Analytical Study of Plasma Equilibrium in Tokamak

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It is certified that the work presented in this thesis entitled "**Analytical Study of Plasma Equilibrium in Tokamak**" by **Syed Sheraz Ahmad**, **Registration No. 17-FBAS/MSPHY/F10** is of sufficient standard in scope and quality for the award of degree of MS Physics from International Islamic University, Islamabad.

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Analytical Study of Plasma Equilibrium in Tokamak

A thesis submitted to

Department of Physics

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as a partial fulfillment for the award of the degree of

MS Physics

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Dedicated to;

My Father and Late Mother

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ABSTRACT

The problem of plasma equilibrium in tokamak is a free boundary problem described by the Grad-Shafranov equation in axisymmetric configurations. We derived the Grad-Shafranov equation and present an analytical solution to the Grad-Shafranov equation by using Solov'ev profile. This solution has a number of degree of freedom in the form of free constants. These constants are to be determined by using the boundary constraints on the plasma surface. The solution can be used to calculate equilibrium in standard tokamak and as well as in spherical tokamak, depending upon the boundary constraint. Graphical comparison of conventional and spherical tokamak has been carried out with the result showing the supremacy of the spherical tokamak. At this point analysis is analytically complete and is ready to be implemented numerically to any toroidal configuration.

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CHAPTER: 1

Introduction

1.1 A Brief History of Plasma Physics:

In the mid of 19th century the great Czech medical scientist, Johannes Purkinje first time used the Greek word plasma (means something "molded") to denote the clear fluid which remain after the removal of all corpuscular material in the blood.

In 1927 an American chemist Irving Langmuir was the first who used the word "Plasma" to describe an ionized gas. But this is unlike the blood where there is really a fluid medium carrying the corpuscular material, there is actually no fluid medium entraining the electrons, ions and neutrals in an ionized gas.[1-2].

1.2 What is Plasma?

In simple words we can say that plasma is an ionized gas. When we heat up a solid the thermal motion of the atoms break the crystal lattice structure and convert into liquid. When a liquid is heat up the atoms evaporated from the surface and convert into a gas. When a gas is heat up due to the collision of the atoms with each other the electron is knocked out and forms plasma which is the fourth state of matter. But in the third case the phase transition does not occur therefore some of the scientist are disagree with statement that plasma is fourth state of matter. The charge separation between ions and electron give rise to the electric fields and the flow of charge particle give rise to current and magnetic field [3]. But it is to be noted that every ionized gas cannot be called plasma. There are some conditions for an ionized gas to be plasma.

Plasma is electrically neutral medium of positive and negative particles but the overall charge of plasma is roughly zero. The temperature of the plasma is very high. At low temperature the ions and the electron will recombine and the plasma will convert into ordinary gas. Examples of Plasma are:

1) Welding Arcs
2) Stars and Neon signs

3) Flash of lighting bulb
4) Slight amount of ionization in a rocket
5) Flame of candle

1.3 Definition of Plasma:

Plasma comes from Greek which means something molded. A useful definition of Plasma is as follows:

"Plasma is a quasineutral gas of charged and neutral particles which exhibits collective behavior".

Now we have to define "quasineutrality" and "collective behavior".

1.3.1 Quasineutrality:

The plasma is quasineutral, which means neutral enough so that:

$$n_i = n_e = n.$$

Where n is common density called plasma density.

But not so neutral that all the electromagnetic forces vanish. But outside the plasma it seems to be neutral, because of equal number of positively charged ions and negatively charged electrons. This phenomenon is known as quasineutrality of plasma. [4]

1.3.2 Collective Behavior:

In order to understand the Collective Behavior let us consider the force acting on molecule of ordinary air. Since the molecules are neutral therefore there is no net force on them. And the gravitational force is negligible. Thus the molecule move without any disturbance until they make collision with each other and due these collisions the motion of particle is controlled.

But in case of plasma the situation is totally different because of the existence of charge particles. These charge particles exert long range Coulomb force on each other due to this long range force plasma has large storage of information of possible motions. In plasma the long range coulomb forces are so much larger than the force due to ordinary collision. Therefore we neglect the forces due to ordinary collisions.

When a single particle is disturbed the whole plasma will be disturbed. And this is called Collective behavior of plasma [4].

1.4 Debye Shielding:

Let we put an electric field in plasma by introducing two charge balls connected to a battery. The balls will interact with the particles of opposite charges and cloud of ions will surround around the negative ball and electrons will surround around the positive ball. If the temperature is finite the particle at outer boundary of cloud where the electric is weak, have thermal energy enough to escape from the cloud. The boundary of the cloud then reaches at the radius where the potential is roughly equal to the thermal energy KT of the particles and the shielding is not completed. This process is called Debye shielding. And the distance over which a significant charge separation occurs is called Debye length. [4]

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{n_e q_e^2}} \qquad (1.1)$$

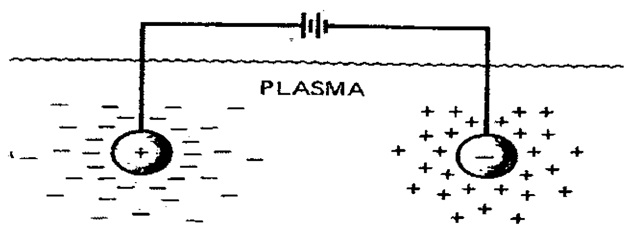


Figure: 1.1 Debye shielding. [4]

1.5 Plasma Parameter:

The Debye shielding is valid only when there is enough particles in the charge cloud. If there is one or two charge particles in the cloud region then it is clear that the shielding is impossible. The Debye shielding is valid only when the number of particles N_D in the Debye sphere is large enough. This means $N_D >>> 1$.[4]

$$N_D = n\frac{3}{4}\pi\lambda_D^3 = 1.38 \times 10^6 \frac{T^{3/2}}{n^{1/2}} \quad \text{(T in kelvin)} \qquad (1.2)$$

In addition the collective behavior $\lambda_D <<< L$

1.6 Conditions of Plasma:

The conditions for an ionized gas to be plasma are:

1: It should be dense enough that λ_D is much smaller than L the dimension of the system.ie. $\lambda_D \ll L$.

Where λ_D Debye is length and L is the dimension of the system

2: In addition to $\lambda_D \ll L$, for an ionized gas to be Plasma requires $N_D \gg 1$. Means the number should be greater than 1.

Where N_D is the number of particles in the Debye sphere.

1: The third condition for plasma is $\omega\tau > 1$.

Where ω is the plasma frequency and τ is the mean time between collisions.

1.7 Application of Plasma in Thermonuclear Fusion Reaction:

The main application is the control thermonuclear fusion. In 1952 it was proposed that hydrogen bomb fusion reaction can be control to make reactor but nuclear fusion is yet to be controlled. As we know that extremely high temperature in the order of $10^7 K^o$ to $10^9 K^o$ is required for initial excitation of nuclear fusion reaction. In practice we faced real problems to produce such an extreme temperature. The various serious problems that is faced by the scientist and engineers to design a container in which very hot plasma can be confined under high pressure to initiate nuclear fusion.

Nowadays various countries are working on "on magnetic bottle" and we hope that they will succeed to confine plasma in the magnetic bottle up to the required of temperature and pressure for the fusion reaction. A nuclear fusion reactor, if made, will be a great blessing to the humanity. Because nuclear fusion reactors the only source that can solve the energy crisis faced by the global community in the future [4].

1.8 Energy Crises:

Worldwide increasing demand for the energy is a very serious problem for the mankind. Saving energy and the use of renewable energy sources will not be sufficient. On the other hand the reservoirs of the fossil fuels are also limited. The use of these fossil fuels causes the emission of carbon dioxide and other chemicals which are harmful for the mankind. The figure1.1 shows the gap between the demand and delivery of crude oil is rapidly widening [5].

Simply we can say that there are so many energy problems facing by this developed world and these problems will become worse in the future. Each of the existing energy source facing a couple of difficulties like, limited reservoirs, Co_2 production, emission of toxic material, waste disposed and high cost. The only possible solution for this generous problem of energy is the nuclear fusion.[6]

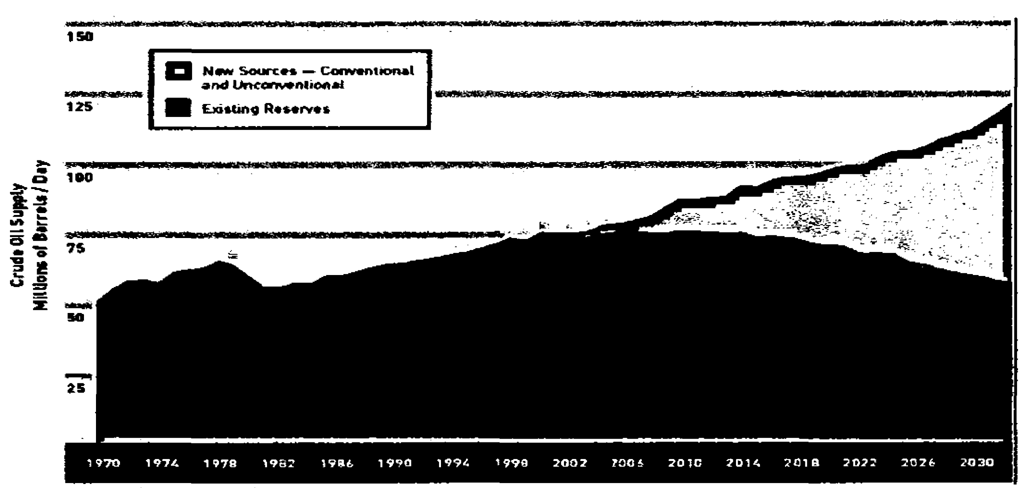


Figure: 1.2 The widening gap between oil delivery and demand.

1.9 The Role of the Fusion Energy:

The main aim of nuclear fusion reaction is the production of electricity. The fusion reaction involves the merging of two light elements mainly (H) and its isotopes tritium deuterium. In the Sun the main reaction is the nuclear fusion reaction of hydrogen which powers the Sun. There is about 1 atom of deuterium for every 6700 atoms of hydrogen in the naturally occurring sea water. If we use the naturally occurring deuterium to power the fusion reactors it can produces enough energy to full fill the need of the whole world for about 2 billion years at the present rate of energy consumption. [6]

The easiest fusion reaction is D-T fusion reaction because it requires the lowest energies and the isotopes can be easily extracted from the sea water. A significant amount of deuterium can be easily obtained from the sea water. But the tritium is the unstable and radioactive isotope of hydrogen having the lifetime of about 12 years; therefore it does not occur naturally. During the D-T reaction a large number of neutrons produce, these neutrons are then used to obtained tritium from lithium. A significant amount of energy is also released during the D-T reaction.

It can be written as:

$$D + T \rightarrow \alpha + n + 17.6Mev$$

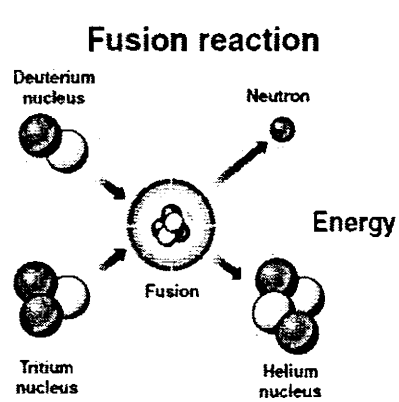


Figure: 1.3. Nuclear fusion

As the deuterium can be obtained from the naturally occurring sea water but the tritium does not therefore we produce the tritium artificially by neutron capture in tritium. The tritium production reactions are

$$n + {}^{6}Li \rightarrow T + {}^{4}He$$

$$n + {}^{7}Li \rightarrow T + {}^{4}He + n$$

The first reaction is suitable for slow neutrons while the second reaction is more feasible for fast neutrons. When we place lithium around the fusion chamber the neutron will release during the D-I fusion reaction are than can be used to produce tritium introducing the possibility of a fusion reactor breading its own fuel.

For ignition of a fusion reaction the energy of the two nuclei should be enough high to overcome the Coulomb repulsive force between the nuclei and must be enough close that the nuclear attractive force became dominant. Therefore the fuel of the fusion reaction must be heated to high temperature. The desired temperature for the D-I fusion reaction is a least 5×10^{7}kat at such a high temperature the gas occurs as a macroscopically neutral assembly of electron and ions called plasma.

Heating of plasma to the desired thermonuclear temperature and confining it adequately such that the net positive energy balance could be attained are the true leading disputes that decides the logical viability of the nuclear fusion. Because of magnetic confinement of plasma there occurs an impressive experimental progress in the current years. [7]

1.10 Magnetic Confinement of Plasma:

In order to get energy from the plasma we have to confine the plasma at extremely high temperature probably equal to the temperature at the core of the sun. But how it is possible? There is no material that can sustain itself in contact with plasma at such high temperature. Fortunately unlike other ordinary gases plasma is a good conductor of electricity and the motion of the plasma particle can be controlled by magnetic field. As the plasma consists of charge particle that can experience the magnetic force, therefore we can easily confine the plasma by using the magnetic field. In the absence of the magnetic field the plasma particle will move randomly in different directions striking the walls of the vessels but when we apply a uniform magnetic field the charge particle will gyrate around the magnetic field lines in spiral paths. the negative charged electrons will gyrates in one direction while the ions will gyrate in opposite direction in this way the motion of the plasma particle in the presence of the magnetic field is restricted and do not hit the walls of the vessel.[8-9]

There are two types of magnetic confinement system.

1. The Magnetic Mirror (open system)
2. The Toroidal system (closed system).

1.11 The Magnetic Mirror:

The magnetic mirror is an old machine designed for the plasma confinement. The idea of the magnetic mirror was based on the fact that the charge particles gyrates around the magnetic field lines and try to repel when enter a region of high magnetic field. The magnetic mirror configuration can be produce by a number of field coils wounded around straight open cylindrical shaped tube. To provide a strong magnetic field at the ends the coils are wounded closer at the ends than in the middle. Thus the ends of the cylindrical tube where the magnetic field is stronger constitute a magnetic mirror.

The magnetic mirror confines the charge particles of plasma with large velocity components in the direction perpendicular to axial field lines. But due to the collision the charge particles move from one magnetic line of force to another. In this way the charge particles move across the magnetic field lines of forces and eventually trap and hit the walls of the vessel. This motion of the charge particles across the magnetic field is called plasma diffusion. [10]

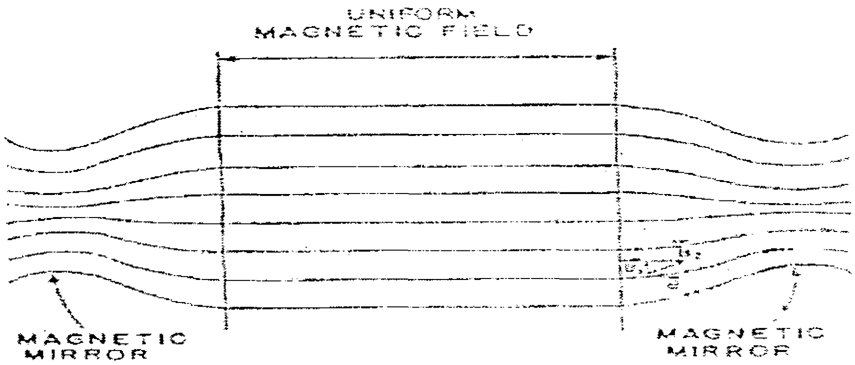


Figure: 1.4. Magnetic Mirror Configuration

1.12 Closed Confinement System Toroidal:

In closed confinement system the magnetic field lines are arranged in such a way that the particles completely endure inside the confinement section. The torus is the simplest arrangement as shown in the figure. A set of coils is positioned in order to create a toroidal field. The particle that flows beside the lines of the closed toroidal field will persist inside the toroidal confinement section. The nonuniformity and curve nature of the toroidal field give rise to such forces that act upon the charge particles to yields drift motions that are directed outward and it would compel the particles to touch the walls of the container, if not compensated. To compensate the drifts of the particles, caused due to the toroidal magnetic field, a poloidal magnetic field is applied upon the toroidal magnetic field. This idea, of the toroidal confinement system, is used in Tokomak configuration. [7]

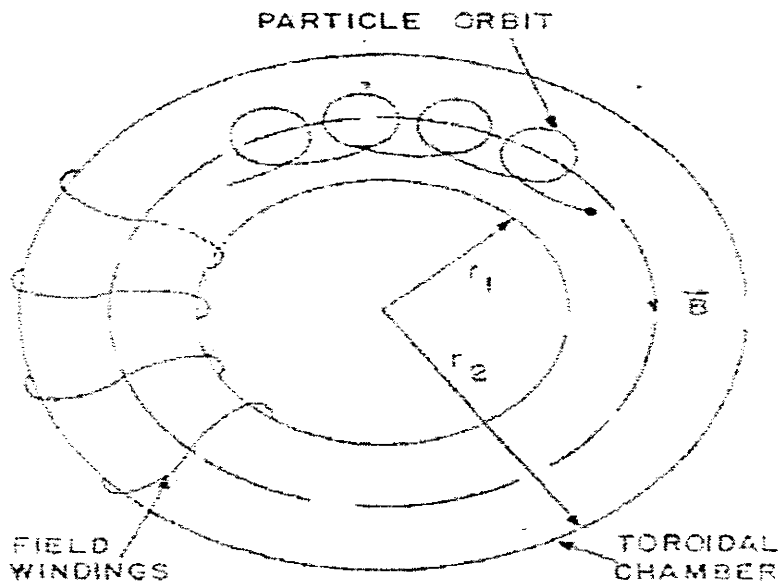


Figure: 1.5 Simple Toroidal Magnetic Field Configuration

1.13 Tokamak:

The word Tokamak is an abbreviation for the Russian word "Torodal naya Kamera Magnitnoi Katushki" which means Toroidal Chamber and Magnetic coil.

The Tokamak is the most effective machine developed to attain the safe condition for the controlled thermonuclear fusion reaction. It is a toroidal shaped device (like a care tire) in which plasma is, contained in a vacuum vessel, confined by winding magnetic fields. In Tokamak the main magnetic field is the toroidal magnetic field which is created by a series of coils evenly spaced around the torus. The toroidal field is not sufficient to confine the plasma. The toroidal field is stronger at the center which causes the plasma particles to drift away towards container walls. In order to balance the plasma pressure by the magnetic forces it necessary to introduces the poloidal magnetic field. The poloidal field is produced by the plasma current flowing in the toroidal direction. The poloidal magnetic field combines with the toroidal magnetic field to create the magnetic fields lines that spiral around the torus and counteracts the drifting effect of on the plasma. A schematic configuration of Tokamak is shown in the figure.

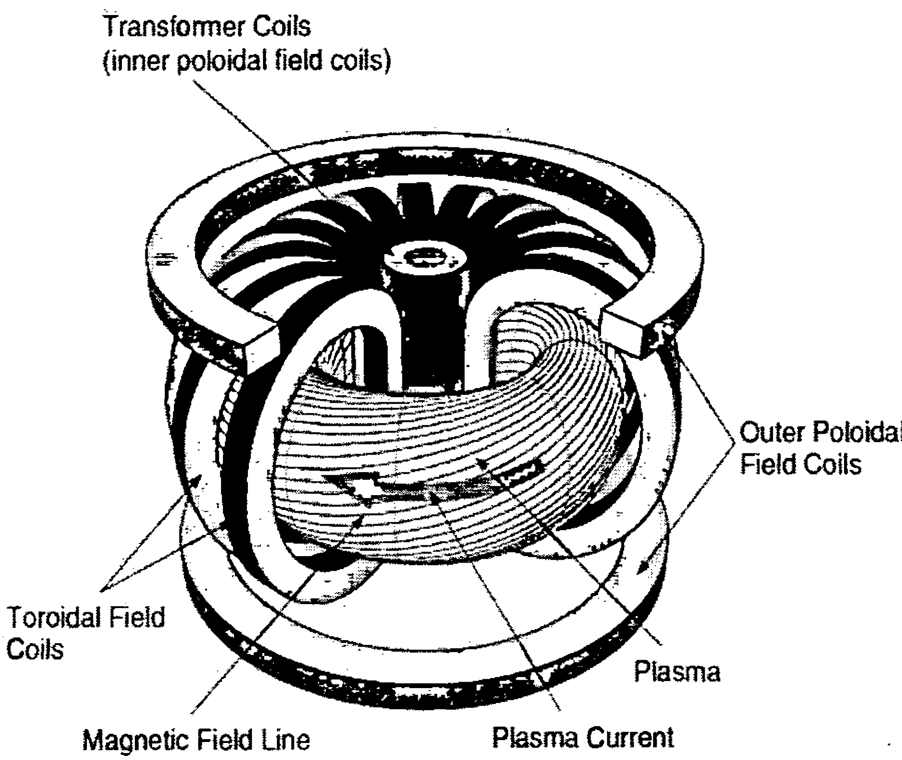


Figure: 1.6. Tokamak Configuration.

1.14 Methods of Plasma Heating

In Tokamak the plasma is heated by the plasma current induced by the primary coil. This type of heating is known as ohmic heating. The amount of heat produced by ohmic process depends on the current and the resistance of the plasma. But unfortunately is the temperature raises the resistance drops and makes this type of heating less effective. the maximum temperature that can be achieved by ohmic heating is up to 20-30 million K, which is twice the temperature in the core of the sun but not sufficient to start up the reactor .in Tokamak the second method used for the plasma heating is the injection of high energy neutral atoms into the plasma which are immediately ionized .these ions are then trapped by the magnetic field and gave some energy to the plasma particle by making collision with them and thus the overall temperature rises. The third method for the plasma heating is the magnetic compression method. In this method the plasma is compressed by increasing magnetic field. In Tokamak the compression of plasma can be achieved by moving the plasma to the region where the magnetic field is higher and in this way the plasma is heated up. In Tokamak the plasma can also be heated by radio frequency heating method. In this method high frequency waves are injected into the plasma by means of oscillators these waves transfer their energy to certain particles which then transfer the energy to the other particles of the plasma by making collision. And thus the overall plasma gets hot. [11]

Layout of the Thesis

The work done in this thesis is organized as follows. In first chapter of this thesis, Brief History of Plasma, Definition of Plasma, Debye Shielding, Plasma Parameter, Conditions of Plasma, Application of Plasma in Thermonuclear Reaction, Energy Crises, The Role of the Fusion Energy, Magnetic Confinement of Plasma, Magnetic Mirror, Closed Confinement system, Tokamak and Methods of Heating Plasma inside the Tokamak are briefly overviewed.

In the second chapter Tokamak Equilibrium and different type of parameters have been discussed. These parameters play very important role in understanding of tokamak equilibrium. The third chapter included a short term discussion of Spherical Tokamak and is tried to compare it with conventional tokamak. In the forth chapter we have derived the Grad Shafranov equation and presented an analytical solution of the Grad Shafranov equation by using Solov'ev profile.

CAHPTER: 2

Tokamak Equilibrium

2.1 Tokamak Equilibrium:

There are two aspects of Tokamak equilibrium the first one is the internally balance between the plasma pressure and the magnetic field forces the second one is the shape and position of plasma which is determined and controlled by currents flowing in the external coils as we have described that the main component is the toroidal magnetic field which is generated by the toroidal current flowing in external coils and the toroidal magnetic field which is smaller than the toroidal magnetic field which is produced by the plasma current. The total toroidal field consist of this internally created field combines with the field due to toroidal current in the primary winding and other coils used to shape and control the plasma and finally the toroidal current in plasma is used to modify toroidal magnetic field. Ampere's law is used to obtain the basic shape of toroidal field B_ϕ. Now by taking the line integral over a closed circuit inside the toroidal field coils and ignoring the small toroidal current we get

$$2\pi R B_\phi = \mu_o I_T \qquad (2.1.1)$$

Where R is the major radius and I_T is the poloidal current in the coils

$$B_\phi = \frac{1}{R}$$

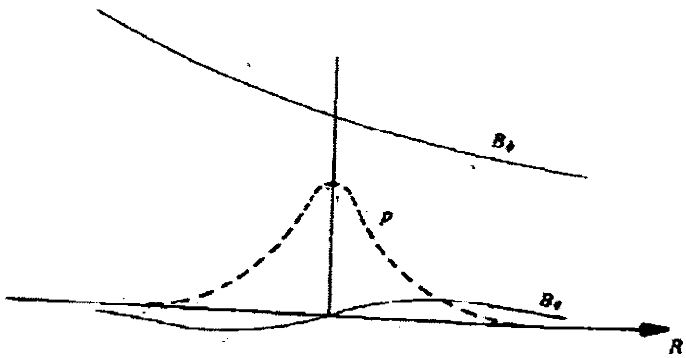


Figure: 2.1. Basic radial variation of poloidal magnetic field.

This redial fall is a great significance in the present Tokamak.

Now the change in the poloidal magnetic field across the plasma is (Taking a as minor radius)

$$\Delta B_\phi = B_{\phi o}\left(\frac{R_o}{R_o - a} - \frac{R_o}{R_o + a}\right)$$

$$\approx B_{\phi o}\frac{2a}{R_o} \qquad (2.1.2)$$

Where B_ϕ is the magnetic field at the mid of the plane $R = R_0$. This change of the poloidal field across the plasma has a very great effect on the trajectories of the plasma particle. The poloidal magnetic field distribution is dependent upon the toroidal current profile .Using electrical conductivity the steady state current profile can be determined and due to this the electron temperature increases as $T_e^{3/2}$. Hence the current is maximum at the central region where the temperature is highest.

An outward force across the minor radius is exerted by the plasma pressure and inward force exerted by the poloidal magnetic field. The magnetic pressure of the toroidal magnetic field taken up the imbalance between their two forces in Tokamak the resulting magnetic field lines follow a helical path due the combination of poloidal and toroidal magnetic field and generate a set of infinite nested magnetic field line wind the torus. They follow a helical path as shown in the figure. And the magnetic field line changes its direction from surface to surface. For the stability of plasma the shearing of the magnetic field has very important implication. On each surface the average twist of the magnetic field line is characterized by the safety factor q. which gives the measurement of the pitch of helical field line. And radial rate of change of q gives the shear.

The motion of the particle is to complex. The helical motion along the magnetic field is the basic component. The particles make frequent collision at low temperature, thus they can be regarded as a fluid. And the collisions are less frequent at low temperature therefore the toroidal geometry of the magnetic field effect the particles orbits.

The plasma equilibrium can be partially obtained by using the externally imposed conditions such as the net current, the applied energy and the toroidal magnetic field [12].

2.2 Flux Function

In case of axisymmetric (independent of toroidal angle) equilibrium the magnetic field lines in nested toroidal magnetic surface is shown in the figure.

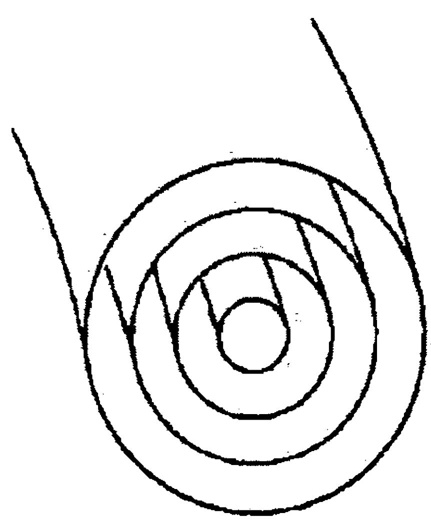


Figure:2.1 Magnetic flux surfaces forming a set of nested toroids.

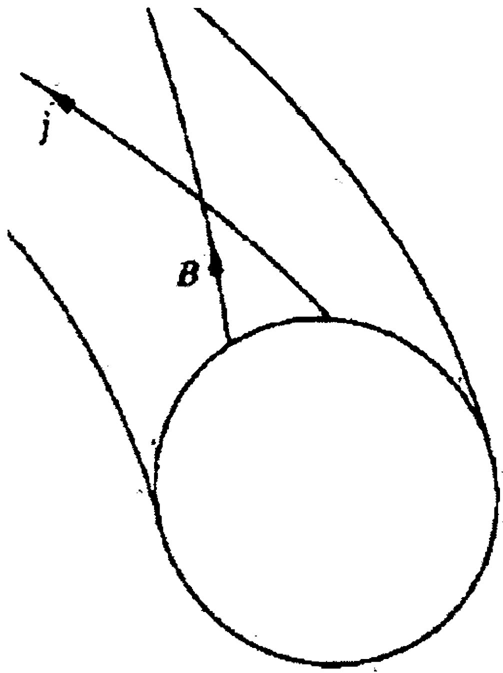


Figure: 2.3 Magnetic field lines and current lines lie in magnet surfaces.

The basic condition of the equilibrium in Tokamak is that the net forces on the plasma should be zero at all points. And for this it is necessary that the magnetic forces should balance by the pressure that is

$$J\times B = \Delta p \qquad (2.2.1)$$

Thus from the above equation it is clear that $B.\Delta P = 0$. This means that along the magnetic field lines there is no pressure gradient and the magnetic surfaces are at constant pressure

Equation (2.2.1) also gives that $J.\Delta p = 0$ And consequently the current also lies on the magnetic surfaces.

For Tokamak equilibrium it is worth to introduce the toroidal magnetic flux function ψ. The poloidal magnetic flux function can be determined from the poloidal flux lying in each magnetic surface. And therefore this flux function constant on that surface this satisfies.

$$B.\Delta\,\psi=0. \qquad (2.2.2)$$

2.3 Safety Factor

The safety factor q has a great importance in determining the stability. The large values of the safety factor q leads to the greater stability. Each magnetic field lie has specific values of q in case of axisymmetric equilibrium. As the magnetic field line goes around the torus on its associated magnetic surface it follows the helical path. If the magnetic line has a specific location in the poloidal plane at some toroidal angle, after changing the toroidal angle, it will regain that position. For this field lines the q value can be defined as

$$q=\Delta\phi/2\pi \qquad (2.3.1)$$

The value of q will be equal to 1 if, after completing exactly one rotation around the torus, a magnetic field line comes back to its starting position. And the value of q will be high if the magnetic field lines move slowly in the poloidal direction.[12] if q = m/n then after m toroidal and n poloidal rotation around the torus the field line join itself. As shown in the figure.

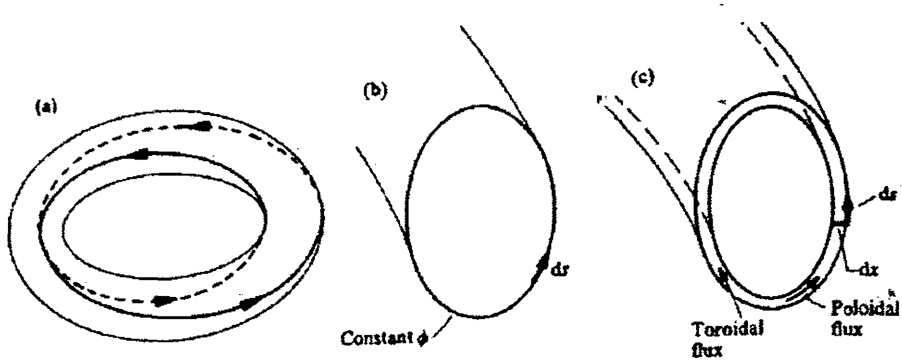


Figure:2.4 (a) field lie on q = 2 surface. (b) Poloidal integration path. (c) Flux anomalous containing toroidal flux $d\phi$ and poloidal flux $d\psi$.

Now we have to use the equation of line to calculate the value of q.

$$\frac{Rd\phi}{ds} = \frac{B_\Phi}{B_P} \qquad (2.3.2)$$

In the above equation ds represent the distance covered in the poloidal direction moving through the toroidal angle. B_P and B_ϕ represent the poloidal and toroidal magnetic fields respectively. Thus equation (2.3.1) can be proceeded as

$$q = \frac{1}{2\pi}\oint \frac{1}{R}\frac{B_\varphi}{B_P}\, ds \qquad (2.3.3)$$

The integration is taken out over a single poloidal circuit around the flux service as shown in the figure (b). Equation (2.3.3) clarify that for all the lines of magnetic field on the magnetic surface the value of q is same. Thus we can say that q is the function of flux that is q = q (ψ). In case of large aspect ratio Tokamak of the circular cross-section equation (2.3.3) becomes

$$q = \frac{rB_\phi}{R_o B_P}$$

Where r represents the minor radius of the flux function and the toroidal magnetic field should be constant.

In case of magnetic fluxes they safety factor q can be written as

$$q = \frac{d\phi}{d\psi}$$

The rate of change of toroidal flux with the poloidal flux gives us the safety factor q.

2.4 Plasma Beta

The ratio represents the efficiency of confinement of plasma by the magnetic field.

$$\beta = \frac{P}{B^2\ /2\mu_o} \qquad (2.4.1)$$

For a magnetic field the thermonuclear power got is an important quality for a reactor. The rate of reaction is proportional to n^2 (σv) which is generally not expressed as a pressure. However in the temperature range imagined for a reactor 10-15KeV, (σv) is coarsely related to T^2 and the thermonuclear power is than proportional to the resulting form of β, called β^* is defined by

$$\beta^* = \frac{\left(\frac{\int P^2 d\tau}{\int d\tau}\right)^{1/2}}{B_0^2/2\mu_o} \tag{2.4. 2}$$

In the above equation B_0 represent the toroidal magnetic field and the integral should be taken over the whole volume of plasma. Now it is more suitable to use the toroidal magnetic field for vacuum at the geometric center of plasma. The average value of β is defined as

$$<\beta>= \frac{(\int P d\tau / \int d\tau)}{B_0^2/2\mu_o}$$

Now the expression for poloidal β *is* given by

$$\beta_P = \frac{(\int P ds / \int ds)}{B_a^2/2\mu_o} \tag{2.4.3}$$

In the above equation the integral are surface integrals over the whole poloidal cross-section and $B_a = \mu_0$ I/l

Where l represents the length of poloidal parameter of plasma And I represent the plasma current. For large aspect ratio circular plasma $l = 2\pi a$ so equation (2.4.3) becomes

$$\beta_P = \frac{\int P ds}{\mu_o I^2/8\pi} \tag{2.4.4}$$

Now in equation (2.4.3), we use P (the volume average) rather than cross section average for an alternative.

Now taking the circular cross section and then taking the integral of the numerator of equation (2.4.4) by parts, we get

$$\beta_P = \frac{8\pi^2}{\mu_o I^2}\int_0^a \frac{dP}{dr} r^2 dr$$

Now putting the value of dP/dr from the approximation pressure balance equation we get.

$$\frac{dp}{dr} + \frac{d}{dr}\left(\frac{B_\varphi^2}{2\mu_0}\right) + \frac{B_\theta}{\mu_o r}\frac{d}{r}(rB_\theta) = 0$$

After integrating we get

$$\beta_P = 1 + \frac{1}{(aB_{\theta a})^2} \int_o^a \frac{dB_\varphi^2}{dr} r^2 dr \qquad (2.4.5)$$

So it is clear from the above equation that there will be no azimuthal currents if the integrand is zero, then $\beta_P = 1$.

If $\frac{dB_\varphi^2}{dr} > 0$, then $B_P > 1$.

On the other hand if $\frac{dB^2}{dr} < 0$, the magnetic pressure $\frac{B_\varphi^2}{2\mu_0}$ dislodges a part of the plasma pressure and $\beta_P < 1$. As shown in the figure.

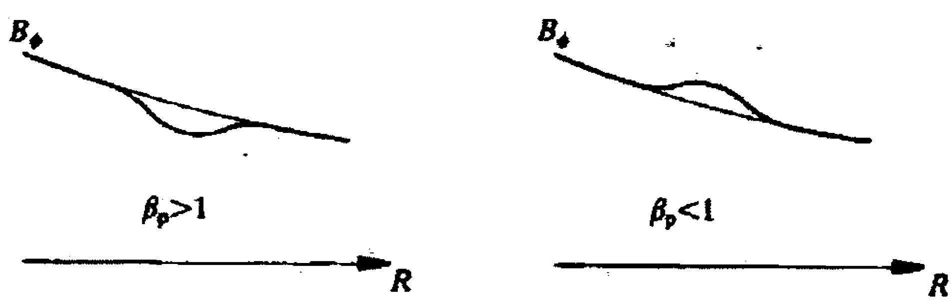


Figure: 2.5 Profiles of B_φ for cases B_P>1 and $B_P < 1$.

2.5 Large Aspect Ratio:

The equilibrium in Tokamak requires a comparatively simple form of low- β, plasma of circular cross-section with large aspect ratio. The ordering of quantities in terms of inverse aspect ratio, $\varepsilon = a/R$ is

$$B_\varphi = B_{\varphi o} \left(\frac{R_o}{R}\right) (1 + 0(\varepsilon^2)) \qquad B_\theta \sim \varepsilon B_{\varphi o}$$

$J_{\varphi} \sim \frac{\varepsilon B_{\varphi o}}{\mu_o a}$ $J_{\theta} \sim \frac{\varepsilon^2 B_{\varphi o}^2}{\mu_o a}$

$P \sim \frac{\varepsilon^2 B_{\varphi o}^2}{\mu_o (B-\varepsilon^2)}$ $B_p \sim 1$

Where $B\varphi_o$is the toroidal magnetic field for vacuum at R_o. Now the pressure balances equation of cylinder.

$$\frac{dp}{dr} = j_{\varphi} B_{\theta} - j_{\theta o} \qquad (2.5.1)$$

Where $j_{\varphi}(r)$ and p (r) with p (a) = 0 specifying the equilibrium. Now the azimuthal field given by ampere's law is.

$$\mu_O j_{\varphi} = -\frac{1}{r}\frac{d}{dr}(rB_{\theta})$$

And j_{θ} can be determined by using equation (2.5.1). The flux surfaces get the shape of non-concentric circles when we include the toroidal effect as shown in the figure.

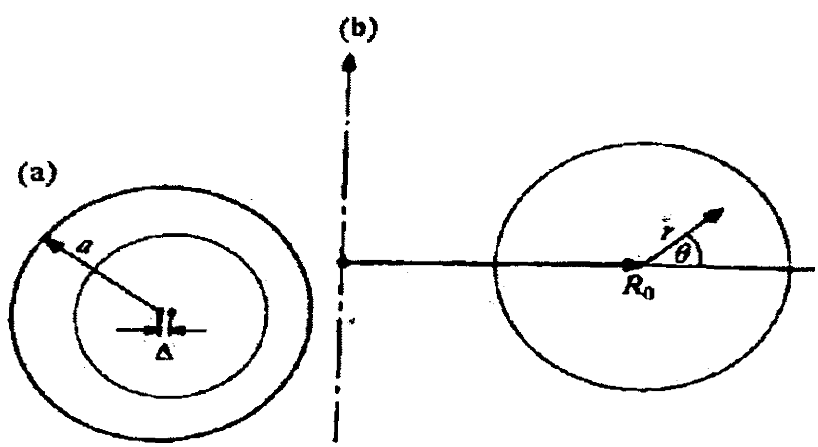


Figure: 2.6 (a) showing surface of circular flux moved by a distance Δ with respect to the outer flux surfaces the center of which is at distance R_o.

Using the coordinate (r,θ) with center at major radius R_o, the G.S equation can be written as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi-\frac{1}{R_o+r\,cos\theta}\left(cos\theta-sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\psi$$

$$=-\mu_o(R_o+rcos\theta)^2P'(\psi)-\mu_o^2f(\psi)f'(\psi) \qquad (2.5.2)$$

Now expanding ψ we get

$$\psi=\psi_o(r)+\psi_1(r,\theta)$$

The flux function ψ given by equation (2.5.2) and (2.5.1).

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_o}{dr}\right)=-\mu_oR_o^2P'(\psi_o)-\mu_o^2\,f(\psi_o)f'(\psi_o) \qquad (2.5.3)$$

And the first part of equation (2) is satisfied by ψ_1

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi-\frac{cos\theta}{R_o}\frac{d\psi_o}{dr}$$

$$=-\mu_oR_o^2P'(\psi_o)\psi_1-\mu_o^2(\,f(\psi_o)\,f'(\psi_o))'\psi_o-2\mu_oR_orcos\theta P'(\psi_o)$$

$$=-\frac{d}{dr}\left(\mu_oR_o^2P'(\psi_o)+\mu_o^2\,f(\psi_o)f'(\psi_o)\right)\frac{dr}{d\psi_o}\psi_1-2\mu_oR_orcos\theta P'(\psi_o)$$

(2.5.4)

If we displace the flux surface ψ by a distance $\Delta(\psi_o(r)),\psi$ then

$$\psi=\psi_o+\psi_1$$

$$=\psi_o-\Delta(r)\frac{\partial\psi_o}{\partial R}$$

$$=\psi_o-\Delta(r)cos\theta\frac{d\psi_o}{dr} \qquad (2.5.5)$$

Now put equation (2.5.5) in (2.5.4).

$$-\Delta\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_o}{dr}\right)\right)-\frac{1}{r}\left(\frac{dr}{d\psi_o}\right)\frac{d}{dr}\left(r\left(\frac{d\psi_o}{dr}\right)^2\frac{d\Delta}{dr}\right)-\frac{1}{R_o}\frac{d\psi_o}{dr}$$

$$=\Delta\frac{d}{dr}(\mu_o R_o^2 P'(\psi_o)+\mu_o^2 f(\psi_o)f'(\psi_o)-2\mu_o R_o r\frac{dp_o}{dr}\frac{dr}{d\psi_o} \qquad (2.5.6)$$

Now from equation (2.5.3) the first terms on the two sides of equation (2.5.6) cancel and leaving

$$\frac{d}{dr}\left(r\bar{B}_{\theta o}^2\frac{d\Delta}{dr}\right)=\frac{r}{R}\left(2\mu_o r\frac{dP_o}{dr}-\mu B_{\theta o}^2\right) \qquad (2.5.7)$$

We have used the definition of flux function to replace $\frac{d\psi_o}{dr}/R_o$ by $B_{\theta o.}$

2.6 Shafranov Shift:

From the above discussion it has been cleared that the centers of the magnetic field surfaces are expatriate with respect to the center of the bounding surfaces .tis surfaces is denoted by $\Delta(r)$

And can be calculated by solving equation (2.5.7). The axis displacement $\Delta_s = \Delta(0)$ is known as the Shafranov shift.

The Shafranov shift is dependent on the particular forms of P_o(r) and $B_{\theta o}$(r) and equation (2.5.7) should have to be solved for each case. However by using simple analytical forms some indications of the behavior can be obtained. Now by dropping the subscript zero we can write [12].

$$P=\hat{P}\,(1-\frac{r^2}{a^2})$$

And

$$j=\hat{J}(1-\frac{r^2}{a^2})^{\,2} \qquad (2.6.1)$$

Equation (2.5.7) of the previous topic can be written as:

$$\frac{d\Delta}{dr}=-\frac{1}{RB_\theta^2}(\frac{r^3}{a^2}B_P B_{\theta a}^2+\frac{1}{r}\int_o^r B_{\theta r}^2 dr) \qquad (2.6.2)$$

Where $B_{\theta a} = B_{\theta(a)}$ and the poloidal beta B_P can be define by

$$B_P = \frac{P}{B_{\theta a}^2/2\mu_o} = \frac{4\mu_o \int_o^a Prdr}{a^2 B_{\theta a}^2} \qquad (2.6.3)$$

And

$B_P = \mu_o \hat{P}/B_{\theta a}^2$ In this case now by using Ampere's law to equation (2.6.3) we get

$$B_\theta = B_{\theta a} = \frac{1-(1-\frac{r^2}{a^2})^{v+1}}{r/a} \qquad (2.6.4)$$

Now using equation (2.6.4) for B_θ we can integrate equation (2.6.2) numerically to get Δs as a function of B_P and v. However, if we use internal inductance l_i rather than v, it will more efficient

Thus

$$l_i = \frac{B_\theta^2}{B_{\theta a}^2} = 2\int_o^a \frac{B_\theta^2 rdr}{a^2 B_{\theta a}^2} \qquad (2.6.5)$$

The graph of l_i is shown in figure.1. And practical fit with 2% is:

$$l_i = \ln(1.65 + 0.89v)$$

The parameter v is related to the central q valve and ratio of edge through $\frac{q}{q_o} = v + 1$ and this ratio is shown in the figure. The figure shows the calculated valve the Shafranov shift in the form of contours of equal $\left(\frac{R}{a}\right) \Delta s/a$ in the (B_P, l_i) plane.

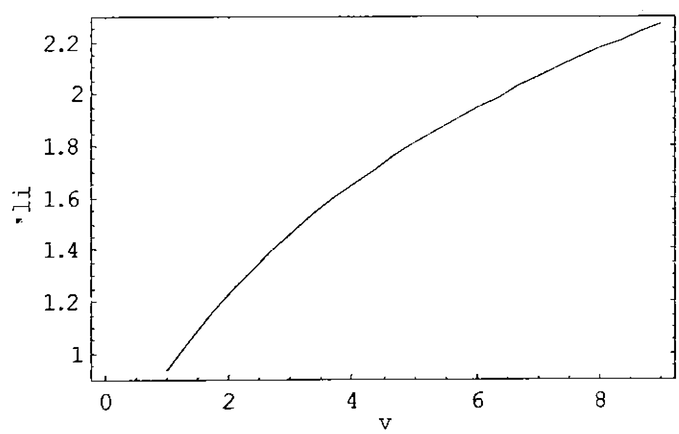


Figure: 2.7 Graph of the internal inductance of the current

CHAPTER:3

Spherical Tokamak

3.1 Spherical Tokamak:

In 1984 Martin Peng suggest an alternative arrangement of the magnetic coils that results in reduction of the aspect ratio. He suggest to use a single large conductor in the mid-section of the reactor instead of the magnetic coils which minimize the magnitude of the hole at center nearly to zero and thus reduces the aspect ratio to 1.2.

The design of the spherical Tokamak also includes the advance plasma shaping. In the spherical Tokamak the D-shaped cross section of plasma is used. If a D-shaped cross section is taken on the right and a overturned cross section on the leftward closed to each other such that their vertical sides touch each other it give a circular shape. And in three dimensions the external surface is somewhat spherical. Therefore such type of configuration was named the Spherical Tokamak as shown in the figure bellow. [13]

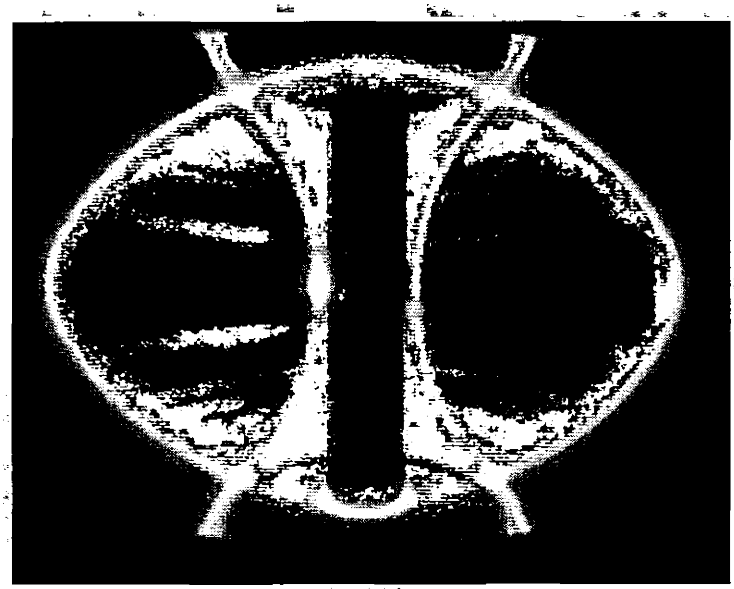


Figure: 3.1. Spherical Tokamak.

The main feature of the spherical Tokamak configuration is the tremendously tight aspect ratio. One the main incentive of the spherical Tokamak configuration is to achieve the MHD β limit scaling $\beta \sim \epsilon$. As we know that β_{crit} should increases when the aspect ratio becomes tighter. Similarly the higher the stability of β_s greater will be the attractiveness of a fusion reactor by allowing the use of lower toroidal magnetic field. In fact when one compare the spherical Tokamak to the standard Tokamak it is not quit clear that a spherical tokamak would lead to a more attractive reactor or not. However spherical tokamak has alternative application for which it would be better suited as a volume neutron source. The basis for the above discussion is as follows.

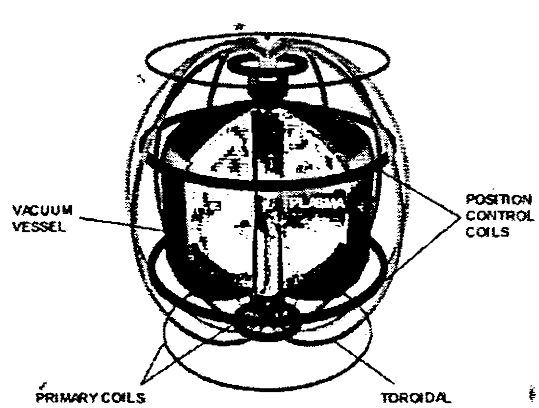
As we know that for a fusion reaction high plasma pressure is required. And also we know that the limiting toroidal B_{max} on the inner leg of the toroidal field magnet results to a larger decrease in B_φ at the center of the plasma and at the center of plasma β is define due to the strong 1/R effect at tight aspect ratio. And the ultimate result is that in spherical tokamak the maximum attainable pressure is usually less than in a standard tokamak.

As we require a much large toroidal current to achieve a safety factor in a tight aspect ratio device. Therefore in spherical tokamak the issue of the current driven is so difficult. [6]

3.2 Spherical Tokamak Configuration:

The spherical Tokamak devices contain a toroidal vacuum container surrounded by a chain of magnets. Logically a series of rings, around the external of the container bound, one set of magnets, but are actually linked by a mutual conductor in the middle. The pillar at the central is also usually used to company the solenoid. That forms the inductive coil for the ohmic heating system (and pinch current).

Figure: 3.2 spherical tokamak configurations.



The official design of the spherical tokamak can be seen in the figure 3.4. The central pillar which is spiral into a solenoid is made up of copper, returns slabs, used to produce the toroidal field, are usually made up of upright copper wire and a metal ring linking the two and provides motorized sustenance to the assembly [14].

3.3 List of Some Operational ST Machines:

- MAST, UK
- NSTX, US
- Globus-M, Russia
- START ENEA, Italy
- TST-2, Japan
- SUNIST, China
- PEGASUS, United State
- ETE, Brazil
- GUTTA, Russia
- KTM, Kazakhstan
- GLAST, Pakistan

3.4 Stability with in the Spherical Tokamak:

In 1970s and 80s the developments in plasma physics perform stronger job in understanding of stability problems, and this leads to a successions of "scaling laws". These laws are used for determining rough operating number across a large variety of systems. On the critical beta of a reactor configuration the Troyon's effort is a great achievement in the field of plasma physics. Troyon's effort offers a beta limit where working reactors will start to perceive important instabilities, and prove how this limit balances with size, configuration, magnetic field and plasma current.

The Troyon's study covered a wide range of configurations but did not give any explanation of the tight aspect ratio spherical tokamak, because the spherical tokamak was not under consideration at that time. Later on a group at the *Princeton Plasma Physics Laboratory* started to work with several enhanced definitions of critical plasma parameters.

They proved that the simple dependency of the β_{crit}on the aspect ratio persist even for extremely tight aspect ratio. They also included the dependency of q_* related with the kink modes. Combining the above results we can determine the ideal q_* and corresponding β_{max} of aspect ratio and elongation. [6]

First we have to introduce the improved definitions of the critical plasma parameter. Now according to Troyon and coworkers replacing the existing definition β by

$$\beta = \frac{2\mu_o < P >}{B_o^2} \rightarrow \frac{2\mu_o < P >}{< B^2 >}$$

Where $< >$ represent the volume averaged value. It is quite clear that in the new definition the vacuum magnetic energy B_o^2 is swapped by the total magnetic energy $< B^2 > = < B_\varphi^2 + B_P^2 >$. And these definitions coincide in case of large aspect ratio $\epsilon \rightarrow 0$. Now, a new definition of the safety factor which is lunched with a slightly different dependence on the elongation:

$$q_* = \frac{2\pi B_o a^2}{\mu_o R_o I} k \rightarrow \frac{2\pi B_o a^2}{\mu_o R_o I}\left(\frac{1 + k^2}{2}\right)$$

In the above definitions the first one is for the case of conventional tokamak and second one is for case of spherical tokamak. Now we can compare these two definitions.

In case of conventional tokamak, from the dependence of q factor on κ, taking the range of κ, 1 ~ 3. It is quite clear from the figure 3.5 that for $\kappa = 3$ q approaches to 6.2.

Similarly taking the same range of κ in the case for spherical tokamak it is quite clear form the figure: 3.6 that at $\kappa = 3$ q approaches to 11.

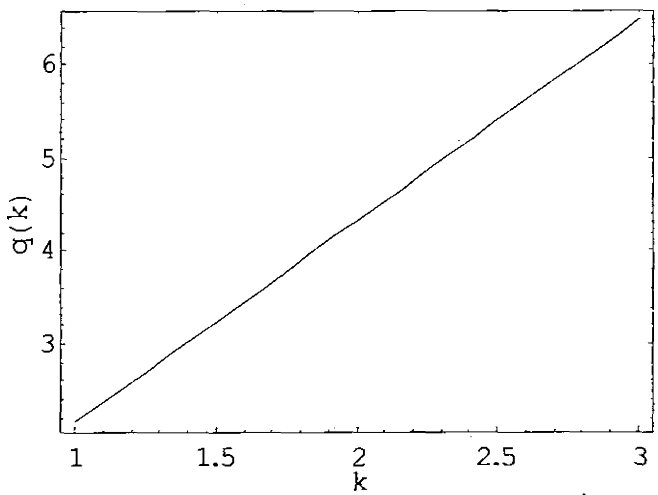


Figure: 3.3 Relation between κ and q for convention tokamak

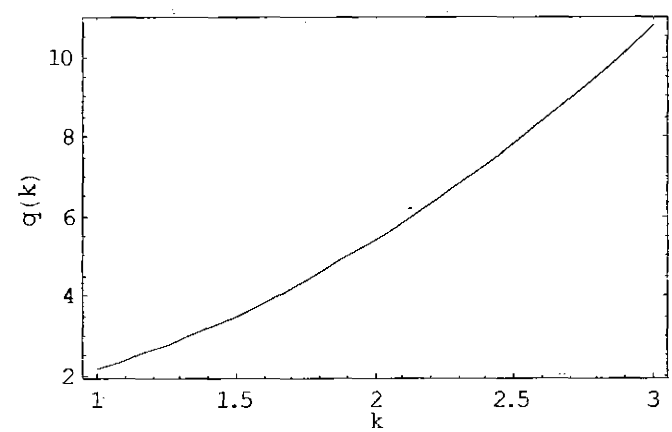


Figure: 3.4 Relation between κ and q for spherical tokamak

Now the comparative study of the conventional and spherical tokamak clarify that the spherical tokamak is more competent and provides more stability to the tokamak plasma as compare to the convention tokamak as shown in the figure 3.7 given below.

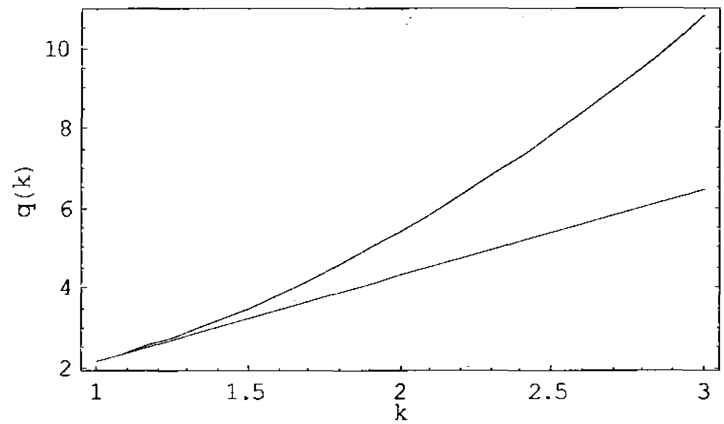


Figure: 3.5 comparison of conventional and spherical tokamak. The upper line represents the spherical tokamak while the lower line represents conventional tokamak.

As we know that the relation between $\beta, \kappa,$ and ϵ is given by

$$\beta = 0.072(\frac{1+k^2}{2})\epsilon\ ;$$

Now in case of conventional tokamak the aspect ratio A is usually 3 and thus the inverse aspect ratio ϵ is 0.33. Using these valves in the above equation and taking the range of κ $1 \rightarrow 4$, we get the graph as shown in the figure 3.8.

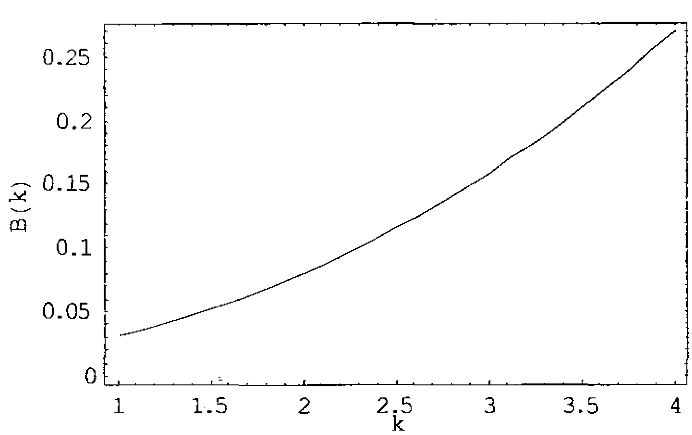


Figure: 3.6. Relation between β and κ for conventional tokamak

It is quite clear in the figure 3.7 that in case of conventional tokamak the maximum valve of β is 0.25 for $\kappa = 4$.

Now considering the same case for the spherical tokamak the aspect ratio A = 1.25 and inverse aspect ratio $\epsilon = 0.8$ we get the graph as shown in the figure 3.8.

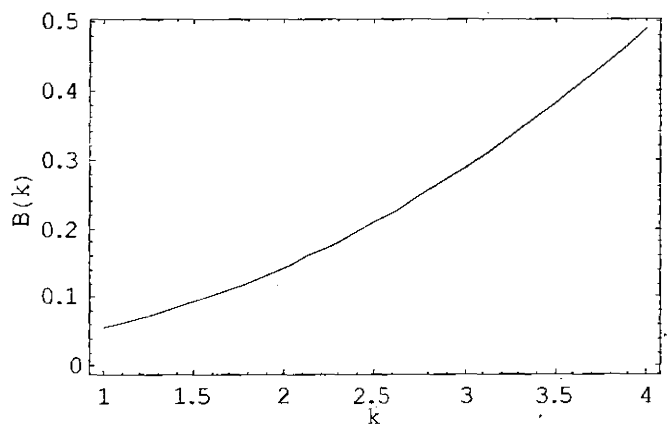


Figure: 3.7. Relation between β and κ for spherical tokamak

It is clear from the figure 3.8 that in case of the spherical tokamak i.e. $\epsilon = 0.8$ q get the value 0.5 for $\kappa = 4$.

Now the comparison of the above graphs is shown in the figure: 3.9. It is quite clear from the figure 3.9 that for the same range of κ the spherical tokamak get higher valve of β than conventional tokamak. And it is obvious that higher valve of β provide much equilibrium to the tokamak plasma.

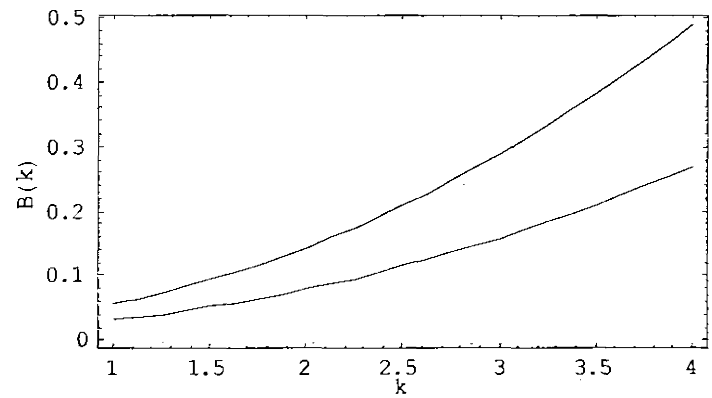


Figure: 3.8. Comparison of ST and conventional tokamak w.r.t β. The upper line represents the spherical tokamak while the lower line represents conventional tokamak

So it will be more convenient to say that, from equilibrium point of view the spherical tokamak is more efficient configuration as compare to the conventional tokamak.

3.5 Advantages of ST

Spherical Tokomaks have two major advantages as compare to the conventional tokamak configuration.

The first is practical. In the Spherical Tokamak configuration the toroidal magnets are nearer to the surface of the plasma. This closeness cause a great reduction in the quantity of energy required to empower the magnets to acquire any specific value of magnetic field inside the plasma. As magnets are smaller and cost less this reduces the price of the reactor. There is no need to use superconducting magnets because the gains are so much greater and this leads to even better cost reductions.

The second advantage is related to the stability of plasma. A large number of instabilities were faced by the configuration of a useful system, in the initial stages of fusion research. In *1954* a meeting was arranged by *Edward Teller* discovering some of these disputes. He sensed that if they followed the convex lines of magnetic force rather than the concave, the plasmas would be intrinsically more stable. It was not clear at the time but after some time the meanings of these terms become obvious.

In most of the tokamak machine, the plasma is constrained to track helical magnetic lines. Due to this the plasma is forced to move to the inner area from the outer confinement area tracking a concave line. They are being strapped to the outer section while traveling inside, following a convex line. According to the *Teller's* thoughts, in the inner side of the reactor the plasma is naturally much stable. The actual limits that vary over the volume of the plasma, proposed by the 'safety factor' q,

In case of ancient circular cross-section tokamak, the plasma passes slightly less time on the inner side surface of plasma than the outer side, due to smaller radii. While in the case of modified tokamak with D-shaped plasma, the particles spend more time on the inner surface of the plasma because the inner side is significantly. But in case of Spherical Tokamak the particles pass much of their time on the inner side of the plasma surface. And this adds to improve the stability [6-14].

3.6 Disadvantages of ST

There are three disadvantages of the spherical tokamak compared to conventional tokamak. The first problem is that the net plasma pressure is lesser in Spherical Tokamak as compare to conventional tokamak configuration, due to higher beta. It is because of the value of the magnetic field on the inner side of the plasma surface, B_{max}. In both the designs the value of the magnetic field is theoretically same, but in case of spherical tokamak the aspect is very high, and a dramatic variation occurs in the effective field over the plasma volume.

The second problem is somewhat surprising, which can be considered as an advantage as well as a disadvantage. The size of ST is so smaller, especially at the middle; therefore there is slight or even no possibility for superconducting magnets. But it is not a big problem for the configuration because for ST configuration the field provided by the conventional copper coiled magnets is enough. However, there is a great chance of power dissipation in the central column. The maximum and possible field is about *7.5 T*, which is smaller than the probable field in a conventional design. This imposes an additional constrain on the permissible plasma pressures. However, the price of the system decreases due to the nonexistence of the superconducting magnets.

In conclusion, in order to sustain an extraordinary toroidal current, strongly looped magnetic fields and extremely asymmetrical plasma cross sections is required. And large extent of secondary heating systems should be needed, such as neutral beam injection. These are vigorously costly; therefore the ST configuration relies on high bootstrap current for efficient working. Fortunately, triangularity and high elongation are the topographies which are the causes of these currents. So there is possibility for the spherical tokamak to be more efficient in this regard. This is a field of dynamic research [6-14]

CHAPTER: 4

Equilibrium of Tokamak and Analytic Solutions of the Grad-Shafranov Equation:

4.1 MHD Equations for Tokamak Equilibrium:

Plasma equilibrium can well describe by the following equation;

$$\mathbf{J} \times \boldsymbol{B} = \nabla$$

$$\nabla . \boldsymbol{B} = 0 \qquad (4.1)$$

$$\nabla \times \boldsymbol{B} = \mu_o \mathbf{J}$$

For toroidally axisymmetric configuration, the above set of equations should be reduces to a single two-dimensional, nonlinear, elliptic Partial differential equation, whose solution comprises all the necessary material about nature of equilibrium. This equation is known as Grad-Shafranov equation and can be written as follows:

$$R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = -\mu_o R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} \qquad (4.2)$$

To describe the axisymmetric MHD equilibria we will initially use the cylindrical coordinates (R,ϕ ,Z) where ϕ is the angel of symmetry and R is the measure of the distance to the axis of symmetry (the major radius of the toroidal system).

As nowadays there are several accurate and fast numerical Grad-Shafranov solvers are available but analytical solution is very important from theoretical point of view.

In several plasma confinement concepts, such as the tokomak and the stellarator for instance, the inverse aspect ratio can be used as an expansion parameter in equation. (4.2). Analytic solutions can be obtained by expanding equation (4.2) order by order. This method has led to a very deep analytic understanding of static equilibrium in tokomaks.

4.2 Equilibrium Equations in Fusion Plasmas:

In all circumstances of fusion interest, we ignoring the inertial term, and we can emphasis on static equilibria, v= **0**, for which the equilibrium momentum equation takes the form:

$$\mathbf{J} \times \boldsymbol{B} = \nabla p \tag{4.3}$$

Which is the well-known equation expressing the balance between the magnetic force **J**×**B** and the pressure gradient.

This is perceptibly not satisfactory to determine the equilibrium. The remaining equations are obtained from the Maxwell's equations, consistent with the ideal MHD ordering: $\nabla.\boldsymbol{B}=0$, and $\nabla \times \boldsymbol{B}=\mu_o\mathbf{J}$. Thus, ideal MHD equilibrium are obtained from the following set of equations

$$\begin{aligned} \nabla.\boldsymbol{B} &= 0 \\ \nabla \times \boldsymbol{B} &= \mu_o \mathbf{J} \\ \mathbf{J} \times \boldsymbol{B} &= \nabla \mathrm{p} \end{aligned} \tag{4.4}$$

And now we have to show that for toroidally axisymmetric plasmas, all the information contained in the seven equations given by eq. (4.4) can be articulated in a single equation for one variable: the Grad-Shafranov equation.

Although the computation of plasma equilibrium in magnetic confinement ideas is frequently considered a part of ideal MHD theory, the equilibrium described by equation.(4.4) is in fact similar with descriptions of the plasma which are more precise than ideal MHD, and valid in rules where ideal MHD is not, where the plasma ions are collision less.

As we have $\nabla.\boldsymbol{B}=0$ and $\nabla \times \boldsymbol{B}=\mu_o\mathbf{J}$ are the equations of magnetostatics.

They are clearly exact equations when $\frac{\partial}{\partial t}=0$.

Now considering the second order moment of electrons' and ions' Maxwell-Boltzmann equations and then adding them we get:

$$\rho\frac{dV}{dt} + \nabla.(\prod_i + \prod_e) = \mathbf{J} \times \boldsymbol{B} - \nabla \mathrm{p} \quad (4.5)$$

But in steady state case: $\frac{dV}{dt} \simeq 0$, and also we know that the viscosity tensors $\prod_i \simeq 0$, $\prod_e \simeq 0$ in equilibrium.

Thus equation (4.5) becomes:

$$\mathbf{J} \times \boldsymbol{B} \simeq \nabla \mathrm{p} \quad (4.6)$$

Simply we can say that equation (4.4) is valid far beyond the limits of ideal MHD.

4.3 The Grad-Shafranov Equation:

Now we will show how the set of equations (4.4) can be reduces, for toroidally axisymmetric configuration, to a single two-dimensional, nonlinear, elliptic partial differential equation, whose solution contains all the necessary information to fully determine the nature of the equilibrium. This was first discovered by Lüst and Schlüter, Grad and Rubin, and Shafranov in the years 1957 to 1959 [15-16-17]. In this section, we will rederive this equation, now known as the Grad-Shafranov equation (GS equation),

In order to describe the toroidal axisymmetric geometries we will initially use the cylindrical coordinates (R,ϕ,Z) where ϕ is the angel of symmetry, i.e. $\frac{\partial}{\partial\phi} = 0$, and R is the measure of the distance to the axis of symmetry (the major radius of the toroidal system). As shown in the figure (4.1)

Top view **Cross section**

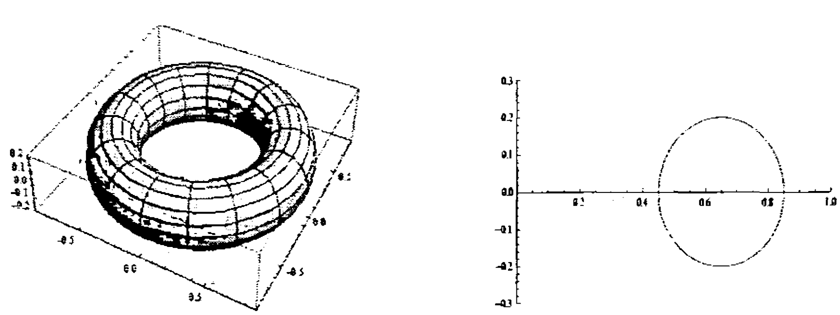


Fig.4.1. Geometry for toroidally axisymmetric equilibria and cylindrical coordinates.

Starting with the first equation in (4.4): $\nabla.\boldsymbol{B} = 0$ because of the toroidal axisymmetric nature this equation does not give any information about $\mathbf{B}_\emptyset$ the component of the magnetic field, which is called the toroidal magnetic field. However, it gives us a very suitable way of writing the poloidal magnetic field $\mathbf{B}_p$ which is in the (R,Z) plane.

Now as we have:

$$\nabla.\mathbf{B} = 0 \tag{4.7}$$

But

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Here **A** is the vector potential. And in axisymmetric case only $A_\emptyset$ appears in the expressions for B_R and B_Z :

$$B_R = -\frac{\partial A_\emptyset}{\partial Z} = -\frac{1}{R}\frac{\partial(RA_\emptyset)}{\partial Z},\quad B_Z = \frac{1}{R}\frac{\partial(RA_\emptyset)}{\partial R} \tag{4.8}$$

Now introducing stream function ψ, defined by $\psi = RA_\emptyset$, so

$$\mathbf{B} = B_\emptyset \mathbf{e}_\emptyset + \frac{1}{R}\nabla\,\psi \times \mathbf{e}_\emptyset \tag{4.9}$$

Where $\mathbf{e}_\emptyset$ is the unit vector in the $\emptyset$ direction, $\mathbf{e}_\emptyset = R\nabla\emptyset$.

The stream function is actually the poloidal flux ψ_P normalized by dividing by a factor 2π. the poloidal flux is defined as:

$$\psi_P = \int \mathbf{B}_P \,.\, \mathbf{dS} \tag{4.10}$$

Where $\mathbf{dS}$ is an infinitesimal surface element.

Now to calculate the poloidal flux through the area of a ring shaped surface in the plane Z=0, prolonging from the magnetic axis located at R=R_a, to an arbitrary ψ contour at R=R_b, we find:

$$\psi_P = \int_0^{2\pi} d\,\emptyset \int_{R_a}^{R_b} dRR\,B_Z(R, Z = 0)$$

$$\psi_P = \int_0^{2\pi} d\,\emptyset \int_{R_a}^{R_b} dR \frac{\partial \psi}{\partial R}$$

$$\psi_P = 2\pi[\psi(R_b, 0) - \psi(R_a, 0)] \quad (4.11)$$

As it is clear from equation (4.8) that ψ is defined with in an arbitrary integration constant, therefore we choose arbitrary constant so that $\psi(R_a, 0) = 0$.

So equation (4.11) becomes:

$$\psi_P = 2\pi\,\psi \quad (4.12)$$

Now using Ampere's law, $\nabla \times \boldsymbol{B} = \mu_o \mathbf{J}$, to obtain an expression for **J** in terms of stream function ψ.

The poloidal current:

$$\mu_o \mathrm{J}_p = \frac{1}{\mathrm{R}} * \nabla(\mathrm{RB}_\emptyset) \times \mathbf{e}_\emptyset \quad (4.13)$$

Now the toroidal current is:

$$\mu_o \mathrm{J}_\emptyset = \frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R}$$

$$\mu_o \mathrm{J}_\emptyset = -\frac{1}{\mathrm{R}}\left[\mathrm{R}\frac{\partial}{\partial \mathrm{R}}\left(\frac{1}{\mathrm{R}}\frac{\partial \psi}{\partial \mathrm{R}}\right) + \frac{\partial^2 \psi}{\partial Z^2}\right]$$

$$\mu_o \mathbf{J}_\emptyset = -\frac{1}{R}\Delta^* \psi \quad (4.14)$$

Where Δ^* is the elliptic operator and given by:

$$\Delta^* X = R^2 \nabla.\left(\frac{\nabla X}{R^2}\right) = R\frac{\partial}{\partial \mathrm{R}}\left(\frac{1}{\mathrm{R}}\frac{\partial \mathrm{X}}{\partial \mathrm{R}}\right) + \frac{\partial^2 \mathrm{X}}{\partial Z^2} \quad (4.15)$$

Now the projection of momentum equation $\mathbf{J} \times \boldsymbol{B} = \nabla \mathrm{p}$ onto the three vectors **B**, **J**, and $\nabla\psi$.

• Projection onto **B**

It is clear that the left-hand side of the momentum equation is perpendicular to **B**, due to axisymmetric, ∇p has only Z and R components, so that the result of the projection is

$$\frac{1}{R}(\nabla \psi \times \mathbf{e}_\emptyset).\nabla \mathrm{p} = 0$$

$$\Rightarrow \quad \mathbf{e}_\emptyset.\nabla \psi \times \nabla \mathrm{p} = 0$$

$$\Rightarrow \quad \nabla\psi \times \nabla p = 0$$

3

It means $\nabla\psi \times \nabla p$ has only $\emptyset$ component, so

$$\Rightarrow \quad p = p(\psi) \qquad (4.16)$$

Equation (4.16) shows that p depends on ψ only and it is a surface quantity.

• Projection onto **J**

As **B** and **J** play similar role in the momentum equation therefore projection onto **J** leads to the following equation:

$$\frac{1}{R}\nabla(RB_{\emptyset}) \times \mathbf{e}_{\emptyset}.\nabla p = 0$$

$$\Rightarrow \mathbf{e}_{\emptyset}.\nabla(RB_{\emptyset}) \times \nabla p = 0 \qquad (4.17)$$

But as we have from equation (4.16)

$$p = p(\psi)$$

So we can write

$$\boldsymbol{\nabla} p = \boldsymbol{\nabla}\psi$$

So equation (4.17) becomes:

$$\mathbf{e}_{\emptyset}.\nabla(RB_{\emptyset}) \times \boldsymbol{\nabla}\psi = 0 \qquad (4.18)$$

This is the similar situation as in equation (4.16) and in the same way we can conclude that

$$RB_{\emptyset}=F(\psi) \qquad (4.19)$$

The quantity $RB_{\emptyset}$ is a surface quantity like p and only depends on ψ. The quantity F has a physical interpretation with ψ : it is net poloidal current flowing in the toroidal field coils and plasma. And can be normalized by dividing by a factor -2π . Now to show this we will calculate the flux of the poloidal current density through a disk-shaped surface lying in the plane Z = 0, extending from R = 0 to an arbitrary ψ contour at R= R_b. We find:

$$I_p = \int \mathbf{J}_p \,.\, \mathbf{dS}$$

$$I_p = -\int_0^{2\pi}\int_0^{R_b} RJ_Z\, dRd\emptyset(\mathrm{R}, \mathrm{Z}=0)$$

$$I_p = -\int_0^{2\pi} d\emptyset \int_0^{R_b} dRRJ_Z\ (\mathrm{R},\mathrm{Z}=0)$$

$$I_p = -2\pi\int_0^{R_b} dRR(\frac{1}{R}\frac{\partial(\mathrm{RB}_\emptyset)}{\partial R})$$

$$I_p = -2\pi\int_0^{R_b} dR\frac{\partial F}{\partial R} = -2\pi\ \mathrm{F}\ (\psi\ ) \qquad (4.20)$$

The −ve sign shows the fact that the element of surface **dS** oriented in the +Z direction.

• Projection onto $\nabla\psi$

Now to calculate J×**B** we have

$$\mathbf{J}\times\mathbf{B} = (J_\emptyset\mathbf{e}_\emptyset + \mathbf{J}_p)\times(B_\emptyset\mathbf{e}_\emptyset + \mathbf{B}_p)$$

In the above equation it is clear that the cross product of the two toroidal components vanishes. And also

$$\mathbf{J}_p\times\mathbf{B}_p = 0$$

It means only the cross terms between poloidal and toroidal components will contribute.

So

$$J_\emptyset\mathbf{e}_\emptyset\times\mathbf{B}_p = \frac{J_\emptyset}{R}\nabla\psi$$

Now using the value of $J_\emptyset$ from equation (4.14) we get

$$J_\emptyset\mathbf{e}_\emptyset\times\mathbf{B}_p = -\frac{1}{\mu_o R^2}\Delta^*\ \psi\nabla\psi$$

Similarly

$$J_p\times B_\emptyset\mathbf{e}_\emptyset = -\frac{1}{\mu_o R^2}F\frac{dF}{d\psi}\nabla\psi \qquad (4.21)$$

Now for toroidal axisymmetry, the momentum equation can be written as:

$$-\frac{1}{\mu_o R^2}\Delta^*\ \psi\nabla\psi - \frac{1}{\mu_o R^2}F\frac{dF}{d\psi}\nabla\psi = \frac{\mathrm{dp}}{\mathrm{d}\psi}\nabla\psi \qquad (4.22)$$

It is clear from the above equation that the only nontrivial information in force balance equation is contained in the $\nabla\psi$ component.

Equation (4.22) can be written in the form

$$\Delta^* \psi = \mu_o R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} \tag{4.23}$$

This is the second-order, nonlinear, elliptic partial differential equation. Which is usually called Grad-Shafranov equation [18,19].

4.4 The Grad-Shafranov Equation with Solov'ev Profiles:

The GS equation equation.(4.23) can be place in a non-dimensional form through the normalization $R = R_0 x,\ \ Z = R_0 y, and\ \psi = \psi_o \psi$.

Where R_0 is the major radius of plasma and ψ_o is the arbitrary constant

$$x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2\psi}{\partial y^2} = -\mu_o \frac{{R_0}^4}{\psi_0^2} x^2 \frac{dp}{d\psi} - \frac{{R_0}^2}{\psi_0^2} F \frac{dF}{d\psi} \tag{4.24}$$

The selections for p and F corresponding to the Solov'ev profiles are given by [20].

$$-\mu_o \frac{R^4}{\psi_0^2}\frac{dp}{d\psi} = C$$

$$-\frac{R^2}{\psi_0^2} F \frac{dF}{d\psi} = A \tag{4.25}$$

Where A and C are constants. Since ψ_o is an arbitrary constant, so we can write $A + C = 1$

According to these conditions, the GS can be written as

$$x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2\psi}{\partial y^2} = (C)x^2 + A$$

But $C = 1 - A$ so the above equation becomes;

$$x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial\psi}{\partial x}\right)+\frac{\partial^2\psi}{\partial y^2}=(1-A)x^2+A \tag{4.26}$$

Now we will calculate equilibria in various magnetic geometries for particular values of A corresponding to a range of β values.

The solution to equation (4.26) is of the form $\psi(x,y)=\psi_p(x,y)+\psi_H(x,y)$

Where ψ_p is the particular and ψ_H is the homogenous solution. The particular solution can be written as :

$$\psi_p(x,y)=\frac{x^2}{8}+A\left(\frac{1}{2}x^2\ln x-\frac{x^4}{8}\right) \tag{4.27}$$

Now the homogenous solution satisfies:

$$x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial\psi_H}{\partial x}\right)+\frac{\partial^2\psi_H}{\partial y^2}=0 \tag{4.28}$$

Now we will present here the detail of a general arbitrary degree polynomial-like solution to this equation for plasma with up-down symmetry which has been derived by Zheng*et. al.* in [21].

Now assume that there exists a general solution of the form

$$\psi_H(x,y)=\sum_{n=0,2\ldots}\sum_{k=0}^{n/2}G(n,k,x)y^{n-2k} \tag{4.29}$$

Where G is a function which has not been yet calculated but expected to have the same form as that of the particular solution ψ_p. Now, if (4.29) is the solution, it will obviously satisfy the equation (4.28). Now putting (4.29) in (4.28) and identifying the terms where y has the same exponent for a given n, so we get the following relation on the index k, for a given n:

$$x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial G(n,0,x)}{\partial x}\right)=0 \tag{4.30}$$

$$x\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial G(n,k,x)}{\partial x}\right)=-(n-2k+1)(n-2k+2)G(n,k-1,x) \qquad (k\neq 0)$$

For $k=0$ there are two solution to equation to the equation (4.30)

$$G1(n,0,x)=1 \quad \textit{and} \quad G2(n.0,x)=x^2$$

Thus we can write

$$G(n,k,x) = c_{n1}G1(n,k,x) + c_{n2}G2(n,k,x) \quad (4.31)$$

Where c_{n1} and c_{n2} are free constants.

Now if G1 and G2 take the general forms:

$$G1(n,0,x) = 1$$

$$G1(n,k,>0,x) = (-1)^k \frac{n!}{(n-2k)!} \frac{1}{2^{2k}k!(k-1)!} x^{2k}\left(2\ln x + \frac{1}{k} - 2\sum_{j=1}^{k}\frac{1}{j}\right) \quad (4.32)$$

$$G2(n,k,x) = (-1)^k \frac{n!}{(n-2k)!} \frac{1}{2^{2k}k!\,(k+1)!} x^{2k+2}$$

Then they will satisfy the recurrence relation (4.30) so that the solution assumed in (4.29) will certainly solve the differential equation (4.28).

We need to shorten the series such that the highest degree polynomials appearing are R^6 and Z^6. The series have been truncated in the previous studies at R^4 and Z^4.The full solution for up-down symmetric ψ containing the most general polynomial and polynomial-like solution for ψ_H satisfying equation (4.28) which is consistent with our truncation condition is given by.

$$\psi(x,y) = \frac{x^4}{8} + A\left(\frac{1}{2}x^2\ln x - \frac{x^4}{8}\right) + c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + c_4\psi_4 + c_5 + c_6\psi_6 + c_7\psi_7$$

$$\psi_1 = 1$$

$$\psi_2 = x^2$$

$$\psi_3 = y^2 - x^2\ln x$$

$$\psi_4 = x^4 - 4x^2y^2$$

$$\psi_5 = 2y^4 - 9x^2y^2 + 3x^4\ln x - 12x^2y^2\ln x \quad (4.33)$$

$$\psi_6 = x^6 - 12x^4y^2 + 8x^2y^4$$

$$\psi_7 = 8y^6 - 140x^2y^4 + 75x^4y^2 - 15x^6\ln x + 180x^4y^2\ln x - 120x^2y^4\ln x$$

Equation (4.33) is the exact solution to the G-S equation that explains all the configurations of interest of the up-down symmetry.

Our next assignment is to determine the unknown c_n that appears in equation (4.33).

4.5 The Boundary Constraints

Consider first the case where the plasma surface is smooth. A good option for these properties is to contest the function and its first and second derivative at three test points: the inner equatorial point, the outer equatorial point, and the high point (see Fig. 4.2 for the geometry). While this might appear to need nine free constants (i.e. three conditions at each of the three points), two are jobless because of the up-down symmetry. As it is clear how to specify the function and its first derivative at each test point but the choice for the second derivative is less understandable. In order to specify the second derivatives we make use of a well-known analytic model for a smooth, elongated, "D" shaped cross section, which precisely defines all the configurations of interest. The boundary of this cross-section is given by the following parametric equations

$$x = 1 + \varepsilon \cos(\tau + \alpha \sin \tau)$$

$$y = \varepsilon k \sin(\tau) \qquad (4.34)$$

Where τ the parameter which covers the range $0 \leq \tau \leq 2\pi$. Also, $\varepsilon = {}^{a}/_{R_0}$ is the inverse aspect ratio, $sin\alpha = \delta$ is the triangularity, and κ is the elongation.

The geometrical representation of these three parameters is shown in figure.4.2

The triangularity is limited to the range $\delta \leq \sin(1) \approx 0.841$. The idea is so simple: we will match the curvature of the plasma surface determined by our solution with the curvature of the model surface (4.32) at each test point.

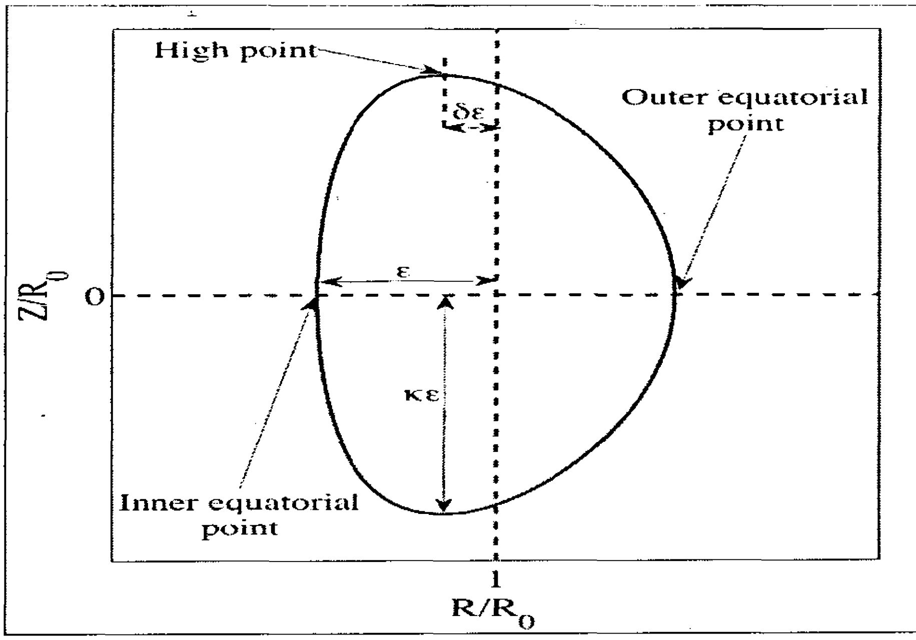


Fig: 4.2. Geometry of the problem and definition of the normalized geometric parameters ε, κ, and z.

Along the surface of constant ψ , we have, by definition,

$$d\psi = \psi_x dx + \psi_y dy = 0 \tag{4.35}$$

First we have to obtained expression for the curvature at each point in terms of the partial derivatives of ψ at these points by using the above equality.

For the inner and outer equatorial points, we can write

$$\frac{d^2x}{dy^2} = -\frac{d}{dy}\left(\frac{\psi_y}{\psi_x}\right) = -\frac{\partial}{\partial y}\left(\frac{\psi_y}{\psi_x}\right) \quad \text{(Since } \frac{dx}{dy} = 0 \text{ at the two points)}$$

$$\frac{d^2x}{dy^2} = -\frac{\psi_{yy}}{\psi_x} \quad \text{(Since } \psi_y = 0 \text{ at the two points)} \tag{4.36}$$

Similarly, at the top point, we have

$$\frac{d^2y}{dx^2} = -\frac{\psi_{xx}}{\psi_y} \tag{4.37}$$

The second step is to calculate $\frac{d^2x}{dy^2}$ and $\frac{d^2y}{dx^2}$ for the model surface (4.34), so that we can compare the curvatures. After some mindless algebra we have from equation (4.34)

$$\frac{d^2x}{dy^2} = -\frac{1}{\varepsilon^2\kappa^2 cos^3\tau}[sin\tau\ sin\ (\tau\ + \alpha sin\tau\tau\ )(2\alpha cos\tau + \ 1) + (1 + \alpha cos\tau\ )^2\ cos\tau\ cos\ \ (\tau\ +\ \alpha sin\tau\ )] \quad (4.38)$$

$$\frac{d^2y}{dx^2} = -\frac{\kappa}{\varepsilon}\frac{sin\tau\ sin(\tau + \alpha sin\tau) + (1 + \alpha cos\tau)^2 cos\tau\ \cos(\tau + \alpha sin\tau)}{(1 + \alpha cos\tau)^3 sin^3(\tau + \alpha sin\tau)}$$

At the three points of interest, these expressions simplify significantly

$$\left[\frac{d^2x}{dy^2}\right]_{\tau=0} = -\frac{(1+\alpha)^2}{\varepsilon\kappa^2} = N_1 \qquad \text{Outer equatorial point}$$

$$\left[\frac{d^2x}{dy^2}\right]_{\tau=\pi} = -\frac{(1-\alpha)^2}{\varepsilon\kappa^2} = N_2 \qquad \text{Inner equatorial point} \qquad (4.39)$$

$$\left[\frac{d^2x}{dy^2}\right]_{\tau=\pi/2} = -\frac{(1+\alpha)^2}{\varepsilon\ cos\alpha^2} = N_3 \qquad \text{High point}$$

To simplify the expressions we named the three different curvatures N_1, N_2 and N_3.

We are now introducing the seven geometric constraints, considering that the free additive constant related to the flux function is selected so that $\psi = 0$ on the plasma surface (this implies that $\psi < 0$ in plasma):

$\psi(1 + \varepsilon, 0) = 0$ Outer equatorial point

$\psi(1 - \varepsilon, 0) = 0$ Inner equatorial point

$\psi(1 - \delta\varepsilon, \kappa\varepsilon) = 0$ High point

$\psi_x(1 - \delta\varepsilon, \kappa\varepsilon) = 0$ High point maximum (4.40)

$\psi_{yy}(1 + \varepsilon, 0) = -N_1\psi_x(1 + \varepsilon, 0)$ Outer equatorial point curvature

$\psi_{yy}(1 - \varepsilon, 0) = -N_2\psi_x(1 - \varepsilon, 0)$ Inner equatorial point curvature

$\psi_{xx}(1 - \delta\varepsilon, \kappa\varepsilon) = -N_3\psi_y(1 - \delta\varepsilon, \kappa\varepsilon)$ High point curvature

For a given value of A the conditions given by equation (4.40) condense to a set of seven linear inhomogeneous algebraic equations for the unknown c_n. This is a trivial numerical problem. We have found that even with only three test points the outer flux surface causing from our analytic solution for ψ is charming and remarkably close to the surface given by equation (4.34) over the entire range of geometric parameters. A similar formulation put on to the circumstances where the plasma surface has a double null divertor X-point. Here, we can imagine that the smooth model surface actually relates to the 95% flux surface. The location of the X-point usually occurs to some extent higher and slightly closer to the inboard side of the plasma. Precisely we assume a 10% shift so that $x_{sep} = 1 - 1.1\delta\varepsilon$ and $y_{sep} = 1.1\kappa\varepsilon$. In terms of the boundary constraints, there is efficiently only one change. At the X-point we can no longer enforce the second derivative curvature constraint but instead need that both the tangential and normal magnetic field dies out. The conditions at the inboard and outboard equatorial points are left unaffected. The end result is that if one tries to find an equilibrium solution where the plasma surface relates to a double null diverter and the 95% surface has an approximate elongation κ and triangularity δ then the constraint conditions determining the c_n are given by

$\psi(1+\varepsilon, 0) = 0$	Outer equatorial point	
$\psi(1-\varepsilon, 0) = 0$	Inner equatorial point	
$\psi(x_{sep}, y_{sep}) = 0$	High point	
$\psi_x(x_{sep}, y_{sep}) = 0$	$B_{normal} = 0$ at the high point	(4.41)
$\psi_y(x_{sep}, y_{sep}) = 0$	$B_{tangential} = 0$ at the high point	
$\psi_{yy}(1+\varepsilon, 0) = -N_1\psi_x(1+\varepsilon, 0)$	Outer equatorial point curvature	
$\psi_{yy}(1-\varepsilon, 0) = -N_2\psi_x(1-\varepsilon, 0)$	Inner equatorial point curvature	

Now our next step is to evaluate the critical figures of merit that describe the plasma equilibrium.

4.6 The Plasma Figures of Merits:

To describe the properties of Solov'ev MHD equilibrium we have four figures of merit. These are described as follows.

Total plasma beta $\beta = \frac{2\mu_o<p>}{B_o^2+\bar{B}_P^2}$

Toroidal plasma beta $\beta_t = \frac{2\mu_o<p>}{B_o^2}$

Poloidal plasma beta $\beta_p = \frac{2\mu_o<p>}{\bar{B}_P^2}$ (4.42)

Kink safety factor $q_* = \frac{\varepsilon B_o}{\bar{B}_o}$

Where B_o is the vacuum toroidal field at R= R_o. And $\bar{B}_p$ is the average poloidal magnetic field on the plasma surface

$$\bar{B}_p = \frac{\oint B_p dl_p}{\oint dl_p} = \frac{\int \mu_o J_\varphi ds_\varphi}{\oint dl_p} = \frac{\mu_o I}{R_o C_p} \quad (4.43)$$

Where C_p is taken as the normalized poloidal circumference of the plasma surface

$$C_p = \frac{1}{R_o}\oint dl_p = 2\int_{1-\varepsilon}^{1+\varepsilon}\left[1+\left(\frac{dx}{dy}\right)^2\right]^{1/2} dx \quad (4.44)$$

Lastly <p> is the volume averaged pressure

$$< p > = \frac{\int p dr}{\int dr} \quad (4.45)$$

Our aim is to derive explicit expression for the figures of merit in terms of the geometric parameters $\varepsilon, \kappa, and\ \delta,$ and ψ, A. For this we require the quantities $p\ and\ F^2 = R^2 B_\varphi^2$ which can be obtained equation (4.25) and using the fact that $\psi = 0$ on the plasma surface.

$$P(x,y) = -\frac{\psi_0^2}{\mu_0 R_o^4}(1-A)\psi$$

$$B_\varphi^2(x,y) = \frac{R_0^2}{R^2}\left[B_o^2 - \frac{2\psi_o^2}{R_0^2}A\psi\right] \tag{4.46}$$

During the evaluation of the figures of merit the normalized quantity ${\psi_o}/{aR_oB_o}$ often appears in the results. So it is more worth to replace this quantity with an equivalent quantity q_* which, after some calculation can be written as

$$\frac{1}{q_*} = \left[\frac{\psi_o}{aR_oB_o}\right]\frac{1}{C}\int\frac{dxdy}{x}\,[A+(1-A)x^2]$$

When we describe the MHD equilibrium there are some natural combinations of the figures of merit that appear which then depend only on the free parameter A and the geometry. This is suitable for determining general scaling relations.

Using this vision the required form of the figures of merit can be written as

$$\beta_p(\varepsilon,\kappa,\delta,A) = -2(1-A)\frac{C_p^2}{V}\left[\int\psi x dxdy\right]\left\{\int\frac{dxdy}{x}[A+(1-A)x^2\right\}^{-2}$$

$$\beta_t = \frac{\varepsilon^2\beta_p}{q_*^2} \tag{4.47}$$

$$\beta = \frac{\varepsilon^2\beta_p}{q_*^2+\varepsilon^2}$$

Where

$$V = \frac{1}{2\pi R_o^3}\int dr = \int xdxdy \tag{4.48}$$

is the normalized plasma volume.

The above set of equations can be solved numerically and applied to any magnetic confinement configuration e.g. spherical tokamak etc. However the numerical solution of these equations is beyond the scope this thesis.

Chapter 5

Summary, Conclusion and Future Suggestions

In the first section of this thesis we discussed properties of spherical tokamak. And also present a brief comparison of spherical tokamak and conventional tokamak with respect to stability and equilibrium. We conclude that the spherical tokamak is more efficient than conventional tokamak because of the greater q value, small aspect ratio and high β limits.

In the second part we reduced the set MHD equations to a single two-dimensional, nonlinear, elliptical partial differential equation usually known as the Grad Shafranov equation then present an analytical solution to the Grad Shafranov equation by using Solov'ev profile. This solution has a number of degree of freedom in the form of free constant. And these constant are to be determined by using the boundary constraints on the plasma surface. This solution can be used to calculate equilibrium in standard tokamak and spherical tokamak, depending upon the boundary constraint. At this point analysis is analytically complete and is ready to be implemented numerically to any toroidal configuration. For future work it is recommended that numerical implementation of this work may be performend.

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