
2. $P_1 \cup P_2 = (\max\{m_1, m_2\}, \min\{i_1, i_2\}, \min\{n_1, n_2\})$
3. $P_1 \cap P_2 = (\min\{m_1, m_2\}, \min\{i_1, i_2\}, \max\{n_1, n_2\})$

4. $P_1^c = (n_1, i_1, m_1)$
5. $P_1 \oplus P_2 = (m_1 + m_2 - m_1 m_2, i_1 i_2, n_1 n_2)$
6. $P_1 \otimes P_2 = (m_1 m_2, i_1 + i_2 - i_1 i_2, n_1 + n_2 - n_1 n_2)$
7. $\tau P_1 = (1 - (1 - m_1)^\tau, i_1^\tau, n_1^\tau), \quad \tau > 0$
8. $P_1^\tau = (\, m_1^\tau, 1 - (1 - i_1)^\tau, 1 - (1 - n_1)^\tau), \quad \tau > 0$

1.5.3. Definition [84]

For any PFN $P = (m, i, n)$, the score function is defined as,

$$SC(P) = m - n$$

and the accuracy function is defined as

$$AC(P) = m + i + n$$

The PFN which have greater score value will be superior to other. If the score of two PFNs is equal, then we rank them using accuracy value and a number is called superior if it has greater accuracy. If again accuracy values of two PFNs become equal, then both numbers are considered as similar.

1.5.4. Definition [15]

An IvPFS on X consists of membership, abstinence and non-membership functions defined as $\mathcal{P} = \{\langle x, [m_L(x), m_U(x)], [i_L(x), i_U(x)], [n_L(x), n_U(x)]\rangle \mid x \in X\}$ such that $m_L, m_U, i_L, i_U, n_L, n_U: X \to [0,1]$ with the condition $0 \le m_U(x) + i_U(x) + n_U(x) \le 1 \; \forall \; x \in X$. Further, the degree of refusal of x in $\mathcal{P}$ is $r(x) = [r_L(x), r_U(x)] = \left[1 - \left(m_U(x) + i_U(x) + n_U(x)\right), 1 - \left(m_L(x) + i_L(x) + n_L(x)\right)\right]$ and the triplet $(m, i, n) = ([m_L, m_U], [i_L, i_U], [n_L, n_U])$ stands for interval-valued PFN.

1.6. Spherical Fuzzy Set

Mahmood et al. [16] proposed the notion of the SFS based on three grades membership "m", abstinence "i" and non-membership "n" of an object with the condition that the square sum of m, i and n degree must belongs to closed unit interval. Some operations are also defined for SFSs. To rank SFN, score and accuracy functions are defined.

1.6.1. Definition [16]

A SFS on X consists of membership, abstinence and non-membership functions defined as $\mathcal{S} = \{(x, m(x), i(x), n(x)) \mid x \in X\}$ such that $m, i, n: X \to [0,1]$ with a condition $0 \leq m^2(x) + i^2(x) + n^2(x) \leq 1\ \forall\, x \in X$. Further, refusal garde of x in $\mathcal{S}$ is $r(x) = \sqrt{1 - \left(m^2(x) + i^2(x) + n^2(x)\right)}$ and the triplet (m, i, n) stands for spherical fuzzy number (SFN).

1.6.2. Definition [16]

Some operations on SFNs $\mathcal{S}_1 = (m_1, i_1, n_1)$ and $\mathcal{S}_2 = (m_2, i_2, n_2)$ are defined as

1. $\mathcal{S}_1 \subseteq \mathcal{S}_2$ iff $m_1 \leq m_2, i_1 \leq i_2, n_1 \geq n_2$
2. $\mathcal{S}_1 \cup \mathcal{S}_2 = (\max\{m_1, m_2\}, \min\{i_1, i_2\}, \min\{n_1, n_2\})$
3. $\mathcal{S}_1 \cap \mathcal{S}_2 = (\min\{m_1, m_2\}, \min\{i_1, i_2\}, \max\{n_1, n_2\})$
4. $\mathcal{S}_1^c = (n_1, i_1, m_1)$

1.6.3. Definition [16]

For any SFN $\mathcal{S} = (m, i, n)$, the score function is defined as,

$$SC(\mathcal{S}) = m^2 - n^2$$

and the accuracy function is defined as

$$AC(\mathcal{S}) = m^2 + i^2 + n^2$$

The SFN which have greater score value will be superior to other. If the score of two SFNs is equal, then we rank them using accuracy value and a number is called superior if it has greater accuracy. If again accuracy values of two SFNs become equal, then both numbers are considered as similar.

1.7. T-Spherical Fuzzy Set

Mahmood et al. [16] proposed the notion of the T-SFS based on three grades membership "m", abstinence "i" and non-membership "n" of an object with the condition that the sum of t^{th} power of m, i and n degree must belongs to closed unit interval. Some operations are also defined for T-SFSs. To rank T-SFN, score and accuracy functions are defined.

1.7.1. Definition [16]

A T-SFS on X consists of membership, abstinence and non-membership functions defined as $\mathcal{T} = \{(x, m(x), i(x), n(x)) \mid x \in X\}$ such that $m, i, n: X \to [0,1]$ with the condition $0 \leq m^t(x) + i^t(x) + n^t(x) \leq 1 \ \forall \ x \in X$ and $t \in Z^+$. Further, refusal garde of x in $\mathcal{T}$ is $r(x) = \sqrt[t]{1 - \left(m^t(x) + i^t(x) + n^t(x)\right)}$ and the triplet (m, i, n) stands for T-spherical fuzzy number (T-SFN).

1.7.2. Definition [16]

Some operations on T-SFNs $\mathcal{T}_1 = (m_1, i_1, n_1)$ and $\mathcal{T}_2 = (m_2, i_2, n_2)$ are defined as

1. $\mathcal{T}_1 \subseteq \mathcal{T}_2$ iff $m_1 \leq m_2, i_1 \leq i_2, n_1 \geq n_2$
2. $\mathcal{T}_1 \cup \mathcal{T}_2 = (\max\{m_1, m_2\}, \min\{i_1, i_2\}, \min\{n_1, n_2\})$
3. $\mathcal{T}_1 \cap \mathcal{T}_2 = (\min\{m_1, m_2\}, \min\{i_1, i_2\}, \max\{n_1, n_2\})$
4. $\mathcal{T}_1^c = (n_1, i_1, m_1)$
5. $\mathcal{T}_1 \otimes \mathcal{T}_2 = \left(m_1 m_2, i_1 i_2, \sqrt[t]{n_1^t + n_2^t - n_1^t n_2^t}\right)$
6. $\mathcal{T}_1^\tau = \left(m_1^\tau, \sqrt[t]{1 - (1 - i_1^t)^\tau}, \sqrt[t]{1 - (1 - n_1^t)^\tau}\right), \quad \tau > 0$

1.7.3. Definition [16]

For any T-SFN $\mathcal{T} = (m, i, n)$, the score function is defined as,

$$SC(\mathcal{T}) = m^t - n^t$$

and the accuracy function is defined as

$$AC(\mathcal{T}) = m^t + i^t + n^t$$

The T-SFN which have greater score value will be superior to other. If the score of two T-SFNs is equal, then we rank them using accuracy value and a number is called superior if it has greater accuracy. If again accuracy values of two T-SFNs become equal, then both numbers are considered as similar.

1.7.4. Definition [24]

An IvT-SFS on X consists of membership, abstinence and non-membership functions defined as $T = \{\langle x, [m_L(x), m_U(x)], [i_L(x), i_U(x)], [n_L(x), n_U(x)]\rangle \mid x \in X\}$ such that $m_L, m_U, i_L, i_U, n_L, n_U: X \to [0,1]$ with the condition $0 \leq \left(m_U(x)\right)^t + \left(i_U(x)\right)^t +$

$\left(n_U(x)\right)^t \leq 1 \; \forall \, x \in X$. Further, the degree of refusal of x in T is $r(x) = [r_L(x), r_U(x)] =$

$$\left[\sqrt[t]{1-\left(\left(m_U(x)\right)^t+\left(i_U(x)\right)^t+\left(n_U(x)\right)^t\right)}, \sqrt[t]{1-\left(\left(m_L(x)\right)^t+\left(i_L(x)\right)^t+\left(n_L(x)\right)^t\right)}\right]$$

and the pair $([m_L, m_U], [i_L, i_U], [n_L, n_U])$ stands for interval-valued T-SFN.

1.8. Some Other Related Notions

In this section, some other related notions namely fuzzy measure and probability function are defined in it.

1.8.1. Definition [66]

A fuzzy measure $\Theta: 2^X \rightarrow [0,1]$ on a finite set X is defined as

i. $\Theta(\phi) = 0; \Theta(X) = 1$

ii. For all $X_1, X_2 \subseteq X$, if $X_1 \subseteq X_2$ then $\Theta(X_1) \leq \Theta(X_2)$

The possible orderings of elements of X are presented by the permutation of X with k elements forms a group X_n.

1.8.2. Definition [66]

The probability function P_ρ on X defined by

$$P_\rho\left(x_{\rho(1)}\right) = \Theta\left(\left\{x_{\rho(1)}\right\}\right), \ldots\ldots\ldots,$$

$$P_\rho\left(x_{\rho(j)}\right) = \Theta\left(\left\{x_{\rho(1)}, x_{\rho(2)}, \ldots \ldots, x_{\rho(j)}\right\}\right) - \Theta\left(\left\{x_{\rho(1)}, x_{\rho(2)}, \ldots \ldots, x_{\rho(j-1)}\right\}\right), \ldots\ldots.,$$

$$P_\rho\left(x_{\rho(k)}\right) = 1 - \Theta\left(\left\{x_{\rho(1)}, x_{\rho(2)}, \ldots \ldots, x_{\rho(k-1)}\right\}\right),$$

$$\Theta\left(\left\{x_{\rho(0)}\right\}\right) \equiv 0.$$

where $\rho = \left(\rho(1), \rho(2), \ldots \ldots, \rho(k)\right) \in X_n$ are called associated probabilities and $\left\{P_\rho\right\}_{\rho \in X_n}$ is associated probability class of Θ.

Chapter 2

Some Similarity Measures for Interval-valued Picture Fuzzy Sets and Their Applications in Decision Making

SMs, distance measures and entropy measures are some common tools considered to be applied to some interesting real-life phenomena including pattern recognition, decision making, medical diagnosis and clustering. Further, interval-valued picture fuzzy sets (IvPFSs) are effective and useful to describe the fuzzy information. Therefore, this chapter aims to develop some similarity measures for IvPFSs due to the significance of describing the grades of PFS in terms of intervals. Several types, cosine similarity measures, cotangent SMs, set-theoretic and grey SMs, four types of dice SMs and generalized dice SMs are developed. All the developed SMs are validated, and their properties are demonstrated. Two well-known problems including mineral field recognition problem and MADM problem are solved using the newly developed SMs. The superiorities of developed SMs over the SMs of PFS, IvIFS and IFS are demonstrated through a comparison and numerical examples.

2.1. Similarity Measures

In this section, some SMs like cosine SMs, cosine SMs based on cosine and cotangent functions, grey SMs, set-theoretic SMs, dice SMs and generalized dice SMs are defined for IvPFSs. Some basic properties of these SMs are also defined. In this chapter, if stated otherwise we use $m_1 = [m_{1L}, m_{1U}]$, $m_2 = [m_{2L}, m_{2U}]$, $m_3 = [m_{3L}, m_{3U}]$, $i_1 = [i_{1L}, i_{1U}]$, $i_2 = [i_{2L}, i_{2U}]$, $i_3 = [i_{3L}, i_{3U}]$, $n_1 = [n_{1L}, n_{1U}]$, $n_2 = [n_{2L}, n_{2U}]$, $n_3 = [n_{3L}, n_{3U}]$, $\mathcal{P}_1 = (m_1, i_1, n_1)$, $\mathcal{P}_2 = (m_2, i_2, n_2)$ and $\mathcal{P}_3 = (m_3, i_3, n_3)$.

2.1.1. Cosine Similarity Measures for IvPFSs

In this subsection, some cosine SMs and weighted cosine SMs for IvPFSs are defined and some basic properties of these SMs are also discussed.

2.1.1.1. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, an interval-valued picture fuzzy cosine SM (IvPFCSM) between these two IvPFNs is defined as

$IvPFCSM^1(\mathcal{P}_1,\mathcal{P}_2)$

$$= \frac{1}{k}\sum_{j=1}^{k}\left(\frac{\begin{array}{c} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{array}}{\sqrt{\begin{array}{c} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{array}}\sqrt{\begin{array}{c} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +m_{2U}^2(x_j) + i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{array}}}\right)$$

For $k = 1$ the above equation becomes correlation coefficient between IvPFSs.

2.1.1.2. Theorem

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, cosine SM fulfils the following properties:

i. $0 \leq IvPFCSM^1(\mathcal{P}_1,\mathcal{P}_2) \leq 1.$
ii. $IvPFCSM^1(\mathcal{P}_1,\mathcal{P}_2) = IvPFCSM^1(\mathcal{P}_2,\mathcal{P}_1)$
iii. $For\ \mathcal{P}_1 = \mathcal{P}_2, IvPFCSM^1(\mathcal{P}_1,\mathcal{P}_2) = 1.$

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFCSM^1(\mathcal{P}_1,\mathcal{P}_2)$ belongs to [0, 1].

(ii) Holds trivially.

(iii) If $\mathcal{P}_1 = \mathcal{P}_2$ then $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$ and $n_{1U} = n_{2U}$.

Then

$$IvPFCSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\left(\frac{m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)}{m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)}\right)$$

$= \frac{1}{k}\sum_{j=1}^{k} 1 = \frac{1}{k}k = 1.$

2.1.1.3. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, weighted cosine SM between these two IvPFNs is defined as

$IvPFWCSM^1(\mathcal{P}_1,\mathcal{P}_2)$

$$= \sum_{j=1}^{k} w_j \left(\frac{\begin{array}{c} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{array}}{\sqrt{\begin{array}{c} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{array}} \sqrt{\begin{array}{c} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +m_{2U}^2(x_j) + i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{array}}} \right)$$

Where $w = (w_1, \dots, w_k)$ is a weight vector (WV) satisfies $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

2.1.1.4. Theorem

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, the weighted cosine SM fulfils the following properties:

i. $0 \leq IvPFWCSM^1(\mathcal{P}_1, \mathcal{P}_2) \leq 1$
ii. $IvPFWCSM^1(\mathcal{P}_1, \mathcal{P}_2) = IvPFWCSM^1(\mathcal{P}_2, \mathcal{P}_1)$
iii. $For\ \mathcal{P}_1 = \mathcal{P}_2, IvPFWCSM^1(\mathcal{P}_1, \mathcal{P}_2) = 1$

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFWCSM^1(\mathcal{P}_1, \mathcal{P}_2)$ belongs to [0, 1] .

(ii) Holds trivially.

(iii) If $\mathcal{P}_1 = \mathcal{P}_2$ then $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$ and $n_{1U} = n_{2U}$.

Then

$$IvPFWCSM^1(\mathcal{P}_1, \mathcal{P}_2)$$
$$= \sum_{j=1}^{k} w_j \left(\frac{m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)}{m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)} \right)$$

$= \sum_{j=1}^{k} w_j = 1$.

2.1.1.5. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, the cosine and weighted cosine SM based on four functions, between $\mathcal{P}_1$ and $\mathcal{P}_2$ are defined as

$$IvPFCSM^2(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \frac{\begin{array}{c} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{array}}{\sqrt{\begin{array}{c} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +r_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{array}}\sqrt{\begin{array}{c} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +r_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{array}}}$$

and

$$IvPFWCSM^2(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j \frac{\begin{array}{c} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{array}}{\sqrt{\begin{array}{c} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +r_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{array}}\sqrt{\begin{array}{c} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +r_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{array}}}$$

Where WV $w = (w_1, \dots, w_k)^T$ is with a condition that for $j = 1,2,\dots,k$ $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

2.1.1.6. Theorem

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, cosine and weighted cosine SMs based on four functions satisfy the following properties:

i. $0 \le IvPFCSM^2(\mathcal{P}_1,\mathcal{P}_2) \le 1$.
ii. $IvPFCSM^2(\mathcal{P}_1,\mathcal{P}_2) = IvPFCSM^2(\mathcal{P}_2,\mathcal{P}_1)$
iii. $For\ \mathcal{P}_1 = \mathcal{P}_2, IvPFCSM^2(\mathcal{P}_1,\mathcal{P}_2) = 1$.

2.1.2. Cosine Similarity Measures for IvPFSs Based on Cosine Function

In this subsection some cosine SMs based on cosine function and some weighted cosine SMs based on cosine function for IvPFSs are defined. Some basic properties of these SMs are also discussed.

2.1.2.1. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, cosine SMs based on cosine function between these two IvPFNs are defined as

$$IvPFCsSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\cos\left\{\frac{\pi}{2}\begin{bmatrix} |m_{1L}-m_{2L}|\vee|i_{1L}-i_{2L}|\vee|n_{1L}-n_{2L}| \\ \vee|m_{1U}-m_{2U}|\vee|i_{1U}-i_{2U}|\vee|n_{1U}-n_{2U}| \end{bmatrix}\right\}$$

$$IvPFCsSM^2(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\cos\left\{\frac{\pi}{4}\begin{bmatrix} |m_{1L}-m_{2L}|+|i_{1L}-i_{2L}|+|n_{1L}-n_{2L}| \\ +|m_{1U}-m_{2U}|+|i_{1U}-i_{2U}|+|n_{1U}-n_{2U}| \end{bmatrix}\right\}$$

Further, cosine SMs using four functions membership, abstinence, non-membership and refusal are defined as

$$IvPFCsSM^3(\mathcal{P}_1,\mathcal{P}_2)$$
$$= \frac{1}{k}\sum_{j=1}^{k}\cos\left\{\frac{\pi}{2}\begin{bmatrix} |m_{1L}-m_{2L}|\vee|i_{1L}-i_{2L}|\vee|n_{1L}-n_{2L}|\vee|r_{1L}-r_{2L}| \\ \vee|m_{1U}-m_{2U}|\vee|i_{1U}-i_{2U}|\vee|n_{1U}-n_{2U}|\vee|r_{1U}-r_{2U}| \end{bmatrix}\right\}$$

$$IvPFCsSM^4(\mathcal{P}_1,\mathcal{P}_2)$$
$$= \frac{1}{k}\sum_{j=1}^{k}\cos\left\{\frac{\pi}{4}\begin{bmatrix} |m_{1L}-m_{2L}|+|i_{1L}-i_{2L}|+|n_{1L}-n_{2L}|+|r_{1L}-r_{2L}| \\ +|m_{1U}-m_{2U}|+|i_{1U}-i_{2U}|+|n_{1U}-n_{2U}|+|r_{1U}-r_{2U}| \end{bmatrix}\right\}$$

2.1.2.2. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, all IvPFCsSMs satisfy the following properties for $t = 1, 2, 3, 4$.

i. $0 \le IvPFCsSM^t(\mathcal{P}_1,\mathcal{P}_2) \le 1$.

ii. $IvPFCsSM^t(\mathcal{P}_1,\mathcal{P}_2) = IvPFCsSM^t(\mathcal{P}_2,\mathcal{P}_1)$

iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFCsSM^t(\mathcal{P}_1,\mathcal{P}_2) = 1$.

iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFCsSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFCsSM^t(\mathcal{P}_1,\mathcal{P}_2)$ and $IvPFCsSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFCsSM^t(\mathcal{P}_2,\mathcal{P}_3)$

Proof: (i) Since value of cosine function lies in [0, 1], so it is obvious that value of $IvPFCsSM^t(\mathcal{P}_1,\mathcal{P}_2)$ lies in [0, 1] for all $t = 1,2,3,4$.

(ii) Trivially hold.

(iii) For $\mathcal{P}_1 = \mathcal{P}_2$, $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$, $n_{1U} = n_{2U}$, $r_{1L} = r_{2L}$ and $r_{1U} = r_{2U}$. This shows that

$|m_{1L} - m_{2L}| = 0$, $|i_{1L} - i_{2L}| = 0$, $|n_{1L} - n_{2L}| = 0$, $|m_{1U} - m_{2U}| = 0$, $|i_{1U} - i_{2U}| = 0$, $|n_{1U} - n_{2U}| = 0$.

So $IvPFCsSM^1(\mathcal{P}_1, \mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \cos\{0\} = \frac{1}{k}\sum_{j=1}^{k} 1 = 1$.

Similarly, for $t = 2,3,4$, the others can also be proved.

(iv) For $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, $m_{1L} \leq m_{2L} \leq m_{3L}$ also $m_{1U} \leq m_{2U} \leq m_{3U}$

Similarly, $i_{1L} \leq i_{2L} \leq i_{3L}$, $i_{1U} \leq i_{2U} \leq i_{3U}$, $n_{1L} \geq n_{2L} \geq n_{3L}$ and $n_{1U} \geq n_{2U} \geq n_{3U}$.

For $j = 1,2, \dots, k$ we have

$$|m_{1L} - m_{2L}| \leq |m_{1L} - m_{3L}|$$

$$|i_{1L} - i_{2L}| \leq |i_{1L} - i_{3L}|$$

$$|n_{1L} - n_{2L}| \leq |n_{1L} - n_{3L}|$$

$$|m_{1U} - m_{2U}| \leq |m_{1U} - m_{3U}|$$

$$|i_{1U} - i_{2U}| \leq |i_{1U} - i_{3U}|$$

$$|n_{1U} - n_{2U}| \leq |n_{1U} - n_{3U}|$$

As cosine function is decreasing in $\left[0, \frac{\pi}{2}\right]$, so $IvPFCsSM^1(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFCsSM^1(\mathcal{P}_1, \mathcal{P}_2)$ and also by following same method it can be proved that $IvPFCsSM^1(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFCsSM^1(\mathcal{P}_2, \mathcal{P}_3)$.

Similarly, for $t = 2,3,4$, the others can also be proved.

2.1.2.3. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, weighted cosine SMs based on cosine function between these two IvPFNs are defined as

$$IvPFWCsSM^1(\mathcal{P}_1, \mathcal{P}_2) = \sum_{j=1}^{k} w_j \cos\left\{\frac{\pi}{2}\begin{bmatrix} |m_{1L} - m_{2L}| \vee |i_{1L} - i_{2L}| \vee |n_{1L} - n_{2L}| \\ \vee |m_{1U} - m_{2U}| \vee |i_{1U} - i_{2U}| \vee |n_{1U} - n_{2U}| \end{bmatrix}\right\}$$

$$IvPFWCsSM^2(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j \cos\left\{\frac{\pi}{4}\begin{bmatrix} |m_{1L}-m_{2L}|+|i_{1L}-i_{2L}|+|n_{1L}-n_{2L}| \\ +|m_{1U}-m_{2U}|+|i_{1U}-i_{2U}|+|n_{1U}-n_{2U}| \end{bmatrix}\right\}$$

Further, the weighted cosine SMs using four functions membership, abstinence, non-membership and refusal are defined as

$$IvPFWCsSM^3(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j \cos\left\{\frac{\pi}{2}\begin{bmatrix} |m_{1L}-m_{2L}|\vee|i_{1L}-i_{2L}|\vee|n_{1L}-n_{2L}|\vee|r_{1L}-r_{2L}| \\ \vee|m_{1U}-m_{2U}|\vee|i_{1U}-i_{2U}|\vee|n_{1U}-n_{2U}|\vee|r_{1U}-r_{2U}| \end{bmatrix}\right\}$$

$$IvPFWCsSM^4(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j \cos\left\{\frac{\pi}{4}\begin{bmatrix} |m_{1L}-m_{2L}|+|i_{1L}-i_{2L}|+|n_{1L}-n_{2L}|+|r_{1L}-r_{2L}| \\ +|m_{1U}-m_{2U}|+|i_{1U}-i_{2U}|+|n_{1U}-n_{2U}|+|r_{1U}-r_{2U}| \end{bmatrix}\right\}$$

Where WV $w = (w_1, \dots, w_k)^T$ is with a condition that for $j = 1,2,\dots,k$ $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

2.1.2.4. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, all IvPFWCsSMs satisfy the following properties for $t = 1, 2, 3, 4$.

i. $0 \le IvPFWCsSM^t(\mathcal{P}_1,\mathcal{P}_2) \le 1$.
ii. $IvPFWCsSM^t(\mathcal{P}_1,\mathcal{P}_2) = IvPFWCsSM^t(\mathcal{P}_2,\mathcal{P}_1)$
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFWCsSM^t(\mathcal{P}_1,\mathcal{P}_2) = 1$.
iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFWCsSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFWCsSM^t(\mathcal{P}_1,\mathcal{P}_2)$ and $IvPFWCsSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFWCsSM^t(\mathcal{P}_2,\mathcal{P}_3)$

Proof: (i) Since value of cosine function lies in [0, 1], so it is obvious that value of $IvPFWCsSM^t(\mathcal{P}_1,\mathcal{P}_2)$ lies in [0, 1] for all $t = 1,2,3,4$.

(ii) Trivially hold.

(iii) For $\mathcal{P}_1 = \mathcal{P}_2$, $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$, $n_{1U} = n_{2U}$, $r_{1L} = r_{2L}$ and $r_{1U} = r_{2U}$. This shows that

$|m_{1L} - m_{2L}| = 0$, $|i_{1L} - i_{2L}| = 0$, $|n_{1L} - n_{2L}| = 0$, $|m_{1U} - m_{2U}| = 0$, $|i_{1U} - i_{2U}| = 0$, $|n_{1U} - n_{2U}| = 0$.

So

$$IvPFWCsSM^1(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j \cos\{0\} = \sum_{j=1}^{k} w_j = 1$$

Similarly, for $t = 2,3,4$, they can also be proved.

(iv) For $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, $m_{1L} \leq m_{2L} \leq m_{3L}$ also $m_{1U} \leq m_{2U} \leq m_{3U}$

Similarly, $i_{1L} \leq i_{2L} \leq i_{3L}$, $i_{1U} \leq i_{2U} \leq i_{3U}$, $n_{1L} \geq n_{2L} \geq n_{3L}$ and $n_{1U} \geq n_{2U} \geq n_{3U}$.

For $j = 1,2,\ldots,k$ we have

$$|m_{1L} - m_{2L}| \leq |m_{1L} - m_{3L}|$$

$$|i_{1L} - i_{2L}| \leq |i_{1L} - i_{3L}|$$

$$|n_{1L} - n_{2L}| \leq |n_{1L} - n_{3L}|$$

$$|m_{1U} - m_{2U}| \leq |m_{1U} - m_{3U}|$$

$$|i_{1U} - i_{2U}| \leq |i_{1U} - i_{3U}|$$

$$|n_{1U} - n_{2U}| \leq |n_{1U} - n_{3U}|$$

As cosine function is decreasing in $\left[0,\frac{\pi}{2}\right]$, so $IvPFWCsSM^1(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFWCsSM^1(\mathcal{P}_1,\mathcal{P}_2)$ and also by following same method it can be proved that $IvPFWCsSM^1(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFWCsSM^1(\mathcal{P}_2,\mathcal{P}_3)$.

Similarly, for $t = 2,3,4$, the others can also be proved.

2.1.3. Similarity Measures for IvPSs Based on Cotangent Function

In this subsection, we proposed some cotangent SMs based on cotangent function and some weighted cotangent SMs based on cotangent function for IvPFSs, and some basic properties of these SMs are also discussed.

2.1.3.1. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, a cotangent SM based on cotangent function between these two IvPFNs is defined as

$IvPFCtSM^1(\mathcal{P}_1,\mathcal{P}_2) =$

$$\frac{1}{k}\sum_{j=1}^{k}\cot\left\{\frac{\pi}{4}+\frac{\pi}{4}\begin{bmatrix}|m_{1L}-m_{2L}|\vee|i_{1L}-i_{2L}|\vee|n_{1L}-n_{2L}|\\ \vee|m_{1U}-m_{2U}|\vee|i_{1U}-i_{2U}|\vee|n_{1U}-n_{2U}|\end{bmatrix}\right\}$$

Further, then cotangent SMs using four functions membership, abstinence, non-membership and refusal is defined as

$IvPFCtSM^2(\mathcal{P}_1,\mathcal{P}_2) =$

$$\frac{1}{k}\sum_{j=1}^{k}\cot\left\{\frac{\pi}{4}+\frac{\pi}{4}\begin{bmatrix}|m_{1L}-m_{2L}|\vee|i_{1L}-i_{2L}|\vee|n_{1L}-n_{2L}|\vee|r_{1L}-r_{2L}|\\ \vee|m_{1U}-m_{2U}|\vee|i_{1U}-i_{2U}|\vee|n_{1U}-n_{2U}|\vee|r_{1U}-r_{2U}|\end{bmatrix}\right\}$$

2.1.3.2. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, all IvPFCtSMs satisfy the following properties for $t = 1, 2$.

i. $0 \leq IvPFCtSM^t(\mathcal{P}_1,\mathcal{P}_2) \leq 1$.
ii. $IvPFCtSM^t(\mathcal{P}_1,\mathcal{P}_2) = IvPFCtSM^t(\mathcal{P}_2,\mathcal{P}_1)$
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFCtSM^t(\mathcal{P}_1,\mathcal{P}_2) = 1$.
iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFCtSM^t(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFCtSM^t(\mathcal{P}_1,\mathcal{P}_2)$ and $IvPFCtSM^t(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFCtSM^t(\mathcal{P}_2,\mathcal{P}_3)$

2.1.3.3. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, a weighted cotangent SM based on cotangent function between these two IvPFNs is defined as

$$IvPFWCtSM^1(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j\cot\left\{\frac{\pi}{4}+\frac{\pi}{4}\begin{bmatrix}|m_{1L}-m_{2L}|\vee|i_{1L}-i_{2L}|\vee|n_{1L}-n_{2L}|\\ \vee|m_{1U}-m_{2U}|\vee|i_{1U}-i_{2U}|\vee|n_{1U}-n_{2U}|\end{bmatrix}\right\}$$

Further, then weighted cotangent SM using four functions membership, abstinence, non-membership and refusal is defined as

$$IvPFWCtSM^2(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j\cot\left\{\frac{\pi}{4}+\frac{\pi}{4}\begin{bmatrix}|m_{1L}-m_{2L}|\vee|i_{1L}-i_{2L}|\vee|n_{1L}-n_{2L}|\\ \vee|r_{1L}-r_{2L}|\vee|m_{1U}-m_{2U}|\vee\\ |i_{1U}-i_{2U}|\vee|n_{1U}-n_{2U}|\vee|r_{1U}-r_{2U}|\end{bmatrix}\right\}$$

Where WV $w = (w_1, \dots, w_k)^T$ is with a condition that for $j = 1,2, \dots, k$ $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

2.1.3.4. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, all IvPFWCtSMs satisfy the following properties for $t = 1, 2$.

i. $0 \leq IvPFWCtSM^t(\mathcal{P}_1, \mathcal{P}_2) \leq 1$.
ii. $IvPFWCtSM^t(\mathcal{P}_1, \mathcal{P}_2) = IvPFWCtSM^t(\mathcal{P}_2, \mathcal{P}_1)$
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFWCtSM^t(\mathcal{P}_1, \mathcal{P}_2) = 1$.
iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFWCtSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFWCtSM^t(\mathcal{P}_1, \mathcal{P}_2)$ and $IvPFWCtSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFWCtSM^t(\mathcal{P}_2, \mathcal{P}_3)$

2.1.4. Set-theoretic Similarity Measures and Grey Similarity Measures for IvPFSs

In this subsection, set-theoretic SM, Grey SM and weighted set-theoretic SM, weighted Grey SM for IvPFSs are defined, and some basic properties of these SMs are also discussed.

2.1.4.1. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, an interval-valued picture fuzzy set-theoretic SM (IvPFStSM) between these IvPFNs is defined as

$$IvPFStSM^1(\mathcal{P}_1, \mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \frac{\begin{matrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{matrix}}{max\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) & m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)' & +m_{2U}^2(x_j) + i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

2.1.4.2. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, the IvPFStSM satisfies the following properties

i. $0 \leq IvPFStSM^1(\mathcal{P}_1, \mathcal{P}_2) \leq 1$.
ii. $IvPFStSM^1(\mathcal{P}_1, \mathcal{P}_2) = IvPFStSM^1(\mathcal{P}_2, \mathcal{P}_1)$
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFStSM^1(\mathcal{P}_1, \mathcal{P}_2) = 1$.

iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_2)$ and $IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFStSM^1(\mathcal{P}_2,\mathcal{P}_3)$

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_2)$ belongs to [0, 1] .

(ii) Holds trivially.

(iii) If $\mathcal{P}_1 = \mathcal{P}_2$ then $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$ and $n_{1U} = n_{2U}$.

Then

$IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_2)$

$$= \frac{1}{k}\sum_{j=1}^{k} \frac{m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)}{m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)}$$

$$= \frac{1}{k}\sum_{j=1}^{k} 1$$

$$= \frac{1}{k}k = 1$$

(iv) For $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, $m_{1L} \leq m_{2L} \leq m_{3L}$ also $m_{1U} \leq m_{2U} \leq m_{3U}$

Similarly, $i_{1L} \leq i_{2L} \leq i_{3L}$, $i_{1U} \leq i_{2U} \leq i_{3U}$, $n_{1L} \geq n_{2L} \geq n_{3L}$ and $n_{1U} \geq n_{2U} \geq n_{3U}$

$$m_{1L}m_{3L} + i_{1L}i_{3L} + n_{1L}n_{3L} + m_{1U}m_{3U} + i_{1U}i_{3U} + n_{1U}n_{3U} \leq m_{1L}m_{2L} + i_{1L}i_{2L} + n_{1L}n_{2L} + m_{1U}m_{2U} + i_{1U}i_{2U} + n_{1U}n_{2U}$$

And

$$\max\begin{pmatrix} m_{1L}^2 + i_{1L}^2 + n_{1L}^2 + m_{1U}^2 + i_{1U}^2 + n_{1U}^2, \\ m_{3L}^2 + i_{3L}^2 + n_{3L}^2 + m_{3U}^2 + i_{3U}^2 + n_{3U}^2 \end{pmatrix} \geq \max\begin{pmatrix} m_{1L}^2 + i_{1L}^2 + n_{1L}^2 + m_{1U}^2 + i_{1U}^2 + n_{1U}^2, \\ m_{2L}^2 + i_{2L}^2 + n_{2L}^2 + m_{2U}^2 + i_{2U}^2 + n_{2U}^2 \end{pmatrix}$$

So clearly, $IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_2)$

Similarly, $IvPFStSM^1(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFStSM^1(\mathcal{P}_2,\mathcal{P}_3)$

2.1.4.3. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, an interval-valued picture fuzzy weighted set-theoretic SM (IvPFWStSM) between these IvPFNs is defined as

$$IvPFWStSM^1(\mathcal{P}_1,\mathcal{P}_2)$$
$$=\sum_{j=1}^{k} w_j \frac{\begin{matrix} m_{1L}(x_j)m_{2L}(x_j)+i_{1L}(x_j)i_{2L}(x_j)+n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j)+i_{1U}(x_j)i_{2U}(x_j)+n_{1U}(x_j)n_{2U}(x_j)\end{matrix}}{max\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j)+n_{1L}^2(x_j) & m_{2L}^2(x_j)+i_{2L}^2(x_j)+n_{2L}^2(x_j) \\ +m_{1U}^2(x_j)+i_{1U}^2(x_j)+n_{1U}^2(x_j), & +m_{2U}^2(x_j)+i_{2U}^2(x_j)+n_{2U}^2(x_j)\end{pmatrix}}$$

Where WV $w=(w_1,\dots,w_k)^T$ is with the condition that for $j=1,2,\dots,k$ $w_j\in[0,1]$ and $\sum_{j=1}^{k} w_j=1$.

2.1.4.4. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, the IvPFWStSM satisfies the following properties

i. $0\le IvPFWStSM^1(\mathcal{P}_1,\mathcal{P}_2)\le 1$.
ii. $IvPFWStSM^1(\mathcal{P}_1,\mathcal{P}_2)=IvPFWStSM^1(\mathcal{P}_2,\mathcal{P}_1)$.
iii. For $\mathcal{P}_1=\mathcal{P}_2$, $IvPFWStSM^1(\mathcal{P}_1,\mathcal{P}_2)=1$.
iv. Consider $\mathcal{P}_1\subseteq\mathcal{P}_2\subseteq\mathcal{P}_3$, then $IvPFWStSM^1(\mathcal{P}_1,\mathcal{P}_3)\le IvPFWStSM^1(\mathcal{P}_1,\mathcal{P}_2)$ and $IvPFWStSM^2(\mathcal{P}_1,\mathcal{P}_3)\le IvPFWStSM^1(\mathcal{P}_2,\mathcal{P}_3)$.

2.1.4.5. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, an interval-valued picture fuzzy grey SM (IvPFGSM) between these IvPFNs is defined as

$$IvPFGSM^1(\mathcal{P}_1,\mathcal{P}_2)$$
$$=\frac{1}{3k}\sum_{j=1}^{k}\left(\begin{matrix}\frac{\begin{matrix}\Delta m_{L(min)}+\Delta m_{U(min)}+\\ \Delta m_{L(max)}+\Delta m_{U(max)}\end{matrix}}{\Delta m_L+\Delta m_U+\Delta m_{L(max)}+\Delta m_{U(max)}}+\frac{\begin{matrix}\Delta i_{L(min)}+\Delta i_{U(min)}+\\ \Delta i_{L(max)}+\Delta i_{U(max)}\end{matrix}}{\Delta i_L+\Delta i_U+\Delta i_{L(max)}+\Delta i_{U(max)}}\\ +\frac{\Delta n_{L(min)}+\Delta n_{U(min)}+\Delta n_{L(max)}+\Delta n_{U(max)}}{\Delta n_L+\Delta n_U+\Delta n_{L(max)}+\Delta n_{U(max)}}\end{matrix}\right)$$

Where $\Delta m_{L(min)}=\min\{|m_{1L}-m_{2L}|\}$, $\Delta m_{U(min)}=\min\{|m_{1U}-m_{2U}|\}$, $\Delta m_L=|m_{1L}-m_{2L}|$, $\Delta m_U=|m_{1U}-m_{2U}|$, $\Delta m_{L(max)}=\max\{|m_{1L}-m_{2L}|\}$, $\Delta m_{U(max)}=\max\{|m_{1U}-m_{2U}|\}$, $\Delta i_{L(min)}=\min\{|i_{1L}-i_{2L}|\}$, $\Delta i_{U(min)}=\min\{|i_{1U}-i_{2U}|\}$, $\Delta i_L=$

$|i_{1L} - i_{2L}|$, $\Delta i_U = |i_{1U} - i_{2U}|$, $\Delta i_{L(max)} = \max\{|i_{1L} - i_{2L}|\}$, $\Delta i_{U(max)} = \max\{|i_{1U} - i_{2U}|\}$, $\Delta n_{L(min)} = \min\{|n_{1L} - n_{2L}|\}$, $\Delta n_{U(min)} = \min\{|n_{1U} - n_{2U}|\}$, $\Delta n_L = |n_{1L} - n_{2L}|$, $\Delta n_U = |n_{1U} - n_{2U}|$, $\Delta n_{L(max)} = \max\{|n_{1L} - n_{2L}|\}$, $\Delta n_{U(max)} = \max\{|n_{1U} - n_{2U}|\}$.

2.1.4.6. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, the IvPFGSM satisfies the following properties:

i. $0 \leq IvPFGSM^1(\mathcal{P}_1, \mathcal{P}_2) \leq 1$.

ii. $IvPFGSM^1(\mathcal{P}_1, \mathcal{P}_2) = IvPFGSM^1(\mathcal{P}_2, \mathcal{P}_1)$

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFGSM^1(\mathcal{P}_1, \mathcal{P}_2)$ belongs to [0, 1] .

(ii) Holds trivially.

2.1.4.7. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, an interval-valued picture fuzzy weighted grey SM (IvPFWGSM) between these IvPFNs is defined as

$IvPFWGSM^1(\mathcal{P}_1, \mathcal{P}_2)$

$$= \frac{1}{3}\sum_{j=1}^{k} w_j \left(\begin{array}{c} \left(\dfrac{\begin{array}{c}\Delta m_{L(min)} + \Delta m_{U(min)} + \\ \Delta m_{L(max)} + \Delta m_{U(max)}\end{array}}{\begin{array}{c}\Delta m_L + \Delta m_U + \Delta m_{L(max)} \\ + \Delta m_{U(max)}\end{array}} \right) + \left(\dfrac{\begin{array}{c}\Delta i_{L(min)} + \Delta i_{U(min)} + \\ \Delta i_{L(max)} + \Delta i_{U(max)}\end{array}}{\begin{array}{c}\Delta i_L + \Delta i_U + \Delta i_{L(max)} \\ + \Delta i_{U(max)}\end{array}} \right) \\ + \left(\dfrac{\Delta n_{L(min)} + \Delta n_{U(min)} + \Delta n_{L(max)} + \Delta n_{U(max)}}{\Delta n_L + \Delta n_U + \Delta n_{L(max)} + \Delta n_{U(max)}} \right) \end{array} \right)$$

Where WV $w = (w_1, \ldots, w_k)^T$ is with a condition that for $j = 1,2, \ldots, k$ $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$ and $\Delta m_{L(min)} = \min\{|m_{1L} - m_{2L}|\}$, $\Delta m_{U(min)} = \min\{|m_{1U} - m_{2U}|\}$, $\Delta m_L = |m_{1L} - m_{2L}|$, $\Delta m_U = |m_{1U} - m_{2U}|$, $\Delta m_{L(max)} = \max\{|m_{1L} - m_{2L}|\}$, $\Delta m_{U(max)} = \max\{|m_{1U} - m_{2U}|\}$, $\Delta i_{L(min)} = \min\{|i_{1L} - i_{2L}|\}$, $\Delta i_{U(min)} = \min\{|i_{1U} - i_{2U}|\}$, $\Delta i_L = |i_{1L} - i_{2L}|$, $\Delta i_U = |i_{1U} - i_{2U}|$, $\Delta i_{L(max)} = \max\{|i_{1L} - i_{2L}|\}$, $\Delta i_{U(max)} = \max\{|i_{1U} - i_{2U}|\}$, $\Delta n_{L(min)} = \min\{|n_{1L} - n_{2L}|\}$, $\Delta n_{U(min)} = \min\{|n_{1U} - n_{2U}|\}$, $\Delta n_L = |n_{1L} - n_{2L}|$, $\Delta n_U = |n_{1U} - n_{2U}|$, $\Delta n_{L(max)} = \max\{|n_{1L} - n_{2L}|\}$, $\Delta n_{U(max)} = \max\{|n_{1U} - n_{2U}|\}$.

2.1.4.8. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, the IvPFWGSM satisfies the following properties:

i. $0 \leq IvPFWGSM^1(\mathcal{P}_1, \mathcal{P}_2) \leq 1$.

ii. $IvPFWGSM^1(\mathcal{P}_1, \mathcal{P}_2) = IvPFWGSM^1(\mathcal{P}_2, \mathcal{P}_1)$.

2.1.5. Some Dice Similarity Measures for IvPFSs

In this subsection, some dice SMs and weighted dice SMs for IvPFSs are defined. Some basic properties of these SMs are discussed.

2.1.5.1. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, some dice SMs for these IvPFNs is defined as

$$IvPFDSM^1(\mathcal{P}_1, \mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \frac{2\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +m_{2U}^2(x_j) + i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

$$IvPFDSM^2(\mathcal{P}_1, \mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \frac{2\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +r_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{pmatrix} + \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +r_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{pmatrix}}$$

$$IvPFDSM^3(\mathcal{P}_1, \mathcal{P}_2) = \frac{\sum_{j=1}^{k} 2\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\sum_{j=1}^{k}\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + \sum_{j=1}^{k}\begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +m_{2U}^2(x_j) + i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

$$IvPFDSM^4(\mathcal{P}_1,\mathcal{P}_2)$$
$$= \frac{2\sum_{j=1}^{k}\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\sum_{j=1}^{k}\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +r_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{pmatrix} + \sum_{j=1}^{k}\begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +r_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{pmatrix}}$$

2.1.5.2. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, all IvPFDSMs satisfy the following properties for $t = 1,2,3,4$:

i. $0 \le IvPFDSM^t(\mathcal{P}_1,\mathcal{P}_2) \le 1$.
ii. $IvPFDSM^t(\mathcal{P}_1,\mathcal{P}_2) = IvPFDSM^t(\mathcal{P}_2,\mathcal{P}_1)$
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFDSM^t(\mathcal{P}_1,\mathcal{P}_2) = 1$.
iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFDSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFDSM^t(\mathcal{P}_1,\mathcal{P}_2)$ and $IvPFDSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFDSM^t(\mathcal{P}_2,\mathcal{P}_3)$

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFDSM^1(\mathcal{P}_1,\mathcal{P}_2)$ belongs to [0, 1].

(ii) Holds trivially.

(iii) If $\mathcal{P}_1 = \mathcal{P}_2$ then $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$ and $n_{1U} = n_{2U}$.

Then

$$IvPFDSM^1(\mathcal{P}_1,\mathcal{P}_2)$$
$$= \frac{1}{k}\sum_{j=1}^{k}\frac{2\left(m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)\right)}{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + \\ m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + \\ m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}$$

$$= \frac{1}{k}\sum_{j=1}^{k}\frac{2\left(m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)\right)}{2\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + \\ m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}$$

$$= \frac{1}{k}\sum_{j=1}^{k} 1$$

$$= \frac{1}{k} k = 1.$$

(iv) For $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, $m_{1L} \leq m_{2L} \leq m_{3L}$ also $m_{1U} \leq m_{2U} \leq m_{3U}$

Similarly, $i_{1L} \leq i_{2L} \leq i_{3L}$, $i_{1U} \leq i_{2U} \leq i_{3U}$, $n_{1L} \geq n_{2L} \geq n_{3L}$ and $n_{1U} \geq n_{2U} \geq n_{3U}$

Now,

$$2\begin{pmatrix} m_{1L}m_{3L} + i_{1L}i_{3L} + n_{1L}n_{3L} \\ +m_{1U}m_{3U} + i_{1U}i_{3U} + n_{1U}n_{3U} \end{pmatrix} \leq 2\begin{pmatrix} m_{1L}m_{2L} + i_{1L}i_{2L} + n_{1L}n_{2L} \\ +m_{1U}m_{2U} + i_{1U}i_{2U} + n_{1U}n_{2U} \end{pmatrix}$$

And

$$\begin{pmatrix} m_{1L}^2 + i_{1L}^2 + n_{1L}^2 \\ +m_{1U}^2 + i_{1U}^2 + n_{1U}^2 \end{pmatrix} + \begin{pmatrix} m_{3L}^2 + i_{3L}^2 + n_{3L}^2 \\ +m_{3U}^2 + i_{3U}^2 + n_{3U}^2 \end{pmatrix} \geq \begin{pmatrix} m_{1L}^2 + i_{1L}^2 + n_{1L}^2 \\ +m_{1U}^2 + i_{1U}^2 + n_{1U}^2 \end{pmatrix} + \begin{pmatrix} m_{2L}^2 + i_{2L}^2 + n_{2L}^2 \\ +m_{2U}^2 + i_{2U}^2 + n_{2U}^2 \end{pmatrix}$$

Clearly $IvPFDSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFDSM^t(\mathcal{P}_1, \mathcal{P}_2)$

Similarly, $IvPFDSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFDSM^t(\mathcal{P}_2, \mathcal{P}_3)$

2.1.5.3. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, some weighted dice SMs between these IvPFNs are defined as

$$IvPFWDSM^1(\mathcal{P}_1, \mathcal{P}_2) = \sum_{j=1}^{k} w_j \frac{2\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +m_{2U}^2(x_j) + i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

$IvPFWDSM^2(\mathcal{P}_1,\mathcal{P}_2)$

$$= \sum_{j=1}^{k} w_j \frac{2\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +r_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{pmatrix} + \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +r_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{pmatrix}}$$

$IvPFWDSM^3(\mathcal{P}_1,\mathcal{P}_2)$

$$= \frac{\sum_{j=1}^{k} 2w_j^2 \begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + \sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) \\ +n_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

$IvPFWDSM^4(\mathcal{P}_1,\mathcal{P}_2)$

$$= \frac{2\sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +r_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{pmatrix} + \sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +r_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{pmatrix}}$$

Where WV $w = (w_1, \dots, w_k)^T$ is with a condition that for $j = 1,2, \dots, k$ $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

2.1.5.4. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, all IvPFWDSMs satisfy the following properties for $t = 1,2,3,4$:

i. $0 \le IvPFWDSM^t(\mathcal{P}_1,\mathcal{P}_2) \le 1$.
ii. $IvPFWDSM^t(\mathcal{P}_1,\mathcal{P}_2) = IvPFWDSM^t(\mathcal{P}_2,\mathcal{P}_1)$
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFWDSM^t(\mathcal{P}_1,\mathcal{P}_2) = 1$.
iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFWDSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFWDSM^t(\mathcal{P}_1,\mathcal{P}_2)$ and $IvPFWDSM^t(\mathcal{P}_1,\mathcal{P}_3) \le IvPFWDSM^t(\mathcal{P}_2,\mathcal{P}_3)$

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFGDSM^1(\mathcal{P}_1, \mathcal{P}_2)$ belongs to [0, 1].

(ii) Holds trivially.

(iii) If $\mathcal{P}_1 = \mathcal{P}_2$ then $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$ and $n_{1U} = n_{2U}$.

Then

$$IvPFWDSM^1(\mathcal{P}_1, \mathcal{P}_2)$$

$$= \sum_{j=1}^{k} w_j \frac{2\left(m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)\right)}{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + \\ m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + \\ m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}$$

$$= \sum_{j=1}^{k} w_j \frac{2\left(m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j)\right)}{2\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) + \\ m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}$$

$$= \sum_{j=1}^{k} w_j = 1.$$

(iv) For $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, $m_{1L} \le m_{2L} \le m_{3L}$ also $m_{1U} \le m_{2U} \le m_{3U}$

Similarly, $i_{1L} \le i_{2L} \le i_{3L}$, $i_{1U} \le i_{2U} \le i_{3U}$, $n_{1L} \ge n_{2L} \ge n_{3L}$ and $n_{1U} \ge n_{2U} \ge n_{3U}$

Now,

$$2\begin{pmatrix} m_{1L}m_{3L} + i_{1L}i_{3L} + n_{1L}n_{3L} \\ +m_{1U}m_{3U} + i_{1U}i_{3U} + n_{1U}n_{3U} \end{pmatrix} \le 2\begin{pmatrix} m_{1L}m_{2L} + i_{1L}i_{2L} + n_{1L}n_{2L} \\ +m_{1U}m_{2U} + i_{1U}i_{2U} + n_{1U}n_{2U} \end{pmatrix}$$

And

$$\begin{pmatrix} m_{1L}^2 + i_{1L}^2 + n_{1L}^2 \\ +m_{1U}^2 + i_{1U}^2 + n_{1U}^2 \end{pmatrix} + \begin{pmatrix} m_{3L}^2 + i_{3L}^2 + n_{3L}^2 \\ +m_{3U}^2 + i_{3U}^2 + n_{3U}^2 \end{pmatrix}$$
$$\ge \begin{pmatrix} m_{1L}^2 + i_{1L}^2 + n_{1L}^2 \\ +m_{1U}^2 + i_{1U}^2 + n_{1U}^2 \end{pmatrix} + \begin{pmatrix} m_{2L}^2 + i_{2L}^2 + n_{2L}^2 \\ +m_{2U}^2 + i_{2U}^2 + n_{2U}^2 \end{pmatrix}$$

Clearly $IvPFWDSM^1(\mathcal{P}_1, \mathcal{P}_3) \le IvPFWDSM^1(\mathcal{P}_1, \mathcal{P}_2)$

Similarly, $IvPFWDSM^1(\mathcal{P}_1, \mathcal{P}_3) \le IvPFWDSM^1(\mathcal{P}_2, \mathcal{P}_3)$

Similarly, all the properties can be proved for $t = 2,3,4$.

2.1.5.5. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, some generalized dice SMs between these IvPFNs are defined as

$$IvPFGDSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \frac{\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\lambda\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + (1-\lambda)\begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) \\ +n_{2L}^2(x_j) + m_{2U}^2(x_j) \\ +i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

$$IvPFGDSM^2(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \frac{\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) + r_{1L}(x_j)r_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\lambda\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +r_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{pmatrix} + (1-\lambda)\begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) + n_{2L}^2(x_j) \\ +r_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{pmatrix}}$$

$$IvPFGDSM^3(\mathcal{P}_1,\mathcal{P}_2) = \frac{\sum_{j=1}^{k}\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\lambda\sum_{j=1}^{k}\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + (1-\lambda)\sum_{j=1}^{k}\begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) \\ +n_{2L}^2(x_j) + m_{2U}^2(x_j) + \\ i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

$$IvPFGDSM^4(\mathcal{P}_1,\mathcal{P}_2) = \frac{\sum_{j=1}^{k}\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\lambda\sum_{j=1}^{k}\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + r_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) \\ +n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{pmatrix} + (1-\lambda)\sum_{j=1}^{k}\begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) \\ +n_{2L}^2(x_j) + r_{2L}^2(x_j) \\ +m_{2U}^2(x_j) + i_{2U}^2(x_j) \\ +n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{pmatrix}}$$

Where $0 \le \lambda \le 1$.

2.1.5.6. Theorem

For any three IvPFNs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ then all IvPFGDSMs satisfy the following properties for $t = 1,2,3,4$:

i. $0 \leq IvPFGDSM^t(\mathcal{P}_1, \mathcal{P}_2) \leq 1$.
ii. $IvPFGDSM^t(\mathcal{P}_1, \mathcal{P}_2) = IvPFGDSM^t(\mathcal{P}_2, \mathcal{P}_1)$.
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFGDSM^t(\mathcal{P}_1, \mathcal{P}_2) = 1$.
iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFGDSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFGDSM^t(\mathcal{P}_1, \mathcal{P}_2)$ and $IvPFGDSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFGDSM^t(\mathcal{P}_2, \mathcal{P}_3)$.

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFGDSM^1(\mathcal{P}_1, \mathcal{P}_2)$ belongs to [0, 1].

(ii) Holds trivially.

(iii) If $\mathcal{P}_1 = \mathcal{P}_2$ then $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$ and $n_{1U} = n_{2U}$

Then,

$$IvPFGDSM^1(\mathcal{P}_1, \mathcal{P}_2)$$

$$= \frac{1}{k}\sum_{j=1}^{k} \frac{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}{\lambda \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + (1-\lambda) \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) \\ +i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}$$

$$= \frac{1}{k}\sum_{j=1}^{k} \frac{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) + n_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}{\begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) + \\ i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix}}$$

$$= \frac{1}{k} k = 1$$

(iv) For $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, $m_{1L} \leq m_{2L} \leq m_{3L}$ also $m_{1U} \leq m_{2U} \leq m_{3U}$

Similarly, $i_{1L} \leq i_{2L} \leq i_{3L}$, $i_{1U} \leq i_{2U} \leq i_{3U}$, $n_{1L} \geq n_{2L} \geq n_{3L}$ and $n_{1U} \geq n_{2U} \geq n_{3U}$

Now,

$$\begin{pmatrix} m_{1L}m_{3L}+i_{1L}i_{3L}+n_{1L}n_{3L} \\ +m_{1U}m_{3U}+i_{1U}i_{3U}+n_{1U}n_{3U} \end{pmatrix} \leq \begin{pmatrix} m_{1L}m_{2L}+i_{1L}i_{2L}+n_{1L}n_{2L} \\ +m_{1U}m_{2U}+i_{1U}i_{2U}+n_{1U}n_{2U} \end{pmatrix}$$

And

$$\lambda\begin{pmatrix} m_{1L}^2+i_{1L}^2 \\ +n_{1L}^2+m_{1U}^2+ \\ i_{1U}^2+n_{1U}^2 \end{pmatrix}+(1-\lambda)\begin{pmatrix} m_{3L}^2+i_{3L}^2 \\ +n_{3L}^2+m_{3U}^2 \\ +i_{3U}^2+n_{3U}^2 \end{pmatrix} \geq \lambda\begin{pmatrix} m_{1L}^2+i_{1L}^2 \\ +n_{1L}^2+m_{1U}^2+ \\ i_{1U}^2+n_{1U}^2 \end{pmatrix}+(1-\lambda)\begin{pmatrix} m_{2L}^2+i_{2L}^2 \\ +n_{2L}^2+m_{2U}^2 \\ +i_{2U}^2+n_{2U}^2 \end{pmatrix}$$

So, $IvPFGDSM^t(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFGDSM^t(\mathcal{P}_1,\mathcal{P}_2)$

Similarly, $IvPFGDSM^t(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFGDSM^t(\mathcal{P}_2,\mathcal{P}_3)$

2.1.5.7. Definition

For any two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$, some weighted generalized dice SMs between IvPFNs are defined as

$$IvPFWGDSM^1(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j \frac{\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j)+i_{1L}(x_j)i_{2L}(x_j)+n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j)+i_{1U}(x_j)i_{2U}(x_j)+n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\lambda\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j) \\ +n_{1L}^2(x_j)+m_{1U}^2(x_j)+ \\ i_{1U}^2(x_j)+n_{1U}^2(x_j) \end{pmatrix}+(1-\lambda)\begin{pmatrix} m_{2L}^2(x_j)+i_{2L}^2(x_j) \\ +n_{2L}^2(x_j)+m_{2U}^2(x_j)+ \\ i_{2U}^2(x_j)+n_{2U}^2(x_j) \end{pmatrix}}$$

$$IvPFWGDSM^2(\mathcal{P}_1,\mathcal{P}_2) = \sum_{j=1}^{k} w_j \frac{\begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j)+i_{1L}(x_j)i_{2L}(x_j)+n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j)+m_{1U}(x_j)m_{2U}(x_j)+ \\ i_{1U}(x_j)i_{2U}(x_j)+n_{1U}(x_j)n_{2U}(x_j)+r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\lambda\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j) \\ +n_{1L}^2(x_j)+r_{1L}^2(x_j) \\ +m_{1U}^2(x_j)+i_{1U}^2(x_j) \\ +n_{1U}^2(x_j)+r_{1U}^2(x_j) \end{pmatrix}+(1-\lambda)\begin{pmatrix} m_{2L}^2(x_j)+i_{2L}^2(x_j) \\ +n_{2L}^2(x_j)+r_{2L}^2(x_j) \\ +m_{2U}^2(x_j)+i_{2U}^2(x_j) \\ +n_{2U}^2(x_j)+r_{2U}^2(x_j) \end{pmatrix}}$$

$IvPFWGDSM^3(\mathcal{P}_1, \mathcal{P}_2)$

$$= \frac{\sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +m_{1U}(x_j)m_{2U}(x_j) + i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{pmatrix}}{\lambda \sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) \\ +i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{pmatrix} + (1-\lambda) \sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) \\ +n_{2L}^2(x_j) + m_{2U}^2(x_j) \\ +i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{pmatrix}}$$

$IvPFWGDSM^4(\mathcal{P}_1, \mathcal{P}_2)$

$$= \frac{\sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) + n_{1L}(x_j)n_{2L}(x_j) \\ +r_{1L}(x_j)r_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) + \\ i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) + r_{1U}(x_j)r_{2U}(x_j) \end{pmatrix}}{\lambda \sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + r_{1L}^2(x_j) \\ +m_{1U}^2(x_j) + i_{1U}^2(x_j) \\ +n_{1U}^2(x_j) + r_{1U}^2(x_j) \end{pmatrix} + (1-\lambda) \sum_{j=1}^{k} w_j^2 \begin{pmatrix} m_{2L}^2(x_j) + i_{2L}^2(x_j) \\ +n_{2L}^2(x_j) + r_{2L}^2(x_j) \\ +m_{2U}^2(x_j) + i_{2U}^2(x_j) \\ +n_{2U}^2(x_j) + r_{2U}^2(x_j) \end{pmatrix}}$$

Where WV $w = (w_1, \dots, w_k)^T$ have a condition that for $j = 1,2, \dots, k$ $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

2.1.5.8. Theorem

For any three IvPFNs $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$, all IvPFWGDSMs satisfy the following properties for $t = 1,2,3,4$:

i. $0 \leq IvPFWGDSM^t(\mathcal{P}_1, \mathcal{P}_2) \leq 1$.
ii. $IvPFWGDSM^t(\mathcal{P}_1, \mathcal{P}_2) = IvPFWGDSM^t(\mathcal{P}_2, \mathcal{P}_1)$
iii. For $\mathcal{P}_1 = \mathcal{P}_2$, $IvPFWGDSM^t(\mathcal{P}_1, \mathcal{P}_2) = 1$.
iv. Consider $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, then $IvPFWGDSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFWGDSM^t(\mathcal{P}_1, \mathcal{P}_2)$ and $IvPFWGDSM^t(\mathcal{P}_1, \mathcal{P}_3) \leq IvPFWGDSM^t(\mathcal{P}_2, \mathcal{P}_3)$

Proof: (i) As membership, abstinence and non-membership of both IvPFNs belong to [0, 1], so it is obvious that $IvPFWGDSM^1(\mathcal{P}_1, \mathcal{P}_2)$ belongs to [0, 1].

(ii) Holds trivially.

(iii) If $\mathcal{P}_1 = \mathcal{P}_2$ then $m_{1L} = m_{2L}$, $m_{1U} = m_{2U}$, $i_{1L} = i_{2L}$, $i_{1U} = i_{2U}$, $n_{1L} = n_{2L}$ and $n_{1U} = n_{2U}$

Then,

$$IvPFWGDSM^1(\mathcal{P}_1,\mathcal{P}_2)$$
$$=\sum_{j=1}^{k} w_j \frac{\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j)+n_{1L}^2(x_j) \\ +m_{1U}^2(x_j)+i_{1U}^2(x_j)+n_{1U}^2(x_j)\end{pmatrix}}{\lambda\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j) \\ +n_{1L}^2(x_j)+m_{1U}^2(x_j)+ \\ i_{1U}^2(x_j)+n_{1U}^2(x_j)\end{pmatrix}+(1-\lambda)\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j) \\ +n_{1L}^2(x_j)+m_{1U}^2(x_j) \\ +i_{1U}^2(x_j)+n_{1U}^2(x_j)\end{pmatrix}}$$

$$=\sum_{j=1}^{k} w_j \frac{\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j)+n_{1L}^2(x_j) \\ +m_{1U}^2(x_j)+i_{1U}^2(x_j)+n_{1U}^2(x_j)\end{pmatrix}}{\begin{pmatrix} m_{1L}^2(x_j)+i_{1L}^2(x_j) \\ +n_{1L}^2(x_j)+m_{1U}^2(x_j)+ \\ i_{1U}^2(x_j)+n_{1U}^2(x_j)\end{pmatrix}}$$

$$=\sum_{j=1}^{k} w_j = 1$$

(iv) For $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \mathcal{P}_3$, $m_{1L} \leq m_{2L} \leq m_{3L}$ also $m_{1U} \leq m_{2U} \leq m_{3U}$

Similarly, $i_{1L} \leq i_{2L} \leq i_{3L}$, $i_{1U} \leq i_{2U} \leq i_{3U}$, $n_{1L} \geq n_{2L} \geq n_{3L}$ and $n_{1U} \geq n_{2U} \geq n_{3U}$

Now,

$$\begin{pmatrix} m_{1L}m_{3L}+i_{1L}i_{3L}+n_{1L}n_{3L} \\ +m_{1U}m_{3U}+i_{1U}i_{3U}+n_{1U}n_{3U}\end{pmatrix} \leq \begin{pmatrix} m_{1L}m_{2L}+i_{1L}i_{2L}+n_{1L}n_{2L} \\ +m_{1U}m_{2U}+i_{1U}i_{2U}+n_{1U}n_{2U}\end{pmatrix}$$

And

$$\lambda\begin{pmatrix} m_{1L}^2+i_{1L}^2 \\ +n_{1L}^2+m_{1U}^2+ \\ i_{1U}^2+n_{1U}^2\end{pmatrix}+(1-\lambda)\begin{pmatrix} m_{3L}^2+i_{3L}^2 \\ +n_{3L}^2+m_{3U}^2 \\ +i_{3U}^2+n_{3U}^2\end{pmatrix}$$
$$\geq \lambda\begin{pmatrix} m_{1L}^2+i_{1L}^2 \\ +n_{1L}^2+m_{1U}^2+ \\ i_{1U}^2+n_{1U}^2\end{pmatrix}+(1-\lambda)\begin{pmatrix} m_{2L}^2+i_{2L}^2 \\ +n_{2L}^2+m_{2U}^2 \\ +i_{2U}^2+n_{2U}^2\end{pmatrix}$$

So, $IvPFWGDSM^t(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFWGDSM^t(\mathcal{P}_1,\mathcal{P}_2)$

Similarly, $IvPFWGDSM^t(\mathcal{P}_1,\mathcal{P}_3) \leq IvPFWGDSM^t(\mathcal{P}_2,\mathcal{P}_3)$

2.2. Applications for Strategy Decision Making and Mineral Fields Recognition

In this section, applications for strategy decision making and mineral fields recognition are developed with the help of numerical examples that show the reliability of proposed SMs.

2.2.1. Numerical Example for Strategy Decision Making

A company wants to launch a new product and board of governors have to decide one strategy. For this purpose, there are three strategies to be selected shown as follows.

1. g_1: Make a product for rich persons
2. g_2: Make a product for every persons
3. g_3: Make a product for poor persons

In order to do the best selection, it is necessary to compare these three strategies with popular product in the existing market, so we give a best strategy g: a popular product in the existing market.

In addition, in order to evaluate these strategies, there are the following five attributes (which WV $= (0.25,0.2,0.15,0.18,0.22)^T$) to be used.

1. S_1: Risk of loss
2. S_2: Barriers in the development of business
3. S_3: Impact on society
4. S_4: Impact on environment
5. S_6: Growth analysis

The decision maker gives the evaluation values for strategies according to attributes which are shown in Table 3.

Table 3 Decision values for Strategy Decision Making

	g_1	g_2	g_3	g
S_1	$\begin{pmatrix}[0.26,0.31], \\ [0.12,0.24], \\ [0.21,0.39]\end{pmatrix}$	$\begin{pmatrix}[0.32,0.37], \\ [0.15,0.28], \\ [0.05,0.12]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.46], \\ [0.1,0.15], \\ [0.31,0.36]\end{pmatrix}$	$\begin{pmatrix}[0.05,0.1], \\ [0.18,0.29], \\ [0.43,0.57]\end{pmatrix}$

S_2	$\begin{pmatrix}[0.25,0.46],\\ [0.03,0.13],\\ [0.17,0.23]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.35],\\ [0.09,0.17],\\ [0.37,0.47]\end{pmatrix}$	$\begin{pmatrix}[0.41,0.56],\\ [0.03,0.09],\\ [0.14,0.27]\end{pmatrix}$	$\begin{pmatrix}[0.45,0.53],\\ [0.1,0.17],\\ [0.01,0.13]\end{pmatrix}$
S_3	$\begin{pmatrix}[0.08,0.26],\\ [0.16,0.37],\\ [0.02,0.29]\end{pmatrix}$	$\begin{pmatrix}[0.25,0.31],\\ [0.21,0.29],\\ [0.3,0.39]\end{pmatrix}$	$\begin{pmatrix}[0.07,0.16],\\ [0.24,0.32],\\ [0.47,0.51]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.41],\\ [0.07,0.17],\\ [0.11,0.26]\end{pmatrix}$
S_4	$\begin{pmatrix}[0.2,0.4],\\ [0.1,0.3],\\ [0.1,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.25],\\ [0.13,0.19],\\ [0.41,0.53]\end{pmatrix}$	$\begin{pmatrix}[0.17,0.21],\\ [0.07,0.14],\\ [0.51,0.61]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.28],\\ [0.12,0.24],\\ [0.06,0.36]\end{pmatrix}$
S_5	$\begin{pmatrix}[0.48,0.57],\\ [0.22,0.3],\\ [0.0,0.07]\end{pmatrix}$	$\begin{pmatrix}[0.31,0.41],\\ [0.02,0.09],\\ [0.39,0.47]\end{pmatrix}$	$\begin{pmatrix}[0.35,0.39],\\ [0.11,0.23],\\ [0.06,0.21]\end{pmatrix}$	$\begin{pmatrix}[0.19,0.31],\\ [0.04,0.08],\\ [0.49,0.59]\end{pmatrix}$

The SM of three alternatives g_1, g_2 and g_3 with g with respect to WV $w = (0.25,0.2,0.15,0.18,0.22)^T$ are calculated by using the formulas of SMs, which are shown in Table 4.

Table 4 Similarity Measures for Strategy Decision Making

SM's	(g_1, g)	(g_2, g)	(g_3, g)
$IvPFWCSM^1$	**0.7961**	0.7794	0.7898
$IvPFWCSM^2$	**0.8341**	0.7811	0.7758
$IvPFWCsSM^1$	**0.8931**	0.8720	0.8416
$IvPFWCsSM^2$	0.6643	**0.7347**	0.6754
$IvPFWCsSM^3$	**0.8931**	0.8551	0.8404
$IvPFWCsSM^4$	**0.6101**	0.5558	0.4932
$IvPFWCtSM^1$	**0.6574**	0.6195	0.5695
$IvPFWCtSM^2$	**0.6254**	0.5909	0.5679
$IvPFWStSM^1$	**0.7111**	0.6576	0.6405

$IvPFWGSM^1$	0.7843	0.7937	**0.8168**
$IvPFWDSM^1$	**0.7886**	0.7621	0.7625
$IvPFWDSM^2$	**0.8317**	0.7791	0.7744
$IvPFWDSM^3$	0.7396	0.7592	**0.7677**
$IvPFWDSM^4$	**0.2772**	0.2166	0.2738
$IvPFWGDSM^1$	0.7537	0.7948	**0.8021**
$IvPFWGDSM^2$	**0.8187**	0.7477	0.7578
$IvPFWGDSM^3$	0.6997	0.7457	**0.7665**
$IvPFWGDSM^4$	**0.2716**	0.2078	0.2668

From Table 4, we can know the different similarity definitions can get the different SMs, however, in 18 SMs, there are 13 SMs in which $(g_1, g)$is the biggest, there are one SM in which $(g_2, g)$is the biggest, and there are four SMs in which $(g_3, g)$is the biggest.

So we can get g_1 is best option for company is to launch product for rich persons.

2.2.2. Numerical Example for Mineral Fields Recognition

Let us consider three kinds of mineral fields g_1, g_2 and g_3. Each of them is featured by five minerals $\{s_1, s_2, s_3, s_4, s_5\}$ and the WV of minerals is $(0.25,0.2,0.15,0.18,0.22)^T$.

Now consider an existing best mineral field g and we have to check that which field is most similar to g. Experts evaluate each field under the consideration of five minerals as listed in Table 5.

Table 5 Decision Values for Mineral Fields Recognition

	g_1	g_2	g_3	g
s_1	$\begin{pmatrix}[0.37,0.49],\\ [0.03,0.11],\\ [0.34,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.33],\\ [0.13,0.20],\\ [0.11,0.19]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.35],\\ [0.07,0.18],\\ [0.22,0.32]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.28],\\ [0.07,0.15],\\ [0.31,0.50]\end{pmatrix}$

s_2	$\begin{pmatrix}[0.07,0.23],\\ [0.11,0.29],\\ [0.21,0.33]\end{pmatrix}$	$\begin{pmatrix}[0.13,0.31],\\ [0.02,0.13],\\ [0.22,0.44]\end{pmatrix}$	$\begin{pmatrix}[0.26,0.44],\\ [0.02,0.08],\\ [0.16,0.27]\end{pmatrix}$	$\begin{pmatrix}[0.33,0.51],\\ [0.02,0.17],\\ [0.20,0.21]\end{pmatrix}$
s_3	$\begin{pmatrix}[0.27,0.36],\\ [0.09,0.19],\\ [0.13,0.18]\end{pmatrix}$	$\begin{pmatrix}[0.09,0.19],\\ [0.17,0.31],\\ [0.22,0.36]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.19],\\ [0.21,0.32],\\ [0.36,0.41]\end{pmatrix}$	$\begin{pmatrix}[0.17,0.37],\\ [0.04,0.14],\\ [0.22,0.36]\end{pmatrix}$
s_4	$\begin{pmatrix}[0.09,0.43],\\ [0.12,0.21],\\ [0.14,0.35]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.21],\\ [0.08,0.13],\\ [0.24,0.49]\end{pmatrix}$	$\begin{pmatrix}[0.13,0.19],\\ [0.08,0.22],\\ [0.48,0.58]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.24],\\ [0.11,0.21],\\ [0.36,0.49]\end{pmatrix}$
s_5	$\begin{pmatrix}[0.16,0.48],\\ [0.14,0.30],\\ [0.01,0.11]\end{pmatrix}$	$\begin{pmatrix}[0.13,0.34],\\ [0.01,0.23],\\ [0.31,0.42]\end{pmatrix}$	$\begin{pmatrix}[0.28,0.38],\\ [0.10,0.20],\\ [0.14,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.15,0.26],\\ [0.09,0.17],\\ [0.43,0.56]\end{pmatrix}$

The SMs of three alternatives with g with respect to WV $w = (0.25,0.2,0.15,0.18,0.22)^T$ are calculated by using the formulas of SMs, which are shown in Table 6.

Table 6 Similarity Measures for Mineral Fields Recognition

SM's	(g_1, g)	(g_2, g)	(g_3, g)
$IvPFWCSM^1$	0.8154	0.9028	**0.9382**
$IvPFWCSM^2$	0.8413	0.9100	**0.9255**
$IvPFWCsSM^1$	0.8983	0.9436	**0.9565**
$IvPFWCsSM^2$	0.7963	0.8998	**0.9112**
$IvPFWCsSM^3$	0.8963	0.9342	**0.9472**
$IvPFWCsSM^4$	0.6449	0.8118	**0.8305**
$IvPFWCtSM^1$	0.6468	0.7273	**0.7607**
$IvPFWCtSM^2$	0.6093	0.6986	**0.7379**
$IvPFWStSM^1$	0.6911	0.7701	**0.7952**
$IvPFWGSM^1$	0.7426	0.7702	**0.8132**

$IvPFWDSM^1$	0.8030	0.8877	**0.9252**
$IvPFWDSM^2$	0.8396	0.9092	**0.9244**
$IvPFWDSM^3$	0.8010	0.8902	**0.9252**
$IvPFWDSM^4$	0.3039	0.2822	**0.3195**
$IvPFWGDSM^1$	0.7736	0.8122	**0.8990**
$IvPFWGDSM^2$	0.8584	0.9129	**0.9260**
$IvPFWGDSM^3$	0.7702	0.8067	**0.8851**
$IvPFWGDSM^4$	0.3122	0.2832	**0.3200**

From Table 6, we can obtain that the g_3 is most similar to g, so we can select the g_3.

2.3. Advantages

In this section, we explain the advantages of the proposed SMs.

2.3.1. Some special cases

We prove the generalization of proposed works. For this, we consider two IvPFNs $\mathcal{P}_1$ and $\mathcal{P}_2$

$$IvPFCSM^1(\mathcal{P}_1, \mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k} \frac{\begin{gathered} m_{1L}(x_j)m_{2L}(x_j) + i_{1L}(x_j)i_{2L}(x_j) \\ +n_{1L}(x_j)n_{2L}(x_j) + m_{1U}(x_j)m_{2U}(x_j) \\ +i_{1U}(x_j)i_{2U}(x_j) + n_{1U}(x_j)n_{2U}(x_j) \end{gathered}}{\sqrt{\begin{gathered} m_{1L}^2(x_j) + i_{1L}^2(x_j) \\ +n_{1L}^2(x_j) + m_{1U}^2(x_j) \\ +i_{1U}^2(x_j) + n_{1U}^2(x_j) \end{gathered}}\sqrt{\begin{gathered} m_{2L}^2(x_j) + i_{2L}^2(x_j) \\ +n_{2L}^2(x_j) + m_{2U}^2(x_j) \\ +i_{2U}^2(x_j) + n_{2U}^2(x_j) \end{gathered}}} \tag{2.3.1.1}$$

1. When lower and upper value of intervals becomes equal, then the equation (2.3.1.1) becomes SM for PFSs.

$$PFCSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\frac{m_1(x_j)m_2(x_j)+i_1(x_j)i_2(x_j)+n_1(x_j)n_2(x_j)}{\sqrt{m_1^2(x_j)+i_1^2(x_j)+n_1^2(x_j)}\sqrt{m_2^2(x_j)+i_2^2(x_j)+n_2^2(x_j)}}$$

2. For $i_1 = [0,0]$ and $i_2 = [0,0]$, the equation (2.3.1.1) becomes SM for interval valued intuitionistic fuzzy number.

$$IvIFCSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\frac{\begin{array}{c}m_{1L}(x_j)m_{2L}(x_j)+n_{1L}(x_j)n_{2L}(x_j)+\\ m_{1U}(x_j)m_{2U}(x_j)+n_{2U}(x_j)n_{2U}(x_j)\end{array}}{\sqrt{\begin{array}{c}m_{1L}^2(x_j)+n_{1L}^2(x_j)\\ +m_{1U}^2(x_j)+n_{1U}^2(x_j)\end{array}}\sqrt{\begin{array}{c}m_{2L}^2(x_j)+n_{2L}^2(x_j)\\ +m_{2U}^2(x_j)+n_{2U}^2(x_j)\end{array}}}$$

3. For $i_1 = [0,0]$ and $i_2 = [0,0]$ and the upper and lower values of membership and non-membership intervals become equal, then the equation (2.3.1.1) becomes SM for intuitionistic fuzzy number

$$IFCSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\frac{m_1(x_j)m_2(x_j)+n_1(x_j)n_2(x_j)}{\sqrt{m_1^2(x_j)+n_1^2(x_j)}\sqrt{m_2^2(x_j)+n_2^2(x_j)}}$$

4. For $i_1 = [0,0], n_1 = [0,0]$ and $i_2 = [0,0]$, $n_2 = [0,0]$, the equation (2.3.1.1) becomes SM for IvFN.

$$IvFCSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\frac{m_{1L}(x_j)m_{2L}(x_j)+m_{1U}(x_j)m_{2U}(x_j)}{\sqrt{m_{1L}^2(x_j)+m_{1U}^2(x_j)}\sqrt{m_{2L}^2(x_j)+m_{2U}^2(x_j)}}$$

5. For $i_1 = [0,0], n_1 = [0,0]$ and $i_2 = [0,0]$, $n_2 = [0,0]$ and the upper and lower values of membership intervals become equal, then the equation (2.3.1.1) becomes SM for FN

$$FCSM^1(\mathcal{P}_1,\mathcal{P}_2) = \frac{1}{k}\sum_{j=1}^{k}\frac{m_1(x_j)m_2(x_j)}{\sqrt{m_1^2(x_j)+m_2^2(x_j)}}$$

Similarly, we can reduce all other similarities in interval-valued intuitionistic, intuitionistic and picture fuzzy environment.

So, we can know the proposed SMs are more general than some existing SMs.

2.3.2. Comparative Study

The main advantage of proposed works is that the existing SMs cannot handle the information given in IvPFNs, but they can handle the information given in intuitionistic, interval-valued intuitionistic and picture fuzzy environment. Hence the proposed SMs are more generalized than those of existing SMs.

2.3.2.1. Example

Here, an example for interval-valued intuitionistic fuzzy information has been taken from [29] and solved by the proposed SMs. A company wants to invest its capital in some business and they have four alternatives $\{g_1, g_2, g_3, g_4\}$ and must select one from these alternatives. So They evaluate these alternatives on the base of three attributes $\{s_1, s_2, s_3\}$ with a WV $(0.35,0.25,0.40)^T$, the evaluation values are shown in table 7.

Now we can use the SM of each alternative with the ideal alternative to select the best one. The given data is listed in Table 7.

Table 7 Decision Makers for Comparative Study

	s_1	s_2	s_3
g_1	$\begin{pmatrix}[0.4,0.5],\\ [0.0,0.0]\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.6],\\ [0.0,0.0]\\ [0.2,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.1,0.3],\\ [0.0,0.0]\\ [0.5,0.6]\end{pmatrix}$
g_2	$\begin{pmatrix}[0.6,0.7],\\ [0.0,0.0]\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.7],\\ [0.0,0.0]\\ [0.2,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.4,0.7],\\ [0.0,0.0]\\ [0.1,0.2]\end{pmatrix}$
g_3	$\begin{pmatrix}[0.3,0.6],\\ [0.0,0.0]\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.6],\\ [0.0,0.0]\\ [0.3,0.4]\end{pmatrix}$	$\begin{pmatrix}[0.5,0.6],\\ [0.0,0.0]\\ [0.1,0.3]\end{pmatrix}$
g_4	$\begin{pmatrix}[0.7,0.8],\\ [0.0,0.0]\\ [0.1,0.2]\end{pmatrix}$	$\begin{pmatrix}[0.6,0.7],\\ [0.0,0.0]\\ [0.1,0.3]\end{pmatrix}$	$\begin{pmatrix}[0.3,0.4],\\ [0.0,0.0]\\ [0.1,0.2]\end{pmatrix}$

By using the above information, the interval-valued intuitionistic fuzzy cosine SM (IvIFCSM) can be found as given

$$IvIFCSM^1(g_1, g) = 0.5645,$$

$$IvIFCSM^1(g_2, g) = 0.8637,$$

$$IvIFCSM^1(g_3, g) = 0.7768,$$

$$IvIFCSM^1(g_4, g) = 0.7801.$$

These results are similar as in [29]. So this proves the effectiveness of the proposed work.

Chapter 3

Algorithm for T-Spherical fuzzy multi attribute decision-making based on improved interactive aggregation operators

The objective of this chapter is to present some new improved aggregation operators for the T-SFS, which is an extension of the several existing sets such as IFS, PyFS, q-ROPFS and PFS. In it, some new improved operational laws and their corresponding properties are studied. Further, based on these laws, we propose some geometric aggregation operators and studied their various relationships. Desirable properties as well as some special cases of the proposed operators are studied. Then, based on these proposed operators, we present a decision-making approach to solve the multi-attribute decision making problems. The reliability of the presented decision-making method is explored with the help of numerical example and compared the proposed results with several prevailing studies result. Finally, the superiority of the proposed approach is explained with a counter example to show the advantages of the proposed work.

3.1. Proposed operational laws and aggregation operators

This section is divided into two subsections. Subsection one presents the improved operations laws for the T-SFSs while other presents some improved geometric aggregation operators under the T-SFS environment.

3.1.1. Improved Operational laws

In this subsection, we present some new improved operations laws by incorporating the features of the degree of refusal into the analysis.

3.1.1.1. Definition

Let $\mathcal{T}_1 = (m_1, i_1, n_1)$ and $\mathcal{T}_2 = (m_2, i_2, n_2)$ be two T-SFNs. Then, the proposed operational laws are defined as

1. $\mathcal{T}_1 \otimes_i \mathcal{T}_2 =$

$$\left(\begin{array}{c}\sqrt[t]{(1-n_1^t)(1-n_2^t)-(1-m_1^t-i_1^t-n_1^t)(1-m_2^t-i_2^t-n_2^t)-i_1^t i_2^t},\\ \sqrt[t]{1-(1-i_1^t)(1-i_2^t)},\ \sqrt[t]{1-(1-n_1^t)(1-n_2^t)}\end{array}\right)$$

2. $\mathcal{T}_1^{\tau} = \left(\begin{array}{c}\sqrt[t]{(1-n_1^t)^{\tau}-(1-m_1^t-i_1^t-n_1^t)^{\tau}-(i_1^t)^{\tau}},\\ \sqrt[t]{1-(1-i_1^t)^{\tau}},\sqrt[t]{1-(1-n_1^t)^{\tau}}\end{array}\right)$

For two T-SFNs $\mathcal{T}_1 = (m_1, i_1, n_1)$ and $\mathcal{T}_2 = (m_2, i_2, n_2)$, new operations of multiplication can be construed from four aspects such as between;

1. Two non-membership functions of different T-SFNs.
2. Two membership functions of different T-SFNs.
3. Membership and non-membership functions of different T-SFNs.
4. Two neutral functions of different T-SFNs.

These multiplication rules are of the form:

1. $E(n_1, n_2) = n_1 . n_2$. Therefore, $n_{\mathcal{T}_1 \otimes \mathcal{T}_2} = \sqrt[t]{(n_1^t + n_2^t - n_1^t n_2^t)}$ is considered as probability non-membership (PN) function operator i.e.

$$PN(n_1, n_2) = \sqrt[t]{n_1^t + n_2^t - n_1^t n_2^t}$$

2. $E(m_1, m_2) = (m_1 + i_1).(m_2 + i_2)$. Therefore,

$m_{\mathcal{T}_1 \otimes \mathcal{T}_2} = \sqrt[t]{1-(1-(m_1^t+i_1^t)(1-(m_2^t+i_2^t))}$ is considered as probability membership (PM) function operator i.e.

$$PM(m_1, m_2) = \sqrt[t]{1-\big(1-(m_1^t+i_1^t)\big)\big(1-(m_2^t+i_2^t)\big)}$$

3. $I(n_1, m_2) = \sqrt[t]{(m_2^t + i_2^t) n_1^t . I(n_1, m_2)}$ is considered as probability heterogeneous (PH) function operator i.e.

$$PH(n_1, m_2) = \sqrt[t]{m_2^t n_1^t + i_2^t n_1^t}$$

4. $I(i_1, i_2) = i_1 . i_2$. Therefore, $i_{\mathcal{T}_1 \otimes \mathcal{T}_2} = \sqrt[t]{(i_1^t + i_2^t - i_1^t i_2^t). i_{\mathcal{T}_1 \otimes \mathcal{T}_2}}$ is considered as probability neutral (PNe) function operator, i.e.

$$PNe(i_1, i_2) = \sqrt[t]{i_1^t + i_2^t - i_1^t i_2^t}$$

From the proposed laws, it is observed that the several existing laws can be considered as a special case of it. For instance,

1. For $t = 2$, above operations become valid for SFNs.
2. For $t = 1$, above operations become valid for PFNs.
3. For $t = 2$ and $i = 0$, above operations become valid for PyFNs.
4. For $t = 1$ and $i = 0$, above operations become valid for IFNs.

Further, it is observed that the above defined PN, PH satisfies the following properties.

3.1.1.2. Theorem

Let $\mathcal{T}_1 = (m_1, i_1, n_1), \mathcal{T}_2 = (m_2, i_2, n_2), \mathcal{T}_3 = (m_3, i_3, n_3)$ and $\mathcal{T}_4 = (m_4, i_4, n_4)$ be four T-SFNs. Then, we have

1. Boundedness: $PN(1,1) = 1, PN(0,0) = 0, 0 \le PN(n_1, n_2) \le 1$.
2. Monotonicity: If $n_1 \le n_3$ and $n_2 \le n_4$. Then $PN(n_1, n_2) \le PN(n_3, n_4)$.
3. Commutativity: $PN(n_1, n_2) = PN(n_2, n_1)$.

Proof:

1. For two T-SFNs, $\mathcal{T}_1$ and $\mathcal{T}_2$, and by definition of PN, we have $PN(n_1, n_2) = \sqrt[t]{n_1^t + n_2^t - n_1^t n_2^t}$. Thus, we have PN(1,1) =1 and PN(0,0)=0. Further, since $n_1, n_2 \in [0,1]$ and $t \in Z$ which implies that $n_1^t + n_2^t - n_1^t n_2^t = 1 - (1 - n_1^t)(1 - n_2^t) \le 1$. Also, $PN(n_1, n_2) \ge 0$. Therefore, $0 \le PN(n_1, n_2) \le 1$.

2. Since $n_1 \le n_3$ and $n_2 \le n_4$. Thus for any $t \in Z$, we get $1 - n_1^t \ge 1 - n_3^t$ and $1 - n_2^t \ge 1 - n_4^t$ and hence $1 - (1 - n_1^t)(1 - n_2^t) \le 1 - (1 - n_3^t)(1 - n_4^t)$. Thus, $PN(n_1, n_2) \le PN(n_3, n_4)$ holds.

3. Holds trivial.

3.1.1.3. Theorem

Let $\mathcal{T}_1 = (m_1, i_1, n_1), \mathcal{T}_2 = (m_2, i_2, n_2), \mathcal{T}_3 = (m_3, i_3, n_3)$ and $\mathcal{T}_4 = (m_4, i_4, n_4)$ be four T-SFN. Then

1. Boundedness: $PH(1,0,1) = 1, PH(0,0,0) = 0, 0 \le PH(m_1, i_1, n_1) \le 1$.

2. Monotonicity: If $m_1 \leq m_3, i_1 \leq i_3$ and $n_2 \leq n_4$. Then $PH(m_1, i_1, n_2) \leq PN(m_3, i_3, n_4)$ and if $n_1 \leq n_3, i_2 \leq i_4$ and $m_2 \leq m_4$. Then $PH(n_1, i_2, n_2) \leq PH(n_3, i_4, m_4)$
3. Commutativity: $PH(m_1, i_1, n_1) = PH(n_1, i_1, m_1)$.

3.1.1.4. Theorem

If $\mathcal{T}_1$ and $\mathcal{T}_2$be two T-SFNs and $\lambda > 0$ be a real number, then $\mathcal{T}_1 \otimes \mathcal{T}_2$ $and \mathcal{T}_1^{\lambda}$ are also T-SFNs.

3.1.1.5. Theorem

Let $\mathcal{T}_1 = (m_1, i_1, n_1)$, $\mathcal{T}_2 = (m_2, i_2, n_2)$ be a T-SFNs, $\lambda, \lambda_1, \lambda_2 > 0$ be real numbers. Then we have

1. $\mathcal{T}_1 \otimes_i \mathcal{T}_2 = \mathcal{T}_2 \otimes_i \mathcal{T}_1$
2. $(\mathcal{T}_1 \otimes_i \mathcal{T}_2)^{\lambda} = \mathcal{T}_1^{\lambda} \otimes_i \mathcal{T}_2^{\lambda}$
3. $\mathcal{T}_1^{\lambda_1} \otimes_i \mathcal{T}_1^{\lambda_2} = \mathcal{T}_1^{\lambda_1 + \lambda_2}$.

3.1.2. Aggregation operators

In this section, based on the above proposed operational laws, we have proposed some series of geometric interactive improved aggregation operators namely, T-spherical fuzzy weighted geometric interactive averaging (T-SFWGIA) operator, T-spherical fuzzy ordered weighted geometric interactive averaging (T-SFOWGIA) operator and T-spherical fuzzy hybrid geometric interactive averaging (T-SFHGIA) operator under the T-SFS environment.

3.1.2.1. Definition

For any collection $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ $(j = 1,2,3, \ldots, k)$ of T-SFNs. If the mapping

$$T - SFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k) = \otimes_{i\, j=1}^{k} \mathcal{T}_j^{w_j}$$

then $T - SFWGIA_w$ is called T-SFWGIA operator. where $w = (w_1, w_2, \ldots, w_k)^T$ is the WV of $\mathcal{T}_j$ with $w_j \in (0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

3.1.2.2. Theorem

For any collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ $(j = 1,2,3, \dots, k)$, the aggregated values obtained by using Definition 3.1.1.1 is still T-SFNs and is given by

$$T - SFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{k}(1-n_j^t)^{w_j} - \prod_{j=1}^{k}(1-m_j^t-i_j^t-n_j^t)^{w_j} - \prod_{j=1}^{k}(i_j^t)^{w_j}}, \\ \sqrt[t]{1-\prod_{j=1}^{k}(1-i_j^t)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{k}(1-n_j^t)^{w_j}} \end{array} \right)$$

Proof: For any collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ $(j = 1,2,3, \dots, k)$, we shall proof the result by induction on k.

For $k = 1$, we have

$$T - SFWGIA_w(\mathcal{T}_1) = \mathcal{T}_1^{w_1} = (m_1, i_1, n_1)$$

$$= \left(\sqrt[t]{(1-n_1^t)^1 - \left(1-(m_1^t+i_1^t+n_1^t)\right)^1 - (i_1^t)^1}, \sqrt[t]{1-1+(i_1^t)^1}, \sqrt[t]{1-1+(n_1^t)^1} \right)$$

Thus, hold for $k = 1$. Now, result holds for $k = l$

$$T - SFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_l) = \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{l}(1-n_j^t)^{w_j} - \prod_{j=1}^{l}(1-m_j^t-i_j^t-n_j^t)^{w_j} - \prod_{j=1}^{l}(i_j^t)^{w_j}}, \\ \sqrt[t]{1-\prod_{j=1}^{l}(1-i_j^t)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{l}(1-n_j^t)^{w_j}} \end{array} \right)$$

Then for $k = l + 1$, we have

$$T - SFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{l+1}) = \otimes_{i\,j=1}^{\,l+1} \mathcal{T}_j^{w_j}$$

$$= T - SFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_l) \otimes_i \mathcal{T}_{l+1}^{w_{l+1}}$$

$$=\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{l}\left(1-n_j^t\right)^{w_j}-\prod_{j=1}^{l}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}-\prod_{j=1}^{l}(i_j^t)^{w_j}},\\ \sqrt[t]{1-\prod_{j=1}^{l}\left(1-i_j^t\right)^{w_j}},\sqrt[t]{1-\prod_{j=1}^{l}\left(1-n_j^t\right)^{w_j}}\end{array}\right)$$

$$\otimes_i\left(\begin{array}{c}\sqrt[t]{\left(1-n_j^t\right)^{w_j}-\left(1-\mathrm{m_j^t}-\mathrm{i_j^t}-n_j^t\right)^{\mathrm{w_j}}-(i_j^t)^{w_j}},\\ \sqrt[t]{1-\left(1-i_j^t\right)^{w_j}},\sqrt[t]{1-\left(1-n_j^t\right)^{w_j}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{l+1}\left(1-n_j^t\right)^{w_j}-\prod_{j=1}^{l+1}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}-\prod_{j=1}^{l+1}(i_j^t)^{w_j}},\\ \sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-i_j^t\right)^{w_j}},\sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-n_j^t\right)^{w_j}}\end{array}\right)$$

So, the result holds for $k=l+1$. Therefore, by mathematical induction, result holds for all $k\in Z^{+}$.

3.1.2.3. Theorem

If $\mathcal{T}_j=(m_j,i_j,n_j),\ \ j=1,\dots,k$ be T-SFNs. Then the aggregated value using the T-SFWGIA operator is also T-SFN.

Proof: Since $\mathcal{T}_j=(m_j,i_j,n_j)$ be a T-SFN, $j=1,\dots,k$, we have $0\le m_j,i_j,n_j\le 1$. So $0\le m_j^t,i_j^t,n_j^t\le 1$ and $0\le m_j^t+i_j^t+n_j^t\le 1$. Then

$$0\ \le\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}-\prod_{j=1}^{k}(i_j^t)^{w_j}\le 1$$

$$0\ \le 1-\prod_{j=1}^{k}(1-i_j^t)^{w_j}\le 1$$

$$0\ \le 1-\prod_{j=1}^{k}(1-n_j^t)^{w_j}\le 1$$

Now

$$\sqrt[t]{\begin{array}{c}\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j^t+i_j^t+n_j^t\right)\right)^{w_j}-\prod_{j=1}^{k}(i_j^t)^{w_j}+\\ 1-\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}+1-\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}\end{array}}$$

$$=\sqrt[t]{2-\prod_{j=1}^{k}\left(1-\left(m_j^t+i_j^t+n_j^t\right)\right)^{w_j}-\prod_{j=1}^{k}(i_j^t)^{w_j}-\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}}\in[0,1]$$

Thus, $T-SFWGIA_w(\mathcal{T}_1,\ldots,\mathcal{T}_k)$ is T-SFN.

Further, it is observed that the proposed operator satisfies certain properties which are listed as follows.

3.1.2.4. Theorem

If all T-SFNs $\mathcal{T}_j$ $(j=1,2,\ldots,k)$ are equal to $\mathcal{T}_0$ where $\mathcal{T}_0$ is another T-SFN then

$$T-SFWGIA_w(\mathcal{T}_1,\ldots,\mathcal{T}_k)=\mathcal{T}_0$$

Proof: Assume that $\mathcal{T}_j=\mathcal{T}_0=(m_0,i_0,n_0)$ is a T-SFN $\forall j$. Then, by definition of T-SFWGIA operator, we have

$$T-SFWGIA_w(\mathcal{T}_1,\mathcal{T}_2,\ldots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j^t+i_j^t+n_j^t\right)\right)^{w_j}-\prod_{j=1}^{k}(i_j^t)^{w_j}},\\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}},\sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{\left(1-n_j^t\right)^{\sum_{j=1}^{k}w_j}-\left(1-\left(m_j^t+i_j^t+n_j^t\right)\right)^{\sum_{j=1}^{k}w_j}-(i_j^t)^{\sum_{j=1}^{k}w_j}},\\ \sqrt[t]{1-\left(1-i_j^t\right)^{\sum_{j=1}^{k}w_j}},\sqrt[t]{1-\left(1-n_j^t\right)^{\sum_{j=1}^{k}w_j}}\end{array}\right)$$

$$=(m_0,i_0,n_0)$$

$$=\mathcal{T}_0$$

3.1.2.5. Theorem

If $\mathcal{T}_j = (m_j, i_j, n_j)$ is a T-SFN and

$\mathcal{T}^L = (max\{0, (min(m_j + i_j + n_j) - min\, i_j - max\, n_j)\}, min\, i_j, max\, n_j)$,

$\mathcal{T}^U = \left((max(m_j + i_j + n_j) - max\, i_j - min\, n_j), max\, i_j, min\, n_j\right)$. Then, we have

$$\mathcal{T}^L \leq T - SFWGIA_w(\mathcal{T}_1, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

3.1.2.6. Theorem

For a collection of two different T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ and $\mathcal{T}_j' = \left(m_j', i_j', n_j'\right)$ for all $j = 1,2, \dots, k$ which satisfy the following inequalities if $n_j \geq n_j', i_j \leq i_j'$ and $m_j^t + i_j^t + n_j^t \leq \left(m_j'\right)^t + \left(i_j'\right)^t + \left(n_j'\right)^t\ \forall j$, then we have

$$T - SFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T - SFWGIA_w(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

Proof: Since $n_j \geq n_j'$, we have

$$\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_j^t\right)^{w_j}} \geq \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - \left(n_j'\right)^t\right)^{w_j}}$$

and $i_j \leq i_j'$,

$$\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i_j^t\right)^{w_j}} \leq \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - \left(i_j'\right)^t\right)^{w_j}}$$

As, $m_j^t + i_j^t + n_j^t \leq \left(m_j'\right)^t + \left(i_j'\right)^t + \left(n_j'\right)^t\ \forall j$ we have

$$\left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{k}\left(1 - n_j^t\right)^{w_j} - \prod_{j=1}^{k}\left(1 - \left(m_j^t + i_j^t + n_j^t\right)\right)^{\mathrm{w_j}} - \prod_{j=1}^{k}(i_j^t)^{w_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i_j^t\right)^{w_j}}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_j^t\right)^{w_j}} \end{array}\right)$$

$$\leq \left(\sqrt[t]{\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t\right)^{w_j} - \prod_{j=1}^{k}\left(1-\left(\left(m_j'\right)^t+\left(i_j'\right)^t+\left(n_j'\right)^t\right)\right)^{\mathrm{w_j}} - \prod_{j=1}^{k}\left(i_j'\right)^{t w_j}}, \atop \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(i_j'\right)^t\right)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t\right)^{w_j}} \right)$$

Therefore, we have

$$T-SFWGIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFWGIA_w(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

3.1.2.7. Definition

For any collection $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$, $(j=1,2,,\dots,k)$ of T-SFNs. The $T-SFOWGIA_{\omega}\colon \Omega^n \to \Omega$ is a mapping defined as

$$T-SFOWGIA_{\omega}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \otimes i_{j=1}^{k}\, \mathcal{T}_{\sigma(j)}^{\omega_j}$$

then $T-SFOWGIA_{\omega}$ is called T-SFOWGIA operator, where $\omega=(\omega_1,\omega_2,\dots,\omega_k)^T$ is the associated WV of $\mathcal{T}_j$ with $\omega_j \in (0,1]$ and $\sum_{j=1}^{k}\omega_j=1$ and σ is the permutation of $\{1,2,\dots,k\}$ such that $\sigma(j-1)\geq\sigma(j)$.

3.1.2.8. Theorem

For any collection $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$, $(j=1,2,\dots,k)$ of T-SFNs. Then

$$T-SFOWGIA_{\omega}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$= \left(\sqrt[t]{\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\left(m_{\sigma(j)}^t+i_{\sigma(j)}^t+n_{\sigma(j)}^t\right)\right)^{\omega_j} - \prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\omega_j}}, \atop \sqrt[t]{1-\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^t\right)^{\omega_j}}, \quad \sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\omega_j}} \right)$$

3.1.2.9. Theorem

If $\mathcal{T}_j=(m_j,i_j,n_j)$ is a T-SFN, $j=1,\dots,k$. Then the aggregated value using the T-SFOWGIA operator is also T-SFN.

Proof: Since $\mathcal{T}_{\sigma(j)} = (m_{\sigma(j)}, i_{\sigma(j)}, n_{\sigma(j)})$ be a T-SFN, $j = 1,2, \dots, k$, we have $0 \leq m_{\sigma(j)}, i_{\sigma(j)}, n_{\sigma(j)} \leq 1$. So $0 \leq m_{\sigma(j)}^t, i_{\sigma(j)}^t, n_{\sigma(j)}^t \leq 1$ and $0 \leq m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t \leq 1$. Then

$$0 \leq \prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k} \left(1 - \left(m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t\right)\right)^{\omega_j} - \prod_{j=1}^{k} (i_{\sigma(j)}^t)^{\omega_j} \leq 1$$

$$0 \leq 1 - \prod_{j=1}^{k} \left(1 - i_{\sigma(j)}^t\right)^{\omega_j} \leq 1$$

$$0 \leq 1 - \prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\omega_j} \leq 1$$

Now

$$\sqrt[t]{\begin{array}{c}\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k} \left(1 - \left(m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t\right)\right)^{\omega_j} - \prod_{j=1}^{k} (i_{\sigma(j)}^t)^{\omega_j} + \\ 1 - \prod_{j=1}^{k} \left(1 - i_{\sigma(j)}^t\right)^{\omega_j} + 1 - \prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\omega_j}\end{array}}$$

$$= \sqrt[t]{2 - \prod_{j=1}^{k} \left(1 - \left(m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t\right)\right)^{\omega_j} - \prod_{j=1}^{k} (i_{\sigma(j)}^t)^{\omega_j} - \prod_{j=1}^{k} \left(1 - i_{\sigma(j)}^t\right)^{\omega_j}}$$

$$\in [0,1]$$

Thus, $T - SFOWGIA_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$ is T-SFN.

3.1.2.10. Theorem

$T - SFOWGIA_{\omega}(\mathcal{T}_1, \dots, \mathcal{T}_k) = \mathcal{T}_0$ if $\mathcal{T}_j = \mathcal{T}_0 = (m_j, i_j, n_j)$ is a T-SFN $\forall j$.

Proof: we have

$T - SFOWGIA_{\omega}(\mathcal{T}_1, \dots, \mathcal{T}_k) =$

$$\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-\left(m_{\sigma(j)}^{t}+i_{\sigma(j)}^{t}+n_{\sigma(j)}^{t}\right)\right)^{\omega_j}-\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\omega_j}},\\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^{t}\right)^{\omega_j}},\quad \sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\omega_j}}\end{array}\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{\left(1-n_{\sigma(j)}^{t}\right)^{\sum_{j=1}^{k}\omega_j}-\left(1-\left(m_{\sigma(j)}^{t}+i_{\sigma(j)}^{t}+n_{\sigma(j)}^{t}\right)\right)^{\sum_{j=1}^{k}\omega_j}-\left(i_{\sigma(j)}^{t}\right)^{\sum_{j=1}^{k}\omega_j}},\\ \sqrt[t]{1-\left(1-i_{\sigma(j)}^{t}\right)^{\sum_{j=1}^{k}\omega_j}},\quad \sqrt[t]{1-\left(1-n_{\sigma(j)}^{t}\right)^{\sum_{j=1}^{k}\omega_j}}\end{array}\right)$$

$$=\left(m_{\sigma(0)}, i_{\sigma(0)}, n_{\sigma(0)}\right)$$

$$=\mathcal{T}_0$$

3.1.2.11. Theorem

If $\mathcal{T}_j=(m_j,i_j,n_j)$ is a T-SFN and

$$\mathcal{T}^L=(max\{0,(min(m_j+i_j+n_j)-min\ i_j-max\ n_j)\},min\ i_j,max\ n_j),$$

$$\mathcal{T}^U=\left((max(m_j+i_j+n_j)-max\ i_j-min\ n_j),max\ i_j,min\ n_j\right).$$ Then

$$\mathcal{T}^L\leq T-SFOWGIA_{\omega}(\mathcal{T}_1,\ldots,\mathcal{T}_k)\leq\mathcal{T}^U$$

3.1.2.12. Theorem

$T-SFOWGIA_{\omega}(\mathcal{T}_1,\ldots,\mathcal{T}_k)=T-SFOWGIA_{\omega}(\mathcal{T}_1',\ldots,\mathcal{T}_k')$ if $\mathcal{T}_j=(m_j,i_j,n_j)$ is any permutation of $\mathcal{T}_j'=(m_j',i_j',n_j')$ where $j=1,2,\ldots,k$.

Proof:

$T-SFOWGIA_{\omega}(\mathcal{T}_1',\mathcal{T}_2',\ldots,\mathcal{T}_k')=$

$$\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{k}\left(1-\left(n_{\sigma(j)}'\right)^{t}\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-\left(\left(m_{\sigma(j)}'\right)^{t}+\left(i_{\sigma(j)}'\right)^{t}+\left(n_{\sigma(j)}'\right)^{t}\right)\right)^{\omega_j}-\prod_{j=1}^{k}\left(i_{\sigma(j)}'\right)^{t\omega_j}},\\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(i_{\sigma(j)}'\right)^{t}\right)^{\omega_j}},\sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(n_{\sigma(j)}'\right)^{t}\right)^{\omega_j}}\end{array}\right)$$

$T - SFOWGIA_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) =$

$$\left(\sqrt[t]{\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k} \left(1 - \left(m_{\sigma(j)}^t + i_{\sigma(j)}^t + n_{\sigma(j)}^t\right)\right)^{\omega_j} - \prod_{j=1}^{k} (i_{\sigma(j)}^t)^{\omega_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - i_{\sigma(j)}^t\right)^{\omega_j}}, \sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\omega_j}} \right)$$

If $\mathcal{T}_j{}' = (m_j', i_j', n_j')$ is any permutation of $\mathcal{T}_j = (m_j, i_j, n_j)$ then we have $\mathcal{T}_{\sigma(j)}' = \mathcal{T}_{\sigma(j)}$. Thus $T - SFOWGIA_{\omega}(\mathcal{T}_1{}', \dots, \mathcal{T}_k') = T - SFOWGIA_{\omega}(\mathcal{T}_1, \dots, \mathcal{T}_k)$

3.1.2.13. Definition

For any collection $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ of T-SFNs $(j = 1,2, \dots, k)$. If the mapping

$$T - SFHGA_{w,\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \otimes_{j=1}^{k} \left(\tilde{\mathcal{T}}_{\sigma(j)}\right)^{\omega_j}$$

then $T - SFHGA_{w,\omega}$ is called T-SFHGA operator. where $\tilde{\mathcal{T}}_j = \left(\mathcal{T}_j\right)^{nw_j}$, $\omega = (\omega_1, \dots, \omega_k)^T$ is the associated WV of $\mathcal{T}_j$ with $\omega_j \in (0,1]$ and $\sum_{j=1}^{k} \omega_j = 1$ and $w = (w_1, \dots, w_k)^T$ is the WV of $\mathcal{T}_j$ with $w_j \in (0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

3.1.2.14. Definition

For any collection $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ $(j = 1,2, \dots, k)$ of T-SFNs. If the mapping

$$T - SFHGIA_{w,\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \otimes_{i\,j=1}^{\,k} \tilde{\mathcal{T}}_{\sigma(j)}^{\omega_j}$$

then $T - SFHGIA_{w,\omega}$ is called T-SFHGIA operator. where $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$ is the WV of $\mathcal{T}_j$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{k} \omega_j = 1$ and $w = (w_1, \dots, w_k)^T$ is the WV of $\mathcal{T}_j$ with $w_j \in (0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

3.1.2.15. Theorem

For any collection $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ $(j = 1,2, \dots, k)$ of T-SFNs. Then

$T - SFHGIA_{w,\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$

$$= \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{k}\left(1-\tilde{n}_{\sigma(j)}^{t}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\left(\tilde{m}_{\sigma(j)}^{t}+\tilde{\imath}_{\sigma(j)}^{t}+\tilde{n}_{\sigma(j)}^{t}\right)\right)^{\omega_{\mathrm{j}}} - \prod_{j=1}^{k}\left(\tilde{\imath}_{\sigma(j)}^{t}\right)^{\omega_j}}, \\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\tilde{\imath}_{\sigma(j)}^{t}\right)^{\omega_j}}, \quad \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\tilde{n}_{\sigma(j)}^{t}\right)^{\omega_j}} \end{array} \right)$$

The following example demonstrates these aggregation operators.

3.1.2.16. Example

Let $\mathcal{T}_1 = (0.3,0.8,0.1)$, $\mathcal{T}_2 = (0.4,0.3,0.6)$, $\mathcal{T}_3 = (0.7,0.1,0.5)$, $\mathcal{T}_4 = (0.9,0.4,0.1)$ and $\mathcal{T}_5 = (0.2,0.6,0.7)$ are T-SFN. The WV for $\mathcal{T}_j$ $(j = 1,2,..,5)$ is $w = (0.18,0.22,0.16,0.21,0.23)^T$. With loss of generality, we use $t = 2$ for all calculations.

Firstly, we utilize T-SFHGIA operators on this data to aggregate it.

$$\mathcal{T}_1 = \left(\begin{array}{c} \sqrt{(1-0.1^2)^{5\times0.18} - \left(1-(0.3^2+0.8^2+0.1^2)\right)^{5\times0.18} - (0.8^2)^{5\times0.18}}, \\ \sqrt{1-(1-0.8^2)^{5\times0.18}}, \sqrt{1-(1-0.1^2)^{5\times0.18}} \end{array} \right)$$

$$= (0.1559, 0.7754, 0.0949)$$

$$\mathcal{T}_2 = \left(\begin{array}{c} \sqrt{(1-0.6^2)^{5\times0.22} - \left(1-(0.4^2+0.3^2+0.6^2)\right)^{5\times0.22} - (0.3^2)^{5\times0.22}}, \\ \sqrt{1-(1-0.3^2)^{5\times0.22}}, \sqrt{1-(1-0.6^2)^{5\times0.22}} \end{array} \right)$$

$$= (0.4317, 0.3139, 0.6228)$$

$$\mathcal{T}_3 = \left(\begin{array}{c} \sqrt{(1-0.5^2)^{5\times0.16} - \left(1-(0.7^2+0.1^2+0.5^2)\right)^{5\times0.16} - (0.1^2)^{5\times0.16}}, \\ \sqrt{1-(1-0.1^2)^{5\times0.16}}, \sqrt{1-(1-0.5^2)^{5\times0.16}} \end{array} \right)$$

$$= (0.6629, 0.0895, 0.4534)$$

$$\mathcal{T}_4 = \left(\begin{array}{c} \sqrt{(1-0.1^2)^{5\times0.21} - \left(1-(0.9^2+0.4^2+0.1^2)\right)^{5\times0.21} - (0.4^2)^{5\times0.21}}, \\ \sqrt{1-(1-0.4^2)^{5\times0.21}}, \sqrt{1-(1-0.1^2)^{5\times0.21}} \end{array} \right)$$

$$= (0.9094,0.4090,0.1024)$$

$$\mathcal{T}_5 = \left(\begin{array}{c} \sqrt{(1-0.7^2)^{5\times 0.23} - \left(1-(0.2^2+0.6^2+0.7^2)\right)^{5\times 0.23} - (0.6^2)^{5\times 0.23}}, \\ \sqrt{1-(1-0.6^2)^{5\times 0.23}}, \sqrt{1-(1-0.7^2)^{5\times 0.23}} \end{array} \right)$$

$$= (0.2705, 0.6336, 0.7342)$$

The score values corresponding to these aggregated numbers are $SC(\mathcal{T}_1) = 0.0153$, $SC(\mathcal{T}_2) = -0.2016$, $SC(\mathcal{T}_3) = 0.2338$, $SC(\mathcal{T}_4) = 0.8166$, $SC(\mathcal{T}_5) = -0.4658$. Based on score values we have the following arrangement of data.

$\mathcal{T}_{\sigma(1)} = (0.9094, 0.4090, 0.1024)$, $\mathcal{T}_{\sigma(2)} = (0.6629, 0.0895, 0.4534)$, $\mathcal{T}_{\sigma(3)} = (0.1559, 0.7754, 0.0949)$,

$\mathcal{T}_{\sigma(4)} = (0.4317, 0.3139, 0.6228)$, $\mathcal{T}_{\sigma(5)} = (0.2705, 0.6336, 0.7342)$

By using normal distribution-based method, we find $\omega = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$ and by the definition of T-SFHGIA operator we have

$$T - SFHGIA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \tilde{\mathcal{T}}_3, \tilde{\mathcal{T}}_4, \tilde{\mathcal{T}}_5) = (0.4688, 0.5643, 0.4792)$$

3.1.2.17. Theorem

If $\mathcal{T}_j = (m_j, i_j, n_j)$ is a T-SFN, $j = 1, \ldots, k$. Then the aggregated value using the T-SFHGIA operator is also T-SFN.

3.1.2.18. Theorem

$T - SFHGIA_{w,\omega}(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k) = \mathcal{T}_0$ if $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ is a T-SFN $\forall j$.

3.1.2.19. Theorem

If $\mathcal{T}_j = (m_j, i_j, n_j)$ is a T-SFN and

$\mathcal{T}^L = (max\{0, (min(m_j + i_j + n_j) - min\, i_j - max\, n_j)\}, min\, i_j, max\, n_j)$,

$\mathcal{T}^U = \left((max(m_j + i_j + n_j) - max\, i_j - min\, n_j), max\, i_j, min\, n_j\right)$. Then

$$\mathcal{T}^L \leq T - SFHGIA_{w,\omega}(\mathcal{T}_1, \ldots, \mathcal{T}_k) \leq \mathcal{T}^U$$

3.1.2.20. Theorem

$T - SFHGIA_{w,\omega}(\mathcal{T}_1', \dots, \mathcal{T}_k') = T - SFHGIA_{w,\omega}(\mathcal{T}_1, \dots, \mathcal{T}_k)$ if $\mathcal{T}_j' = (m_j', i_j', n_j')$ is any permutation of $\mathcal{T}_j = (m_j, i_j, n_j)$ where $j = 1,2,\dots,k$.

Whenever membership and neutral number of one T-SFN become zero then the membership and abstinence value is not accounted in aggregation [10]. While geometric interaction averaging operators developed in our manuscript overcome this problem. An example will describe it more clearly.

3.1.2.21. Example

Let $\mathcal{T}_1 = (0.7,0.5,0.6), \mathcal{T}_2 = (0.9,0.5,0.4), \mathcal{T}_3 = (0,0,0.1), \mathcal{T}_4 = (0.5,0.3,0.4)$ and $\mathcal{T}_5 = (0.6,0.4,0.5)$ are T-SFN. The WV for $\mathcal{T}_j$ $(j = 1,2,..,5)$ is $w = (0.18,0.22,0.16,0.21,0.23)^T$.

Solution: First we will find T-SFHGA operator.

As, $0.7 + 0.5 + 0.6 = 1.8 \notin [0,1]$, $0.7^2 + 0.5^2 + 0.6^2 = 1.1 \notin [0,1]$ but $0.7^3 + 0.5^3 + 0.6^3 = 0.684 \in [0,1]$

Similarly, $\mathcal{T}_2$ and $\mathcal{T}_4$ satisfy the condition for $t = 3$.

$$\tilde{\mathcal{T}}_1 = \left(\sqrt[3]{(0.7^3 + 0.5^3)^{5\times0.18} - (0.5^3)^{5\times0.18}}, 0.5^{5\times0.18}, \sqrt[3]{1 - (1 - 0.6^3)^{5\times0.18}}\right)$$

$$= (0.7054,0.5359,0.5816)$$

$$\tilde{\mathcal{T}}_2 = \left(\sqrt[3]{(0.9^3 + 0.5^3)^{5\times0.22} - (0.5^3)^{5\times0.22}}, 0.5^{5\times0.22}, \sqrt[3]{1 - (1 - 0.4^3)^{5\times0.22}}\right)$$

$$= (0.9041,0.4665,0.4125)$$

$$\tilde{\mathcal{T}}_3 = \left(\sqrt[3]{(0^3 + 0^3)^{5\times0.16} - (0^3)^{5\times0.16}}, 0^{5\times0.16}, \sqrt[3]{1 - (1 - 0.1^3)^{5\times0.16}}\right)$$

$$= (0,0,0.0928)$$

$$\tilde{\mathcal{T}}_4 = \left(\sqrt[3]{(0.5^3 + 0.3^3)^{5\times0.21} - (0.3^3)^{5\times0.21}}, 0.3^{5\times0.21}, \sqrt[3]{1 - (1 - 0.4^3)^{5\times0.21}}\right)$$

$$= (0.4874,0.2885,0.4063)$$

$$\tilde{\mathcal{T}}_5 =$$

$$\left(\sqrt[3]{(0.6^3+0.4^3)^{5\times0.23}-(0.4^3)^{5\times0.23}}, 0.4^{5\times0.23}, \sqrt[3]{1-(1-0.5^3)^{5\times0.23}}\right)$$

$$= (0.5738, 0.3486, 0.0.5221)$$

Scores values for these aggregated numbers are obtained as $SC(\tilde{\mathcal{T}}_1) = 0.1543$, $SC(\tilde{\mathcal{T}}_2) = 0.6689$, $SC(\tilde{\mathcal{T}}_3) = -0.0008$, $SC(\tilde{\mathcal{T}}_4) = 0.0487$, $SC(\tilde{\mathcal{T}}_5) = 0.0466$ and based on these score values, we have

$\tilde{\mathcal{T}}_{\sigma(1)} = (0.9041, 0.4665, 0.4125)$, $\tilde{\mathcal{T}}_{\sigma(2)} = (0.7054, 0.5359, 0.5816)$,
$\tilde{\mathcal{T}}_{\sigma(3)} = (0.4874, 0.2885, 0.4063)$, $\tilde{\mathcal{T}}_{\sigma(4)} = (0.5738, 0.3486, 0.0.5221)$, $\tilde{\mathcal{T}}_{\sigma(5)} = (0, 0, 0.0928)$

By using normal distribution-based method, we find $\omega = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$ and by the definition of T-SFHGA operator, we find

$$T - SFHGA_{w,\omega}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5) = (0, 0, 0.4803)$$

This type of aggregated value seems meaningless as whenever membership and abstinence value is zero in any one of the T-SFN it will make the value of membership and non-membership as zero in whole aggregated value. This shows that geometric aggregation operator of T-SFSs [10] does not possess the ability of aggregating such type of information effectively.

On the other hand, the proposed new geometric interactive aggregation operators can process any type of information effectively. Now the Example 3.1.2.21 is solved using proposed new aggregation operators in order to justify its effectiveness. For it, we aggregate the data using T-SFHGIA operator,

$$\mathcal{T}_1 = \left(\begin{array}{c}\sqrt[3]{(1-0.6^3)^{5\times0.18}-\left(1-(0.7^3+0.5^3+0.6^3)\right)^{5\times0.18}-(0.5^3)^{5\times0.18}},\\ \sqrt[3]{1-(1-0.5^3)^{5\times0.18}}, \sqrt[3]{1-(1-0.6^3)^{5\times0.18}}\end{array}\right)$$

$$= (0.6656, 0.5359, 0.5816)$$

$$\mathcal{T}_2 = \left(\begin{array}{c}\sqrt[3]{(1-0.4^3)^{5\times0.22}-\left(1-(0.9^3+0.5^3+0.4^3)\right)^{5\times0.22}-(0.5^3)^{5\times0.22}},\\ \sqrt[3]{1-(1-0.5^3)^{5\times0.22}}, \sqrt[3]{1-(1-0.4^3)^{5\times0.22}}\end{array}\right)$$

$$= (0.9144, 0.4665, 0.4125)$$

$$\mathcal{T}_3 = \left(\begin{array}{c} \sqrt[3]{(1-0.1^3)^{5\times 0.16} - \left(1-(0^3+0^3+0.1^3)\right)^{5\times 0.16} - (0^3)^{5\times 0.16}}, \\ \sqrt[3]{1-(1-0^3)^{5\times 0.16}}, \sqrt[3]{1-(1-0.1^3)^{5\times 0.16}} \end{array} \right)$$

$$= (0, 0, 0.0928)$$

$$\mathcal{T}_4 = \left(\begin{array}{c} \sqrt[3]{(1-0.4^3)^{5\times 0.21} - \left(1-(0.5^3+0.3^3+0.4^3)\right)^{5\times 0.21} - (0.3^3)^{5\times 0.21}}, \\ \sqrt[3]{1-(1-0.3^3)^{5\times 0.21}}, \sqrt[3]{1-(1-0.4^3)^{5\times 0.21}} \end{array} \right)$$

$$= (0.5141, 0.2885, 0.4063)$$

$$\mathcal{T}_5 = \left(\begin{array}{c} \sqrt[3]{(1-0.5^3)^{5\times 0.23} - \left(1-(0.6^3+0.4^3+0.5^3)\right)^{5\times 0.23} - (0.4^3)^{5\times 0.23}}, \\ \sqrt[3]{1-(1-0.4^3)^{5\times 0.23}}, \sqrt[3]{1-(1-0.5^3)^{5\times 0.23}} \end{array} \right)$$

$$= (0.6422, 0.3486, 0.0.5221)$$

The score values of these numbers are obtained as $SC(\mathcal{T}_1) = 0.0981$, $SC(\mathcal{T}_2) = 0.6943$, $SC(\mathcal{T}_3) = -0.0008$, $SC(\mathcal{T}_4) = 0.0688$, $SC(\mathcal{T}_5) = 0.1225$ and based on score values we have the following arrangement.

$\mathcal{T}_{\sigma(1)} = (0.9144, 0.4665, 0.4125)$, $\mathcal{T}_{\sigma(2)} = (0.6422, 0.3486, 0.0.5221)$, $\mathcal{T}_{\sigma(3)} = (0.6656, 0.5359, 0.5816)$, $\mathcal{T}_{\sigma(4)} = (0.5141, 0.2885, 0.4063)$, $\mathcal{T}_{\sigma(5)} = (0, 0, 0.0928)$

Now, by using definition of T-SFHGIA operator, we find

$$T - SFHGIA_{w,\omega}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5) = (0.8375, 0.4223, 0.4928)$$

Clearly the aggregated value obtained using T-SFHGIA operator is improved than the one obtained in using aggregated operators in [10] as it incorporates the zero values occurring in the membership and abstinence of T-SFNs efficiently. The analysis of both results proves the significance of proposed aggregation operators.

3.2. MADM approach based on proposed operators

Consider a decision making problem which consists a set of alternatives $Y = \{y_1, y_2, \dots, y_l\}$ and set of attributes $Z = \{z_1, z_2, \dots, z_q\}$ with weighted vector $w = \left(w_1, w_2, \dots, w_q\right)^T$, where $w_k \in (0,1]$ and $\Sigma_{k=1}^{q} w_k = 1$. Suppose every alternative y_j is represented by T-SFNs $\mathcal{T}_{jk} = \left(m_{jk}, i_{jk}, n_{jk}\right)$ which show that by which degree alternative satisfy, neutral and not satisfy the given attribute. Then, the following steps of the MADM approach based on the proposed operators are summarized as follows:

Step 1: Find the value of t for which information of decision matrix lies in T-SF environment.

Step 2: Assume the WV $w = \left(w_1, \dots, w_q\right)^T$ of $\mathcal{T}_{j1}, \mathcal{T}_{j2}, \dots, \mathcal{T}_{jq}$. where $w_k \in (0,1]$ and $\Sigma_{k=1}^{q} w_k = 1$ we get $\mathcal{T}_{jk} = \mathcal{T}_{jk}^{lw_k}$.

Step 3: By calculating the scores of each attribute of all alternatives, we find

$$\mathcal{T}_{\sigma(j1)}, \mathcal{T}_{\sigma(j2)}, \dots, \mathcal{T}_{\sigma(jk)}$$

Step 4: By using normal-distribution based method we find w and then aggregate the data using T-SFHGIA operator.

Step 5: Find the scores of all alternatives.

Step 6: With the help of score values, we find the best option.

3.2.1. Numerical Example

The above mentioned approach has been illustrated with a real life decision-making problem under the T-SFS environment and obtained results have been compared with the other existing results.

3.2.1.1. Case study

Jharkhand is the eastern state of the India, which has the 40 percent mineral resources of the country and second leading state of the mineral wealth after Chhattisgarh state. It is also known for its vast forest resources. Jamshedpur, Bokaro and Dhanbad cities of the Jharkhand are famous for industries in all over the world. After that, it is the widespread poverty state of the India because it is the primarily a rural state as 76 percent of the population live in the villages which depend on the agriculture and wages. Only 30 percent villages are connected by roads while only 55 percent villages have accessed to electricity and other facilities. But in the today's life, everyone is changing fast to himself for a better life, therefore, everyone moves to the urban cities for a better job. To stop this emigration, Jharkhand government wants to set up the industries based on the agriculture in the rural areas. For this, the government has been organized

``MOMENTUM JHARKHAND'' global investor submit 2017 in Ranchi to invite the companies for investment in the rural areas. Government announced the various facilities for setup the five food processing plants in the rural areas and consider the five attributes required for company selection to setup them, namely, project cost (Q_1), technical capability (Q_2), financial status (Q_3), company background (Q_4) and other factors (Q_5). The three companies taken as in the form of the alternatives, namely, Surya Food and Agro Pvt. Ltd. (s_1), Mother Dairy Fruit and Vegetable Pvt. Ltd. (s_2) and Parle Products Ltd. (s_3) interested for these projects. Then the main object of the government is to choose the best company among them for the task. In order to fulfill it, a decision maker evaluated these and gives their preferences in the term of T-SFS and their preferences values are summarized in the form of decision-matrix shown in Table 8 as follows.

Table 8 Input information related to each alternative

	Q_1	Q_2	Q_3	Q_4	Q_5
s_1	(0.7,0.5,0.6)	(0.9,0.5,0.4)	(0.4,0.2,0.1)	(0.5,0.3,0.4)	(0.6,0.4,0.5)
s_2	(0.5,0.4,0.6)	(0.7,0.2,0.3)	(0.5,0.3,0.6)	(0.4,0.1,0.6)	(0.5,0.2,0.4)
s_3	(0.4,0.1,0.2)	(0.5,0.4,0.1)	(0,0,0.5)	(0.6,0.2,0.2)	(0.6,0.1,0.5)

The given problem is solved using two approaches. First it is solved using new interactive operators showing their applicability. Then it is solved using geometric aggregation operators proposed in [10] showing their failure.

Solution using proposed operators:

Step 1: With some calculations, it is found that all the values in Table 8 are T-SFNs for $t = 3$.

Step 2: By taking $w = (0.18,0.22,0.16,0.21,0.23)^T$ we find $\mathcal{T}_{jk}$ and their values are summarized as below in Table 9.

Table 9 Aggregated values

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	$\begin{pmatrix}0.6656,\\0.5359,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.9144,\\0.4665,\\0.4125\end{pmatrix}$	$\begin{pmatrix}0.3333,\\0.2759,\\0.0928\end{pmatrix}$	$\begin{pmatrix}0.5141,\\0.2825,\\0.4063\end{pmatrix}$	$\begin{pmatrix}0.6422,\\0.3486,\\0.5221\end{pmatrix}$

j = 2	$\begin{pmatrix}0.4520,\\0.4384,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.7194,\\0.1703,\\0.3095\end{pmatrix}$	$\begin{pmatrix}0.4212,\\0.3817,\\0.5614\end{pmatrix}$	$\begin{pmatrix}0.4053,\\0.0891,\\0.6086\end{pmatrix}$	$\begin{pmatrix}0.5264,\\0.1571,\\0.4184\end{pmatrix}$
j = 3	$\begin{pmatrix}0.3843,\\0.1259,\\0.1931\end{pmatrix}$	$\begin{pmatrix}0.5397,\\0.3650,\\0.1032\end{pmatrix}$	$\begin{pmatrix}0,\\0,\\0.4662\end{pmatrix}$	$\begin{pmatrix}0.6104,\\0.1845,\\0.2033\end{pmatrix}$	$\begin{pmatrix}0.6209,\\0.0708,\\0.5221\end{pmatrix}$

Step 3: Now we have to find the score of each attribute of all alternatives and their computed values are given as below in Table 10.

Table 10 Score Values

	k = 1	k = 2	k = 3	k = 4	k = 5
j=1	0.0981	0.6943	0.0362	0.0688	0.1225
j=2	−0.1043	0.3426	−0.1021	−0.1589	0.0726
j=3	0.0495	0.1561	−0.1013	0.2190	0.0970

By comparing the score values, we have

$$SC(\mathcal{T}_{12}) > SC(\mathcal{T}_{15}) > SC(\mathcal{T}_{11}) > SC(\mathcal{T}_{14}) > SC(\mathcal{T}_{13})$$

$$SC(\mathcal{T}_{22}) > SC(\mathcal{T}_{25}) > SC(\mathcal{T}_{23}) > SC(\mathcal{T}_{21}) > SC(\mathcal{T}_{24})$$

$$SC(\mathcal{T}_{34}) > SC(\mathcal{T}_{32}) > SC(\mathcal{T}_{35}) > SC(\mathcal{T}_{31}) > SC(\mathcal{T}_{33})$$

Based on above score analysis, we find $\mathcal{T}_{\sigma(jk)}$ and summarized as in Table 11

Table 11 Ordered Aggregated values

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	$\begin{pmatrix}0.9144,\\0.5150,\\0.4125\end{pmatrix}$	$\begin{pmatrix}0.6422,\\0.9857,\\0.5221\end{pmatrix}$	$\begin{pmatrix}0.6656,\\0.4838,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.5141,\\0.3048,\\0.4063\end{pmatrix}$	$\begin{pmatrix}0.3333,\\0.1857,\\0.0928\end{pmatrix}$
j = 2	$\begin{pmatrix}0.7194,\\0.2064,\\0.3095\end{pmatrix}$	$\begin{pmatrix}0.5264,\\0.9987,\\0.4184\end{pmatrix}$	$\begin{pmatrix}0.4212,\\0.2787,\\0.5614\end{pmatrix}$	$\begin{pmatrix}0.4520,\\0.9804,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.4053,\\0.1016,\\0.6086\end{pmatrix}$
j = 3	$\begin{pmatrix}0.6104,\\0.2033,\\0.2033\end{pmatrix}$	$\begin{pmatrix}0.5397,\\0.4125,\\0.1032\end{pmatrix}$	$\begin{pmatrix}0.6209,\\0.9999,\\0.5221\end{pmatrix}$	$\begin{pmatrix}0.3843,\\0.0966,\\0.1931\end{pmatrix}$	$\begin{pmatrix}0,\\0,\\0.4662\end{pmatrix}$

Step 4: By using normal distribution-based method, we get $\omega = (0.1117,0.2365,0.3036,0.2365,0.1117)^T$ and by using the defined aggregation operators, we have.

$$\mathcal{T}_1 = T - SFHGIA_{w,\omega}(\mathcal{T}_{11},\mathcal{T}_{12},\mathcal{T}_{13},\mathcal{T}_{14},\mathcal{T}_{15})$$

$$= \left(\begin{array}{c} \sqrt[3]{\prod_{j=1}^{5}\left(1-n^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}\right)^{\omega_j} - \prod_{j=1}^{5}\left(1-\left(m^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}+i^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}+n^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}\right)\right)^{\omega_j} - \prod_{j=1}^{5}(i^3_{\tilde{\mathcal{T}}_{\sigma(1k)}})^{\omega_j}}, \\ \sqrt[3]{1-\prod_{j=1}^{5}\left(1-i^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}\right)^{\omega_j}}, \quad \sqrt[3]{1-\prod_{j=1}^{5}\left(1-n^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}\right)^{\omega_j}} \end{array} \right)$$

$$= (0.9380,0.4264,0.4928)$$

$$\mathcal{T}_2 = T - SFHGIA_{w,\omega}(\mathcal{T}_{21},\mathcal{T}_{22},\mathcal{T}_{23},\mathcal{T}_{24},\mathcal{T}_{25})$$

$$= \left(\begin{array}{c} \sqrt[3]{\prod_{j=1}^{5}\left(1-n^3_{\tilde{\mathcal{T}}_{\sigma(2k)}}\right)^{\omega_j} - \prod_{j=1}^{5}\left(1-\left(m^3_{\tilde{\mathcal{T}}_{\sigma(2k)}}+i^3_{\tilde{\mathcal{T}}_{\sigma(2k)}}+n^3_{\tilde{\mathcal{T}}_{\sigma(2k)}}\right)\right)^{\omega_j} - \prod_{j=1}^{5}(i^3_{\tilde{\mathcal{T}}_{\sigma(2k)}})^{\omega_j}}, \\ \sqrt[3]{1-\prod_{j=1}^{5}\left(1-i^3_{\tilde{\mathcal{T}}_{\sigma(2k)}}\right)^{\omega_j}}, \sqrt[3]{1-\prod_{j=1}^{5}\left(1-n^3_{\tilde{\mathcal{T}}_{\sigma(2k)}}\right)^{\omega_j}} \end{array} \right)$$

$$= (0.9420,0.3390,0.5296)$$

$$\mathcal{T}_3 = T - SFHGIA_{w,\omega}(\mathcal{T}_{31},\mathcal{T}_{32},\mathcal{T}_{33},\mathcal{T}_{34},\mathcal{T}_{35})$$

$$= \left(\begin{array}{c} \sqrt[3]{\prod_{j=1}^{5}\left(1-n^3_{\tilde{\mathcal{T}}_{\sigma(3k)}}\right)^{\omega_j} - \prod_{j=1}^{5}\left(1-\left(m^3_{\tilde{\mathcal{T}}_{\sigma(3k)}}+i^3_{\tilde{\mathcal{T}}_{\sigma(3k)}}+n^3_{\tilde{\mathcal{T}}_{\sigma(3k)}}\right)\right)^{\omega_\mathrm{j}} - \prod_{j=1}^{5}(i^3_{\tilde{\mathcal{T}}_{\sigma(3k)}})^{\omega_j}}, \\ \sqrt[3]{1-\prod_{j=1}^{5}\left(1-i^3_{\tilde{\mathcal{T}}_{\sigma(3k)}}\right)^{\omega_j}}, \sqrt[3]{1-\prod_{j=1}^{5}\left(1-n^3_{\tilde{\mathcal{T}}_{\sigma(3k)}}\right)^{\omega_j}} \end{array} \right)$$

$$= (0.9779,0.9713,0.3906)$$

Step 5: The score values of three alternatives based on their aggregated values are computed as $SC(\mathcal{T}_1) = 0.7056$, $SC(\mathcal{T}_2) = 0.6874$, and $SC(\mathcal{T}_3) = 0.8813$.

Step 6: By comparing score values, we get.

$$SC(\mathcal{T}_3) > SC(\mathcal{T}_1) > SC(\mathcal{T}_2)$$

The comparison of score values indicate that $\mathcal{T}_3$ has a greater score value. So, third Company is the best option. Thus, by using the new geometric interaction averaging operators a MADM problem is successfully solved.

Solution using aggregation operators proposed in [10]:

Step 1: The input preferences related to each alternative is summarized in Table 8 for $t = 3$.

Step 2: By using WV $w = (0.18,0.22,0.16,0.21,0.23)^T$ we find $\mathcal{T}'_{jk}$ as follows in Table 12

Table 12 Aggregated values

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	$\begin{pmatrix}0.7054,\\0.5359,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.9041,\\0.4665,\\0.4125\end{pmatrix}$	$\begin{pmatrix}0.4655,\\0.2759,\\0.0928\end{pmatrix}$	$\begin{pmatrix}0.4874,\\0.2825\\0.4063\end{pmatrix}$	$\begin{pmatrix}0.5738,\\0.3486,\\0.5221\end{pmatrix}$
j = 2	$\begin{pmatrix}0.5180,\\0.4384,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.6776,\\0.1703,\\0.3095\end{pmatrix}$	$\begin{pmatrix}0.7330,\\0.3817,\\0.5614\end{pmatrix}$	$\begin{pmatrix}0.3826,\\0.0891,\\0.6086\end{pmatrix}$	$\begin{pmatrix}0.4553,\\0.1517,\\0.4184\end{pmatrix}$
j = 3	$\begin{pmatrix}0.4370,\\0.1259,\\0.1931\end{pmatrix}$	$\begin{pmatrix}0.4811,\\0.3650,\\0.1032\end{pmatrix}$	$\begin{pmatrix}0,\\0,\\0.4662\end{pmatrix}$	$\begin{pmatrix}0.5863,\\0.1845,\\0.2033\end{pmatrix}$	$\begin{pmatrix}0.5563,\\0.0708,\\0.5221\end{pmatrix}$

Step 3: Now we have to find the score of each attribute of all alternatives as listed in Table 13

Table 13 Score values

	k = 1	k = 2	k = 3	k = 4	k = 5
j = 1	0.1543	0.6689	0.1000	0.0487	0.0466
j = 2	−0.0577	0.2815	0.2169	−0.1695	0.0212
j = 3	0.0762	0.1103	−0.1013	0.1932	0.0298

By comparing the score values, we have

$$SC(\mathcal{T}'_{12}) > SC(\mathcal{T}'_{11}) > SC(\mathcal{T}'_{13}) > SC(\mathcal{T}'_{14}) > SC(\mathcal{T}'_{15})$$

$$SC(\mathcal{T}'_{22}) > SC(\mathcal{T}'_{23}) > SC(\mathcal{T}'_{25}) > SC(\mathcal{T}'_{21}) > SC(\mathcal{T}'_{24})$$

$$SC(\mathcal{T}'_{34}) > SC(\mathcal{T}'_{32}) > SC(\mathcal{T}'_{31}) > SC(\mathcal{T}'_{35}) > SC(\mathcal{T}'_{33})$$

Based on above score analysis, we find $\mathcal{T}'_{\sigma(jk)}$ as listed in Table 14

Table 14 Ordered Aggregated values

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
$j = 1$	$\begin{pmatrix}0.9041,\\0.4665,\\0.4125\end{pmatrix}$	$\begin{pmatrix}0.7054,\\0.5359,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.4655,\\0.2759,\\0.0928\end{pmatrix}$	$\begin{pmatrix}0.4874,\\0.2825,\\0.4063\end{pmatrix}$	$\begin{pmatrix}0.5738,\\0.3486,\\0.5221\end{pmatrix}$
$j = 2$	$\begin{pmatrix}0.6776,\\0.1703,\\0.3095\end{pmatrix}$	$\begin{pmatrix}0.7330,\\0.3817,\\0.5614\end{pmatrix}$	$\begin{pmatrix}0.4553,\\0.1571,\\0.4184\end{pmatrix}$	$\begin{pmatrix}0.5180,\\0.4384,\\0.5816\end{pmatrix}$	$\begin{pmatrix}0.3826,\\0.0891,\\0.6086\end{pmatrix}$
$j = 3$	$\begin{pmatrix}0.5863,\\0.1845,\\0.2033\end{pmatrix}$	$\begin{pmatrix}0.4811,\\0.3650,\\0.1032\end{pmatrix}$	$\begin{pmatrix}0.4370,\\0.1259,\\0.1931\end{pmatrix}$	$\begin{pmatrix}0.5563,\\0.0708,\\0.5221\end{pmatrix}$	$\begin{pmatrix}0,\\0,\\0.4662\end{pmatrix}$

Step 4: By using normal distribution-based method, we get $\omega = (0.1117,0.2365,0.3036,0.2365,0.1117)^T$ and by using the defined aggregation operators, we have.

$$\mathcal{T}'_1 = T - SFHGIA_{w,\omega}(\mathcal{T}'_{11}, \mathcal{T}'_{12}, \mathcal{T}'_{13}, \mathcal{T}'_{14}, \mathcal{T}'_{15})$$

$$= \left(\begin{array}{c} \sqrt[3]{\prod_{j=1}^{5}\left(m^3_{\tilde{\mathcal{T}}_{\sigma(1k)}} + i^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}\right)^{\omega_j} - \prod_{j=1}^{5}(i^3_{\tilde{\mathcal{T}}_{\sigma(1k)}})^{\omega_j}}, \prod_{j=1}^{5}\left(i_{\tilde{\mathcal{T}}_{\sigma(1k)}}\right)^{\omega_j}, \\ \sqrt[3]{1 - \prod_{j=1}^{5}\left(1 - n^3_{\tilde{\mathcal{T}}_{\sigma(1k)}}\right)^{\omega_j}} \end{array}\right)$$

$$= (0.5750, 0.3533, 0.4473)$$

$$\mathcal{T}'_2 = T - SFHGIA_{w,\omega}(\mathcal{T}'_{21}, \mathcal{T}'_{22}, \mathcal{T}'_{23}, \mathcal{T}'_{24}, \mathcal{T}'_{25})$$

$$= \left(\sqrt[3]{\prod_{j=1}^{5} \left(m^3_{\tilde{\mathcal{T}}_{\sigma(2k)}} + i^3_{\tilde{\mathcal{T}}_{\sigma(2k)}} \right)^{\omega_j} - \prod_{j=1}^{5} (i^3_{\tilde{\mathcal{T}}_{\sigma(2k)}})^{\omega_j}}, \prod_{j=1}^{5} \left(i_{\tilde{\mathcal{T}}_{\sigma(2k)}} \right)^{\omega_j}, \atop \sqrt[3]{1 - \prod_{j=1}^{5} \left(1 - n^3_{\tilde{\mathcal{T}}_{\sigma(2k)}} \right)^{\omega_j}} \right)$$

$= \ (0.5384,0.1970,0.5721)$

$\mathcal{T}_3' = \ T - SFHGIA_{w,\omega}(\mathcal{T}_{31}', \mathcal{T}_{32}', \mathcal{T}_{33}', \mathcal{T}_{34}', \mathcal{T}_{35}')$

$$= \left(\sqrt[3]{\prod_{j=1}^{5} \left(m^3_{\tilde{\mathcal{T}}_{\sigma(3k)}} + i^3_{\tilde{\mathcal{T}}_{\sigma(3k)}} \right)^{\omega_j} - \prod_{j=1}^{5} (i^3_{\tilde{\mathcal{T}}_{\sigma(3k)}})^{\omega_j}}, \prod_{j=1}^{5} \left(i_{\tilde{\mathcal{T}}_{\sigma(3k)}} \right)^{\omega_j}, \atop \sqrt[3]{1 - \prod_{j=1}^{5} \left(1 - n^3_{\tilde{\mathcal{T}}_{\sigma(3k)}} \right)^{\omega_j}} \right)$$

$= \ (0, 0, 0.3692)$

This seems meaningless because membership and abstinence of only one T-SFN is zero but existing operator make a whole aggregated value zero.

Step 5: The score values are

$$SC(\mathcal{T}_1) = 0.1006$$

$$SC(\mathcal{T}_2) = -0.0312$$

$$SC(\mathcal{T}_3) = -0.0503$$

Step 6: By comparing score values, we get.

$$SC(\mathcal{T}_1) > SC(\mathcal{T}_2) > SC(\mathcal{T}_3)$$

From above example, the applicability of proposed operators can easily be checked by comparing the results obtained using new and existing geometric aggregation operators. It is noticed that whenever membership and abstinence of one TSFN becomes zero, then the aggregated valued using existing aggregation operators seems impractical. However, the aggregated value using new geometric interactive aggregation operators seems significant and consistent.

3.3. Advantages of the proposed work

In this section, we prove the generalization of proposed work over the existing literature. Here we observed that under some certain conditions the proposed work reduces to existing work which shows the superiority of our proposed work.

Consider the T-SFWGIA operator defined as

$$\boldsymbol{T-SFWGIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)} = \left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j^t+i_j^t+n_j^t\right)\right)^{w_j}-\prod_{j=1}^{k}(i_j^t)^{w_j}}, \\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}\end{array}\right) \tag{3.3.1}$$

1. If we take $t=2$ the equation (3.3.1) reduces to spherical fuzzy weighted geometric interaction averaging operator (SFWGIA operator) and we have

$$SFWGIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \left(\begin{array}{c}\sqrt{\prod_{j=1}^{k}\left(1-n_j^2\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j^2+i_j^2+n_j^2\right)\right)^{\mathrm{w_j}}-\prod_{j=1}^{k}(i_j^2)^{w_j}}, \\ \sqrt{1-\prod_{j=1}^{k}\left(1-i_j^2\right)^{w_j}}, \sqrt{1-\prod_{j=1}^{k}\left(1-n_j^2\right)^{w_j}}\end{array}\right)$$

2. If we take $t=1$ the equation (3.3.1) reduces to picture fuzzy weighted geometric interaction averaging operator (PFWGIA operator) and we have

$$PFWGIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \left(\begin{array}{c}\prod_{j=1}^{k}\left(1-n_j\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j+i_j+n_j\right)\right)^{\mathrm{w_j}}-\prod_{j=1}^{k}\left(i_j\right)^{w_j}, \\ 1-\prod_{j=1}^{k}\left(1-i_j\right)^{w_j}, 1-\prod_{j=1}^{k}\left(1-n_j\right)^{w_j}\end{array}\right)$$

3. If we take $t = 2$ and $i = 0$ the equation (3.3.1) reduces to Pythagorean fuzzy weighted geometric interaction averaging operator (PyFWGIA operator) and we have

$$PyFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \begin{pmatrix} \sqrt{\prod_{j=1}^{k}\left(1 - n_j^2\right)^{w_j} - \prod_{j=1}^{k}\left(1 - \left(m_j^2 + n_j^2\right)\right)^{\mathrm{w_j}}}, \\ \sqrt{1 - \prod_{j=1}^{k}\left(1 - n_j^2\right)^{w_j}} \end{pmatrix}$$

4. If we take $t = 1$ and $i = 0$ the equation (3.3.1) reduces to intuitionistic fuzzy weighted geometric interaction averaging operator (IFWGIA operator) and we have

$$IFWGIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \begin{pmatrix} \prod_{j=1}^{k}\left(1 - n_j\right)^{w_j} - \prod_{j=1}^{k}\left(1 - \left(m_j + n_j\right)\right)^{\mathrm{w_j}}, \\ 1 - \prod_{j=1}^{k}\left(1 - n_j\right)^{w_j} \end{pmatrix}$$

Similarly, T-SFOWGIA and T-SFHGIA operators can be reduced to the existing operators. All this clearly indicated that our proposed work can be used in the problems of existing literature, but the operators of existing literature are unable to deal with problems of T-spherical fuzzy information. For example, if we look at Example 3.1.2.21, it can be seen that none of the existing operators can be applied to such problems where information is in the form of T-SFNs.

3.3.1. Comparative analysis

The significance of proposed new geometric operators lies in the fact that the result obtained by using these operations are justifiable than those developed earlier i.e., [16] and [82, 84]. Such operators could not deal with situations where if membership and abstinence value of any number becomes zero then the membership and abstinence value of their aggregated value is also zero. Hence the existing operations of PFSs and T-SFSs did possess the capability of dealing with any kinds of information. But on the other hand the new geometric operations of T-SFSs can deal with any type of data justifiably. This point is demonstrated in example.

The second main advantage of our proposed work is that it has the ability of aggregate the data available in the form of IFSs, PyFSs, PFSs and SFSs. But conversely, the existing operators could not handle the data provided in T-spherical fuzzy environment. For example if we look at Example 3.1.2.21, its data is purely in the form of T-SFNs based on four grades of membership, abstinence, non-membership and refusal degree with $t = 3$, which shows that the aggregation operators of IFSs, PyFSs, PFSs and SFSs could not aggregate this data. But if we look at Example 3.2.1 its data is in the form of IFNs and our proposed operators easily aggregated this type of data with $t = 1$ and $i = 0$.

Hence by all means the proposed work has superiority over the existing work.

3.3.1.1. Example

Let $\mathcal{T}_1 = (0,0.5), \mathcal{T}_2 = (0.5,0.4), \mathcal{T}_3 = (0.4,0.2), \mathcal{T}_4 = (0.3,0.3)$ and $\mathcal{T}_5 = (0.7,0.1)$ ∈IFN. The WV for $\mathcal{T}_j (j = 1,2,..,5)$ is $w = (0.18,0.22,0.16,0.21,0.23)^T$.

$$\mathcal{T}_1 = ((1-0.5)^{5\times 0.18} - \left(1-(0+0.5)\right)^{5\times 0.18}, 1-(1-0.5)^{5\times 0.18})$$

$$= (0,0.5796)$$

$$\mathcal{T}_2 = ((1-0.4)^{5\times 0.22} - \left(1-(0.5+0.4)\right)^{5\times 0.22}, 1-(1-0.4)^{5\times 0.22})$$

$$= (0.5039,0.3183)$$

$$\mathcal{T}_3 = ((1-0.2)^{5\times 0.16} - \left(1-(0.4+0.2)\right)^{5\times 0.16}, 1-(1-0.2)^{5\times 0.16})$$

$$= (0.4000,0.2000)$$

$$\mathcal{T}_4 = ((1-0.3)^{5\times 0.21} - \left(1-(0.3+0.3)\right)^{5\times 0.21}, 1-(1-0.3)^{5\times 0.21})$$

$$= (0.2870,0.2746)$$

$$\mathcal{T}_5 = ((1-0.1)^{5\times 0.23} - \left(1-(0.7+0.1)\right)^{5\times 0.23}, 1-(1-0.1)^{5\times 0.23})$$

$$= (0.7203,0.1094)$$

Scores values are

$$SC(\mathcal{T}_1) = -0.5796, \quad SC(\mathcal{T}_2) = 0.1856, \quad SC(\mathcal{T}_3) = 0.2000,$$
$$SC(\mathcal{T}_4) = 0.0125, \quad SC(\mathcal{T}_5) = 0.6109.$$

Thus, $SC(\mathcal{T}_5) > SC(\mathcal{T}_3) > SC(\mathcal{T}_2) > SC(\mathcal{T}_4) > SC(\mathcal{T}_1)$ and we have

$$\mathcal{T}_{\sigma(1)} = (0.7203,0.1094)$$

$$\mathcal{T}_{\sigma(2)} = (0.4000,0.2000)$$

$$\mathcal{T}_{\sigma(3)} = (0.5039,0.3183)$$

$$\mathcal{T}_{\sigma(4)} = (0.2870,0.2746)$$

$$\mathcal{T}_{\sigma(5)} = (0,0.5796)$$

By using normal distribution-based method, we find

$\omega = (0.1117,0.2365,0.3036,0.2365,0.1117)^T$.

Now, by using definition of T-SFHGIA operator, we find

$$T - SFHGIA_{w,\omega}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5) = (0.4093,0.2919)$$

Here we get the same result as in [62, 64]. Thus, the proposed new operators have the capability to solve problems that lies in the existing structures.

Chapter 4

Multi-attribute decision making process with immediate probabilistic interactive averaging aggregation operators of T-spherical fuzzy sets and its application in the selection of solar cells

The objective of this chapter is to present some new interactive averaging aggregation operators by assigning associate probabilities for the T-SFSs. T-SFS is a generalization of the several existing theories such as IFSs and PFSs to handle the imprecise information. Under such environment, we developed some series of averaging interactive aggregation operators under the features that each element is represented with T-SFNs. Various properties of the proposed operators are also investigated. Further, to rank the different T-SFNs, we exhibit the new score functions and state their some properties. To demonstrate the presented algorithm, a decision making process algorithm is presented with T-SFS features. To save non-renewable resources and to protect environment the use of renewable sources is important. Solar energy is one of the best renewable sources of energy and also an environment-friendly source so the selection of solar cells is typically a MADM problem. So the applicability of the developed algorithm is demonstrated with a numerical example in the selection of the solar cells and comparison of their performance with the several existing approaches.

4.1. New Score function of T-SFSs

This section is shown that existing score function [16] for T-SFSs have some shortcomings and a new score function is proposed to overcome these shortcoming.

4.1.1. Definition

For any T-SFN $\mathcal{T} = (m, i, n)$ the new score function is defined as

$$SC(\mathcal{T}) = m^t - i^t - n^t + \left(\frac{e^{m^t - i^t - n^t}}{e^{m^t - i^t - n^t} + 1} - \frac{1}{2}\right) r^t, \qquad (4.1.1)$$

Where $r = \sqrt[t]{1 - (m^t + i^t + n^t)}$.

A T-SFN $\mathcal{T}_1 = (m_1, i_1, n_1)$ is said to be superior than another T-SFN $\mathcal{T}_2 = (m_2, i_2, n_2)$ if score of $\mathcal{T}_1$ is greater than $\mathcal{T}_2$. If score of both numbers is equal then the superiority is checked by the comparison of their refusal degree, the number $\mathcal{T}_1$ is superior if its refusal degree is smaller than $\mathcal{T}_2$ and numbers will be similar if their refusal degree is same.

4.1.2. Remark

The following special cases are concluded from the proposed score value, defined in Eq. (4.1.1), as

1. Eq. (4.1.1) reduces to score value for SFSs when $t = 2$;
2. Eq. (4.1.1) become valid for PFSs if $t = 1$;
3. Eq. (4.1.1) become valid for PyFSs if $t = 2$ and $i = 0$;
4. Eq. (4.1.1) reduces to score value for IFSs if $t = 1$ and $i = 0$.

4.2. Some T-Spherical Fuzzy Averaging Operators

In this section some operations for T-SFSs are defined and with the help of these operations some T-spherical fuzzy aggregation operators are proposed. This section is further divided into three subsections. In first subsection some averaging aggregation operators are proposed and some basic properties of these operators are also discussed. In second subsection some interactive averaging aggregation operators along with some basic properties are proposed. In third subsection superiority of interactive averaging aggregation operators over averaging aggregation operators is explained with the help of an example.

4.2.1. T-Spherical Fuzzy Averaging Aggregation operators

In this subsection some averaging aggregation operators e.g. T-SFOWA, IP-T-SFOWA, T-SGCA, Ass. IP-T-SFOWA operators are proposed with some of their basic properties.

4.2.1.1. Definition

Consider collection of T-SFNs, $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, then T-SFOWA operator is defined as

$$T - SFOWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \oplus_{j=1}^{k} \left(w_j \mathcal{T}_{\sigma(j)}\right)$$

where $w=(w_1,\dots,w_k)^T$ is a weight vector with a conditions that all weight vectors must belong to $[0,1]$ and the sum of all weights is equal to 1 and $\sigma=\big(\sigma(1),\sigma(2),\dots,\sigma(k)\big)$ is the permutation such that $SC\big(\mathcal{T}_{\sigma(j-1)}\big)\geq SC\big(\mathcal{T}_{\sigma(j)}\big)$.

4.2.1.2. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j=\big(m_j,i_j,n_j\big)$, then

$T-SFOWA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$

$$=\left(\sqrt[t]{1-\prod_{j=1}^{k}\big(1-m_{\sigma(j)}^{t}\big)^{w_j}},\prod_{j=1}^{k}i_{\sigma(j)}^{w_j},\prod_{j=1}^{k}n_{\sigma(j)}^{w_j}\right),$$

Proof: The above result is proved by using mathematical induction,

For $k=1$,

$$T-SFOWA_w(\mathcal{T}_1)=\left(\sqrt[t]{1-(1-m_1^t)},i_1,n_1\right)$$

$$=(m_1,i_1,n_1)$$

Thus results hold for $k=1$. Now consider that the results hold for $k=l$,

$T-SFOWA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_l)$

$$=\left(\sqrt[t]{1-\prod_{j=1}^{l}\big(1-m_{\sigma(j)}^{t}\big)^{w_j}},\prod_{j=1}^{l}i_{\sigma(j)}^{w_j},\prod_{j=1}^{l}n_{\sigma(j)}^{w_j}\right)$$

Then to prove that result hold for $k=l+1$,

$T-SFOWA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_{l+1})$

$$=\left(\sqrt[t]{1-\prod_{j=1}^{l}\big(1-m_{\sigma(j)}^{t}\big)^{w_j}},\prod_{j=1}^{l}i_{\sigma(j)}^{w_j},\prod_{j=1}^{l}n_{\sigma(j)}^{w_j}\right)$$

$$\oplus\left(\sqrt[t]{1-\big(1-m_{\sigma(j)}^{t}\big)^{w_j}},i_{\sigma(j)}^{w_j},n_{\sigma(j)}^{w_j}\right)$$

$$= \left(\sqrt[t]{1 - \prod_{j=1}^{l+1} \left(1 - m_{\sigma(j)}^{t}\right)^{w_j}}, \prod_{j=1}^{l+1} i_{\sigma(j)}^{w_j}, \prod_{j=1}^{l+1} n_{\sigma(j)}^{w_j} \right)$$

This proves that the results hold for all $k \in Z^{+}$.

4.2.1.3. Theorem

If $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2, \dots, k$ then $T - SFOWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

Proof: As $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ then

$$T - SFOWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \left(\sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t}\right)^{w_j}}, \prod_{j=1}^{k} i_{\sigma(j)}^{w_j}, \prod_{j=1}^{k} n_{\sigma(j)}^{w_j} \right)$$

$$= \left(\sqrt[t]{1 - \left(1 - m_{\sigma(j)}^{t}\right)^{\sum_{j=1}^{k} w_j}}, \left(i_{\sigma(j)}\right)^{\sum_{j=1}^{k} w_j}, \left(n_{\sigma(j)}\right)^{\sum_{j=1}^{k} w_j} \right)$$

$$= (m_0, i_0, n_0) = \mathcal{T}_0$$

4.2.1.4. Theorem

For a collection of any two different T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ and $\mathcal{T}_j{}' = \left(m_j', i_j', n_j'\right)$ $(j = 1,2, \dots, k)$ such that $m_j \le m_j'$, $i_j \le i_j'$ and $n_j \ge n_j'$ for all j, then

$$T - SFOWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \le T - SFOWA_w(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

Proof: As $m_j \le m_j'$, $i_j \le i_j'$ and $n_j \ge n_j'$ for all j. This implies that

$$\sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t}\right)^{w_j}} \le \sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - \left(m_{\sigma(j)}'\right)^{t}\right)^{w_j}}$$

$$\prod_{j=1}^{k} i_{\sigma(j)}^{w_j} \le \prod_{j=1}^{k} \left(i_{\sigma(j)}'\right)^{w_j}$$

$$\prod_{j=1}^{k} n_{\sigma(j)}^{w_j} \ge \prod_{j=1}^{k} \left(n_{\sigma(j)}'\right)^{w_j}$$

$$\left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{w_j}},\prod_{j=1}^{k} i_{\sigma(j)},\prod_{j=1}^{k} n_{\sigma(j)}\right)$$

$$\leq\left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}\right)^{w_j}},\prod_{j=1}^{k}\left(i'_{\sigma(j)}\right)^{w_j},\prod_{j=1}^{k}\left(n'_{\sigma(j)}\right)^{w_j}\right)$$

4.2.1.5. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ for all $j=1,2,\dots,k$ such that $\mathcal{T}_L=\min_j\{\mathcal{T}_j\}$ and $\mathcal{T}_U=\max_j\{\mathcal{T}_j\}$. Then

$$\mathcal{T}_L\leq T-SFOWA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)\leq\mathcal{T}_U$$

4.2.1.6. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$, then IP-T-SFOWA operator is defined as

$$IP-T-SFOWA_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\bigoplus_{j=1}^{k}\left(\lambda'_j\mathcal{T}_{\sigma(j)}\right)$$

where $w=(w_1,\dots,w_k)^T$ is a weight vector with a condition that all weight vectors belong to $[0,1]$ and the sum of all weights must be equal to 1. λ_j is probability for each $\mathcal{T}_j$, λ'_j is an immediate probability of $\mathcal{T}_{\sigma(j)}$ and $\lambda'_j=\frac{(w_j\lambda_j)}{\sum_{j=1}^{k}w_j\lambda_j}$ and $\sigma=\left(\sigma(1),\dots,\sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right)\geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

4.2.1.7. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$, then

$$IP-T-SFOWA_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\left(\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_j}},\prod_{j=1}^{k} i_{\sigma(j)}^{\lambda'_j},\prod_{j=1}^{k} n_{\sigma(j)}^{\lambda'_j}\right)$$

Proof: The above result is proved by using mathematical induction,

For $k=1$,

$$IP-T-SFOWA_P(\mathcal{T}_1)=\left(\sqrt[t]{1-(1-m_1^t)},i_1,n_1\right)$$

$$= (m_1, i_1, n_1)$$

Thus results hold for $k = 1$. Now consider that the results hold for $k = l$,

$IP - T - SFOWA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_l)$

$$= \left(\sqrt[t]{1 - \prod_{j=1}^{l} \left(1 - m_{\sigma(j)}^t\right)^{\lambda'_j}}, \prod_{j=1}^{l} i_{\sigma(j)}^{\lambda'_j}, \prod_{j=1}^{l} n_{\sigma(j)}^{\lambda'_j} \right)$$

Then to prove that result hold for $k = l + 1$,

$IP - T - SFOWA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{l+1})$

$$= \left(\sqrt[t]{1 - \prod_{j=1}^{l} \left(1 - m_{\sigma(j)}^t\right)^{\lambda'_j}}, \prod_{j=1}^{l} i_{\sigma(j)}^{\lambda'_j}, \prod_{j=1}^{l} n_{\sigma(j)}^{\lambda'_j} \right)$$

$$\oplus \left(\sqrt[t]{1 - \left(1 - m_{\sigma(j)}^t\right)^{\lambda'_j}}, i_{\sigma(j)}^{\lambda'_j}, n_{\sigma(j)}^{\lambda'_j} \right)$$

$$= \left(\sqrt[t]{1 - \prod_{j=1}^{l+1} \left(1 - m_{\sigma(j)}^t\right)^{\lambda'_j}}, \prod_{j=1}^{l+1} i_{\sigma(j)}^{\lambda'_j}, \prod_{j=1}^{l+1} n_{\sigma(j)}^{\lambda'_j} \right)$$

This proves that the results hold for all $k \in Z^+$.

4.2.1.8. Theorem

If $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2, \dots, k$ then $IP - T - SFOWA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

Proof: As $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ then

$$IP - T - SFOWA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \left(\sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t\right)^{\lambda'_j}}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda'_j}, \prod_{j=1}^{k} n_{\sigma(j)}^{\lambda'_j} \right)$$

$$= \left(\sqrt[t]{1 - \left(1 - m_{\sigma(j)}^t\right)^{\sum_{j=1}^{k} \lambda'_j}}, i_{\sigma(j)}^{\sum_{j=1}^{k} \lambda'_j}, n_{\sigma(j)}^{\sum_{j=1}^{k} \lambda'_j} \right)$$

$$= (m_0, i_0, n_0) = \mathcal{T}_0$$

4.2.1.9. Theorem

For a collection of any two different T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ and $\mathcal{T}_j{}' = (m_j', i_j', n_j')$ $(j = 1,2, \dots, k)$ such that $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$ for all j. Then

$$IP - T - SFOWA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq IP - T - SFOWA_P(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

Proof: As $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$ for all j. This implies that

$$\sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t\right)^{\lambda_j'}} \leq \sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - \left(m_{\sigma(j)}'\right)^t\right)^{\lambda_j'}}$$

$$\prod_{j=1}^{k} \left(i_{\sigma(j)}\right)^{\lambda_j'} \leq \prod_{j=1}^{k} \left(i_{\sigma(j)}'\right)^{\lambda_j'}$$

$$\prod_{j=1}^{k} \left(n_{\sigma(j)}\right)^{\lambda_j'} \geq \prod_{j=1}^{k} \left(n_{\sigma(j)}'\right)^{\lambda_j'}$$

$$\left(\sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t\right)^{\lambda_j'}}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda_j'}, \prod_{j=1}^{k} n_{\sigma(j)}^{\lambda_j'}\right)$$
$$\leq \left(\sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - \left(m_{\sigma(j)}'\right)^t\right)^{\lambda_j'}}, \prod_{j=1}^{k} \left(i_{\sigma(j)}'\right)^{\lambda_j'}, \prod_{j=1}^{k} \left(n_{\sigma(j)}'\right)^{\lambda_j'}\right)$$

4.2.1.10. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2, \dots, k$ such that $\mathcal{T}_L = \min_j \{\mathcal{T}_j\}$ and $\mathcal{T}_U = \max_j \{\mathcal{T}_j\}$. Then

$$\mathcal{T}_L \leq IP - T - SFOWA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}_U$$

4.2.1.11. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j = (m_j, i_j, n_j)$ on a set of states of nature $X = \{x_1, \dots, x_k\}$, then T-SFCA operator with respect to fuzzy measure Θ is defined as

$$T - SFCA_\Theta(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigoplus_{j=1}^{k} \left(\lambda_j \mathcal{T}_{\sigma(j)}\right)$$

where $\lambda_j = \Theta\left(\{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(j)}\}\right) - \Theta\left(\{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(j-1)}\}\right)$, $\Theta\left(\{x_{\sigma(0)}\}\right) \equiv 0$ and σ is the permutation. $\sigma = \left(\sigma(1), \sigma(2), \dots, \sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right) \geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

4.2.1.12. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, then

$$T - SFCA_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \left(\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^{t}\right)^{\lambda_j}}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda_j}, \prod_{j=1}^{k} n_{\sigma(j)}^{\lambda_j}\right)$$

Further, it is observed that T-SFCA operator also fulfils the properties as defined in theorems 4.2.1.3. – 4.2.1.5., so we omit here their proofs.

4.2.1.13. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, the Ass. IP-T-SFOWA operators is defined as

$$Ass.IP - T - SFOWA_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigvee_{\rho \in X_n} \left[\oplus_{j=1}^{k} \left(\lambda'_{\rho(j)} \mathcal{T}_{\sigma(j)}\right)\right]$$

and

$$Ass.IP - T - SFOWA_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigwedge_{\rho \in X_n} \left[\oplus_{j=1}^{k} \left(\lambda'_{\rho(j)} \mathcal{T}_{\sigma(j)}\right)\right]$$

where $w = (w_1, \dots, w_k)^T$ is a weight vector with a condition that all weight vectors belong to $[0,1]$ and the sum of all weights must be equal to 1. $\sigma = \left(\sigma(1), \dots, \sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right) \geq SC\left(\mathcal{T}_{\sigma(j)}\right)$. For each associated probability P_ρ: $\lambda'_{\rho(j)} = \frac{\left(w_j \lambda_{\rho(j)}\right)}{\sum_{j=1}^{k} w_j \lambda_{\rho(j)}}$, $\lambda_{\rho(j)} \equiv P_\rho\left(\mathcal{T}_{\sigma(j)}\right)$ is an associated immediate probability and $\vee$ $=$maximum and $\wedge$=minimum.

4.2.1.14. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, then

$$Ass.IP-T-SFOWA_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$
$$=\left(\begin{array}{c}\sqrt[t]{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)},\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\\ \min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\end{array}\right)$$

and

$$Ass.IP-T-SFOWA_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$
$$=\left(\begin{array}{c}\sqrt[t]{1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)},\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right),\\ \max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(n_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)\end{array}\right)$$

Further, it is observed that $Ass.IP-T-SFOWA_{\vee}$ and $Ass.IP-T-SFOWA_{\wedge}$ operator also fulfils the properties as defined in theorems 4.2.1.3. – 4.2.1.5., so we omit here their proofs.

4.2.1.15. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ on a set of states of nature $X=\{x_1,\dots,x_k\}$, then T-spherical fuzzy conjugate Choquet averaging (T-SFCCA) operator with respect to fuzzy measure Θ is defined as

$$T-SFCCA_{\Theta}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\left(\oplus_{j=1}^{k}\left(\lambda_j\left(\mathcal{T}_j\right)^{c}\right)\right)^{c}$$

where $\lambda_j=\Theta\left(\left\{x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(j)}\right\}\right)-\Theta\left(\left\{x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right)\equiv 0$ and $\sigma=\left(\sigma(1),\sigma(2),\dots,\sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right)\geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

4.2.1.16. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$, then

$$T-SFCCA_{\Theta}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \left(\prod_{j=1}^{k}\left(m_{\sigma(j)}\right)^{\lambda_j}, \prod_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda_j}, \sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda_j}}\right)$$

4.2.2. T-Spherical Fuzzy Interactive Aggregation operators

In this subsection some interactive averaging aggregation operators e.g. T-SFOWIA, IP-T-SFOWIA, T-SGCIA and Ass. IP-T-SFOWIA operators along with some of their basic properties are proposed.

4.2.2.1. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, then T-SFOWIA operator is defined as

$$T-SFOWIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \oplus i_{j=1}^{k}\left(w_j\mathcal{T}_{\sigma(j)}\right)$$

where $w = (w_1, w_2, \dots, w_k)^T$ is a weight vector with a conditions that all weight vectors must belong to $[0,1]$ and the sum of all weights is equal to 1 and $\sigma = \left(\sigma(1), \sigma(2), \dots, \sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right) \geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

4.2.2.2. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, then

$$T-SFOWIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \left(\frac{\sqrt[t]{1-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^{t}\right)^{w_j}},}{\sqrt[t]{\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}-i_{\sigma(j)}^{t}-n_{\sigma(j)}^{t}\right)^{w_j}-\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{w_j}}}\right),$$

Proof: The above result is proved by using mathematical induction,

For $k = 1$,

$$T - SFOWIA_w(\mathcal{T}_1)$$

$$= \left(\sqrt[t]{1-(1-m_1^t)}, \sqrt[t]{1-(1-i_1^t)}, \sqrt[t]{(1-m_1^t)-(1-m_1^t-i_1^t-n_1^t)-i_1^t}\right)$$

$$= (m_1, i_1, n_1)$$

Thus results hold for $k = 1$. Now consider that the results hold for $k = l$,

$$T - SFOWIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_l)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^t\right)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{l}\left(1-i_{\sigma(j)}^t\right)^{w_j}}, \\ \sqrt[t]{\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^t\right)^{w_j} - \prod_{j=1}^{l}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{w_j} - \prod_{j=1}^{l}\left(i_{\sigma(j)}^t\right)^{w_j}} \end{array} \right)$$

Then to prove that result hold for $k = l + 1$,

$$T - SFOWIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{l+1})$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^t\right)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{l}\left(1-i_{\sigma(j)}^t\right)^{w_j}}, \\ \sqrt[t]{\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^t\right)^{w_j} - \prod_{j=1}^{l}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{w_j} - \prod_{j=1}^{l}\left(i_{\sigma(j)}^t\right)^{w_j}} \end{array} \right)$$

$$\oplus_i \left(\begin{array}{c} \sqrt[t]{1-\left(1-m_{\sigma(j)}^t\right)^{w_j}}, \sqrt[t]{1-\left(1-i_{\sigma(j)}^t\right)^{w_j}}, \\ \sqrt[t]{\left(1-m_{\sigma(j)}^t\right)^{w_j} - \left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{w_j} - \left(i_{\sigma(j)}^t\right)^{w_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-m_{\sigma(j)}^t\right)^{w_j}}, \sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-i_{\sigma(j)}^t\right)^{w_j}}, \\ \sqrt[t]{\prod_{j=1}^{l+1}\left(1-m_{\sigma(j)}^t\right)^{w_j} - \prod_{j=1}^{l+1}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{w_j} - \prod_{j=1}^{l+1}\left(i_{\sigma(j)}^t\right)^{w_j}} \end{array} \right)$$

This proves that the results hold for all $k \in Z^+$.

4.2.2.3. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j = (m_j, i_j, n_j)$, then IP-T-SFOWIA operator is defined as

$$IP - T - SFOWIA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \oplus i_{j=1}^{k} \left(\lambda_j' \mathcal{T}_{\sigma(j)}\right)$$

where $w = (w_1, w_2, \dots, w_k)^T$ is a weight vector with a condition that all weight vectors belong to $[0,1]$ and the sum of all weights must be equal to 1. λ_j is probability for each $\mathcal{T}_j$, λ_j is an immediate probability of $\mathcal{T}_{\sigma(j)}$ and $\lambda_j' = \frac{(w_j \lambda_j)}{\sum_{j=1}^{k} w_j \lambda_j}$ and $\sigma = (\sigma(1), \dots, \sigma(k))$ is the permutation such that $SC(\mathcal{T}_{\sigma(j-1)}) \geq SC(\mathcal{T}_{\sigma(j)})$.

4.2.2.4. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$, then

$$IP - T - SFOWIA_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$

$$= \begin{pmatrix} \sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t\right)^{\lambda_j'}}, \sqrt[t]{1 - \prod_{j=1}^{k} \left(1 - i_{\sigma(j)}^t\right)^{\lambda_j'}}, \\ \sqrt[t]{\prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t\right)^{\lambda_j'} - \prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda_j'} - \prod_{j=1}^{k} \left(i_{\sigma(j)}^t\right)^{\lambda_j'}} \end{pmatrix}$$

4.2.2.5. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j = (m_j, i_j, n_j)$ on a set of states of nature $X = \{x_1, \dots, x_k\}$, then T-SFCIA operator with respect to fuzzy measure Θ is defined as

$$T - SFCIA_\Theta(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \oplus i_{j=1}^{k} \left(\lambda_j \mathcal{T}_{\sigma(j)}\right)$$

where $\lambda_j = \Theta(\{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(j)}\}) - \Theta(\{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(j-1)}\})$, $\Theta(\{x_{\sigma(0)}\}) \equiv 0$ and σ is the permutation. $\sigma = (\sigma(1), \sigma(2), \dots, \sigma(k))$ is the permutation such that $SC(\mathcal{T}_{\sigma(j-1)}) \geq SC(\mathcal{T}_{\sigma(j)})$.

4.2.2.6. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$, then

$$T - SFCIA_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$
$$= \left(\begin{array}{c} \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^t\right)^{\lambda_j}}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i_{\sigma(j)}^t\right)^{\lambda_j}}, \\ \sqrt[t]{\prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^t\right)^{\lambda_j} - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda_j} - \prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\lambda_j}} \end{array} \right)$$

4.2.2.7. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, then Ass.IP-T-SFOWIA operators is defined as

$$Ass.IP - T - SFOWIA_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigvee_{\rho \in X_n} \left[\oplus_{i\,j=1}^{\;k} \left(\lambda'_{\rho(j)} \mathcal{T}_{\sigma(j)}\right)\right]$$

And

$$Ass.IP - T - SFOWIA_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigwedge_{\rho \in X_n} \left[\oplus_{i\,j=1}^{\;k} \left(\lambda'_{\rho(j)} \mathcal{T}_{\sigma(j)}\right)\right]$$

where $w = (w_1, w_2, \dots, w_k)^T$ is a weight vector with a condition that all weight vectors belong to $[0,1]$ and the sum of all weights must be equal to 1. $\sigma = \left(\sigma(1), \dots, \sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right) \geq SC\left(\mathcal{T}_{\sigma(j)}\right)$. For each associated probability P_ρ: $\lambda'_{\rho(j)} = \frac{\left(w_j \lambda_{\rho(j)}\right)}{\sum_{j=1}^{k} w_j \lambda_{\rho(j)}}$, $\lambda_{\rho(j)} \equiv P_\rho\left(\mathcal{T}_{\sigma(j)}\right)$ is an associated immediate probability and $\vee$ =maximum and $\wedge$=minimum.

4.2.2.8. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$, then

$Ass.IP-T-SFOWIA_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$

$$=\left(\begin{array}{c}\sqrt[t]{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)},\sqrt[t]{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)},\\ \sqrt[t]{\begin{array}{c}\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}-i_{\sigma(j)}^{t}-n_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)\\ -\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)\end{array}}\end{array}\right)$$

and

$Ass.IP-T-SFOWIA_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$

$$=\left(\begin{array}{c}\sqrt[t]{1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)},\sqrt[t]{1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)},\\ \sqrt[t]{\begin{array}{c}\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}-i_{\sigma(j)}^{t}-n_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)\\ -\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)\end{array}}\end{array}\right)$$

4.2.2.9. Definition

Consider a collection of T-SFNs, $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ on a set of states of nature $X=\{x_1,\dots\dots,x_k\}$, then T-spherical fuzzy conjugate Choquet interactive averaging (T-SFCCIA) operator with respect to fuzzy measure Θ is defined as

$$T-SFCCIA_{\Theta}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\left(\oplus_{i_{j=1}}^{k}\left(\lambda_j\left(\mathcal{T}_j\right)^{c}\right)\right)^{c}$$

where $\lambda_j=\Theta\left(\left\{x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(j)}\right\}\right)-\Theta\left(\left\{x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right)\equiv 0$ and $\sigma=\left(\sigma(1),\sigma(2),\dots,\sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right)\geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

4.2.2.10. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$, then

$$T - SFCCIA_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$
$$= \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda_j} - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda_j} - \prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\lambda_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i_{\sigma(j)}^t\right)^{\lambda_j}}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda_j}} \end{array} \right)$$

Also, it is observed that all the above defined operators also satisfied the properties as defined in Properties 4.2.1.3. – 4.2.1.5.

4.2.2.11. Remark

If fuzzy measure and probability of T-SFSs become equal and furthermore probabilities of all T-SFNs become equal then Ass.IP-T-SFOWIA operator reduces to T-SFOWA operator.

4.2.3. Comparison between aggregation operators and interactive aggregation operators

In this section, the superiority of interactive averaging aggregation operators over averaging aggregation operators is explained with the help of an example. It is also explained that under some conditions the averaging aggregation operators fail while interactive averaging aggregation operators overcome this shortcoming.

4.2.3.1. Example

Consider T-SFNs, $g_1 = (0.63, 0.0, 0.0)$, $g_2 = (0.68, 0.25, 0.81)$ and $g_3 = (0.0, 0.51, 0.93)$ having a weight vector $w = \{0.25, 0.40, 0.35\}$, fuzzy measures will be

$\Theta(\phi) = 0$, $\Theta(\{g_1\}) = 0.125$, $\Theta(\{g_2\}) = 0.200$, $\Theta(\{g_3\}) = 0.175$, $\Theta(\{g_1, g_2\}) = 0.325$, $\Theta(\{g_1, g_3\}) = 0.300$, $\Theta(\{g_2, g_3\}) = 0.375$, $\Theta(\{g_1, g_2, g_3\}) = 1$.

Immediate probabilities for all possible permutations and the associated immediate probabilities for all possible permutations are given in Table 15.

Table 15 Immediate Probabilities & Associated Immediate Probability

Immediate Probabilities

	λ'_1	λ'_2	λ'_3
$\sigma = (g_1, g_2, g_3)$	0.125	0.200	0.675
$\sigma = (g_1, g_3, g_2)$	0.125	0.700	0.175
$\sigma = (g_2, g_1, g_3)$	0.200	0.200	0.600
$\sigma = (g_2, g_3, g_1)$	0.700	0.200	0.100
$\sigma = (g_3, g_1, g_2)$	0.125	0.700	0.175
$\sigma = (g_3, g_2, g_1)$	0.700	0.125	0.175
	Associated Immediate Probability		
	$\lambda'_{\rho(1)}$	$\lambda'_{\rho(2)}$	$\lambda'_{\rho(3)}$
$\sigma = (g_1, g_2, g_3)$	0.0899	0.2302	0.6798
$\sigma = (g_1, g_3, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_2, g_1, g_3)$	0.1470	0.2353	0.6176
$\sigma = (g_2, g_3, g_1)$	0.6034	0.2759	0.1207
$\sigma = (g_3, g_1, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_3, g_2, g_1)$	0.6114	0.1747	0.2140

As for $t = 1$, $0.68 + 0.25 + 0.81 = 1.74 \notin [0,1]$,

As for $t = 2$, $0.68^2 + 0.25^2 + 0.81^2 = 1.181 \notin [0,1]$

As for $t = 3$, $0.68^3 + 0.25^3 + 0.81^3 = 0.861 \in [0,1]$

Similarly for $t = 3$, $\mathcal{T}_1$ and $\mathcal{T}_3$ are T-SFNs.

Then aggregated values of averaging aggregation operators defined in Definitions 4.2.1.1., 4.2.1.6., 4.2.1.11., and 4.2.1.13., will be

$$T - SFOWA_w(g_1, g_2, g_3) = (0.5846, 0.0, 0.0)$$

$$IP - T - SFOWA_P(g_1, g_2, g_3) = (0.5452, 0.0, 0.0)$$

$$T - SFCA_\Theta(g_1, g_2, g_3) = (0.4752, 0.0, 0.0)$$

$$Ass.\ IP-T-SFOWA_{\vee}(g_1,g_2,g_3)=(0.6423,0.0,0.0)$$

$$Ass.\ IP-T-SFOWA_{\wedge}(g_1,g_2,g_3)=(0.4742,0.0,0.0)$$

The above aggregation results seem meaningless as these averaging operators do not aggregate abstinence and non-membership value because one of the abstinence and non-membership value of given data is zero. So the results obtained through these averaging aggregation operators are not valid. To overcome this shortcoming we used interactive averaging aggregation operators. The results obtained by using interactive operators defined in Definitions 4.2.2.1., 4.2.2.3., 4.2.2.5., and 4.2.2.7. will be

$$T-SFOWIA_w(g_1,g_2,g_3)=(0.5846,0.3793,0.8617)$$

$$IP-T-SFOWIA_P(g_1,g_2,g_3)=(0.5452,0.4205,0.8973)$$

$$T-SFCIA_{\Theta}(g_1,g_2,g_3)=(0.4752,0.4554,0.9260)$$

$$Ass.\ IP-T-SFOWIA_{\vee}(g_1,g_2,g_3)=(0.6423,0.4571,0.7420)$$

$$Ass.\ IP-T-SFOWIA_{\wedge}(g_1,g_2,g_3)=(0.4742,0.2772,0.9279)$$

The proposed interactive operators aggregate all membership, abstinence and non-membership values. This shows the superiority of interactive aggregation operators and the results obtained using these interactive operators are more reliable.

4.3. Algorithm for MADM based on proposed operators

In this section an algorithm was developed to solve MADM problem using the proposed averaging aggregation and interactive aggregation operators and a well-known MADM example is solved by using the algorithm.

Consider a set of alternatives $D=\{d_1,d_2,\dots,d_l\}$ and set of attributes $G=\{g_1,g_2,\dots,g_k\}$ having a weight vector $w=\{w_1,w_2,\dots,w_k\}$, set of probabilities associated with them is $P=\{\lambda_1,\lambda_2,\dots,\lambda_k\}$ and associated immediate probabilities are $P_{\rho(j)}=\left\{\lambda'_{\rho(1)},\lambda'_{\rho(2)},\dots,\lambda'_{\rho(k)}\right\}$. The weight vector and set of probabilities have a same condition that the sum of weights and probabilities must equal to 1 and weights and probabilities belong to closed unit interval. The fuzzy measure Θ have been calculated for all subsets of $\{d_1,d_2,\dots,d_l\}$. Then to find the finest alternative among the feasible one, we summarized the following steps based on the proposed aggregation operators.

Step 1. Rate the given alternatives under the different set of attributes by an expert in terms of T-spherical fuzzy numbers and are summarized in the decision matrix as follows:

$$\mathcal{T} = \begin{pmatrix} (m_{11}, i_{11}, n_{11}) & (m_{12}, i_{12}, n_{12}) & \ldots\ldots & (m_{1k}, i_{ik}, n_{ik}) \\ \ldots & \ldots & \ldots\ldots & \ldots \\ (m_{l1}, i_{l1}, n_{l1}) & (m_{l2}, i_{l2}, n_{l2}) & \ldots\ldots & (m_{lk}, i_{lk}, n_{lk}) \end{pmatrix}$$

Step 2: Normalize the data, if required by converting the cost type ratings into the benefit type by using the following equation

$$r_{lk} = \begin{cases} (m_{lk}, i_{lk}, n_{lk}) \ ; & for\ benefit\ type\ attributes \\ (n_{lk}, i_{lk}, m_{lk}) \ ; & for\ cost\ type\ attributes \end{cases}$$

and hence obtained the normalized decision matrix $R = (r_{lk})$.

Step 3: Find the value of t for which information matrix R lie in T-spherical fuzzy environment i.e., to find the smallest value of t which satisfy the condition $m_{lk}^t + i_{lk}^t + n_{lk}^t \leq 1$ for all l, k.

Step 4. Utilize the normalized data and the value of t, aggregate all the numbers into the collective ones by using the aggregation operators such as IP-T-SFOWA, T-SFCA, Ass.IP-T-SFOWA etc. The resultant number is denoted by $\mathcal{T}_k = (m_k, i_k, n_k)$ for $k = 1,2, \ldots, l$.

Step 5. Compute the score value of the obtained number $\mathcal{T}_k = (m_k, i_k, n_k)$ by using equation

$$SC(\mathcal{T}_k) = m_k^t - i_k^t - n_k^t + \left(\frac{e^{m_k^t - i_k^t - n_k^t}}{e^{m_k^t - i_k^t - n_k^t + 1}} - \frac{1}{2} \right) r_k^t,$$

Where $r_k^t = \sqrt[t]{1 - m_k^t - i_k^t - n_k^t}$

Step 6. Rank the alternatives based on the score values and hence select the best one.

4.3.1. Numerical Example

To save the non-renewable energy resources and the environment, the use of renewable energy plays a significant role in the aspect of the production of electricity. Solar cells are the best renewable sources of energy. There are several types of solar cells but few of them are studied in our application. The solar cells made with inorganic semiconductors like crystalline silicon solar cell, a solar cell with advanced III-V thin layer, amorphous silicon solar cell, cadmium telluride solar cell, etc. are expensive and their use has been confined to a few technological options. While on the other hand, the solar cells made with organic semiconductors like a dye-sensitized solar cell, etc. can be processed on large surfaces at a relatively low temperature but they have some serious problems that the degradation of their compounds (plastics) and also they provide less efficiency about 5-11%.

A MARCO company is situated in Islamabad Pakistan. This factory manufactures PVC pipes and plastic water tanks. Due to the load shedding of electricity, the company is unable to meet the demand. To overcome the deficit of supply and demand the company wants to generate electricity using solar energy. For which they have to select the best solar cell that increases production or efficiency, minimizes cost, and at the same time confers high maturity and reliability. They have a set of alternatives $D = \{d_1, d_2, d_3, d_4, d_5\}$ where:

d_1: Amorphous Silicon Solar Cell;

d_2: Dye-sensitized Solar Cell;

d_3: Cadmium Telluride Solar Cell;

d_4: Solar cell with advanced III-V thin layer with tracking systems for solar concentration;

d_5: Crystalline Silicon Solar Cell.

Experts have evaluate these alternatives under consideration of following attributes $G = \{g_1, g_2, g_3\}$

g_1: Cost;

g_2: Efficiency in energy conversion;

g_3: Heat tolerance.

The experts give information in T-spherical fuzzy numbers after evaluation as in Table 16.

Table 16 Decision Matrix

	g_1	g_2	g_3
d_1	(0.51,0.42,0.87)	(0.54,0.21,0.44)	(0.47,0.36,0.81)
d_2	(0.53,0.33,0.84)	(0.00,0.23,0.47)	(0.67,0.11,0.55)
d_3	(0.39,0.26,0.77)	(0.73,0.38,0.59)	(0.64,0.41,0.52)
d_4	(0.00,0.34,0.93)	(0.91,0.27,0.56)	(0.66,0.19,0.79)
d_5	(0.22,0.46,0.78)	(0.69,0.52,0.42)	(0.59,0.41,0.72)

Assume that the g_1 is the cost type attribute, so we normalize the given information by converting the cost type into benefit type and obtained the normalized decision matrix given in Table 17.

Table 17 Normalized Decision Matrix

	g_1	g_2	g_3
d_1	(0.87,0.42,0.51)	(0.54,0.21,0.44)	(0.47,0.36,0.81)
d_2	(0.84,0.33,0.53)	(0.00,0.23,0.47)	(0.67,0.11,0.55)
d_3	(0.77,0.26,0.39)	(0.73,0.38,0.59)	(0.64,0.41,0.52)
d_4	(0.93,0.34,0.00)	(0.91,0.27,0.56)	(0.66,0.19,0.79)
d_5	(0.78,0.46,0.22)	(0.69,0.52,0.42)	(0.59,0.41,0.72)

As for $t = 1,\ 0.87 + 0.42 + 0.51 = 1.8 \notin [0,1]$,

As for $t = 2,\ 0.87^2 + 0.42^2 + 0.51^2 = 1.19 \notin [0,1]$

As for $t = 3,\ 0.87^3 + 0.42^3 + 0.51^3 = 0.865 \in [0,1]$

Similarly for $t = 3$ all the information given in Table 17 are T-SFNs. The interaction of states of nature and weights of given attributes is in Table 18.

Table 18 Interaction of states of nature and weights

	g_1	g_2	g_3	Risk importance
g_1	-	0.150	0.100	0.250

g_2	0.150	-	0.250	0.400
g_3	0.100	0.250	-	0.350

With the help of Table 18, the interaction between attributes will be $I_{g_1} = 0.250$, $I_{g_2} = 0.400$, $I_{g_3} = 0.350$, $I_{g_1 g_2} = 0.150$, $I_{g_1 g_3} = 0.100$, $I_{g_2 g_3} = 0.250$. The fuzzy measure can be calculated with the help of interaction using the following relationship

$$\Theta(\{g_j\}) = I_{g_j} - \frac{1}{2} \sum_{g \in G\{g_j\}} I_{g_j g}$$

$$\Theta(\{g_j, g_k\}) = I_{g_j} + I_{g_k} - \frac{1}{2} \sum_{g \in G\{g_j, g_k\}} I_{g_j g} - \frac{1}{2} \sum_{g \in G\{g_j, g_k\}} I_{g_k g}$$

$$j, k = 1,2,3,4. j \neq k$$

$$\Theta(\phi) = 0, \Theta(G) = 1.$$

The fuzzy measures will be $\Theta(\phi) = 0$, $\Theta(\{g_1\}) = 0.125$, $\Theta(\{g_2\}) = 0.200$, $\Theta(\{g_3\}) = 0.175$, $\Theta(\{g_1, g_2\}) = 0.325$, $\Theta(\{g_1, g_3\}) = 0.300$, $\Theta(\{g_2, g_3\}) = 0.375$, $\Theta(G) = 1$. The immediate probabilities for every possible permutation are summarized in Table 19. An associated immediate probability for every possible permutation is summarized in Table 20.

Table 19 Immediate Probability

	λ'_1	λ'_2	λ'_3
$\sigma = (g_1, g_2, g_3)$	0.125	0.200	0.675
$\sigma = (g_1, g_3, g_2)$	0.125	0.700	0.175
$\sigma = (g_2, g_1, g_3)$	0.200	0.200	0.600
$\sigma = (g_2, g_3, g_1)$	0.700	0.200	0.100
$\sigma = (g_3, g_1, g_2)$	0.125	0.700	0.175
$\sigma = (g_3, g_2, g_1)$	0.700	0.125	0.175

Table 20 Associated Immediate Probability

	$\lambda'_{\rho(1)}$	$\lambda'_{\rho(2)}$	$\lambda'_{\rho(3)}$

$\sigma = (g_1, g_2, g_3)$	0.0899	0.2302	0.6798
$\sigma = (g_1, g_3, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_2, g_1, g_3)$	0.1470	0.2353	0.6176
$\sigma = (g_2, g_3, g_1)$	0.6034	0.2759	0.1207
$\sigma = (g_3, g_1, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_3, g_2, g_1)$	0.6114	0.1747	0.2140

The aggregated values by proposed operators (Definitions 4.2.1.1., 4.2.1.6., 4.2.1.11., and 4.2.1.13.) are shown in Table 21.

Table 21 Aggregated values using aggregation operator

	d_1	d_2	d_3	d_4	d_5
$T - SFOWA_w$	$\begin{pmatrix}0.6790,\\0.3016,\\0.5652\end{pmatrix}$	$\begin{pmatrix}0.6658,\\0.1944,\\0.5117\end{pmatrix}$	$\begin{pmatrix}0.7154,\\0.4963,\\0.5090\end{pmatrix}$	$\begin{pmatrix}0.8718\\0.2529,\\0.0\end{pmatrix}$	$\begin{pmatrix}0.6911,\\0.4641,\\0.4315\end{pmatrix}$
$IP - T - SFOWA_P$	$\begin{pmatrix}0.6074,\\0.3002,\\0.6088\end{pmatrix}$	$\begin{pmatrix}0.6328,\\0.1662,\\0.5163\end{pmatrix}$	$\begin{pmatrix}0.6967,\\0.3764,\\0.5257\end{pmatrix}$	$\begin{pmatrix}0.8397,\\0.2330,\\0.0\end{pmatrix}$	$\begin{pmatrix}0.6630,\\0.4545,\\0.5076\end{pmatrix}$
$T - SFCA_\Theta$	$\begin{pmatrix}0.5994,\\0.3295,\\0.6766\end{pmatrix}$	$\begin{pmatrix}0.6679,\\0.1462,\\0.5305\end{pmatrix}$	$\begin{pmatrix}0.6812,\\0.3815,\\0.5145\end{pmatrix}$	$\begin{pmatrix}0.7988,\\0.2192,\\0.0\end{pmatrix}$	$\begin{pmatrix}0.6466,\\0.4362,\\0.5574\end{pmatrix}$
$Ass.IP - T - SFOWA_\vee$	$\begin{pmatrix}0.7982,\\0.2432,\\0.4925\end{pmatrix}$	$\begin{pmatrix}0.7748,\\0.1439,\\0.4872\end{pmatrix}$	$\begin{pmatrix}0.7474,\\0.3050,\\0.4459\end{pmatrix}$	$\begin{pmatrix}0.9111,\\0.2171,\\0.0\end{pmatrix}$	$\begin{pmatrix}0.7423,\\0.4376,\\0.3034\end{pmatrix}$
$\mathrm{Ass.}IP - T - SFOWA_\wedge$	$\begin{pmatrix}0.5748,\\0.3600,\\0.6752\end{pmatrix}$	$\begin{pmatrix}0.5008,\\0.2616,\\0.5287\end{pmatrix}$	$\begin{pmatrix}0.6787,\\0.3867,\\0.5581\end{pmatrix}$	$\begin{pmatrix}0.7954,\\0.2974,\\0.0\end{pmatrix}$	$\begin{pmatrix}0.6411,\\0.4950,\\0.5717\end{pmatrix}$

This seems meaningless because the averaging aggregation operators cannot aggregate the non-membership value of d_4 so valid aggregate of d_4 is not obtained.

The score values of aggregated operators are listed in Table 22, and the corresponding score values the ranking of alternatives is shown in the Table 23.

Table 22 Score values of Table 21

	d_1	d_2	d_3	d_4	d_5
$T - SFOWA_w$	0.1176	0.1754	0.1226	0.6967	0.1687
$IP - T - SFOWA_P$	-0.0324	0.1279	0.1556	0.6352	0.0749

$T-SFCA_{\Theta}$	-0.1445	0.1655	0.1397	0.5579	0.0159
$Ass.IP-T-SFOWA_{\vee}$	0.4077	0.3822	0.3353	0.7877	0.3326
$\text{Ass}.IP-T-SFOWA_{\wedge}$	-0.1832	-0.0472	0.0901	0.5320	-0.0493

Table 23 Ranking order of the alternatives

Operators	Rankings
$T-SFOWA_w$	$d_4 \geq d_2 \geq d_5 \geq d_3 \geq d_1$
$IP-T-SFOWA_P$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$
$T-SFCA_{\Theta}$	$d_4 \geq d_2 \geq d_3 \geq d_5 \geq d_1$
$Ass.IP-T-SFOWA_{\vee}$	$d_4 \geq d_1 \geq d_2 \geq d_3 \geq d_5$
$\text{Ass}.IP-T-SFOWA_{\wedge}$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$

The ranking results in Table 23 are not accurate because non-membership value of d_4 in averaging aggregation operators has not been aggregated.

Now the aggregated values of all proposed interactive aggregation operators are shown in Table 24, the corresponding score values are listed in Table 25, and the ranking of alternatives is summarized in Table 26.

Table 24 Aggregated Values by interactive aggregation operators

	d_1	d_2	d_3	d_4	d_5
$T-SFOWIA_w$	$\begin{pmatrix}0.6790,\\0.3389,\\0.6681\end{pmatrix}$	$\begin{pmatrix}0.6658,\\0.2433,\\0.5577\end{pmatrix}$	$\begin{pmatrix}0.7154,\\0.3700,\\0.5365\end{pmatrix}$	$\begin{pmatrix}0.8718\\0.2722,\\0.5942\end{pmatrix}$	$\begin{pmatrix}0.6911,\\0.4721,\\0.5452\end{pmatrix}$
$IP-T-SFOWIA_P$	$\begin{pmatrix}0.6074,\\0.3315,\\0.7050\end{pmatrix}$	$\begin{pmatrix}0.6328,\\0.2138,\\0.5496\end{pmatrix}$	$\begin{pmatrix}0.6967,\\0.3857,\\0.5452\end{pmatrix}$	$\begin{pmatrix}0.8397,\\0.2508,\\0.6502\end{pmatrix}$	$\begin{pmatrix}0.6630,\\0.4639,\\0.6019\end{pmatrix}$
$T-SFCIA_{\Theta}$	$\begin{pmatrix}0.5994,\\0.3498,\\0.7428\end{pmatrix}$	$\begin{pmatrix}0.6679,\\0.1991,\\0.5552\end{pmatrix}$	$\begin{pmatrix}0.6812,\\0.3911,\\0.5303\end{pmatrix}$	$\begin{pmatrix}0.7988,\\0.2384,\\0.6881\end{pmatrix}$	$\begin{pmatrix}0.6466,\\0.4436,\\0.6443\end{pmatrix}$
$Ass.IP-T-SFOWIA_{\vee}$	$\begin{pmatrix}0.7982,\\0.3853,\\0.5960\end{pmatrix}$	$\begin{pmatrix}0.7748,\\0.2936,\\0.5204\end{pmatrix}$	$\begin{pmatrix}0.7474,\\0.3940,\\0.4757\end{pmatrix}$	$\begin{pmatrix}0.9111,\\0.3109,\\0.4613\end{pmatrix}$	$\begin{pmatrix}0.7423,\\0.5008,\\0.4089\end{pmatrix}$
$\text{Ass}.IP-T-SFOWIA_{\wedge}$	$\begin{pmatrix}0.5748,\\0.2763,\\0.7477\end{pmatrix}$	$\begin{pmatrix}0.5008,\\0.1912,\\0.5637\end{pmatrix}$	$\begin{pmatrix}0.6787,\\0.3249,\\0.5729\end{pmatrix}$	$\begin{pmatrix}0.7954,\\0.2338,\\0.6963\end{pmatrix}$	$\begin{pmatrix}0.6411,\\0.4456,\\0.6486\end{pmatrix}$

Table 25 Score Values of Table 24

	d_1	d_2	d_3	d_4	d_5
$T - SFOWIA_w$	-0.0262	0.1211	0.1783	0.4442	0.0691
$IP - T - SFOWIA_P$	-0.1846	0.0887	0.1318	0.3012	-0.0291
$T - SFCIA_\Theta$	-0.2568	0.1344	0.1199	0.1768	-0.0922
$Ass.IP - T - SFOWIA_\vee$	0.2528	0.3262	0.2744	0.6456	0.2363
Ass.$IP - T - SFOWIA_\wedge$	-0.2723	-0.0709	0.1008	0.1584	-0.1070

Table 26 Ranking Orderings

Operators	Rankings
$T - SFOWIA_w$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$
$IP - T - SFOWIA_P$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$
$T - SFCIA_\Theta$	$d_4 \geq d_2 \geq d_3 \geq d_5 \geq d_1$
$Ass.IP - T - SFOWIA_\vee$	$d_4 \geq d_2 \geq d_3 \geq d_1 \geq d_5$
Ass.$IP - T - SFOWIA_\wedge$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$

It is observed from the Tables 23 to 26, T-SFOWA and T-SFOWIA operators do not reflect interactions between some states of nature. While the T-SFCA and T-SFCIA operators reflect interactions between some states of nature but Ass. IP-T-SFOWA and Ass. IP-TSFOWIA operators reflect interactions among all states of nature.

4.4. Advantages

In this section, the advantages of proposed operators over existing operators are discussed and some conditions are also discussed under which the proposed operators become valid for existing operators.

Consider Ass.IP-T-SFOWIA operators

$$\boldsymbol{Ass.IP-T-SFOWIA_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)}$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \min\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right)}, \\ \sqrt[t]{1 - \min\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right)}, \\ \sqrt[t]{\begin{array}{c} \min\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right) - \\ \min\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t} - i_{\sigma(j)}^{t} - n_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right) \\ - \min\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right) \end{array}} \end{array} \right) \quad (4.4.1)$$

and

$$\boldsymbol{Ass.IP-T-SFOWIA_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)}$$

$$= \left(\begin{array}{c} \sqrt[t]{1 - \max\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right)}, \\ \sqrt[t]{1 - \max\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right)}, \\ \sqrt[t]{\begin{array}{c} \max\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right) - \\ \max\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(1 - m_{\sigma(j)}^{t} - i_{\sigma(j)}^{t} - n_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right) \\ - \max\limits_{\rho \in X_n} \left(\prod\limits_{j=1}^{k} \left(i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}} \right) \end{array}} \end{array} \right) \quad (4.4.2)$$

1. For $t = 2$, Eqs. (4.4.1) and (4.4.2) reduces to associate immediate probability spherical fuzzy ordered weighted interaction averaging (Ass.IP-SFOWIA) operator

$$Ass.IP-SFOWIA_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\sqrt{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\\ \sqrt{\begin{array}{c}\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\\ \min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2-i_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)\end{array}}\end{array}\right)$$

and

$$Ass.IP-SFOWIA_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt{1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\sqrt{1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\\ \sqrt{\begin{array}{c}\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\\ \max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2-i_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)\end{array}}\end{array}\right)$$

2. For $t=1$, Eqs. (4.4.1) and (4.4.2) reduces to associate immediate probability picture fuzzy ordered weighted interaction averaging (Ass.IP-PFOWIA) operator

$$Ass.IP-PFOWIA_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\\ \min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\\ \min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}-i_{\sigma(j)}-n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\end{array}\right)$$

and

$$Ass.IP-PFOWIA_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c} 1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}\right)^{\lambda'_j}\right), 1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}\right)^{\lambda'_j}\right), \\ \max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}\right)^{\lambda'_j}\right)- \\ \max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}-i_{\sigma(j)}-n_{\sigma(j)}\right)^{\lambda'_j}\right)-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda'_j}\right) \end{array}\right)$$

3. For $t=2$ and $i=0$, Eqs. (4.4.1) and (4.4.2) reduces to associate immediate probability Pythagorean fuzzy ordered weighted interaction averaging (Ass.IP-PyFOWIA) operator

$$Ass.IP-PyFOWIA_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c} \sqrt{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)}, \\ \sqrt{\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)} \end{array}\right)$$

and

$$Ass.IP-PyFOWIA_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c} \sqrt{1-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)}, \\ \sqrt{\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\max_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)} \end{array}\right)$$

4. For $t=1$ and $i=0$, Eqs. (4.4.1) and (4.4.2) reduces to Ass.IP-IFOWIA operator

$$Ass.IP - IFOWIA_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k)$$
$$= \left(\begin{array}{c} 1 - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right), \\ \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)} - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) \end{array} \right)$$

and

$$Ass.IP - IFOWIA_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k)$$
$$= \left(\begin{array}{c} 1 - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right), \\ \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)} - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) \end{array} \right)$$

Similarly we can reduce T-SFOWIA, IP-T-SFOWIA, T-SFCIA and T-SFCCIA operators. Another advantage of the proposed operators is that they aggregate that information where the existing operator fails.

Next we investigate a comparison analysis between proposed method and existing work. The existing operators have some limitations that the existing operators cannot handle the information given in PyFSs, PFSs, SFSs and T-SFSs. The proposed operators are most generalized that they can handle the information given in IFSs, PyFSs, PFSs, SFSs and T-SFSs. Here with the help of an example discussed in [66], it is shown that the proposed operators can solve the information given in IFSs.

4.4.1. Example

Consider a normalized decision matrix in which information is given in IFNs given in Table 27.

Table 27 Decision Matrix for Example 4.4.1.

	d_1	d_2	d_3
g_1	$(0.60, 0.30)$	$(0.50, 0.20)$	$(0.60, 0.35)$
g_2	$(0.60, 0.30)$	$(0.50, 0.20)$	$(0.20, 0.00)$
g_3	$(0.31, 0.00)$	$(0.50, 0.20)$	$(0.60, 0.35)$
g_4	$(0.20, 0.00)$	$(0.50, 0.20)$	$(0.60, 0.30)$

g_5	$(0.70, 0.30)$	$(0.40, 0.20)$	$(0.80, 0.10)$
g_6	$(0.60, 0.30)$	$(0.80, 0.20)$	$(0.50, 0.20)$

The given information can be written as T-spherical fuzzy information and hence summarized their values in Table 2

Table 28 Decision Matrix in T-SF information

	d_1	d_2	d_3
g_1	$(0.60, 0.00, 0.30)$	$(0.50, 0.00, 0.20)$	$(0.60, 0.00, 0.35)$
g_2	$(0.60, 0.00, 0.30)$	$(0.50, 0.00, 0.20)$	$(0.20, 0.00, 0.00)$
g_3	$(0.31, 0.00, 0.00)$	$(0.50, 0.00, 0.20)$	$(0.60, 0.00, 0.35)$
g_4	$(0.20, 0.00, 0.00)$	$(0.50, 0.00, 0.20)$	$(0.60, 0.00, 0.30)$
g_5	$(0.70, 0.00, 0.30)$	$(0.40, 0.00, 0.20)$	$(0.80, 0.00, 0.10)$
g_6	$(0.60, 0.00, 0.30)$	$(0.80, 0.00, 0.20)$	$(0.50, 0.00, 0.20)$

The weight vector for attributes will be $w = \{0.25, 0.40, 0.35\}$ and fuzzy measures will be as defined in [52]:

$\Theta(\phi) = 0$, $\Theta(\{d_1\}) = 0.175$, $\Theta(\{d_2\}) = 0.125$, $\Theta(\{d_3\}) = 0.100$, $\Theta(\{d_1, d_2\}) = 0.500$, $\Theta(\{d_1, d_3\}) = 0.425$, $\Theta(\{d_2, d_3\}) = 0.475$, $\Theta(\{d_1, d_2, d_3\}) = 1$.

Immediate probabilities and associated immediate probabilities for all possible orders are computed given in Table 29 and Table 30, respectively.

Table 29 Immediate Probabilities

	λ'_1	λ'_2	λ'_3
$\sigma = (d_1, d_2, d_3)$	0.175	0.325	0.500
$\sigma = (d_1, d_3, d_2)$	0.175	0.575	0.250
$\sigma = (d_2, d_1, d_3)$	0.375	0.125	0.500
$\sigma = (d_2, d_3, d_1)$	0.525	0.175	0.350
$\sigma = (d_3, d_1, d_2)$	0.325	0.575	0.100
$\sigma = (d_3, d_2, d_1)$	0.525	0.375	0.100

Table 30 Associated Immediate Probability

	$\lambda'_{\rho(1)}$	$\lambda'_{\rho(2)}$	$\lambda'_{\rho(3)}$
$\sigma = (d_1, d_2, d_3)$	0.1885	0.3500	0.4615
$\sigma = (d_1, d_3, d_2)$	0.1815	0.5963	0.2222
$\sigma = (d_2, d_1, d_3)$	0.4038	0.1346	0.4615
$\sigma = (d_2, d_3, d_1)$	0.5526	0.1316	0.3158

$\sigma = (d_3, d_1, d_2)$	0.3297	0.5833	0.0870
$\sigma = (d_3, d_2, d_1)$	0.5326	0.3804	0.0870

As $0.6 + 0.0 + 0.3 = 0.9 \in [0, 1]$ similarly for $t = 1$, all values lie in T-SFSs. So here $t = 1$ is taken. Then the aggregated values for $t = 1$ are summarized in Table 31.

Table 31 Aggregated Values

	g_1	g_2	g_3	g_4	g_5	g_6
T-SFOWA	$\begin{pmatrix}0.57,\\0.00,\\0.27\end{pmatrix}$	$\begin{pmatrix}0.47,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.48,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.45,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.66,\\0.00,\\0.19\end{pmatrix}$	$\begin{pmatrix}0.66,\\0.00,\\0.23\end{pmatrix}$
T-SFCA	$\begin{pmatrix}0.57,\\0.00,\\0.26\end{pmatrix}$	$\begin{pmatrix}0.54,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.42,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.51,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.72,\\0.00,\\0.16\end{pmatrix}$	$\begin{pmatrix}0.73,\\0.00,\\0.23\end{pmatrix}$
Ass.IP-T-SFOW$A_\vee$	$\begin{pmatrix}0.588,\\0.00,\\0.244\end{pmatrix}$	$\begin{pmatrix}0.537,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.521,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.528,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.725,\\0.00,\\0.154\end{pmatrix}$	$\begin{pmatrix}0.718,\\0.00,\\0.211\end{pmatrix}$
Ass.IP-T-SFOW$A_\wedge$	$\begin{pmatrix}0.543,\\0.00,\\0.305\end{pmatrix}$	$\begin{pmatrix}0.404,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.418,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.385,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.584,\\0.00,\\0.228\end{pmatrix}$	$\begin{pmatrix}0.611,\\0.00,\\0.255\end{pmatrix}$

The score values of aggregated values are given in Table 32 and the ranking of all alternatives through score values or accuracy function is represented in Table 33.

Table 32 Score Values

	g_1	g_2	g_3	g_4	g_5	g_6
T-SFOWA	0.3119	0.5312	0.5412	0.5108	0.4873	0.4416
T-SFCA	0.3231	0.6006	0.4800	0.5712	0.5764	0.5049
Ass.IP-T-SFOW$A_\vee$	0.3583	0.5977	0.5820	0.5889	0.5878	0.5158
ASS.IP-T-SFOW$A_\wedge$	0.2470	0.4634	0.4779	0.4435	0.3054	0.3678

Table 33 Rankings of Alternatives

Operators	Rankings

T-SFOWA	$g_3 > g_2 > g_4 > g_5 > g_6 > g_1$
T-SFCA	$g_2 > g_5 > g_4 > g_6 > g_3 > g_1$
Ass.IP-T-SFOW$A_{\vee}$	$g_2 > g_4 > g_5 > g_3 > g_6 > g_1$
ASS.IP-T-SFOW$A_{\wedge}$	$g_3 > g_2 > g_4 > g_6 > g_5 > g_1$

From above example it is clear that the results obtained from proposed operators are similar to existing operators. This proves that the proposed operators are generalizations of existing operator.

Chapter 5

Methods for multi-attribute decision making in T-spherical fuzzy sets using associated immediate probability interactive geometric aggregation operators

In this chapter, associated immediate probability geometric aggregation operators have been developed for T-SFSs and associated immediate probability interactive geometric aggregation operators are proposed. Then a comparison between these operators is developed with the help of an example. The existing score function for T-SFSs does not involve abstinence so a new score function is developed which provides a better comparison between any two T-SFNs. Then to check the reliability of proposed operators an application for MADM problem is developed. The advantages of proposed work are also discussed in which it is shown that under some conditions the proposed operators can be reduced to other tools of uncertainty. The comparison between existing and proposed work is also developed with the help of an example.

5.1. New Score Function and Geometric Operators for T-Spherical Fuzzy Sets

This section is further divided into four subsections. In first subsection a new score function is proposed which involve abstinence while existing score function does not involve abstinence. Second subsection have some T-spherical geometric aggregation operators while third subsection have some T-spherical geometric interactive aggregation operators. A comparison between geometric aggregation operators and geometric interactive aggregation operators is discussed in fourth subsection.

5.1.1. Score function

The score function defined in [16] does not involve abstinence so better comparison is not done by using this score function. To overcome this shortcoming a new score function is proposed in this subsection in which abstinence is involved.

5.1.1.1. Definition

The new score function for any T-SFN $\mathcal{T} = (m, i, n)$ is defined as

$$SF(\mathcal{T}) = m^t - i^t - n^t$$

$$AC(\mathcal{T}) = m^t + i^t + n^t$$

The T-SFN which have greater score value will be superior to other. If the score of two T-SFNs is equal, then we rank them using accuracy value and a number is called superior if it has greater accuracy. If again accuracy values of two T-SFNs become equal, then both numbers are considered as similar.

5.1.1.2. Remark

a) The proposed score function reduced to SFSs for $t = 2$

b) The proposed score function reduced to PFSs for $t = 1$

c) The proposed score function reduced to PyFSs for $t = 2$ and $i = 0$

d) The proposed score function reduced to IFSs for $t = 1$ and $i = 0$

5.1.2. T-Spherical Fuzzy Geometric Aggregation operators

In this subsection different geometric aggregation operators are proposed using a tool of uncertainty called T-SFSs. Here IP-T-SFOWG, T-SFCG, Ass.IP-T-SFOWG operators are proposed and some basic properties of all these operators are also discussed.

5.1.2.1. Definition

The IP-T-SFOWG operator for a collection of T-SFNs, $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ is defined as

$$IP - T - SFOWG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \otimes_{j=1}^{k} \left(\mathcal{T}_{\sigma(j)}^{\lambda_j'}\right),$$

where $w = (w_1, \dots, w_k)^T$ is a WV with a condition that all WVs belong to $[0,1]$ and the sum of all weights must be equal to 1. λ_j is probability for each $\mathcal{T}_j$, λ_j' is an immediate probability of $\mathcal{T}_{\sigma(j)}$ and $\lambda_j' = \frac{(w_j \lambda_j)}{\sum_{j=1}^{k} w_j \lambda_j}$ and $\sigma = \left(\sigma(1), \dots, \sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right) \geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

5.1.2.2. Theorem

For a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ for all $j = 1,2, \dots, k$, then

$$IP-T-SFOWG_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\left(\prod_{j=1}^{k} m_{\sigma(j)}^{\lambda'_j},\prod_{j=1}^{k} i_{\sigma(j)}^{\lambda'_j},\sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_j}}\right),$$

Proof: By using mathematical induction,

For $k=1$,

$$IP-T-SFOWG_P(\mathcal{T}_1)=\left(m_1,i_1,\sqrt[t]{1-(1-n_1^t)}\right)$$

$$=(m_1,i_1,n_1)$$

Now let us assume that the results hold for $k=l$,

$$IP-T-SFOWG_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_l)=\left(\prod_{j=1}^{l} m_{\sigma(j)}^{\lambda'_j},\prod_{j=1}^{l} i_{\sigma(j)}^{\lambda'_j},\sqrt[t]{1-\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_j}}\right)$$

Then to prove that result hold for $k=l+1$,

$$IP-T-SFOWG_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_{l+1})$$

$$=\left(\prod_{j=1}^{l} m_{\sigma(j)}^{\lambda'_j},\prod_{j=1}^{l} i_{\sigma(j)}^{\lambda'_j},\sqrt[t]{1-\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_j}}\right)$$

$$\otimes\left(m_{\sigma(j)}^{\lambda'_j},i_{\sigma(j)}^{\lambda'_j},\sqrt[t]{1-\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_j}}\right)$$

$$=\left(\prod_{j=1}^{l+1} m_{\sigma(j)}^{\lambda'_j},\prod_{j=1}^{l+1} i_{\sigma(j)}^{\lambda'_j},\sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_j}}\right)$$

This proves that the results hold for all $k\in Z^{+}$.

5.1.2.3. Theorem

IP-T-SFOWG operator satisfies the following conditions:

i. If $\mathcal{T}_j=\mathcal{T}_0=(m_0,i_0,n_0)$ for all $j=1,2,\dots,k$ then $IP-T-SFOWG_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\mathcal{T}_0$.

ii. For any two T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ and $\mathcal{T}_j{}' = (m_j', i_j', n_j')$ such that $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$. Then

$$IP-T-SFOWG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq IP-T-SFOWG_P(\mathcal{T}_1{}', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

iii. For a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2,\dots,k$ such that $\mathcal{T}_L = \left(\min_j(m_j), \min_j(i_j), \max_j(n_j)\right)$ is minimal element and $\mathcal{T}_U = \left(\max_j(m_j), \max_j(i_j), \min_j(n_j)\right)$ is maximal element. Then

$$\mathcal{T}_L \leq IP-T-SFOWG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}_U$$

Proof: i. As $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ then

$$IP-T-SFOWG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \left(\prod_{j=1}^{k} m_{\sigma(j)}^{\lambda_j'}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda_j'}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda_j'}}\right)$$

$$= \left(m_{\sigma(j)}^{\sum_{j=1}^{k}\lambda_j'}, i_{\sigma(j)}^{\sum_{j=1}^{k}\lambda_j'}, \sqrt[t]{1 - \left(1 - n_{\sigma(j)}^t\right)^{\sum_{j=1}^{k}\lambda_j'}}\right)$$

$$= (m_0, i_0, n_0) = \mathcal{T}_0$$

ii. As $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$. This implies that

$$\prod_{j=1}^{k} m_{\sigma(j)} \leq \prod_{j=1}^{k} m_{\sigma(j)}' \ ,$$

$$\prod_{j=1}^{k} i_{\sigma(j)} \leq \prod_{j=1}^{k} i_{\sigma(j)}' \ ,$$

$$\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda_j'}} \geq \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - \left(n_{\sigma(j)}'\right)^t\right)^{\lambda_j'}}$$

$$\left(\prod_{j=1}^{k} m_{\sigma(j)}^{\lambda_j'}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda_j'}, \sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda_j'}}\right)$$

$$\leq \left(\prod_{j=1}^{k}\left(m_{\sigma(j)}'\right)^{\lambda_j'}, \prod_{j=1}^{k}\left(i_{\sigma(j)}'\right)^{\lambda_j'}, \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(n_{\sigma(j)}'\right)^{t}\right)^{\lambda_j'}}\right)$$

iii. Proof is straightforward.

5.1.2.4. Definition

The T-SFCG operator for a collection of T-SFNs, $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ with respect to fuzzy measure Θ is defined as

$$T-SFCG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \otimes_{j=1}^{k}\left(\mathcal{T}_{\sigma(j)}^{\lambda_j}\right)$$

where $\lambda_j = \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \dots \dots, x_{\sigma(j)}\right\}\right) - \Theta\left(\left\{x_{\sigma(1)}, x_{\sigma(2)}, \dots \dots, x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right) \equiv 0$ and σ is the permutation. $\sigma = \left(\sigma(1), \sigma(2), \dots \dots, \sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right) \geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

5.1.2.5. Theorem

For a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ for all $j = 1,2, \dots, k$, then

$$T-SFCG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \left(\prod_{j=1}^{k} m_{\sigma(j)}^{\lambda_j}, \prod_{j=1}^{k} i_{\sigma(j)}^{\lambda_j}, \sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda_j}}\right)$$

5.1.2.6. Theorem

T-SFCG operators satisfies the following properties:

i. If $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2, \dots, k$ then $T-SFCG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

ii. For any two T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ and $\mathcal{T}_j' = \left(m_j', i_j', n_j'\right)$ such that $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$. Then

$$T-SFCG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T-SFCG_{\Theta}(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

iii. For a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2, \dots, k$ such that $\mathcal{T}_L = \left(\min_j(m_j), \min_j(i_j), \max_j(n_j)\right)$ is minimal element and $\mathcal{T}_U = \left(\max_j(m_j), \max_j(i_j), \min_j(n_j)\right)$ is maximal element. Then

$$\mathcal{T}_L \leq T - SFCG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}_U$$

5.1.2.7. Definition

The Ass.IP-T-SFOWG operators for a collection of T-SFNs, $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2, \dots, k$, is defined as

$$Ass.IP - T - SFOWG_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigvee_{\rho \in X_n} \left[\otimes_{j=1}^{k} \left(\mathcal{T}_{\sigma(j)}^{\lambda'_{\rho(j)}}\right)\right]$$

and

$$Ass.IP - T - SFOWG_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigwedge_{\rho \in X_n} \left[\otimes_{j=1}^{k} \left(\mathcal{T}_{\sigma(j)}^{\lambda'_{\rho(j)}}\right)\right],$$

where $w = (w_1, \dots, w_k)^T$ is a WV with a condition that all WVs belong to $[0,1]$ and the sum of all weights must be equal to 1. $\sigma = (\sigma(1), \dots \dots, \sigma(k))$ is the permutation such that $SC(\mathcal{T}_{\sigma(j-1)}) \geq SC(\mathcal{T}_{\sigma(j)})$. For each associated probability P_ρ: $\lambda'_{\rho(j)} = \frac{(w_j \lambda_{\rho(j)})}{\sum_{j=1}^{k} w_j \lambda_{\rho(j)}}$, $\lambda_{\rho(j)} \equiv P_\rho(\mathcal{T}_{\sigma(j)})$ is an associated immediate probability and $\vee$=maximum and $\wedge$ =minimum.

5.1.2.8. Theorem

For a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2, \dots, k$, then

$$Ass.IP - T - SFOWG_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$
$$= \left(\max_{\rho \in X_n}\left(\prod_{j=1}^{k}(m_{\sigma(j)})^{\lambda'_{\rho(j)}}\right), \max_{\rho \in X_n}\left(\prod_{j=1}^{k}(i_{\sigma(j)})^{\lambda'_{\rho(j)}}\right), \sqrt[t]{1 - \max_{\rho \in X_n}\left(\prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)}\right)$$

and

$Ass.IP-T-SFOWG_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$

$$=\left(\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(m_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right),\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right),\sqrt[t]{1-\min_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_{\rho(j)}}\right)}\right)$$

5.1.2.9. Theorem

Ass.IP-T-SFOWG operators satisfies the following properties:

i. If $\mathcal{T}_j=\mathcal{T}_0=(m_0,i_0,n_0)$ for all $j=1,2,\dots,k$ then $Ass.IP-T-SFOWG_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\mathcal{T}_0$ and $Ass.IP-T-SFOWG_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\mathcal{T}_0$

ii. For any two T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ and $\mathcal{T}_j'=\left(m_j',i_j',n_j'\right)$ such that $m_j\leq m_j'$, $i_j\leq i_j'$ and $n_j\geq n_j'$. Then

$$Ass.IP-T-SFOWG_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)\leq Ass.IP-T-SFOWG_{\vee}(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

and

$$Ass.IP-T-SFOWG_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)\leq Ass.IP-T-SFOWG_{\wedge}(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

iii. For a collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ for all $j=1,2,\dots,k$ such that $\mathcal{T}_L=\left(\min_j\left(m_j\right),\min_j\left(i_j\right),\max_j\left(n_j\right)\right)$ is minimal element and $\mathcal{T}_U=\left(\max_j\left(m_j\right),\max_j\left(i_j\right),\min_j\left(n_j\right)\right)$ is maximal element. Then

$$\mathcal{T}_L\leq Ass.IP-T-SFOWG_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)\leq\mathcal{T}_U$$

and

$$\mathcal{T}_L\leq Ass.IP-T-SFOWG_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)\leq\mathcal{T}_U$$

5.1.3. T-Spherical Fuzzy Interactive Aggregation operators

In this subsection some geometric interactive aggregation operators are proposed by using a tool of uncertainty called T-SFSs. In this subsection IP-T-SFOWIG, T-SGCIG and Ass.IP-T-SFOWIG operators are proposed and some basic properties of these operators are also discussed.

5.1.3.1. Definition

The IP-T-SFOWIG operator for a collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ is defined as

$$IP-T-SFOWIG_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\otimes i_{j=1}^{k}\left(\mathcal{T}_{\sigma(j)}^{\lambda_j'}\right),$$

where $w=(w_1,\dots,w_k)^T$ is a WV with a condition that all WVs belong to $[0,1]$ and the sum of all weights must be equal to 1. λ_j is probability for each $\mathcal{T}_j$, λ_j' is an immediate probability of $\mathcal{T}_{\sigma(j)}$ and $\lambda_j'=\frac{(w_j\lambda_j)}{\sum_{j=1}^{k}w_j\lambda_j}$ and $\sigma=\big(\sigma(1),\dots,\sigma(k)\big)$ is the permutation such that $SC\big(\mathcal{T}_{\sigma(j-1)}\big)\geq SC\big(\mathcal{T}_{\sigma(j)}\big)$.

5.1.3.2. Theorem

For a collection of T-SFNs $\mathcal{T}_j=\big(m_j,i_j,n_j\big)$, then

$$IP-T-SFOWIG_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$
$$=\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\lambda_j'}-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{\lambda_j'}-\prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\lambda_j'}},\\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^t\right)^{\lambda_j'}},\sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\lambda_j'}}\end{array}\right)$$

Proof: By using mathematical induction,

For $k=1$,

$$IP-T-SFOWIG_P(\mathcal{T}_1)$$
$$=\left(\sqrt[t]{(1-n_1^t)-(1-m_1^t-i_1^t-n_1^t)-i_1^t},\sqrt[t]{1-(1-i_1^t)},\sqrt[t]{1-(1-n_1^t)}\right)$$
$$=(m_1,i_1,n_1)$$

Now assume that the results hold for $k=l$,

$$IP-T-SFOWIG_P(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_l)$$
$$=\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^t\right)^{\lambda_j'}-\prod_{j=1}^{l}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{\lambda_j'}-\prod_{j=1}^{l}\left(i_{\sigma(j)}^t\right)^{\lambda_j'}},\\ \sqrt[t]{1-\prod_{j=1}^{l}\left(1-i_{\sigma(j)}^t\right)^{\lambda_j'}},\sqrt[t]{1-\prod_{j=1}^{l}\left(1-n_{\sigma(j)}^t\right)^{\lambda_j'}}\end{array}\right)$$

Then to prove that result hold for $k = l + 1$,

$IP - T - SFOWIG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_{l+1})$

$$= \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{l} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{l} \left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{l} \left(i_{\sigma(j)}^t\right)^{\lambda'_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{l} \left(1 - i_{\sigma(j)}^t\right)^{\lambda'_j}}, \sqrt[t]{1 - \prod_{j=1}^{l} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j}} \end{array} \right)$$

$$\otimes_i \left(\begin{array}{c} \sqrt[t]{\left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j} - \left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda'_j} - \left(i_{\sigma(j)}^t\right)^{\lambda'_j}}, \\ \sqrt[t]{1 - \left(1 - i_{\sigma(j)}^t\right)^{\lambda'_j}}, \sqrt[t]{1 - \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{l+1} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{l+1} \left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{l+1} \left(i_{\sigma(j)}^t\right)^{\lambda'_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{l+1} \left(1 - i_{\sigma(j)}^t\right)^{\lambda'_j}}, \sqrt[t]{1 - \prod_{j=1}^{l+1} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j}} \end{array} \right)$$

This proves that the results hold for all $k \in Z^+$.

5.1.3.3. Theorem

IP-T-SFOWIG operators satisfies the following properties:

i. If $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2, \dots, k$ then $IP - T - SFOWIG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

ii. For any two T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ and $\mathcal{T}_j{}' = \left(m'_j, i'_j, n'_j\right)$ such that $m_j \leq m'_j$, $i_j \leq i'_j$ and $n_j \geq n'_j$. Then

$$IP - T - SFOWIG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq IP - T - SFOWIG_P(\mathcal{T}'_1, \mathcal{T}'_2, \dots, \mathcal{T}'_k)$$

iii. For a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ for all $j = 1,2, \dots, k$ such that $\mathcal{T}_L = \left(\min_j\left(m_j\right), \min_j\left(i_j\right), \max_j\left(n_j\right)\right)$ is minimal element and $\mathcal{T}_U = \left(\max_j\left(m_j\right), \max_j\left(i_j\right), \min_j\left(n_j\right)\right)$ is maximal element. Then

$$\mathcal{T}_L \leq IP - T - SFOWIG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}_U$$

Proof: i. As $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ then

$$IP - T - SFOWIG_P(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\lambda'_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i_{\sigma(j)}^t\right)^{\lambda'_j}}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{\left(1 - n_{\sigma(j)}^t\right)^{\sum_{j=1}^{k}\lambda'_j} - \left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\sum_{j=1}^{k}\lambda'_j} - \left(i_{\sigma(j)}^t\right)^{\sum_{j=1}^{k}\lambda'_j}}, \\ \sqrt[t]{1 - \left(1 - i_{\sigma(j)}^t\right)^{\sum_{j=1}^{k}\lambda'_j}}, \sqrt[t]{1 - \left(1 - n_{\sigma(j)}^t\right)^{\sum_{j=1}^{k}\lambda'_j}} \end{array} \right)$$

$$= (m_0, i_0, n_0) = \mathcal{T}_0$$

ii. As $m_j \leq m'_j$, $i_j \leq i'_j$ and $n_j \geq n'_j$. This implies that

$$\sqrt[t]{\prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\lambda'_j}}$$

$$\leq \sqrt[t]{\prod_{j=1}^{k}\left(1 - \left(n'_{\sigma(j)}\right)^t\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(1 - \left(m'_{\sigma(j)}\right)^t - \left(i'_{\sigma(j)}\right)^t - \left(n'_{\sigma(j)}\right)^t\right)^{\lambda'_j} - \prod_{j=1}^{k}\left(\left(i'_{\sigma(j)}\right)^t\right)^{\lambda'_j}},$$

$$\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i_{\sigma(j)}^t\right)^{\lambda'_j}} \leq \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - \left(i'_{\sigma(j)}\right)^t\right)^{\lambda'_j}},$$

$$\sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^t\right)^{\lambda'_j}} \geq \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - \left(n'_{\sigma(j)}\right)^t\right)^{\lambda'_j}}$$

$$\left(\begin{array}{c}\sqrt[t]{\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_j}-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}-i_{\sigma(j)}^{t}-n_{\sigma(j)}^{t}\right)^{\lambda'_j}-\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\lambda'_j}},\\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^{t}\right)^{\lambda'_j}},\sqrt[t]{1-\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^{t}\right)^{\lambda'_j}}\end{array}\right)$$

$$\leq\left(\begin{array}{c}\sqrt[t]{\begin{array}{c}\prod_{j=1}^{k}\left(1-\left(n'_{\sigma(j)}\right)^{t}\right)^{\lambda'_j}-\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}-\left(i'_{\sigma(j)}\right)^{t}-\left(n'_{\sigma(j)}\right)^{t}\right)^{\lambda'_j}\\ -\prod_{j=1}^{k}\left(\left(i'_{\sigma(j)}\right)^{t}\right)^{\lambda'_j}\end{array}},\\ \sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(i'_{\sigma(j)}\right)^{t}\right)^{\lambda'_j}},\sqrt[t]{1-\prod_{j=1}^{k}\left(1-\left(n'_{\sigma(j)}\right)^{t}\right)^{\lambda'_j}}\end{array}\right)$$

iii. Proof is straightforward.

5.1.3.4. Definition

The T-SFCIG operator for a collection of T-SFNs, $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ with respect to fuzzy measure Θ is defined as

$$T-SFCIG_{\Theta}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\otimes i_{j=1}^{k}\left(\mathcal{T}_{\sigma(j)}^{\lambda_j}\right)$$

where $\lambda_j=\Theta\left(\left\{x_{\sigma(1)},x_{\sigma(2)},\dots\dots,x_{\sigma(j)}\right\}\right)-\Theta\left(\left\{x_{\sigma(1)},x_{\sigma(2)},\dots\dots,x_{\sigma(j-1)}\right\}\right)$, $\Theta\left(\left\{x_{\sigma(0)}\right\}\right)\equiv 0$ and σ is the permutation. $\sigma=\left(\sigma(1),\sigma(2),\dots\dots,\sigma(k)\right)$ is the permutation such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right)\geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

5.1.3.5. Theorem

For a collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ for all $j=1,2,\dots,k$, then

$$T - SFCIG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$
$$= \left(\begin{array}{c} \sqrt[t]{\prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^{t}\right)^{\lambda_j} - \prod_{j=1}^{k}\left(1 - m_{\sigma(j)}^{t} - i_{\sigma(j)}^{t} - n_{\sigma(j)}^{t}\right)^{\lambda_j} - \prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\lambda_j}}, \\ \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - i_{\sigma(j)}^{t}\right)^{\lambda_j}}, \sqrt[t]{1 - \prod_{j=1}^{k}\left(1 - n_{\sigma(j)}^{t}\right)^{\lambda_j}} \end{array} \right)$$

5.1.3.6. Theorem

T-SFCIG operators satisfies the following properties:

i. If $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2, \dots, k$ then $T - SFCIG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

ii. For any two T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ and $\mathcal{T}_j{}' = \left(m_j', i_j', n_j'\right)$ such that $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$. Then

$$T - SFCIG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T - SFCIG_{\Theta}(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

iii. For a collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ for all $j = 1,2, \dots, k$ such that $\mathcal{T}_L = \left(\min_j\left(m_j\right), \min_j\left(i_j\right), \max_j\left(n_j\right)\right)$ is minimal element and $\mathcal{T}_U = \left(\max_j\left(m_j\right), \max_j\left(i_j\right), \min_j\left(n_j\right)\right)$ is maximal element. Then

$$\mathcal{T}_L \leq T - SFCIG_{\Theta}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}_U$$

5.1.3.7. Definition

The Ass.IP-T-SFOWIG operators for a collection of T-SFNs, $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ is defined as

$$Ass.IP - T - SFOWIG_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigvee_{\rho \in X_n}\left[\otimes i_{j=1}^{k}\left(\mathcal{T}_{\sigma(j)}^{\lambda'_{\rho(j)}}\right)\right]$$

and

$$Ass.IP - T - SFOWIG_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \bigwedge_{\rho \in X_n}\left[\otimes i_{j=1}^{k}\left(\mathcal{T}_{\sigma(j)}^{\lambda'_{\rho(j)}}\right)\right]$$

where $w = (w_1, \ldots, w_k)^T$ is a WV with a condition that all WVs belong to $[0,1]$ and the sum of all weights must be equal to 1. $\sigma = (\sigma(1), \ldots, \sigma(k))$ is the permutation such that $SC(\mathcal{T}_{\sigma(j-1)}) \geq SC(\mathcal{T}_{\sigma(j)})$. For each associated probability P_ρ: $\lambda'_{\rho(j)} = \frac{(w_j \lambda_{\rho(j)})}{\sum_{j=1}^{k} w_j \lambda_{\rho(j)}}$, $\lambda_{\rho(j)} \equiv P_\rho(\mathcal{T}_{\sigma(j)})$ is an associated immediate probability and $\vee$=maximum and $\wedge$ =minimum.

5.1.3.8. Theorem

For a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2, \ldots, k$, then

$$Ass.IP - T - SFOWIG_\vee(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \sqrt[t]{\begin{array}{c} \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right) \\ - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right) - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right) \end{array}}, \\ \sqrt[t]{1 - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right)}, \sqrt[t]{1 - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right)} \end{array} \right)$$

and

$$Ass.IP - T - SFOWIG_\wedge(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \sqrt[t]{\begin{array}{c} \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right) \\ - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)}^t - i_{\sigma(j)}^t - n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right) - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right) \end{array}}, \\ \sqrt[t]{1 - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right)}, \sqrt[t]{1 - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}} \right)} \end{array} \right)$$

5.1.3.9. Theorem

Ass.IP-T-SFOWIG operators satisfies the following properties:

i. If $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2,\dots,k$ then $Ass.IP-T-SFOWIG_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$ and $Ass.IP-T-SFOWIG_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$

ii. For any two T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ and $\mathcal{T}_j{}' = (m_j', i_j', n_j')$ such that $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$. Then

$$Ass.IP-T-SFOWIG_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq Ass.IP-T-SFOWIG_{\vee}(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

and

$$Ass.IP-T-SFOWIG_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq Ass.IP-T-SFOWIG_{\wedge}(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

iii. For a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2,\dots,k$ such that $\mathcal{T}_L = \left(\min_j(m_j), \min_j(i_j), \max_j(n_j)\right)$ is minimal element and $\mathcal{T}_U = \left(\max_j(m_j), \max_j(i_j), \min_j(n_j)\right)$ is maximal element. Then

$$\mathcal{T}_L \leq Ass.IP-T-SFOWIG_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}_U$$

and

$$\mathcal{T}_L \leq Ass.IP-T-SFOWIG_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}_U$$

5.1.3.10. Remark

If fuzzy measure and probability of T-SFSs become equal and furthermore probabilities of all T-SFNs become equal then Ass.IP-T-SFOWIG operators become equal to T-SFOWG operator

5.1.4. Comparison between aggregation operators and interactive aggregation operators

In this subsection the superiority of interactive averaging aggregation operators over averaging aggregation operators is explained with the help of an example. It is also explained that under some conditions the averaging aggregation operators fail while interactive averaging aggregation operators overcome this shortcoming.

5.1.4.1. Example

Consider T-SFNs, $g_1 = (0.94, 0.25, 0.41)$, $g_2 = (0.00, 0.35, 0.47)$ and $g_3 = (0.74, 0.00, 0.39)$ having a WV $w = \{0.45, 0.40, 0.15\}$, fuzzy measures will be

$\Theta(\phi) = 0,\quad \Theta(\{g_1\}) = 0.225,\quad \Theta(\{g_2\}) = 0.200,\quad \Theta(\{g_3\}) = 0.075,$ $\Theta(\{g_1, g_2\}) = 0.425,\quad \Theta(\{g_1, g_3\}) = 0.300,\quad \Theta(\{g_2, g_3\}) = 0.275,$ $\Theta(\{g_1, g_2, g_3\}) = 1.$

Immediate probabilities for all possible permutations are listed in Table 34

Table 34 Immediate Probabilities

	λ'_1	λ'_2	λ'_3
$\sigma = (g_1, g_2, g_3)$	0.125	0.200	0.675
$\sigma = (g_1, g_3, g_2)$	0.125	0.700	0.175
$\sigma = (g_2, g_1, g_3)$	0.200	0.200	0.600
$\sigma = (g_2, g_3, g_1)$	0.700	0.200	0.100
$\sigma = (g_3, g_1, g_2)$	0.125	0.700	0.175
$\sigma = (g_3, g_2, g_1)$	0.700	0.125	0.175

and associated immediate probabilities for all possible permutation are listed in Table 35

Table 35 Associated Immediate Probabilities

	$\lambda'_{\rho(1)}$	$\lambda'_{\rho(2)}$	$\lambda'_{\rho(3)}$
$\sigma = (g_1, g_2, g_3)$	0.0899	0.2302	0.6798
$\sigma = (g_1, g_3, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_2, g_1, g_3)$	0.1470	0.2353	0.6176
$\sigma = (g_2, g_3, g_1)$	0.6034	0.2759	0.1207
$\sigma = (g_3, g_1, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_3, g_2, g_1)$	0.6114	0.1747	0.2140

As for $t = 1,\; 0.94 + 0.25 + 0.41 = 1.6 \notin [0, 1]$, As for $t = 2$, $0.94^2 + 0.25^2 + 0.41^2 = 1.11 \notin [0, 1]$

As for $t = 3$, $0.94^3 + 0.25^3 + 0.41^3 = 0.92 \in [0, 1]$

So for $t = 3$, the given information lie in T-spherical fuzzy environment.

Then the aggregate of all averaging aggregation operators defined in Definition 5.1.2.1., 5.1.2.4., and 5.1.2.7 will be

$$IP - T - SFOWG_P(g_1, g_2, g_3) = (0.0, 0.0, 0.4244)$$

$$T - SFCG_\Theta(g_1, g_2, g_3) = (0.0, 0.0, 0.4253)$$

$$Ass.IP - T - SFOWG_\vee(g_1, g_2, g_3) = (0.0, 0.0, 0.4176)$$

$$Ass.IP - T - SFOWG_\wedge(g_1, g_2, g_3) = (0.0, 0.0, 0.4568)$$

This seems meaningless as geometric aggregation operators ignore all membership and abstinence values as one of their value become zero. That's why results obtain through these aggregation operators are not valid. To overcome this shortcoming we used interactive averaging aggregation operators. The results obtained by using interactive operators defined in definitions 5.1.3.1., 5.1.3.4., and 5.1.3.7 will be

$$IP - T - SFOWIG_P(g_1, g_2, g_3) = (0.8513, 0.2663, 0.4244)$$

$$T - SFCIG_\Theta(g_1, g_2, g_3) = (0.8625, 0.2698, 0.4253)$$

$$Ass.IP - T - SFOWIG_\vee(g_1, g_2, g_3) = (0.9276, 0.2626, 0.4176)$$

$$Ass.IP - T - SFOWIG_\wedge(g_1, g_2, g_3) = (0.6897, 0.3291, 0.4568)$$

The proposed interactive operators aggregate all membership, abstinence and non-membership values. This shows the superiority of interactive aggregation operators and the results obtained using these interactive operators are more reliable.

5.2. Algorithm for MADM based on proposed operators

In this section an algorithm was developed to solve MADM problem using the proposed aggregation and interactive aggregation operators and a well-known MADM example is solved by using the algorithm.

Consider a set of alternatives $D = \{d_1, d_2, \dots, d_l\}$ and set of attributes $G = \{g_1, g_2, \dots, g_k\}$ having a WV $w = \{w_1, w_2, \dots, w_k\}$, set of probabilities associated with them is $\mathcal{T} = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and associate immediate probabilities are $\mathcal{T}' = \{\lambda'_{\rho(1)}, \lambda'_{\rho(2)}, \dots, \lambda'_{\rho(k)}\}$. The WV and set of probabilities have a same condition that the sum of weights and probabilities must equal to 1 and weights and probabilities belong to closed unit interval. The fuzzy measure Θ have been calculated for all subsets of $\{d_1, d_2, \dots, d_l\}$. Then to solve MADM problem we have to follow the following steps.

Step 1. Find the value of t for which information lie in T-spherical fuzzy environment.

Step 2. Aggregate the data using proposed operators.

Step 3. Find the score of aggregated values.

Step 4. With the help of score values choose the best option.

5.2.1. Numerical Example

A manufacturing company wants to select a supplier among five alternatives $D = \{d_1, d_2, d_3, d_4, d_5\}$, the experts evaluate these alternatives under consideration of following three attributes

g_1: Price

g_2: Delivery Compliance

g_3: Technological capability

The experts evaluate these alternatives with respect to attributes and give information in membership, abstinence and non-membership as in Table 36.

Table 36 Decision Matrix

	g_1	g_2	g_3
d_1	(0.47,0.25,0.88)	(0.34,0.19,0.51)	(0.51,0.29,0.77)
d_2	(0.23,0.41,0.75)	(0.61,0.15,0.49)	(0.53,0.22,0.48)
d_3	(0.89,0.33,0.43)	(0.74,0.46,0.49)	(0.84,0.36,0.60)
d_4	(0.91,0.33,0.42)	(0.55,0.23,0.71)	(0.88,0.39,0.61)
d_5	(0.59,0.43,0.51)	(0.66,0.22,0.36)	(0.63,0.41,0.72)

As for $t = 1$, $0.47 + 0.25 + 0.88 = 1.6 \notin [0, 1]$, so the given information does not lie in picture fuzzy environment,

As for $t = 2$, $0.47^2 + 0.25^2 + 0.88^2 = 1.0578 \notin [0, 1]$, so the given information also does not lie in spherical fuzzy environment,

As for $t = 3$, $0.47^3 + 0.25^3 + 0.88^3 = 0.8009 \in [0, 1]$

Similarly for $t = 3$ all the information given in Table 36 are T-SFNs so the information lie in T-spherical fuzzy environment for $t = 3$.

The interaction of states of nature and weights of given attributes is as follows in Table 37

Table 37 States of Nature and Weights

	g_1	g_2	g_3	Risk importance
g_1	-	0.150	0.250	0.400
g_2	0.150	-	0.100	0.250
g_3	0.250	0.100	-	0.350

Where $I_{g_1} = 0.400$, $I_{g_2} = 0.250$, $I_{g_3} = 0.350$ will be weights of g_1, g_2 and g_3 respectively and $I_{g_1g_2} = 0.150$, $I_{g_1g_3} = 0.250$, $I_{g_2g_3} = 0.100$ will be the interactions between attributes. The fuzzy measure can be measured using the following relationship

$$\Theta(\{g_j\}) = I_{g_j} - \frac{1}{2} \sum_{g \in G\{g_j\}} I_{g_j g}$$

$$\Theta(\{g_j, g_k\}) = I_{g_j} + I_{g_k} - \frac{1}{2} \sum_{g \in G\{g_j, g_k\}} I_{g_j g} - \frac{1}{2} \sum_{g \in G\{g_j, g_k\}} I_{g_k g} \qquad j, k = 1,2,3,4. j \neq k$$

$\Theta(\phi) = 0, \Theta(G) = 1.$

The fuzzy measures will be

$\Theta(\phi) = 0,\quad \Theta(\{g_1\}) = 0.200,\quad \Theta(\{g_2\}) = 0.125,\quad \Theta(\{g_3\}) = 0.175,$
$\Theta(\{g_1, g_2\}) = 0.325,\quad \Theta(\{g_1, g_3\}) = 0.375,\quad \Theta(\{g_2, g_3\}) = 0.300,\quad \Theta(G) = 1.$

The immediate probabilities for every possible permutation are listed in Table 38

Table 38 Immediate Probabilities

	λ'_1	λ'_2	λ'_3
$\sigma = (g_1, g_2, g_3)$	0.125	0.200	0.675
$\sigma = (g_1, g_3, g_2)$	0.125	0.700	0.175
$\sigma = (g_2, g_1, g_3)$	0.200	0.200	0.600
$\sigma = (g_2, g_3, g_1)$	0.700	0.200	0.100
$\sigma = (g_3, g_1, g_2)$	0.125	0.700	0.175
$\sigma = (g_3, g_2, g_1)$	0.700	0.125	0.175

Associated immediate probabilities are listed in Table 39 for every possible permutation

Table 39 Associated Immediate Probabilities

	$\lambda'_{\rho(1)}$	$\lambda'_{\rho(2)}$	$\lambda'_{\rho(3)}$
$\sigma = (g_1, g_2, g_3)$	0.0899	0.2302	0.6798
$\sigma = (g_1, g_3, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_2, g_1, g_3)$	0.1470	0.2353	0.6176
$\sigma = (g_2, g_3, g_1)$	0.6034	0.2759	0.1207
$\sigma = (g_3, g_1, g_2)$	0.0839	0.7517	0.1644
$\sigma = (g_3, g_2, g_1)$	0.6114	0.1747	0.2140

Now the aggregate the all aggregation operators are shown in Table 40

Table 40 Aggregated Values

	d_1	d_2	d_3	d_4	d_5
$IP-T-SFOWG_P$	$\begin{pmatrix}0.4826,\\0.2698,\\0.7931\end{pmatrix}$	$\begin{pmatrix}0.4429,\\0.2453,\\0.5800\end{pmatrix}$	$\begin{pmatrix}0.8416,\\0.3607,\\0.5623\end{pmatrix}$	$\begin{pmatrix}0.8501,\\0.3579,\\0.5920\end{pmatrix}$	$\begin{pmatrix}0.6232,\\0.3920,\\0.6684\end{pmatrix}$
$T-SFCG_\Theta$	$\begin{pmatrix}0.4266,\\0.2358,\\0.7695\end{pmatrix}$	$\begin{pmatrix}0.4259,\\0.2333,\\0.6119\end{pmatrix}$	$\begin{pmatrix}0.8162,\\0.3836,\\0.5150\end{pmatrix}$	$\begin{pmatrix}0.7459,\\0.3030,\\0.6174\end{pmatrix}$	$\begin{pmatrix}0.6276,\\0.3297,\\0.5696\end{pmatrix}$
$Ass.IP-T-SFOWG_\vee$	$\begin{pmatrix}0.4744,\\0.2664,\\0.6988\end{pmatrix}$	$\begin{pmatrix}0.4992,\\0.3402,\\0.5504\end{pmatrix}$	$\begin{pmatrix}0.8680,\\0.4119,\\0.4623\end{pmatrix}$	$\begin{pmatrix}0.8676,\\0.3517,\\0.5044\end{pmatrix}$	$\begin{pmatrix}0.6408,\\0.4033,\\0.5226\end{pmatrix}$
$Ass.IP-T-SFOWG_\wedge$	$\begin{pmatrix}0.7321,\\0.2182,\\0.8535\end{pmatrix}$	$\begin{pmatrix}0.6579,\\0.1941,\\0.7124\end{pmatrix}$	$\begin{pmatrix}0.9226,\\0.3442,\\0.5676\end{pmatrix}$	$\begin{pmatrix}0.8725,\\0.2743,\\0.6602\end{pmatrix}$	$\begin{pmatrix}0.8444,\\0.2826,\\0.6694\end{pmatrix}$

The score values of aggregated operators are listed in Table 41

Table 41 Score Values

	d_1	d_2	d_3	d_4	d_5
$IP-T-SFOWG_P$	-0.4060	-0.1230	0.3714	0.3610	-0.1168
$T-SFCG_\Theta$	-0.3911	-0.1645	0.3507	0.1517	0.0265
$Ass.IP-T-SFOWG_\vee$	-0.2534	-0.0817	0.4854	0.4811	0.0547
$Ass.IP-T-SFOWG_\wedge$	-0.2397	-0.0841	0.5616	0.3557	0.2796

With respect to score values the ranking of alternatives are listed in Table 42

Table 42 Rankings

Operators	Rankings
$IP-T-SFOWG_P$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$

$T-SFCG_{\Theta}$	$d_4 \geq d_2 \geq d_3 \geq d_5 \geq d_1$
$Ass.IP-T-SFOWG_{\vee}$	$d_4 \geq d_1 \geq d_2 \geq d_3 \geq d_5$
$Ass.IP-T-SFOWG_{\wedge}$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$

Now the aggregate the all interactive aggregation operators is shown in Table 43

Table 43 Aggregated Values

	d_1	d_2	d_3	d_4	d_5
$IP-T-SFOWIG_P$	$\begin{pmatrix}0.5021,\\0.2751,\\0.7931\end{pmatrix}$	$\begin{pmatrix}0.5050,\\0.2874,\\0.5800\end{pmatrix}$	$\begin{pmatrix}0.8478,\\0.3656,\\0.5623\end{pmatrix}$	$\begin{pmatrix}0.8835,\\0.3680,\\0.5920\end{pmatrix}$	$\begin{pmatrix}0.6380,\\0.4047,\\0.6684\end{pmatrix}$
$T-SFCIG_{\Theta}$	$\begin{pmatrix}0.4905,\\0.2466,\\0.7695\end{pmatrix}$	$\begin{pmatrix}0.5201,\\0.3011,\\0.6119\end{pmatrix}$	$\begin{pmatrix}0.8359,\\0.3970,\\0.5150\end{pmatrix}$	$\begin{pmatrix}0.8419,\\0.3247,\\0.6174\end{pmatrix}$	$\begin{pmatrix}0.6516,\\0.3709,\\0.5696\end{pmatrix}$
$Ass.IP-T-SFOWIG_{\vee}$	$\begin{pmatrix}0.5033,\\0.2300,\\0.6988\end{pmatrix}$	$\begin{pmatrix}0.5611,\\0.2556,\\0.5504\end{pmatrix}$	$\begin{pmatrix}0.8742,\\0.3504,\\0.4623\end{pmatrix}$	$\begin{pmatrix}0.8950,\\0.2985,\\0.5044\end{pmatrix}$	$\begin{pmatrix}0.7284,\\0.3291,\\0.5226\end{pmatrix}$
$Ass.IP-T-SFOWIG_{\wedge}$	$\begin{pmatrix}0.4684,\\0.2737,\\0.8535\end{pmatrix}$	$\begin{pmatrix}0.3957,\\0.3796,\\0.7124\end{pmatrix}$	$\begin{pmatrix}0.8071,\\0.4239,\\0.5676\end{pmatrix}$	$\begin{pmatrix}0.7830,\\0.3659,\\0.6602\end{pmatrix}$	$\begin{pmatrix}0.6174,\\0.4164,\\0.6694\end{pmatrix}$

The score values of aggregated operators are listed in Table 44

Table 44 Score Values

	d_1	d_2	d_3	d_4	d_5
$IP-T-SFOWIG_P$	-0.3930	-0.0901	0.3828	0.4322	-0.1052
$T-SFCIG_{\Theta}$	-0.3526	-0.1157	0.3849	0.3272	0.0408
$Ass.IP-T-SFOWIG_{\vee}$	-0.2258	-0.0068	0.5262	0.5618	0.2080

$Ass.IP-T-SFOWIG_{\wedge}$	-0.5394	-0.3543	0.2668	0.1432	-0.1367

With respect to score values the ranking of alternatives are as defined in Table 45

Table 45 Rankings

Operators	Rankings
$IP-T-SFOWIG_{P}$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$
$T-SFCIG_{\Theta}$	$d_4 \geq d_2 \geq d_3 \geq d_5 \geq d_1$
$Ass.IP-T-SFOWIG_{\vee}$	$d_4 \geq d_2 \geq d_3 \geq d_1 \geq d_5$
$Ass.IP-T-SFOWIG_{\wedge}$	$d_4 \geq d_3 \geq d_2 \geq d_5 \geq d_1$

IP-T-SFOWG, T-SFCG, IP-T-SFOWIG and T-SFCIG operators aggregate only one order of T-SFNs at a time while Ass.IP-T-SFOWG and Ass.IP-T-SFOWIG operators aggregate all possible orders of T-SFNs at a time. This indicates that T-SFCG and T-SFCIG are special cases of Ass.IP-T-SFOWG and Ass.IP-T-SFOWIG operators respectively.

5.3. Advantages and Comparative Analysis

In this section some advantages of proposed work are discussed and a comparative study of proposed and existing work is also developed.

5.3.1. Advantages

In this subsection, some conditions are discussed under which the proposed operators can reduced to existing operators.

Consider Ass.IP-T-SFOWIG operators

$$Ass.IP-T-SFOWIG_{\vee}(\mathcal{J}_1,\mathcal{J}_2,\dots,\mathcal{J}_k)$$

$$=\left(\sqrt[t]{\begin{array}{c}\max\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)- \\ \max\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)\end{array}}, \atop \sqrt[t]{1-\max\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)}, \quad \sqrt[t]{1-\max\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)}\right) \quad (5.3.1.1)$$

and

$$Ass.IP-T-SFOWIG_{\wedge}(\mathcal{J}_1,\mathcal{J}_2,\dots,\mathcal{J}_k)$$

$$=\left(\sqrt[t]{\begin{array}{c}\min\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right) \\ -\min\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)\end{array}}, \atop \sqrt[t]{1-\min\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-i_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)}, \quad \sqrt[t]{1-\min\limits_{\rho\in X_n}\left(\prod_{j=1}^{k}\left(1-n_{\sigma(j)}^t\right)^{\lambda'_{\rho(j)}}\right)}\right) \quad (5.3.1.2)$$

1. For $t=2$, eq. (5.3.1.1) and (5.3.1.2) reduces to associate immediate probability spherical fuzzy ordered weighted interaction geometric (Ass.IP-SFOWIG) operator

$$Ass.IP-SFOWIG_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt{\begin{array}{c}\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\\ \max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m_{\sigma(j)}^2-i_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)\end{array}}\\ \sqrt{1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\quad\sqrt{1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)}\end{array}\right)$$

and

$$Ass.IP-SFOWIG_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt{\begin{array}{c}\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\\ \min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m_{\sigma(j)}^2-i_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)\end{array}},\\ \sqrt{1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\quad\sqrt{1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)}\end{array}\right)$$

2. For $t=1$, eq. (5.3.1.1) and (5.3.1.2) reduces to associate immediate probability picture fuzzy ordered weighted interactive geometric (Ass.IP-PFOWIG) operator

$$Ass.IP-PFOWIG_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\\ \max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m_{\sigma(j)}-i_{\sigma(j)}-n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\\ 1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\quad 1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\end{array}\right)$$

and

$$Ass.IP-PFOWIG_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\begin{array}{c}\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\\ \min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m_{\sigma(j)}-i_{\sigma(j)}-n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\end{array}\\ 1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-i_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right),\quad 1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}}\right)\end{array}\right)$$

3. For $t=2$ and $i=0$, eq. (5.3.1.1) and (5.3.1.2) reduces to associate immediate probability Pythagorean fuzzy ordered weighted interactive geometric (Ass.IP-PyFOWIG) operator

$$Ass.IP-PyFOWIG_{\vee}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt{\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\\ \sqrt{1-\max\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)}\end{array}\right)$$

and

$$Ass.IP-PyFOWIG_{\wedge}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt{\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-m_{\sigma(j)}^2-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)},\\ \sqrt{1-\min\limits_{\rho\in X_n}\left(\prod\limits_{j=1}^{k}\left(1-n_{\sigma(j)}^2\right)^{\lambda'_{\rho(j)}}\right)}\end{array}\right)$$

4. For $t=1$ and $i=0$, eq. (5.3.1.1) and (5.3.1.2) reduces to Ass.IP-IFOWIG operator

$$Ass.IP - IFOWIG_{\vee}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)} - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right), \\ 1 - \max_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) \end{array} \right)$$

and

$$Ass.IP - IFOWIG_{\wedge}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - m_{\sigma(j)} - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right), \\ 1 - \min_{\rho \in X_n} \left(\prod_{j=1}^{k} \left(1 - n_{\sigma(j)}\right)^{\lambda'_{\rho(j)}} \right) \end{array} \right)$$

Similarly we can reduce all aggregation operators defined in section 5.1.2 and 5.1.3. Another advantage of the proposed operators is that they aggregate that information where the existing operators fails.

5.3.2. Comparative Analysis

A comparison analysis between existing and proposed work has been established in this section. Here an example has been taken in which information is given in IFNs and solved by using proposed operators which shows that the proposed operators can solve the information given in IFSs, PyFSs, PFSs and SFSs but the existing operators cannot solve the information given in PyFSs, PFSs, SFSs and T-SFSs. So the proposed operators are the generalization of existing work.

5.3.2.1. Example

Consider a decision matrix in which information is given in IFNs as listed in Table 46

Table 46 Decision Matrix

	d_1	d_2	d_3
g_1	$(0.60, 0.30)$	$(0.50, 0.20)$	$(0.60, 0.35)$
g_2	$(0.60, 0.30)$	$(0.50, 0.20)$	$(0.20, 0.00)$

g_3	$(0.31, 0.00)$	$(0.50, 0.20)$	$(0.60, 0.35)$
g_4	$(0.20, 0.00)$	$(0.50, 0.20)$	$(0.60, 0.30)$
g_5	$(0.70, 0.30)$	$(0.40, 0.20)$	$(0.80, 0.10)$
g_6	$(0.60, 0.30)$	$(0.80, 0.20)$	$(0.50, 0.20)$

The given information can be written in T-spherical fuzzy information as in Table 47

Table 47 Decision Matrix in T-SF information

	d_1	d_2	d_3
g_1	$(0.60, 0.00, 0.30)$	$(0.50, 0.00, 0.20)$	$(0.60, 0.00, 0.35)$
g_2	$(0.60, 0.00, 0.30)$	$(0.50, 0.00, 0.20)$	$(0.20, 0.00, 0.00)$
g_3	$(0.31, 0.00, 0.00)$	$(0.50, 0.00, 0.20)$	$(0.60, 0.00, 0.35)$
g_4	$(0.20, 0.00, 0.00)$	$(0.50, 0.00, 0.20)$	$(0.60, 0.00, 0.30)$
g_5	$(0.70, 0.00, 0.30)$	$(0.40, 0.00, 0.20)$	$(0.80, 0.00, 0.10)$
g_6	$(0.60, 0.00, 0.30)$	$(0.80, 0.00, 0.20)$	$(0.50, 0.00, 0.20)$

The WV for attributes will be $w = \{0.25, 0.40, 0.35\}$ and fuzzy measures will be as

$\Theta(\phi) = 0$, $\Theta(\{d_1\}) = 0.175$, $\Theta(\{d_2\}) = 0.125$, $\Theta(\{d_3\}) = 0.100$, $\Theta(\{d_1, d_2\}) = 0.500$, $\Theta(\{d_1, d_3\}) = 0.425$, $\Theta(\{d_2, d_3\}) = 0.475$, $\Theta(\{d_1, d_2, d_3\}) = 1$.

Immediate probabilities for all possible permutations are listed in Table 48

Table 48 Immediate Probabilities

	λ'_1	λ'_2	λ'_3
$\sigma = (d_1, d_2, d_3)$	0.175	0.325	0.500
$\sigma = (d_1, d_3, d_2)$	0.175	0.575	0.250
$\sigma = (d_2, d_1, d_3)$	0.375	0.125	0.500
$\sigma = (d_2, d_3, d_1)$	0.525	0.175	0.350

$\sigma = (d_3, d_1, d_2)$	0.325	0.575	0.100
$\sigma = (d_3, d_2, d_1)$	0.525	0.375	0.100

Associated immediate probabilities for all possible permutation are listed in Table 49

Table 49 Associated Immediate Probabilities

	$\lambda'_{\rho(1)}$	$\lambda'_{\rho(2)}$	$\lambda'_{\rho(3)}$
$\sigma = (d_1, d_2, d_3)$	0.1885	0.3500	0.4615
$\sigma = (d_1, d_3, d_2)$	0.1815	0.5963	0.2222
$\sigma = (d_2, d_1, d_3)$	0.4038	0.1346	0.4615
$\sigma = (d_2, d_3, d_1)$	0.5526	0.1316	0.3158
$\sigma = (d_3, d_1, d_2)$	0.3297	0.5833	0.0870
$\sigma = (d_3, d_2, d_1)$	0.5326	0.3804	0.0870

As $0.6 + 0.0 + 0.3 = 0.9 \in [0, 1]$ similarly for $t = 1$, all values lie in TSFSs. So here $t = 1$ is taken.

Then the aggregate of all aggregation operators for $t = 1$ are listed in Table 50

Table 50 Aggregated Values

	g_1	g_2	g_3	g_4	g_5	g_6
T-SFCG	$\begin{pmatrix}0.57,\\0.00,\\0.26\end{pmatrix}$	$\begin{pmatrix}0.54,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.42,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.51,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.72,\\0.00,\\0.16\end{pmatrix}$	$\begin{pmatrix}0.73,\\0.00,\\0.23\end{pmatrix}$
Ass.IP-T-SFOW$G_\vee$	$\begin{pmatrix}0.588,\\0.00,\\0.244\end{pmatrix}$	$\begin{pmatrix}0.537,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.521,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.528,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.725,\\0.00,\\0.154\end{pmatrix}$	$\begin{pmatrix}0.718,\\0.00,\\0.211\end{pmatrix}$
Ass.IP-T-SFOW$G_\wedge$	$\begin{pmatrix}0.543,\\0.00,\\0.305\end{pmatrix}$	$\begin{pmatrix}0.404,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.418,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.385,\\0.00,\\0.00\end{pmatrix}$	$\begin{pmatrix}0.584,\\0.00,\\0.228\end{pmatrix}$	$\begin{pmatrix}0.611,\\0.00,\\0.255\end{pmatrix}$

The score values of aggregated values in Table 51 will be

Table 51 Score Values

	g_1	g_2	g_3	g_4	g_5	g_6
T-SFCG	0.3231	0.6006	0.4800	0.5712	0.5764	0.5049
Ass.IP-T-SFOW$G_\vee$	0.3583	0.5977	0.5820	0.5889	0.5878	0.5158
Ass.IP-T-SFOW$G_\wedge$	0.2470	0.4634	0.4779	0.4435	0.3054	0.3678

The ranking of all alternatives through score values or accuracy function are listed in Table 52

Table 52 Rankings

Operators	Rankings
T-SFCG	$g_2 > g_5 > g_4 > g_6 > g_3 > g_1$
Ass.IP-T-SFOW$G_\vee$	$g_2 > g_4 > g_5 > g_3 > g_6 > g_1$
Ass.IP-T-SFOW$G_\wedge$	$g_3 > g_2 > g_4 > g_6 > g_5 > g_1$

From above example it is clear that the results obtained from proposed operators are similar to existing operators. This proves that the proposed operators are generalizations of existing operators.

Chapter 6

T-Spherical Fuzzy Einstein Hybrid Aggregation Operators and Their Applications in Multi-Attribute Decision Making Problems

T-SFS is a recently developed model that copes with imprecise and uncertain events of real-life with the help of four functions having no restrictions. This chapter aim to define some improved algebraic operations for T-SFSs known as Einstein sum, Einstein product and Einstein scalar multiplication based on Einstein t-norms and t-conorms. Then some geometric and averaging aggregation operators have been established based on defined Einstein operations. The validity of the defined aggregation operators has been investigated thoroughly. The MADM method is described in the environment of T-SFSs and is supported by a comprehensive numerical example using the proposed Einstein aggregation tools. As consequences of the defined aggregation operators, the same concept of Einstein aggregation operators has been proposed for q-ROPFSs, SFSs, PyFSs, PFSs, and IFSs. To signify the importance of proposed operators, a comparative analysis of proposed and existing studies is developed, and the results are analyzed numerically. The advantages of the proposed study are demonstrated numerically over the existing literature with the help of examples.

6.1. Einstein Operations for T-SFS

In this section, some Einstein operators for T-SFS are proposed with the help of Einstein sum and Einstein product. Some special cases of proposed operators are also discussed in the remark.

6.1.1. Definition

Let $\mathcal{T}_1 = (m_1, i_1, n_1)$ and $\mathcal{T}_2 = (m_2, i_2, n_2)$ be two T-SFNs. Then their Einstein operations are defined as follows:

i). $\mathcal{T}_1 \leq \mathcal{T}_2 \Rightarrow m_1 \leq m_2,\ i_1 \leq i_2,\ n_1 \geq n_2$

ii). $$\mathcal{T}_1 \otimes_E \mathcal{T}_2 = \left(\sqrt[t]{\frac{m_1^t m_2^t}{1+(1-m_1^t)(1-m_2^t)}},\ \sqrt[t]{\frac{i_1^t i_2^t}{1+(1-i_1^t)(1-i_2^t)}},\ \sqrt[t]{\frac{n_1^t + n_2^t}{1+n_1^t n_2^t}}\right)$$

iii). $$\mathcal{T}_1 \oplus_E \mathcal{T}_2 = \left(\sqrt[t]{\frac{m_1^t + m_2^t}{1+(m_1^t+i_1^t)(m_2^t+i_2^t)-i_1^t i_2^t}},\ \sqrt[t]{\frac{i_1^t i_2^t}{1+(1-i_1^t)(1-i_2^t)}},\ \sqrt[t]{\frac{n_1^t n_2^t}{1+(1-n_1^t)(1-n_2^t)}}\right)$$

iv). $\tau\mathcal{T}_1 = \left(\sqrt[t]{\frac{(1+m_1^t)^\tau-(1-m_1^t)^\tau}{(1+m_1^t)^\tau+(1-m_1^t)^\tau}}, \sqrt[t]{\frac{(2i_1^t)^\tau}{(2-i_1^t)^\tau+(i_1^t)^\tau}}, \sqrt[t]{\frac{(2n_1^t)^\tau}{(2-n_1^t)^\tau+(n_1^t)^\tau}}\right), \quad \tau > 0$

v). $\mathcal{T}_1^\tau = \left(\sqrt[t]{\frac{2(m_1^t)^\tau}{(2+m_1^t)^\tau+(m_1^t)^\tau}}, \sqrt[t]{\frac{(2i_1^t)^\tau}{(2-i_1^t)^\tau+(i_1^t)^\tau}}, \sqrt[t]{\frac{(1+n_1^t)^\tau-(1-n_1^t)^\tau}{(1+n_1^t)^\tau+(1-n_1^t)^\tau}}\right), \quad \tau > 0$

6.1.2. Remark

i. For $t = 2$, above operations become valid for SFSs

ii. For $t = 1$, above operations become valid for PFSs

iii. For $i = 0$, above operations become valid for q-ROPFSs

iv. For $t = 2$ and $i = 0$, above operations become valid for PyFSs

v. For $t = 1$ and $i = 0$, above operations become valid for IFSs

6.2. T-Spherical Fuzzy Einstein Hybrid Averaging Operators

In this section, by using Einstein operations, T-SF Einstein weighted averaging (T-SFEWA) operators, T-SF Einstein ordered weighted averaging (T-SFEOWA) operators, T-SF Einstein hybrid averaging (T-SFEHA) operators are defined and some of their properties are also discussed.

6.2.1. Definition

For any collection $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2,3, \dots, k$ of T-SFS,

$$T - SFEWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \oplus_{E\, j=1}^{\;k} w_j\mathcal{T}_j$$

$$= \left(\begin{array}{c}\sqrt[t]{\frac{\prod_{j=1}^k(1+m_j^t)^{w_j}-\prod_{j=1}^k(1-m_j^t)^{w_j}}{\prod_{j=1}^k(1+m_j^t)^{w_j}+\prod_{j=1}^k(1-m_j^t)^{w_j}}}, \sqrt[t]{\frac{2\prod_{j=1}^k(i_j^t)^{w_j}}{\prod_{j=1}^k(2-i_j^t)^{w_j}+\prod_{j=1}^k(i_j^t)^{w_j}}}, \\ \sqrt[t]{\frac{2\prod_{j=1}^k(n_j^t)^{w_j}}{\prod_{j=1}^k(2-n_j^t)^{w_j}+\prod_{j=1}^k(n_j^t)^{w_j}}}\end{array}\right)$$

is called $T - SFEWA$ operator with WV $w = (w_1, w_2, \dots, w_k)^T$ of $\mathcal{T}_j$, where $w_j \in (0,1]$ and $\sum_{j=1}^k w_j = 1$.

6.2.2. Theorem

If $\mathcal{T}_j = \mathcal{T}_0$ for all j, then $T - SFEWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

Proof: Since $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2,3, \dots, k$ and $\sum_{j=1}^k w_j = 1$. Then

$$T - SFEWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{\prod_{j=1}^{k}(1+m_0^t)^{w_j} - \prod_{j=1}^{k}(1-m_0^t)^{w_j}}{\prod_{j=1}^{k}(1+m_0^t)^{w_j} + \prod_{j=1}^{k}(1-m_0^t)^{w_j}}}, \sqrt[t]{\frac{2\prod_{j=1}^{k}(i_0^t)^{w_j}}{\prod_{j=1}^{k}(2-i_0^t)^{w_j} + \prod_{j=1}^{k}(i_0^t)^{w_j}}}, \\ \sqrt[t]{\frac{2\prod_{j=1}^{k}(n_0^t)^{w_j}}{\prod_{j=1}^{k}(2-n_0^t)^{w_j} + \prod_{j=1}^{k}(n_0^t)^{w_j}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{(1+m_0^t)^{\sum_{j=1}^{k} w_j} - (1-m_0^t)^{\sum_{j=1}^{k} w_j}}{(1+m_0^t)^{\sum_{j=1}^{k} w_j} + (1-m_0^t)^{\sum_{j=1}^{k} w_j}}}, \sqrt[t]{\frac{2(i_0^t)^{\sum_{j=1}^{k} w_j}}{(2-i_0^t)^{\sum_{j=1}^{k} w_j} + (i_0^t)^{\sum_{j=1}^{k} w_j}}}, \\ \sqrt[t]{\frac{2(n_0^t)^{\sum_{j=1}^{k} w_j}}{(2-n_0^t)^{\sum_{j=1}^{k} w_j} + (n_0^t)^{\sum_{j=1}^{k} w_j}}} \end{array} \right)$$

$= (m_0, i_0, n_0) = \mathcal{T}_0$.

6.2.3. Theorem

For a collection of T-SFNs $\mathcal{T}_j$ for all $j = 1,2,3, \dots, k$ and $\mathcal{T}^L = \min_j \mathcal{T}_j$, and $\mathcal{T}^U = \max_j \mathcal{T}_j$. Then

$$\mathcal{T}^L \leq T - SFEWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

Proof: As $\mathcal{T}^L = \min_j \mathcal{T}_j = \left(\min m_j, \min i_j, \max n_j\right)$ and $\mathcal{T}^U = \max_j \mathcal{T}_j = \left(\max m_j, \max i_j, \min n_j\right)$. Then

$$\min m_j \leq m_j \leq \max m_j$$

$$\min m_j^t \leq m_j^t \leq \max m_j^t$$

$$1 + \min m_j^t \leq 1 + m_j^t \leq 1 + \max m_j^t$$

$$\left(1 + \min m_j^t\right)^{w_j} \leq \prod_{j=1}^{k}\left(1 + m_j^t\right)^{w_j} \leq \left(1 + \max m_j^t\right)^{w_j}$$

$$\Rightarrow \sqrt[t]{\frac{\left(1+\min m_j^t\right)^{w_j}-\left(1-\min m_j^t\right)^{w_j}}{\left(1+\min m_j^t\right)^{w_j}+\left(1-\min m_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{\prod_{j=1}^k\left(1+m_j^t\right)^{w_j}-\prod_{j=1}^k\left(1-m_j^t\right)^{w_j}}{\prod_{j=1}^k\left(1+m_j^t\right)^{w_j}+\prod_{j=1}^k\left(1-m_j^t\right)^{w_j}}}$$

$$\leq \sqrt[t]{\frac{\left(1+\max m_j^t\right)^{w_j}-\left(1-\max m_j^t\right)^{w_j}}{\left(1+\max m_j^t\right)^{w_j}+\left(1-\max m_j^t\right)^{w_j}}}$$

Now, $\min i_j \leq i_j \leq \max i_j$

$$\min i_j^t \leq i_j^t \leq \max i_j^t$$

$$2\min\left(i_j^t\right)^{w_j} \leq 2\prod_{j=1}^k\left(i_j^t\right)^{w_j} \leq 2\max\left(i_j^t\right)^{w_j}$$

$$\Rightarrow \sqrt[t]{\frac{2\min\left(i_j^t\right)^{w_j}}{\left(2-\min i_j^t\right)^{w_j}+\min\left(i_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{2\prod_{j=1}^k(i_j^t)^{w_j}}{\prod_{j=1}^k\left(2-i_j^t\right)^{w_j}+\prod_{j=1}^k\left(i_j^t\right)^{w_j}}}$$

$$\leq \sqrt[t]{\frac{2\max\left(i_j^t\right)^{w_j}}{\left(2-\max i_j^t\right)^{w_j}+\max\left(i_j^t\right)^{w_j}}}$$

Similarly, $\max n_j \geq n_j \geq \min n_j$

$$\max n_j^t \geq n_j^t \geq \min n_j^t$$

$$2\max\left(n_j^t\right)^{w_j} \geq 2\prod_{j=1}^k\left(n_j^t\right)^{w_j} \geq 2\min\left(n_j^t\right)^{w_j}$$

$$\Rightarrow \sqrt[t]{\frac{2\max\left(n_j^t\right)^{w_j}}{\left(2-\max n_j^t\right)^{w_j}+\max\left(n_j^t\right)^{w_j}}} \geq \sqrt[t]{\frac{2\prod_{j=1}^k(n_j^t)^{w_j}}{\prod_{j=1}^k\left(2-n_j^t\right)^{w_j}+\prod_{j=1}^k\left(n_j^t\right)^{w_j}}}$$

$$\geq \sqrt[t]{\frac{2\min\left(n_j^t\right)^{w_j}}{\left(2-\min n_j^t\right)^{w_j}+\min\left(n_j^t\right)^{w_j}}}$$

$$\Rightarrow \mathcal{T}^L \leq T-SFEWA_w(\mathcal{T}_1,\mathcal{T}_2,\ldots,\mathcal{T}_k) \leq \mathcal{T}^U$$

6.2.4. Theorem

For any two T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ and $\mathcal{T}_j' = (m_j', i_j', n_j')$ such that $\mathcal{T}_j \leq \mathcal{T}_j'$ for all $j = 1,2,3,\dots,k$. Then

$$T - SFEWA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T - SFEWA_w(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

Proof: As $\mathcal{T}_j \leq \mathcal{T}_j'$, which means $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$.

As, $m_j \leq m_j' \;\Rightarrow\; m_j^t \leq (m_j')^t$

$$\Rightarrow 1 + m_j^t \leq 1 + (m_j')^t$$

$$\prod_{j=1}^{k}(1 + m_j^t)^{w_j} \leq \prod_{j=1}^{k}\left(1 + (m_j')^t\right)^{w_j}$$

$$\sqrt[t]{\frac{\prod_{j=1}^{k}(1 + m_j^t)^{w_j} - \prod_{j=1}^{k}(1 - m_j^t)^{w_j}}{\prod_{j=1}^{k}(1 + m_j^t)^{w_j} + \prod_{j=1}^{k}(1 - m_j^t)^{w_j}}}$$

$$\leq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1 + (m_j')^t\right)^{w_j} - \prod_{j=1}^{k}\left(1 - (m_j')^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1 + (m_j')^t\right)^{w_j} + \prod_{j=1}^{k}\left(1 - (m_j')^t\right)^{w_j}}}$$

As, $i_j \leq i_j' \;\Rightarrow\; i_j^t \leq (i_j')^t$

$$\Rightarrow 2\prod_{j=1}^{k}(i_j^t)^{w_j} \leq 2\prod_{j=1}^{k}\left((i_j')^t\right)^{w_j}$$

$$\sqrt[t]{\frac{2\prod_{j=1}^{k}(i_j^t)^{w_j}}{\prod_{j=1}^{k}(2 - i_j^t)^{w_j} + \prod_{j=1}^{k}(i_j^t)^{w_j}}} \leq \sqrt[t]{\frac{2\prod_{j=1}^{k}\left((i_j')^t\right)^{w_j}}{\prod_{j=1}^{k}\left(2 - (i_j')^t\right)^{w_j} + \prod_{j=1}^{k}\left((i_j')^t\right)^{w_j}}}$$

Similarly, $n_j \geq n_j' \;\Rightarrow\; n_j^t \geq (n_j')^t$

$$\Rightarrow 2\prod_{j=1}^{k}(n_j^t)^{w_j} \geq 2\prod_{j=1}^{k}\left((n_j')^t\right)^{w_j}$$

$$\sqrt[t]{\frac{2\prod_{j=1}^{k}(n_j^t)^{w_j}}{\prod_{j=1}^{k}\left(2-n_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(n_j^t\right)^{w_j}}} \geq \sqrt[t]{\frac{2\prod_{j=1}^{k}\left(\left(n_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\left(n_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(\left(n_j'\right)^t\right)^{w_j}}}$$

$$\Rightarrow T-SFEWA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFEWA_w(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

6.2.5. Definition

For any collection $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ for all $j=1,2,3,\dots,k$ of T-SFS. Then

$$T-SFEOWA_\omega(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\oplus_{E_{j=1}}^{k}\omega_j\mathcal{T}_{\sigma(j)}$$

$$=\left(\begin{array}{c}\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+m_{\sigma(j)}^t\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+m_{\sigma(j)}^t\right)^{\omega_j}+\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t\right)^{\omega_j}}},\sqrt[t]{\frac{2\prod_{j=1}^{k}(i_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}\left(2-i_{\sigma(j)}^t\right)^{\omega_j}+\prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\omega_j}}},\\ \sqrt[t]{\frac{2\prod_{j=1}^{k}(n_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}\left(2-n_{\sigma(j)}^t\right)^{\omega_j}+\prod_{j=1}^{k}\left(n_{\sigma(j)}^t\right)^{\omega_j}}}\end{array}\right)$$

then $T-SFEOWA_\omega$ is called $T-SFEOWA$ operator with associated WV $\omega=(\omega_1,\omega_2,\dots,\omega_k)^T$ of $\mathcal{T}_j$, where $\omega_j\in(0,1]$ and $\sum_{j=1}^{k}\omega_j=1$. $\sigma(j)$ is the permutation with respect to score value such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right)\geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

In next theorems, idempotency, boundedness, and monotonicity properties are proved for the above operator.

6.2.6. Theorem

If $\mathcal{T}_j=\mathcal{T}_0$ for all $j=1,2,3,\dots,k$, then $T-SFEOWA_\omega(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\mathcal{T}_0$.

Proof: Since $\mathcal{T}_j=\mathcal{T}_0=(m_0,i_0,n_0)$ for all $j=1,2,3,\dots,k$ and $\sum_{j=1}^{k}\omega_j=1$. Then

$$T-SFEOWA_\omega(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt[t]{\frac{\prod_{j=1}^{k}(1+m_0^t)^{\omega_j}-\prod_{j=1}^{k}(1-m_0^t)^{\omega_j}}{\prod_{j=1}^{k}(1+m_0^t)^{\omega_j}+\prod_{j=1}^{k}(1-m_0^t)^{\omega_j}}},\sqrt[t]{\frac{2\prod_{j=1}^{k}(i_0^t)^{\omega_j}}{\prod_{j=1}^{k}(2-i_0^t)^{\omega_j}+\prod_{j=1}^{k}(i_0^t)^{\omega_j}}},\\ \sqrt[t]{\frac{2\prod_{j=1}^{k}(n_0^t)^{\omega_j}}{\prod_{j=1}^{k}(2-n_0^t)^{\omega_j}+\prod_{j=1}^{k}(n_0^t)^{\omega_j}}}\end{array}\right)$$

$=$

$$\left(\sqrt[t]{\frac{(1+m_0^t)^{\sum_{j=1}^k \omega_j}-(1-m_0^t)^{\sum_{j=1}^k \omega_j}}{(1+m_0^t)^{\sum_{j=1}^k \omega_j}+(1-m_0^t)^{\sum_{j=1}^k \omega_j}}}, \sqrt[t]{\frac{2(i_0^t)^{\sum_{j=1}^k \omega_j}}{(2-i_0^t)^{\sum_{j=1}^k \omega_j}+(i_0^t)^{\sum_{j=1}^k \omega_j}}}, \sqrt[t]{\frac{2(n_0^t)^{\sum_{j=1}^k \omega_j}}{(2-n_0^t)^{\sum_{j=1}^k \omega_j}+(n_0^t)^{\sum_{j=1}^k \omega_j}}}\right)$$

$= (m_0, i_0, n_0) = \mathcal{T}_0$.

6.2.7. Theorem

For a collection of T-SFNs $\mathcal{T}_j$ for all $j = 1,2,3, \dots, k$ and $\mathcal{T}^L = \min_j \mathcal{T}_j$, and $\mathcal{T}^U = \max_j \mathcal{T}_j$. Then

$$\mathcal{T}^L \leq T - SFEOWA_\omega(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

Proof: As $\mathcal{T}^L = \min_j \mathcal{T}_{\sigma(j)} = \left(\min m_{\sigma(j)}, \min i_{\sigma(j)}, \max n_{\sigma(j)}\right)$ and $\mathcal{T}^U = \max_j \mathcal{T}_{\sigma(j)} = \left(\max m_{\sigma(j)}, \max i_{\sigma(j)}, \min n_{\sigma(j)}\right)$. Then

$$\min m_{\sigma(j)} \leq m_{\sigma(j)} \leq \max m_{\sigma(j)}$$

$$\min m_{\sigma(j)}^t \leq m_{\sigma(j)}^t \leq \max m_{\sigma(j)}^t$$

$$1 + \min m_{\sigma(j)}^t \leq 1 + m_{\sigma(j)}^t \leq 1 + \max m_{\sigma(j)}^t$$

$$\left(1 + \min m_{\sigma(j)}^t\right)^{\omega_j} \leq \prod_{j=1}^k \left(1 + m_{\sigma(j)}^t\right)^{\omega_j} \leq \left(1 + \max m_{\sigma(j)}^t\right)^{\omega_j}$$

$$\Rightarrow \sqrt[t]{\frac{\left(1 + \min m_{\sigma(j)}^t\right)^{\omega_j} - \left(1 - \min m_{\sigma(j)}^t\right)^{\omega_j}}{\left(1 + \min m_{\sigma(j)}^t\right)^{\omega_j} + \left(1 - \min m_{\sigma(j)}^t\right)^{\omega_j}}}$$

$$\leq \sqrt[t]{\frac{\prod_{j=1}^k \left(1 + m_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^k \left(1 - m_{\sigma(j)}^t\right)^{\omega_j}}{\prod_{j=1}^k \left(1 + m_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^k \left(1 - m_{\sigma(j)}^t\right)^{\omega_j}}}$$

$$\leq \sqrt[t]{\frac{\left(1 + \max m_{\sigma(j)}^t\right)^{\omega_j} - \left(1 - \max m_{\sigma(j)}^t\right)^{\omega_j}}{\left(1 + \max m_{\sigma(j)}^t\right)^{\omega_j} + \left(1 - \max m_{\sigma(j)}^t\right)^{\omega_j}}}$$

Now, $\min i_{\sigma(j)} \leq i_{\sigma(j)} \leq \max i_{\sigma(j)}$

$$\min i_{\sigma(j)}^t \leq i_{\sigma(j)}^t \leq \max i_{\sigma(j)}^t$$

$$2\min\left(i_{\sigma(j)}^{t}\right)^{\omega_j} \leq 2\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\omega_j} \leq 2\max\left(i_{\sigma(j)}^{t}\right)^{\omega_j}$$

$$\Rightarrow \sqrt[t]{\frac{2\min\left(i_{\sigma(j)}^{t}\right)^{\omega_j}}{\left(2-\min i_{\sigma(j)}^{t}\right)^{\omega_j}+\min\left(i_{\sigma(j)}^{t}\right)^{\omega_j}}} \leq \sqrt[t]{\frac{2\prod_{j=1}^{k}(i_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}\left(2-i_{\sigma(j)}^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\omega_j}}}$$

$$\leq \sqrt[t]{\frac{2\max\left(i_{\sigma(j)}^{t}\right)^{\omega_j}}{\left(2-\max i_{\sigma(j)}^{t}\right)^{\omega_j}+\max\left(i_{\sigma(j)}^{t}\right)^{\omega_j}}}$$

Similarly, $\max n_{\sigma(j)} \geq n_{\sigma(j)} \geq \min n_{\sigma(j)}$

$$\max n_{\sigma(j)}^{t} \geq n_{\sigma(j)}^{t} \geq \min n_{\sigma(j)}^{t}$$

$$2\max\left(n_{\sigma(j)}^{t}\right)^{\omega_j} \geq 2\prod_{j=1}^{k}\left(n_{\sigma(j)}^{t}\right)^{\omega_j} \geq 2\min\left(n_{\sigma(j)}^{t}\right)^{\omega_j}$$

$$\Rightarrow \sqrt[t]{\frac{2\max\left(n_{\sigma(j)}^{t}\right)^{\omega_j}}{\left(2-\max n_{\sigma(j)}^{t}\right)^{\omega_j}+\max\left(n_{\sigma(j)}^{t}\right)^{\omega_j}}} \geq \sqrt[t]{\frac{2\prod_{j=1}^{k}(n_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}\left(2-n_{\sigma(j)}^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(n_{\sigma(j)}^{t}\right)^{\omega_j}}}$$

$$\geq \sqrt[t]{\frac{2\min\left(n_{\sigma(j)}^{t}\right)^{\omega_j}}{\left(2-\min n_{\sigma(j)}^{t}\right)^{\omega_j}+\min\left(n_{\sigma(j)}^{t}\right)^{\omega_j}}}$$

$$\Rightarrow \mathcal{T}^{L} \leq T-SFEOWA_{\omega}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq \mathcal{T}^{U}$$

6.2.8. Theorem

For any two T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ and $\mathcal{T}_j{}'=\left(m_j',i_j',n_j'\right)$ such that $\mathcal{T}_j \leq \mathcal{T}_j{}'$ for all $j=1,2,3,\dots,k$. Then

$$T-SFEOWA_{\omega}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFEOWA_{\omega}(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

Proof: As $\mathcal{T}_{\sigma(j)} \leq \mathcal{T}_{\sigma(j)}'$, which means $m_{\sigma(j)} \leq m_{\sigma(j)}'$, $i_{\sigma(j)} \leq i_{\sigma(j)}'$ and $n_{\sigma(j)} \geq n_{\sigma(j)}'$.

As, $m_{\sigma(j)} \leq m_{\sigma(j)}' \;\Rightarrow\; m_{\sigma(j)}^{t} \leq \left(m_{\sigma(j)}'\right)^{t}$

$$\Rightarrow 1+m_{\sigma(j)}^{t} \leq 1+\left(m_{\sigma(j)}'\right)^{t}$$

$$\prod_{j=1}^{k}\left(1+m_{\sigma(j)}^{t}\right)^{\omega_j} \leq \prod_{j=1}^{k}\left(1+\left(m'_{\sigma(j)}\right)^{t}\right)^{\omega_j}$$

$$\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+m_{\sigma(j)}^{t}\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+m_{\sigma(j)}^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^{t}\right)^{\omega_j}}}$$

$$\leq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\left(m'_{\sigma(j)}\right)^{t}\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\left(m'_{\sigma(j)}\right)^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(1-\left(m'_{\sigma(j)}\right)^{t}\right)^{\omega_j}}}$$

As, $i_{\sigma(j)} \leq i'_{\sigma(j)} \;\Rightarrow\; i_{\sigma(j)}^{t} \leq \left(i'_{\sigma(j)}\right)^{t}$

$$\Rightarrow 2\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\omega_j} \leq 2\prod_{j=1}^{k}\left(\left(i'_{\sigma(j)}\right)^{t}\right)^{\omega_j}$$

$$\sqrt[t]{\frac{2\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(2-i_{\sigma(j)}^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(i_{\sigma(j)}^{t}\right)^{\omega_j}}}$$

$$\leq \sqrt[t]{\frac{2\prod_{j=1}^{k}\left(\left(i'_{\sigma(j)}\right)^{t}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\left(i'_{\sigma(j)}\right)^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(\left(i'_{\sigma(j)}\right)^{t}\right)^{\omega_j}}}$$

Similarly, $n_{\sigma(j)} \geq n'_{\sigma(j)} \;\Rightarrow\; n_{\sigma(j)}^{t} \geq \left(n'_{\sigma(j)}\right)^{t}$

$$\Rightarrow 2\prod_{j=1}^{k}\left(n_{\sigma(j)}^{t}\right)^{\omega_j} \geq 2\prod_{j=1}^{k}\left(\left(n'_{\sigma(j)}\right)^{t}\right)^{\omega_j}$$

$$\sqrt[t]{\frac{2\prod_{j=1}^{k}\left(n_{\sigma(j)}^{t}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(2-n_{\sigma(j)}^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(n_{\sigma(j)}^{t}\right)^{\omega_j}}}$$

$$\geq \sqrt[t]{\frac{2\prod_{j=1}^{k}\left(\left(n'_{\sigma(j)}\right)^{t}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\left(n'_{\sigma(j)}\right)^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(\left(n'_{\sigma(j)}\right)^{t}\right)^{\omega_j}}}$$

$$\Rightarrow T-SFEOWA_{\omega}(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFEOWA_{\omega}(\mathcal{T}'_1,\mathcal{T}'_2,\dots,\mathcal{T}'_k)$$

6.2.9. Definition

For any collection $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ for all $j=1,2,3,\dots,k$ of T-SFNs. The mapping

$$T - SFEHA_{w,\omega}\left(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k\right) = \oplus_E{}_{j=1}^{k} \omega_j \tilde{\mathcal{T}}_{\sigma(j)}$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}}, \sqrt[t]{\frac{2\prod_{j=1}^{k}(\tilde{\imath}_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{\imath}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{\imath}_{\sigma(j)}^t\right)^{\omega_j}}}, \\ \sqrt[t]{\frac{2\prod_{j=1}^{k}(\tilde{n}_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{n}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{n}_{\sigma(j)}^t\right)^{\omega_j}}} \end{array} \right)$$

is called T-SFEHA operator, where $\tilde{\mathcal{T}}_j = kw_j\mathcal{T}_j$. Let $w = (w_1, w_2, \dots, w_k)^T$ is the WV and $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$ is the associated WV of $\tilde{\mathcal{T}}_j$ with w_j, $\omega_j \in (0,1]$ and $\sum_{j=1}^{k} \omega_j = 1$, $\sum_{j=1}^{k} w_j = 1$.

T-SFEHA operator first weights the T-spherical fuzzy values then rearrange them and measure the ordered T-spherical fuzzy values, so T-SFEHA operator is generalization of T-SFEWA and T-SFEOWA operator. For this reason, T-SFEHA operator will also be idempotent, monotone, and bounded.

6.3. T-Spherical Fuzzy Einstein Hybrid Geometric Operators

In this section, by using Einstein operations, T-SF Einstein weighted geometric (T-SFEWG) operators, T-SF Einstein ordered weighted geometric (T-SFEOWG) operators, T-SF Einstein hybrid geometric (T-SFEHG) operators are defined and some of their properties are also discussed.

6.3.1. Definition

For any collection $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ for all $j = 1,2,3, \dots, k$ of T-SFNs. The mapping

$$T - SFEWG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \otimes_E{}_{j=1}^{k} \mathcal{T}_j^{w_j}$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{2\prod_{j=1}^{k}\left(m_j^t\right)^{w_j}}{\prod_{j=1}^{k}(2-m_j^t)^{w_j} + \prod_{j=1}^{k}(m_j^t)^{w_j}}}, \sqrt[t]{\frac{2\prod_{j=1}^{k}(i_j^t)^{w_j}}{\prod_{j=1}^{k}\left(2-i_j^t\right)^{w_j} + \prod_{j=1}^{k}\left(i_j^t\right)^{w_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^{k}(1+n_j^t)^{w_j} - \prod_{j=1}^{k}(1-n_j^t)^{w_j}}{\prod_{j=1}^{k}(1+n_j^t)^{w_j} + \prod_{j=1}^{k}(1-n_j^t)^{w_j}}} \end{array} \right)$$

where $w = (w_1, w_2, \dots, w_k)^T$ is the WV of $\mathcal{T}_j$ for all $j = 1,2,3, \dots, k$ such that $w_j \in (0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

In next theorems, idempotency, boundedness, and monotonicity properties are proved for the above operator.

6.3.2. Theorem

If $\mathcal{T}_j = \mathcal{T}_0$ for all $j = 1,2,3, \dots, k$, then $T - SFEWG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

Proof: Since $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all $j = 1,2,3, \dots, k$ and $\sum_{j=1}^{k} w_j = 1$. Then

$$T - SFEWG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$
$$= \left(\begin{array}{c} \sqrt[t]{\dfrac{2\prod_{j=1}^{k}(m_0^t)^{w_j}}{\prod_{j=1}^{k}(2-m_0^t)^{w_j} + \prod_{j=1}^{k}(m_0^t)^{w_j}}}, \sqrt[t]{\dfrac{2\prod_{j=1}^{k}(i_0^t)^{w_j}}{\prod_{j=1}^{k}(2-i_0^t)^{w_j} + \prod_{j=1}^{k}(i_0^t)^{w_j}}}, \\ \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1+n_0^t)^{w_j} - \prod_{j=1}^{k}(1-n_0^t)^{w_j}}{\prod_{j=1}^{k}(1+n_0^t)^{w_j} + \prod_{j=1}^{k}(1-n_0^t)^{w_j}}} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{\dfrac{2(m_0^t)^{\sum_{j=1}^{k} w_j}}{(2-m_0^t)^{\sum_{j=1}^{k} w_j} + (m_0^t)^{\sum_{j=1}^{k} w_j}}}, \sqrt[t]{\dfrac{2(i_0^t)^{\sum_{j=1}^{k} w_j}}{(2-i_0^t)^{\sum_{j=1}^{k} w_j} + (i_0^t)^{\sum_{j=1}^{k} w_j}}}, \\ \sqrt[t]{\dfrac{(1+n_0^t)^{\sum_{j=1}^{k} w_j} - (1-n_0^t)^{\sum_{j=1}^{k} w_j}}{(1+n_0^t)^{\sum_{j=1}^{k} w_j} + (1-n_0^t)^{\sum_{j=1}^{k} w_j}}} \end{array} \right)$$

$= (m_0, i_0, n_0) = \mathcal{T}_0$.

6.3.3. Theorem

For a collection of T-SFNs $\mathcal{T}_j$ for all $j = 1,2,3, \dots, k$ and $\mathcal{T}^L = \min\limits_j \mathcal{T}_j$, and $\mathcal{T}^U = \max\limits_j \mathcal{T}_j$. Then

$$\mathcal{T}^L \leq T - SFEWG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

Proof: As $\mathcal{T}^L = \min\limits_j \mathcal{T}_j = \left(\min m_j, \min i_j, \max n_j\right)$ and $\mathcal{T}^U = \max\limits_j \mathcal{T}_j = \left(\max m_j, \max i_j, \min n_j\right)$. Then

$$\min m_j \leq m_j \leq \max m_j$$

$$\min m_j^t \leq m_j^t \leq \max m_j^t$$

$$2\min\left(m_j^t\right)^{w_j} \leq 2\prod_{j=1}^{k}\left(m_j^t\right)^{w_j} \leq 2\max\left(m_j^t\right)^{w_j}$$

$$\Rightarrow \sqrt[t]{\frac{2\min\left(m_j^t\right)^{w_j}}{\left(2-\min m_j^t\right)^{w_j}+\min\left(m_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{2\prod_{j=1}^{k}(m_j^t)^{w_j}}{\prod_{j=1}^{k}\left(2-m_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(m_j^t\right)^{w_j}}}$$

$$\leq \sqrt[t]{\frac{2\max\left(m_j^t\right)^{w_j}}{\left(2-\max m_j^t\right)^{w_j}+\max\left(m_j^t\right)^{w_j}}}$$

Now, $\min i_j \leq i_j \leq \max i_j$

$$\min i_j^t \leq i_j^t \leq \max i_j^t$$

$$2\min\left(i_j^t\right)^{w_j} \leq 2\prod_{j=1}^{k}\left(i_j^t\right)^{w_j} \leq 2\max\left(i_j^t\right)^{w_j}$$

$$\Rightarrow \sqrt[t]{\frac{2\min\left(i_j^t\right)^{w_j}}{\left(2-\min i_j^t\right)^{w_j}+\min\left(i_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{2\prod_{j=1}^{k}(i_j^t)^{w_j}}{\prod_{j=1}^{k}\left(2-i_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(i_j^t\right)^{w_j}}}$$

$$\leq \sqrt[t]{\frac{2\max\left(i_j^t\right)^{w_j}}{\left(2-\max i_j^t\right)^{w_j}+\max\left(i_j^t\right)^{w_j}}}$$

Similarly, $\max n_j \geq n_j \geq \max n_j$

$$\max n_j^t \geq n_j^t \geq \min n_j^t$$

$$1+\max n_j^t \geq 1+n_j^t \geq 1+\min n_j^t$$

$$\left(1+\max n_j^t\right)^{w_j} \geq \prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j} \geq \left(1+\min n_j^t\right)^{w_j}$$

$$\Rightarrow \sqrt[t]{\frac{\left(1+\max n_j^t\right)^{w_j}-\left(1-\max n_j^t\right)^{w_j}}{\left(1+\max n_j^t\right)^{w_j}+\left(1-\max n_j^t\right)^{w_j}}} \geq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}}$$

$$\geq \sqrt[t]{\frac{\left(1+\min n_j^t\right)^{w_j}-\left(1-\min n_j^t\right)^{w_j}}{\left(1+\min n_j^t\right)^{w_j}+\left(1-\min n_j^t\right)^{w_j}}}$$

$$\Rightarrow \mathcal{T}^L \leq T-SFEWG_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq \mathcal{T}^U$$

6.3.4. Theorem

For any two T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ and $\mathcal{T}_j{}'=\left(m_j',i_j',n_j'\right)$ such that $\mathcal{T}_j \leq \mathcal{T}_j{}'$ for all $j=1,2,3,\dots,k$. Then

$$T-SFEWG_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFEWG_w(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

Proof: As $\mathcal{T}_j \leq \mathcal{T}_j{}'$, which means $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$.

As, $m_j \leq m_j' \;\Rightarrow\; m_j^t \leq \left(m_j'\right)^t$

$$\Rightarrow 2\prod_{j=1}^{k}\left(m_j^t\right)^{w_j} \leq 2\prod_{j=1}^{k}\left(\left(m_j'\right)^t\right)^{w_j}$$

$$\sqrt[t]{\frac{2\prod_{j=1}^{k}(m_j^t)^{w_j}}{\prod_{j=1}^{k}\left(2-m_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(m_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{2\prod_{j=1}^{k}\left(\left(m_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\left(m_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(\left(m_j'\right)^t\right)^{w_j}}}$$

As, $i_j \leq i_j' \;\Rightarrow\; i_j^t \leq \left(i_j'\right)^t$

$$\Rightarrow 2\prod_{j=1}^{k}\left(i_j^t\right)^{w_j} \leq 2\prod_{j=1}^{k}\left(\left(i_j'\right)^t\right)^{w_j}$$

$$\sqrt[t]{\frac{2\prod_{j=1}^{k}(i_j^t)^{w_j}}{\prod_{j=1}^{k}\left(2-i_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(i_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{2\prod_{j=1}^{k}\left(\left(i_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(2-\left(i_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(\left(i_j'\right)^t\right)^{w_j}}}$$

Similarly, $n_j \geq n_j' \;\Rightarrow\; n_j^t \geq \left(n_j'\right)^t$

$$\Rightarrow 1+n_j^t \geq 1+\left(n_j'\right)^t$$

$$\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j} \geq \prod_{j=1}^{k}\left(1+\left(n_j'\right)^t\right)^{w_j}$$

$$\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}}$$

$$\geq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\left(n_j'\right)^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\left(n_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t\right)^{w_j}}}$$

$$\Rightarrow T-SFEWG_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFEWG_w(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

6.3.5. Definition

For any collection $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ for all $j=1,2,3,\dots,k$ of T-SFNs. The mapping

$$T-SFEOWG_\omega(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \otimes_{E}{}_{j=1}^{k} \mathcal{T}_{\sigma(j)}^{\omega_j}$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{2\prod_{j=1}^{k}\left(m_{\sigma(j)}^t\right)^{\omega_j}}{\prod_{j=1}^{k}(2-m_{\sigma(j)}^t)^{\omega_j}+\prod_{j=1}^{k}(m_{\sigma(j)}^t)^{\omega_j}}}, \sqrt[t]{\frac{2\prod_{j=1}^{k}(i_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}\left(2-i_{\sigma(j)}^t\right)^{\omega_j}+\prod_{j=1}^{k}\left(i_{\sigma(j)}^t\right)^{\omega_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^{k}(1+n_{\sigma(j)}^t)^{\omega_j}-\prod_{j=1}^{k}(1-n_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1+n_{\sigma(j)}^t)^{\omega_j}+\prod_{j=1}^{k}(1-n_{\sigma(j)}^t)^{\omega_j}}} \end{array} \right)$$

where $\omega=(\omega_1,\omega_2,\dots,\omega_k)^T$ is the associated WV of $\mathcal{T}_j$ for all $j=1,2,3,\dots,k$ such that $\omega_j \in (0,1]$ and $\sum_{j=1}^{k}\omega_j=1$ and $\sigma(j)$ is permutation with respect to score value such that $SC\left(\mathcal{T}_{\sigma(j-1)}\right) \geq SC\left(\mathcal{T}_{\sigma(j)}\right)$.

In next theorems, idempotency, boundedness, and monotonicity properties are proved for the above operator.

6.3.6. Theorem

If $\mathcal{T}_j=\mathcal{T}_0$ for all $j=1,2,3,\dots,k$, then $T-SFEOWG_\omega(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\mathcal{T}_0$.

6.3.7. Theorem

For a collection of T-SFNs $\mathcal{T}_j$ for all $j=1,2,3,\dots,k$ and $\mathcal{T}^L=\min_j \mathcal{T}_j$, and $\mathcal{T}^U=\max_j \mathcal{T}_j$. Then

$$\mathcal{T}^L \leq T - SFEOWG_\omega(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

6.3.8. Theorem

For any two T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ and $\mathcal{T}_j' = (m_j', i_j', n_j')$ such that $\mathcal{T}_j \leq \mathcal{T}_j'$ for all $j = 1,2,3, \dots, k$. Then

$$T - SFEOWG_\omega(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T - SFEOWG_\omega(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

6.3.9. Definition

For any collection $\mathcal{T}_j = (m_j, i_j, n_j)$ for all $j = 1,2,3, \dots, k$ of T-SFNs. The mapping

$$T - SFEHG_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k) = \otimes_E{}_{j=1}^{k} \tilde{\mathcal{T}}_{\sigma(j)}^{\omega_j}$$

$$= \left(\begin{array}{c} \sqrt[t]{\dfrac{2\prod_{j=1}^{k}\left(\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}{\prod_{j=1}^{k}(2 - \tilde{m}_{\sigma(j)}^t)^{\omega_j} + \prod_{j=1}^{k}(\tilde{m}_{\sigma(j)}^t)^{\omega_j}}}, \sqrt[t]{\dfrac{2\prod_{j=1}^{k}(\tilde{\imath}_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}\left(2 - \tilde{\imath}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{\imath}_{\sigma(j)}^t\right)^{\omega_j}}}, \\ \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1 + \tilde{n}_{\sigma(j)}^t)^{\omega_j} - \prod_{j=1}^{k}(1 - \tilde{n}_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1 + \tilde{n}_{\sigma(j)}^t)^{\omega_j} + \prod_{j=1}^{k}(1 - \tilde{n}_{\sigma(j)}^t)^{\omega_j}}} \end{array} \right)$$

is called T-SFEHG operator, where $\tilde{\mathcal{T}}_j = \mathcal{T}_j^{kw_j}$. Let $w = (w_1, w_2, \dots, w_k)^T$ is the WV and $\omega = (\omega_1, \omega_2, \dots, \omega_k)^T$ is the associated WV of $\tilde{\mathcal{T}}_j$ with $w_j, \omega_j \in (0,1]$ and $\sum_{j=1}^{k} \omega_j = 1, \sum_{j=1}^{k} w_j = 1$.

T-SFEHG operator first weights the T-spherical fuzzy values then rearrange them and measure the ordered T-spherical fuzzy values, so T-SFEHG operator is generalization of T-SFEWG and T-SFEOWG operator. For this reason, T-SFEHG operator will also be idempotent, monotone, and bounded.

6.4. An approach to MADM with T-spherical fuzzy information

Let $D = \{d_1, d_2, d_3, \dots d_l\}$ be a set of alternatives and $E = \{e_1, e_2, e_3, \dots e_k\}$ be a set of attributes. The selection of best alternative is carried out using the aggregation tools proposed under the WV $w = \{w_1, w_2, w_3, \dots w_l\}$ such that $w_j \in (0,1]$ and $\sum_{j=1}^{l} w_j = 1$. The WV is chosen to weigh the arguments of decision makers. The detailed steps of decision making process are illustrated as follows.

Step 1. Find a value of t for which the values lie in T-SF information means that find the exponent t (which is finite natural number) such that the sum of the t^{th} power of all m, i and n values belong to [0, 1].

Step 2. Find $\tilde{\mathcal{T}}_j = kw_j\mathcal{T}_j$ (or $\tilde{\mathcal{T}}_j = \mathcal{T}_j^{kw_j}$).

Step 3. Find scores values and by using these score values we reorder them in a descending order.

Step 4. Aggregate these ordered values using T-SFEHA (or T-SFEHG) operators.

Step 5. By finding scores we choose the best option.

The flow chart of a proposed algorithm is given below:

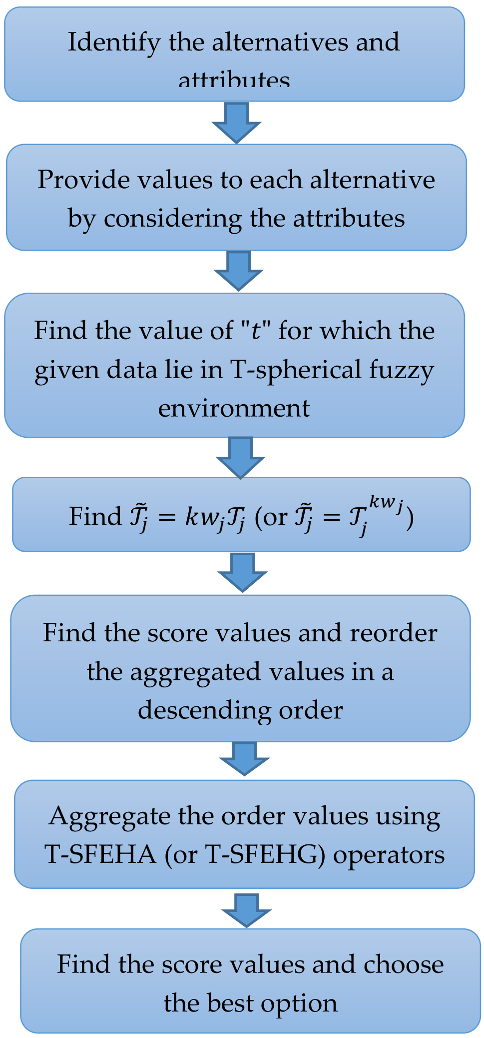

Figure 2 Flow Chart

6.4.1. Example

A company wants to extend his business and board of governors decided to invest their money in one of the best options from three business options

i. b_1: Food company
ii. b_2: Mobile phone company
iii. b_3: Construction company

They assess the given companies on the basis of the following attributes.

i. G_1: Growth analysis
ii. G_2: Risk analysis
iii. G_3: Environmental impact analysis
iv. G_4: Development of society
v. G_5: Social-political impact

The experts evaluate the given attributes under the consideration of given attributes as follows in Table 53:

Table 53 Decision Matrix

	G_1	G_2	G_3	G_4	G_5
b_1	$(0.5, 0.3, 0.4)$	$(0.9, 0.4, 0.5)$	$(0.7, 0.5, 0.2)$	$(0.8, 0.5, 0.5)$	$(0.2, 0.2, 0.8)$
b_2	$(0.2, 0.4, 0.7)$	$(0.4, 0.1, 0.2)$	$(0.9, 0.2, 0.5)$	$(0.3, 0.2, 0.6)$	$(0.5, 0.3, 0.7)$
b_3	$(0.6, 0.2, 0.4)$	$(0.3, 0.5, 0.7)$	$(0.7, 0.2, 0.4)$	$(0.5, 0.1, 0.2)$	$(0.4, 0.3, 0.5)$

Step 1: As, $0.9 + 0.4 + 0.5 = 1.8 \notin [0,1]$, $0.9^2 + 0.4^2 + 0.5^2 = 1.22 \notin [0,1]$ but $0.9^3 + 0.4^3 + 0.5^3 = 0.918 \in [0,1]$. Similarly, sum of cube of all other values lie in [0, 1]. So for $t = 3$ all values in Table 53 are T-SFNs. This clearly indicates that the given information cannot be handled by the existing AOs of IFSs, PyFSs, PFSs as well as SFSs.

Step 2: By taking the WV w$= (0.25, 0.20, 0.15, 0.18, 0.22)^T$, we find T-SFEWA values as

$$\begin{pmatrix} \sqrt[3]{\frac{(1+0.5^3)^{5\times0.25}-(1-0.5^3)^{5\times0.25}}{(1+0.5^3)^{5\times0.25}+(1-0.5^3)^{5\times0.25}}}=0.5381, \\ \sqrt[3]{\frac{2\times(0.3^3)^{5\times0.25}}{(2-0.3^3)^{5\times0.25}+(0.3^3)^{5\times0.25}}}=0.2104, \\ \sqrt[3]{\frac{2\times(0.4^3)^{5\times0.25}}{(2-0.4^3)^{5\times0.25}+(0.4^3)^{5\times0.25}}}=0.3029 \end{pmatrix}$$

Similarly, we can find all other values as follow in Table 54

Table 54 T-SFEWA values

	G_1	G_2	G_3	G_4	G_5
b_1	$\begin{pmatrix}0.5381,\\0.2104,\\0.3029\end{pmatrix}$	$\begin{pmatrix}0.9,\\0.4,\\0.5\end{pmatrix}$	$\begin{pmatrix}0.6398,\\0.6144,\\0.3155\end{pmatrix}$	$\begin{pmatrix}0.7770,\\0.5437,\\0.5437\end{pmatrix}$	$\begin{pmatrix}0.2064,\\0.1665,\\0.7788\end{pmatrix}$
b_2	$\begin{pmatrix}0.2154,\\0.3029,\\0.6258\end{pmatrix}$	$\begin{pmatrix}0.4,\\0.1,\\0.2\end{pmatrix}$	$\begin{pmatrix}0.8440,\\0.3155,\\0.6144\end{pmatrix}$	$\begin{pmatrix}0.2896,\\0.2401,\\0.6384\end{pmatrix}$	$\begin{pmatrix}0.5160,\\0.2604,\\0.6698\end{pmatrix}$
b_3	$\begin{pmatrix}0.6444,\\0.1264,\\0.3029\end{pmatrix}$	$\begin{pmatrix}0.3,\\0.5,\\0.7\end{pmatrix}$	$\begin{pmatrix}0.6398,\\0.3155,\\0.5240\end{pmatrix}$	$\begin{pmatrix}0.4829,\\0.1288,\\0.2401\end{pmatrix}$	$\begin{pmatrix}0.4129,\\0.2604,\\0.4591\end{pmatrix}$

Step 3: Scores of each attribute of all alternatives using $SC(\mathcal{T}) = m^3(x) - n^3(x)$ will be

Table 55 Score Values

	G_1	G_2	G_3	G_4	G_5
b_1	0.1280	0.6040	0.2305	0.3083	-0.4636
b_2	-0.2350	0.0560	0.3692	-0.2359	-0.1632
b_3	0.2398	-0.3160	0.1180	0.0988	-0.0264

Based on above score analysis, we order the values of Table 56 as:

Table 56 Ordered T-SFEWA values

	G_1	G_2	G_3	G_4	G_5

$b_{\sigma(1)}$	$\begin{pmatrix}0.9,\\0.4,\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7770,\\0.5437,\\0.5437\end{pmatrix}$	$\begin{pmatrix}0.6398,\\0.6144,\\0.3155\end{pmatrix}$	$\begin{pmatrix}0.5381,\\0.2104,\\0.3029\end{pmatrix}$	$\begin{pmatrix}0.2064,\\0.1665,\\0.7788\end{pmatrix}$
$b_{\sigma(2)}$	$\begin{pmatrix}0.8440,\\0.3155,\\0.6144\end{pmatrix}$	$\begin{pmatrix}0.4,\\0.1,\\0.2\end{pmatrix}$	$\begin{pmatrix}0.5160,\\0.2604,\\0.6698\end{pmatrix}$	$\begin{pmatrix}0.2154,\\0.3029,\\0.6258\end{pmatrix}$	$\begin{pmatrix}0.2896,\\0.2401,\\0.6384\end{pmatrix}$
$b_{\sigma(3)}$	$\begin{pmatrix}0.6444,\\0.1264,\\0.3029\end{pmatrix}$	$\begin{pmatrix}0.6398,\\0.3155,\\0.5240\end{pmatrix}$	$\begin{pmatrix}0.4829,\\0.1288,\\0.2401\end{pmatrix}$	$\begin{pmatrix}0.4129,\\0.2604,\\0.4591\end{pmatrix}$	$\begin{pmatrix}0.3,\\0.5,\\0.7\end{pmatrix}$

Step 4: With the help of normal distribution-based method, we get $\omega = (0.112,0.236,0.304,0.236,0.112)^T$ and find T-SFEHA values as,

$$\left(\sqrt[3]{\frac{\begin{pmatrix}(1+0.9^3)^{0.112}\times(1+0.7770^3)^{0.236}\times(1+0.6398^3)^{0.304}\\ \times(1+0.5381^3)^{0.236}\times(1+0.2064^3)^{0.112}\end{pmatrix}-\begin{pmatrix}(1-0.9^3)^{0.112}\times(1-0.7770^3)^{0.236}\times(1-0.6398^3)^{0.304}\\ \times(1-0.5381^3)^{0.236}\times(1-0.2064^3)^{0.112}\end{pmatrix}}{\begin{pmatrix}(1+0.9^3)^{0.112}\times(1+0.7770^3)^{0.236}\times(1+0.6398^3)^{0.304}\\ \times(1+0.5381^3)^{0.236}\times(1+0.2064^3)^{0.112}\end{pmatrix}+\begin{pmatrix}(1-0.9^3)^{0.112}\times(1-0.7770^3)^{0.236}\times(1-0.6398^3)^{0.304}\\ \times(1-0.5381^3)^{0.236}\times(1-0.2064^3)^{0.112}\end{pmatrix}}}\right) = 0.6914$$

Similarly, all other values can also be find

$$\tilde{b}_{\sigma(1)} = (0.6914, 0.3859,0.4178)$$

$$\tilde{b}_{\sigma(2)} = (0.5182, 0.2182,0.4960)$$

$$\tilde{b}_{\sigma(3)} = (0.5277, 0.2188,0.3922)$$

Step 5: Now we have to find the score values

$SC\left(\tilde{b}_{\sigma(1)}\right) = 0.2576,\ SC\left(\tilde{b}_{\sigma(2)}\right) = 0.0172,\ SC\left(\tilde{b}_{\sigma(3)}\right) = 0.0866$

$$SC(\tilde{b}_{\sigma(1)}) > SC(\tilde{b}_{\sigma(3)}) > SC(\tilde{b}_{\sigma(2)})$$

Since the score value of b_1 is highest so Food Company is the best option for investment.

Now, we check their validity by using Einstein hybrid geometric operators.

By taking WV $w = (0.25, 0.20, 0.15, 0.18,0.22)^T$, find T-SFEWG values as listed in Table 57,

Table 57 T-SFEWG values

	G_1	G_2	G_3	G_4	G_5
b_1	$\begin{pmatrix}0.4032,\\0.2104,\\0.4308\end{pmatrix}$	$\begin{pmatrix}0.9,\\0.4,\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7773,\\0.6144,\\0.1817\end{pmatrix}$	$\begin{pmatrix}0.8211,\\0.5437,\\0.4829\end{pmatrix}$	$\begin{pmatrix}0.1665,\\0.1665,\\0.8206\end{pmatrix}$
b_2	$\begin{pmatrix}0.1264,\\0.3029,\\0.7485\end{pmatrix}$	$\begin{pmatrix}0.4,\\0.1,\\0.2\end{pmatrix}$	$\begin{pmatrix}0.9262,\\0.3155,\\0.4546\end{pmatrix}$	$\begin{pmatrix}0.3453,\\0.2401,\\0.5799\end{pmatrix}$	$\begin{pmatrix}0.4591,\\0.2604,\\0.7206\end{pmatrix}$
b_3	$\begin{pmatrix}0.5108,\\0.1264,\\0.4308\end{pmatrix}$	$\begin{pmatrix}0.3,\\0.5,\\0.7\end{pmatrix}$	$\begin{pmatrix}0.7773,\\0.3155,\\0.3635\end{pmatrix}$	$\begin{pmatrix}0.5437,\\0.1288,\\0.1931\end{pmatrix}$	$\begin{pmatrix}0.3581,\\0.2604,\\0.5160\end{pmatrix}$

Scores of each attribute of all alternatives will be as in Table 58

Table 58 Score Values

	G_1	G_2	G_3	G_4	G_5
b_1	-0.0144	0.6040	0.4636	0.4411	-0.5479
b_2	-0.4173	0.0560	0.7005	-0.1538	-0.2774
b_3	0.0534	-0.3160	0.4216	0.1536	-0.0914

Based on above score analysis, we find the ordered values of Table 59 as:

Table 59 Ordered T-SFEWG values

	G_1	G_2	G_3	G_4	G_5
$b_{\sigma(1)}$	$\begin{pmatrix}0.9,\\0.4,\\0.5\end{pmatrix}$	$\begin{pmatrix}0.7773,\\0.6144,\\0.1817\end{pmatrix}$	$\begin{pmatrix}0.8211,\\0.5437,\\0.4829\end{pmatrix}$	$\begin{pmatrix}0.4032,\\0.2104,\\0.4308\end{pmatrix}$	$\begin{pmatrix}0.1665,\\0.1665,\\0.8206\end{pmatrix}$
$b_{\sigma(2)}$	$\begin{pmatrix}0.9262,\\0.3155,\\0.4546\end{pmatrix}$	$\begin{pmatrix}0.4,\\0.1,\\0.2\end{pmatrix}$	$\begin{pmatrix}0.3453,\\0.2401,\\0.5799\end{pmatrix}$	$\begin{pmatrix}0.4591,\\0.2604,\\0.7206\end{pmatrix}$	$\begin{pmatrix}0.1264,\\0.3029,\\0.7485\end{pmatrix}$
$b_{\sigma(3)}$	$\begin{pmatrix}0.7773,\\0.3155,\\0.3635\end{pmatrix}$	$\begin{pmatrix}0.5437,\\0.1288,\\0.1931\end{pmatrix}$	$\begin{pmatrix}0.5108,\\0.1264,\\0.4308\end{pmatrix}$	$\begin{pmatrix}0.3581,\\0.2604,\\0.5160\end{pmatrix}$	$\begin{pmatrix}0.3,\\0.5,\\0.7\end{pmatrix}$

With the help of normal distribution-based method, we get $\omega = (0.112,0.236,0.304,0.236,0.112)^T$ and find T-SFEHG values as,

$$\tilde{b}_{\sigma(1)} = (0.6121, 0.8737, 0.8837)$$

$$\tilde{b}_{\sigma(2)} = (0.4325, 0.8056, 0.9297)$$

$$\tilde{b}_{\sigma(3)} = (0.5078, 0.8111, 0.8663)$$

Step 5: Now we have to find the score values

$SC(\tilde{b}_{\sigma(1)}) = -0.4608, SC(\tilde{b}_{\sigma(2)}) = -0.7227, SC(\tilde{b}_{\sigma(3)}) = -0.5192$

$$SC(\tilde{b}_{\sigma(1)}) > SC(\tilde{b}_{\sigma(3)}) > SC(\tilde{b}_{\sigma(2)})$$

Here again the score value of alternative b_1 is high. So, Food Company is the best option for investment**.** Here it is important to discuss that the information given in Table 53 is purely T-SFNs; therefore, it cannot be aggregated using the existing approaches of IFSs [94, 95], PyFSs [96, 97], q-ROPFSs [79] as well as PFSs [82, 84]. On the other hand, the work proposed in this manuscript can deal with all the existing problems that lie in the environment of IFSs, PyFSs, q-ROPFSs and PFSs which is clearly demonstrated in Section 6.5.

6.5. Comparative Analysis

In this section, a comparative study is done in which it is shown that the proposed operators can be reduced to existing operators under some condition which proves the superiority of proposed operators. An example is taken from [94] and it is proved that the proposed operators provide the same result.

Consider the T-SFEHA defined as

$$\boldsymbol{T - SFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k)} = \left(\begin{array}{c} \sqrt[t]{\dfrac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j}}}, \\ \sqrt[t]{\dfrac{2\prod_{j=1}^{k}(\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{\imath}_{\sigma(j)}^{t}\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{\imath}_{\sigma(j)}^{t}\right)^{\omega_j}}}, \\ \sqrt[t]{\dfrac{2\prod_{j=1}^{k}(\tilde{n}_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{n}_{\sigma(j)}^{t}\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{n}_{\sigma(j)}^{t}\right)^{\omega_j}}} \end{array} \right) \quad (6.5.1)$$

1. For $t = 2$ the equation 6.5.1 reduces to spherical fuzzy Einstein hybrid averaging operators (SFEHA operator) i.e.

$$SFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k)$$

$$= \left(\begin{array}{c} \sqrt{\frac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j}}}, \sqrt{\frac{2\prod_{j=1}^{k}(\tilde{\imath}_{\sigma(j)}^2)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{\imath}_{\sigma(j)}^2\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{\imath}_{\sigma(j)}^2\right)^{\omega_j}}}, \\ \sqrt{\frac{2\prod_{j=1}^{k}(\tilde{n}_{\sigma(j)}^2)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{n}_{\sigma(j)}^2\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{n}_{\sigma(j)}^2\right)^{\omega_j}}} \end{array} \right)$$

2. For $t = 1$ the equation 6.5.1 reduces to picture fuzzy Einstein hybrid averaging operators (PFEHA operator) i.e.

$$PFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k)$$

$$= \left(\begin{array}{c} \frac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}\right)^{\omega_j}}, \frac{2\prod_{j=1}^{k}(\tilde{\imath}_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{\imath}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{\imath}_{\sigma(j)}\right)^{\omega_j}}, \\ \frac{2\prod_{j=1}^{k}(\tilde{n}_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{n}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{n}_{\sigma(j)}\right)^{\omega_j}} \end{array} \right)$$

3. For $i = 0$ the equation 6.5.1 reduces to q-ROPF Einstein hybrid averaging operators (q-ROPFEHA operator) i.e.

$$q - ROPFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k)$$

$$= \left(\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}}, \sqrt[t]{\frac{2\prod_{j=1}^{k}(\tilde{n}_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{n}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{n}_{\sigma(j)}^t\right)^{\omega_j}}} \right)$$

4. For $t = 2$ and $i = 0$ the equation 6.5.1 reduces to PyF Einstein hybrid averaging operators (PyFEHA operator) i.e.

$$PyFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k)$$

$$= \left(\sqrt{\frac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j}}}, \sqrt{\frac{2\prod_{j=1}^{k}(\tilde{n}_{\sigma(j)}^2)^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{n}_{\sigma(j)}^2\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{n}_{\sigma(j)}^2\right)^{\omega_j}}} \right)$$

5. For $t = 1$ and $i = 0$ the equation 6.5.1 reduces to IF Einstein hybrid averaging operators (IFEHA operator) i.e.

$$IFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k)$$

$$= \left(\frac{\prod_{j=1}^{k}\left(1+\widetilde{m}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\widetilde{m}_{\sigma(j)}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\widetilde{m}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\widetilde{m}_{\sigma(j)}\right)^{\omega_j}}, \frac{2\prod_{j=1}^{k}(\tilde{n}_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{k}\left(2-\tilde{n}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(\tilde{n}_{\sigma(j)}\right)^{\omega_j}} \right)$$

Similarly, we can reduce T-SFEWA operator, T-SFEOWA operator, T-SFEWG operator, T-SFEOWG operator and T-SFEHG operators.

6.5.1. Example

Consider a decision matrix having five alternatives $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$ and evaluate under four attributes $\{G_1, G_2, G_3, G_4\}$

The experts evaluate the alternatives on the basis of given attributes as

Table 60 Decision Matrix

	G_1	G_2	G_3	G_4
$\mathcal{A}_1$	(0.4,0.5)	(0.5,0.4)	(0.2,0.7)	(0.2,0.5)
$\mathcal{A}_2$	(0.6,0.4)	(0.6,0.3)	(0.6,0.3)	(0.3,0.6)
$\mathcal{A}_3$	(0.5,0.5)	(0.4,0.5)	(0.4,0.4)	(0.5,0.4)
$\mathcal{A}_4$	(0.7,0.2)	(0.5,0.4)	(0.2,0.5)	(0.3,0.7)
$\mathcal{A}_5$	(0.5,0.3)	(0.3,0.4)	(0.6,0.2)	(0.4,0.4)

Above decision matrix can be written in T-SFSs environment as in Table 61

Table 61 Decision Matrix in the form of T-SFNs

	G_1	G_2	G_3	G_4
$\mathcal{A}_1$	(0.4,0,0.5)	(0.5,0,0.4)	(0.2,0,0.7)	(0.2,0,0.5)
$\mathcal{A}_2$	(0.6,0,0.4)	(0.6,0,0.3)	(0.6,0,0.3)	(0.3,0,0.6)
$\mathcal{A}_3$	(0.5,0,0.5)	(0.4,0,0.5)	(0.4,0,0.4)	(0.5,0,0.4)
$\mathcal{A}_4$	(0.7,0,0.2)	(0.5,0,0.4)	(0.2,0,0.5)	(0.3,0,0.7)
$\mathcal{A}_5$	(0.5,0,0.3)	(0.3,0,0.4)	(0.6,0,0.2)	(0.4,0,0.4)

With a WV $\omega = (0.2,0.1, 0.3,0.4)^T$. Then by using eq. (1) we get

Table 62 T-SFEWA values

	G_1	G_2	G_3	G_4
$\mathcal{A}_1$	$(0.3265,0,0.5109)$	$(0.2163,0,0.4814)$	$(0.2386,0,0.7406)$	$(0.3135,0,0.4458)$
$\mathcal{A}_2$	$(0.5039,0,0.4319)$	$(0.2704,0,0.4396)$	$(0.6814,0,0.2548)$	$(0.4584,0,0.6213)$
$\mathcal{A}_3$	$(0.4132,0,0.5109)$	$(0.1679,0,0.5171)$	$(0.4687,0,0.3659)$	$(0.7059,0,0.2975)$
$\mathcal{A}_4$	$(0.6004,0,0.2561)$	$(0.2163,0,0.4814)$	$(0.2386,0,0.4850)$	$(0.4584,0,0.8210)$
$\mathcal{A}_5$	$(0.4132,0,0.3478)$	$(0.1232,0,0.4814)$	$(0.6814,0,0.1535)$	$(0.5901,0,0.2975)$

Then by using score function we order them as in Table 63

Table 63 Ordered T-SFEWA values

	G_1	G_2	G_3	G_4
$\mathcal{A}_{\sigma(1)}$	$(0.3135,0,0.4458)$	$(0.3265,0,0.5109)$	$(0.2163,0,0.4814)$	$(0.2386,0,0.7406)$
$\mathcal{A}_{\sigma(2)}$	$(0.6814,0,0.2548)$	$(0.5039,0,0.4319)$	$(0.4584,0,0.6213)$	$(0.2704,0,0.4396)$
$\mathcal{A}_{\sigma(3)}$	$(0.7059,0,0.2975)$	$(0.4687,0,0.3659)$	$(0.4132,0,0.5109)$	$(0.1679,0,0.5171)$
$\mathcal{A}_{\sigma(4)}$	$(0.6004,0,0.2561)$	$(0.2386,0,0.4850)$	$(0.2163,0,0.4814)$	$(0.4584,0,0.8210)$
$\mathcal{A}_{\sigma(5)}$	$(0.6814,0,0.1535)$	$(0.5901,0,0.2975)$	$(0.4132,0,0.3478)$	$(0.1232,0,0.4814)$

By using eq. (3), we get

$$\tilde{\mathcal{A}}_{\sigma(1)} = (0.2434,0,0.5477)$$

$$\tilde{\mathcal{A}}_{\sigma(2)} = (0.4534,0,0.4360)$$

$$\tilde{\mathcal{A}}_{\sigma(3)} = (0.4273,0,0.4119)$$

$$\tilde{\mathcal{A}}_{\sigma(4)} = (0.3109,0,0.5072)$$

$$\tilde{\mathcal{A}}_{\sigma(5)} = (0.4440,0,0.2941)$$

The score values of aggregated values will be

$SC(\tilde{\mathcal{A}}_{\sigma(1)}) = -0.3043$, $SC(\tilde{\mathcal{A}}_{\sigma(2)}) = 0.0174$, $SC(\tilde{\mathcal{A}}_{\sigma(3)}) = 0.0154$, $SC(\tilde{\mathcal{A}}_{\sigma(4)}) = -0.1964$, $SC(\tilde{\mathcal{A}}_{\sigma(5)}) = 0.1499$.

This shows that A_5 is most desirable alternative. Similarly, the above example can be aggregated by using T-SFEHG operator.

6.5.2. Example

Consider the information is given in T-spherical fuzzy environment for $t = 3$ as in Table 64:

Table 64 Decision Matrix

	G_1	G_2	G_3	G_4	G_5
$\mathcal{A}_1$	$(0.5, 0.3, 0.4)$	$(0.9, 0.4, 0.5)$	$(0.7, 0.5, 0.2)$	$(0.8, 0.5, 0.5)$	$(0.2, 0.2, 0.8)$
$\mathcal{A}_2$	$(0.2, 0.4, 0.7)$	$(0.4, 0.1, 0.2)$	$(0.9, 0.2, 0.5)$	$(0.3, 0.2, 0.6)$	$(0.5, 0.3, 0.7)$
$\mathcal{A}_3$	$(0.6, 0.2, 0.4)$	$(0.3, 0.5, 0.7)$	$(0.7, 0.2, 0.4)$	$(0.5, 0.1, 0.2)$	$(0.4, 0.3, 0.5)$

Then some aggregation operators e. g. T-spherical fuzzy weighted averaging (T-SFWA) operators, T-spherical fuzzy hybrid geometric (T-SFHG) operators, T-spherical fuzzy weighted interactive averaging (T-SFWIA), T-spherical fuzzy hybrid interactive geometric (T-SFHIG) operators, T-SFEWA operators and T-SFEWG operators are used to solve given data. The aggregated values will be as in Table 65:

Table 65 Aggregated values of Table 64

	T-SFWA operators	T-SFHG operators [10]	T-SFWIA operators	T-SFHIG operators	T-SFEWA operators	T-SFEWG operators
$\mathcal{A}_1$	$\begin{pmatrix}0.7284,\\0.3440,\\0.4570\end{pmatrix}$	$\begin{pmatrix}0.5855,\\0.3824,\\0.5216\end{pmatrix}$	$\begin{pmatrix}0.7284,\\0.3995,\\0.6010\end{pmatrix}$	$\begin{pmatrix}0.9132,\\0.7872,\\0.5216\end{pmatrix}$	$\begin{pmatrix}0.6914,\\0.3859,\\0.4178\end{pmatrix}$	$\begin{pmatrix}0.6121,\\0.8737,\\0.8837\end{pmatrix}$
$\mathcal{A}_2$	$\begin{pmatrix}0.6015,\\0.2264,\\0.5039\end{pmatrix}$	$\begin{pmatrix}0.4723,\\0.2102,\\0.6030\end{pmatrix}$	$\begin{pmatrix}0.6015,\\0.2927,\\0.6121\end{pmatrix}$	$\begin{pmatrix}0.9111,\\0.8905,\\0.6030\end{pmatrix}$	$\begin{pmatrix}0.5182,\\0.2182,\\0.4960\end{pmatrix}$	$\begin{pmatrix}0.4325,\\0.8056,\\0.9297\end{pmatrix}$
$\mathcal{A}_3$	$\begin{pmatrix}0.5367,\\0.2318,\\0.4148\end{pmatrix}$	$\begin{pmatrix}0.5164,\\0.1959,\\0.5770\end{pmatrix}$	$\begin{pmatrix}0.5367,\\0.3286,\\0.5440\end{pmatrix}$	$\begin{pmatrix}0.9579,\\0.8506,\\0.5770\end{pmatrix}$	$\begin{pmatrix}0.5277,\\0.2188,\\0.3922\end{pmatrix}$	$\begin{pmatrix}0.5078,\\0.8111,\\0.8663\end{pmatrix}$

The scores of the aggregated data obtained in Table 65 are given in Table 66 as follows:

Table 66 Score Values

	T-SFWA operators	T-SFHG operators [16]	T-SFWIA operators	T-SFHIG operators	T-SFEWA operators	T-SFEWG operators
$\mathcal{A}_1$	0.2909	0.0588	0.1693	0.6196	0.2576	-0.4608
$\mathcal{A}_2$	0.0897	-0.1140	-0.0118	0.5371	0.0172	-0.7227
$\mathcal{A}_3$	0.0832	-0.0544	-0.0064	0.6868	0.0866	-0.5192

The geometrical comparison of the score values obtained using different aggregation techniques is depicted in Figure 3 where the blue stars denote the score values of the A_1 using different aggregation operators while the orange and grey stars denote the score values of the alternatives A_2 and A_3 respectively.

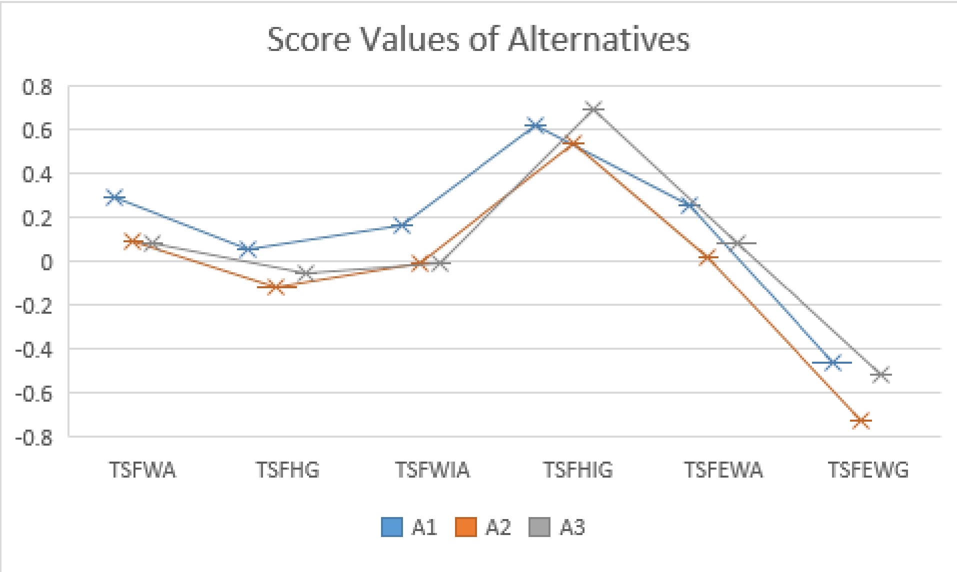

Figure 3 (Score values of alternatives using different aggregation operators)

The demonstration of the ranking results observed in Figure 3 are described in Table 67.

Table 67 Rankings

Aggregation Operators	Reference	Rankings
T-SFWA operators	Chapter 4	$\mathcal{A}_1 > \mathcal{A}_2 > \mathcal{A}_3$

T-SFHG operators	[16]	$\mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2$
T-SFWIA operators	Chapter 4	$\mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2$
T-SFHIG operators	Chapter 3	$\mathcal{A}_3 > \mathcal{A}_1 > \mathcal{A}_2$
T-SFEWA operators	This Chapter	$\mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2$
T-SFEWG operators	This Chapter	$\mathcal{A}_1 > \mathcal{A}_3 > \mathcal{A}_2$

6.5.3 Advantages.

The advantages of proposed work over existing work are discussed in this section. The advantages of our work are as follows:

1. T-SFS is superior to IFS, PyFS, q-ROPFS, PFS and SFS which is claimed and proved Example 6.4.1 and 6.5.1.
2. T-spherical fuzzy Einstein AOs are more flexible than Einstein aggregation operators of IFSs, PyFSs and, PFS. This flexibility is shown in Section 6.5., where few restrictions on the proposed operator reduce them to Einstein operators of IFSs, PyFSs, q-ROPFSs, PFSs, and SFSs.

Proposed operators can solve all the problems that are discussed in [82, 84, 94-97] but the existing operators cannot solve the problems when the information is given in T-SFNs.

Chapter 7

Some T-spherical fuzzy Einstein interactive aggregation operators and their application to selection of photovoltaic cells

In this chapter, some new averaging and geometric operators in the T-SF environment are proposed. First new operational laws are defined then on the basis of these laws Einstein geometric interaction operators and Einstein averaging interactive aggregation operators are proposed. Some basic properties of these operators are also discussed. Then proposed operators are applied to the MADM problem to check their reliability. The advantages of proposed aggregation operators are also discussed. The superiority of proposed operators over existing work is checked with the help of an example.

7.1. Einstein Interaction Operations for T-SFS

Existing Einstein operations (Chapter 6) have some limitations that they fail under some condition. So, we proposed some new Einstein interaction operations on which we define some new aggregation operators. If $\mathcal{T}_1 = (m_1, i_1, n_1)$ and $\mathcal{T}_2 = (m_2, i_2, n_2)$ are two T-SFSs then their Einstein interaction operations are as follows:

vi). $$\mathcal{T}_1 \otimes_{Ei} \mathcal{T}_2 = \left(\begin{array}{c} \sqrt[t]{\frac{2\left((1-n_1^t-i_1^t)(1-n_2^t-i_2^t)-(1-m_1^t-i_1^t-n_1^t)(1-m_2^t-i_2^t-n_2^t)\right)}{(1+n_1^t)(1+n_2^t)+(1-n_1^t)(1-n_2^t)}}, \\ \sqrt[t]{\frac{(1+i_1^t)(1+i_2^t)-(1-i_1^t)(1-i_2^t)}{(1+i_1^t)(1+i_2^t)+(1-i_1^t)(1-i_2^t)}}, \sqrt[t]{\frac{(1+n_1^t)(1+n_2^t)-(1-n_1^t)(1-n_2^t)}{(1+n_1^t)(1+n_2^t)+(1-n_1^t)(1-n_2^t)}} \end{array} \right)$$

vii). $$\mathcal{T}_1 \oplus_{Ei} \mathcal{T}_2 = \left(\begin{array}{c} \sqrt[t]{\frac{(1+m_1^t)(1+m_2^t)-(1-m_1^t)(1-m_2^t)}{(1+m_1^t)(1+m_2^t)+(1-m_1^t)(1-m_2^t)}}, \sqrt[t]{\frac{(1+i_1^t)(1+i_2^t)-(1-i_1^t)(1-i_2^t)}{(1+i_1^t)(1+i_2^t)+(1-i_1^t)(1-i_2^t)}}, \\ \sqrt[t]{\frac{2\left((1-m_1^t-i_1^t)(1-m_2^t-i_2^t)-(1-m_1^t-i_1^t-n_1^t)(1-m_2^t-i_2^t-n_2^t)\right)}{(1+m_1^t)(1+m_2^t)+(1-m_1^t)(1-m_2^t)}} \end{array} \right)$$

viii). $$\tau\mathcal{T}_1 = \left(\begin{array}{c} \sqrt[t]{\frac{(1+m_1^t)^\tau-(1-m_1^t)^\tau}{(1+m_1^t)^\tau+(1-m_1^t)^\tau}}, \sqrt[t]{\frac{(1+i_1^t)^\tau-(1-i_1^t)^\tau}{(1+m_1^t)^\tau+(1-m_1^t)^\tau}}, \\ \sqrt[t]{\frac{2\left((1-m_1^t-i_1^t)^\tau-(1-m_1^t-i_1^t-n_1^t)^\tau\right)}{(1+m_1^t)^\tau+(1-m_1^t)^\tau}} \end{array} \right), \quad \tau > 0$$

ix). $$(\mathcal{T}_1)^{\tau} = \begin{pmatrix} \sqrt[t]{\frac{2\left(\left(1-n_1^t-i_1^t\right)^{\tau}-\left(1-m_1^t-i_1^t-n_1^t\right)^{\tau}\right)}{\left(1+n_1^t\right)^{\tau}+\left(1-n_1^t\right)^{\tau}}}, \\ \sqrt[t]{\frac{(1+i_1^t)^{\tau}-\left(1-i_1^t\right)^{\tau}}{(1+i_1^t)^{\tau}+(1-i_1^t)^{\tau}}}, \sqrt[t]{\frac{(1+n_1^t)^{\tau}-\left(1-n_1^t\right)^{\tau}}{(1+n_1^t)^{\tau}+(1-n_1^t)^{\tau}}} \end{pmatrix}, \quad \tau > 0$$

7.1.1. Remarks

1. The defined operations will reduced to SFSs for $t = 2$.
2. The defined operations will reduced to PFSs for $t = 1$.
3. The defined operations will reduced to q-ROFSs for $i = 0$.
4. The defined operations will reduced to PyFSs for $t = 2$ and $i = 0$.
5. The defined operations will reduced to IFSs for $t = 1$ and $i = 0$.
6. The defined operations will reduced to FSs for $t = 1$, $i = 0$ and $n = 0$.

7.2. T-Spherical Fuzzy Einstein Hybrid Geometric Interaction Operators

In this section, on basis of new proposed Einstein operations we defined geometric interaction operator in the environment of T-SFS and some of its basic properties are also discussed like monotonicity, boundedness and idempotency. The validity of proposed work is checked with the help of an example.

7.2.1. Definition

For any collection of T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ $(j = 1,2,3, \dots, k)$. The mapping

$$T - SFEWIG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \otimes_{Ei}{}_{j=1}^{k} \ \mathcal{T}_j^{w_j}$$

$$= \begin{pmatrix} \sqrt[t]{\frac{2\left(\prod_{j=1}^k (1-n_j^t-i_j^t)^{w_j} - \prod_{j=1}^k \left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}\right)}{\prod_{j=1}^k (1+n_j^t)^{w_j} + \prod_{j=1}^k (1-n_j^t)^{w_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^k (1+i_j^t)^{w_j} - \prod_{j=1}^k (1-i_j^t)^{w_j}}{\prod_{j=1}^k (1+i_j^t)^{w_j} + \prod_{j=1}^k (1-i_j^t)^{w_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^k (1+n_j^t)^{w_j} - \prod_{j=1}^k (1-n_j^t)^{w_j}}{\prod_{j=1}^k (1+n_j^t)^{w_j} + \prod_{j=1}^k (1-n_j^t)^{w_j}}} \end{pmatrix}$$

Where $w = (w_1, \dots, w_k)^T$ is the WV of $\mathcal{T}_j$ with $w_j \in [0,1]$ and $\sum_{j=1}^k w_j = 1$.

7.2.2. Theorem

If all $\mathcal{T}_j = \mathcal{T}_0$, then $T - SFEWIG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

Proof: Let $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all j then

$$T - SFEWIG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{2\left(\prod_{j=1}^{k}(1-n_j^t-i_j^t)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}\right)}{\prod_{j=1}^{k}(1+n_j^t)^{w_j}+\prod_{j=1}^{k}(1-n_j^t)^{w_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^{k}(1+i_j^t)^{w_j}-\prod_{j=1}^{k}(1-i_j^t)^{w_j}}{\prod_{j=1}^{k}(1+i_j^t)^{w_j}+\prod_{j=1}^{k}(1-i_j^t)^{w_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^{k}(1+n_j^t)^{w_j}-\prod_{j=1}^{k}(1-n_j^t)^{w_j}}{\prod_{j=1}^{k}(1+n_j^t)^{w_j}+\prod_{j=1}^{k}(1-n_j^t)^{w_j}}} \end{array} \right)$$

$$T - SFEWIG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k)$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{2\left((1-n_j^t-i_j^t)^{\sum_{j=1}^{k}w_j}-\left(1-m_j^t-i_j^t-n_j^t\right)^{\sum_{j=1}^{k}w_j}\right)}{(1+n_j^t)^{\sum_{j=1}^{k}w_j}+(1-n_j^t)^{\sum_{j=1}^{k}w_j}}}, \\ \sqrt[t]{\frac{(1+i_j^t)^{\sum_{j=1}^{k}w_j}-(1-i_j^t)^{\sum_{j=1}^{k}w_j}}{(1+i_j^t)^{\sum_{j=1}^{k}w_j}+(1-i_j^t)^{\sum_{j=1}^{k}w_j}}}, \\ \sqrt[t]{\frac{(1+n_j^t)^{\sum_{j=1}^{k}w_j}-(1-n_j^t)^{\sum_{j=1}^{k}w_j}}{(1+n_j^t)^{\sum_{j=1}^{k}w_j}+(1-n_j^t)^{\sum_{j=1}^{k}w_j}}} \end{array} \right)$$

$$= (m_0, i_0, n_0) = \mathcal{T}_0$$

7.2.3. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j$ $(j = 1,2, \dots, k)$ with $\mathcal{T}^L = \min_j \mathcal{T}_j$, and $\mathcal{T}^U = \max_j \mathcal{T}_j$. Then

$$\mathcal{T}^L \leq T - SFEWIG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

7.2.4. Theorem

Consider any two T-SFNs $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ and $\mathcal{T}_j{}' = \left(m_j', i_j', n_j'\right)$ $(j = 1,2, \dots, k)$ such that $\mathcal{T}_j \leq \mathcal{T}_j{}'$ for all j. Then

$$T - SFEWIG_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T - SFEWIG_w(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

Proof: Let $\mathcal{T}_j \leq \mathcal{T}_j{}'$ then $m_j \leq m_j'$, $i_j \leq i_j'$ and $n_j \geq n_j'$. Then by using basic information

$$\sqrt[t]{\frac{2\left(\prod_{j=1}^{k}\left(1-n_j^t-i_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}\right)}{\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}}$$

$$\leq \sqrt[t]{\frac{2\left(\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t-\left(i_j'\right)^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j'\right)^t-\left(i_j'\right)^t-\left(n_j'\right)^t\right)^{w_j}\right)}{\prod_{j=1}^{k}\left(1+\left(n_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t\right)^{w_j}}}$$

$$\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+i_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+i_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\left(i_j'\right)^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(i_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\left(i_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-\left(i_j'\right)^t\right)^{w_j}}}$$

$$\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+n_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-n_j^t\right)^{w_j}}}$$

$$\geq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\left(n_j'\right)^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\left(n_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-\left(n_j'\right)^t\right)^{w_j}}}$$

This shows that

$$T-SFEWIG_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFEWIG_w(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

7.2.5. Definition

For any collection of T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ $(j=1,2,3,\dots,k)$. The mapping

$$T-SFEOWIG_\omega(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) = \otimes_{Ei}{}_{j=1}^{k}\ \mathcal{T}_{\sigma(j)}^{\omega_j}$$

$$= \left(\begin{array}{c} \sqrt[t]{\frac{2\left(\prod_{j=1}^{k}(1-n_{\sigma(j)}^t-i_{\sigma(j)}^t)^{\omega_j}-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{\omega_j}\right)}{\prod_{j=1}^{k}(1+n_{\sigma(j)}^t)^{\omega_j}+\prod_{j=1}^{k}(1-n_{\sigma(j)}^t)^{\omega_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^{k}(1+i_{\sigma(j)}^t)^{\omega_j}-\prod_{j=1}^{k}(1-i_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1+i_{\sigma(j)}^t)^{\omega_j}+\prod_{j=1}^{k}(1-i_{\sigma(j)}^t)^{\omega_j}}}, \\ \sqrt[t]{\frac{\prod_{j=1}^{k}(1+n_{\sigma(j)}^t)^{\omega_j}-\prod_{j=1}^{k}(1-n_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1+n_{\sigma(j)}^t)^{\omega_j}+\prod_{j=1}^{k}(1-n_{\sigma(j)}^t)^{\omega_j}}} \end{array}\right)$$

Where $\omega=(\omega_1,\dots,\omega_k)^T$ is the associated WV of $\mathcal{T}_j$ with $\omega_j\in[0,1]$ and $\sum_{j=1}^{k}\omega_j=1$, and $\sigma(j)$ is any permutation of $(1,2,\dots,k)$ such that $\tilde{\mathcal{T}}_{\sigma(j-1)}\geq\tilde{\mathcal{T}}_{\sigma(j)}$.

7.2.6. Theorem

If all $\mathcal{T}_j = \mathcal{T}_0$, then $T - SFEOWIG_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

7.2.7. Theorem

Consider a collection of $T - SFNs$ $\mathcal{T}_j$ $(j = 1,2, \dots, k)$ with $\mathcal{T}^L = \min_j \mathcal{T}_j$, and $\mathcal{T}^U = \max_j \mathcal{T}_j$. Then

$$\mathcal{T}^L \leq T - SFEOWIG_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

7.2.8. Theorem

For any two T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)$ and $\mathcal{T}_j{}' = (m_j', i_j', n_j')$ for all $(j = 1,2, \dots, k)$such that $\mathcal{T}_j \leq \mathcal{T}_j{}'$ for all j. Then

$$T - SFEOWIG_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T - SFEOWIG_{\omega}(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

7.2.9. Definition

For any collection $\mathcal{T}_j = (m_j, i_j, n_j)$ $(j = 1,2,3, \dots, k)$ of T-SFNs. The mapping

$$T - SFEHIG_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k) = \otimes_{Ei}{}_{j=1}^{k} \tilde{\mathcal{T}}_{\sigma(j)}^{\omega_j}$$

$$= \left(\begin{array}{c} \sqrt[t]{\dfrac{2\left(\prod_{j=1}^{k}(1 - \tilde{n}_{\sigma(j)}^t - \tilde{\imath}_{\sigma(j)}^t)^{\omega_j} - \prod_{j=1}^{k}\left(1 - \tilde{m}_{\sigma(j)}^t - \tilde{\imath}_{\sigma(j)}^t - \tilde{n}_{\sigma(j)}^t\right)^{\omega_j}\right)}{\prod_{j=1}^{k}(1 + n_{\sigma(j)}^t)^{\omega_j} + \prod_{j=1}^{k}(1 - n_{\sigma(j)}^t)^{\omega_j}}}, \\ \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1 + \tilde{\imath}_{\sigma(j)}^t)^{\omega_j} - \prod_{j=1}^{k}(1 - \tilde{\imath}_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1 + \tilde{\imath}_{\sigma(j)}^t)^{\omega_j} + \prod_{j=1}^{k}(1 - \tilde{\imath}_{\sigma(j)}^t)^{\omega_j}}}, \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1 + \tilde{n}_{\sigma(j)}^t)^{\omega_j} - \prod_{j=1}^{k}(1 - \tilde{n}_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1 + \tilde{n}_{\sigma(j)}^t)^{\omega_j} + \prod_{j=1}^{k}(1 - \tilde{n}_{\sigma(j)}^t)^{\omega_j}}} \end{array} \right)$$

is called T-SFEHIG operator, where $\tilde{\mathcal{T}}_j = (\mathcal{T}_j)^{kw_j}$. Let $w = (w_1, \dots, w_k)^T$ is the WV and $\omega = (\omega_1, \dots, \omega_k)^T$ is the associated WV of $\mathcal{T}_j$ with condition that both weight and associated WV belong to closed unit interval and their sum is equal to 1.

Hybrid aggregation operators first aggregate the given data considering their attributes then rearrange them in a specific order. After that, they aggregate the data considering their order. This means that hybrid operators are a generalization of weighted and ordered weighted operators. So T-SFEHIG operator will satisfy idempotent, monotone and bounded property.

7.2.10. Example

Consider T-SFNs $\mathcal{T}_1 = (0.7,0.3,0.2)$, $\mathcal{T}_2 = (0.9,0.1,0.6)$, $\mathcal{T}_3 = (0.4,0.6,0.8)$, $\mathcal{T}_4 = (0.1,0.5,0.7)$, $\mathcal{T}_5 = (0.0,0.0,0.8)$ with a WV $w = (0.25, 0.20, 0.15, 0.18,0.22)^T$.

Solution: First of all we find the aggregated value of these T-SFNs by using T-spherical fuzzy Einstein hybrid geometric aggregation (T-SFEHG) operator from Chapter 6 to find out the drawbacks of given operators. For this purpose first of all we have to calculate the value of t for which the given data lie in T-SF information.

As, $0.9 + 0.1 + 0.6 = 1.6$

For $t = 2$, $0.9^2 + 0.1^2 + 0.6^2 = 1.18$

For $t = 3$, $0.9^3 + 0.1^3 + 0.6^3 = 0.946$

Similarly, for $t = 3$ all the given data lie in the T-spherical fuzzy information.

By using T-spherical fuzzy Einstein weighted geometric operator we shall be able to find these values,

$\mathcal{T}_1 = (0.6388, 0.3232,0.5381)$

$\mathcal{T}_2 = (0.9,0.1,0.6)$

$\mathcal{T}_3 = (0.4163, 0.5464,0.7370)$

$\mathcal{T}_4 = (0.1050, 0.4829,0.6776)$

$\mathcal{T}_5 = (0.0,0.0,0.8206)$

Their scores values will be

$SC(\mathcal{T}_1) = 0.1048$, $SC(\mathcal{T}_2) = 0.5130$, $SC(\mathcal{T}_3) = -0.3282$, $SC(\mathcal{T}_4) = -0.3099$, $SC(\mathcal{T}_5) = -0.5525$

Now using score value, the aggregated values obtained by using T-SFEWG operators are rearranged in descending order. Then these ordered values are again aggregated by using T-SFEHG operator with associated WV will be $\omega = (0.112,0.236,0.304,0.236,0.112)$

$\tilde{\mathcal{T}}_{\sigma(1)} = (0.9,0.1,0.6)$

$\tilde{\mathcal{T}}_{\sigma(2)} = (0.6388, 0.3232, 0.5381)$

$\tilde{\mathcal{T}}_{\sigma(3)} = (0.1050, 0.4829, 0.6776)$

$\tilde{\mathcal{T}}_{\sigma(4)} = (0.4163, 0.5464, 0.7370)$

$\tilde{\mathcal{T}}_{\sigma(5)} = (0.0, 0.0, 0.8206)$

Now again measure T-SFEHG operator

$$T - SFEHG_{w,\omega}(\mathcal{T}_1, \dots, \mathcal{T}_5) = (0.0, 0.8525, 0.9882)$$

From the above result, it is noticed that when i or n value of one T-SFN is zero then the T-SFEHG operator cannot aggregate the whole membership value. This shows a big flaw in the T-SFEHG operator. This means that the results obtained from T-SFEHG operators are not reliable. Now by T-SFEHIG operator, we shall show that the proposed operator will overcome this drawback.

By using T-SFEWIG operator we shall be able to find

$\mathcal{T}_1 = (0.7393, 0.3232, 0.2154)$

$\mathcal{T}_2 = (0.9, 0.1, 0.6)$

$\mathcal{T}_3 = (0.4131, 0.5464, 0.7370)$

$\mathcal{T}_4 = (0.0988, 0.4829, 0.6776)$

$\mathcal{T}_5 = (0.0, 0.0, 0.8206)$

Their scores values will be

$SC(\mathcal{T}_1) = 0.3941$, $SC(\mathcal{T}_2) = 0.5130$, $SC(\mathcal{T}_3) = -0.3299$, $SC(\mathcal{T}_4) = -0.3101$, $SC(\mathcal{T}_5) = -0.5525$

Now using score value, the aggregated values obtained by using T-SEWIG operators are rearranged in descending order. Then these ordered values are again aggregated by using T-SFEHIG operator with associated WV will be $\omega = (0.112, 0.236, 0.304, 0.236, 0.112)^T$

$\tilde{\mathcal{T}}_{\sigma(1)} = (0.9, 0.1, 0.6)$

$\tilde{\mathcal{T}}_{\sigma(2)} = (0.7393, 0.3232, 0.2154)$

$\tilde{\mathcal{T}}_{\sigma(3)} = (0.0988, 0.4829, 0.6776)$

$\tilde{\mathcal{T}}_{\sigma(4)} = (0.4131, 0.5464, 0.7370)$

$\tilde{\mathcal{T}}_{\sigma(5)} = (0.0, 0.0, 0.8206)$

Now again measure T-SFEHG operator

$$T - SFEHIG_{w,\omega}\left(\tilde{\mathcal{T}}_1, \dots, \tilde{\mathcal{T}}_5\right) = (0.6878, 0.4329, 0.6591)$$

This shows that the T-SFEIG operator aggregate the membership value.

7.3. T-Spherical Fuzzy Einstein Hybrid Averaging Interaction Operators

In this section, on basis of new proposed Einstein operations we define averaging interaction operators in T-spherical fuzzy environment and some basic properties are also discussed.

7.3.1. Definition

Consider a collection of T-SFS $\mathcal{T}_j = \left(m_j, i_j, n_j\right)$ $(j = 1,2,3, \dots, k)$. Then

$$T - SFEWIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \oplus_{Ei}{}_{j=1}^{k} w_j \mathcal{T}_j$$

$$= \left(\begin{array}{c} \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1+m_j^t)^{w_j} - \prod_{j=1}^{k}(1-m_j^t)^{w_j}}{\prod_{j=1}^{k}(1+m_j^t)^{w_j} + \prod_{j=1}^{k}(1-m_j^t)^{w_j}}}, \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1+i_j^t)^{w_j} - \prod_{j=1}^{k}(1-i_j^t)^{w_j}}{\prod_{j=1}^{k}(1+i_j^t)^{w_j} + \prod_{j=1}^{k}(1-i_j^t)^{w_j}}}, \\ \sqrt[t]{\dfrac{2\left(\prod_{j=1}^{k}\left(1-m_j^t-i_j^t\right)^{w_j} - \prod_{j=1}^{k}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}\right)}{\prod_{j=1}^{k}\left(1+m_j^t\right)^{w_j} + \prod_{j=1}^{k}\left(1-m_j^t\right)^{w_j}}} \end{array} \right)$$

then $T - SFEWIA_w$ is called T-SFEWIA operator with WV $w = (w_1, w_2, \dots, w_k)^T$ of $\mathcal{T}_j$ with $w_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$.

7.3.2. Theorem

If all $\mathcal{T}_j = \mathcal{T}_0$, then $T - SFEWIA_w(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

Proof: Let $\mathcal{T}_j = \mathcal{T}_0 = (m_0, i_0, n_0)$ for all j then

$$T-SFEWIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt[t]{\frac{\prod_{j=1}^{k}(1+m_j^t)^{w_j}-\prod_{j=1}^{k}(1-m_j^t)^{w_j}}{\prod_{j=1}^{k}(1+m_j^t)^{w_j}+\prod_{j=1}^{k}(1-m_j^t)^{w_j}}},\sqrt[t]{\frac{\prod_{j=1}^{k}(1+i_j^t)^{w_j}-\prod_{j=1}^{k}(1-i_j^t)^{w_j}}{\prod_{j=1}^{k}(1+i_j^t)^{w_j}+\prod_{j=1}^{k}(1-i_j^t)^{w_j}}},\\ \sqrt[t]{\frac{2\left(\prod_{j=1}^{k}(1-m_j^t-i_j^t)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}\right)}{\prod_{j=1}^{k}(1+m_j^t)^{w_j}+\prod_{j=1}^{k}(1-m_j^t)^{w_j}}}\end{array}\right)$$

$$T-SFEWIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)$$

$$=\left(\begin{array}{c}\sqrt[t]{\frac{(1+m_j^t)^{\sum_{j=1}^{k}w_j}-(1-m_j^t)^{\sum_{j=1}^{k}w_j}}{(1+m_j^t)^{\sum_{j=1}^{k}w_j}+(1-m_j^t)^{\sum_{j=1}^{k}w_j}}},\sqrt[t]{\frac{(1+i_j^t)^{\sum_{j=1}^{k}w_j}-(1-i_j^t)^{\sum_{j=1}^{k}w_j}}{(1+i_j^t)^{\sum_{j=1}^{k}w_j}+(1-i_j^t)^{\sum_{j=1}^{k}w_j}}},\\ \sqrt[t]{\frac{2\left((1-m_j^t-i_j^t)^{\sum_{j=1}^{k}w_j}-\left(1-m_j^t-i_j^t-n_j^t\right)^{\sum_{j=1}^{k}w_j}\right)}{(1+m_j^t)^{\sum_{j=1}^{k}w_j}+(1-m_j^t)^{\sum_{j=1}^{k}w_j}}}\end{array}\right)$$

$$=(m_0,i_0,n_0)=\mathcal{T}_0$$

7.3.3. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j$ $(j=1,2,\dots,k)$ with $\mathcal{T}^L=\min_j \mathcal{T}_j$, and $\mathcal{T}^U=\max_j \mathcal{T}_j$. Then

$$\mathcal{T}^L\leq T-SFEWIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)\leq \mathcal{T}^U$$

7.3.4. Theorem

For any two T-SFNs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ and $\mathcal{T}_j{}'=\left(m_j',i_j',n_j'\right)$ such that $\mathcal{T}_j\leq\mathcal{T}_j{}'$ for all j. Then

$$T-SFEWIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)\leq T-SFEWIA_w(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

Proof: Let $\mathcal{T}_j\leq\mathcal{T}_j{}'$ then $m_j\leq m_j'$, $i_j\leq i_j'$ and $n_j\geq n_j'$. Then by using this basic information

$$\sqrt[t]{\frac{2\left(\prod_{j=1}^{k}\left(1-m_j^t-i_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^t-i_j^t-n_j^t\right)^{w_j}\right)}{\prod_{j=1}^{k}\left(1+m_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-m_j^t\right)^{w_j}}}$$

$$\leq \sqrt[t]{\frac{2\left(\prod_{j=1}^{k}\left(1-\left(m_j'\right)^t-\left(i_j'\right)^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j'\right)^t-\left(i_j'\right)^t-\left(n_j'\right)^t\right)^{w_j}\right)}{\prod_{j=1}^{k}\left(1+\left(m_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-\left(m_j'\right)^t\right)^{w_j}}}$$

$$\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+i_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+i_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-i_j^t\right)^{w_j}}} \leq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\left(i_j'\right)^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(i_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\left(i_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-\left(i_j'\right)^t\right)^{w_j}}}$$

$$\sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+m_j^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-m_j^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+m_j^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-m_j^t\right)^{w_j}}}$$

$$\geq \sqrt[t]{\frac{\prod_{j=1}^{k}\left(1+\left(m_j'\right)^t\right)^{w_j}-\prod_{j=1}^{k}\left(1-\left(m_j'\right)^t\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\left(m_j'\right)^t\right)^{w_j}+\prod_{j=1}^{k}\left(1-\left(m_j'\right)^t\right)^{w_j}}}$$

This shows that

$$T-SFEWIA_w(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k) \leq T-SFEWIA_w(\mathcal{T}_1',\mathcal{T}_2',\dots,\mathcal{T}_k')$$

7.3.5. Definition

Consider a collection of T-SFSs $\mathcal{T}_j=\left(m_j,i_j,n_j\right)$ $(j=1,2,3,\dots,k)$. Then

$$T-SFEOWIA_\omega(\mathcal{T}_1,\mathcal{T}_2,\dots,\mathcal{T}_k)=\bigoplus_{Ei}{}_{j=1}^{k}\,\omega_j\mathcal{T}_{\sigma(j)}$$

$$=\left(\begin{array}{c}\sqrt[t]{\frac{\prod_{j=1}^{k}(1+m_{\sigma(j)}^t)^{\omega_j}-\prod_{j=1}^{k}(1-m_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1+m_{\sigma(j)}^t)^{\omega_j}+\prod_{j=1}^{k}(1-m_{\sigma(j)}^t)^{\omega_j}}},\sqrt[t]{\frac{\prod_{j=1}^{k}(1+i_{\sigma(j)}^t)^{\omega_j}-\prod_{j=1}^{k}(1-i_{\sigma(j)}^t)^{\omega_j}}{\prod_{j=1}^{k}(1+i_{\sigma(j)}^t)^{\omega_j}+\prod_{j=1}^{k}(1-i_{\sigma(j)}^t)^{\omega_j}}},\\ \sqrt[t]{\frac{2\left[\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t-i_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{\omega_j}\right]}{\prod_{j=1}^{k}\left(1+m_{\sigma(j)}^t\right)^{\omega_j}+\prod_{j=1}^{k}\left(1-m_{\sigma(j)}^t\right)^{\omega_j}}}\end{array}\right)$$

then $T-SFEOWIA_\omega$ is called T-SFEOWIA operator with associated WV $\omega=(\omega_1,\omega_2,\dots,\omega_k)^T$ of $\mathcal{T}_j$ with $\omega_j\in[0,1]$ and $\sum_{j=1}^{k}\omega_j=1$. Where $\sigma(j)$ is any permutation of $(1,2,\dots,k)$ such that $\tilde{\mathcal{T}}_{\sigma(j-1)}\geq\tilde{\mathcal{T}}_{\sigma(j)}$.

7.3.6. Theorem

If for all $\mathcal{T}_j = \mathcal{T}_0$, then $T - SFEOWA_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) = \mathcal{T}_0$.

7.3.7. Theorem

Consider a collection of T-SFNs $\mathcal{T}_j\ (j = 1,2, \dots, k)$ with $\mathcal{T}^L = \min\limits_j \mathcal{T}_j$, and $\mathcal{T}^U = \max\limits_j \mathcal{T}_j$. Then

$$\mathcal{T}^L \leq T - SFEOWIA_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq \mathcal{T}^U$$

7.3.8. Theorem

Consider any two $T - SFNs$ $\mathcal{T}_j = (m_j, i_j, n_j)$ and $\mathcal{T}_j{}' = (m_j', i_j', n_j')$ such that $\mathcal{T}_j \leq \mathcal{T}_j{}'$ for all j. Then

$$T - SFEOWIA_{\omega}(\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_k) \leq T - SFEOWIA_{\omega}(\mathcal{T}_1', \mathcal{T}_2', \dots, \mathcal{T}_k')$$

7.3.9. Definition

Consider a collection of T-SFNs $\mathcal{T}_j = (m_j, i_j, n_j)\ (j = 1,2,3, \dots, k)$. The mapping

$$T - SFEHIA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k) = \oplus_{Ei}{}_{j=1}^{k}\, \omega_j \tilde{\mathcal{T}}_{\sigma(j)}$$

$$= \left(\begin{array}{c} \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1+\tilde{m}_{\sigma(j)}^{t})^{\omega_j} - \prod_{j=1}^{k}(1-\tilde{m}_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}(1+\tilde{m}_{\sigma(j)}^{t})^{\omega_j} + \prod_{j=1}^{k}(1-\tilde{m}_{\sigma(j)}^{t})^{\omega_j}}}, \\ \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1+\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j} - \prod_{j=1}^{k}(1-\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}(1+\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j} + \prod_{j=1}^{k}(1-\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j}}}, \\ \sqrt[t]{\dfrac{2\left(\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}-\tilde{\imath}_{\sigma(j)}^{t}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}-\tilde{\imath}_{\sigma(j)}^{t}-\tilde{n}_{\sigma(j)}^{t}\right)^{\omega_j}\right)}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j}}} \end{array} \right)$$

called T-SFEHIA operator. where $\tilde{\mathcal{T}}_j = k w_j \mathcal{T}_j$ and $w_j, \omega_j \in [0,1]$ and $\sum_{j=1}^{k} w_j = 1$, $\sum_{j=1}^{k} \omega_j = 1$.

Hybrid aggregation operators first aggregate the given data considering their attributes then rearrange them in a specific order. After that, they aggregate the data considering their order. This means that hybrid operators are a generalization of weighted and ordered weighted operators. So T-SFEHIA operator will satisfy idempotent, monotone and bounded property.

7.3.10. Example

Consider five T-SFNs $\mathcal{T}_1 = (0.9,0.3,0.4)$, $\mathcal{T}_2 = (0.6,0.3,0.2)$, $\mathcal{T}_3 = (0.3,0.8,0.6)$, $\mathcal{T}_4 = (0.4,0.5,0.8)$, $\mathcal{T}_5 = (0.6,0.0,0.0)$ with a WV $w = (0.25, 0.20, 0.15, 0.18,0.22)^T$.

Solution: First of all we find the aggregated value of these T-SFNs by using T-spherical fuzzy Einstein hybrid averaging aggregation (T-SFEHA) operator from Chapter 6 to find out the drawbacks of given operators. For this purpose first of all we have to find the value of t for which the given data lie in T-spherical fuzzy environment.

As, $0.3 + 0.8 + 0.6 = 1.7$

For $t = 2$, $0.3^2 + 0.8^2 + 0.6^2 = 1.09$

For $t = 3$, $0.3^3 + 0.8^3 + 0.6^3 = 0.755$

Similarly, for $t = 3$ all the given data lie in the T-spherical fuzzy environment.

By using T-spherical fuzzy Einstein weighted averaging operator we shall be able to find these values,

$\mathcal{T}_1 = (0.9362, 0.2104,0.3029)$

$\mathcal{T}_2 = (0.6,0.3,0.2)$

$\mathcal{T}_3 = (0.2726, 0.8527,0.6984)$

$\mathcal{T}_4 = (0.3862, 0.5437,0.8211)$

$\mathcal{T}_5 = (0.6187,0.0,0.0)$

Their scores values will be

$SC(\mathcal{T}_1) = 0.7927$, $SC(\mathcal{T}_2) = 0.2080$, $SC(\mathcal{T}_3) = -0.3203$, $SC(\mathcal{T}_4) = -0.4961$, $SC(\mathcal{T}_5) = 0.2368$

Now using score value, the aggregated values obtained by using T-SFEWA operators are rearranged in descending order. Then these ordered values are again aggregated by using T-SFEHA operator with associated WV will be $\omega = (0.112,0.236,0.304,0.236,0.112)$

$\tilde{\mathcal{T}}_{\sigma(1)} = (0.9362,0.2104,0.3029)$

$\tilde{\mathcal{T}}_{\sigma(2)} = (0.6187,0.0,0.0)$

$\tilde{\mathcal{T}}_{\sigma(3)} = (0.6, 0.3,0.2)$

$\tilde{\mathcal{T}}_{\sigma(4)} = (0.2726, 0.8527,0.6984)$

$\tilde{\mathcal{T}}_{\sigma(5)} = (0.3862,0.5437,0.8211)$

Now again measure T-SFEHA operator

$$T - SFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \dots, \tilde{\mathcal{T}}_5) = (0.6187,0.0,0.0)$$

From the above result, it is noticed that when abstinence or non-membership value of one T-SFN is zero then the T-SFEHA operator cannot aggregate the whole abstinence and non-membership value. This shows a big flaw in the T-SFEHA operator. This means that the results obtained from T-SFEHA operators are not reliable. Now by T-SFEIA operator, we shall show that the proposed operator will overcome this drawback.

By using T-SFEIA operator we shall be able to find

$\mathcal{T}_1 = (0.9362, 0.4308,0.2739)$

$\mathcal{T}_2 = (0.6,0.3,0.2)$

$\mathcal{T}_3 = (0.2726, 0.7370,0.5956)$

$\mathcal{T}_4 = (0.3862, 0.4829,0.7888)$

$\mathcal{T}_5 = (0.6187,0.0,0.0)$

Their scores values will be

$SC(\mathcal{T}_1) = 0.7999$, $SC(\mathcal{T}_2) = 0.2080$, $SC(\mathcal{T}_3) = -0.1910$, $SC(\mathcal{T}_4) = -0.4333$, $SC(\mathcal{T}_5) = 0.2368$

Now using score value, the aggregated values obtained by using T-SFEWIA operators are rearranged in descending order. Then these ordered values are again aggregated by using T-SFEHIA operator with associated WV will be $\omega = (0.112,0.236,0.304,0.236,0.112)$

$\tilde{\mathcal{T}}_{\sigma(1)} = (0.9362,0.4308,0.2739)$

$\tilde{\mathcal{T}}_{\sigma(2)} = (0.6187,0.0,0.0)$

$\tilde{\mathcal{T}}_{\sigma(3)} = (0.6, 0.3,0.2)$

$\tilde{\mathcal{T}}_{\sigma(4)} = (0.2726, 0.7370,0.5956)$

$\tilde{\mathcal{T}}_{\sigma(5)} = (0.3862,0.4829,0.7888)$

Now again measure T-SFEHIA operator

$T - SFEHIA_{w,\omega}(\tilde{\mathcal{T}}_1, \dots, \tilde{\mathcal{T}}_5) = (0.6372,0.5055,0.3978)$

This shows that the T-SFEIA operator aggregate the membership value.

7.4. An algorithm for MADM with T-spherical fuzzy information

Consider a set of alternatives $D = \{d_1, d_2, d_3, \dots, d_l\}$ and a set of attributes $M = \{m_1, m_2, m_3, \dots, m_k\}$ having a WV $w = \{w_1, w_2, w_3, \dots, w_l\}$ where $w_j \in [0,1]$ and $\sum_{m=1}^{l} w_m = 1$. For making a decision we have to follow these steps.

Step 1. Calculate t for which the values lie in T-spherical information.

Step 2. Aggregate the given alternatives according to attributes by T-SFEWIA (or T-SFEWIG) operators using some WV.

Step 3. Find scores values and with the help of score value we reorder them in descending order.

Step 4. Aggregate these ordered values using T-SFEHIA (or T-SFEHIG) operator.

Step 5. Using score values find out the best option.

7.4.1. Example

A company wants to maximize its profit and board of governors decided to reduce their expenses. They observe that the cost of electricity is one of the major expense and they can reduce it if they started to generate electricity using solar energy. They have three options of photovoltaic cells that they may use in their solar plant.

i. d_1: Monocrystalline Photovoltaic Cell

ii. d_2: Polycrystalline Photovoltaic Cell

iii. d_3: Thin Film Photovoltaic Cell

They assess the given photovoltaic cell on the base of following attributes.

i. M_1: Heat Tolerance

ii. M_2: Cost

iii. M_3: Reliability

iv. M_4: Efficiency

v. M_5: Ability of charge separation

Table 68 Decision Matrix

	M_1	M_2	M_3	M_4	M_5
d_1	$(0.4, 0.1, 0.7)$	$(0.5, 0.2, 0.4)$	$(0.8, 0.3, 0.7)$	$(0.4, 0.8, 0.5)$	$(0.9, 0.5, 0.2)$
d_2	$(0.7, 0.4, 0.3)$	$(0.2, 0.4, 0.7)$	$(0.9, 0.3, 0.6)$	$(0.3, 0.2, 0.8)$	$(0.4, 0.7, 0.5)$
d_3	$(0.4, 0.7, 0.5)$	$(0.6, 0.6, 0.1)$	$(0.6, 0.9, 0.2)$	$(0.8, 0.1, 0.1)$	$(0.5, 0.6, 0.2)$

Step 1: After some calculation we found $t = 3$ at which all values in Table 68 are T-SFNs.

Step 2: By taking $w = (0.25, 0.20, 0.15, 0.18, 0.22)^T$ we find T-SFEWIA values of given data, as listed in Table 69.

Table 69 Aggregated values

	M_1	M_2	M_3	M_4	M_5
d_1	$\begin{pmatrix} 0.4308, \\ 0.1077, \\ 0.7367 \end{pmatrix}$	$\begin{pmatrix} 0.5, \\ 0.2, \\ 0.4 \end{pmatrix}$	$\begin{pmatrix} 0.7370, \\ 0.2726, \\ 0.7172 \end{pmatrix}$	$\begin{pmatrix} 0.3862, \\ 0.7770, \\ 0.5022 \end{pmatrix}$	$\begin{pmatrix} 0.9164, \\ 0.5160, \\ 0.1906 \end{pmatrix}$
d_2	$\begin{pmatrix} 0.7485, \\ 0.4308, \\ 0.3061 \end{pmatrix}$	$\begin{pmatrix} 0.2, \\ 0.4, \\ 0.7 \end{pmatrix}$	$\begin{pmatrix} 0.8440, \\ 0.2726, \\ 0.6677 \end{pmatrix}$	$\begin{pmatrix} 0.2896, \\ 0.1931, \\ 0.7820 \end{pmatrix}$	$\begin{pmatrix} 0.4129, \\ 0.7206, \\ 0.5037 \end{pmatrix}$
d_3	$\begin{pmatrix} 0.4308, \\ 0.7485, \\ 0.5064 \end{pmatrix}$	$\begin{pmatrix} 0.6, \\ 0.6, \\ 0.1 \end{pmatrix}$	$\begin{pmatrix} 0.5464, \\ 0.8440, \\ 0.2430 \end{pmatrix}$	$\begin{pmatrix} 0.7770, \\ 0.0965, \\ 0.0829 \end{pmatrix}$	$\begin{pmatrix} 0.5160, \\ 0.6187, \\ 0.2031 \end{pmatrix}$

Step 3: Scores of each alternative with respect to all attributes are shown in Table 70

Table 70 Score Values

	M_1	M_2	M_3	M_4	M_5
d_1	-0.3199	0.0610	0.0314	-0.0690	0.7626
d_2	0.3906	-0.3350	0.3035	-0.4538	-0.0574
d_3	-0.0499	0.2150	0.1488	0.4685	0.1290

By comparing the score values, we have

$$SC(\mathcal{T}_{15}) > SC(\mathcal{T}_{12}) > SC(\mathcal{T}_{13}) > SC(\mathcal{T}_{14}) > SC(\mathcal{T}_{11})$$

$$SC(\mathcal{T}_{21}) > SC(\mathcal{T}_{23}) > SC(\mathcal{T}_{25}) > SC(\mathcal{T}_{22}) > SC(\mathcal{T}_{24})$$

$$SC(\mathcal{T}_{34}) > SC(\mathcal{T}_{32}) > SC(\mathcal{T}_{33}) > SC(\mathcal{T}_{35}) > SC(\mathcal{T}_{31})$$

Based on above score analysis, the data is arranged in descending order and the aggregated values of ordered data is as listed in Table 71

Table 71 Ordered Aggregated Values

	M_1	M_2	M_3	M_4	M_5
d_1	$\begin{pmatrix}0.9164,\\0.5160,\\0.1906\end{pmatrix}$	$\begin{pmatrix}0.5,\\0.2,\\0.4\end{pmatrix}$	$\begin{pmatrix}0.7370,\\0.2726,\\0.7172\end{pmatrix}$	$\begin{pmatrix}0.3862,\\0.7770,\\0.5022\end{pmatrix}$	$\begin{pmatrix}0.4308,\\0.1077,\\0.7367\end{pmatrix}$
d_2	$\begin{pmatrix}0.7485,\\0.4308,\\0.3061\end{pmatrix}$	$\begin{pmatrix}0.8440,\\02726,\\0.6677\end{pmatrix}$	$\begin{pmatrix}0.4129,\\0.7206,\\0.5037\end{pmatrix}$	$\begin{pmatrix}0.2,\\0.4,\\0.7\end{pmatrix}$	$\begin{pmatrix}0.2896,\\0.1931,\\0.7820\end{pmatrix}$
d_3	$\begin{pmatrix}0.7770,\\0.0965,\\0.0829\end{pmatrix}$	$\begin{pmatrix}0.6,\\0.6,\\0.1\end{pmatrix}$	$\begin{pmatrix}0.5464,\\0.8440,\\0.2430\end{pmatrix}$	$\begin{pmatrix}0.5160,\\0.6187,\\0.2031\end{pmatrix}$	$\begin{pmatrix}0.4308,\\0.7485,\\0.5064\end{pmatrix}$

Step 4: Associated WV will be $\omega = (0.112{,}0.236{,}0.304{,}0.236{,}0.112)^T$ and by T-SFEHIA operators, we have

$$\tilde{\mathcal{T}}_1 = (0.6596, 0.5227{,}0.4668)$$

$$\tilde{\mathcal{T}}_2 = (0.6176, 0.5291{,}0.5276)$$

$$\tilde{\mathcal{T}}_3 = (0.5826, 0.7075{,}0.2290)$$

Step 5: Now we have to find the score values

$$SC(\tilde{\mathcal{T}}_1) = 0.1853$$

$$SC(\tilde{\mathcal{T}}_2) = 0.0887$$

$$SC(\tilde{\mathcal{T}}_3) = 0.1858$$

$$SC(\tilde{\mathcal{T}}_3) > SC(\tilde{\mathcal{T}}_1) > SC(\tilde{\mathcal{T}}_2)$$

Since the score value of d_3 is highest so thin film photovoltaic cell is best option.

Now, we check their validity by using Einstein hybrid geometric interaction operators.

By taking $w = (0.25, 0.20, 0.15, 0.18, 0.22)^T$ we find T-SFEWIG values of given data, as listed in Table 72.

Table 72 Aggregated Values

	M_1	M_2	M_3	M_4	M_5
d_1	$\begin{pmatrix}0.4117,\\0.1077,\\0.7485\end{pmatrix}$	$\begin{pmatrix}0.5,\\0.2,\\0.4\end{pmatrix}$	$\begin{pmatrix}0.7998,\\0.2726,\\0.6398\end{pmatrix}$	$\begin{pmatrix}0.4008,\\0.7770,\\0.4829\end{pmatrix}$	$\begin{pmatrix}0.9051,\\0.5160,\\0.2064\end{pmatrix}$
d_2	$\begin{pmatrix}0.7347,\\0.4308,\\0.3232\end{pmatrix}$	$\begin{pmatrix}0.2,\\0.4,\\0.7\end{pmatrix}$	$\begin{pmatrix}0.9071,\\0.2726,\\0.5464\end{pmatrix}$	$\begin{pmatrix}0.2984,\\0.1931,\\0.7770\end{pmatrix}$	$\begin{pmatrix}0.4034,\\0.7206,\\0.5160\end{pmatrix}$
d_3	$\begin{pmatrix}0.4064,\\0.7485,\\0.5381\end{pmatrix}$	$\begin{pmatrix}0.6,\\0.6,\\0.1\end{pmatrix}$	$\begin{pmatrix}0.6434,\\0.8440,\\0.1817\end{pmatrix}$	$\begin{pmatrix}0.7807,\\0.0965,\\0.0965\end{pmatrix}$	$\begin{pmatrix}0.5103,\\0.6187,\\0.2064\end{pmatrix}$

Scores of Table 72 are shown in Table 73

Table 73 Score Values

	M_1	M_2	M_3	M_4	M_5
d_1	-0.3495	0.0610	0.2497	-0.0482	0.7327
d_2	0.3628	-0.3350	0.5833	-0.4425	-0.0717
d_3	-0.0887	0.2150	0.2603	0.4749	0.1241

By comparing the score values, we have

$$SC(\mathcal{T}_{15}) > SC(\mathcal{T}_{13}) > SC(\mathcal{T}_{12}) > SC(\mathcal{T}_{14}) > SC(\mathcal{T}_{11})$$

$$SC(\mathcal{T}_{23}) > SC(\mathcal{T}_{21}) > SC(\mathcal{T}_{25}) > SC(\mathcal{T}_{22}) > SC(\mathcal{T}_{24})$$

$$SC(\mathcal{T}_{34}) > SC(\mathcal{T}_{33}) > SC(\mathcal{T}_{32}) > SC(\mathcal{T}_{35}) > SC(\mathcal{T}_{31})$$

Based on above score analysis, the data is arranged in descending order and the aggregated values of ordered data is as listed in Table 74

Table 74 Ordered Aggregated Values

	M_1	M_2	M_3	M_4	M_5
d_1	$\begin{pmatrix}0.9051,\\0.5160,\\0.2064\end{pmatrix}$	$\begin{pmatrix}0.7998,\\0.2726,\\0.6398\end{pmatrix}$	$\begin{pmatrix}0.5,\\0.2,\\0.4\end{pmatrix}$	$\begin{pmatrix}0.4008,\\0.7770,\\0.4829\end{pmatrix}$	$\begin{pmatrix}0.4117,\\0.1077,\\0.7485\end{pmatrix}$
d_2	$\begin{pmatrix}0.9071,\\0.2726,\\0.5464\end{pmatrix}$	$\begin{pmatrix}0.7347,\\0.4308,\\0.3232\end{pmatrix}$	$\begin{pmatrix}0.4034,\\0.7206,\\0.5160\end{pmatrix}$	$\begin{pmatrix}0.2,\\0.4,\\0.7\end{pmatrix}$	$\begin{pmatrix}0.2984,\\0.1931,\\0.7770\end{pmatrix}$
d_3	$\begin{pmatrix}0.7807,\\0.0965,\\0.0965\end{pmatrix}$	$\begin{pmatrix}0.6434,\\0.8440,\\0.1817\end{pmatrix}$	$\begin{pmatrix}0.6,\\0.6,\\0.1\end{pmatrix}$	$\begin{pmatrix}0.5103,\\0.6187,\\0.2064\end{pmatrix}$	$\begin{pmatrix}0.4064,\\0.7485,\\0.5381\end{pmatrix}$

Associated WV will be $\omega = (0.112,0.236,0.304,0.236,0.112)^T$ and by T-SFEHIG operators, we have

$$\tilde{\mathcal{T}}_1 = (0.5445, 0.5217,0.5419)$$

$$\tilde{\mathcal{T}}_2 = (0.6830, 0.5376,0.5913)$$

$$\tilde{\mathcal{T}}_3 = (0.7556, 0.6879,0.2780)$$

Step 5: Now we have to find the score values

$$SC\left(\tilde{\mathcal{T}}_1\right) = 0.0023$$

$$SC\left(\tilde{\mathcal{T}}_2\right) = 0.1080$$

$$SC\left(\tilde{\mathcal{T}}_3\right) = 0.4099$$

$$SC(\tilde{\mathcal{T}}_3) > SC(\tilde{\mathcal{T}}_2) > SC(\tilde{\mathcal{T}}_1)$$

Here again the score value of alternative d_3 is high. So, thin film photovoltaic cell is best option.

7.5. Advantages

In this section we prove that our work is more generalized than that of existing work. In our proposed work experts are free in giving the values to alternatives according to given attributes not only this proposed work is also valid under those conditions where the existing work fail. Here we reduced the proposed work in intuitionistic, Pythagorean, q-rung orthopair, picture and spherical fuzzy environments. It prove that proposed work is valid for all those environments.

Consider the T-SFEHIA defined as

$$T-SFEHIA_{w,\omega}\left(\tilde{\mathcal{T}}_1,\tilde{\mathcal{T}}_2,\ldots,\tilde{\mathcal{T}}_k\right)$$

$$=\left(\begin{array}{c}\sqrt[t]{\dfrac{\prod_{j=1}^{k}(1+\tilde{m}_{\sigma(j)}^{t})^{\omega_j}-\prod_{j=1}^{k}(1-\tilde{m}_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}(1+\tilde{m}_{\sigma(j)}^{t})^{\omega_j}+\prod_{j=1}^{k}(1-\tilde{m}_{\sigma(j)}^{t})^{\omega_j}}},\\ \sqrt[t]{\dfrac{\prod_{j=1}^{k}(1+\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j}-\prod_{j=1}^{k}(1-\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j}}{\prod_{j=1}^{k}(1+\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j}+\prod_{j=1}^{k}(1-\tilde{\imath}_{\sigma(j)}^{t})^{\omega_j}}},\\ \sqrt[t]{\dfrac{2\left(\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}-\tilde{\imath}_{\sigma(j)}^{t}\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}-\tilde{\imath}_{\sigma(j)}^{t}-\tilde{n}_{\sigma(j)}^{t}\right)^{\omega_j}\right)}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j}+\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{t}\right)^{\omega_j}}}\end{array}\right) \tag{7.5.1}$$

1. For $t=2$ the equation (7.5.1) reduces to SF Einstein hybrid interaction averaging operators (SFEHIA operator) i.e.

$$SFEHIA_{w,\omega}\left(\tilde{\mathcal{T}}_1,\tilde{\mathcal{T}}_2,\ldots,\tilde{\mathcal{T}}_k\right)$$

$$=\left(\begin{array}{c}\sqrt{\dfrac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^{2}\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{2}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^{2}\right)^{\omega_j}+\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{2}\right)^{\omega_j}}},\\ \sqrt{\dfrac{\prod_{j=1}^{k}(1+\tilde{\imath}_{\sigma(j)}^{2})^{\omega_j}-\prod_{j=1}^{k}(1-\tilde{\imath}_{\sigma(j)}^{2})^{\omega_j}}{\prod_{j=1}^{k}(1+\tilde{\imath}_{\sigma(j)}^{2})^{w_j}+\prod_{j=1}^{k}(1-\tilde{\imath}_{\sigma(j)}^{2})^{\omega_j}}},\\ \sqrt{\dfrac{2\left(\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{2}-\tilde{\imath}_{\sigma(j)}^{2}\right)^{\omega_j}-\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{2}-\tilde{\imath}_{\sigma(j)}^{2}-\tilde{n}_{\sigma(j)}^{2}\right)^{\omega_j}\right)}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^{2}\right)^{\omega_j}+\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^{2}\right)^{\omega_j}}}\end{array}\right)$$

2. For $t = 1$ the equation (7.5.1) reduces to PF Einstein hybrid interaction averaging operators (PFEHIA operator) i.e.

$$PFEHA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k) = \left(\begin{array}{c} \dfrac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}\right)^{\omega_j}}, \dfrac{\prod_{j=1}^{k}\left(1+\tilde{\imath}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{\imath}_{\sigma(j)}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{\imath}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{\imath}_{\sigma(j)}\right)^{\omega_j}}, \\ \dfrac{2\left(\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}-\tilde{\imath}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}-\tilde{\imath}_{\sigma(j)}-\tilde{n}_{\sigma(j)}\right)^{\omega_j}\right)}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}\right)^{\omega_j}} \end{array} \right)$$

3. For $i = 0$ the equation (7.5.1) reduces to q-ROPF Einstein hybrid interaction averaging operators (q-ROFEHIA operator) i.e.

$$q-ROFEHIA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k) = \left(\begin{array}{c} \sqrt[t]{\dfrac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}}, \\ \sqrt[t]{\dfrac{2\left(\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t-n_{\sigma(j)}^t\right)^{\omega_j}\right)}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^t\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^t\right)^{\omega_j}}} \end{array} \right)$$

4. For $t = 2$ and $i = 0$ the equation (7.5.1) reduces to PyF Einstein hybrid interaction averaging operators (PyFEHIA operator) i.e.

$$PyFEHIA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k) = \left(\begin{array}{c} \sqrt{\dfrac{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j}}}, \\ \sqrt{\dfrac{2\left(\prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2-\tilde{n}_{\sigma(j)}^2\right)^{\omega_j}\right)}{\prod_{j=1}^{k}\left(1+\tilde{m}_{\sigma(j)}^2\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\tilde{m}_{\sigma(j)}^2\right)^{\omega_j}}} \end{array} \right)$$

5. For $t = 1$ and $i = 0$ the equation (7.5.1) reduces to IF Einstein hybrid interaction averaging operators (IFEHA operator) i.e.

$$IFEHIA_{w,\omega}(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_k) = \left(\begin{array}{c} \dfrac{\prod_{j=1}^{k}\left(1+\widetilde{m}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\widetilde{m}_{\sigma(j)}\right)^{\omega_j}}{\prod_{j=1}^{k}\left(1+\widetilde{m}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\widetilde{m}_{\sigma(j)}\right)^{\omega_j}}, \\ \dfrac{2\left(\prod_{j=1}^{k}\left(1-\widetilde{m}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{k}\left(1-\widetilde{m}_{\sigma(j)}-\tilde{n}_{\sigma(j)}\right)^{\omega_j}\right)}{\prod_{j=1}^{k}\left(1+\widetilde{m}_{\sigma(j)}\right)^{\omega_j} + \prod_{j=1}^{k}\left(1-\widetilde{m}_{\sigma(j)}\right)^{\omega_j}} \end{array} \right)$$

Similarly we can reduce T-SFEWIA operator, T-SFEOWIA operator, T-SFEWIG operator, T-SFEOWIG operator and T-SFEHIG operator.

7.6. Comparative Study

The proposed aggregation operators can aggregate the data given in FS, IFS, PyFS, q-ROPFS, PFS and SFS environments but the converse is not possible. Here with the help of an example it is proved that the proposed aggregation operator can aggregate the data given in IFSs.

7.6.1. Example

Let IFNs $\mathcal{T}_1 = (0.2,0.5)$, $\mathcal{T}_2 = (0.7,0.1)$, $\mathcal{T}_3 = (0.3,0.4)$, $\mathcal{T}_4 = (0.6,0.2)$ and $\mathcal{T}_5 = (0.5,0.5)$ with a WV $w = (0.25, 0.20, 0.15, 0.18,0.22)^T$.

Solution. We can write these IFNs in the form of T-SFNs as $\mathcal{T}_1 = (0.2,0,0.5)$, $\mathcal{T}_2 = (0.7,0,0.1)$, $\mathcal{T}_3 = (0.3,0,0.4)$, $\mathcal{T}_4 = (0.6,0,0.2)$ and $\mathcal{T}_5 = (0.5,0,0.5)$. Then by using T-SFEWIA operator we shall be able to find these values,

$\mathcal{T}_1 = (0.2481,0,0.5312)$

$\mathcal{T}_2 = (0.7,0,0.1)$

$\mathcal{T}_3 = (0.2281, 0,0.3630)$

$\mathcal{T}_4 = (0.5538, 0,0.2071)$

$\mathcal{T}_5 = (0.5401,0,0.4599)$

Their scores values will be

$SC(\mathcal{T}_1) = -0.2831$, $SC(\mathcal{T}_2) = 0.6$, $SC(\mathcal{T}_3) = -0.1350$, $SC(\mathcal{T}_4) = 0.3467$, $SC(\mathcal{T}_5) = 0.0801$

Now using score value, the aggregated values obtained by using T-SFEWIA operators are rearranged in descending order. Then these ordered values are again aggregated by using T-SFEHIA operator with associated WV will be $\omega = (0.112,0.236,0.304,0.236,0.112)^T$

$\tilde{\mathcal{T}}_{\sigma(1)} = (0.7,0,0.1)$

$\tilde{\mathcal{T}}_{\sigma(2)} = (0.5538, 0,0.2071)$

$\tilde{\mathcal{T}}_{\sigma(3)} = (0.5401,0,0.4599)$

$\tilde{\mathcal{T}}_{\sigma(4)} = (0.2281, 0,0.3630)$

$\tilde{\mathcal{T}}_{\sigma(5)} = (0.2481,0,0.5312)$

Now again measure T-SFEHIA operator

$$T - SFEHIA_{w,\omega}\left(\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2, \dots, \tilde{\mathcal{T}}_5\right) = (0.4709,0,0.2645)$$

Here it is proved that the information given in IFNs can be solved by using T-SFEHIA operator. Similarly we can solve the information given in IFNs by using T-SEHIG operator and the information given in any other fuzzy structure can also be aggregated using the proposed operators.

Chapter 8

Exponential Similarity Measures for T-Spherical Fuzzy Sets and Their Applications in Decision Making

SMs are common tools that are considered to be applied to some interesting phenomena in real life including pattern recognition, decision making, etc. In this chapter, some SMs based on cosine function and some SMs based on exponential function are developed for T-SFSs. The basic properties of developed SMs are also discussed. By using developed SMs, two well-known problems; pattern recognition and strategy decision making problems are solved. The superiority of developed SMs over SMs in SFS, PFS, q-ROPFS, PyFS, and IFS is demonstrated through a comparison. A numerical example is also discussed to prove the superiority of the proposed work.

8.1. Similarity Measures

In this section some cosine SMs based on cosine function are proposed and also some exponential SMs are proposed. Some basic properties of proposed SMs are also studied.

8.1.1. Similarity Measures Based on Cosine Function

In this subsection some SMs and weighted SMs based on cosine function are proposed and basic properties of these SMs are also discussed.

8.1.1.1. Definition

Consider two T-SFSs on domain X, $\mathcal{T}_1 = \left\{\left(x_j, m_1(x_j), i_1(x_j), n_1(x_j)\right) \mid x_j \in X\right\}$ and $\mathcal{T}_2 = \left\{\left(x_j, m_2(x_j), i_2(x_j), n_2(x_j)\right) \mid x_j \in X\right\}$ where $(j = 1,2,\ldots,k)$ then four cosine SMs based on cosine function can be calculated as

$$CSM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\cos\left[\frac{\pi}{2}\left(\begin{array}{c}\left|m_1^t(x_j) - m_2^t(x_j)\right| \vee \left|i_1^t(x_j) - i_2^t(x_j)\right| \\ \vee\left|n_1^t(x_j) - n_2^t(x_j)\right|\end{array}\right)\right]$$

$$CSM^2(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\cos\left[\frac{\pi}{4}\left(\begin{array}{c}\left|m_1^t(x_j) - m_2^t(x_j)\right| + \left|i_1^t(x_j) - i_2^t(x_j)\right| \\ +\left|n_1^t(x_j) - n_2^t(x_j)\right|\end{array}\right)\right]$$

$$CSM^3(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\cos\left[\frac{\pi}{2}\begin{pmatrix} \left|m_1^t(x_j) - m_2^t(x_j)\right| \vee \left|i_1^t(x_j) - i_2^t(x_j)\right| \\ \vee\left|n_1^t(x_j) - n_2^t(x_j)\right| \vee \left|r_1^t(x_j) - r_2^t(x_j)\right| \end{pmatrix}\right]$$

$$CSM^4(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\cos\left[\frac{\pi}{4}\begin{pmatrix} \left|m_1^t(x_j) - m_2^t(x_j)\right| + \left|i_1^t(x_j) - i_2^t(x_j)\right| \\ +\left|n_1^t(x_j) - n_2^t(x_j)\right| + \left|r_1^t(x_j) - r_2^t(x_j)\right| \end{pmatrix}\right]$$

Where the symbol "$\vee$" means the maximum operation.

The SMs defined above will fulfil the following properties:

1. $0 \leq CSM^t(\mathcal{T}_1,\mathcal{T}_2) \leq 1$
2. $CSM^t(\mathcal{T}_1,\mathcal{T}_2) = 1$ if and only if $\mathcal{T}_1 = \mathcal{T}_2$
3. $CSM^t(\mathcal{T}_1,\mathcal{T}_2) = CSM^t(\mathcal{T}_2,\mathcal{T}_1)$
4. If $\mathcal{T}_3 = \left\{\left(x_j, m_3(x_j), i_3(x_j), n_3(x_j)\right) \mid x_j \in X\right\}$ is a T-SFS on X and $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$ then $CSM^t(\mathcal{T}_1,\mathcal{T}_3) \leq CSM^t(\mathcal{T}_1,\mathcal{T}_2)$ and $CSM^t(\mathcal{T}_1,\mathcal{T}_3) \leq CSM^t(\mathcal{T}_2,\mathcal{T}_3)$.

where $t = 1,2,3,4$.

Proof: 1. As cosine function always lie in $[0,1]$ interval so it is obvious that $0 \leq CSM^t(\mathcal{T}_1,\mathcal{T}_2) \leq 1$ for all $t = 1,2,3,4$.

2. Let us assume $\mathcal{T}_1 = \mathcal{T}_2$, then it means $m_1^t(x_j) = m_2^t(x_j)$, $i_1^t(x_j) = i_2^t(x_j)$, and $n_1^t(x_j) = n_2^t(x_j)$.

$$\Rightarrow \left|m_1^t(x_j) - m_2^t(x_j)\right| = \left|i_1^t(x_j) - i_2^t(x_j)\right| = \left|n_1^t(x_j) - n_2^t(x_j)\right| = 0$$

$$CSM^t(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\cos[0]$$

$$CSM^t(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}1 = \frac{1}{k}\times k = 1$$

Conversely assume $CSM^t(\mathcal{T}_1,\mathcal{T}_2) = 1$, then

$$\left|m_1^t(x_j) - m_2^t(x_j)\right| = \left|i_1^t(x_j) - i_2^t(x_j)\right| = \left|n_1^t(x_j) - n_2^t(x_j)\right| = 0$$

which means $m_1^t(x_j) = m_2^t(x_j)$, $i_1^t(x_j) = i_2^t(x_j)$, and $n_1^t(x_j) = n_2^t(x_j)$. Thus $\mathcal{T}_1 = \mathcal{T}_2$.

3. Consider $t = 1$,

$$CSM^1(\mathcal{T}_1, \mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k} \cos\left[\frac{\pi}{2}\begin{pmatrix} |m_1^t(x_j) - m_2^t(x_j)| \vee |i_1^t(x_j) - i_2^t(x_j)| \\ \vee |n_1^t(x_j) - n_2^t(x_j)| \end{pmatrix}\right]$$

$$= \frac{1}{k}\sum_{j=1}^{k} \cos\left[\frac{\pi}{2}\begin{pmatrix} |m_2^t(x_j) - m_1^t(x_j)| \vee |i_2^t(x_j) - i_1^t(x_j)| \\ \vee |n_2^t(x_j) - n_1^t(x_j)| \end{pmatrix}\right] = CSM^1(\mathcal{T}_2, \mathcal{T}_1)$$

Similarly, this can be proved for $t = 2,3,4$.

4. Consider $t = 1$, If $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$ then $m_1^t(x_j) \leq m_2^t(x_j) \leq m_3^t(x_j)$, $i_1^t(x_j) \leq i_2^t(x_j) \leq i_3^t(x_j)$, and $n_1^t(x_j) \geq n_2^t(x_j) \geq n_3^t(x_j)$. Thus

$$|m_1^t(x_j) - m_2^t(x_j)| \leq |m_1^t(x_j) - m_3^t(x_j)|$$

$$|m_2^t(x_j) - m_3^t(x_j)| \leq |m_1^t(x_j) - m_3^t(x_j)|$$

$$|i_1^t(x_j) - i_2^t(x_j)| \leq |i_1^t(x_j) - i_3^t(x_j)|$$

$$|i_2^t(x_j) - i_3^t(x_j)| \leq |i_1^t(x_j) - i_3^t(x_j)|$$

$$|n_1^t(x_j) - n_2^t(x_j)| \leq |n_1^t(x_j) - n_3^t(x_j)|$$

$$|n_2^t(x_j) - n_3^t(x_j)| \leq |n_1^t(x_j) - n_3^t(x_j)|$$

So $CSM^t(\mathcal{T}_1, \mathcal{T}_3) \leq CSM^t(\mathcal{T}_1, \mathcal{T}_2)$, and $CSM^t(\mathcal{T}_1, \mathcal{T}_3) \leq CSM^t(\mathcal{T}_2, \mathcal{T}_3)$.

8.1.1.2. Definition

Consider two T-SFSs on domain X, $\mathcal{T}_1 = \left\{\left(x_j, m_1(x_j), i_1(x_j), n_1(x_j)\right) \mid x_j \in X\right\}$ and $\mathcal{T}_2 = \left\{\left(x_j, m_2(x_j), i_2(x_j), n_2(x_j)\right) \mid x_j \in X\right\}$ where $(j = 1,2, \dots, k)$ and corresponding WVs to the decision criteria will be $w = (w_1, w_2, \dots, w_k)^T$. Then four cosine SMs based on cosine function can be calculated as

$$WCSM^1(\mathcal{T}_1, \mathcal{T}_2) = \sum_{j=1}^{k} w_j \cos\left[\frac{\pi}{2}\begin{pmatrix} |m_1^t(x_j) - m_2^t(x_j)| \vee |i_1^t(x_j) - i_2^t(x_j)| \\ \vee |n_1^t(x_j) - n_2^t(x_j)| \end{pmatrix}\right]$$

$$WCSM^2(\mathcal{T}_1, \mathcal{T}_2) = \sum_{j=1}^{k} w_j \cos\left[\frac{\pi}{4}\begin{pmatrix} |m_1^t(x_j) - m_2^t(x_j)| + |i_1^t(x_j) - i_2^t(x_j)| \\ + |n_1^t(x_j) - n_2^t(x_j)| \end{pmatrix}\right]$$

$$WCSM^3(\mathcal{T}_1,\mathcal{T}_2) = \sum_{j=1}^{k} w_j \cos\left[\frac{\pi}{2}\begin{pmatrix} |m_1^t(x_j) - m_2^t(x_j)| \vee |i_1^t(x_j) - i_2^t(x_j)| \\ \vee |n_1^t(x_j) - n_2^t(x_j)| \vee |r_1^t(x_j) - r_2^t(x_j)| \end{pmatrix}\right]$$

$$WCSM^4(\mathcal{T}_1,\mathcal{T}_2) = \sum_{j=1}^{k} w_j \cos\left[\frac{\pi}{4}\begin{pmatrix} |m_1^t(x_j) - m_2^t(x_j)| + |i_1^t(x_j) - i_2^t(x_j)| \\ +|n_1^t(x_j) - n_2^t(x_j)| + |r_1^t(x_j) - r_2^t(x_j)| \end{pmatrix}\right]$$

Where the symbol "$\vee$" means the maximum operation.

The weighted SMs defined above will fulfil the following properties:

1. $0 \leq WCSM^t(\mathcal{T}_1,\mathcal{T}_2) \leq 1$
2. $WCSM^t(\mathcal{T}_1,\mathcal{T}_2) = 1$ if and only if $\mathcal{T}_1 = \mathcal{T}_2$
3. $WCSM^t(\mathcal{T}_1,\mathcal{T}_2) = WCSM^t(\mathcal{T}_2,\mathcal{T}_1)$
4. If $\mathcal{T}_3 = \left\{\left(x_j, m_3(x_j), i_3(x_j), n_3(x_j)\right) \mid x_j \in X\right\}$ is a T-SFS on X and $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$ then $WCSM^t(\mathcal{T}_1,\mathcal{T}_3) \leq WCSM^t(\mathcal{T}_1,\mathcal{T}_2)$ and $WCSM^t(\mathcal{T}_1,\mathcal{T}_3) \leq WCSM^t(\mathcal{T}_2,\mathcal{T}_3)$.

8.1.2. Similarity Measures Based on Exponential Function

In this subsection SMs based on exponential function were proposed. Some basic properties of these SMs are also discussed. Further these proposed are extended to weighted SMs.

8.1.2.1. Definition

Consider two T-SFSs on domain X, $\mathcal{T}_1 = \left\{\left(x_j, m_1(x_j), i_1(x_j), n_1(x_j)\right) \mid x_j \in X\right\}$ and $\mathcal{T}_2 = \left\{\left(x_j, m_2(x_j), i_2(x_j), n_2(x_j)\right) \mid x_j \in X\right\}$ where $(j = 1,2,\dots,k)$ then four SMs based on exponential function can be calculated as

$$ESM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-(|m_1^t(x_j)-m_2^t(x_j)| \vee |i_1^t(x_j)-i_2^t(x_j)| \vee |n_1^t(x_j)-n_2^t(x_j)|)} - 1\right]$$

$$ESM^2(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-\begin{pmatrix} |m_1^t(x_j)-m_2^t(x_j)| \vee |i_1^t(x_j)-i_2^t(x_j)| \\ \vee |n_1^t(x_j)-n_2^t(x_j)| \vee |r_1^t(x_j)-r_2^t(x_j)| \end{pmatrix}} - 1\right]$$

$$ESM^3(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-\frac{1}{2}(|m_1^t(x_j)-m_2^t(x_j)| + |i_1^t(x_j)-i_2^t(x_j)| + |n_1^t(x_j)-n_2^t(x_j)|)} - 1\right]$$

$$ESM^4(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-\frac{1}{2}\left(\substack{|m_1^t(x_j)-m_2^t(x_j)|+|i_1^t(x_j)-i_2^t(x_j)| \\ +|n_1^t(x_j)-n_2^t(x_j)|+|r_1^t(x_j)-r_2^t(x_j)|}\right)} - 1\right]$$

Where the symbol "$\vee$" means the maximum operation.

The SMs defined above will fulfil the following properties:

1. $0 \leq ESM^t(\mathcal{T}_1,\mathcal{T}_2) \leq 1$
2. $ESM^t(\mathcal{T}_1,\mathcal{T}_2) = 1$ if and only if $\mathcal{T}_1 = \mathcal{T}_2$
3. $ESM^t(\mathcal{T}_1,\mathcal{T}_2) = ESM^t(\mathcal{T}_2,\mathcal{T}_1)$
4. If $\mathcal{T}_3 = \left\{\left(x_j, m_3(x_j), i_3(x_j), n_3(x_j)\right) \mid x_j \in X\right\}$ is a T-SFS on X and $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$ then $ESM^t(\mathcal{T}_1,\mathcal{T}_3) \leq ESM^t(\mathcal{T}_1,\mathcal{T}_2)$ and $ESM^t(\mathcal{T}_1,\mathcal{T}_3) \leq ESM^t(\mathcal{T}_2,\mathcal{T}_3)$.

where $t = 1,2,3,4$.

Proof. 1. Trivially holds

2. Let us assume $\mathcal{T}_1 = \mathcal{T}_2$, then it means $m_1^t(x_j) = m_2^t(x_j)$, $i_1^t(x_j) = i_2^t(x_j)$, and $n_1^t(x_j) = n_2^t(x_j)$.

$$\Rightarrow |m_1^t(x_j) - m_2^t(x_j)| = |i_1^t(x_j) - i_2^t(x_j)| = |n_1^t(x_j) - n_2^t(x_j)| = 0$$

$$ESM^t(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k} 2^1 - 1$$

$$ESM^t(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k} 1 = \frac{1}{k} \times k = 1$$

Conversely assume $ESM^t(\mathcal{T}_1,\mathcal{T}_2) = 1$, then

$$|m_1^t(x_j) - m_2^t(x_j)| = |i_1^t(x_j) - i_2^t(x_j)| = |n_1^t(x_j) - n_2^t(x_j)| = 0$$

which means $m_1^t(x_j) = m_2^t(x_j)$, $i_1^t(x_j) = i_2^t(x_j)$, and $n_1^t(x_j) = n_2^t(x_j)$. Thus $\mathcal{T}_1 = \mathcal{T}_2$.

3. Consider $t = 1$,

$$ESM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-(|m_1^t(x_j)-m_2^t(x_j)| \vee |i_1^t(x_j)-i_2^t(x_j)| \vee |n_1^t(x_j)-n_2^t(x_j)|)} - 1\right]$$

$$=\frac{1}{k}\sum_{j=1}^{k}\left[2^{1-\left(|m_2^t(x_j)-m_1^t(x_j)|\vee|i_2^t(x_j)-i_1^t(x_j)|\vee|n_2^t(x_j)-n_1^t(x_j)|\right)}-1\right]=ESM^1(\mathcal{T}_2,\mathcal{T}_1)$$

Similarly, this can be proved for $t = 2,3,4$.

4. Consider $t = 1$, If $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$ then $m_1^t(x_j) \le m_2^t(x_j) \le m_3^t(x_j)$, $i_1^t(x_j) \le i_2^t(x_j) \le i_3^t(x_j)$, and $n_1^t(x_j) \ge n_2^t(x_j) \ge n_3^t(x_j)$. Thus

$$|m_1^t(x_j)-m_2^t(x_j)| \le |m_1^t(x_j)-m_3^t(x_j)|$$

$$|m_2^t(x_j)-m_3^t(x_j)| \le |m_1^t(x_j)-m_3^t(x_j)|$$

$$|i_1^t(x_j)-i_2^t(x_j)| \le |i_1^t(x_j)-i_3^t(x_j)|$$

$$|i_2^t(x_j)-i_3^t(x_j)| \le |i_1^t(x_j)-i_3^t(x_j)|$$

$$|n_1^t(x_j)-n_2^t(x_j)| \le |n_1^t(x_j)-n_3^t(x_j)|$$

$$|n_2^t(x_j)-n_3^t(x_j)| \le |n_1^t(x_j)-n_3^t(x_j)|$$

So $ESM^t(\mathcal{T}_1,\mathcal{T}_3) \le ESM^t(\mathcal{T}_1,\mathcal{T}_2)$, and $ESM^t(\mathcal{T}_1,\mathcal{T}_3) \le ESM^t(\mathcal{T}_2,\mathcal{T}_3)$.

8.1.2.2. Definition

Consider two T-SFSs on domain X, $\mathcal{T}_1 = \left\{\left(x_j, m_1(x_j), i_1(x_j), n_1(x_j)\right) \mid x_j \in X\right\}$ and $\mathcal{T}_2 = \left\{\left(x_j, m_2(x_j), i_2(x_j), n_2(x_j)\right) \mid x_j \in X\right\}$ where $(j = 1,2,\dots,k)$ and corresponding WVs to the decision criteria will be $w = (w_1, w_2, \dots, w_k)^T$. Then four SMs based on exponential function can be calculated as

$$WESM^1(\mathcal{T}_1,\mathcal{T}_2)=\sum_{j=1}^{k} w_j\left[2^{1-\left(|m_1^t(x_j)-m_2^t(x_j)|\vee|i_1^t(x_j)-i_2^t(x_j)|\vee|n_1^t(x_j)-n_2^t(x_j)|\right)}-1\right]$$

$$WESM^2(\mathcal{T}_1,\mathcal{T}_2)=\sum_{j=1}^{k} w_j\left[2^{1-\left(\begin{array}{c}|m_1^t(x_j)-m_2^t(x_j)|\vee|i_1^t(x_j)-i_2^t(x_j)|\\ \vee|n_1^t(x_j)-n_2^t(x_j)|\vee|r_1^t(x_j)-r_2^t(x_j)|\end{array}\right)}-1\right]$$

$$WESM^3(\mathcal{T}_1,\mathcal{T}_2)=\sum_{j=1}^{k} w_j\left[2^{1-\frac{1}{2}\left(|m_1^t(x_j)-m_2^t(x_j)|+|i_1^t(x_j)-i_2^t(x_j)|+|n_1^t(x_j)-n_2^t(x_j)|\right)}-1\right]$$

$$WESM^4(\mathcal{T}_1,\mathcal{T}_2) = \sum_{j=1}^{k} w_j \left[2^{1-\frac{1}{2}\left(\substack{|m_1^t(x_j)-m_2^t(x_j)|+|i_1^t(x_j)-i_2^t(x_j)| \\ +|n_1^t(x_j)-n_2^t(x_j)|+|r_1^t(x_j)-r_2^t(x_j)|} \right)} - 1 \right]$$

Where the symbol "$\vee$" means the maximum operation.

The weighted SMs defined above will fulfil the following properties:

1. $0 \leq WESM^t(\mathcal{T}_1,\mathcal{T}_2) \leq 1$
2. $WESM^t(\mathcal{T}_1,\mathcal{T}_2) = 1$ if and only if $\mathcal{T}_1 = \mathcal{T}_2$
3. $WESM^t(\mathcal{T}_1,\mathcal{T}_2) = WESM^t(\mathcal{T}_2,\mathcal{T}_1)$
4. If $\mathcal{T}_3 = \left\{ \left(x_j, m_3(x_j), i_3(x_j), n_3(x_j) \right) \mid x_j \in X \right\}$ is a T-SFS on X and $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3$ then $WESM^t(\mathcal{T}_1,\mathcal{T}_3) \leq WESM^t(\mathcal{T}_1,\mathcal{T}_2)$ and $WESM^t(\mathcal{T}_1,\mathcal{T}_3) \leq WESM^t(\mathcal{T}_2,\mathcal{T}_3)$.

where $t = 1,2,3,4$.

8.2. Application for Pattern Recognition and MADM problems

In this section, the reliability of proposed SMs is checked by developing an application for pattern recognition and MADM problems.

8.2.1. Numerical Example for Pattern Recognition

Let us consider three known patterns $X = \{x_1, x_2, x_3\}$ which are characterized by the T-SFSs as:

$$\mathcal{T}_1 = \{(0.81,0.37,0.63),(0.71,0.08,0.57),(0.87,0.24,0.56),(0.39,0.13,0.74)\}$$

$$\mathcal{T}_2 = \{(0.77,0.56,0.19),(0.91,0.25,0.39),(0.72,0.49,0.62),(0.56,0.12,0.47)\}$$

$$\mathcal{T}_3 = \{(0.64,0.58,0.47),(0.65,0.14,0.55),(0.63,0.07,0.57),(0.78,0.34,0.51)\}$$

And one unknown Pattern

$$\mathcal{T}_4 = \{(0.79,0.48,0.51),(0.84,0.11,0.5),(0.69,0.22,0.55),(0.47,0.1,0.74)\}$$

Here all values lie in T-spherical fuzzy environment for $t = 3$. In order to find the pattern of $\mathcal{T}_4$, the proposed SMs are calculated as listed in Table 75. Where $l = 1,2,3$

Table 75 Similarity Measures

	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$

$CSM^1(\mathcal{T}_l, \mathcal{T}_4)$	0.9327	**0.9811**	0.9102
$CSM^2(\mathcal{T}_l, \mathcal{T}_4)$	0.9757	**0.9857**	0.9631
$CSM^3(\mathcal{T}_l, \mathcal{T}_4)$	0.9302	**0.9694**	0.9102
$CSM^4(\mathcal{T}_l, \mathcal{T}_4)$	0.9268	**0.9647**	0.8967
$ESM^1(\mathcal{T}_l, \mathcal{T}_4)$	0.7158	**0.8391**	0.6874
$ESM^2(\mathcal{T}_l, \mathcal{T}_4)$	0.7124	**0.8030**	0. 6874
$ESM^3(\mathcal{T}_l, \mathcal{T}_4)$	0.8169	**0.8572**	0.7913
$ESM^4(\mathcal{T}_l, \mathcal{T}_4)$	0.6997	**0.7872**	0.6654

Their rankings are listed in Table 76.

Table 76 Rankings

	Rankings
$CSM^1(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$
$CSM^2(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$
$CSM^3(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$
$CSM^4(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$
$ESM^1(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$
$ESM^2(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$
$ESM^3(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$
$ESM^4(\mathcal{T}_l, \mathcal{T}_4)$	$\mathcal{T}_2 \geq \mathcal{T}_1 \geq \mathcal{T}_3$

As SM between $\mathcal{T}_2$ and $\mathcal{T}_4$ is greater so they belong to same pattern.

8.2.2. Numerical Example Strategy Decision Making Problem

A company wants to launch a new product and owner of a company have to choose one strategy from the following three strategies:

$\mathcal{s}_1$: Make product for poor persons

s_2: Make product for rich persons

s_3: Make product for both poor and rich person

The decision maker have to evaluate these strategies under the consideration of following attributes:

q_1: Barriers in the development

q_2: Brisk of loss

q_3: Growth analysis

q_4: Impact on environment

having a WV $w = (0.2,0.25,0.35,0.2)^T$. Decision maker evaluate these alternatives with respect to given attributes and provide the data in T-spherical fuzzy environment as listed in Table 77:

Table 77 Decision Matrix

	q_1	q_2	q_3	q_4
s_1	$(0.75,0.14,0.33)$	$(0.63,0.08,0.29)$	$(0.47,0.34,0.64)$	$(0.57,0.11,0.36)$
s_2	$(0.81,0.25,0.41)$	$(0.59,0.12,0.35)$	$(0.59,0.42,0.19)$	$(0.66,0.26,0.44)$
s_3	$(0.77,0.29,0.22)$	$(0.71,0.18,0.36)$	$(0.61,0.1,0.49)$	$(0.50,0.37,0.40)$
s	$(0.66,0.23,0.31)$	$(0.91,0.07,0.39)$	$(0.57,0.30,0.30)$	$(0.59,0.17,0.41)$

The SMs of three alternatives $\{s_1, s_2, s_3\}$ with respect to s will be as listed in Table 78

Table 78 Weighted Similarity Measures

	s_1	s_2	s_3
$WCSM^1(s_l, s_4)$	0.8974	0.9153	**0.9497**
$WCSM^2(s_l, s_4)$	0.9572	0.9696	**0.9748**
$WCSM^3(s_l, s_4)$	0.8789	0.8959	**0.9497**
$WCSM^4(s_l, s_4)$	0.8706	0.8913	**0.9357**
$WESM^1(s_l, s_4)$	0.6875	0.7376	**0.7601**

$WESM^2(\mathcal{s}_l, \mathcal{s}_4)$	0.6602	0.6973	**0.7601**
$WESM^3(\mathcal{s}_l, \mathcal{s}_4)$	0.7854	0.8150	**0.8189**
$WESM^4(\mathcal{s}_l, \mathcal{s}_4)$	0.6467	0.6845	**0.7259**

The rankings of alternatives are listed in Table 79

Table 79 Rankings

	Rankings
$WCSM^1(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$
$WCSM^2(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$
$WCSM^3(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$
$WCSM^4(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$
$WESM^1(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$
$WESM^2(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$
$WESM^3(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$
$WESM^4(\mathcal{s}_l, \mathcal{s}_4)$	$\mathcal{s}_3 \geq \mathcal{s}_2 \geq \mathcal{s}_1$

Since SM between $\mathcal{s}_3$ and $\mathcal{s}$ is greater so $\mathcal{s}_3$ is better strategy for a company to adopt.

8.3. Comparative Analysis

In this section some conditions are discussed under which the proposed SMs can reduced to other tools of uncertainty like SFS, PFS, q-ROPFS, PyFS and IFSs. Superiority of proposed SMs is proved with the help of an example.

For any two T-SFSs on domain X, $\mathcal{T}_1 = \left\{\left(x_j, m_1(x_j), i_1(x_j), n_1(x_j)\right) \mid x_j \in X\right\}$ and $\mathcal{T}_2 = \left\{\left(x_j, m_2(x_j), i_2(x_j), n_2(x_j)\right) \mid x_j \in X\right\}$ where $(j = 1,2, \dots, k)$ then consider SMs based on exponential as

$$ESM^1(\mathcal{T}_1, \mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-\left(\begin{array}{c}|m_1^t(x_j)-m_2^t(x_j)| \vee |i_1^t(x_j)-i_2^t(x_j)| \\ \vee |n_1^t(x_j)-n_2^t(x_j)|\end{array}\right)} - 1\right] \quad (8.3.1)$$

1. For $t = 2$, the equation (8.3.1) becomes valid for SFSs

$$ESM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-(|m_1^2(x_j)-m_2^2(x_j)|\vee|i_1^2(x_j)-i_2^2(x_j)|\vee|n_1^2(x_j)-n_2^2(x_j)|)} - 1\right]$$

2. For $t = 1$, the equation (8.3.1) becomes valid for PFSs

$$ESM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-(|m_1(x_j)-m_2(x_j)|\vee|i_1(x_j)-i_2(x_j)|\vee|n_1(x_j)-n_2(x_j)|)} - 1\right]$$

3. For $i = 0$, the equation (8.3.1) become valid for q-ROPFSs

$$ESM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-(|m_1^t(x_j)-m_2^t(x_j)|\vee|n_1^t(x_j)-n_2^t(x_j)|)} - 1\right]$$

4. For $i = 0, t = 2$ the equation (8.3.1) become valid for PyFSs

$$ESM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-(|m_1^2(x_j)-m_2^2(x_j)|\vee|n_1^2(x_j)-n_2^2(x_j)|)} - 1\right]$$

5. For $i = 0, t = 1$ the equation (8.3.1) become valid for IFSs

$$ESM^1(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{k}\sum_{j=1}^{k}\left[2^{1-(|m_1(x_j)-m_2(x_j)|\vee|n_1(x_j)-n_2(x_j)|)} - 1\right]$$

Similarly all the proposed SMs can be reduced for other fuzzy structures like SFSs, PFSs, q-ROPFSs, PyFSs and IFSs by follow the conditions defined above.

8.3.1. Example

Consider two TSFNs $\mathcal{T}_1 = (0.801,0.401,0.701), \mathcal{T}_2 = (0.8,0.4,0.7)$

The different SMs calculate their similarity as listed in Table 80:

Table 80 Different Similarity Measures

	Similarity Measures
$SM^1(\mathcal{T}_1,\mathcal{T}_2)$ [51]	1
$SM^2(\mathcal{T}_1,\mathcal{T}_2)$ [51]	0.9985
$SM^3(\mathcal{T}_1,\mathcal{T}_2)$ [51]	1
$CSM^1(\mathcal{T}_1,\mathcal{T}_2)$	1
$CSM^2(\mathcal{T}_1,\mathcal{T}_2)$	1

$CSM^3(\mathcal{T}_1,\mathcal{T}_2)$	1
$CSM^4(\mathcal{T}_1,\mathcal{T}_2)$	1
$ESM^1(\mathcal{T}_1,\mathcal{T}_2)$	0.9973
$\mathrm{ESM}^2(\mathcal{T}_1,\mathcal{T}_2)$	0.9973
$\mathrm{ESM}^3(\mathcal{T}_1,\mathcal{T}_2)$	0.9946
$\mathrm{ESM}^4(\mathcal{T}_1,\mathcal{T}_2)$	0.9946

In Table 6, different SMs were calculated of given data. It is clear that the existing SMs proposed in [51] and cosine SMs based on cosine function does not differentiate between $\mathcal{T}_1$ and $\mathcal{T}_2$ but the proposed SMs based on exponential function differentiate between $\mathcal{T}_1$ and $\mathcal{T}_2$. So this proves the accuracy of proposed ESMs.

Conclusion

The whole thesis can be concluded as:

In chapter 1, the basic definitions of different fuzzy structures are defined and some basic operations on these fuzzy frameworks are also discussed. In chapter 2, some SMs for interval-valued picture fuzzy information are proposed, include cosine SMs, SMs using cosine function, SMs using cotangent function, Set-theoretic SM, Grey SM, dice and generalized dice SMs for IvPFSs, and some basic properties of all these SMs are also discussed, then the proposed SMs are applied to decision making problems with the help of numerical examples. In addition, advantages of proposed works are also discussed.

In chapter 3, some new product and power operations for T-SFS are introduced and based on new operations, some new geometric aggregation operations are defined. The generalization of new work is proved by using examples and remarks. Some properties of proposed operators are investigated and supported with examples. The new operators are applied in MADM process and results are studied. A comparison of new work is established with existing literature and its advantages over the existing work are discussed.

In chapter 4, an extension of existing immediate probability, Choquet averaging and associated immediate probability averaging operators are developed by utilizing the concept of T-SFSs. In it, it is pointed out that the existing operators have some limitations and decision makers are not free to make a decision freely, and they fail to work when the information is given in PyFSs, PFSs, SFSs and T-SFSs. To overcome this shortcoming, some averaging aggregation operators are defined in most generalized tool of uncertainty called T-SFSs but they also fail under some conditions. To overcome this defect, some interactive averaging operators are defined and a comparison between these proposed operators is developed with the help of an example. The existing score values have shortcoming that they do not involve abstinence so new score function is proposed in which all degrees are involved and with the help of this new score function the different aggregated values are compared. To check the reliability an application of MADM problem is developed. The advantages of proposed work are also discussed. The comparative study of existing and proposed operators is also developed with the help of an example.

In chapter 5, some geometric and interactive geometric operators are developed by utilizing the concept of T-SFSs. In it, the main focus is on Ass.IP-T-SFOWG and Ass.IP-T-SFOWIG operators because associated immediate probability geometric aggregation operators reflect the interaction among all subsets of states of nature. As well as the proposed work has another advantage that these operators are proposed by utilizing the concept of T-SFSs so we can reduce the proposed operators under some conditions to SFSs, PFSs, PyFSs, IFSs. A comparison between geometric and interactive geometric operators is also developed. The superiority of any two TSFNs is checked using newly developed score function because the existing score function does not involve abstinence. An algorithm is established for MADM problem and a numerical example is also solved using that algorithm. Some conditions that reduce proposed operators to other fuzzy structures are discussed in the advantages section and a comparative study of proposed and existing work is also established.

In chapter 6, some Einstein operations are defined for T-SFSs and based on these operations some improved Einstein averaging aggregation operators and Einstein geometric aggregation operators are defined. Some properties of these aggregation operators are also discussed. The validity of proposed operators is checked with the help of the MADM problem. The comparative analysis between existing and proposed work is also discussed in which some conditions are studied under which the proposed operators can be reduced to other tools of uncertainty like IFSs, PyFSs, q-ROPFSs, PFSs, SFSs. Some examples are also discussed in which the superiority of proposed operators is proved. The advantages of proposed operators are also discussed.

In chapter 7, some new Einstein interactive operational laws are proposed. On the basis of these operational laws T-spherical Einstein interactive geometric and T-spherical Einstein interactive averaging operators are proposed. We validate these operators with the help of an application in MADM. After that, some conditions are discussed on which the proposed operators can reduce to other fuzzy frameworks. A comparison of proposed and existing work is also established and explained using an example.

In chapter 8, some cosine and weighted cosine SMs based on cosine function are proposed for most generalized fuzzy structure called T-SFS. Along with this some SMs based on the exponential function are also proposed for T-SFSs. Pattern recognition problems and strategic decision making problems are investigated using proposed SMs. Then a comparative analysis is developed in which some conditions are discussed through which the proposed SMs can be reduced to other fuzzy structures. An example is also discussed in which it is proved that the proposed SMs based on exponential function has much better distinguishability than existing SMs.

In future, there is scope to extend the proposed work to different frameworks and applied these aggregation operators and SMs to different fields. We also aim to generalize these operators and SMs in the field of T-spherical fuzzy soft sets.

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