Bayesian Estimation of Exponentiated Pareto Distribution under Different Loss Functions

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2015

1. Bayesian Statistical decision theory

Bayesian Estimation of Exponentiated Pareto Distribution under Different Loss Functions

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A Dissertation

Submitted in the Partial Fulfillment of theRequirements

for the Degree of

MASTER OF SCIENCE INSTATISTICS

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Certificate

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Reg. No. 02-FBAS/MSST/F12

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***MS IN STATISTICS***

We accept the dissertation as conforming to the required standard.

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2015

DEDICATION

- *To my parents, for their endless support, patience and prayers.*

- *To my respectable teachers, due to their guidance,constant source of knowledge and inspiration I have achieved this remarkable position.*

- *To my colleagues, who helped and courage me.*

Forwarding Sheet

The thesis entitled **Bayesian Estimation of Exponentiated Pareto Distribution under Different Loss Functions** submitted by **Khuram Shehzad Khan** (Registration No. 02 - FBAS /MSST /F12) in the partial fulfillment of MS degree in Statistics has been completed under our guidance and supervision. We are satisfied with the quality of his research work and allow him to submit this thesis for further process to graduate with Master of Science in Statistics from Department of Mathematics and Statistics as per International Islamic university, Islamabad rules and regulations.

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Acknowledgements

All the intrinsic worth and praise to almighty Allah, the most merciful, the cordial and beneficent, and the most gracious. First and foremost I would like to thank my Allah and His Prophet (Peace Be Upon Him) that I have succeeded in my task to do the research work in MS. Statistics. It is a truth that a single person cannot accomplish a work without the help of others compassionate and experienced persons

I am very thankful to my **supervisor Prof. Dr. Irshad Ahmad Arshad** for paying me invaluable, unparallel and untiring effort for the preparation and completion of my research work. He helps me a lot to accomplish this task with success May God bless him for all progress and prosperity he deserves (Ameen). I am also thankful to my **co-supervisor Dr. Kamran Abbas** for him friendly cooperation, perfect guidance, encouragement, suggestions, inspection and keen interest in my research work and without his support it would be a very tough job for me to complete my studies successfully.

I extend my heart full gratitude to my beloved and sweet **parents**, **brothers** and **sisters**, the soothing shadow for me in the desert of life. May Allah bless them with the best of His own choice in both the worlds.

At the end, I offer my thanks to all my class fellows and friends specially Raja Rashad Munir, Raja Afif Ahmad, Atique-ur-Rehman Awan, Muhammad Aslam and Masror Naqvi to provide me enjoyable, pleasant knowledgeable and joyful company.

Khuram Shehzad Khan

Declaration

I hereby declare that this thesis, neither as a whole nor a part of it, has been copied out from any source. It is further declared that I have prepared this dissertation entirely on the basis of my personal efforts made under the supervision **Prof. Dr. Irshad Ahmad Arshad** and specially under co-supervision **Dr. Kamran Abbas**. No portion of the work, presented in this dissertation has been submitted in the support of any application for any degree or qualification of this or any other learning institute.

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Contents

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List of Abbreviations

EPD	Exponentiated Pareto Distribution
GPD	Generalized Pareto distribution
MLE	Maximum Likelihood Estimator
MSE	Mean Square Error
SELF	Squared Error Loss Function
GELF	General Entropy Loss Function
PLF	Precautionary Loss Function
LINEX	Linear Exponential
BLGP	Bayesian LINEX Gamma Prior
BLLP	Bayesian LINEX Levy Prior
BLUP	Bayesian LINEX Uniform Prior
BGEGP	Bayesian General Entropy Gamma Prior
BGELP	Bayesian General Entropy Levy Prior
BGEUP	Bayesian General Entropy Uniform Prior
BSEGP	Bayesian Squared Error Gamma Prior
BSELP	Bayesian Squared Error Levy Prior
BSEUP	Bayesian Squared Error Uniform Prior
BPGF	Bayesian Precautionary Gamma Prior
BPLP	Bayesian Precautionary Levy Prior
BPUP	Bayesian Precautionary Uniform Prior
BFGS	Broyden Fletcher Goldfarb Shanno
Gba	Gross Building Area
CI	Credible Interval

Abstract

The present research develops Bayesian and non-Bayesian estimators for the parameters of Exponentiated Pareto distribution (EPD) based on complete samples. The Bayesian estimators of EPD cannot be obtained in closed form. For this purpose, it is recommended Lindley's approximation to compute the approximate Bayesian estimators using Gamma and Levy priors as informative priors. As for the case of non-informative priors, we utilize uniform prior. This is done with respect to symmetric loss function (Squared Error) and asymmetric loss functions (linear exponential (LINEX), General Entropy and Precautionary). The Bayesian estimators are then compared with their maximum likelihood estimators (MLEs) for complete samples using simulation study.

As for the case of complete samples, it is concluded that Bayesian estimators perform better than MLEs in terms of mean squared error (MSEs). However, MSEs of Bayesian estimators are notably smaller than other estimators. Also for large sample size the Bayesian estimators and MLEs become closer in terms of MSEs. Further, the performances of Bayesian and MLEs become better when the sample size increases. Generally, the Bayesian estimators under different loss functions are closed to the true values of parameters of EPD by increasing sample size. A Monte Carlo simulation study is carried out to compare the performances of different methods. The real data set is also provided to illustrate the results for complete samples data.

For censored samples, Type-II censoring scheme is used. Under this censoring scheme, different sample sizes and percentages of failures are fixed for calculating the MLEs and their 90% and 95% confidence intervals (CIs) for both parameters of EPD.

From the results of Type-II censoring, it is concluded that MSEs of MLEs for both shape parameters of EPD decreases due to increasing sample size. It is also observed that the length of CIs becomes narrow by increasing simple size for fixed level of Type-II censoring. This indicated that the MLEs are consistent and approaches true parameter values. Further, it is seen that length of Cls becomes narrow by increasing percentages of failures (r) and sample sizes (n) this indicated that the accuracy of the results. Furthermore, results of MLEs are calculated through simulation method. A real data set is also used for illustration to compare the MLEs for censored samples.

Keywords: Exponentiated Pareto distribution, Maximum likelihood estimators, Bayesian estimators, Lindley's approximation, Type-II censoring, LINEX loss function, Squared Error loss function, General Entropy loss function, Precautionary loss function, Type-II censoring.

Chapter 1

Introduction

The Exponentiated Pareto distribution (EPD) was introduced by Gupta *et al.* (1998) in the same settings that the Generalized Exponential distribution extends the Exponential distribution. They showed that the EPD can be used quite effectively in analyzing many lifetime data. EPD can be defined by raising the cumulative distribution function of a Pareto distribution to a positive power and used in life testing experiment, when units are lost from that experiment while they are still alive e.g., testing of experiment of cancer patients. Cummulative distribution function (CDF) of EPD is defined as:

$$F(x, \alpha, \theta) = [1 - (1 + x_i)^{-\alpha}]^{\theta}, \; x > 0, \;\; \alpha, \theta > 0, \tag{1}$$

where α and θ are both shape parameters. The probability density function (PDF) is:

$$f(x, \alpha, \theta) = \alpha\, \theta\, [1 - (1 + x_i)^{-\alpha}]^{\theta-1}\, (1 + x_i)^{-(\alpha+1)}, \; x > 0, \;\; \alpha, \theta > 0. \tag{2}$$

Moreover, survival function of EPD can be expressed as:

$$S(x) = 1 - F(x, \alpha, \theta) = 1 - [1 - (1 + x_i)^{-\alpha}]^{\theta}, \; x > 0, \;\; \alpha, \; \theta > 0, \tag{3}$$

when $\theta = 1$ then equation (2) becomes,

$$f(x, \alpha, 1) = \alpha\ (1 + x_i)^{-(\alpha+1)}, \ x > 0, \ \ \alpha > 0, \tag{4}$$

which is Standard Pareto distribution of second kind. And a hazard function is:

$$h(x, \alpha, \theta) = \frac{\alpha\ \theta\ [1 - (1 + x_i)^{-\alpha}]^{(\theta-1)}\ (1 + x_i)^{-(\alpha+1)}}{1 - [1 - (1 + x_i)^{-\alpha}]^{\theta}}. \tag{5}$$

EPD is widely used for skewed data, modeling and reliability theory. It is also used for analyzing the skewed data and a model for the distribution of income. Further, it has also played a vital role for investigations of population sizes, accurance of natural resources, insurance risks and business failures. Applications of the EPD in various fields are given in Green *et al.* (1994) and Zaharim *et al.* (2008).

Several estimation methods have been proposed to estimate the parameters of life time models. Selection of suitable method to estimate the parameters of one life time model might not be necessarily as efficient and in predicting for another model is given by Al-Baidhani and Sinclair (1987). The method of maximum likelihood is widely popular in terms of theoretical prospective and the least squares method is easier then other methods computationally.

Afify (2010) obtained Bayesian and classical estimators for two parameters of EPD, when samples are available from complete samples, Type-I and Type-II censoring scheme. Bayesian estimators were developed under SELF as well as LINEX loss function by taking non-informative type of priors. Furthermore, comparisons are made for the performance of estimators on the basis of their simulated risks obtained under SELF as well as LINEX loss function. In addition, different shapes of EPD can be observed using different values of α and θ from the following figures.

Figure 1(a): Graph of Cumulative distribution function with different values of parameters of EPD.

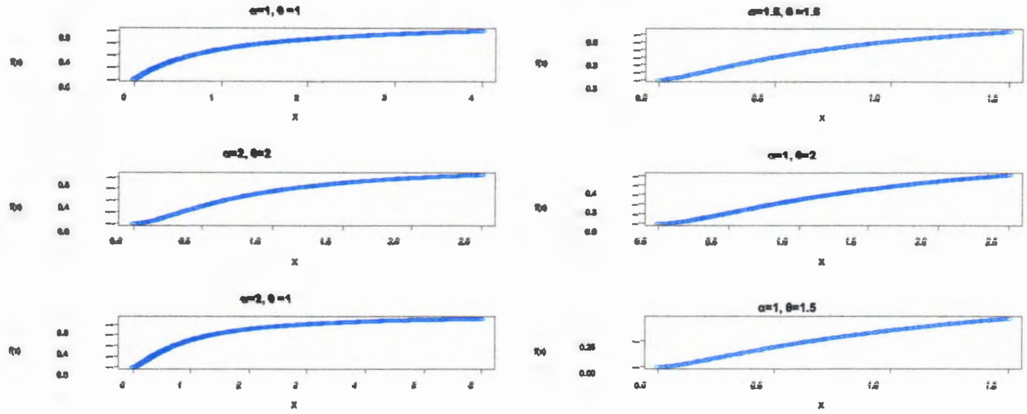


Figure 1(b): Graph of Probability density function with different values of parameters of EPD.

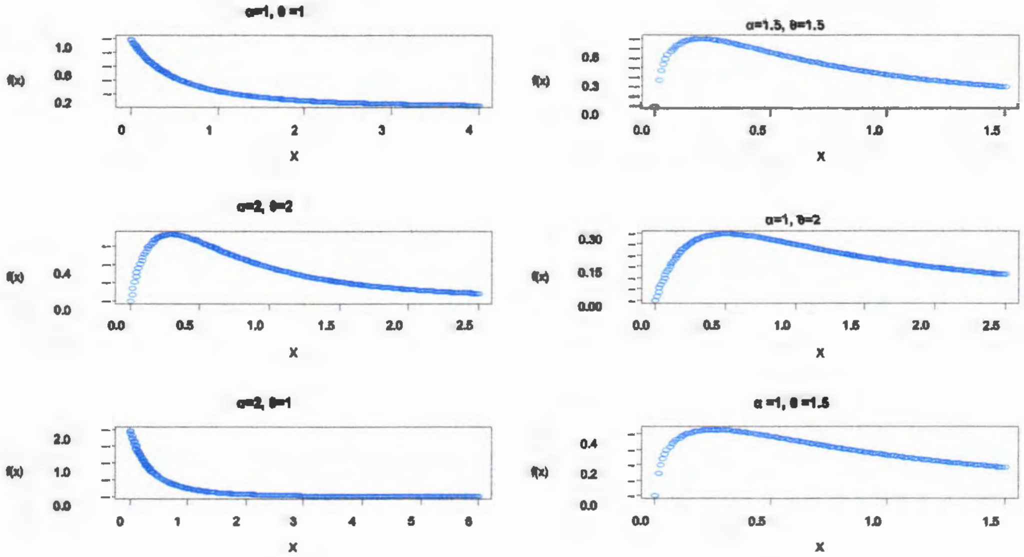


Figure 1(c): Graph of Survival function with different values of parameters of EPD.

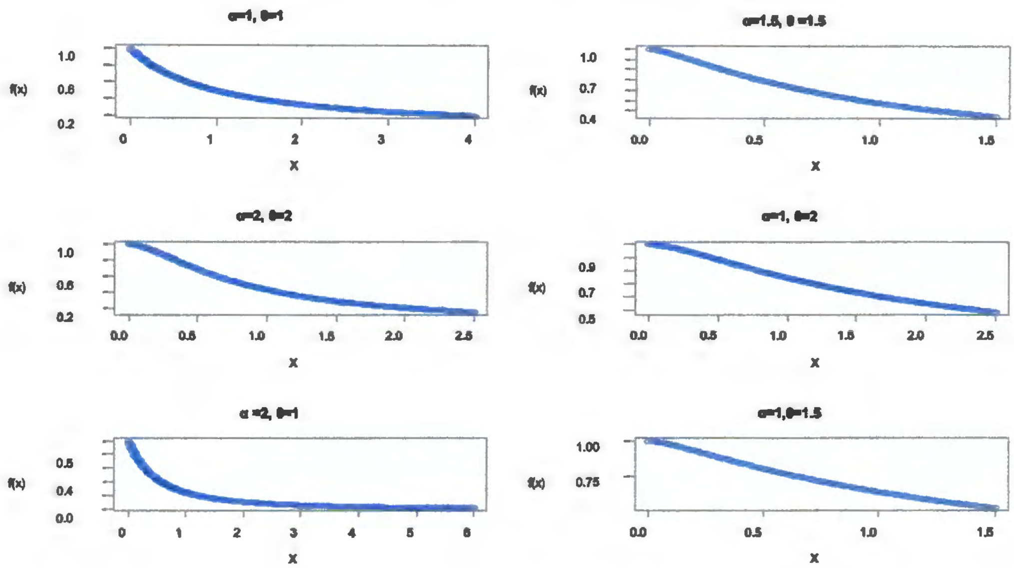
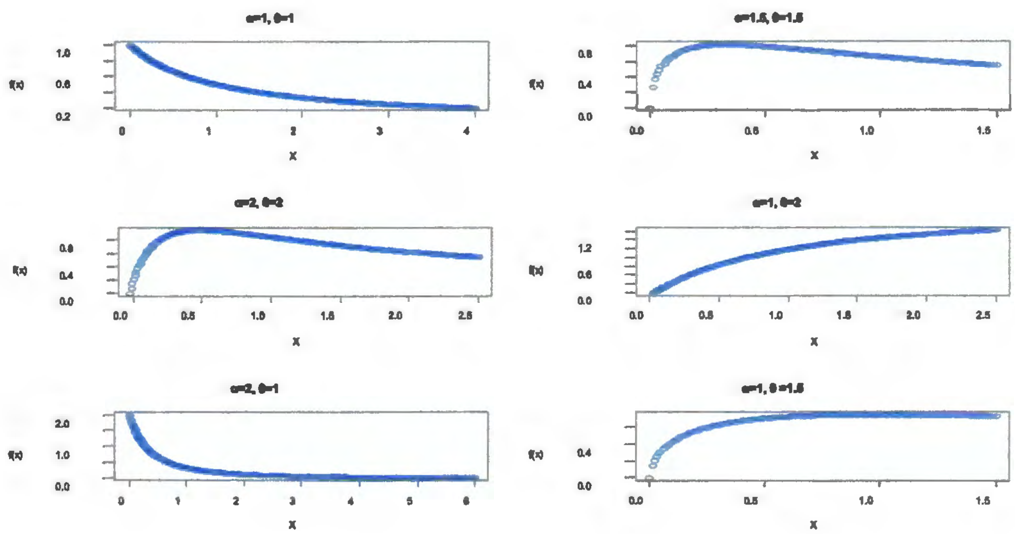


Figure 1(d): Graph of Hazard function with different values of parameters of EPD.



Abstract

The present research develops Bayesian and non-Bayesian estimators for the parameters of Exponentiated Pareto distribution (EPD) based on complete samples. The Bayesian estimators of EPD cannot be obtained in closed form. For this purpose, it is recommended Lindley's approximation to compute the approximate Bayesian estimators using Gamma and Levy priors as informative priors. As for the case of non-informative priors, we utilize uniform prior. This is done with respect to symmetric loss function (Squared Error) and asymmetric loss functions (linear exponential (LINEX), General Entropy and Precautionary). The Bayesian estimators are then compared with their maximum likelihood estimators (MLEs) for complete samples using simulation study.

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The shape parameters of distribution determines the shape of distribution function. Shape parameter defines how our data is distributed but does not affect the location and scale of our distribution. Figures 1(a)-1(d) shows the effect of the different values of the shape parameters of EPD and it is observed from figures 1(a)-1(d) that shapes of graph of CDF, PDF, Survival function and Hazard function are mostly same when $\alpha = 1, 1.5$ and 2 and $\theta = 1.5$ and 2 but different bend in shapes when $\alpha = 1$ and 2 and $\theta = 1.5$. Hence, it is concluded that when there is change in values of α then this highly effect on shapes of graphs of CDF, PDF, Survival and Hazard function when α and θ are 1, 1.5 and 2.

Hossain and Zimmer (2003) studied about the comparison of estimation method for complete and censored samples based on Weibull distribution. Similarly, Hossain and Howlader (1996) gave comparison about least square estimators and the method of MLEs for complete samples. Moreover, The estimators of the parameters of EPD were obtained by Shawky and Abu-Zinadah (2009) under different estimation methods for complete sample case.

A great deal of research has been done on estimating the parameters of the EPD using both classical and Bayesian techniques. Gupta *et al.* (1998) showed that the EPD can be used quite effectively in analyzing many lifetime data. Moreover, Pandey and Rao (2009) also studied about the shape parameters of the Generalized Pareto distribution (GPD) by using quasi, inverted gamma and uniform prior distributions using LINEX loss function, PLF and GELF.

As expected the Bayesian estimators of the unknown parameters cannot be obtained in closed forms so, we suggest Lindley's approximation to compute the approximate Bayesian estimators of EPD. Hassan and Basheikh (2012) used Bayesian and non-Bayesian estimation of reliability of S-out-of-K system. They assumed both stress and strength had an EPD with common and known shape parameter and studied about the Bayesian estimation under SELF and LINEX loss function by using Lindley's

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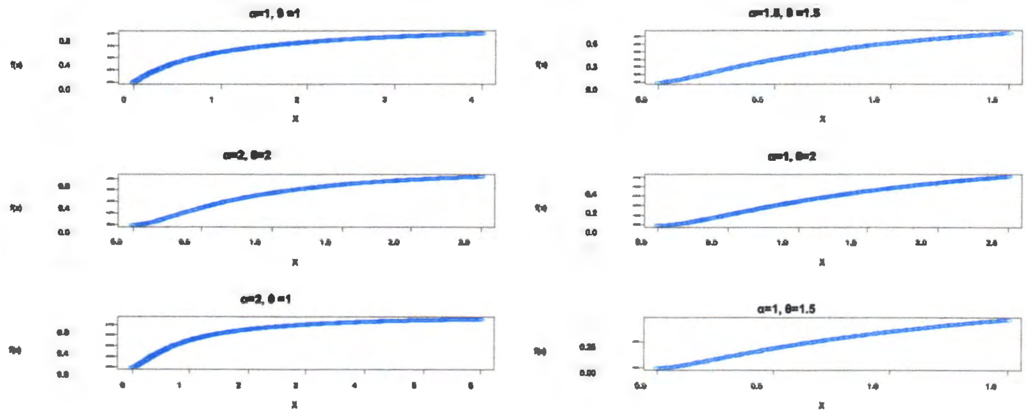

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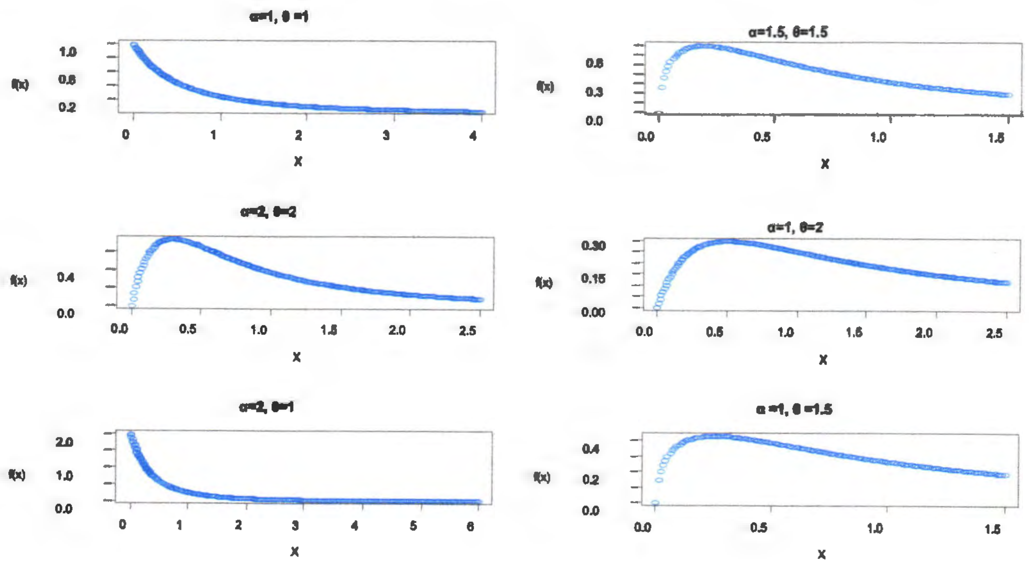


Figure 1(c): Graph of Survival function with different values of parameters of EPD.

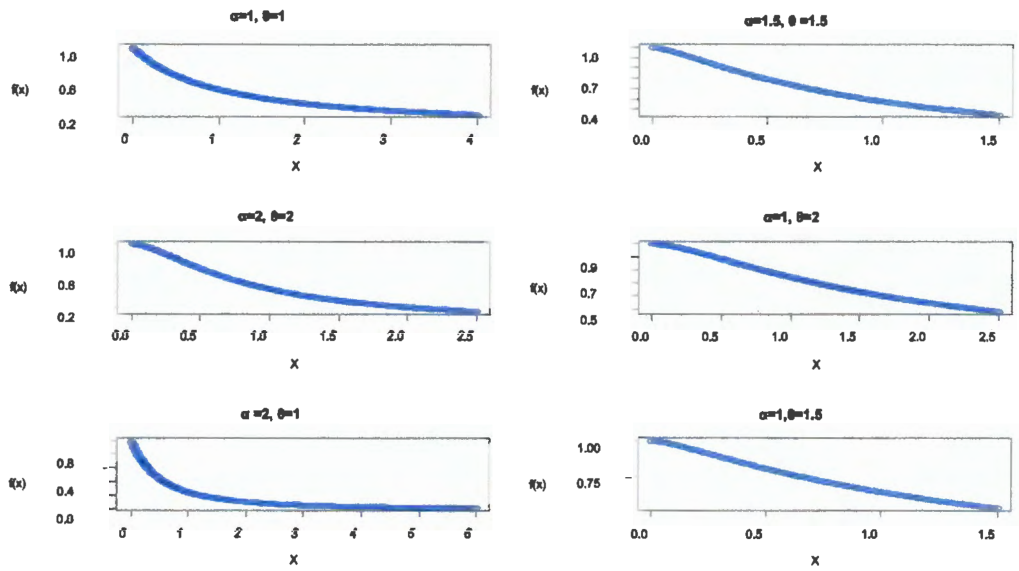
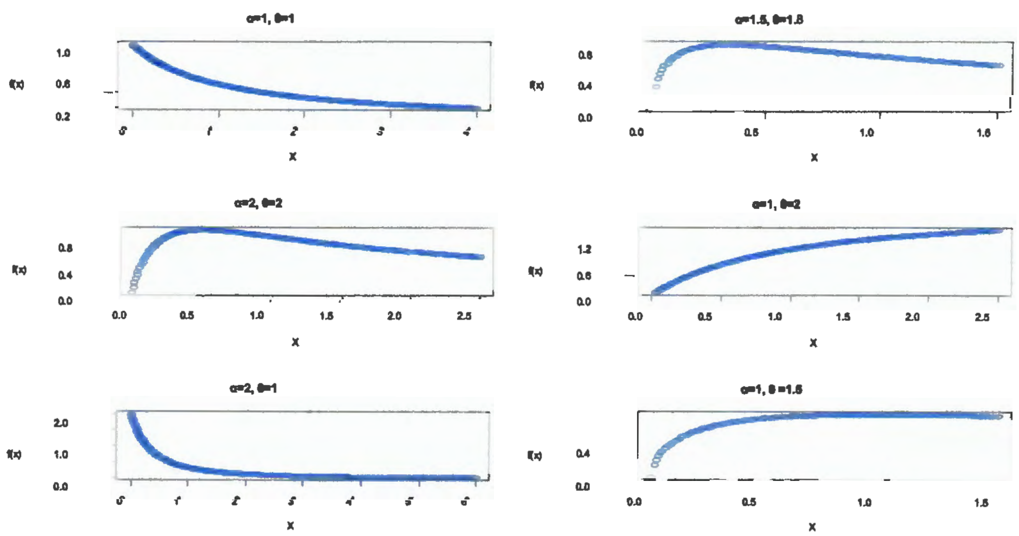


Figure 1(d): Graph of Hazard function with different values of parameters of EPD.



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Chapter 2

Review of Literature

Lot of research work is already being conducted relevant to this objective of research on EPD with different loss functions. The most related review of literature is as under:

Singh *et al.* (2013) proposed MLEs and Bayesian estimators of parameters of EPD under GELF and SELF for Progressive Type-II censored data with binomial removals. The method of MLEs and corresponding Bayesian estimators were compared in terms of their risks based on simulated samples from EPD.

Shams (2013) introduced Kumaraswamy Generalized EPD and analyzed that this distribution could have a decreasing and upside-down bathtub failure rate function depending on the value of its parameters. He included some special sub-model like EPD and its original form. Some structural properties of the proposed distribution were studied including explicit expressions for the moments. Further, he provided the density functions of the order Statistics and obtained their moments. Also MLEs was used for estimating the model parameters. Moreover, the observed Fisher information matrix was also found for above mentioned distribution. The real data was provided to illustrate the theoretical results in the complete data.

AL-Hussaini and Hussein (2011) obtained Bayesian predictive probability density function when the underlying population distribution was Exponentiated by taking subjective prior. The corresponding predictive survival function was obtained and then used in constructing predictive interval by taking one and two sample schemes.

Ghafoori *et al.* (2011) considered some well known and useful models for obtaining prediction bounds as well as Bayes predictive estimations under SELF for the S-thorder statistic in a future random sample drawn from the parent population. independently and with an arbitrary progressive censoring scheme.

Ali *et al.* (2010) obtained the method of MLEs of the threshold parameter β with known parameters α and c for the EPD. Further they obtained the MLEs of the tail probability of the EPD. Finally, they considered MLEs of reliability in two independent EPD.

Afify (2010) discussed Bayesian estimators and MLEs for two parameters of EPD when samples were available from complete samples, Type-I and Type-II censored samples. Bayesian estimators were developed under SELF as well as under LINEX loss function by using non-informative type of priors for the parameters of EPD. The performance of the proposed estimators was compared on the basis of their simulated risks obtained under SELF as well as under LINEX loss function.

Abu-Zinadah (2010) applied Bayesian and the methods of MLEs method for estimating the parameters of EPD. He used reliability and hazard functions of the model under complete samples and Type-II censored samples. He also used Lindley's approximation for obtaining the Bayesian estimators under SELF and LINEX loss function.

Chapter 3

Materials and Methods

In this chapter, we derived MLEs, observed Fisher information matrix and different Bayesian estimation methods using Lindley's approximation taking informative and non-informative types of priors for complete samples. As for the case of censored samples, the MLEs and CIs for the parameters under Type-II censored are derived.

3.1 Classical Estimation of parameters

In this estimation approach, parameters are considered to be fixed and unknown quantity. Hassan and Basheikh (2012) compared the MLEs with other classical estimation methods and concluded that MLEs is the best method among others. Ali *et al.* (2010) also concluded that MLEs is the best method as compared to other classical estimation methods.

3.1.1 MLEs for complete samples

Suppose that $x_1, x_2, x_3, \ldots, x_n$ be the set of n random lifetimes whose lifetimes have EPD with parameters α and θ. The likelihood function of equation (2) is:

$$L(\alpha,\theta) = \alpha^n\,\theta^n \prod_{i=1}^{n}\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1}\prod_{i=1}^{n}(1+x_i)^{-(\alpha+1)}. \tag{6}$$

The log-likelihood function is:

$$\log L(\alpha,\theta) = n\log(\alpha)+n\log(\theta)+(\theta-1)\sum_{i=1}^{n}\log(1-(1+x_i)^{-\alpha})-(\alpha+1)\sum_{i=1}^{n}\log(1+x_i), \tag{7}$$

where

$$\frac{\partial \log L(\alpha,\theta)}{\partial\alpha} = \frac{n}{\alpha}+(\theta-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}\log(1+x_i)}{1-(1+x_i)^{-\alpha}} - \sum_{i=1}^{n}\log(1+x_i), \tag{8}$$

$$\frac{\partial \log L(\alpha,\theta)}{\partial\theta} = \frac{n}{\theta}+\sum_{i=1}^{n}\log(1-(1+x_i)^{-\alpha}). \tag{9}$$

The MLEs of α and θ say $\hat{\alpha}$ and $\hat{\theta}$ can be obtained as the solution of

$$\frac{n}{\alpha}+(\theta-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}\log(1+x_i)}{1-(1+x_i)^{-\alpha}} - \sum_{i=1}^{n}\log(1+x_i) = 0, \tag{10}$$

and

$$\frac{n}{\theta}+\sum_{i=1}^{n}\log(1-(1+x_i)^{-\alpha}) = 0. \tag{11}$$

As normal equations (10) and (11) cannot be expressed in closed form, therefore for this purpose we use Newton-Rophson method to compute MLEs. The Observed Fisher information matrix is obtained by taking the second and partial derivatives of (8) and (9) with respect to α and θ. Therefore, the observed Fisher information matrix may be written as:

$$I_{(\alpha,\theta)} = \begin{pmatrix} -\frac{\partial^2 \log L(\alpha,\theta)}{\partial\alpha^2} & -\frac{\partial^2 \log L(\alpha,\theta)}{\partial\alpha\,\partial\theta} \\ -\frac{\partial^2 \log L(\alpha,\theta)}{\partial\theta\,\partial\alpha} & -\frac{\partial^2 \log L(\alpha,\theta)}{\partial\theta^2} \end{pmatrix},$$

where

$$\frac{\partial^2 \log L(\alpha,\theta)}{\partial\alpha^2} = -\frac{n}{\alpha^2} - (\theta-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}(\log(1+x_i))^2}{(1-(1+x_i)^{-\alpha})^2}, \tag{12}$$

$$\frac{\partial^2 \log L(\alpha,\theta)}{\partial\theta^2} = -\frac{n}{\theta^2}, \tag{13}$$

$$\frac{\partial^2 \log L(\alpha,\theta)}{\partial\alpha\,\partial\theta} = \frac{\partial^2 \log L(\alpha,\theta)}{\partial\theta\,\partial\alpha} = \sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}(\log(1+x_i))}{1-(1+x_i)^{-\alpha}}. \tag{14}$$

3.2 Bayesian Estimation

The fundamental difference between Bayesian and classical estimation is that the parameters are considered random variables whereas these are fixed and unknown quantities in classical estimation.

In this section, we developed the Bayesian estimators of EPD based on complete and censored samples under different loss functions by taking informative and non-informative priors. In order to obtain the Bayesian estimators, we assume that the parameters α and θ are random variables and independently distributed. For Bayesian estimations we need prior distributions for the parameters. For obtaining the Bayesian estimators, we mainly used LINEX loss function, GELF, SELF and PLF by taking Gamma, Levy and Uniform priors.

3.2.1 Prior distributions

The prior knowledge of parameter before the data collected is changed into quantitative form. Such a quantitative form is called the prior distribution. The parameters of prior distribution are acknowledged as hyper parameters generally represented by small English alphabets. The selection of prior distribution is an important part of Bayesian statistical inference because the selection of an inappropriate prior may leads to the misleading results. The prior distribution is multiplied by likelihood function then normalized to obtain the posterior distribution of given parameters which is elementary tool in Bayesian statistical inference to compute the Bayesian estimators.

3.2.2 Types of prior distributions

The prior distribution have commonly classified into two types: informative prior (IP) and non-informative priors (NIP). For the selection of informative priors we considered Gamma and Levy priors and for non-informative type of priors we take Uniform prior.

3.2.3 Gamma prior (GP)

A random variable X is said to have a gamma distribution with the shape parameter $\alpha > 0$ and the scale parameter $\theta > 0$, if X has the following probability density function

$$f(x, \alpha, \theta) = \frac{\theta^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\theta\, x}, \quad \alpha,\, \theta > 0, \tag{15}$$

and it is denoted as X $\sim$ gamma (α, θ).
Let α and θ are the independent GP distributions as Gamma(a_1, b_1) and Gamma(a_2, b_2), then joint prior distribution for α and θ is:

$$\pi(\alpha, \theta) \propto \alpha^{a_1 - 1}\, e^{-b_1\, \alpha}\, \theta^{a_2 - 1}\, e^{-b_2\, \theta}, \tag{16}$$

where a_1, a_2, b_1 and b_2 are known and non-negative. Also log function and derivatives of log function with respect to α and θ are given in Appendix A.

3.2.4 Levy prior (LP)

The Levy distribution is defines as:

$$f(\alpha, \theta) = \sqrt{\frac{\alpha}{2\,\pi}}\;\theta^{-3/2}\,e^{-\alpha/2\,\theta}, \quad \alpha,\,\theta > 0, \tag{17}$$

joint prior distribution for α and θ is:

$$\pi(\alpha, \theta) \propto \alpha^{1/2}\,\theta^{-3/2}\,e^{-\alpha/2\,\theta}, \tag{18}$$

and the log function and derivatives of log function with respect to α and θ are defined in Appendix A.

3.2.5 Uniform prior (UP)

According to Laplace (1812) the UP is the most widely used non-informative prior. Bayes (1763) give the knowledge of UP. In the case of restricted parametric range, UP are easy to recognize. For the parameter θ, with range 0 to 1, the UP is:

$$\pi(\theta) = 1, \quad 0 < \theta \leq 1. \tag{19}$$

For the parameter range 0 to ∞ it is:

$$\pi(\theta) \propto 1, \quad 0 < \theta \leq \infty. \tag{20}$$

And for the parameter range $-\infty$ to ∞, the UP is:

$$\pi(\theta) \propto 1, \quad -\infty < \theta \leq \infty, \tag{21}$$

here (20) and (21) are also known as improper priors and defined in Appendix A.

3.2.6 Posterior distribution

We can calculate posterior distribution by Density Kernal and Posterior method. In Density Kernal method we just take product of likelihood function of EPD and marginal distribution of α and θ respectively which is taken from GP, LP and UP. General form of this method is as under

$$P(\alpha \mid x) \propto L(\alpha, \theta) \times f(\alpha), \tag{22}$$

$$P(\theta \mid x) \propto L(\alpha, \theta) \times f(\theta). \tag{23}$$

Posterior method can be defined as it is the ratio between product of marginal distribution of a parameter which is taken from any prior distribution and likelihood function of any distribution and integral of product of prior distribution and likelihood function of any distribution as:

$$P(\alpha \mid x) = \frac{P(\alpha)L(x,\alpha)}{\int_0^\infty P(\alpha)L(x,\alpha)} \tag{24}$$

$$P(\theta \mid x) = \frac{P(\theta)L(x,\theta)}{\int_0^\infty P(\theta)L(x,\theta)} \tag{25}$$

3.2.7 Lindley's approximation

Lindley's approximation firstly familiarized by Lindley's (1980) for computing the approximate Bayes estimators. As expected the Bayes estimates of the unknown parameters cannot be obtained in nice closed forms as:

$$\hat{g}(\alpha,\theta) = \frac{\int_0^\infty \int_0^\infty g(\alpha,\theta)\ \log(\alpha,\theta) \times \pi_2(\theta) \times \pi_1(\alpha)\, \partial\alpha\, \partial\theta}{\int_0^\infty \int_0^\infty \log(\alpha,\theta) \times \pi_2(\theta) \times \pi_1(\alpha)\, \partial\alpha\, \partial\theta}, \tag{26}$$

so we suggest Lindley's approximations to compute the approximate Bayes estimates of the unknown parameters of EPD. The general form of this procedure is:

$$\hat{g} = \hat{g}(\hat{\alpha}, \hat{\theta}) + \frac{1}{2}\left[\sum_{i=1}^{2}\sum_{j=1}^{2} v_{ij}\tau_{ij} + L_{30}B_{12} + L_{03}B_{21} + L_{21}C_{12} + L_{12}C_{21}\right] + W_1 A_{12} + W_2 A_{21},$$

where detail of all terms and explicit expressions of this procedure are provided in the Appendix B.

3.2.8 LINEX loss function

The LINEX loss function was firstly introduced by Varian (1975) and this loss function is expressed as:

$$L(\triangle) \propto \exp(k\triangle) + k\triangle - 1 \,,\; k \neq 0, \tag{27}$$

where $\triangle = \hat{\theta} - \theta$ is scalar estimation. Under this loss function the Bayes estimator of $\hat{\theta}$ is given as:

$$\hat{\theta}_{LINEX} = -\frac{1}{k}\log\left[E_{\theta}(e^{-k\,\theta})\right]. \tag{28}$$

The sign and magnitude of shape parameter k reproduce the direction and degree of asymmetry respectively. If $k > 0$ the overestimation is more serious than underestimation. If $k \cong 0$ then LINEX loss is approximately equal to SELF and therefore almost symmetric. We will assume that α and θ are independently distributed.

Bayesian estimators of α and θ under LINEX loss function using GP and LP respectively are:

$$\hat{\alpha}_{BLGP} = -\frac{1}{k}\log\Bigg[e^{-k\,\hat{\alpha}} + \frac{1}{2}\Bigg\{(k^2\,e^{-k\,\hat{\alpha}})\,\tau_{11} - \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})\,(\log(1+x_i))^3\,(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{11}^2 - \left(\frac{2\,n}{\hat{\theta}^3}\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{21}\tau_{22} + \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\,(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(3\,k\,e^{-k\,\hat{\alpha}})\,\tau_{11}\tau_{12}\Bigg\} - \left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{11} - \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{12}\Bigg], \tag{29}$$

$$\hat{\theta}_{BLGP} = -\frac{1}{k}\log\Bigg[e^{-k\,\hat{\theta}} + \frac{1}{2}\Bigg\{(k^2\,e^{-k\,\hat{\theta}})\,\tau_{22} - \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})\,(\log(1+x_i))^3\,(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\,e^{-k\,\hat{\theta}}) \times\,\tau_{11}\,\tau_{12} - \left(\frac{2\,n}{\hat{\theta}^3}\right)(k\,e^{-k\,\hat{\theta}})\,\tau_{22}^2 + \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\,(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(k\,e^{-k\,\hat{\theta}}) \times\,(\tau_{11}\,\tau_{22} + 2\,\tau_{12}^2)\Bigg\} - \left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)(k\,e^{-k\,\hat{\theta}})\,\tau_{21} - \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)(k\,e^{-k\,\hat{\theta}})\,\tau_{22}\Bigg]. \tag{30}$$

$$\hat{\alpha}_{BLLP} = -\frac{1}{k}\log\Bigg[e^{-k\,\hat{\alpha}} + \frac{1}{2}\Bigg\{(k^2\,e^{-k\,\hat{\alpha}})\,\tau_{11} - \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{11}^2 - \left(\frac{2\,n}{\hat{\theta}^3}\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{21}\,\tau_{22} + \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(3\,k\,e^{-k\,\hat{\alpha}})\,\tau_{11} \times\,\tau_{12}\Bigg\} - \frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\,\hat{\theta}}\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{11} - \frac{1}{2}\left(\frac{\hat{\alpha}-3\,\hat{\theta}}{\hat{\theta}^2}\right)(k\,e^{-k\,\hat{\alpha}})\,\tau_{12}\Bigg],$$

(31)

$$
\begin{aligned}
\hat{\theta}_{BLLP} = -\frac{1}{k}\log\Bigg[e^{-k\,\hat{\theta}} + \frac{1}{2}\Bigg\{&(k^2\, e^{-k\,\hat{\theta}})\;\tau_{22} \\
&- \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\, e^{-k\,\hat{\theta}})\;\tau_{11}\;\tau_{12} \\
&- \left(\frac{2\,n}{\hat{\theta}^3}\right)(k\, e^{-k\,\hat{\theta}})\;\tau_{22}^2 + \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(k\, e^{-k\,\hat{\theta}})\Bigg(\tau_{11}\;\tau_{22} \\
&+ 2\;\tau_{12}^2\Bigg)\Bigg\} - \frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\;\hat{\theta}}\right)(k\, e^{-k\,\hat{\theta}})\;\tau_{21} - \frac{1}{2}\left(\frac{\hat{\alpha}-3\,\hat{\theta}}{\hat{\theta}^2}\right)(k\, e^{-k\,\hat{\theta}})\;\tau_{22}\Bigg].
\end{aligned}
\tag{32}
$$

Bayesian estimators of α and θ under LINEX loss function using UP are:

$$
\begin{aligned}
\hat{\alpha}_{BLUP} = -\frac{1}{k}\log\Bigg[e^{-k\,\hat{\alpha}} + \frac{1}{2}\Bigg\{&(k^2 e^{-k\,\hat{\alpha}})\;\tau_{11} \\
&- \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\, e^{-k\,\hat{\alpha}})\;\tau_{11}^2 \\
&- \left(\frac{2\,n}{\hat{\theta}^3}\right)(k\, e^{-k\,\hat{\alpha}})\;\tau_{21}\;\tau_{22} + \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(3\;k\, e^{-k\,\hat{\alpha}})\;\tau_{11}\;\tau_{12}\Bigg\}\Bigg],
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\hat{\theta}_{BLUP} = -\frac{1}{k}\log\Bigg[e^{-k\,\hat{\theta}} + \frac{1}{2}\Bigg\{&(k^2\, e^{-k\,\hat{\theta}})\;\tau_{22} \\
&- \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\, e^{-k\,\hat{\theta}})\;\tau_{11}\;\tau_{12} \\
&- \left(\frac{2\,n}{\hat{\theta}^3}\right)(k\, e^{-k\,\hat{\theta}})\;\tau_{22}^2 + \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}\;(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(k\, e^{-k\,\hat{\theta}}) \\
&\times\left(\tau_{11}\tau_{22} + 2\;\tau_{12}^2\right)\Bigg\}\Bigg].
\end{aligned}
\tag{34}
$$

3.2.9 General Entropy Loss Function (GELF)

This loss function clearly estimates the natural parameter which is the canonical form of the exponential family. The Calabria and Pulcini (1996) defined GELF as:

$$\ell\left(\theta, \hat{\theta}\right) = E\left(\left(\frac{\hat{\theta}}{\theta}\right)^{k} - k\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right). \tag{35}$$

The constant involved in (35), is its shape parameter. It reflects disappearance from symmetry. When k > 0, it considers positive error to be more serious than negative error and converse for k < 0.

The Bayes estimator $\hat{\theta}$ of θ under GELF is given by

$$\hat{\theta}_{BG} = \left[E_{\theta}(\theta)^{-k}\right]^{-\frac{1}{k}}. \tag{36}$$

provided $E_{\theta}(\theta)^{-k}$ exit and is finite. Bayesian estimators of α and θ under GELF using GP are:

$$\begin{aligned}
\hat{\alpha}_{BGEGP} = & \left[\hat{\alpha}^{-k} + \frac{1}{2}\left\{k(k+1)\,\hat{\alpha}^{-(k+2)}\,\tau_{11} - \left(\frac{2\,n}{\hat{\alpha}^{3}} + (\hat{\theta}-1)\right.\right.\right.\\
& \times \left.\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^{3}(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^{3}}\right)(k\,\hat{\alpha}^{-(k+1)})\,\tau_{11}^{2} - \left(\frac{2\,n}{\hat{\theta}^{3}}\right)\\
& \times (k\,\hat{\alpha}^{-(k+1)})\,\tau_{21}\,\tau_{22} - \left.\left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^{2}}{(1-(1+x_i)^{-\hat{\alpha}})^{2}}\right)(3\,k\,\hat{\alpha}^{-(k+1)})\,\tau_{11}\,\tau_{12}\right\}\\
& \left.-\left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)(k\,\hat{\alpha}^{-(k+1)})\,\tau_{11} - \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)(k\,\hat{\alpha}^{-(k+1)})\,\tau_{12}\right]^{-\frac{1}{k}},
\end{aligned} \tag{37}$$

Bayesian estimators of α and θ under GELF using UP are:

$$\hat{\alpha}_{BGEUP} = \left[\hat{\alpha}^{-k} + \frac{1}{2}\left\{k(k+1)\,\hat{\alpha}^{-(k+2)}\,\tau_{11} - \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\right.\right.\right.$$
$$\left.\times \sum_{i=1}^{n} \frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\,\hat{\alpha}^{-(k+1)})\,\tau_{11}^2 - \left(\frac{2\,n}{\hat{\theta}^3}\right)$$
$$\left.\left.\times\,(k\,\hat{\alpha}^{-(k+1)})\tau_{21}\tau_{22} + \left(\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(3\,k\,\hat{\alpha}^{-(k+1)})\,\tau_{11}\,\tau_{12}\right\}\right]^{-\frac{1}{k}}, \tag{41}$$

$$\hat{\theta}_{BGEUP} = \left[\hat{\theta}^{-k} + \frac{1}{2}\left\{k(k+1)\,\hat{\theta}^{-(k+2)}\,\tau_{22} - \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\right.\right.\right.$$
$$\left.\times \sum_{i=1}^{n} \frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)(k\,\hat{\theta}^{-(k+1)})\,\tau_{11}\,\tau_{12} - \left(\frac{2\,n}{\hat{\theta}^3}\right)$$
$$\times\,(k\,\hat{\theta}^{-(k+1)})\,\tau_{22}^2 + \left(\sum_{i=1}^{n} \frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(k\,\hat{\theta}^{-(k+1)})(\tau_{11}\,\tau_{22}$$
$$\left.\left. +\,2\,\tau_{12}^2)\right\}\right]^{-\frac{1}{k}}. \tag{42}$$

3.2.10 Squared Error Loss Function (SELF)

This loss function was proposed by Legendre (1805) to develop least squares theory. The most common loss function used for Bayesian estimation is SELF also known as quadratic loss function. The squared error loss denotes the error in using to estimate any parameter. This loss function is just the squared value of the root mean square error and general form of this loss function is mostly in the form of quadratic as:

$$L(\alpha,\theta) = \left(\frac{\alpha-\theta}{\alpha}\right)^2 \tag{43}$$

General form of this loss function is:

$\theta = E_q\,(\alpha|x)$.

Bayesian estimators of α and θ under SELF using GP are:

$$\hat{\alpha}_{BSEGP} = \Bigg[\hat{\alpha} + \frac{1}{2}\Bigg\{\left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)$$
$$\times\,\tau_{11}^2 + \left(\frac{2\,n}{\hat{\theta}^3}\right)\tau_{21}\,\tau_{22} - \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(3\,\tau_{11}\,\tau_{12})\Bigg\}$$
$$+\left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)\tau_{11} + \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)\tau_{12}\Bigg], \tag{44}$$

$$\hat{\theta}_{BSEGP} = \Bigg[\hat{\theta} + \frac{1}{2}\Bigg\{\left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)$$
$$\times\,\tau_{11}\,\tau_{12} + \left(\frac{2\,n}{\hat{\theta}^3}\right)\tau_{22}^2 - \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(\tau_{11}\tau_{22} + 2\,\tau_{12}^2)\Bigg\}$$
$$+\left(\frac{a_1-1}{\hat{\alpha}} - b_1\right)\tau_{21} + \left(\frac{a_2-1}{\hat{\theta}} - b_2\right)\tau_{22}\Bigg]. \tag{45}$$

Bayesian estimators of α and θ under SELF using LP are:

$$\hat{\alpha}_{BSELP} = \Bigg[\hat{\alpha} + \frac{1}{2}\Bigg\{\left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\right)\tau_{11}^2$$
$$+\left(\frac{2\,n}{\hat{\theta}^3}\right)\tau_{21}\,\tau_{22} - \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(3\,\tau_{11}\tau_{12})\Bigg\}$$
$$+\frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\,\hat{\theta}}\right)\tau_{11} + \frac{1}{2}\left(\frac{\hat{\alpha}-3\,\hat{\theta}}{\hat{\theta}^2}\right)\tau_{12}\Bigg]; \tag{46}$$

$$\hat{\theta}_{BSELP} = \Bigg[\hat{\theta} + \frac{1}{2}\Bigg\{\Bigg(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\Bigg)$$

$$\times\,\tau_{11}\,\tau_{12} + \left(\frac{2\,n}{\hat{\theta}^3}\right)\tau_{22}^2 - \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(\tau_{11}\tau_{22} + 2\,\tau_{12}^2)\Bigg\}$$

$$+\frac{1}{2}\left(\frac{\hat{\theta}-\hat{\alpha}}{\hat{\alpha}\,\hat{\theta}}\right)\tau_{21} + \frac{1}{2}\left(\frac{\hat{\alpha}-3\,\hat{\theta}}{\hat{\theta}^2}\right)\tau_{22}\Bigg]. \tag{17}$$

Bayesian estimators of α and θ under SELF using UP are:

$$\hat{\alpha}_{BSEUP} = \Bigg[\hat{\alpha} + \frac{1}{2}\Bigg\{\Bigg(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\Bigg)\tau_{11}^2$$

$$+\left(\frac{2\,n}{\hat{\theta}^3}\right)\tau_{21}\,\tau_{22} - \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(3\,\tau_{11}\,\tau_{12})\Bigg\}\Bigg]. \tag{18}$$

$$\hat{\theta}_{BSEUP} = \Bigg[\hat{\theta} + \frac{1}{2}\Bigg\{\Bigg(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{(1+(1+x_i)^{-\hat{\alpha}})(\log(1+x_i))^3(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^3}\Bigg)$$

$$\times\,\tau_{11}\,\tau_{12} + \left(\frac{2\,n}{\hat{\theta}^3}\right)\tau_{22}^2 - \left(\sum_{i=1}^{n}\frac{(1+x_i)^{-\hat{\alpha}}(\log(1+x_i))^2}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(\tau_{11}\,\tau_{22} + 2\,\tau_{12}^2)\Bigg\}\Bigg]. \tag{19}$$

3.2.11 Precautionary loss function (PLF)

Norstrom (1996) introduced an alternative asymmetric loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. This loss function approach infinitely near the origin to prevent under-estimation therefore giving predictable estimators, especially when low failure rates are being estimated. These estimators are very appreciable when underestimation may lead to serious consequences. General form of this loss function is:

$$L(\hat{\theta}, \theta) = \frac{\left(\hat{\theta} - \theta\right)^2}{\hat{\theta}}. \tag{50}$$

after simplifying we have.

$\hat{\theta} = [E(\theta^2)]^{\frac{1}{2}}$.

Bayesian estimators of α and θ under PLF using GP are:

$$\begin{aligned}
\hat{\alpha}_{BPGP} = \Bigg[\hat{\alpha}^2 + \frac{1}{2}\Bigg\{2\ \tau_{11} + \Bigg(\frac{2\ n}{\hat{\alpha}^3} + (\hat{\theta} - 1) \\
\times \sum_{i=1}^{n} \frac{(\log(1+x_i))^3\ (1+x_i)^{-\hat{\alpha}}\ (1+(1+x_i)^{-\hat{\alpha}})}{(1-(1+x_i)^{\hat{\alpha}})^3}\Bigg)(2\ \hat{\alpha})\ \tau_{11}^2 + \left(\frac{4\ n\ \hat{\alpha}}{\hat{\theta}^3}\right)(\tau_{21}\ \tau_{11}) \\
- \left(\sum_{i=1}^{n} \frac{(\log(1+x_i))^2 (1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(6\ \hat{\alpha})\ \tau_{11}\ \tau_{12}\Bigg\} + \left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right)(2\ \hat{\alpha}\ \tau_{11}) \\
+ \left(\frac{a_2 - 1}{\hat{\theta}} - b_2\right)(2\ \hat{\alpha})\ \tau_{12}\Bigg]^{\frac{1}{2}}.
\end{aligned} \tag{51}$$

$$\begin{aligned}
\hat{\theta}_{BPGP} = \Bigg[\hat{\theta}^2 + \frac{1}{2}\Bigg\{2\ \tau_{22} + \Bigg(\frac{2\ n}{\hat{\alpha}^3} + (\hat{\theta} - 1) \\
\times \sum_{i=1}^{n} \frac{(\log(1+x_i))^3 (1+x_i)^{-\hat{\alpha}} (1+(1+x_i)^{-\hat{\alpha}})}{(1-(1+x_i)^{\hat{\alpha}})^3}\Bigg)(2\ \hat{\theta})\ \tau_{12}\ \tau_{11} + \left(\frac{4\ n}{\hat{\theta}^2}\right)\ \tau_{22}^2 \\
- \left(\sum_{i=1}^{n} \frac{(\log(1+x_i))^2 (1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^2}\right)(2\ \hat{\theta}\ (\tau_{11}\ \tau_{22} + 2\ \tau_{12}^2))\Bigg\} + \left(\frac{a_1 - 1}{\hat{\alpha}} - b_1\right)(2\ \hat{\theta})\ \tau_{21} \\
+ \left(\frac{a_2 - 1}{\hat{\theta}} - b_2\right)(2\ \hat{\theta})\ \tau_{22}\Bigg]^{\frac{1}{2}}.
\end{aligned} \tag{52}$$

Bayesian estimators of α and θ under PLF using LP and UP respectively are:

$$
\begin{aligned}
\hat{\alpha}_{BPGP} = \Bigg[& \hat{\alpha}^2 + \frac{1}{2}\Bigg\{ 2\,\tau_{11} + \Bigg(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta} - 1) \\
& \times \sum_{i=1}^{n} \frac{\log(1+x_i)^3 (1+x_i)^{-\hat{\alpha}} (1 + (1+x_i)^{-\hat{\alpha}})}{(1-(1+x_i)^{\hat{\alpha}})^3} \Bigg) (2\,\hat{\alpha})\, \tau_{11}^2 + \left(\frac{4\,n\,\hat{\alpha}}{\hat{\theta}^3} \right) \tau_{21} \tau_{11} \\
& - \left(\sum_{i=1}^{n} \frac{(\log(1+x_i))^2 (1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^2} \right) (6\,\hat{\alpha})\, \tau_{11}\, \tau_{12} \Bigg\} + \left(\frac{\hat{\theta} - \hat{\alpha}}{\hat{\alpha}\hat{\theta}} \right) \hat{\alpha}\, \tau_{11} \\
& + \left(\frac{\hat{\alpha} - 3\,\hat{\theta}}{\hat{\theta}^2} \right) \hat{\alpha}\, \tau_{21} \Bigg]^{\frac{1}{2}},
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
\hat{\theta}_{BPGP} = \Bigg[& \hat{\theta}^2 + \frac{1}{2}\Bigg\{ 2\,\tau_{22} + \Bigg(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta} - 1) \\
& \times \sum_{i=1}^{n} \frac{\log(1+x_i)^3 (1+x_i)^{-\hat{\alpha}} (1 + (1+x_i)^{-\hat{\alpha}})}{(1-(1+x_i)^{\hat{\alpha}})^3} \Bigg) 2\,\hat{\theta}(\tau_{12}\,\tau_{11}) + \left(\frac{4\,n}{\hat{\theta}^2} \right) \tau_{22}^2 \\
& - \left(\sum_{i=1}^{n} \frac{(\log(1+x_i))^2 (1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^2} \right) (2\,\hat{\theta}(\tau_{11}\,\tau_{22} + 2\,\tau_{12}^2)) \Bigg\} + \left(\frac{\hat{\theta} - \hat{\alpha}}{\hat{\alpha}\,\hat{\theta}} \right) \hat{\theta}\, \tau_{12} \\
& + \left(\frac{\hat{\alpha} - 3\,\hat{\theta}}{\hat{\theta}^2} \right) \hat{\theta}\, \tau_{22} \Bigg]^{\frac{1}{2}}.
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\hat{\alpha}_{BPUP} = \Bigg[& \hat{\alpha}^2 + \frac{1}{2}\Bigg\{ 2\,\tau_{11} + \Bigg(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta} - 1) \\
& \times \sum_{i=1}^{n} \frac{\log(1+x_i)^3\, (1+x_i)^{-\hat{\alpha}}\, (1 + (1+x_i)^{-\hat{\alpha}})}{(1-(1+x_i)^{\hat{\alpha}})^3} \Bigg) (2\,\hat{\alpha}\,\tau_{11}^2) + \left(\frac{4\,n\,\hat{\alpha}}{\hat{\theta}^3} \right) \tau_{21}\,\tau_{11} \\
& - \left(\sum_{i=1}^{n} \frac{(\log(1+x_i))^2\, (1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^2} \right) (6\,\hat{\alpha})\, \tau_{11}\, \tau_{12} \Bigg\} \Bigg]^{\frac{1}{2}},
\end{aligned}
\tag{55}
$$

$$\hat{\theta}_{BPUP} = \Bigg[\hat{\theta}^2 + \frac{1}{2}\Bigg\{2\,\tau_{22} + \left(\frac{2\,n}{\hat{\alpha}^3} + (\hat{\theta}-1)\sum_{i=1}^{n}\frac{\log(1+x_i)^3\,(1+x_i)^{-\hat{\alpha}}\,(1+(1+x_i)^{-\hat{\alpha}})}{(1-(1+x_i)^{\hat{\alpha}})^3}\right)$$

$$\times\, 2\,\hat{\theta}(\tau_{12}\,\tau_{11}) + \left(\frac{4\,n}{\hat{\theta}^2}\right)\tau_{22}^2 - \left(\sum_{i=1}^{n}\frac{(\log(1+x_i))^2\,(1+x_i)^{-\hat{\alpha}}}{(1-(1+x_i)^{-\hat{\alpha}})^2} - \right)(2\,\hat{\theta}(\tau_{11}\,\tau_{22} + 2\,\tau_{12}^2)\Bigg\}\Bigg]^{\frac{1}{2}}. \tag{56}$$

3.3 MLEs for Type-II censoring

Suppose $X = (X_1 < X_2 < X_3 <, \ldots, < X_r)$ is a Type-II censored sample of size r obtained from a life test on n items whose life times have the EPD with shape parameters α and θ. The likelihood function of r failures and $(n-r)$ censored values is:

$$L(\alpha,\theta) = \frac{n!}{(n-r)!}(\alpha\,\theta)^r \prod_{i=1}^{r}[1-(1+x_i)^{\alpha}]^{\theta-1}\prod_{i=1}^{r}(1+x_i)^{-(\alpha+1)} \times \left[1-\left(1-(1+x_r)^{-\alpha}\right)^{\theta}\right]^{n-r}. \tag{57}$$

the log-likelihood function is:

$$\log L(\alpha,\theta) = r\log(\alpha) + r\log(\theta) + \sum_{i=1}^{r}\log\left[1-(1+x_i)^{-\alpha}\right]^{\theta-1} + \sum_{i=1}^{r}\log\left[1+x_i\right]^{-(\alpha+1)} + \log\left[1-\left(1-(1+x_r)^{-\alpha}\right)^{\theta}\right]^{n-r}. \tag{58}$$

this may be written as:

$$\log L(\alpha,\theta) = r\log(\alpha) + r\log(\theta) + (\theta-1)\sum_{i=1}^{r}\log\left(1-(1+x_i)^{-\alpha}\right) - (\alpha+1)\sum_{i=1}^{r}\log(1+x_i) + (n-r)\log\left[1-\left(1-(1+X_r)^{-\alpha}\right)^{\theta}\right]. \tag{59}$$

Where

$$\frac{\partial \log L(\alpha,\theta)}{\partial \alpha} = \frac{r}{\alpha} + (\theta-1)\sum_{i=1}^{r}\frac{(1+x_i)^{-\alpha}\log(1+x_i)}{1-(1+x_i)^{-\alpha}} - \sum_{i=1}^{r}\log(1+x_i) + \frac{\theta(n-r)\left(1-(1+X_r)^{-\alpha}\right)^{\theta-1}(1+X_r)^{-\alpha}\log(1+X_r)}{1-\left(1-(1+X_r)^{-\alpha}\right)^{\theta}}. \quad (60)$$

$$\frac{\partial \log L(\alpha,\theta)}{\partial \theta} = \frac{r}{\theta} + \sum_{i=1}^{r}\log(1-(1+x_i)^{-\alpha}) - \frac{(n-r)\left(1-(1+X_r)^{-\alpha}\right)^{\theta}\log(1-(1+X_r)^{-\alpha})}{1-\left(1-(1+X_r)^{-\alpha}\right)^{\theta}}. \quad (61)$$

The MLEs of α and θ for $\hat{\alpha}$ and $\hat{\theta}$ can be obtained as the solution of

$$\frac{r}{\alpha} + (\theta-1)\sum_{i=1}^{r}\frac{(1+x_i)^{-\alpha}\log(1+x_i)}{1-(1+x_i)^{-\alpha}} - \sum_{i=1}^{r}\log(1+x_i) + \frac{\theta(n-r)\left(1-(1+X_r)^{-\alpha}\right)^{\theta-1}(1+X_r)^{-\alpha}\log(1+X_r)}{1-\left(1-(1+X_r)^{-\alpha}\right)^{\theta}} = 0. \quad (62)$$

$$\frac{r}{\theta} + \sum_{i=1}^{r}\log(1-(1+x_i)^{-\alpha}) - \frac{(n-r)\left(1-(1+X_r)^{-\alpha}\right)^{\theta}\log(1-(1+X_r)^{-\alpha})}{1-\left(1-(1+X_r)^{-\alpha}\right)^{\theta}} = 0. \quad (63)$$

Since normal equations are not in closed form so we used an alternative method Broyden Fletcher GoldFarb Shanno (BFGS) which is used by Battiti and Masulli

(1990) of Newton Rophson method. And elements of observed Fisher information matrix for censored sample are:

$$\frac{\partial^2 \log L(\alpha,\theta)}{\partial \alpha^2} = -\frac{r}{\alpha^2} - (\theta - 1)\sum_{i=1}^{r} \frac{(1+x_i)^{-\alpha}(\log(1+x_i))^2}{(1-(1+x_i)^{-\alpha})^2}.$$

$$\frac{\partial^2 \log L(\alpha,\theta)}{\partial \theta^2} = -\frac{r}{\theta^2}.$$

$$\frac{\partial^2 \log L(\alpha,\theta)}{\partial \alpha \, \partial \theta} = \sum_{i=1}^{r} \frac{\log(1+x_i)(1+x_i)^{-\alpha}}{1-(1+x_i)^{-\alpha}} - (n-r)\log(1+x_r).$$

Here $\hat{\alpha}$ and $\hat{\theta}$ are the MLEs of EPD for α and θ respectively. And $\hat{\alpha}_B$ and $\hat{\theta}_B$ are Bayesian estimators of EPD.

Chapter 4

Results and Discussion

In this chapter, we will discuss about the results of simulation study. For complete samples results are shown in Tables 1-8 for α and θ. For Type-II censored samples, results of MSEs of MLEs and CIs are also shown in Tables 9-19.

4.1 Simulation study

As we know that performance of the different methods cannot be compared theoretically, hence we performed Monte Carlo Simulation to compare the performance of presented MLEs and Bayesian estimators of EPD using various samples sizes and various parameters values. For Type-II censoring scheme, we have fixed different sample sizes (n) and predetermined different percentages of failures (r). In real data analysis, we have fixed different percentages of failures (r) and sample size (n) = 100 for Type-II censoring scheme for parameters. In all Tables, it is assumed that the parameters are random variables and generated through simulation method.

For computing the estimates, we generated 1000 samples from the EPD using the inverse transformation as $X_i = (1 - U^{1/\theta})^{-1/\alpha} - 1$. Where U_i is uniformly distributed random variable. We replicated the process 1000 times and estimate the MLEs. We have also obtained Bayesian estimators under symmetric and asymmetric loss functions using Lindley's approximation. Further, we used informative and non-informative types of priors for estimating Bayesian estimators. Moreover, comparisons are made in terms of MSEs. And the results are summarized in Tables 1-12. Finally, graphical representations of MSEs of MLEs and Bayesian estimators are also made for complete samples shown in Figures 1-3.

Results for complete samples for α are as under:

Table 1: Average estimates for α and MSEs (within parenthesis).

$n\downarrow$	$\alpha\rightarrow$	1	1.5	2
30	ML	1.0859(0.0859)	1.6031(0.1468)	2.1485(0.2339)
	BLGP	1.0609(0.0745)	1.6021**(0.1297)**	2.1299(0.2335)
	BLLP	1.0700(0.0797)	1.6029(0.1395)	2.1470(0.2337)
	BSEGP	1.0850(0.0803)	1.6030(0.1397)	2.1481(0.2338)
	BSELP	1.2559(0.1697)	1.8708(0.3331)	2.5398(0.6249)
	BGEGP	1.0854(0.0840)	1.6027(0.1393)	2.1374**(0.2308)**
	BGELP	1.2452(0.1518)	1.8067(0.2587)	2.4143(0.4414)
	BPLGP	1.2180(0.1501)	1.8591(0.3404)	2.5640(0.7002)
	BPLLP	1.0698(0.0774)	1.6026(0.1391)	2.1477(0.2333)
	BLUP	1.0761**(0.0605)**	1.6028**(0.1297)**	2.1459(0.2329)
	BSEUP	1.0670(0.0840)	1.6020(0.1358)	2.1471(0.2336)
	BGEUP	1.0759(0.0804)	1.6025(0.1355)	2.1379(0.2330)
	BPLUP	1.0847(0.0861)	1.6023(0.1359)	2.1396(0.2334)
50	ML	1.0495(0.0437)	1.5656(0.0751)	2.0631(0.1117)
	BLGP	1.0384(0.0430)	1.5619**(0.0695)**	2.0623(0.1073)
	BLLP	1.0396(0.0435)	1.5655(0.0748)	2.0630(0.1110)
	BSEGP	1.0490(0.0406)	1.5654(0.0711)	2.0624(0.1090)
	BSELP	1.2473(0.0706)	1.7210(0.1373)	2.2833(0.2208)
	BGEGP	1.0489(0.0436)	1.5642(0.0747)	2.0629(0.1115)
	BGELP	1.1433(0.0659)	1.6856(0.1138)	2.2152(0.1699)
	BPLGP	1.1261(0.0640)	1.7167(0.1408)	2.3018(0.2459)
	BPLLP	1.0480(0.0433)	1.5647(0.0739)	2.0494**(0.1023)**
	BLUP	1.0393**(0.0389)**	1.5652(0.0698)	2.0611(0.1036)
	BSEUP	1.0412(0.0432)	1.5637(0.0745)	2.0625(0.1115)
	BGEUP	1.0470(0.0429)	1.5483(0.0736)	2.0627(0.1028)
	BPLUP	1.0471(0.0434)	1.5618(0.0740)	2.0628(0.1111)
80	ML	1.0371(0.0262)	1.5331(0.0430)	2.0526(0.0672)
	BLGP	1.0362(0.0255)	1.5176(0.0424)	2.0499(0.0588)
	BLLP	1.0365(0.0260)	1.5317(0.0427)	2.0521(0.0627)
	BSEGP	1.0359(0.0230)	1.5312(0.0420)	2.0516(0.0590)

$n \downarrow$	$\alpha \rightarrow$	1	1.5	2
	BSELP	1.0975(0.0371)	1.6275(0.0644)	2.1520(0.1659)
	BGEGP	1.0368(0.0254)	1.5330(0.0426)	2.0488(0.0621)
	BGELP	1.0953(0.0354)	1.6066(0.0563)	2.1464(0.0916)
	BPLGP	1.0854(0.0347)	1.6262(0.0660)	2.2008(0.1213)
	BPLLP	1.0369(0.0258)	1.5327(0.0679)	2.0309(0.0559)
	BLUP	1.0348(0.0256)	1.5209(0.0416)	2.0523(0.0574)
	BSEUP	1.0334(0.0241)	1.5245(0.0423)	2.0514(0.0590)
	BGEUP	1.0313(0.0225)	1.5321(0.0415)	2.0141(0.0506)
	BPLUP	1.0323(**0.0219**)	1.5297(**0.0410**)	2.0202(**0.0497**)
100	ML	1.0309(0.0208)	1.5270(0.0347)	2.0387(0.0495)
	BLGP	1.0300(0.0201)	1.5245(0.0344)	2.0362(0.0448)
	BLLP	1.0304(**0.0200**)	1.5264(0.0346)	2.0376(0.0481)
	BSEGP	1.0307(0.0203)	1.5206(0.0338)	2.0357(0.0483)
	BSELP	1.0789(0.0272)	1.6017(0.0481)	2.1451(0.0757)
	BGEGP	1.0302(0.0201)	1.5244(0.0345)	2.0378(0.0482)
	BGELP	1.0774(0.0262)	1.5855(0.0431)	2.1329(0.0640)
	BPLGP	1.0692(0.0256)	1.6004(0.0490)	2.1355(0.0548)
	BPLLP	1.0305(0.0205)	1.5207(0.0339)	2.0384(0.0477)
	BLUP	1.0304(0.0206)	1.5268(0.0340)	2.0379(0.0479)
	BSEUP	1.0306(0.0205)	1.5237(**0.0335**)	2.0197(**0.0274**)
	BGEUP	1.0303(0.0204)	1.5269(0.0343)	2.0275(0.0374)
	BPLUP	1.0308(0.0207)	1.5234(0.0342)	2.0355(0.0390)
130	ML	1.0309(0.0172)	1.5270(0.0400)	2.0387(0.0416)
	BLGP	1.0300(0.0101)	1.5245(0.0304)	2.0362(0.0448)
	BLLP	1.0304(0.0102)	1.5264(0.0301)	2.0376(0.0481)
	BSEGP	1.0307(**0.0100**)	1.5206(0.0303)	2.0357(0.0479)
	BSELP	1.0789(0.0101)	1.6017(0.0301)	2.1451(0.0482)
	BGEGP	1.0302(0.0120)	1.5244(0.0321)	2.0378(0.0640)
	BGELP	1.0774(**0.0100**)	1.5855(**0.0300**)	2.1329(0.0374)
	BPLGP	1.0692(0.0101)	1.6004(0.0302)	2.1355(0.0483)
	BPLLP	1.0305(0.0101)	1.5207(0.0400)	2.0384(**0.0274**)
	BLUP	1.0304(0.0103)	1.5268(**0.0300**)	2.0379(0.0548)
	BSEUP	1.0306(0.0102)	1.5237(0.0401)	2.0197(0.0477)
	BGEUP	1.0303(0.0104)	1.5269(0.0302)	2.0275(0.0390)
	BPLUP	1.0308(0.0105)	1.5234(0.0302)	2.0355(0.0342)
150	ML	1.0185(0.0127)	1.5302(0.0241)	2.0224(0.0381)
	BLGP	1.0143(0.0124)	1.5258(**0.0209**)	2.0202(0.0295)
	BLLP	1.0154(0.0125)	1.5292(0.0233)	2.0215(0.0368)
	BSEGP	1.0183(0.0126)	1.5259(0.0239)	2.0217(0.0363)
	BSELP	1.0502(0.0156)	1.5801(0.0313)	2.0923(0.0495)
	BGEGP	1.0179(0.0121)	1.5301(0.0236)	2.0222(0.0319)
	BGELP	1.0493(0.0152)	1.5692(0.0289)	2.0713(0.0444)

$n\downarrow$	$\alpha\rightarrow$	1	1.5	2
.	BPLGP	1.0440(0.0149)	1.5796(0.0317)	2.0995(0.0524)
	BPLLP	1.0152(0.0116)	1.5233(0.0240)	2.0215(0.0363)
	BLUP	1.0182(0.0125)	1.5278(0.0211)	2.0301(**0.0291**)
	BSEUP	1.0127(0.0124)	1.5280(0.0232)	2.0356(0.0359)
	BGEUP	1.0119(**0.0114**)	1.5272(0.0249)	2.0220(0.0315)
	BPLUP	1.0179(0.0123)	1.5216(0.0235)	2.0294(0.0319)
200	ML	1.0106(0.0089)	1.5153(0.0155)	2.0190(0.0269)
	BLGP	1.0100(**0.0082**)	1.5092(0.0146)	2.0102(0.0236)
	BLLP	1.0103(0.0087)	1.5093(0.0154)	2.0119(0.0255)
	BSEGP	1.0105(0.0088)	1.5018(0.0153)	2.0189(0.0267)
	BSELP	1.0136(0.0104)	1.5523(0.0188)	2.0714(0.0335)
	BGEGP	1.0104(0.0088)	1.5138(0.0152)	2.0132(**0.0059**)
	BGELP	1.0130(0.0090)	1.5440(0.0176)	2.0557(0.0306)
	BPLGP	1.0192(0.0101)	1.5520(0.0190)	2.0773(0.0352)
	BPLLP	1.0106(0.0088)	1.5124(0.0150)	2.0189(0.0265)
	BLUP	1.0102(0.0088)	1.5107(0.0148)	2.0100(0.0134)
	BSEUP	1.0106(0.0089)	1.5133(**0.0145**)	2.0188(0.0275)
	BGEUP	1.0104(0.0086)	1.5152(0.0155)	2.0131(0.0269)
	BPLUP	1.0101(0.0183)	1.5136(0.0151)	2.0172(0.0249)

Here we discussed about the results of parameter α for complete samples using various sample sizes and different values of α. Averages estimates and MSEs (within parenthesis) are shown in Table 1. It is observed from Table 1 that BLUP gives best estimates for $\alpha = 1$, 1.5 when sample size is 30 and 50. BLGP provide effective result for $\alpha = 1.5$ but BGEGP give superior result for $\alpha = 2$ when sample size is 30. For $\alpha = 1.5$ with sample size 50 and 150, BLGP contribute improved result whereas BPLLP gives better result for $\alpha = 2$ with sample size is 50. Also BPLUP gives best performance for all values of α those are taken with sample size 80. Further, BSEUP also gives outstanding result for $\alpha = 2$ with sample size 100. Furthermore. BLLP. BGEUP and BLGP provide recovering outcome for $\alpha = 1$ when samples size is 100. 150 and 200 respectively. Moreover. BSEUP gives superior result for $\alpha = 1.5$ and 2 with sample size 100 and 200. And results for parameter θ are as under:

Table 2: Average estimates for θ and MSEs (within parenthesis).

$n \downarrow$	$\theta \rightarrow$	1	1.5	2
30	ML	1.1082(0.1009)	1.6770(0.2803)	2.2753(0.5799)
	BLGP	1.0554(0.0964)	1.5970(0.2661)	2.1634(0.5632)
	BLLP	1.0127(0.0890)	1.5315(0.2632)	2.0737(0.5927)
	BSEGP	1.0242(0.0647)	1.4887(0.1430)	1.9250(0.2273)
	BSELP	0.9810(**0.0552**)	1.4161(**0.1227**)	1.8176(**0.2022**)
	BGEGP	1.0368(0.0786)	1.5592(0.2022)	2.0867(0.3731)
	BGELP	0.9936(0.0670)	1.4866(0.1659)	1.9793(0.2945)
	BPLGP	0.9547(0.0715)	1.3857(0.1554)	1.7650(0.2560)
	BPLLP	1.0012(0.0717)	1.4598(0.1513)	1.8749(0.2301)
	BLUP	1.0816(0.1075)	1.6511(0.3082)	2.2555(0.6700)
	BSEUP	1.0495(0.0729)	1.5405(0.1665)	2.0130(0.2704)
	BGEUP	1.0620(0.0880)	1.6110(0.2371)	2.1747(0.4594)
	BPLUP	1.0740(0.0967)	1.5967(0.2256)	2.1016(0.3750)
50	ML	1.0539(0.0507)	1.6006(0.1329)	2.1451(0.2644)
	BLGP	1.0259(0.0496)	1.5553(0.1270)	2.0830(0.2555)
	BLLP	1.0022(0.0474)	1.5175(0.1232)	2.0327(0.2575)
	BSEGP	1.0092(0.0395)	1.5003(0.0926)	1.9660(0.1663)
	BSELP	0.9854(**0.0362**)	1.4604(**0.0849**)	1.9082(**0.1542**)
	BGEGP	1.0136(0.0440)	1.5352(0.1102)	2.0453(0.2095)
	BGELP	0.9897(0.0403)	1.4954(0.0989)	1.9875(0.1857)
	BPLGP	0.9738(0.0415)	1.4486(0.0969)	1.8956(0.1765)
	BPLLP	0.9988(0.0418)	1.4889(0.0973)	1.9521(0.1728)
	BLUP	1.0398(0.0528)	1.5842(0.1390)	2.1311(0.2849)
	BSEUP	1.0229(0.0421)	1.5284(0.1003)	2.0126(0.1813)
	BGEUP	1.0273(0.0468)	1.5633(0.1204)	2.0919(0.2338)
	BPLUP	1.0376(0.0496)	1.5603(0.1196)	2.0638(0.2174)
80	ML	1.0378(0.0269)	1.5657(0.0736)	2.0936(0.1416)
	BLGP	1.0218(0.0262)	1.5397(0.0712)	2.0551(0.1372)
	BLLP	1.0072(0.0251)	1.5168(0.0690)	2.0233(0.1360)
	BSEGP	1.0112(0.0231)	1.5070(0.0594)	1.9894(0.1072)
	BSELP	0.9967(**0.0219**)	1.4830(**0.0564**)	1.9555(**0.1020**)
	BGEGP	1.0132(0.0245)	1.5266(0.0655)	2.0346(0.1225)
	BGELP	0.9987(0.0232)	1.5026(0.0614)	2.0006(0.1138)
	BPLGP	0.9912(0.0234)	1.4785(0.0606)	1.9494(0.1109)
	BPLLP	1.0062(0.0238)	1.5022(0.0613)	1.9832(0.1105)
	BLUP	1.0301(0.0273)	1.5567(0.0754)	2.0833(0.1473)
	BSEUP	1.0194(0.0240)	1.5237(0.0624)	2.0170(0.1130)
	BGEUP	1.0214(0.0255)	1.5433(0.0692)	2.0622(0.1311)
	BPLUP	1.0293(0.0266)	1.5440(0.0695)	2.0471(0.1266)

$n \downarrow$	$\theta \rightarrow$	1	1.5	2
100	ML	1.0270(0.0205)	1.5419(0.0557)	2.0615(0.1014)
	BLGP	1.0144(0.0201)	1.5209(0.0547)	2.0310(0.0995)
	BLLP	1.0029(0.0195)	1.5027(0.0537)	2.0056(0.0991)
	BSEGP	1.0062(0.0183)	1.4966(0.0479)	1.9815(0.0834)
	BSELP	0.9948(**0.0176**)	1.4781(**0.0462**)	1.9553(**0.0808**)
	BGEGP	1.0076(0.0191)	1.5114(0.0514)	2.0158(0.0914)
	BGELP	0.9962(0.0184)	1.4928(0.0491)	1.9895(0.0869)
	BPLGP	0.9962(0.0186)	1.4739(0.0491)	1.9505(0.0865)
	BPLLP	1.0024(0.0188)	1.4926(0.0492)	1.9769(0.0857)
	BLUP	1.0209(0.0208)	1.5341(0.0570)	2.0528(0.1049)
	BSEUP	1.0127(0.0188)	1.5096(0.0495)	2.0029(0.0863)
	BGEUP	1.0140(0.0197)	1.5244(0.0534)	2.0371(0.0960)
	BPLUP	1.0204(0.0203)	1.5248(0.0537)	2.0260(0.0939)
130	ML	1.0227(0.0162)	1.5416(0.0413)	2.0610(0.0817)
	BLGP	1.0137(0.0160)	1.5256(0.0403)	2.0379(0.0798)
	BLLP	1.0049(0.0156)	1.5115(0.0394)	2.0185(0.0789)
	BSEGP	1.0069(0.0148)	1.5070(0.0362)	1.9998(0.0690)
	BSELP	0.9981(**0.0143**)	1.4928(**0.0350**)	1.9796(**0.0668**)
	BGEGP	1.0078(0.0153)	1.5182(0.0383)	2.0259(0.0747)
	BGELP	0.9990(0.0148)	1.5040(0.0368)	2.0057(0.0713)
	BPLGP	0.9956(0.0150)	1.4898(0.0366)	1.9767(0.0702)
	BPLLP	1.0045(0.0151)	1.5041(0.0369)	1.9968(0.0701)
	BLUP	1.0186(0.0164)	1.5356(0.0418)	2.0545(0.0836)
	BSEUP	1.0118(0.0151)	1.5169(0.0373)	2.0161(0.0713)
	BGEUP	1.0127(0.0156)	1.5282(0.0397)	2.0423(0.0779)
	BPLUP	1.0183(0.0162)	1.5287(0.0398)	2.0342(0.0765)
150	ML	1.0176(0.0121)	1.5353(0.0347)	2.0327(0.0687)
	BLGP	1.0096(0.0120)	1.5213(0.0340)	2.0127(0.0680)
	BLLP	1.0021(0.0117)	1.5091(0.0333)	1.9960(0.0677)
	BSEGP	1.0040(0.0113)	1.5055(0.0310)	1.9812(0.0613)
	BSELP	0.9965(**0.0110**)	1.4933(**0.0301**)	1.9642(**0.0603**)
	BGEGP	1.0047(0.0116)	1.5151(0.0326)	2.0030(0.0647)
	BGELP	0.9972(0.0113)	1.5029(0.0315)	1.9860(0.0629)
	BPLGP	0.9942(0.0114)	1.4906(0.0313)	1.9614(0.0628)
	BPLLP	1.0018(0.0114)	1.5030(0.0315)	1.9786(0.0624)
	BLUP	1.0139(0.0122)	1.5299(0.0351)	2.0268(0.0701)
	BSEUP	1.0082(0.0115)	1.5140(0.0318)	1.9950(0.0625)
	BGEUP	1.0089(0.0118)	1.5236(0.0335)	2.0168(0.0666)
	BPLUP	1.0137(0.0121)	1.5240(0.0337)	2.0101(0.0657)

$n\downarrow$	$\theta\rightarrow$	1	1.5	2
200	ML	1.0143(0.0091)	1.5247(0.0245)	2.0393(0.0528)
	BLGP	1.0086(0.0090)	1.5145(0.0242)	2.0247(0.0520)
	BLLP	1.0030(0.0088)	1.5055(0.0239)	2.0122(0.0515)
	BSEGP	1.0042(0.0085)	1.5027(0.0226)	2.0007(0.0474)
	BSELP	0.9986**(0.0084)**	1.4936**(0.0221)**	1.9878**(0.0464)**
	BGEGP	1.0047(0.0087)	1.5097(0.0234)	2.0170(0.0498)
	BGELP	0.9990(0.0085)	1.5006(0.0228)	2.0042(0.0484)
	BPLGP	0.9971(0.0086)	1.4920(0.0228)	1.9863(0.0478)
	BPLLP	1.0028(0.0087)	1.5011(0.0229)	1.9991(0.0480)
	BLUP	1.0117(0.0091)	1.5209(0.0248)	2.0352(0.0535)
	BSEUP	1.0073(0.0087)	1.5091(0.0230)	2.0111(0.0484)
	BGEUP	1.0078(0.0089)	1.5160(0.0239)	2.0275(0.0512)
	BPLUP	1.0116(0.0090)	1.5167(0.0240)	2.0227(0.0506)

Here we discussed about the results of parameter θ of EPD those are found through simulation study for complete samples using different sample sizes and changing values of θ. Averages estimates of θ and MSEs (within parenthesis) are shown in Tables 2. We observed the following results:

Estimating of θ from Tables 2, it is noticed that BSELP provide preeminent outcom from all other Bayesian estimators and MLEs for different values of θ with different sample sizes but MLEs offer paramount results comparatively BLUP for θ = 1. BPLUP also overestimate for $\theta = 1$. Furthermore, BSELP underestimate as compared to all other Bayesian estimation methods for $\theta = 1$, 1.5 and 2. Moreover. BLUP overestimate from all other Bayesian estimation methods for $\theta = 1.5$ and 2 It is also seen that BLUP overestimate among all Bayesian methods and MLEs for θ = 1, 1.5 and 2 with various sample sizes. Further, BPLLP and BSEUP gives same performance, in other words these estimators are same MSEs but not better than BGEGP when $\theta = 1$. BLLP and BPLUP perform equally, BGELP and BPLGP also performed equally for $\theta = 1.5$ when sample size is 100. Furthermore. BSEGP and BGELP, BSEUP and BPLLP contributed equal performance for $\theta = 1$ when sample size is 130.

For large sample size BSELP shows incomparable results comparatively all other Bayesian estimation methods and MLEs for $\theta = 1$, 1.5 and 2. Therefore, it is concluded that BSELP is better selection for estimation of parameter of θ of EPD for $\theta = 1$, 1.5 and 2 with sample size 200. Also BLGP and BPLUP, BSEGP and BGELP, BGEGP, BPLLP and BSEUP, BLUP and MLEs shows same performance but not comparable with BSELP for $\theta = 1$ when sample size 200. MLEs and BLUP also overestimate for $\theta = 1$ when sample size 200. Hence and concluded that MLEs and BLUP are not performed very well as compared to others for $\theta = 1$ when sample size 200. Furthermore, BLLP and BGEUP, BGELP and BPLGP gave same results but not better from other Bayesian estimation methods for $\theta - 1.5$ when sample size 200. Moreover, BGELP and BSEUP are also provide same outcomes for $\theta = 2$ with sample size 200 but not superlative as compared to others Bayesian estimation methods. We can see above all results of MSEs by graphical representation as given in figures 1-3 as under:

Figure 1: Graph of MSEs when α = 1 and 1.5

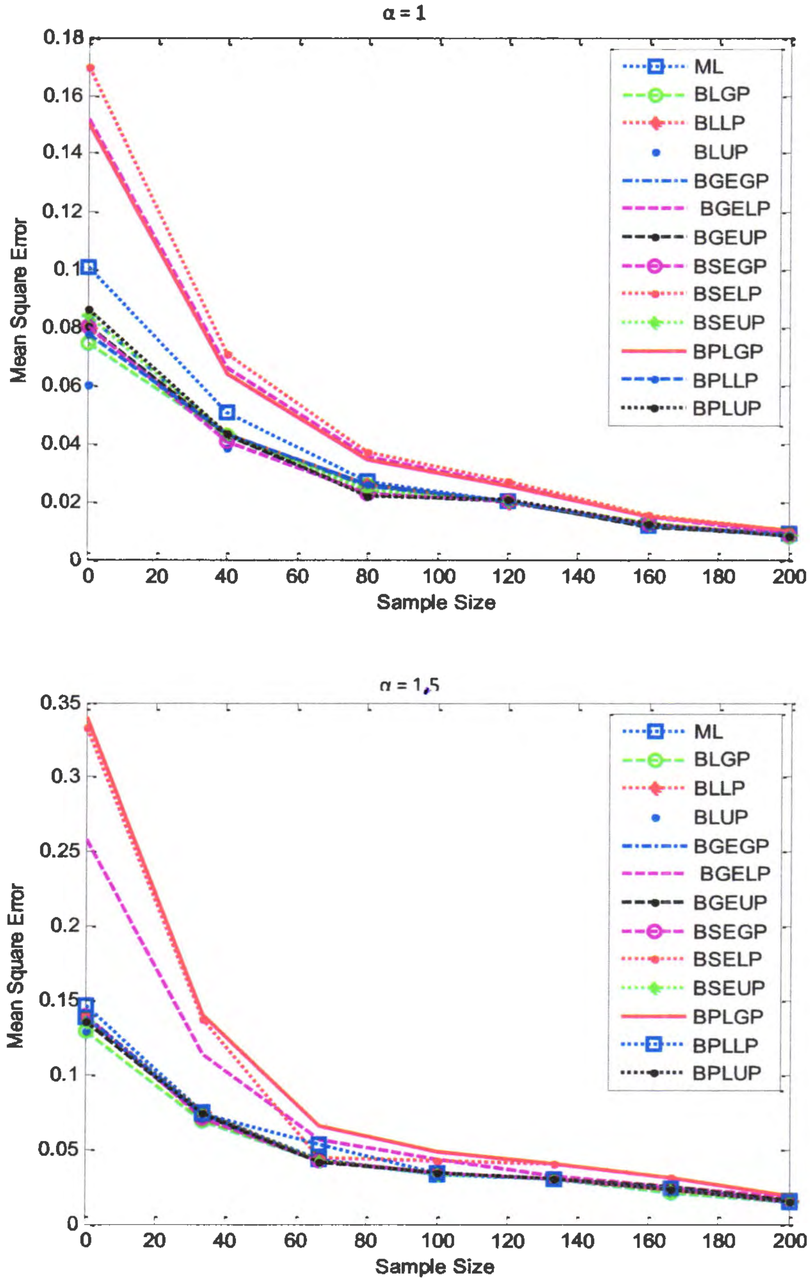
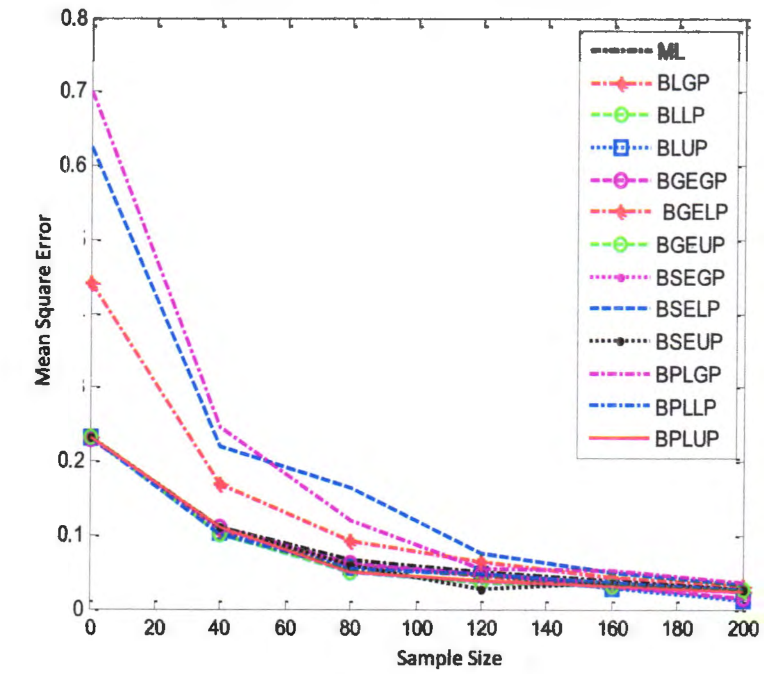


Figure 2: Graph of MSEs when α = 2 and θ = 1

α = 2



θ = 1

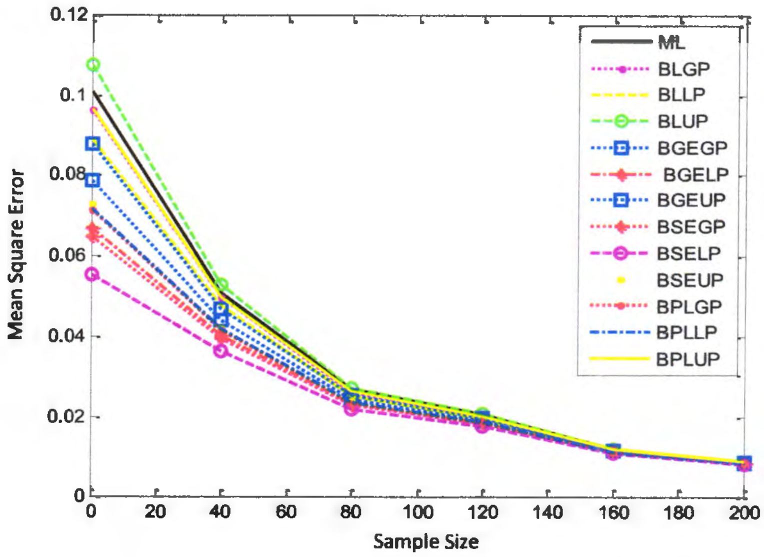
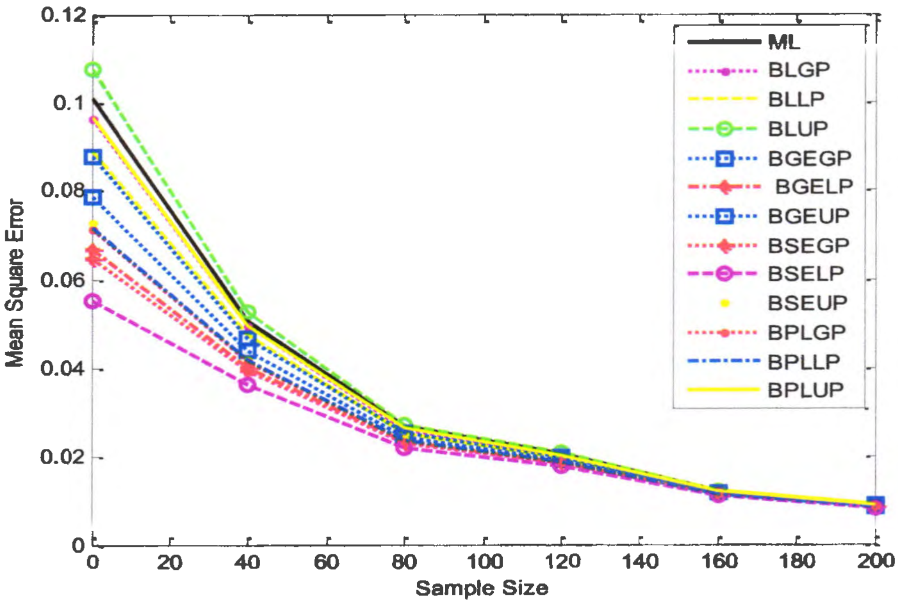
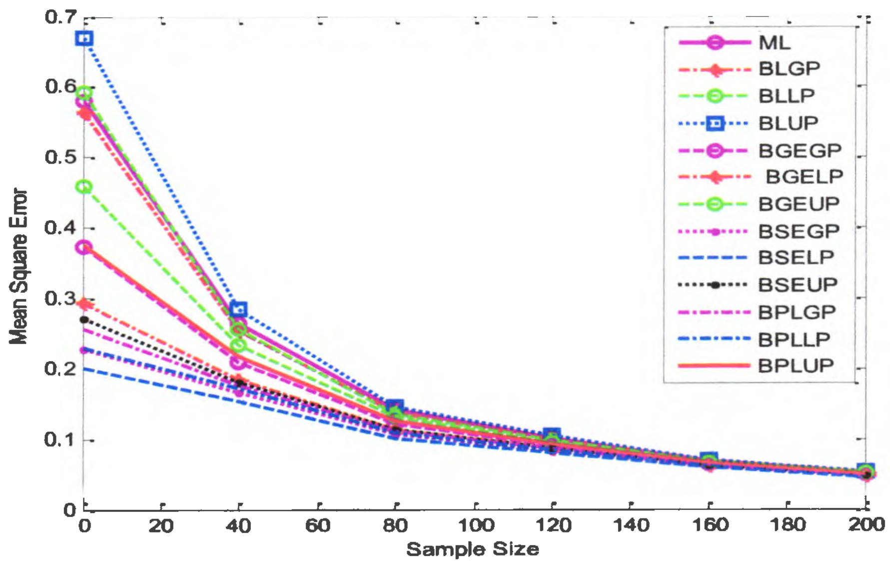


Figure 3: Graph of MSEs when θ = 1.5 and 2

θ = 1.5



θ = 2



In Figures 1-3, title bar showing the values of α and θ, vertical bar represents MSEs of MLEs and different Bayesian estimators and horizontal bar indicates various values of sample sizes.

It is observed that MSEs of MLEs and Bayesian estimators are decreases as sample size increases. This indicates that the consistancy and accuracy of the results. It is also noticed that when sample sizes goes to sufficiently large then MSEs of Bayesian and MLEs are close to each other. Hence it can be concluded that for large sample sizes Bayesian and MLEs are close to the true values of parameters.

4.1.1 Real data analysis for complete samples

In this section, we consider the real data set which is taken from Nichols and Padgett (2006) and is presented in Table 3. The data gives 100 observations on breaking stress of carbon fibres in Gross Building Area (Gba). Here we considered an uncensored data set corresponding censored data set from consisting of 100 observations on breaking stress of carbon fibers (in Gba). This data was obtained from a process producing carbon fibers and it was used in constructing fibrous composite materials.

Table 3: Breaking stress of carbon fibers

0.39	0.81	0.85	0.98	1.08	1.12	1.17	1.18	1.22	1.25	1.36	1.41	1.47
1.57	1.57	1.59	1.59	1.61	1.61	1.69	1.69	1.71	1.73	1.80	1.84	1.84
1.87	1.89	1.92	2.00	2.03	2.03	2.05	2.12	2.17	2.17	2.17	2.35	2.38
2.41	2.43	2.48	2.48	2.50	2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74
2.76	2.77	2.79	2.81	2.81	2.82	2.83	2.85	2.87	2.88	2.93	2.95	2.96
2.97	2.97	3.09	3.11	3.11	3.15	3.15	3.19	3.19	3.22	3.22	3.27	3.28
3.31	3.31	3.33	3.39	3.39	3.51	3.56	3.60	3.65	3.68	3.68	3.68	3.70
3.75	4.20	4.38	4.42	4.70	4.90	4.91	5.08	5.56				

Table 4: Average estimates and MSEs of α and θ.

Estimators	α	MSE	θ	MSE
ML	3.0698	0.0507	27.6963	43.1718
BLGP	3.4019	0.0495	24.2969	39.1880
BLLP	3.4673	0.0504	25.8123	43.1609
BSEGP	3.3778	0.0483	20.1001	40.4333
BSELP	3.4232	**0.0435**	18.7756	**36.1996**
BGEGP	3.1566	0.0502	23.4999	42.8693
BGELP	3.2585	0.0507	22.5835	43.1718
BPLGP	3.6108	0.0523	18.8168	42.1609
BPLLP	4.0944	0.0602	18.1421	41.1287
BLUP	3.3584	0.0501	24.3350	40.4812
BSEUP	3.3458	0.0520	21.2217	42.1367
BGEUP	3.1736	0.0505	24.3360	41.9716
BPLUP	3.5835	0.0493	21.5553	43.1217

It is observed from Table 4 that MSEs of BSELP is lowest from all other Bayesian and non-Bayesian estimators. Hence it is concluded that Bayesian approach using SELF taking LP is best for both α and θ from all other estimation methods. It is also concluded that Bayesian estimation methods are best as compared to MLEs.

4.2 Type-II censoring

Type-II censoring scheme is the most popular censoring scheme used in the reliability and life testing experiments. In this censoring scheme the experiment n items are set first and placed on test and the number of uncensored data r is predetermined. Instead of continuing experiments until all n items have failed, the experiment is terminated when the rth item fails. The remaining $(n-r)$ items are regarded as censored data. This censoring scheme is applied by several scholars as Abu-Zinadah (2010), Ahmadi *et al.* (2010), Balakrishnan and Sandhu (1995), Viveros and Balakrishnan (1994), Thomas and Wilson (1972) and Mann (1971). For this censoring scheme MLEs and their CIs are found with different percentages of failures

(r) and various sample sizes (n). Furthermore, data is generated through simulation study and MLEs are estimated using BFGS algorithem. Average estimates for both parameters, MSEs (within parenthesis) and CIs are shown in Tables 5-12 for Type-II censoring as under:

Table 5: Average estimates for $\alpha = 1$, MSEs (within parenthesis) and CIs.

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
30	30	1.6903(1.8894)	(-2.9394, 6.3201)	(-3.8264, 7.2070)
	40	1.3992(0.9267)	(-4.2430, 7.0416)	(-5.3240, 8.1225)
	50	1.2594(**0.3991**)	(-4.9889, 7.5078)	(-6.1859, 8.7048)
50	30	1.3456(0.7477)	(-4.8665, 7.5578)	(-6.0566, 8.7479)
	40	1.2157(0.3173)	(-6.0077. 8.4392)	(-7.3915, 9.8230)
	50	1.1749(**0.1921**)	(-6.7999, 9.1499)	(-8.3277, 10.6777)
80	30	1.2408(0.3350)	(-6.5777, 9.0594)	(-8.0755, 10.5572)
	40	1.1149(0.1527)	(-8.1803, 10.4103)	(-9.9611, 12.1910)
	50	2.0999(**0.0943**)	(-9.1215, 11.3213)	(-11.0796, 13.2795)
100	30	0.1646(0.2255)	(-7.6784, 10.0077)	(-9.3725, 11.7018)
	40	1.1168(0.1235)	(-9.1015, 11.3351)	(-11.0590, 13.2927)
	50	1.0785(**0.0769**)	(-10.3866, 12.5438)	(-12.5831, 14.7402)
150	30	1.1251(0.1330)	(-9.6859, 11.9363)	(-11.7571, 14.0074)
	40	1.0645(0.0691)	(-11.6167, 13.7458)	(-14.0461, 16.1752)
	50	1.0347(**0.0391**)	(-13.2588, 15.3282)	(-15.9971, 18.0665)
200	30	1.0832(0.0832)	(-11.5162, 13.6826)	(-13.9299, 16.0964)
	40	1.0587(0.0531)	(-13.5253, 15.6428)	(-16.3193, 18.4367)
	50	1.0336(**0.0344**)	(-15.3615, 17.4288)	(-18.5024, 20.5697)
300	30	1.0501(0.0553)	(-14.4978, 16.5982)	(-17.4765, 19.5768)
	40	1.0384(0.0312)	(-16.8005, 18.8773)	(-20.2179, 22.2948)
	50	1.0217(**0.0194**)	(-19.0581, 21.1016)	(-22.9049, 24.9483)
400	30	1.0409(0.0364)	(-16.8630, 18.9449)	(-20.2930, 22.3749)
	40	1.0297(0.0229)	(-19.6205, 21.6800)	(-23.5765, 25.6361)
	50	1.0168(**0.0160**)	(-22.2293, 24.2630)	(-26.6827, 28.7164)
500	30	1.0334(0.0295)	(-19.0140, 21.0809)	(-22.8545, 24.9214)
	40	1.0192(0.0175)	(-22.2527, 24.2912)	(-26.7110, 28.7495)
	50	1.0147(**0.0120**)	(-24.9090, 26.9385)	(-29.8753, 31.9048)

Table 6: Average estimates for $\alpha = 1.5$, MSEs (within parenthesis) and CIs.

$n \downarrow$	$r(\%) \downarrow$	MLE	90%	95%
30	30	2.2138(2.4929)	(-2.0066, 6.4343)	(-2.8151, 7.2428)
	40	2.0352(1.1790)	(-2.6722. 6.7427)	(-3.5740. 7.6445)
	50	1.8540(**0.6281**)	(-3.4028, 7.1110)	(-4.4099, 8.1181)
50	30	1.8572(0.7685)	(-3.6550, 7.3695)	(-4.7110, 8.4255)
	40	1.9162(0.9757)	(-3.5728, 7.4053)	(-4.6243, 8.4568)
	50	1.7698(**0.4592**)	(-4.4812, 8.0209)	(-5.6787, 9.2185)
80	30	1.6951(0.2612)	(-5.2045, 8.5947)	(-6.5263, 9.9163)
	40	1.7163(0.3948)	(-5.3223. 8.7551)	(-6.6707. 10.1035)
	50	1.6637(**0.2323**)	(-6.3025, 9.6301)	(-7.8287, 11.1562)
100	30	1.6145(0.1569)	(-7.1946, 10.4236)	(-8.8822, 12.1112)
	40	1.6926(0.2916)	(-6.1130, 9.4983)	(-7.6084. 10.9937)
	50	1.6312(**0.1717**)	(-7.2866, 10.5490)	(-8.9950, 12.2574)
150	30	1.5790(0.1111)	(-8.2977, 11.4559)	(-10.1898, 13.3480)
	40	1.5911(0.1732)	(-8.1776, 11.3600)	(-10.0491, 13.2314)
	50	1.5770(**0.0979**)	(-9.4206, 12.5746)	(-11.5274, 14.6815)
200	30	1.5662(0.0750)	(-10.4879, 13.6204)	(-12.7971. 15.9296)
	40	1.5761(0.1141)	(-9.6328, 12.7850)	(-11.7801, 14.9324)
	50	1.5627(**0.0745**)	(-11.1255, 14.2510)	(-13.5562, 16.6818)
300	30	1.5459(0.0539)	(-12.4311, 15.5230)	(-15.1087, 18.2006)
	40	1.5612(0.0777)	(-12.1729, 15.2954)	(-14.8040, 17.9265)
	50	1.5337(**0.0469**)	(-14.0948, 17.1622)	(-17.0888. 20.1562)
400	30	1.5205(**0.0340**)	(-15.7098. 18.7509)	(-19.0107, 22.0518)
	40	1.5436(0.0600)	(-14.3210, 17.4083)	(-17.3603, 20.4475)
	50	1.5313(0.0375)	(-16.5065. 19.5693)	(-19.9621. 23.0249)
500	30	1.5209(**0.0235**)	(-18.3116, 21.3535)	(-22.1110, 25.1529)
	40	1.5295(0.0402)	(-16.2577, 19.3168)	(-19.6652, 22.7243)
	50	1.5221(0.0297)	(-18.6125, 21.6567)	(-22.4697, 25.5140)

Table 7: Average estimates for $\alpha = 2$, MSEs (within parenthesis) and CIs.

$n \downarrow$	$r(\%) \downarrow$	MLE	90%	95%
30	30	2.7610(2.5974)	(-1.0330, 6.5551)	(-1.7598, 7.2819)
	40	2.5955(1.6317)	(-1.5998, 6.7910)	(-2.4036, 7.5947)
	50	2.4174(**0.9659**)	(-2.2476, 7.0825)	(-3.1413, 7.9762)
50	30	2.4112(0.9708)	(-2.5286, 7.3511)	(-3.4750, 8.2975)
	40	2.2439(0.5934)	(-3.3735, 7.8615)	(-4.4497. 8.9376)
	50	2.2046(**0.3966**)	(-3.9274, 8.3367)	(-5.1022, 9.5115)
80	30	2.2901(0.5653)	(-3.9949, 8.5753)	(-5.1990. 9.7793)
	40	2.1810(0.3230)	(-4.9496, 9.3116)	(-6.3157. 10.6777)

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
	50	2.1027(**0.1929**)	(-5.6867, 9.8921)	(-7.1790, 11.3844)
100	30	2.1741(0.3695)	(-4.9719, 9.3203)	(-6.3409, 10.6893)
	40	2.1279(0.2299)	(-5.8576, 10.1135)	(-7.3874. 11.6433)
	50	2.1121(**0.1754**)	(-6.5215, 10.7458)	(-8.1755, 12.3997)
150	30	2.1304(0.2297)	(-6.5751, 10.8360)	(-8.2428, 12.5038)
	40	2.0965(0.1506)	(-7.6658, 11.8589)	(-9.5360, 13.7292)
	50	2.0728(**0.1032**)	(-8.5792, 12.7249)	(-10.6199, 14.7655)
200	30	2.1034(0.1846)	(-7.9810, 12.1878)	(-9.9129, 14.1198)
	40	2.0628(0.1054)	(-9.2495, 13.3753)	(-11.4167, 15.5424)
	50	2.0551(**0.0726**)	(-10.2717, 14.3820)	(-12.6332, 16.7435)
300	30	2.0688(0.1026)	(-10.3002, 14.4379)	(-12.6698, 16.8075)
	40	2.0512(0.0669)	(-11.7947. 15.8971)	(-14.4472. 18.5197)
	50	2.0364(**0.0476**)	(-13.0398, 17.1128)	(-15.9281, 20.0019)
400	30	2.0532(0.0799)	(-12.2225, 16.3291)	(-14.9574, 19.0640)
	40	2.0385(0.0480)	(-13.9795, 18.0565)	(-17.0481, 21.1252)
	50	2.0297(**0.0352**)	(-15.3970, 19.4564)	(-18.7355, 22.7949)
500	30	2.0427(0.0605)	(-13.9169, 18.0023)	(-16.0023, 21.0598)
	40	2.0308(0.0407)	(-15.8599. 19.9216)	(-19.2873. 23.3490)
	50	2.0245(**0.0287**)	(-17.4332, 21.4823)	(-21.1608, 25.2099)

Discussion of the results for parameter α are as under:

Tables 5-7 shows that the average estimates for α, MSEs (with in paranthesis) and CLs. For α, we taken different percentages of failures for each sample sizes.

It is seen that when sample size and different percentages of failures increases then MSEs of MLEs decreases. This shows that the consistency of estimators. It is also noticed that width of 90% and 95% CIs are also becomes narrow when we increase sample sizes. This shows that the accuracy of the results. And the results for shape parameter θ are as under:

Table 8: Average estimates for $\theta = 1$, MSEs (within parenthesis) and CIs.

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
30	30	1.4177(0.7703)	(-6.0544, 8.8899)	(-7.4858, 10.3214)
	40	1.3205(0.6341)	(-6.6193, 9.2605)	(-8.1404, 10.7816)
	50	1.2151(**0.2691**)	(-7.1000, 9.5302)	(-8.6930, 11.1232)
50	30	1.1948(0.2439)	(-9.1824, 11.5720)	(-11.1704, 13.5600)
	40	1.1486(0.1540)	(-9.6095, 11.9068)	(-11.6705, 13.9678)
	50	1.1267(**0.1065**)	(-9.7557, 12.0093)	(-11.8405, 14.0941)
80	30	1.1356(0.1201)	(-12.1368, 14.4080)	(-14.6794, 16.9507)
	40	1.0871(0.7770)	(-12.8340, 15.0083)	(-15.5010, 17.6752)
	50	1.0679(**0.0496**)	(-13.0679, 15.2039)	(-15.7760, 17.9120)
100	30	1.0836(0.0748)	(-14.2219, 16.3892)	(-17.1540, 19.3214)
	40	1.0684(0.0489)	(-14.5649, 16.7019)	(-17.5599, 19.6969)
	50	1.0513(**0.0405**)	(-14.9306, 17.0332)	(-17.9923, 20.0950)
150	30	1.0652(0.0415)	(-17.6445, 19.7750)	(-21.2288, 23.3593)
	40	1.0415(0.0293)	(-18.3813, 20.4643)	(-22.1022, 24.1853)
	50	1.0317(**0.0218**)	(-18.6732, 20.7366)	(-22.4481, 24.5116)
200	30	1.0438(0.0264)	(-20.8144, 22.9021)	(-25.0018, 27.0896)
	40	1.0386(0.0228)	(-21.3214, 23.3986)	(-25.6050, 27.6822)
	50	1.0224(**0.0176**)	(-21.8603, 23.9052)	(-26.2440, 28.2890)
300	30	1.0255(0.0171)	(-26.0471, 28.0981)	(-31.2334, 33.2845)
	40	1.0197(0.0132)	(-26.6781, 28.7175)	(-31.9842, 34.0237)
	50	1.0144(**0.0108**)	(-27.0725, 29.1015)	(-32.4533, 34.4822)
400	30	1.0214(0.0121)	(-30.2304, 32.2733)	(-36.2174, 38.2604)
	40	1.0153(0.0088)	(-30.9872, 33.0179)	(-37.1181, 39.1487)
	50	1.0121(**0.0076**)	(-31.4050, 33.4292)	(-37.6153, 39.6395)
500	30	1.0182(0.0094)	(-33.9509, 35.9874)	(-40.6501, 42.6865)
	40	1.0152(0.0073)	(-34.7159, 36.7465)	(-41.5610, 43.5916)
	50	1.0087(**0.0063**)	(-35.3109, 37.3284)	(-42.2688, 44.2863)

Table 9: Average estimates for $\theta = 1.5$, MSEs (within parenthesis)and CIs.

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
30	30	2.3276(5.6170)	(-2.7008, 7.3561)	(-3.6642, 8.3194)
	40	2.1003(1.9077)	(-3.0513, 7.2520)	(-4.0383, 8.2390)
	50	1.9114(**0.9665**)	(-3.5035, 7.3263)	(-4.5408, 8.3637)
50	30	1.8283(0.6449)	(-5.0810, 8.7376)	(-6.4047, 10.0613)
	40	1.9021(0.9634)	(-4.8792, 8.6834)	(-6.1783, 9.9825)
	50	1.7899(**0.4926**)	(-5.2711, 8.8509)	(-6.6238, 10.2036)
80	30	1.7261(0.3260)	(-5.4691, 8.9214)	(-6.8475, 10.2998)
	40	1.6819(0.2794)	(-7.3433, 10.7072)	(-9.0723, 12.4362)

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
	50	1.6630(**0.2045**)	(-7.5150, 10.8411)	(-9.2733, 12.5994)
100	30	1.6288(0.1629)	(-7.7570, 11.0148)	(-9.5551, 12.8129)
	40	1.6560(0.1853)	(-8.4007, 11.7128)	(-10.3273, 13.6394)
	50	1.6248(**0.1355**)	(-8.7140, 11.9637)	(-10.6946, 13.9443)
150	30	1.5839(0.1085)	(-9.0858, 12.2538)	(-11.1299, 14.2978)
	40	1.5807(0.1159)	(-11.1273, 14.2887)	(-13.5618, 16.7232)
	50	1.5762(**0.0806**)	(-11.3055, 14.4580)	(-13.7733, 16.9258)
200	30	1.5667(0.0727)	(-11.5032, 14.6367)	(-14.0071, 17.1405)
	40	1.5612(0.0706)	(-13.1110. 16.2334)	(-15.9218. 19.0442)
	50	1.5575(**0.0564**)	(-13.3800, 16.4952)	(-16.2417, 19.3569)
300	30	1.5471(0.0488)	(-13.6193, 16.7137)	(-16.5248, 19.6192)
	40	1.5536(0.0490)	(-16.3676, 19.4748)	(-19.8008, 22.9080)
	50	1.5348(**0.0361**)	(-16.9085, 19.9781)	(-20.4417, 23.5114)
400	30	1.5273(0.0311)	(-17.1779, 20.2325)	(-20.7614, 23.8160)
	40	1.5315(0.0344)	(-19.3756. 22.4386)	(-23.3808, 26.4438)
	50	1.5327(**0.0269**)	(-19.7212, 22.7868)	(-23.7929, 26.8585)
500	30	1.5235(0.0208)	(-20.0350, 23.0821)	(-24.1651, 27.2122)
	40	1.5341(0.0236)	(-21.8608, 24.9091)	(-26.3407, 29.3891)
	50	1.5170(**0.0202**)	(-22.4407, 25.4748)	(-27.0303, 30.0645)

Table 10: Average estimates for $\theta = 2$, MSEs (within parenthesis) and CIs.

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
30	30	3.1003(6.9538)	(-0.6496, 6.8504)	(-1.3680, 7.5688)
	40	2.8488(4.7047)	(-1.0483, 6.7459)	(-1.7949, 7.4925)
	50	2.6536(**2.4542**)	(-1.3741, 6.6814)	(-2.1458, 7.4531)
50	30	2.4918(1.4203)	(-2.6148, 7.5985)	(-3.5932, 8.5768)
	40	2.3218(0.9420)	(-3.1785, 7.8222)	(-4.2323, 8.8759)
	50	2.2984(**0.6552**)	(-3.1685, 7.7654)	(-4.2158, 8.8127)
80	30	2.3381(0.6797)	(-4.2392, 8.9155)	(-5.4992, 10.1755)
	40	2.2463(0.4474)	(-4.6007, 9.0933)	(-5.9124. 10.4050)
	50	2.1405(**0.2585**)	(-4.9972, 9.2783)	(-6.3646, 10.6457)
100	30	2.2126(0.4254)	(-5.4013, 9.8266)	(-6.8600, 11.2853)
	40	2.1605(0.2680)	(-5.6639, 9.9850)	(-7.1628. 11.4840)
	50	2.1298(**0.2383**)	(-5.8564, 10.1162)	(-7.3864, 11.6461)
150	30	2.1403(0.2255)	(-7.2629, 11.5437)	(-9.0643, 13.3451)
	40	2.1154(0.1782)	(-7.5284, 11.7594)	(-9.3759, 13.6069)
	50	2.0940(**0.1287**)	(-7.6802, 11.8683)	(-9.5527, 13.7408)

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
200	30	2.1153(0.1780)	(-8.8031, 13.0338)	(-10.8948, 15.1255)
	40	2.0704(0.1057)	(-9.1890, 13.3298)	(-11.3460, 15.4868)
	50	2.0770(**0.9190**)	(-9.2233, 13.3773)	(-11.3881, 16.5422)
300	30	2.0749(0.0874)	(-11.3444, 16.4944)	(-13.9152, 18.0652)
	40	2.0599(0.0669)	(-11.6986, 15.8186)	(-14.3344, 18.4544)
	50	2.0423(**0.0582**)	(-11.9550, 16.0396)	(-14.6365, 18.7212)
400	30	2.0535(0.0693)	(-13.5565, 17.6635)	(-16.5470, 20.6540)
	40	2.0476(0.0495)	(-13.8633, 17.9585)	(-16.9114, 21.0066)
	50	2.0353(**0.0402**)	(-14.1149, 18.1856)	(-17.2088, 21.2795)
500	30	2.0481(0.0523)	(-15.4653, 19.5469)	(-18.8190, 22.9007)
	40	2.0322(0.0414)	(-15.8678, 19.9323)	(-19.2970, 23.3615)
	50	2.0229(**0.0328**)	(-16.1196, 20.1655)	(-19.5952, 23.6411)

Discussion of the results for parameter θ are as under:

Tables 8-10 show that the average estimates for θ, MSEs (with in paranthesis) and CLs. For θ, we also taken different percentages of failures for each sample sizes.

We found the results of MLEs and their 90% and 95% CIs using Type-II censoring for parameter θ. It is observed that when sample sizes and percentages of failures increases then MSEs of MLEs decreases for both parameters of EPD. This indicates that the consistency and efficiency of MLEs. Widths of 90% and 95% CIs are also becomes narrow by increasing sample sizes and percentages of failures. This shows the accuracy of the results.

4.2.1 Real data analysis for Type-II censoring

For this censoring scheme same real data set is used for censored samples which is given in Table 3. Results from that data set using this censoring scheme are as under:

Table 11: Average estimates for α, MSEs (within parenthesis) and CIs.

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
100	20	1.8414(0.1648)	(-12.9975, 16.6803)	(-15.8402, 19.5231)
	30	2.0255(0.1157)	(-14.0133, 18.0644)	(-17.0860, 21.1370)
	40	2.0335(0.0838)	(-15.1602, 19.2273)	(-18.6167, 23.0617)
	50	2.2224(0.0718)	(-15.2663, 19.7113)	(-18.6167, 23.0617)
	60	2.4474(0.0652)	(-15.0559, 19.9509)	(-18.4091, 23.3041)
	70	2.6159(0.0598)	(-14.9300, 20.1620)	(-18.2914, 23.5234)
	80	2.8086(0.0565)	(-14.6405, 20.2577)	(-17.9833, 23.6005)
	90	2.9626(**0.0534**)	(-14.4086, 20.3340)	(-17.7365, 23.6619)

Table 12: Average estimates for θ, MSEs (within parenthesis) and CIs.

$n\downarrow$	$r(\%)\downarrow$	MLE	90%	95%
100	20	9.2956(18.7554)	(7.6714, 10.9199)	(7.3602, 11.2310)
	30	10.8110(14.0197)	(9.3539, 12.2682)	(9.0747, 12.5474)
	40	10.8719(13.3055)	(9.3911, 12.3527)	(9.1074, 12.6364)
	50	12.8766(12.6025)	(11.6134, 14.1399)	(11.3713, 14.3819)
	60	15.7714(11.7667)	(14.7345, 16.8083)	(14.5359, 17.0070)
	70	18.3489(11.0371)	(17.4548, 19.2429)	(17.2835, 19.4142)
	80	21.8294(10.3817)	(21.0767, 22.5820)	(20.9325, 22.7262)
	90	25.0818(**10.0093**)	(24.4261, 25.7374)	(24.3005, 25.8630)

Tables 11 and 12 shows that the average estimates for α and θ their MSEs (with in paranthesis) and CIs taking different percentages of failures. It is observed that MSEs of MLEs are decreases when percentages of failures increases. This is congurent to the consistency of the estimators. It is also noticed that 90% and 95% CIs becomes narrow when percentages of failures increases. It indicates that the accuracy of the results.

Chapter 5

Summary and Conclusion

The present research proposes classical and Bayesian approaches to estimate the two unknown shape parameters of EPD for complete samples. And for the case of censored samples MLEs and their CIs are considered for checking the behaviour of parameters of given distribution. MLEs are calculated through BFGS algorithem.

Current research work developed Bayesian estimators of EPD under different loss functions using Lindley's approximation taking informative and non-informative types of priors. Detail of abbreviations of Bayesian estimators are as follows:

Bayesian estimators as Bayesian LINEX Gamma Prior (BLGP), Bayesian LINEX Levy Prior (BLLP), Bayesian LINEX Uniform Prior (BLUP). Bayesian General Entropy Gamma Prior (BGEGP), Bayesian General Entropy Levy Prior (BGELP). Bayesian General Entropy Uniform Prior (BGEUP), Bayesian Squared Error Gamma Prior (BSEGP), Bayesian Squared Error Levy Prior (BSELP). Bayesian Squared Error Uniform Prior (BSEUP), Bayesian Precautionary Gamma Prior (BPLGP). Bayesian Precautionary Levy Prior (BPLLP) and Bayesian Precautionary Uniform Prior (BPLUP).

Results of simulation study illustrated in Table 1-4 for complete samples. Estimates of both shape parameters of EPD revealed that MSEs decreases as sample size increases for different values of parameters, which is congruent to the proposition of consistency of estimators. Bayesian estimators of EPD using Lindley's approximation under

different loss functions based on Gamma, Levy and Uniform priors also divulged that BSELP, BGELP and BPLGP has overestimate for different values of parameters of EPD, so we can say that BSELP, BGELP and BPLGP overestimate the parameters for estimating the θ while for the estimating the α, BSELP, BGELP and BPLGP underestimate the parameters of EPD.

Bayesian estimators of EPD are obtained using Lindley's approximation by taking informative and non-informative types of priors. Therefore, Lindley's approximation is a good alternative for the case in which the Bayesian estimators of EPD cannot be obtained in explicit forms. It appears from this study that the Bayesian estimators using informative priors are superior to the method of MLEs.

From the results of Type-II censoring, MSEs of MLEs for parameters of EPD decreases as sample sizes increases, so that the bias tends to be worse for the larger sample sizes with increasing percentages of failures. It is also observed that the length of CIs becomes narrow when sample size and percentages of failures increases. This indicated that the MLEs are consistent and approaches true parameters value. In terms of MSEs the MLEs are better for all level of Type-II censoring by increasing sample sizes. Also for real data set MSEs of MLEs for both parameters of EPD decreases when we increase percentages of failures. CIs are also becomes narrow by increasing percentages of failures.

Recommendations: Further, this research study can be extend for estimating the Bayesian estimators of EPD under Type-I and Type-II censoring schemes. Credible intervals, Hypothesis testing can also be obtain for complete and Type-I censored samples.

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Appendices

Appendix A

Gamma prior (GP)

Logfunction from Gamma prior may be written as:

$$\log \pi(\alpha, \theta) = (a_1 - 1)\log(\alpha) - b_1\alpha + (a_2 - 1)\log(\theta) - b_2\,\theta,$$

$$\frac{\partial \log \pi(\alpha, \theta)}{\partial \alpha} = \frac{a_1 - 1}{\alpha} - b_1\,,$$

$$\frac{\partial \log \pi(\alpha, \theta)}{\partial \theta} = \frac{a_2 - 1}{\theta} - b_2.$$

Levy prior (LP)

$$\log \pi(\alpha, \theta) = \frac{1}{2}\log(\alpha) - \frac{3}{2}\log(\theta) - \frac{\alpha}{2\,\theta},$$

$$\frac{\partial \log \pi(\alpha, \theta)}{\partial \alpha} = \frac{1}{2}\left(\frac{\theta - \alpha}{\alpha\,\theta}\right), \frac{\partial \log \pi(\alpha, \theta)}{\partial \theta} = \left(\frac{\alpha - 3\,\theta}{2\,\theta^2}\right).$$

Improper prior (IP)

The Improper prior $\pi(\theta)$ is one which satisfied the equation

$$\int_{-\infty}^{\infty} \pi(\theta)d\theta = \infty.$$

For example, $\pi(\theta) \propto 1, \quad -\infty < \theta \leq \infty$, is an improper prior. In Bayesian inference by employing an improper prior, proper posterior distribution can be obtained and inference based on such posterior distribution will be valid.

Appendix B

Detail of all terms of Lindley's approximation for complete samples are:

$\hat{g}(\hat{\alpha}, \hat{\theta})$ is taken from according to the simplified form of any loss function.

$$\sum_{i=1}^{2} \sum_{j=1}^{2} v_{ij} \tau_{ij} = v_{11}\tau_{11} + v_{12}\tau_{12} + v_{21}\tau_{21} + v_{22}\tau_{22}.$$

$$L_{ij} = \frac{\partial^{i+j}\ \theta}{\partial \alpha^{i}\ \partial \theta^{j}}, \quad i,\ j = 0,\ 1,\ 2,\ 3, \quad i + j = 3. \qquad B_{ij} = (v_i\ \tau_{ii} + v_j\ \tau_{ij})\ \tau_{ii}.$$

$$C_{ij} = 3\ v_i\ \tau_{ii}\ \tau_{ij} + v_j\ \left(\tau_{ii}\ \tau_{jj} + 2\ \tau_{ij}^{2}\right), \quad A_{ij} = v_i\ \tau_{ii} + v_j\ \tau_{ji}, \quad i,\ j = 1,\ 2.$$

$$W_1 = \frac{\partial\ \log \pi(\alpha, \theta)}{\partial \alpha}, \quad W_2 = \frac{\partial\ \log \pi(\alpha, \theta)}{\partial \theta}.$$

$$p = -\frac{\partial^2 \log L(\alpha, \theta)}{\partial \alpha^2} = \frac{n}{\alpha^2} + (\theta - 1) \sum_{i=1}^{n} \frac{(1 + x_i)^{-\alpha} (\log(1 + x_i))^2}{(1 - (1 + x_i)^{-\alpha})^2}.$$

$$q = -\frac{\partial^2 \log L(\alpha, \theta)}{\partial \theta^2} = \frac{n}{\theta^2}.$$

$$r = -\frac{\partial^2 \log L(\alpha, \theta)}{\partial \theta\ \partial \alpha} = -\sum_{i=1}^{n} \frac{(1 + x_i)^{-\alpha} (\log(1 + x_i))}{1 - (1 + x_i)^{-\alpha}}.$$

Where explicit expressions are defined as:

$$\tau_{11} = \frac{q}{pq - r^2}; \quad \tau_{22} = \frac{p}{pq - r^2}, \quad \tau_{12} = \tau_{21} = \frac{r}{pq - r^2}, \quad v_1 = \frac{\partial\ g(\alpha, \theta)}{\partial \alpha}$$

$$v_2 = \frac{\partial\ g(\alpha, \theta)}{\partial \theta}, \quad v_{11} = \frac{\partial^2 g(\alpha, \theta)}{\partial \alpha^2}, \quad v_{22} = \frac{\partial^2\ g(\alpha, \theta)}{\partial \theta^2}, \quad v_{12} = \frac{\partial^2\ g(\alpha, \theta)}{\partial \alpha\ \partial \theta}.$$

$$v_{21} = \frac{\partial^2\ g(\alpha,\ \theta)}{\partial \theta\ \partial \alpha}.$$

$$L_{30} = \frac{\partial^3 \log L}{\partial \alpha^3} = \frac{2n}{\hat{\alpha}^3} + (\hat{\theta} - 1) \sum_{i=1}^{n} \frac{(1 + (1 + x_i)^{-\hat{\alpha}})\,(\log(1 + x_i))^3\,(1 + x_i)^{-\hat{\alpha}}}{(1 - (1 + x_i)^{-\hat{\alpha}})^3}$$

$$L_{03} = \frac{\partial^3 \log L}{\partial \theta^3} = \frac{2n}{\hat{\theta}^3}, \qquad L_{12} = \frac{\partial^3 \log L}{\partial \alpha\, \partial \theta^2} = 0.$$

$$L_{21} = \frac{\partial^3 \log L}{\partial \alpha^2\, \partial \theta} = -\sum_{i=1}^{n} \frac{(1 + x_i)^{-\hat{\alpha}}\,(\log(1 + x_i))^2}{(1 - (1 + x_i)^{-\hat{\alpha}})^2}.$$

$$B_{12} = v_1 \tau_{11}^2 + v_2 \tau_{12} \tau_{11}, \quad B_{21} = v_2 \tau_{22}^2 + v_1 \tau_{21} \tau_{22}.$$

$$C_{12} = 3\, v_1\, \tau_{11}\, \tau_{12} + v_2\, (\tau_{11}\, \tau_{22} + 2\, \tau_{12}^2), \; C_{21} = 3\, v_2\, \tau_{22}\, \tau_{21} + v_1\, (\tau_{22}\, \tau_{11} + 2\, \tau_{21}^2)$$

$$A_{12} = v_1\, \tau_{11} + v_2\, \tau_{21}, \qquad A_{21} = v_2\, \tau_{22} + v_1\, \tau_{12} \quad i,\ j = 1,\ 2.$$

Detail of terms for Type-II censoring are:

$$p = -\frac{\partial^2 \log L(\alpha, \theta)}{\partial \alpha^2} = \frac{r}{\alpha^2} + (\theta - 1) \sum_{i=1}^{r} \frac{(1 + x_i)^{-\alpha}(\log(1 + x_i))^2}{(1 - (1 + x_i)^{-\alpha})^2}$$

$$q = -\frac{\partial^2 \log L(\alpha, \theta)}{\partial \theta^2} = \frac{r}{\theta^2}.$$

$$r = -\frac{\partial^2 \log L(\alpha, \theta)}{\partial \theta\, \partial \alpha} = -\sum_{i=1}^{r} \frac{(1 + x_i)^{-\alpha}(\log(1 + x_i))}{1 - (1 + x_i)^{-\alpha}} + (n - r)\log(1 + x_r).$$

$$L_{30} = \frac{\partial^3 \log L}{\partial \alpha^3} = \frac{2r}{\hat{\alpha}^3} + (\hat{\theta} - 1) \sum_{i=1}^{r} \frac{(1 + (1 + x_i)^{-\hat{\alpha}})\,(\log(1 + x_i))^3\,(1 + x_i)^{-\hat{\alpha}}}{(1 - (1 + x_i)^{-\hat{\alpha}})^3}.$$

$$L_{03} = \frac{\partial^3 \log L}{\partial \theta^3} = \frac{2r}{\hat{\theta}^3}, \qquad L_{12} = \frac{\partial^3 \log L}{\partial \alpha\, \partial \theta^2} = 0.$$

$$L_{21} = \frac{\partial^3 \log L}{\partial \alpha^2\, \partial \theta} = -\sum_{i=1}^{r} \frac{(1 + x_i)^{-\hat{\alpha}}\,(\log(1 + x_i))^2}{(1 - (1 + x_i)^{-\hat{\alpha}})^2}.$$