Fixed Points of Locally and Globally Contractive Mappings in Ordered Spaces

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***DOCTOR OF PHILOSOPHY IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD.

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DECLARATION

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1. Prof. Muhammad Arshad
   (Supervisor) ____________________

2. Abdullah Shoaib
   (Student) ____________________

DEDICATED TO….

My parents, teachers, friends and family for supporting and encouraging me.

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CONTENTS

Preface

Let $T : X ! X$ be a mapping. A point x 2 X is called a xed point of T if $x = Tx:$

Let x_0 be an arbitrarily chosen point in $X.$ De ne a sequence $\{x_n\}$ in X by a simple iterative method given by $x_{n+1} = Tx_n,$ where $n \in \{0, 1, 2, \dots\}.$ Such a sequence is called a Picard iterative sequence and its convergence plays an important role in proving existence of a xed point of a mapping $T.$ A self mapping T on a metric space X is said to be a Banach contraction mapping if $d(Tx, Ty) \leq kd(x, y)$ holds for all $x, y \in X$ where $0 \leq k < 1.$ The Banach xed point theorem is commonly known as Banach contraction principle, which states that if X is a complete metric space and T is a Banach contraction mapping on X, then T has a unique xed point in X. This theorem looks simple but plays a fundamental role in the eld of xed point theory and has become even more important because being based on iteration, it can easily be implemented on a computer. The Banach contraction principle implies that T is uniformly continuous on X. It is natural to ask if there is a contractive de nition which does not force T to be continuous. It was answered in a¢ rmative by Kannan [45], who established a xed point theorem for a self mapping T satisfying, $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)],$ for all $x, y \in X,$ where $0 \leq k < \frac{1}{2}.$ Chatterjea [26], proved a similar result for a self mapping satisfying, $d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$ where $0 \leq k < \frac{1}{2}.$ It is important to note that these three theorems are independent of each other and have laid down the foundation of modern xed point theory for contractive type mappings.

Fixed point results of mappings satisfying certain contractive condition on the entire domain have been at the centre of rigorous research activity, for example (see [10, 19, 20, 22, 27, 28, 29, 38, 39, 59, 84, 75]) and they have a wide range of applications in di⁄erent areas such as nonlinear and adaptive control systems, parameterize estimation problems, computing magnetostatic elds in a nonlinear medium and convergence of recurrent networks. (see [53, 69, 82, 83]).

Ran and Reurings [64] proved an analogue of Banach s xed point theorem in a metric space endowed with a partial order and gave applications to matrix equations. In this way, they weakened the usual contractive condition. Subsequently, Nieto et. al. [60] extended the results in [64] for nondecreasing mappings and applied it to obtain a unique solution for a 1st order ordinary di⁄erential equation with periodic boundary conditions. Samet and Vetro [71] generalized the results in ordered metric spaces and introduced the concept of contractive type mappings and established xed point theorems for such mappings in complete metric spaces.

On the other hand, the notion of a partial metric space was introduced by Matthews in [55]. In partial metric spaces, the distance of a point from itself may not be zero. Partial metric spaces have applications in theoretical computer science (see [42]). Altun et. al. [6], Samet et. al. [70] and Paesano et. al. [62] used the idea of partial metric space and partial order and gave some xed point theorems for contractive condition on ordered partial metric spaces. To generalize partial metric space, Hitzler et. al. [37] introduced the concept of a dislocated topologies and its corresponding generalized metric space named as dislocated metric space (metric-like space [8]) and have established a xed point theorem in complete dislocated metric spaces which generalizes the celebrated Banach contraction principle. The notion of dislocated topologies has useful applications in the context of logic programming semantics (see [36]). For further related results see ([1, 8, 44, 46, 47, 50, 54, 66, 72, 81]). Furthermore, dislocated quasi metric space (quasi-metric-like space) (see [1, 23, 73, 84, 85]) generalized the idea of dislocated metric space and quasi-partial metric space(see [33, 48, 57, 74, 76]).

A multivalued function is a set valued function. In the last thirty years, the theory of multivalued functions has advanced in a variety of ways. In 1969, the systematic study of Banach type xed theorems of multivalued mappings had been started with the work of Nadler [57], who proved that a multivalued contractive mapping of a complete metric space X into the family of closed bounded subsets of X has a xed point. His ndings were followed by many authors (see [17, 18]). Asl et al. [14] generalized the notion of ( ) contractive mapping by introducing the concepts of ( ) contractive multifunctions and obtained some xed point results for these multivalued contractive mapping (see also [5, 40, 41]).

This thesis deals with the xed point results of locally, globally, single and multivalued contractive mappings in ordered spaces. This dissertation consists of four chapters. Each chapter begins with a brief introduction which acts as a summery of the material there in.

Chapter 1, is a survey, aimed at clarifying the terminology to be used and recalls basic de nitions and facts.

Chapter 2, is devoted to study the existence of coincidence and common xed points of mappings satisfying generalized contractive conditions. Some xed point results have been established in the frame work of partial metric space. Moreover, we initiate the concept of an ordered 0-complete left/right *K*-sequentially quasi-partial metric space and prove some results for dominated mappings in these spaces.

Chapter 3, deals with an ordered complete left *K*-sequentially as well as right *K*-sequentially dislocated quasi metric spaces. Moreover, we introduce the concept of -dominated mapping. Some coincidence and common xed point results have been established for -dominated mappings in left/right *K*-sequentially dislocated quasi metric spaces.

Chapter 4, deals with the multivalued contractive mappings. We establish xed point results for -admissible multivalued mappings satisfying generalized - -contractive conditions in complete left *K*-sequentially dislocated quasi metric space. A theorem on xed point of multivalued locally contractive mappings in a fuzzy metric space is also established.

I would like to express my sincere gratitude to my supervisor Dr. Muhammad Arshad. Without his sincere pieces of advice and valuable guidance this thesis could never have become a reality. The department of Mathematics remained encouraging and supportive during my Ph.D. studies for which I am grateful. Finally, I thank my family for their a⁄ection and support throughout my research.

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Pakistan.

Chapter 1

Preliminaries

The aim of this chapter is to present some basic concepts and to explain the terminology used throughout this dissertation. Some previously known results are given without proof. Section 1.1, is devoted to the introductory material on the notions of partial and quasi-partial metric spaces. Section 1.2, is concerned with the introduction of dislocated and dislocated quasi metric spaces which are the natural generalizations of metric spaces. Section 1.3, deals with the concepts of fuzzy metric and hausdor⁄ fuzzy metric spaces. Section 1.4, introduces some other basic relevant concepts.

1.1 Partial and Quasi-partial Metric Spaces

1.1.1 De nition [48]

A quasi-partial metric is a function, $q : X X ! R^+$ satisfying:

(i) if $0\ q(x;x) = q(x;y) = q(y;y);$ then $x = y$ (equality),

(ii) $q(x;x)\ q(y;x)$ (small self-distances),

(iii) $q(x;x)\ q(x;y)$ (small self-distances),

(iv) $q(x;z) + q(y;y)\ q(x;y) + q(y;z)$ (triangle inequality), for all $x;y;z$ 2 $X;$ Then the

pair $(X;q)$ is called a quasi-partial metric space.

Note that, if $q(x;y) = q(y;x)$ for all $x;y$ 2 $X;$ then $(X;q)$ becomes a partial metric space $(X;p)$.

Moreover if $q(x;x) = 0;$ for all x 2 $X;$ then $(X;q)$ and $(X;p)$ become a quasi metric

space and a metric space respectively. Also, $p_q(x,y) = \frac{1}{2}[q(x,y) + q(y,x)]$, where $x;y$ 2 X is a partial metric on X. The function $d_{p_q} : X X ! R^+$ de ned by $d_{p_q}(x;y) = q(x;y) + q(y;x)$

$q(x;x)\ q(y;y)$ is a (usual) metric on $X:$ The ball $\overline{B_q(x;\&)};$ where $\overline{B_q(x;\&)} =$ fy 2 $X : q(x;y)$

$\& + q(x;x)$g and $\overline{B_p(x;\&)} =$ fy 2 $X : p(x;y)\ \& + p(x;x)$g are closed balls in a quasi-partial metric

space and a partial metric space respectively, for some x 2 X and $\&> 0:$

1.1.2 Examples

Let $X = [0,\infty)$, then

(i) [55] $p(x,y) = \max\{x,y\}$ for all $x,y \in X$; defines a partial metric p on X.

(ii) $q(x,y) = \max\{y - x, 0\} + x$ for all $x,y \in X$; defines a quasi-partial metric q on X.

1.1.3 Definition [55]

Let (X,p) be a partial metric space, then

(i) A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x)$.

(ii) A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if the $\lim_{n,m\to\infty} p(x_n,x_m)$ exists (and is finite).

(iii) [68] A sequence $\{x_n\}$ in (X,p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$.

The space (X,p) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$, such that $p(x,x) = 0$. If (X,p) is a partial metric space, then $p_s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, $x,y \in X$, is a metric on X.

1.1.4 Lemma [55]

Let (X,p) be a partial metric space, then

(i) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p_s).

(ii) (X,p) is complete if and only if the metric space (X,p_s) is complete.

(iii) [68] Every 0-Cauchy sequence in (X,p) is Cauchy in (X,p_s).

(iv) [68]If (X,p) is complete, then it is 0-complete.

(v) [59] Every closed subset of a 0-complete partial metric space is 0-complete.

Romaguera [68] has given an example which proves that converse assertions of (iii) and (iv) do not hold.

1.1.5 Definition [48]

Let $(X;q)$ be a quasi-partial metric, then (i) A sequence $\{x_n\}$ in $(X;q)$ converges to a point $x \in X$; if and only if $\lim_{n\to\infty} q(x;x_n) = \lim_{n\to\infty} q(x_n;x) = q(x;x)$.

(ii) A sequence $\{x_n\}$ in $(X;q)$ is called a Cauchy sequence, if the $\lim_{n;m\to\infty} q(x_n;x_m)$ and $\lim_{n;m\to\infty} q(x_m;x_n)$ exists, (and are finite).

(iii) The space $(X;q)$ is said to be complete, if every Cauchy sequence $\{x_n\}$ in $(X;q)$ converges to a point $x \in X$, such that $q(x,x) = \lim_{n,m\to\infty} q(x_n,x_m) = \lim_{n,m\to\infty} q(x_m,x_n)$.

1.1.6 Lemma [48]

Let $(X;q)$ be a quasi-partial metric space, let $(X;p_q)$ be the corresponding partial metric space, then these statements are equivalent. (i) The sequence $\{x_n\}$ is Cauchy in $(X;q)$. (ii) The sequence $\{x_n\}$ is Cauchy in $(X;p_q)$. (iii) The sequence $\{x_n\}$ is Cauchy in $(X;d_{p_q})$. These statements are also equivalent. (i) $(X;q)$ is complete. (ii) $(X;p_q)$ is complete. (iii) $(X;d_{p_q})$ is complete.

1.1.7 Definition [76]

Let X be a nonempty set, then $(X;\preceq;q)$ is called an ordered quasi-partial metric space if:

(i) q is a quasi-partial metric on X and (ii) $\preceq$ is a partial order on X.

1.2 Dislocated and Dislocated Quasi Metric Spaces

1.2.1 Definition [84]

Let X be a nonempty set and let $d_q : X \times X \to [0;\infty)$ be a function, called a dislocated quasi metric (or simply d_q-metric) if the following conditions hold for any $x;y;z$ 2 X : (i)

If $d_q(x;y) = d_q(y;x) = 0;$ then $x = y;$

(ii) $d_q(x;y)$ $d_q(x;z) + d_q(z;y):$

The pair $(X;d_q)$ is called a dislocated quasi metric space. It is clear that if $d_q(x;y) = d_q(y;x) = 0$, then from (i), $x = y$. But if $x = y;$ then $d_q(x;y)$ or $d_q(y;x)$ may not be $0:$ It is observed that if $d_q(x;y) = d_q(y;x)$ for all $x;y$ 2 $X;$ then $(X;d_q)$ becomes a dislocated metric space $(X;d_l)$. Moreover, if $d_q(x;x) = 0$ for all x 2 $X;$ then $(X;d_q)$ and $(X;d_l)$ become

a quasi metric space $(X;q)$ and a metric space $(X;d)$ respectively. The ball $\overline{B_{dq}(x;\&)};$ where

$\overline{B_{dq}(x;\&)}$ = fy 2 X : $d_q(x;y)$ &g; is a closed ball in dislocated quasi metric space, for some x 2 X and &> 0: Recently, Sarm and Kumari [72] proved results that establish existence of a topology induced by a dislocated metric and that this topology is metrizable. This topology has as a base the family of sets f$B(x;\&)$ [fxg : x 2 X;&> 0g, where $B(x;\&)$ is an open ball

and $B(x;\&)$ = fy 2 X : $d_l(x;y)$ <&g for some x 2 X and &> 0. Also, $\overline{B_{dq}(x;\&)}$ = fy 2 X : $d_l(x;y)$ &g is a closed ball.

Also, Harandi [8] de ned the concept of metric like space which is similar to dislocated metric space. Each metric-like on X generates a topology on X whose base is the family of open -balls

$$B\ (x;\&) = \text{f}y\ 2\ X : \text{j}\ (x;y) \qquad (x;x)\text{j} <\&\text{g}:$$

1.2.2 Examples

Let $X = Q^+$ [f0g; then

(i)

$$d_l(x;y) = \begin{cases} 0 & \text{if } x = y = 0 \\ 2 & \text{if } x = y \neq 0 \\ 1 & \text{if } x \neq y \end{cases}$$

de nes a dislocated metric d_l on X. :

(ii) $d_q(x;y) = x + 2y$ de nes a dislocated quasi metric d_q on X.

(iii) $d_q(x;y) = x + \max\{x;y\}$ de nes a dislocated quasi metric d_q on X.

1.2.3 De nition [84]

Let $(X;d_q)$ be a dislocated quasi metric space.

(i) A sequence $\{x_n\}$ in $(X;d_q)$ is called Cauchy if $\forall\, \varepsilon > 0$, $\exists\, n_0 \in N$, such that $\forall\, m;n \geq n_0;$ $d_q(x_m;x_n) < \varepsilon$ or $d_q(x_n;x_m) < \varepsilon$: (ii) A sequence $\{x_n\}$ dislocated quasi-converges (for short dq -converges) to x if $\lim_{n\to\infty} d_q(x_n;x) = \lim_{n\to\infty} d_q(x;x_n) = 0$: In this case x is called a dq-limit of $\{x_n\}$:

(iii) A dislocated quasi metric $(X;d_q)$ is called complete if every Cauchy Sequence in it is dq -convergent.

1.3 Fuzzy Metric and Hausdor Fuzzy Metric Spaces

1.3.1 De nition [32]

A binary operation $*$: $[0;1] \times [0;1] \to [0;1]$ is a continuous t-norm if $([0;1]; *)$ is a topological monoid with unit 1, such that

$$a * b \leq c * d \text{ whenever } a \leq c \text{ and } b \leq d \ (a;b;c;d \in [0;1]).$$

1.3.2 De nition [31]

The 3-tuple $(X;M;*)$ is said to be a fuzzy metric space if X an arbitrary set, $*$ is a continuous

t-norm and M is a fuzzy set on X^2 $(0;1)$ satisfying the following conditions:

(i) $M(x;y;t) > 0;$

(ii) $M(x;y;t) = 1$ if and only if $x = y;$

(iii) $M(x;y;t) = M(y;x;t);$

(iv) $M(x;y;t)\ M(y;z;S)\ M(x;z;t + s);$

(v) $M(x;y;:) : (0;1)\ !\ [0;1]$ is continuous $x;y;z\ 2\ X$ and $t;\ S > 0:$

1.3.3 Remark [31]

$M(x;y;t)$ can be thought of as the degree of nearness between x and y with respect to $t:$ We identify $x = y$ with $M(x;y;t) = 1$ for $t > 0$ and $M(x;y;t) = 0$ when $t = 1:$ In this context

we modify the above de nition in order to introduce a Hausdor⁄ topology on the fuzzy metric space.

1.3.4 Lemma [32]

$M(x;y;:)$ is nondecreasing for all $x;y$ in X.

1.3.5 De nition [32]

Let $(X;M;)$ be a fuzzy metric space. (i) A sequence fx_ig in X is said to converge to point $x\ 2\ X$ if $\lim_{i!1} M(x_i;x;t) = 1$, for all $t > 0:$ (ii) A sequence fx_ig in X is said to be Cauchy sequence in X if $\lim_{i!1} M(x_i;x_{i+j};t) = 1$, for all $t > 0;j > 0:$

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is called complete.

George and Veeramani [31] proved that every fuzzy metric $(M;)$ on X generates a topology $_M$ on X which has as a base the family of sets of the form f$B_M(x_0;r;t) : x\ 2\ X;\ 0 < r < 1;\ t > 0g;$

where $B_M(x_0;r;t) =$ f$y\ 2\ X : M(x;y;t) > 1\ rg:$

For a given fuzzy metric space $(X;M;)$, we shall denote $K_0(X);$ the set of non empty compact subsets of $(X;\ _M)$, where $(X;\ _M)$ is a metrizable topological space, generated by fuzzy metric space $(X;M;):$

1.3.6 De nition [67]

Let B be non empty subset of a fuzzy metric space $(X;M;\ )$ for a 2 X and $t > 0;$ then $M(a;B;t) =$

sup f$M(a;b;t)$: b 2 Bg$:$

1.3.7 Lemma [67]

Let $(X;M;\ )$ be a fuzzy metric space, then for each a 2 $X;B$ 2 $K_0(X)$ and $t > 0;$ there is b_0 2 B,

such that $M(a;B;t) = M(a;b_0;t):$

1.3.8 Lemma [67]

Let $(X;M;\ )$ be a fuzzy metric space, then for each a 2 $X;B$ 2 $K_0(X)$ the function t !

$M(a;B;t)$ is continuous on $(0;1):$

1.3.9 Lemma [67]

Let $(X;M;\ )$ be a fuzzy metric space, then for each A 2 $K_0(X)$ and for any non empty subset B of

X and $t > 0;$ then there exists a_0 2 A, such that $\inf_{a2A} M(a;B;t) = M(a_0;B;t):$

1.3.10 De nition [67]

Let $(X;M;\ )$ be a fuzzy metric space. We de ne a function H_M on $K_0(X)$ $K_0(X)$ $(0;1)$

by $H_M(A;B;t) =$ min f$\inf_{a2A} M(a;B;t);$ $\inf_{b2B} M(A;b;t)$g$;$

for all $A;B$ 2 $K_0(X)$ and $t > 0:$

1.3.11 Lemma [67]

Let $(X;M;\ )$ be a fuzzy metric space, then for each a 2 $X;B;C$ 2 $K_0(X)$ and $t;\ S > 0$, then

$$M(a,C,t+s) \geq M(a,B,t) * M(b_a,C,S),$$

where $b_a \in B$ satisfies $M(a,B,t) = M(a,b_a,t).$

1.3.12 Theorem [67]

Let $(X,M,*)$ be a fuzzy metric space, then $(K_0(X),H_M,*)$ is a fuzzy metric space, known as Hausdorff fuzzy metric space on $K_0(X).$

1.4 Some Basic Concepts

1.4.1 Definition [84]

Let X be a non empty set and $T,f : X \to X$. A point $u \in X$ is said to be common fixed point of the pair (T,f) if $Tu = fu = u.$ A point $y \in X$ is called point of coincidence of T and $f,$ if there exists a point $x \in X$, such that $y = Tx = fx$, here x is called coincidence point of T and f. The mappings T,f are said to be weakly compatible if they commute at their coincidence point (i.e. $Tfx = fTx$ whenever $Tx = fx$).

1.4.2 Lemma [22]

Let X be a non empty set and $f : X \to X$ be a function, then there exists $E \subseteq X$, such that $fE = fX$ and $f : E \to X$ is one to one.

1.4.3 Lemma [10]

Let X be a non empty set and the mappings $S,T,f : X \to X$ have a unique point of coincidence v in $X.$ If (S,f) and (T,f) are weakly compatible, then S,T,f have a unique common fixed point.

Let denote the family of all non-decreasing functions : $[0,+\infty) \to [0,+\infty)$, such that

$\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$; for all $t > 0$; where ^{n} is the n^{th} iterate of :

1.4.4 Lemma [69]

If 2 ; then $(t) < t$ for all $t > 0$:

1.4.5 De nition

Let $S : X$! X and : $X X$! [0;+1) be two functions and $A X$. We say that S is

-admissible mapping on A; if $x;y$ 2 A, such that $(x;y)$ 1, then we have $(Sx; Sy)$ 1.

1.4.6 De nition

Let $(X;d)$ be metric space. The ball $B(x;\&)$; where $B(x;\&)$ = fy 2 X : $d(x;y)$ &g is a closed ball in metric space, for some x 2 X and $\& > 0$.

1.4.7 Theorem [52]

Let X be a non empty set and $\overline{B(x_0;r)}$ be a closed subset of X. Let $S : X$! X be a a mapping satisfying:

$$d(Sx;Sy)\ kd(x;y):$$

for all $x;y$ 2 $\overline{B(x_0;r)}$; where $0\ k < 1$, then S has a unique xed point in $\overline{B(x_0;r)}$.

1.4.8 De nition [52]

Let $(X;)$ be a partial ordered set, then $x;y$ 2 X are called comparable if $x\ y$ or $y\ x$ holds.

1.4.9 De nition [3]

Let $(X;)$ be a partially ordered set. A self mapping f on X is called dominated if $fx\ x$ for each x in X:

1.4.10 Example [3]

Let $X = [0;1]$ be endowed with the usual ordering and $f : X ! X$ be de ned by $fx = x^n$ for some n 2

N$:$ Since $fx = x^n\, x$ for all x 2 $X;$ therefore f is a dominated map.

1.4.11 De nition

Let $(X;)$ be a preordered set and $T : X ! X:$ If $A\ X;x;y$ 2 $A;$ with $x\ y$ implies $Tx\ Ty$, then the mapping T is said to be non-decreasing on A.

1.4.12 De nition [57]

Let $(X;d)$ be a metric space. For $A;B$ 2 $CB(X)$ and $\& > 0;$ the sets $N(\&;A)$ and $E_{A;B}$ are de ned as follows:

$$N(\&;A) = \mathrm{f}x\ 2\ X : d(x;A) < \&\mathrm{g};$$

$$E_{A;B} = \mathrm{f}\& : A\ N(\&;B);B\ N(\&;A)\mathrm{g};$$

where $d(x;A) = \mathrm{inff}d(x;y) : y\ 2\ A\mathrm{g}$. The distance function H on $CB(X)$ induced by d is de ned as

$$H(A;B) = \inf E_{A;B};$$

which is known as Hausdor⁄ metric on $X:$

1.4.13 Lemma [17]

Let $(X;d)$ be a metric space. If $A;B$ 2 $CB(X);$ then for each y 2 A, $d(y;B)\ H(A;B):$

1.4.14 De nition [57]

A mapping $T : X ! CB(X)$ is said to be multivalued contraction if there exists a constant $;\ 0 < 1$, such that for all $x;y$ 2 $X;$

$$H(Tx;Ty)\ d(x;y):$$

Nadler [57] generalized Banach contraction principle and proved the following important xed point result for multivalued contractions.

1.4.15 Theorem [57]

Let $(X;d)$ be a complete metric space, $T : X\,!\,CB(X)$ is a multivalued contraction, then T has a xed point.

Chapter 2

Fixed Point of Contractive Mapping in an Ordered Partial and Quasi-Partial Metric Spaces

2.1 Introduction

The de nitions given in this section have been published in [76, 79].

Recently, many results related to the xed point in complete metric spaces endowed with a partial ordering appeared in the literature. Indeed, they all deal with a monotone mapping (either order-preserving or order-reversing mapping) and such that for some x_0 2 X, either x_0 fx_0 or fx_0 x_0, where f is a self-map on metric space. To obtain unique solution they used an additional restriction that each pair of element has a lower bound and an upper bound. In this chapter, we introduce a new condition of partial order instead of monotone mapping and restriction for uniqueness. We take dominated mapping to approximate the unique solution to non linear functional equations. We will exploit this concept for self, two, three and four, locally and globally, contractive mappings on an ordered complete space X to generalize/improve and extend several classical xed point results. Also, we will not nd common xed points for three or four mappings in a standard way. Instead of usual technique, we will nd common xed points for three or four mappings via common xed point for two mappings.

Our results will not only extend some classical theorems to ordered spaces but also restrict the contractive conditions in a closed ball only. Our analysis is based on the simple observation that xed point results can be deduced from xed point theory of mappings on closed balls. Practically speaking there are many situations in which the mappings are not contractive on the whole space but instead they are contractive on its subsets. However, by imposing a subtle restriction, one can establish the existence of a xed point of such mappings. We feel that this aspect of nding the xed points via closed balls was overlooked and our work will bring a lot of interest into this area. Furthermore, the concept of dominated mappings and weaker conditions in the process of investigating the existence of unique xed point of locally and globally contractive conditions in the settings of ordered metric spaces is applied in this chapter.

Recently, Karap‹nar et. al. [48] introduced the concept of quasi-partial metric space (see also [33, 48, 57, 74, 76]) and generalized the idea of partial metric space (see [2, 6, 15, 55, 62, 70]). Romaguera [68] has given the idea of 0-complete partial metric space. Nashine et. al. [59] used this concept and proved some classical results. Reilly et al. [65] introduced the notion of left (right) K-Cauchy sequence and complete left (right) K-sequentially spaces (see also [21, 30]).

In this chapter, we introduce a new concept of an ordered 0-complete left/right K-sequentially quasi-partial metric space. Some better and interesting results are explored. Our results improve several well-known conventional results. Section 2.2 deals with an ordered 0-complete left/right K-sequentially quasi-partial metric space and the existence of xed points of self mappings satisfying contractive conditions of Banach, Kannan, Chatterjea and Reich type. In section 2.3, coincidence and common xed point results of mappings satisfying contractive conditions of Hardy Roger type in an ordered 0-complete partial metric space are discussed.

Consistent with [76, 79], the following de nitions and results will be needed in the sequel.

2.1.1 De nition [76]

Let $(X;q)$ be a quasi-partial metric space. (i) A sequence $\{x_n\}$ in $(X;q)$ is called 0-Cauchy if

$$\lim_{n;m\to\infty} q(x_n;x_m) = \lim_{n;m\to\infty} q(x_m;x_n) = 0.$$

(ii) The space $(X;q)$ is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$, such that $q(x;x) = 0$.

It is easy to see that every 0-Cauchy sequence in $(X;q)$ is Cauchy in $(X;d_{pq})$ and if $(X;q)$ is complete, then it is 0-complete but the converse assertions do not hold. For example, the space $X = [0;+1) \setminus Q$ with $q(x;y) = \mathrm{j}x\ y\mathrm{j} + \mathrm{j}x\mathrm{j}$ is a 0-complete quasi-partial metric space but it is not complete (since $d_{pq}(x;y) = 2\mathrm{j}x\ y\mathrm{j}$ and $(X;d_{pq})$ is not complete).

2.1.2 De nition [79]

Let $(X;q)$ be a quasi-partial metric space. (i) A sequence fx_ng in $(X;q)$ is called left (right) K-0-Cauchy if 8 $n>m;$ lim $q(x_m;x_n) =$ $_{n;m!1}$

0 (respectively lim $q(x_n;x_m) = 0):$ $_{n;m!1}$

(ii) The space $(X;q)$ is called 0-complete left(right) K-sequentially if every left (right) K0-Cauchy sequence in X converges to a point x 2 X, such that $q(x;x) = 0$.

One can easily observe that every 0-complete quasi-partial metric space is also a 0-complete left K-sequentially quasi-partial metric space but the converse does not hold always. Also, every closed subset of a 0-complete left K-sequentially quasi-partial metric space is a 0-complete left K-sequentially quasi-partial metric space.

2.2 Fixed Points of Reich, Banach, Kannan and Chatterjea Type Mappings in an Ordered $\mathbf{0}$-Complete Left $\boldsymbol{K}$-Sequentially Quasi-Partial Metric Spaces

The results given in this section have been published in [13, 76, 79].

2.2.1 Theorem [79]

Let $(X; ;q)$ be an ordered 0-complete left K-sequentially quasi-partial metric space, S be a self dominated mapping on X and x_0 2 X. Suppose that for $a;b$ 2 $[0;1)$, such that $a + 2b < 1$

and

$$q(Sx,Sy)\ aq(x,y) + b[q(x,Sx) + q(y,Sy)], \tag{2.1}$$

for all comparable elements x,y in $\overline{B_q(x_0,r)}$. Also,

$$q(x_0,Sx_0) \quad (1\ k)[r + q(x_0,x_0)], \tag{2.2}$$

where $k = \dfrac{a+b}{1-b}$. If for a nonincreasing sequence $\{x_n\}$ in $\overline{B_q(x_0,r)}$, $\{x_n\} \to$ implies that x_n, then there exists a point $\hat{w}$ in $\overline{B_q(x_0,r)}$, such that $d_q(\hat{w},\hat{w}) = 0$ and $\hat{w} = S\hat{w}$. Moreover, $\hat{w}$

is unique, if for any $x,y \in \overline{B_q(x_0,r)}$, the set $A_{x,y} = \{z \in \overline{B_q(x_0,r)} : z\ x$ and $z\ y\}$ is non empty and

$$q(x_0,Sx_0) + q(z,Sz)\ q(x_0,z) + q(Sx_0,Sz); \text{ for all } z\ Sx_0. \tag{2.3}$$

Proof. Consider a Picard sequence, $x_{n+1} = Sx_n$ with initial guess x_0. As $x_{n+1} = Sx_n\ x_n$ for all $n \in \{0\} \cup N$. By the inequality (2.2), we have

$$\begin{aligned} q(x_0,x_1) &\quad (1\ k)[r + q(x_0,x_0)] \\ &\quad r + q(x_0,x_0). \end{aligned}$$

Therefore, x_1 belongs to the closed ball. Now, let $x_i \in \overline{B_q(x_0,r)}$ for some $i = 1,2,\ldots,j \in N$. As x_{n+1} x_n, so by using the inequality (2.1), we obtain

$$\begin{aligned} q(x_j,x_{j+1}) &= q(Sx_{j-1},Sx_j) \\ &\quad a[q(x_{j-1},x_j))] + b[q(x_{j-1},x_j) + q(x_j,x_{j+1})], \\ q(x_j,x_{j+1}) &\quad kq(x_{j-1},x_j), \end{aligned}$$

which implies that

$$q(x_j;x_{j+1}) \quad k^2 q(x_{j\,2};x_{j\,1}) \qquad\qquad k^j q(x_0;x_1): \qquad (2.4)$$

Now,

$$q(x_0;x_{j+1}) \quad q(x_0, x_1) + \cdots + q(x_j, x_{j+1}) - [q(x_1, x_1) + \cdots + q(x_j, x_j)]$$

$$q(x_0, x_1)[1 + .. + k^{j-1} + k^j], \quad \text{(by (2.4))}$$

$$q(x_0;x_{j+1}) \quad (1-k)[r + q(x_0, x_0)]\frac{(1-k^{j+1})}{1-k}. \quad \text{(by (2.2))}$$

Thus, x_{j+1} 2 $\overline{B_q(x_0;r)}$. Hence, x_n 2 $\overline{B_q(x_0;r)}$ for all n 2 N. Also, $x_{n+1}\, x_n$ for all n 2 N: It implies that $q(x_n;x_{n+1})\ k^n q(x_0;x_1)$ for all n 2 N:

It follows that

$$q(x_n;x_{n+i}) \quad q(x_n;x_{n+1}) + \qquad + q(x_{n+i\,1};x_{n+i})\ q(x_{n+1};x_{n+1}) \qquad q(x_{n+i\,1};x_{n+i\,1})$$

$$q(x_n;x_{n+i}) \quad k^n q(x_0;x_1)[1 + \qquad + k^{i\,2} + k^{i\,1}]\ !\ 0 \text{ as } n\ !\ 1:$$

Notice that, the sequence fx_ng is a left K-0-Cauchy sequence in $(\overline{B_q(x_0;r)};q)$: As, $\overline{B_q(x_0;r)}$ is closed and so it is 0-complete left K-sequentially quasi-partial metric space. Therefore, there exists a point $w^\wedge$ 2 $\overline{B_q(x_0;r)}$ with

$$q(w;^\wedge w^\wedge) = \lim_{n!1} q(x_n;w^\wedge) = \lim_{n!1} q(w;x^\wedge{}_n) = 0: \quad (2.5)$$

Now,

$$q(w;S^\wedge w^\wedge)\ q(w;x^\wedge{}_n) + q(Sx_{n\,1};Sw^\wedge)\ q(x_n;x_n):$$

On taking limit as n ! 1 and using the fact that $w^\wedge\ x_n\, x_{n\,1}$; when x^n ! $w;^\wedge$ we have

$$q(w;S^\wedge w^\wedge) \quad \lim_{n!1} [q(w;x^\wedge{}_n) + aq(x_{n\,1};w^\wedge) + b\text{f}q(x_{n\,1};Sx_{n\,1}) + q(w;S^\wedge w^\wedge)\text{g}]$$
$$\lim [q(w;x^\wedge{}_n) + aq(x_{n\,1};w^\wedge) + b\text{f}k^{n\,1}q(x_0;x_1) + q(w;S^\wedge w^\wedge)\text{g}]:$$

$n \to 1$

Then by the inequality (2.5), we have

$$(1 b)q(w;S\hat{} w\hat{}) \quad 0:$$

Similarly, we have

$$q(Sw;\hat{} w\hat{}) \quad 0:$$

Hence, $\hat{w} = S\hat{w}$. Now, we will prove that $\hat{w}$ is the unique xed point of S in $\overline{B_q(x_0;r)}$. Let y be another point in $\overline{B_q(x_0;r)}$, such that $y = Sy$. Then, we have

$$q(y;y) \quad = q(Sy;Sy) \; aq(y;y) + b\{q(y;Sy) + q(y;Sy)\};$$

$$(1 \; a \; 2b)q(y;y) \quad 0;$$

and hence, we have

$$q(y;y) = 0: \tag{2.6}$$

Now, if $\hat{w}$ y. Then, we have

$$q(w;y\hat{} \;) = q(Sw;Sy\hat{} \;) \; aq(w;y\hat{} \;) + b[q(w;S\hat{} \; w\hat{}) + q(y;Sy)];$$

$$(1 \; a)q(w;y\hat{} \;) \quad 0. \text{ (by (2.5) and (2.6))}$$

Similarly, $q(y;\hat{w})$ $0:$ This proves that $\hat{w}$ is the only xed point in $B_q(x_0;r)$. Now, it is possible that $\hat{w}$ y and y $w;\hat{}$ then there exists a point z 2 X, such that z $\hat{w}$ and z $y:$ Now, we will prove that $S^n z$ 2 $\overline{B_q(x_0;r)}:$ By assumptions z $\hat{w}$ x_n:: x_0 and hence, we have

$$q(Sx_0;Sz) \quad aq(x_0;z) + b[q(x_0;x_1) + q(z;Sz)]$$

$$aq(x_0;z) + b[q(x_0;z) + q(x_1;Sz)]; \quad \text{by (2.3)}$$

$$q(x_1;Sz) \qquad kq(x_0;z): \tag{2.7}$$

Now, we have

$q(x_0;Sz)$

$q(x_0;x_1) + q(x_1;Sz)\ q(x_1;x_1)\ q(x_0;x_1) +$
$kq(x_0;z);$ by (2.7)

$q(x_0;Sz) \qquad (1\ k)[r + q(x_0;x_0)] + k[r + q(x_0;x_0)] = r:$

$$q(S^n z, S^{n+1} z) \leq aq(S^{n-1} z, S^n z) + b[q(S^{n-1} z, S^n z) + q(S^n z, S^{n+1} z)],$$

which implies that

$q(S^n z;S^{n+1} z) \qquad kq(S^{n\ 1} z;S^n z)$

It follows that $Sz \in B_q(x_0;r)$. Let $S^2z,\dots, S^jz \in B_q(x_0;r)$ for some $j \in N$. As S^jz $S^{j-1}z$

$\dots z\ \hat{w}\ x_n \dots x_0$; then

$$q(x_1;S^{j+1}z) = aq(x_0;S^jz) + b[q(x_0;x_1) + q(S^jz;S^{j+1}z)]$$

$$aq(x_0;S^jz) + b[q(x_0;S^jz) + q(x_1;S^{j+1}z)]; \text{ (by 2.3)}$$

which implies that

$$q(x_1;S^{j+1}z)\ kq(x_0;S^jz)\ k[r + q(x_0;x_0)] \qquad (\text{as } S^jz \in B_q(x_0;r)). \tag{2.8}$$

Now, we have

$$q(x_0;S^{j+1}z) \quad q(x_0;x_1) + q(x_1;S^{j+1}z)$$

$$(1-k)[r + q(x_0;x_0)] + k[r + q(x_0;x_0)] = r$$

It follows that $S^{j+1}z \in B_q(x_0;r)$; and hence, $S^nz \in B_q(x_0;r)$. As, $S^nz\ S^{n-1}z \dots z$ and so

$$k^2q(S^{n-2}z, S^{n-1}z) \leq \dots \leq k^nq(z, Sz) \longrightarrow 0 \text{ as } n \to \infty. \tag{2.9}$$

Now, we have

$$q(w;\hat{y}) = q(Sw;S\hat{y})$$

$$q(Sw;\hat{S}^{n+1}z) + q(S^{n+1}z;Sy)\ q(S^{n+1}z;S^{n+1}z):$$

As, $S^{n-1}z\ \hat{w}$ and $S^{n-1}z\ y$; $S^n\hat{w} = \hat{w}$ and $S^ny = y$ for all $n \in N$: It implies that $S^{n-1}z\ S^n\hat{w}$ and $S^{n-1}z\ S^ny$; for all $n \in N$, then

$$q(w;\hat{y}) \quad aq(w;\hat{S}^nz) + b\{q(w;\hat{S}\hat{w}) + q(S^nz;S^{n+1}z)\}$$

$$+aq(S^nz;y) + b\{q(S^nz;S^{n+1}z) + q(y;Sy)\}:$$

On taking limit as $n \to \infty$; and by using the inequalities (2.6) and (2.9), we have

$$q(w;\hat{y}) \quad \lim_{n\to\infty} [aq(\hat{w}, S^nz) + aq(S^nz, y)]$$

$$\lim_{n\to\infty}[a^2q(w,S^{n-1}\hat{z})+a^2q(S^{n-1}z,y)]$$

...

$$\lim_{n\to\infty}[a^nq(w,S\hat{z})+a^nq(Sz,y)]\to 0.$$

Similarly, $q(y,\hat{w}) \le 0$. Hence, $\hat{w}=y$. ■

2.2.2 Example [79]

Let $X=[0,+\infty)\cap Q$ be endowed with order, $x \preceq y$ if $q(x,x) \le q(y,y)$ and let $q: X\times X \to R^+$

be an ordered 0-complete left K-sequentially quasi-partial metric on X defined by $q(x,y) = \max\{y-x,0\}+x$. Define

$$Sx=\begin{cases}\frac{1}{10}x \text{ if } x\in[0,1]\cap Q\\ x-\frac{4}{9} \text{ if } x\in(1,\infty)\cap Q.\end{cases}$$

Clearly, S is dominated mapping. Take, $a=\frac{7}{45}, b=\frac{1}{5}, x_0=\frac{1}{2}, r=\frac{1}{2}$, then $B_q(x_0,r) = [0,1]\cap Q$; we have $q(x_0,x_0)=\frac{1}{2}$, $k=\frac{a+b}{1-b}=\frac{4}{9}$ with

$$(1-k)[r+q(x_0,x_0)]=(1-\frac{4}{9})[\frac{1}{2}+\frac{1}{2}]=\frac{5}{9}$$

and

$$q(x_0,Sx_0)=q(\frac{1}{2},S(\frac{1}{2}))=q(\frac{1}{2},\frac{1}{20})=\frac{1}{2}<\frac{5}{9}.$$

Also, if $x,y\in(1,1)\cap Q$, then

$$q(Sx,Sy)=\max\{y-\frac{4}{9}-x+\frac{4}{9},0\}+x-\frac{4}{9}=\max\{y-x,0\}+x-\frac{4}{9}.$$

Now, if $x=y$, then

$$x-\frac{4}{9}\ge\frac{5}{9}x.$$

Now, if $x>y$, then

$$x-\frac{4}{9}\ge\frac{16}{45}x+\frac{1}{5}y.$$

Now, if $x<y;$ then

$$y-\frac{4}{9}\geq\frac{16}{45}y+\frac{1}{5}x.$$

So the contractive condition does not hold on the whole space in each case*:*

Now, if $x;y$ 2 $\overline{B_q(x_0;r)}$ \ $Q;$ then

$$\begin{aligned} q(Sx,Sy) &= \max\{\frac{1}{10}y-\frac{1}{10}x,0\}+\frac{1}{10}x=\frac{1}{10}q(x,y)<\frac{7}{45}q(x,y) \\ &< aq(x,y)+b[q(x,Sx)+q(y,Sy)]. \end{aligned}$$

Also,

$$q(x_0;Sx_0)+q(z;Sz)\ q(x_0;z)+q(Sx_0;Sz) \text{ for all } z\ Sx_0:$$

Hence, all the conditions of Theorem 2.2.1 are satis ed. Moreover, 0 is equal to $S(0)$ and $q(0;0)=0:$

In Theorem 2.2.1, the condition for a nonincreasing sequence fx_ng ! implies that x_n *;* the existence of lower bound and the condition (2.3) are imposed to restrict the condition (2.1) only for comparable elements*:* However, the following result relax these restrictions but impose the condition (2.1) for all elements in $\overline{B_q(x_0;r)}:$

2.2.3 Theorem [79]

Let $(X;q)$ be a 0-complete left K-sequentially quasi-partial metric space, $S:X$! X be a self map and x_0 2 X. Suppose that for $a;b$ 2 $[0;1)$, such that $a+2b<1$ with

$$q(Sx;Sy)\ aq(x;y)+b[q(x;Sx)+q(y;Sy)];$$

for all elements $x;y$ in $\overline{B_q(x_0;r)}$ and

$$q(x_0;Sx_0)\qquad(1\ k)[r+q(x_0;x_0)];$$

where $k = \frac{a+b}{1-b}$, then there exists a unique xed point $\hat{w}$ in $\overline{B_q(x_0;r)}$, such that $\hat{w} = S\hat{w}$ and $q(\hat{w};\hat{w}) = 0:$

In Theorem 2.2.1, the condition (2.2) and (2.3) are imposed to restrict the condition (2.1) only for $x;y$ in $\overline{B_q(x_0;r)}$ and Example 2.2.2 explains the utility of these restrictions. However, the following result relax the condition (2.2) and (2.3) but impose the condition (2.1) for all comparable elements in the whole space X.

2.2.4 Theorem [79]

Let $(X; ;q)$ be an ordered 0-complete left K-sequentially quasi-partial metric space, S be a self dominated mapping on X and x_0 2 X. Suppose that for $a;b$ 2 $[0;1)$, such that $a + 2b < 1$

with

$$q(Sx;Sy)\ aq(x;y) + b[q(x;Sx) + q(y;Sy)];$$

for all comparable elements $x;y$ in $X:$
if for a nonincreasing sequence f$x_n$gin $X;$ fx_ng ! implies that x_n, then there exists a point $\hat{w}$ in X, such that $\hat{w} = S\hat{w}$ and $q(\hat{w};\hat{w}) = 0:$ Moreover, $\hat{w}$ is unique, if for any $x;y$ 2 $X;$ the set $A_{x;y}$ = fz 2 X : $z\ x$ and $z\ y$g is non empty$:$

In Theorem 2.2.1, the conditions (2.3) is imposed to obtain unique xed point of a contractive mapping satisfying conditions (2.1). However, the following result relax restriction (2.3) but impose the condition (2.1) for $b = 0:$ Also, we can replace an ordered 0-complete left Ksequentially quasi-partial metric space by an ordered 0-complete quasi-partial metric space to obtain Theorem 10 of [76] as a corollary of Theorem 2.2.1.

2.2.5 Corollary [76]

Let $(X; ;q)$ be an ordered 0-complete quasi-partial metric space, S be a self dominated mapping on X and x_0 2 X. Suppose that there exists a 2 $[0;1)$, such that

$$q(Sx;Sy)\ aq(x;y)$$

for all comparable elements $x;y$ in $\overline{B_q(x_0;r)}$ and

$$q(x_0;Sx_0) \qquad (1\ a)[r + q(x_0;x_0)]:$$

if for a nonincreasing sequence fx_ng in $\overline{B_q(x_0;r)}$; fx_ng ! implies that x_n, then there exists a point w^ in ~~$B_q(x_0;r)$,~~ such that $d_q(w;$^ w^$) = 0$ and w^ $= Sw:$^ Also, w^ is unique, if for any

$x;y$ 2 $\overline{B_q(x_0;r)}$; the set $A_{x;y}$ = fz 2 $\overline{B_q(x_0;r)}$: $z\ x$ and $z\ y$g is non empty:

2.2.6 Corollary [76]

Let $(X;\ ;q)$ be an ordered 0-complete quasi-partial metric space, S be a self dominated mapping on X and x_0 2 X. Suppose there exists k 2 $[0;1)$ with

$$q(Sx;Sy)\ kq(x;y); \text{ for all comparable elements } x;y \text{ in } X:$$

If for a nonincreasing sequence f$x_n$gin X; fx_ng ! implies that x_n, then there exists a point w^ in X, such that w^ $= Sw$^ and $q(w;$^ w^$) = 0:$ Moreover, w^ is unique, if for any $x;y$ 2 X; the set $A_{x;y}$ = fz 2 X : $z\ x$ and $z\ y$g is non empty:

2.2.7 Corollary [76]

Let $(X;q)$ be a 0-complete quasi-partial metric space, $S : X$! X be a map and x_0 2 X.

Suppose there exists k 2 $[0;1)$ with

$$q(Sx;Sy)\ kq(x;y); \text{ for all elements } x;y \text{ in } \overline{B_q(x_0;r)}$$

and

$$q(x_0;Sx_0) \qquad (1\ k)[r + q(x_0;x_0)]$$

then there exists a unique point $\hat{w}$ in $\overline{B_q(x_0;r)}$, such that $\hat{w} = S\hat{w}.$ Further $q(\hat{w};\hat{w}) = 0.$

2.2.8 Remark

By taking $a = 0$ and an ordered 0-complete quasi-partial metric space instead of an ordered 0-complete left K-sequentially quasi-partial metric space in Theorem 2.2.1 and in Theorem 2.2.4, we can obtain Theorem 15 and Theorem 17 of [76].

2.2.9 Corollary [76]

Let $(X; ;q)$ be an ordered 0-complete quasi-partial metric space, S be a self dominated mapping on X and x_0 2 X. Suppose that there exists $b \in [0, \frac{1}{2})$, such that

$$q(Sx;Sy)\ b[q(x;Sx) + q(y;Sy)]$$

for all comparable elements $x;y$ in $\overline{B_q(x_0;r)}$ and

$$q(x_0;Sx_0) \qquad (1\ k)[r + q(x_0;x_0)];$$

where $k = \frac{b}{1-b}.$ If for a nonincreasing sequence $\{x_n\}$ in $\overline{B_q(x_0;r)}$; $\{x_n\} \to$ implies that x_n, then there exists a point $\hat{w}$ in $\overline{B_q(x_0;r)}$, such that $d_q(\hat{w};\hat{w}) = 0$ and $\hat{w} = S\hat{w}.$ Moreover, $\hat{w}$

is unique, if for any $x;y$ 2 $\overline{B_q(x_0;r)}$; the set $A_{x;y} = \{z$ 2 $\overline{B_q(x_0;r)} : z\ x$ and $z\ y\}$ is non empty and

$$q(x_0;Sx_0) + q(z;Sz)\ q(x_0;z) + q(Sx_0;Sz) \text{ for all } z\ Sx_0.$$

2.2.10 Corollary [76]

Let $(X; ;q)$ be an ordered 0-complete quasi-partial metric space, $S : X \to X$ be a dominated map. Suppose that there exists $b \in [0, \frac{1}{2})$, such that

$$q(Sx;Sy)\ b[q(x;Sx) + q(y;Sy)]$$

for all comparable elements $x;y$ in $X:$ If for a nonincreasing sequence fx_ng in $X;$ fx_ng ! implies that x_n, then there exists a point w^ in X, such that w^ = Sw^ and $q(w;$^ w^$) = 0:$ Moreover, w^ is unique, if for any $x;y$ 2 $X;$ the set $A_{x;y}$ = fz 2 X : z x and z yg is non empty$:$

2.2.11 Corollary [13]

Let $(X; ;p)$ be a complete ordered partial metric space, S be a self dominated mapping on X and x_0 2 X. Suppose there exists $k \in [0, \frac{1}{2})$ with

$$p(Sx;Sy)\ k[p(x;Sx) + p(y;Sy)];$$

for all comparable elements $x;y$ in $\overline{B_p(x_0;r)}$ and

$$p(x_0;Sx_0) \quad (1 \quad)[r + p(x_0;x_0)];$$

where $= \frac{k}{1\ k}:$ If for a nonincreasing sequence in $\overline{B_p(x_0;r)};$ fx_ng ! implies that then there exists a point w^ in $\overline{B_p(x_0;r)}$, such that $d_p(w;$^ w^$) = 0$ and w^ = $Sw:$^ Also, if for any $x;y$ 2 $\overline{B_p(x_0;r)};$ the set $A_{x;y}$ = fz 2 $\overline{B_p(x_0;r)}$: z x and z yg is non empty and

$$p(x_0;Sx_0) + p(z;Sz)\ p(x_0;z) + p(Sx_0;Sz);$$

then w^ is unique.

2.2.12 Example [13]

Let $X = R^+ \cup \{0\}$ and $B_p(x_0; r) = [0; 1]$ be endowed with the usual ordering and let p be the complete partial metric on X defined by $p(x; y) = \max\{x; y\}$ for all $x; y \in X$. Let $S : X \to X$ be defined by

$$Sx = \begin{cases} \frac{3x}{70} \text{ if } x \in [0, \frac{1}{2}), \\ \frac{2x}{70} \text{ if } x \in [\frac{1}{2}, 1] \\ x - \frac{1}{2} \text{ if } x \in (1, \infty). \end{cases}$$

Clearly, $Sx \preceq x$ for all $x \in X$ that is,is S dominating map. For all comparable elements with $k = \frac{1}{5} \in [0, \frac{1}{2})$, $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$, $p(x_0, x_0) = \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}$, $\theta = \frac{k}{1-k} = \frac{1}{4}$

$$(1-\theta)[r + p(x_0, x_0)] = (1 - \frac{1}{4})[\frac{1}{2} + \frac{1}{2}] = \frac{3}{4}.$$

$$p(x_0, Sx_0) = p(\frac{1}{2}, S\frac{1}{2}) = p(\frac{1}{2}, \frac{1}{70}) = \max\{\frac{1}{2}, \frac{1}{70}\} = \frac{1}{2} < \frac{3}{4}$$

Also,

$$\begin{aligned} \text{If } x, y &\in (1, \infty),\ p(Sx, Sy) = \max\{x - \frac{1}{2}, y - \frac{1}{2}\} \\ &\geq \frac{1}{5}[x + y] \\ &= \frac{1}{5}[\max\{x, x - \frac{1}{2}\} + \max\{y, y - \frac{1}{2}\}] \\ p(Sx, Sy) &\geq k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

So the contractive condition does not hold on $(1; 1)$. For the closed ball $[0; 1]$; the four cases arrises:

(i) If $x, y \in [0, \frac{1}{2})$, we have

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{3x}{70}, \frac{3y}{70}\} = \frac{3}{70}\max\{x, y\} \\ &\leq \frac{1}{5}[x + y] \\ &= \frac{1}{5}[\max\{x, \frac{3x}{70}\} + \max\{y, \frac{3y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

(ii) For $x \in [0, \frac{1}{2})$, $y \in [\frac{1}{2}, 1]$, we have

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{3x}{70}, \frac{2y}{70}\} = \frac{1}{70}\max\{3x, 2y\} \\ &\leq \frac{1}{5}[x+y] \\ &= \frac{1}{5}[\max\{x, \frac{3x}{70}\} + \max\{y, \frac{2y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

(iii) When $y \in [0, \frac{1}{2})$, $x \in [\frac{1}{2}, 1]$, we have

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{2x}{70}, \frac{3y}{70}\} = \frac{1}{70}\max\{2x, 3y\} \\ &\leq \frac{1}{5}[x+y] = \frac{1}{5}[\max\{x, \frac{2x}{70}\} + \max\{y, \frac{3y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

(iv) And if $x, y \in [\frac{1}{2}, 1]$, we obtain

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{2x}{70}, \frac{2y}{70}\} = \frac{2}{70}\max\{x, y\} \\ &\leq \frac{1}{5}[x+y] \\ &= \frac{1}{5}[\max\{x, \frac{2x}{70}\} + \max\{y, \frac{2y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

Hence, all conditions of the above theorem are satis ed and 0 is the unique xed point of S:

2.2.13 Theorem [76]

Let $(X; ;q)$ be an ordered 0-complete quasi-partial metric space, S be a self dominated mapping on X and x_0 2 X. Suppose that there exists $c \in [0, \frac{1}{2})$, such that

$$q(Sx;Sy)\ c[q(x;Sy) + q(Sx;y)] \tag{2.10}$$

for all comparable elements $x;y$ in $B_q(x_0;r)$ and

$$q(x_0;Sx_0)\ \ (1\ c)[r + q(x_0;x_0)]; \tag{2.11}$$

where $k = \frac{c}{1\ c}$. If for a nonincreasing sequence fx_ng in $\overline{B_q(x_0;r)}$; fx_ng ! implies that

x_n, then there exists a point $\hat{w}$ in $B_q(x_0;r)$, such that $w\hat{} = Sw:\hat{}$ Further $q(w;\hat{}\ w\hat{}) = 0:$

Proof. Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. As $x_{n+1} = Sx_n\ x_n$ for all n 2 f0g [$N:$ By using the inequality (2.11), we have

$$q(x_0;x_1)\ r + q(x_0;x_0):$$

Therefore, x_1 2 $B_q(x_0;r):$ Now, let $x_2;\ ;x_j$ 2 $B_q(x_0;r)$ for some j 2 N. As $x_{n+1}\ x_n;$ so by using the inequality (2.10), we obtain

$$q(x_j;x_{j+1}) \qquad c[q(x_{j\ 1};x_j) + q(x_j;x_{j+1})\ cq(x_{j\ 1};x_j); \qquad q(x_j;x_j) + q(x_j;x_j)]$$

which implies that

$$q(x_j;x_{j+1})\ c_2q(x_{j\ 2};x_{j\ 1}) \qquad c_jq(x_0;x_1):$$

Now,

$$q(x_0;x_{j+1}) \qquad q(x_0,x_1) + \cdots + q(x_j,x_{j+1}) - [q(x_1,x_1) + \cdots + q(x_j,x_j)]$$

$$q(x_0;x_{j+1}) \qquad (1-c)[r + q(x_0,x_0)]\frac{(1-c^{j+1})}{1-c}. \quad \text{(by 2.11)}$$

Thus, x_{j+1} 2 $B_q(x_0;r):$ Hence, x_n 2 $B_q(x_0;r)$ for all n 2 N. Also, $x_{n+1}\ x_n$ for all n 2 $N:$ It implies that

$$q(x_n;x_{n+1})\ c^nq(x_0;x_1) \text{ for all } n \text{ 2 } N:$$

It follows that

$$q(x_n;x_{n+i}) \qquad q(x_n;x_{n+1}) + \qquad + q(x_{n+i\ 1};x_{n+i})$$

$$q(x_n;x_{n+i}) \qquad c^nq(x_0;x_1)[1 + \qquad + c^{i\ 2} + c^{i\ 1}]\ !\ 0 \text{ as } n\ !\ 1:$$

Notice that the sequence fx_ng is a 0-Cauchy sequence in $(B_q(x_0;r);q):$ As $B_q(x_0;r)$ is closed and so is 0-complete Therefore there exists a point $\hat{w} \in B_q(x_0;r)$ with

$$q(\hat{w};\hat{w}) = \lim_{n\to\infty} q(x_n;\hat{w}) = \lim_{n\to\infty} q(\hat{w};x_n) = 0. \quad (2.12)$$

Now,

$$q(\hat{w};S\hat{w}) \leq q(\hat{w};x_n) + q(Sx_{n-1};S\hat{w}) - q(x_n;x_n).$$

On taking limit as $n \to \infty$ and using the fact that $\hat{w} \preceq x_n \preceq x_{n-1}$; when $x_n \to \hat{w}$; we have

$$\begin{aligned} q(\hat{w};S\hat{w}) &\leq \lim_{n\to\infty} [q(\hat{w};x_n) + c\{q(x_{n-1};S\hat{w}) + q(x_n;\hat{w})\}] \\ &\leq \lim_{n\to\infty} [c\{q(x_{n-1};\hat{w}) + q(\hat{w};S\hat{w}) - q(\hat{w};\hat{w}) + q(x_n;\hat{w})\}] \\ (1-c)q(\hat{w};S\hat{w}) &\leq 0. \quad \text{(by 2.12)} \end{aligned}$$

Similarly,

$$q(S\hat{w};\hat{w}) \leq 0.$$

Hence, $\hat{w} = S\hat{w}$. ■

2.2.14 Remark [79]

(i) The above results can easily be proved in an ordered 0-complete right K-sequentially quasipartial metric space. (ii) We can obtain the quasi-metric and metric version of all theorems which are still not present in the literature.

2.3 Common Fixed Points of a Pair of Hardy Rogers Type Mappings in a Closed Ball in Ordered Partial Metric Spaces

The results given in this section have been published in [12, 77].

2.3.1 Theorem [77]

Let $(X; ;p)$ be an ordered 0-complete partial metric space, x_0 2 X, $r>0$ and $S;T : X ! X$ be two dominated mappings. Suppose that for $a;b$ 2 $[0;1);c$ 2 $[0;1)$, such that $a+2b+2c<1$

and

$$p(Sx;Ty)\ ap(x;y)+b[p(x;Sx)+p(y;Ty)]+c[p(y;Sx)+p(x;Ty)]; \qquad (2.13)$$

for all comparable elements $x;y$ in $\overline{B_p(x_0;r)}$ and

$$p(x_0;Sx_0) \quad (1 \qquad)[r+p(x_0;x_0)]; \qquad (2.14)$$

where $= \frac{a+b+c}{1-b-c}$, then there exists a point $\hat{w}$ in $\overline{B_p(x_0;r)}$, such that $d_p(\hat{w};\hat{w})=0$: If for a nonincreasing sequence fx_ng in $\overline{B_p(x_0;r)}$; fx_ng ! implies that x_n, then $\hat{w}=S\hat{w}=T\hat{w}$:

Proof. Choose a point x_1 in X, such that $x_1=Sx_0$: As $Sx_0\ x_0$ and so $x_1\ x_0$: Let $x_2=Tx_1$. Now, $Tx_1\ x_1$ gives $x_2\ x_1$, Continuing this process and having chosen x_n in X, such that

$$x_{2k+1}=Sx_{2k} \text{ and } x_{2k+2}=Tx_{2k+1}; \text{ where } k=0;1;2;::::$$

By using the inequality (2.14), we have

$$p(x_0;x_1)\ r+p(x_0;x_0):$$

Therefore, x_1 2 $\overline{B_p(x_0;r)}$: Let $x_2;::;x_j$ 2 $\overline{B_p(x_0;r)}$ for some j 2 N. If $j=2k+1$, then $x_{2k+1}=Sx_{2k}\ x_{2k}$; where $k=$ $0;1;2;::::\frac{j-1}{2}$: So using the inequality (2.13), we obtain

$p(x_{2k+1};x_{2k+2}) = p(Sx_{2k};Tx_{2k+1})$

$a[p(x_{2k};x_{2k+1})] + b[p(x_{2k};Sx_{2k}) + p(x_{2k+1};Tx_{2k+1})] +$

$c[p(x_{2k};Tx_{2k+1}) + p(x_{2k+1};Sx_{2k})]\ a[p(x_{2k};x_{2k+1})] +$

$b[p(x_{2k};x_{2k+1}) + p(x_{2k+1};x_{2k+2})] + c[p(x_{2k};x_{2k+1}) +$

$p(x_{2k+1};x_{2k+2})];$

which implies that

$$p(x_{2k+1};x_{2k+2})\ p(x_{2k};x_{2k+1}) :: {}^{2k+1}p(x_0;x_1): \quad (2.15)$$

If $j = 2k + 2$, then as $x_1;x_2...;x_j\ 2\ B_p(x_0;r)$ and $x_{2k+2}\ x_{2k+1}$, $(k = 0;1;2;:::; \frac{j}{2}2)$; we obtain

$$^{2k+2}$$

$$p(x_{2k+2};x_{2k+3}) \qquad p(x_0;x_1): \quad (2.16)$$

Thus, from the inequality (2.15) and (2.16), we have

$$p(x_j;x_{j+1}) \qquad {}_jp(x_0;x_1) \text{ for some } j\ 2\ N: \quad (2.17)$$

Now,

$$p(x_0;x_{j+1}) \qquad p(x_0;x_1) + :: + p(x_j;x_{j+1}) \qquad [p(x_1;x_1) + :: + p(x_j;x_j)]$$

$$p(x_0, x_1)[1 + .. + \lambda^{j-1} + \lambda^j], \quad \text{(by 2.17)}$$

$$p(x_0;x_{j+1}) \qquad - \lambda)[r + p(x_0, x_0)]\frac{(1-\lambda^{j+1})}{1-\lambda}. \qquad (1$$

Thus, $x_{j+1}\ 2\ B_p(x_0;r)$. Hence, $x_n\ 2\ B_p(x_0;r)$ for all $n\ 2\ N$. Also, $x_{n+1}\ x_n$ for all $n\ 2\ N$: It implies that

$$p(x_n;x_{n+1}) \qquad {}^n p(x_0;x_1) \text{ for all } n\ 2\ N: \quad (2.18)$$

So, we have

$$p(x_{n+i};x_n) \qquad p(x_{n+i}, x_{n+i\ \ 1}) + .. + p(x_{n+1}, x_n)$$

$$\lambda^{n+i-1}p(x_0, x_1) + .. + \lambda^n p(x_0, x_1)$$

$$\lambda^n p(x_0, x_1)\frac{(1-\lambda^i)}{1-\lambda} \longrightarrow 0 \text{ as } n \to \infty$$

Hence, the sequence $\{x_n\}$ is a 0-Cauchy sequence in $(\overline{B_p(x_0,r)},p)$. As $\overline{B_p(x_0,r)}$ is closed and so is 0-complete partial metric space. Therefore, there exists a point $\hat{w} \in \overline{B_p(x_0,r)}$ with

$$p(\hat{w},\hat{w}) = \lim_{n\to\infty} p(x_n,\hat{w}) = 0. \tag{2.19}$$

Now,

$$p(\hat{w},S\hat{w}) \le p(\hat{w},x_{2n+2}) + p(x_{2n+2},S\hat{w}) - p(x_{2n+2},x_{2n+2}).$$

On taking limit as $n \to \infty$ and by assumptions $\hat{w} \preceq x_n$ as $x_n \to \hat{w}$, therefore, we have

$$\begin{aligned} p(\hat{w},S\hat{w}) &\le \lim_{n\to\infty} [p(\hat{w},x_{2n+2}) + ap(x_{2n+1},\hat{w}) + b\{p(x_{2n+1},Tx_{2n+1}) \\ &\quad + p(\hat{w},S\hat{w})\} + c\{p(x_{2n+1},S\hat{w}) + p(\hat{w},Tx_{2n+1})\}] \\ &\le \lim_{n\to\infty} [p(\hat{w},x_{2n+2}) + ap(x_{2n+1},\hat{w}) + b\{p(x_{2n+1},x_{2n+2}) \\ &\quad + p(\hat{w},S\hat{w})\} + c\{p(x_{2n+1},\hat{w}) + p(\hat{w},S\hat{w}) + p(\hat{w},x_{2n+2})\}]. \end{aligned}$$

By using the inequality (2.18) and (2.19), we obtain

$$(1-b-c)p(\hat{w},S\hat{w}) \le 0,$$

which implies that $\hat{w} = S\hat{w}$. Similarly, from

$$p(\hat{w},T\hat{w}) \le p(\hat{w},x_{2n+1}) + p(x_{2n+1},T\hat{w}) - p(x_{2n+1},x_{2n+1});$$

we can obtain $w^* = Tw^*$. Hence, S and T have a common fixed point in $\overline{B_p(x_0, r)}$. ■

2.3.2 Example [77]

Let $X = [0, +\infty) \cap Q$ be endowed with order, $x \preceq y$ if $p(x, x) \leq p(y, y)$ and let $p : X \times X \to R^+$ be an ordered 0-complete partial metric on X defined by $p(x, y) = \max\{x, y\}$. Define

$$Sx = \begin{cases} \frac{x}{16} \text{ if } x \in [0,1] \cap Q \\ x - \frac{1}{6} \text{ if } x \in (1,\infty) \cap Q \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{5x}{17} \text{ if } x \in [0,1] \cap Q \\ x - \frac{1}{7} \text{ if } x \in (1,\infty) \cap Q. \end{cases}$$

Clearly, S and T are dominated mappings. Take, $a = \frac{1}{5},\ b = \frac{1}{10},\ c = \frac{1}{15},\ x_0 = \frac{1}{2},\ r = \frac{1}{2}$, then $\overline{B_p(x_0, r)} = [0, 1] \cap Q$. We have $p(x_0, x_0) = \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2},\ \lambda = (\frac{a+b+c}{1-b-c}) = \frac{11}{25}$ with

$$(1-\lambda)[r + p(x_0, x_0)] = \frac{14}{25}$$

and

$$p(x_0, Sx_0) = p(\frac{1}{2}, \frac{1}{32}) = \frac{1}{2} < (1-\lambda)[r + p(x_0, x_0)].$$

Also, if $x, y \in (1, \infty) \cap Q$, then

$$\begin{aligned} p(Sx, Ty) &= \max\{x - \frac{1}{6}, y - \frac{1}{7}\} \\ &\geq \frac{1}{5}\max\{x, y\} + \frac{1}{10}[\max\{x, x - \frac{1}{6}\} + \max\{y, y - \frac{1}{7}\}] \\ &\quad + \frac{1}{15}[\max\{x, y - \frac{1}{7}\} + \max\{y, x - \frac{1}{6}\}] \\ &\geq ap(x, y) + b[p(x, Sx) + p(y, Ty)] + c[p(y, Sx) + p(x, Ty)]. \end{aligned}$$

So the contractive condition does not hold on the whole space.

Now, if $x, y \in \overline{B_p(x_0, r)}$, then

$$\begin{aligned} p(Sx, Ty) &= \max\{\frac{x}{16}, \frac{5y}{17}\} \\ &\leq \frac{1}{5}\max\{x, y\} + \frac{1}{10}[\max\{x, \frac{x}{16}\} + \max\{y, \frac{5y}{17}\}] \\ &\quad + \frac{1}{15}[\max\{x, \frac{5y}{17}\} + \max\{y, \frac{x}{16}\}] \\ &= ap(x, y) + b[p(x, Sx) + p(y, Ty)] + c[p(y, Sx) + p(x, Ty)]. \end{aligned}$$

Hence, all the conditions of Theorem 2.3.1 are satis ed. Moreover, 0 is equal to $S(0) = T(0)$ and $p(0;0) = 0:$

If we take $a = b = 0$ in Theorem 2.3.1, then we obtain the following theorem.

2.3.3 Theorem

Let $(X; ;p)$ be an ordered 0-complete partial metric space, x_0 2 X, $r>0$ and $S;T : X$! X be two dominated mappings. Suppose that there exists $c \in [0, \frac{1}{2})$, such that

$$p(Sx;Ty) \quad c[p(y;Sx) + p(x;Ty)];$$

for all comparable elements $x;y$ in $\overline{B_p(x_0;r)}$ and

$$p(x_0;Sx_0) \quad (1 \quad)[r + p(x_0;x_0)];$$

where $= \frac{c}{1-c}$, then there exists a point $w\hat{}$ in $\overline{B_p(x_0;r)}$, such that $d_p(w;\hat{}\ w\hat{}) = 0:$ If for a nonincreasing sequence fx_ng in $\overline{B_p(x_0;r)}$; fx_ng ! implies that x_n, then $w\hat{} = Sw\hat{} = Tw:\hat{}$

2.3.4 Example

Let $X = [0;+1) \setminus Q$ and $p : X^2\ X^2$! R^+ be an ordered 0-complete partial metric on X^2 de ned by

$p((x_1;y_1);(x_2;y_2)) =$ max f$x_1;y_1;x_2;y_2$g. Let X^2 be endowed with order, $(x_1;y_1)\ (x_2;y_2)$ if

$p((x_1;y_1);(x_1;y_1))\ p((x_2;y_2);(x_2;y_2)):$ Let $S;T : X^2$! X^2 be de ned by

$$S(x,y) = \begin{cases} (\frac{x}{7}, \frac{3y}{11}) \text{ if } x,y \in [0,1] \\ \\ (x - \frac{1}{3}, y - \frac{3}{8}) \text{ if } x,y \notin [0,1] \end{cases}$$

and

$$T(x,y) = \begin{cases} (\frac{4x}{15}, \frac{2y}{7}) \text{ if } x, y \in [0,1] \\ \\ (x - \frac{1}{4}, y - \frac{1}{5}) \text{ if } x, y \notin [0,1]. \end{cases}$$

Clearly, S and T are dominated mappings. Let $(x_0, y_0) = (\frac{2}{7}, \frac{4}{7})$, $r = \frac{3}{7}$, then

$$\overline{B_p((x_0;y_0);r)} = f(x;y)\ 2\ X^2 : x;y\ 2\ [0;1]g$$

with $c = \frac{1}{5} \in [0, \frac{1}{2})$, $\lambda = \frac{c}{1-c} = \frac{1}{4}$

$$(1-\lambda)[r + p((x_0, y_0), (x_0, y_0))] = (1 - \frac{1}{4})[\frac{3}{7} + \frac{4}{7}] = \frac{3}{4}$$

$$p((x_0, y_0), S(x_0, y_0)) = \max\{\frac{2}{7}, \frac{4}{7}, \frac{2}{49}, \frac{12}{77}\} = \frac{4}{7} < \frac{3}{4}.$$

Putting $x_1 = y_1 = x_2 = y_2 = 3;$ we obtain

$$p(S(3,3), T(3,3)) = \max\{\frac{8}{3}, \frac{21}{8}, \frac{11}{4}, \frac{14}{5}\} = \frac{14}{5}$$

$$\begin{aligned} & \frac{1}{5}[p((3,3), T(3,3)) + p((3,3), S(3,3))] \\ = & \frac{1}{5}[\max\{3, 3, \frac{11}{4}, \frac{14}{5}\} + \max\{3, 3, \frac{8}{3}, \frac{21}{8}\} = \frac{6}{5} < \frac{14}{5}. \end{aligned}$$

So the contractive condition does not hold on the whole space. Now, if $(x_1;y_1);(x_2;y_2)\ 2$ $\overline{B_p((x_0;y_0);r)};$ then

$$\begin{aligned} p(S(x_1, y_1), T(x_2, y_2)) &= \max\{\frac{x_1}{7}, \frac{3y_1}{11}, \frac{4x_2}{15}, \frac{2y_2}{7}\} \\ &\leq \frac{1}{5}[\max\{x_1, y_1, \frac{4x_2}{15}, \frac{2y_2}{7}\} + \max\{\frac{x_1}{7}, \frac{3y_1}{11}, x_2, y_2\}] \\ &= c[p((x_1, y_1), T(x_2, y_2)) + p(S(x_1, y_1), (x_2, y_2))]. \end{aligned}$$

Hence, all the conditions of Theorem 2.3.3 are satis ed. Moreover, $(0;0)$ is the common xed point of S and $T.$

2.3.5 Theorem [77]

Let $(X; ;p)$ be an ordered 0-complete partial metric space, $x_0\ 2\ X$, $r > 0$ and $S : X\ !\ X$ be two dominated mapping. Suppose that for $a;b;c$ 2 $[0;1)$ 2 $[0;1)$, such that $a+2b+2c<1$ and

$$p(Sx;Sy)\ ap(x;y)+b[p(x;Sx)+p(y;Sy)]+c[p(y;Sx)+p(x;Sy)];$$

for all comparable elements $x;y$ in $\overline{B_p(x_0;r)}$ and

$$p(x_0;Sx_0)\quad (1\qquad)[r+p(x_0;x_0)];$$

where $=\dfrac{a+b+c}{1-b-c}$, then there exists a point $\hat{w}$ in $\overline{B_p(x_0;r)}$, such that $d_p(\hat{w};\hat{w})=0$: If for a nonincreasing sequence fx_ng in $\overline{B_p(x_0;r)}$; fx_ng ! implies that x_n, then $\hat{w}=S\hat{w}$:

Proof. In Theorem 2.3.1 take $T=S$ to get xed point $\hat{w}$ 2 $\overline{B_p(x_0;r)}$, such that $\hat{w}=S\hat{w}$:

■

In Theorem 2.3.1,the condition for a nonincreasing sequence fx_ng ! implies that x_n; is imposed to restrict the condition (2.13) only for comparable elements: However, the following result relax this restriction but impose the condition (2.13) for all elements in $\overline{B_p(x_0;r)}$: In Theorem 2.3.1, the common xed point of S and T may not be unique, whereas without order we can obtain unique xed point of S and T separately, which is proved in the following theorem.

2.3.6 Theorem [77]

Let $(X;p)$ be a 0-complete partial metric space, x_0 2 X, $r>0$ and $S;T:X$! X be two dominated mappings. Suppose that for $a;b;c$ 2 $[0;1)$ 2 $[0;1)$, such that $a+2b+2c<1$ and

$$p(Sx;Ty)\ ap(x;y)+b[p(x;Sx)+p(y;Ty)]+c[p(y;Sx)+p(x;Ty)];$$

for all elements $x;y$ in $B_p(x_0;r)$ and

$$p(x_0;Sx_0) \quad (1 \quad)[r + p(x_0;x_0)];$$

where $= \frac{a+b+c}{1-b-c}$, then there exists a unique point $\hat{w}$ in $\overline{}$ $B_p(x_0;r)$, such that $\hat{w} = S\hat{w} = T\hat{w}$ and $p(\hat{w};\hat{w}) = 0$: Moreover; S and T have noxed point other than $\hat{w}$:

Proof. By following similar arguments of Theorem 2.3.1, we can obtain a point $\hat{w}$ in $\overline{B_p(x_0;r)}$, such that $\hat{w} = S\hat{w} = T\hat{w}$: Let $y = Ty$, then y is the xed point of T and it may not be the xed point of S, then

$$p(\hat{w};y) = p(S\hat{w};Ty)$$
$$ap(\hat{w};y) + b[p(\hat{w};\hat{w}) + p(y;y) + c[p(\hat{w};y) +$$
$$p(y;\hat{w})] \ (a + b + 2c)p(\hat{w};y):$$

This shows that $\hat{w} = y$: Hence, T has no xed point other than $\hat{w}$: Similarly, S has no xed point other than $\hat{w}$: ■

In Theorem 2.3.1, the condition (2.14) is imposed to restrict the condition (2.13) only for $x;y$ in $\overline{B_p(x_0;r)}$ and Example 2.3.2 explains the utility of this restriction. However, the following result relax the condition (2.14) but impose the condition (2.13) for all comparable elements in the whole space X. Moreover, we introduce a weaker restriction to obtain unique common xed point.

2.3.7 Theorem [77]

Let $(X; ;p)$ be an ordered 0-complete partial metric space, x_0 2 X and $S;T : X ! X$ be two dominated mappings. Suppose that there exists there exists $a;b;c$ 2 $[0;1)$, such that $a + 2b + 2c < 1$ and

$$p(Sx;Ty) \ ap(x;y) + b[p(x;Sx) + p(y;Ty)] + c[p(y;Sx) + p(x;Ty)];$$

for all comparable elements $x;y$ in $X:$ If for a nonincreasing sequence fx_ng in $X;$ fx_ng ! implies that x_n, then there exists a point w^ in X, such that w^ = Sw^ = Tw^ and $p(w;^\wedge w^\wedge) = 0:$ Moreover, the point w^ is unique if for any two points $x;y$ in X there exists a point z_0 2 X, such that z_0 w^ and z_0 y.

Proof. By following similar arguments of Theorem 2.3.1, we can obtain a point w^ in X, such that w^ = Sw^ = Tw:^ By Theorem 2.3.4, w^ is unique common xed point for all comparable elements. Now, if w^ and y are not comparable, such that $y = Sy = Ty$, then there exists a point z_0 2 X, such that z_0 w^ and z_0 $y:$ Choose a point z_1 in X, such that $z_1 = Tz_0:$ As Tz_0 z_0 and so z_1 z_0 and let $z_2 = Sz_1$. Now, Sz_1 z_1 gives z_2 z_1, continuing this process and having chosen z_n in X, such that

$$z_{2i+1} = Tz_{2i};\ z_{2i+2} = Sz_{2i+1} \text{ and } z_{2i+1} = Tz_{2i}\ z_{2i} \text{ where } i = 0;1;2;:::$$

It follows that z_{n+1} z_n :: z_0 w:^ Following similar arguments as we have used to prove the inequality (2.18), we have

$$p(z_n;z_{n+1}) \quad {}^n p(z_0;z_1) \text{ for all } n \ 2 \ N: \tag{2.20}$$

As z_0 w^ and z_0 $y;$ it follows that z_n Tw^ and z_n Ty for all n 2 N, then for i 2 $N;$

$$p(Tw;Sz^\wedge{}_{2i\,1}) \quad ap(w;z^\wedge{}_{2i\,1}) + b[p(w;T^\wedge w^\wedge) + p(z_{2i\,1};Sz_{2i\,1})]$$

$$+c[p(w;Sz^\wedge{}_{2i\,1}) + p(z_{2i\,1};Tw^\wedge)];\ (1$$

$$c)p(w;Sz^\wedge{}_{2i\,1})\ (a + c)p(w;z^\wedge{}_{2i\,1}) + bp(z_{2i\,1};z_{2i});\ p(w;Sz^\wedge{}_{2i\,1})$$

$$p(w;z^\wedge{}_{2i\,1}) + p(z_{2i\,1};z_{2i});$$

(where $= \dfrac{a+c}{1-c}$ and $= \dfrac{b}{1-c}$)

$$p(w;Sz^\wedge{}_{2i\,1}) \quad {}^2p(w;z^\wedge{}_{2i\,2}) + p(z_{2i\,2};z_{2i\,1}) + p(z_{2i\,1};z_{2i})$$

...

$2i$ $\qquad$ $2i\,1$

$$p(w;\hat{z}_0) + \quad p(z_0;z_1) +$$

$$+p(z_{2i-2};z_{2i-1}) + p(z_{2i-1};z_{2i}):$$

On taking limit as $i \to \infty$ and by the inequality (2.20), we have

$$p(w;S\hat{z}_{2i-1}) = 0. \tag{2.21}$$

Similarly,

$$p(Sz_{2i-1};y) \to 0 \text{ as } n \to \infty. \tag{2.22}$$

Now, by using the inequality (2.21) and (2.22), we have

$$p(w;\hat{y}) \le p(w;S\hat{z}_{2i-1}) + p(Sz_{2i-1};y) - p(Sz_{2i-1};Sz_{2i-1}) \to 0 \text{ as } n \to \infty.$$

so $\hat{w} = y$. ■

2.3.8 Remark [77]

In Theorem 2.3.1, the common fixed point of S and T may not be unique. However, fixed point is unique in Theorem 2.3.1, if for every pair of elements x,y in $\overline{B_p(x_0,r)}$ there exists a point $z_0 \in \overline{B_p(x_0,r)}$, such that $z_0 \preceq x$ and $z_0 \preceq y$ and the sequence $z_n \in \overline{B_p(x_0,r)}$, such that

$$z_{2i+1} = Tz_{2i};\ z_{2i+2} = Sz_{2i+1}, \text{ where } i = 0,1,2,\ldots.$$

Metric version of Theorem 2.3.1 is given below.

2.3.9 Theorem [77]

Let $(X,\preceq,d)$ be a ordered complete metric space, $x_0 \in X$, $r > 0$ and $S,T : X \to X$ be two dominated mappings. Suppose that for $a,b,c \in [0,1) \in [0,1)$, such that $a + 2b + 2c < 1$ and

$$d(Sx;Ty)\ ad(x;y) + b[d(x;Sx) + d(y;Ty)] + c[d(y;Sx) + d(x;Ty)];$$

for all comparable elements $x;y$ in $\overline{B(x_0;r)}$ and

$$d(x_0;Sx_0) \quad (1 \qquad)r;$$

where $= \frac{a+b+c}{1-b-c}$. If for a nonincreasing sequence fx_ng in $\overline{B(x_0;r)}$; fx_ng !

x_n, then there exists a point $\hat{w}$ in $\overline{B(x_0;r)}$, such that $\hat{w} = S\hat{w} = T\hat{w}$:

Now, we apply our Theorem 2.3.7 to obtain unique common xed point of three mappings in an ordered 0-complete partial metric space.

2.3.10 Theorem [77]

Let $(X; ;p)$ be a ordered partial metric space and $S;T$ self mapping and f be a dominated mapping on X, such that SX [$TX\ fX$ with $Tx;\ Sx\ fx:$ Assume that the following

conditions holds for $a;b;c$ 2 $[0;1)$, such that $a + 2b + 2c < 1$:

$$p(Sx;Ty) \qquad ap(fx;fy) + b[p(fx;Sx) + p(fy;Ty)]$$
$$+c[p(fy;Sx) + p(fx;Ty)]; \tag{2.23}$$

for all comparable elements $fx;fy$ 2 $fX:$

If for a nonincreasing sequence fx_ng in $fX;$ fx_ng ! implies that x_n. Also, for any two points z and

x in fX there exists a point y 2 fX, such that $y\ z;y\ x:$ If the subset fX is

complete and $(T;f);\ (S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz 2 fX, such that $S(fz) = T(fz) = f(fz) = fz$. Moreover $p(fz;fz) = 0:$

Proof. By Lemma 1.4.2, there exists $E\ X$, such that $fE = fX$ and $f : E$! X is one-toone. Now,

since SX [$TX\ fX;$ we de ne two mappings $g;h : fE$! fE by $g(fx) = Sx$ and

$h(fx) = Tx$ respectively. Since f is one-to-one on E, then $g;h$ are well-de ned. As $Sx\ fx$ implies that $g(fx)\ fx$ and $Tx\ fx$ implies that $h(fx)\ fx$ therefore g and h are dominated maps. Let $y_0 = fx_0;$ choose a point y_1 in fX, such that $y_1 = h(y_0):$ As $h(y_0)\ y_0;$ so $y_1\ y_0$ and let $y_2 = g(y_1)$. Now, $g(y_1)\ y_1$ gives $y_2\ y_1$. Continuing this process and having chosen y_n in fX, such that

$$y_{2i+1} = h(y_{2i}) \text{ and } y_{2i+2} = g(y_{2i+1}); \text{ where } i = 0;1;2;::;$$

then $y_{n+1}\ y_n$ for all n 2 N. Note that for $fx;fy$ 2 fX, where fx and fy are comparable and $a;b;c$ 2 $[0;1)$, such that $a + 2b + 2c < 1$, then by using the inequality (2.23), we have

$$p(g(fx);h(fy)) \quad ap(fx;fy) + b[p(fx;g(fx)) + p(fy;h(fy))]$$
$$+c[p(fy;g(fx)) + p(fx;h(fy))]:$$

As fX is a 0-complete space and so that all the conditions of Theorem 2.3.7 are satis ed, we deduce that there exists a unique common xed point fz 2 fX of g and $h:$ Also, $p(fz;fz) = 0:$ The rest of the proof is similar to the proof given in Theorem 4 [12] (see also [22]) and so we write it, as it is in inverted commas.

"Now, $fz = g(fz) = h(fz)$ or $fz = Sz = Tz = fz$. Thus, fz is the point of coincidence of $S;T$ and f. Let v 2 fX be another point of coincidence of $f;S$ and T, then there exists u 2 fX, such that $v = fu = Su = Tu;$ which implies that $fu = g(fu) = h(fu):$ A contradiction, as, fz 2 fX is a unique common xed point of g and $h:$ Hence, $v = fz:$ Thus, $S;T$ and f have a unique point of coincidence fz 2 fX. Now, since $(S;f)$ and $(T;f)$ are weakly compatible, by Lemma 1.4.3 fz is a unique common xed point of $S;T$ and f:" ■

2.3.11 Example

Let $X = [0;4]$ and $x\ y$ if $x\ y$ be the order and let $p : X\ X\ !\ R^+$ be the complete ordered partial metric on X de ned by $p(x;y) = \max\{x;y\}$: De ne $Sx = \frac{x}{11}$, $Tx = \frac{2x}{11}$ and $fx = \frac{3x}{11}$: Clearly, S and T are self mapping and f be a dominated mapping on X, such that $SX \cup TX \subseteq fX$ with $Tx; Sx \preceq fx$: Take, $a = b = c = \frac{1}{7}$: Also, if $x;y \in X$; then

$$\begin{aligned} p(Sx,Ty) &= \max\{\frac{x}{11},\frac{2y}{11}\} \\ &\leq \frac{1}{7}\max\{\frac{3x}{11},\frac{3y}{11}\}+\frac{1}{7}[\max\{\frac{3x}{11},\frac{x}{11}\}+\max\{\frac{3y}{11},\frac{2y}{11}\}] \\ &\quad +\frac{1}{7}[\max\{\frac{3y}{11},\frac{x}{11}\}+\max\{\frac{3x}{11},\frac{2y}{11}\}] \\ &= ap(fx,fy)+b[p(fx,Sx)+p(fy,Ty)] \\ &\quad +c[p(fy,Sx)+p(fx,Ty)]. \end{aligned}$$

Hence, all the conditions of Theorem 2.3.10 are satis ed. Moreover, 0 is equal to $S(0) = T(0) = f(0)$. Also, $p(0;0) = 0$:

One cannot proof the above theorem for mappings satisfying locally contractive conditions in a closed ball in an ordered 0-complete partial metric space in a similar way by using Theorem 2.3.1. In order to prove unique common xed point of three mappings satisfying locally contractive conditions in a closed ball in an ordered 0-complete partial metric space, rst we should prove that S and T have a unique common xed point, in Theorem 2.3.1. Common xed point result of three mappings in a closed ball in 0-complete partial metric space is given below which can be proved with the help of Theorem 2.3.6 in a similar way to that of the above theorem.

2.3.12 Theorem [77]

Let $(X;p)$ be a partial metric space and $S;T$ and f be self mappings on X, such that $SX \cup TX \subseteq fX$. Assume that the following conditions holds:

$$\begin{aligned} p(Sx;Ty) &\leq ap(fx;fy) + b[p(fx;Sx) + p(fy;Ty)] \\ &\quad +c[p(fy;Sx) + p(fx;Ty)]; \end{aligned}$$

for all elements *fx;fy* 2 B_p(*fx*$_0$;*r*); where *a;b;c* 2 [0;1), such that *a* + 2*b* + 2*c*< 1 and

$$p(fx_0;Tx_0) \quad (1 \qquad)[r + p(fx_0;fx_0)];$$

for r > 0 and $= \frac{a+b+c}{1-b-c}$: If the subset *fX* is complete and (*T;f*); (*S;f*) satis es the condition of weakly compatible pair of functions, then there exists *fz* 2 $\overline{B_p(fx_0;r)}$, such that *S*(*fz*) = *T*(*fz*) = *f*(*fz*) = *fz*. Moreover, *p*(*fz;fz*) = 0:

In the following theorem we use Theorem 2.3.6 to establish a new way of nding the existence of a unique common xed point of four mappings on closed ball in 0-complete partial metric

space.

2.3.13 Theorem [77]

Let (*X;p*) be a partial metric space and *S;T;g* and *f* be self mappings on *X*, such that *SX; TX fX* = *gX:* Assume that the following condition holds:

$$p(Sx;Ty) \qquad ap(fx;gy) + b[p(fx;Sx) + p(gy;Ty)]$$
$$+c[p(gy;Sx) + p(fx;Ty)]; \tag{2.24}$$

for all elements *fx;fy* 2 $\overline{B_p(fx_0;r)}$ *fX*; with *a;b;c* 2 [0;1), such that *a* + 2*b* + 2*c*< 1 and

$$p(fx_0;Sx_0) \quad (1 \qquad)[r + p(fx_0;fx_0)]; \tag{2.25}$$

for r > 0 and $= \frac{a+b+c}{1-b-c}$: If the subset *fX* is 0-complete and (*T;g*); (*S;f*) satis es the condition of weakly compatible pair of functions, then there exists *fz* 2 $\overline{B_p(fx_0;r)}$, such that *S*(*fz*) = *T*(*fz*) = *f*(*fz*) = *g*(*fz*) = *fz*. Moreover, *p*(*fz;fz*) = 0:

Proof. By Lemma 1.4.2, there exists $E_1;E_2\ X$, such that $fE_1 = fX = gX = gE_2; f : E_1\ !\ X; g : E_2\ !$ X are one to one. Now, de ne the mappings $A;B : fE_1\ !\ fE_1$ by

$A(fx) = Sx$ and $B(gx) = Tx$ respectively. Since $f;g$ are one to one on $E_1;$ and E_2 respectively, then the mappings $A;B$ are well-de ned. As fx_0 2 $\overline{B_p(fx_0;r)}$ $fX;$ then fx_0 2 $fX:$ Let $y_0 = fx_0;$ choose a point y_1 in fX, such that $y_1 = A(y_0)$ and let $y_2 = B(y_1)$. Continuing this process and having chosen y_n in fX, such that

$$y_{2i+1} = A(y_{2i}) \text{ and } y_{2i+2} = B(y_{2i+1}); \text{ where } i = 0;1;2;:::$$

Following similar arguments of Theorem 2.3.1, y_n 2 $\overline{B_p(fx_0;r)}:$ Also, by the inequality (2.25), we have

$$p(fx_0;A(fx_0)) \quad (1 \quad)[r + p(fx_0;fx_0)]:$$

By using the inequality (2.24), for $fx;gy$ 2 $\overline{B_p(fx_0;r)}$ and $a + 2b + 2c < 1$ we have

$$p(A(fx);B(gy)) \quad ap(fx;gy) + b[p(fx;A(fx)) + p(gy;B(gy))]$$
$$+c[p(gy;A(fx)) + p(fx;B(gy))]:$$

As fX is a 0-complete space, all the conditions of Theorem 2.3.6 are satis ed, we deduce that there exists a unique common xed point fz 2 $\overline{B_p(fx_0;r)}$ of and $B:$ Further and B have no xed point other than $fz:$ Also, $p(fz;fz) = 0:$ The rest of the proof is similar to the proof given in Theorem 2.8 [11] (see also [22]) and so we write it, as it is in inverted commas.

"Now, $fz = A(fz) = B(fz)$ or $fz = Sz = fz:$ Thus, fz is a point of coincidence of f and $S:$ Let w 2 $\overline{B_p(fx_0;r)}$ be another point of coincidence of S and f, then there exists u 2 $\overline{B_p(fx_0;r)}$, such that is $w = fu = Su;$ which implies that $fu = A(fu):$ A contradiction as fz 2 $\overline{B_p(fx_0;r)}$ is a unique xed point of $A:$ Hence, $w = fz:$ Thus, S and f have a unique point of coincidence

fz 2 $\overline{B_p(fx_0;r)}$. Since $(S;f)$ are weakly compatible, by Lemma 1.4.3 fz is a unique common

xed point of S and f. As $fX = gX$, then there exists v 2 X, such that $fz = gv$: Now, as $A(fz)$ =

$B(fz) = fz$ implies that $A(gv) = B(gv) = gv$) $Tv = gv$: Thus, gv is the point of coincidence of T

and g. Now, if $Tx = gx$; then we have $B(gx) = gx$; a contradiction. This implies that $gv = gx$:

As $(T;g)$ are weakly compatible, we obtain that gv is the unique common

xed point for T and g: But $gv = fz$: Thus, $S;T;g$ and f have a unique common xed point fz 2 $\overline{B_p(fx_0;r)}$." ■

One cannot prove the above theorem for an ordered 0-complete partial metric space in a similar way by using Theorem 2.3.7. In order to prove unique common xed point of four mappings in an ordered 0-complete partial metric space, rst we should prove that S and T have no xed point other than $w^\wedge$ in Theorem 2.3.7. Coincidence point results of three and four mappings can be obtained as a corollaries of Theorem 2.3.10 and Theorem 2.3.13.

In the following result, we obtain common xed for a pair of Kannan type contractive dominated mapping in a closed ball. Here, we also prove the uniqueness of the xed point with the weaker conditions. One cannot prove the uniqueness of the xed point in Theorem 2.3.1 with these weaker conditions.

2.3.14 Theorem [12]

Let $(X; ;p)$ be a complete ordered partial metric space, $x_0;x;y$ 2 X, $r > 0$ and $S;T : X$! X be two dominated mappings. Suppose that there exists $t \in [0, \frac{1}{2})$, such that following conditions hold:

$$p(Sx;Ty) \quad t[p(x;Sx) + p(y;Ty)]; \text{ for all } (x;y) \text{ in } (\overline{B_p(x_0;r)} \quad \overline{B_p(x_0;r)}) \setminus r; \tag{2.26}$$

and

$$p(x_0;Sx_0) \quad (1 \quad)[r + p(x_0;x_0)]; \tag{2.27}$$

where r = f$(x;y)$ 2 X X j x and y are comparableg and $= \frac{t}{1-t}$, then there exists a point w^, such that $d_p(w;\hat{}\ w\hat{}) = 0:$ Also, if for a nonincreasing sequence fx_ng in $\overline{B_p(x_0;r)}$; fx_ng ! implies that x_n, then $w\hat{} = Sw\hat{} = Tw:\hat{}$ Moreover, w^ is unique, if for any $x;y$ 2 $\overline{B_p(x_0;r)}$; the set $A_{x;y}$ = fz_0 2 $\overline{B_p(x_0;r)}$: z_0 x and z_0 yg is non empty and

$$p(x_0;Sx_0) + p(z;Tz)\ p(x_0;z) + p(Sx_0;Tz) \tag{2.28}$$

for all z 2 $\overline{B_p(x_0;r)}$, such that z $Sx_0:$

Proof. Take $a = c = 0$ in Theorem 2.3.1, we obtain a point w^, such that $d_p(w;\hat{}\ w\hat{}) = 0$ and $w\hat{} = Sw\hat{} = Tw:\hat{}$ Let y be another point in $\overline{B_p(x_0;r)}$, such that $y = Sy = Ty:$ If $w\hat{}\ y$, then

$$\begin{aligned} p(w;y\hat{} \quad) &= p(Sw;Ty\hat{} \quad) \\ & t[p(w;S\hat{}\ w\hat{}) + p(y;Ty)] \\ &= t[p(w;\hat{}\ w\hat{}) + p(y;y)] \\ & tp(y;y)\ p(y;y): \end{aligned}$$

Using the fact that $p(y;y)\ p(w;y\hat{} \quad);$ we have $w\hat{} = y$. Now, if $w\hat{}\ y$, then there exists a point z_0 2 $\overline{B_p(x_0;r)}$, such that $(z_0;w\hat{})$ 2 r and $(z_0;y)$ 2 r$:$ Choose a point z_1 in X, such that $z_1 = Tz_0:$ As $Tz_0\ z_0$ and so $(z_1;z_0)$ 2 r$:$ Let $z_2 = Sz_1\ z_1$ gives $(z_2;z_1)$ 2 r. Continuing this process and having chosen z_n in X, such that

$$z_{2i+1} = Tz_{2i};\ z_{2i+2} = Sz_{2i+1} \text{ and } z_{2i+1} = Tz_{2i}\ z_{2i} \text{ where } i = 0;1;2;::::$$

It follows that $z_{n+1}\ z_n ::\ z_0\ w\hat{}\ x_n ::\ x_0:$ We will prove that z_n 2 $\overline{B_p(x_0;r)}$ for all

n 2 N by using mathematical induction. For $n = 1$: Now, $(x_0;z_0)$ 2 $B_p(x_0;r)$ $B_p(x_0;r)$ \ r

$$p(Sx_0;Tz_0) \quad t[p(x_0;x_1) + p(z_0;Tz_0)]$$

$$t[p(x_0;z_0) + p(x_1;Tz_0)] \quad \text{(by 2.28)}$$

$$p(x_0;z_0) \qquad (2.29)$$

and

$$p(x_0;z_1) \quad p(x_0;x_1) + p(x_1;z_1)\ p(x_1;x_1)$$

$$(1 \quad)[r + p(x_0;x_0)] + p(x_0;z_0) \quad \text{(by 2.27 and 2.29)}$$

$$(1\ )r + (1\ )p(x_0;x_0) + [r + p(x_0;x_0)] \text{ (as } z_0 \text{ 2 } B_p(x_0;r)) \ r + p(x_0;x_0)$$

implies that z_1 2 $B_p(x_0;r)$: Let $z_2;z_3;::;z_j$ 2 $B_p(x_0;r)$ for some j 2 N: Following similar arguments as we have used to prove the inequality (2.18), we have

$$p(z_j;z_{j+1}) \quad {}^j p(z_0;z_1) \text{ for some } j \text{ 2 } N: \qquad (2.30)$$

Note that, if j is odd, then we have

$$p(x_{j+1};z_{j+1}) = p(Tx_j;Sz_j)\ t[p(x_j;Tx_j) + p(z_j;Sz_j)]\ t[\ {}^j p(x_0;x_1) + {}^j p(z_0;z_1)] \text{ (by 2.30) } t\ {}^j[p(x_0;z_0) + p(x_1;z_1)] \text{ (by 2.28) } t\ {}^j[p(x_0;z_0) + p(x_0;z_0)]$$

$$= \ {}^{j+1} p(x_0;z_0): \qquad (2.31)$$

Similarly, if j is even, then we obtain

$$p(x_{j+1};z_{j+1}) \quad {}_{j+1}p(x_0;z_0): \qquad (2.32)$$

Now,

$$p(x_0;z_{j+1})$$

$p(x_0;x_1) + p(x_1;x_2) + ... + p(x_{j+1};z_{j+1})\ p(x_0;x_1) + p(x_0;x_1) + ... +$

$^{j+1}p(x_0;z_0)$ (by 2.31 and 2.32)

$$p(x_0,x_1)[1+\lambda+..\lambda^j]+\lambda^{j+1}[r+p(x_0,x_0)]$$
$$(1-\lambda)[r+p(x_0,x_0)]\frac{(1-\lambda^{j+1})}{1-\lambda}+\lambda^{j+1}r+\lambda^{j+1}p(x_0,x_0)$$

$= r + p(x_0;x_0)$

gives $z_{j+1} 2 \overline{B_p(x_0;r)}:$ Hence, $z_n 2 \overline{B_p(x_0;r)}$ for all n 2 N. Now, the inequality (2.30) can be written as

$$p(z_n;z_{n+1}) \quad {}^n p(z_0;z_1) \text{ for all } n \ 2 \ N: \tag{2.33}$$

As $(z_0;w^{\wedge});(z_0;y)$ 2 $(\overline{B_p(x_0;r)}\ \overline{B_p(x_0;r)}) \setminus$ r and so it follows that $(z_n;w^{\wedge})$ and $(z_n;y)$ are in $(\overline{B_p(x_0;r)}\ \overline{B_p(x_0;r)}) \setminus$ r for all n 2 N, then for i 2 $N;$

$p(w;y^{\wedge}) = p(Tw;Ty^{\wedge})$

$p(Tw;Sz^{\wedge}_{2i\ 1}) + p(Sz_{2i\ 1};Ty)\ p(Sz_{2i\ 1};Sz_{2i\ 1})\ tp(w;^{\wedge}$

$w^{\wedge}) + 2tp(z_{2i\ 1};z_{2i}) + tp(y;y)$

$2t\ ^{2i\ 1}d_l(z_0;z_1) + tp(y;y):$ (by 2.33)

On taking limit as i ! 1; we obtain

$$p(w;y^{\wedge})\ tp(y;y)\ p(y;y):$$

A contradiction, so $w^{\wedge} = y:$ Hence, $w^{\wedge}$ is a unique commonxed point of T and S in $\overline{B_p(x_0;r)}:$ ■

2.3.15 Example [12]

Let $X = R^+$[f0g be endowed with order x y if $p(x;x)$ $p(y;y)$ and let $p : X X$! R^+[f0g be the complete ordered partial metric on X de ned by $p(x;y)$ = maxf$x;y$g and $S;T : X$! X as follows:

$$Sx = \begin{cases} \dfrac{x}{17} \text{ if } x \in [0,1] \\ x - \frac{1}{5} \text{ if } x \in (1,\infty) \end{cases}$$

and

$$Tx = \begin{cases} \dfrac{3x}{17} \text{ if } x \in [0,1] \\ x - \frac{1}{6} \text{ if } x \in (1,\infty). \end{cases}$$

Clearly, S and T are dominated mappings. Take, $t = \dfrac{3}{10} \in [0, \frac{1}{2}),\ x_0 = \frac{1}{2},\ r = \frac{1}{2},$ we have $p(x_0, x_0) = \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2},\ \lambda = \frac{t}{1-t} = \frac{3}{7}$ and $\overline{B_p(x_0;r)} = [0;1]$. Also,

$$(1-\lambda)[r + p(x_0,x_0)] = (1 - \frac{3}{7})[\frac{1}{2} + \frac{1}{2}] = \frac{4}{7}$$

$$p(x_0, Sx_0) = p(\frac{1}{2}, S\frac{1}{2}) = p(\frac{1}{2}, \frac{1}{17}) = \max\{\frac{1}{2}, \frac{1}{17}\} = \frac{1}{2} < \frac{4}{7}$$

Also, if

$$\begin{aligned} x, y &\in (1,\infty),\ p(Sx,Ty) = \max\{x - \frac{1}{5}, y - \frac{1}{6}\} \\ &\geq \frac{3}{10}[x+y] \\ &= \frac{3}{10}[\max\{x, x - \frac{1}{5}\} + \max\{y, y - \frac{1}{6}\}] \\ p(Sx,Ty) &\geq t[p(x,Sx) + p(y,Ty)]. \end{aligned}$$

So the contractive condition does not hold on the whole space. Now, if $x;y$ 2 $\overline{B_p(x_0;r)}$; then

$$\begin{aligned} p(Sx,Ty) &= \max\{\frac{x}{17}, \frac{3y}{17}\} = \frac{1}{17}\max\{x, 3y\} \\ &\leq \frac{3}{10}[x+y] \\ &= \frac{3}{10}[\max\{x, \frac{x}{17}\} + \max\{y, \frac{3y}{17}\}] \\ &= t[p(x,Sx) + p(y,Ty)]. \end{aligned}$$

Also, for all z 2 $B_p(x_0;r)$, such that z $Sx_0;$ then

$$p(x_0;Sx_0) + p(z;Tz)\ p(x_0;z) + p(Sx_0;Tz):$$

Hence, all the conditions of Theorem 2.3.14 are satis ed. Moreover, 0 is the unique common xed point of S and $T:$

In Theorem 2.3.14, the conditions (2.27) and (2.28) are imposed to restrict the condition (2.26) only for $x;y$ in $\overline{B_p(x_0;r)}$ and Example 2.3.15 explains the utility of these restrictions. However, the following result relax the conditions (2.27) and (2.28) but impose the condition (2.26) for all comparable elements in the whole space X.

2.3.16 Theorem [12]

Let $(X; ;p)$ be a complete ordered partial metric space and $S;T : X$! X be two dominated mappings. Suppose that there exists $t \in [0, \frac{1}{2})$, such that following condition holds for $x;y$ 2 X,

$$p(Sx;Ty)\ t[p(x;Sx) + p(y;Ty)]; \text{ for all } (x;y) \text{ in r}:$$

then there exists a point w^, such that $d_p(w;\hat{}\ w\hat{}) = 0:$ Also, if for a nonincreasing sequence fx_ng in $X;$ fx_ng ! implies that x_n, then $w\hat{} = Sw\hat{} = Tw:\hat{}$ Moreover, w^ is unique, if for any $x;y$ 2 $X;$ the set $A_{x;y}$ = fz_0 2 X : $z_0\,x$ and $z_0\,y$g is non empty$:$

In Theorem 2.3.14, the condition for a nonincreasing sequence fx_ng ! implies that $x_n;$ the existence of z_0 and the condition (2.28) are imposed to restrict the condition (2.26) only for comparable elements$:$ However, the following result relax these restrictions but impose the condition (2.26) for all elements in $B_p(x_0;r)$:

2.3.17 Theorem [12]

Let $(X;p)$ be a complete partial metric space, $x_0;x;y$ 2 X, $r>0$ and $S;T : X$! X be two mappings. Suppose that there exists $t \in [0, \frac{1}{2})$, such that following conditions hold

$$p(Sx;Ty)\ t[p(x;Sx) + p(y;Ty)]; \text{ for all } x;y \text{ in } \overline{B_p(x_0;r)}$$

and

$$p(x_0;Sx_0) \quad (1 \quad)[r + p(x_0;x_0)]$$

where $= \frac{t}{1-t}$, then there exists a unique point $\hat{w}$ in $\overline{B_p(x_0;r)}$, such that $\hat{w} = S\hat{w} = T\hat{w}$. Also, $p(\hat{w};\hat{w}) = 0$. Further S and T have no xed point other than $\hat{w}$.

Now, we apply our Theorem 2.3.14 to obtain unique common xed point of three mappings in a closed ball in complete partial ordered metric space. One can easily prove this result.

2.3.18 Theorem [12]

Let $(X; ;p)$ be a ordered partial metric space, $x_0;x;y$ 2 X, $r > 0$ and $S;T$ self mapping and f be a dominated mapping on X, such that SX [$TX\ fX$; $\overline{B_p(fx_0;r)}\ fX$ and $(Tx;fx);(Sx;fx)$ 2 r: Assume that the following conditions hold:

$$p(Sx;Ty)\ k[p(fx;Sx) + p(fy;Ty)]$$

for all $(fx;fy)$ 2 $(\overline{B_p(fx_0;r)}\ \overline{B_p(fx_0;r)})$ \ r; where 0 $k<1=2$;

$$p(fx_0;Sx_0) + p(fy;Ty)\ p(fx_0;fy) + p(Sx_0;Ty)$$

for all fy 2 $B_p(fx_0;r)$, such that $fy\ Sx_0;$

$$p(fx_0;Tx_0) \quad (1 \qquad)[r + p(fx_0;fx_0)]$$

where $= \frac{k}{1-k}:$ If for a nonincreasing sequence fx_ng in $\overline{B_p(fx_0;r)};$ fx_ng ! implies that x_n and if for any $x;z$ 2 $\overline{B_p(fx_0;r)};$ the set $A_{x;z}$ = fy 2 $B_p(fx_0;r)$: $y\ z$ and $y\ x$g is non empty: If the subset fX is complete and $(T;f);\ (S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz 2 $\overline{B_p(fx_0;r)}$, such that $S(fz) = T(fz) = f(fz) = fz:$ Also, $p(fz;fz) = 0:$

Now, we can apply our Theorem 2.3.16 to obtain unique common xed point result of three mappings in complete partial ordered metric space. One can easily prove this result.

2.3.19 Theorem [12]

Let $(X;\ ;p)$ be a ordered partial metric space, $x;y$ 2 X and $S;T$ self mapping and f be a dominated mapping on X, such that SX [$TX\ fX$ and $(Tx;fx);(Sx;fx)$ 2 r: Assume that the following conditions hold:

$$p(Sx;Ty)\ k[p(fx;Sx) + p(fy;Ty)]$$

for all $(fx;fy)$ 2 r; where $0\ k < 1=2:$

If for a nonincreasing sequence fx_ng in $fX;$ fx_ng ! implies that x_n and if for any $z;x$ 2 $fX;$ the set $A_{z;x}$ = fy 2 fX : $y\ z$ and $y\ x$g is non empty: If the subset fX is complete and $(T;f);\ (S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz 2 fX, such that $S(fz) = T(fz) = f(fz) = fz$. Also, $p(fz;fz) = 0:$

Now, we can apply our Theorem 2.3.17 to obtain unique common xed point of three mappings on closed ball in complete partial metric space.

2.3.20 Theorem [12]

Let $(X;p)$ be a partial metric space, $x_0;x;y$ 2 X, $r>0$ and $S;T$ and f be the self mappings on X, such that SX [TX $fX;\overline{B_p(fx_0;r)}$ $fX:$ Assume that the following conditions hold:

$$p(Sx;Ty) \quad k[p(fx;Sx)+p(fy;Ty)]$$

for all $fx;fy$ 2 $\overline{B_p(fx_0;r)}$; where 0 $k<1=2;$ $p(fx_0;Tx_0)$ (

and $= \frac{k}{1-k}:$ If the subset fX is complete and $(T;f);$ $(S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz 2 $\overline{B_p(fx_0;r)}$, such that $S(fz)=T(fz)=f(fz)=fz$. Also, $p(fz;fz)=0:$

In the following theorem, we establish the existence of a unique common xed point of four mappings on closed ball in complete partial metric space. One can easily prove this result by using the technique given in the proof of Theorem 2.3.12.

2.3.21 Theorem [12]

Let $(X;p)$ be a partial metric space, $x_0;x;y$ 2 X, $r>0$ and $S;T;g$ and f be self mappings on X, such that $SX;TX$ $fX=gX$ and $\overline{B_p(fx_0;r)}$ $fX:$ Assume that the following condition holds:

$$p(Sx;Ty) \; k[p(fx;Sx)+p(gy;Ty)]$$

for all $fx;fy$ 2 $\overline{B_p(fx_0;r)}$, where 0 $k<1=2;$ and

$$p(fx_0;Sx_0) \quad (1 \quad)[r+p(fx_0;fx_0)]$$

where $= \frac{k}{1\ k}$: If the subset fX is complete and $(T;f)$; $(S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz in $\overline{B_p(fx_0;r)}$ $S(fz) = T(fz) = f(fz) = g(f) = fz$. Also, $p(fz;fz) = 0$:

In the following theorem, we establish the existence of a unique common xed point of four mappings in complete partial metric space. One can easily prove this result by using the technique given in the proof of Theorem 2.3.12.

2.3.22 Theorem [12]

Let $(X;p)$ be a partial metric space, $x;y$ 2 X and $S;T;g$ and f be self mappings on X, such that $SX;TX\ fX = gX$: Assume that the following condition holds:

$$p(Sx;Ty)\ k[p(fx;Sx) + p(gy;Ty)]$$

for all $fx;fy$ 2 fX, where $0\ k < 1=2$:

If the subset fX is complete and $(T;g)$; $(S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz 2 fX, such that $S(fz) = T(fz) = f(fz) = g(fz) = fz$. Also, $p(fz;fz) = 0$:

We can obtain the unique point of coincidence results as a corollaries of Theorem 2.3.18 to Theorem 2.3.22. Unique point of coincidence result for Theorem 2.3.18 is given below.

2.3.23 Theorem [12]

Let $(X; ;p)$ be a ordered partial metric space, $x_0;x;y$ 2 X, $r > 0$ and $S;T$ self mapping and f be a dominated mapping on X, such that SX [$TX\ fX$; $\overline{B_p(fx_0;r)}$ fX and $(Tx;fx);(Sx;fx)$ 2 r: Assume that the following conditions holds:

$$p(Sx,Ty) \le k[p(fx,Sx) + p(fy,Ty)]$$

for all $(fx,fy) \in (\overline{B_p(fx_0,r)} \times \overline{B_p(fx_0,r)}) \cap \Gamma$; where $0 \le k < 1/2$;

$$p(fx_0,Sx_0) + p(fy,Ty) \le p(fx_0,fy) + p(Sx_0,Ty)$$

for all $fy \in \overline{B_p(fx_0,r)}$, such that $fy \preceq Sx_0$;

$$p(fx_0,Tx_0) \le (1 - \lambda)[r + p(fx_0,fx_0)]$$

where $\lambda = \frac{k}{1-k}$. If for a nonincreasing sequence $\{x_n\}$ in $\overline{B_p(fx_0,r)}$; $\{x_n\} \to$ implies that x_n and if for any $z,x \in \overline{B_p(fx_0,r)}$, the set $A_{z,x} = \{y \in B_p(fx_0,r) : y \preceq z$ and $y \preceq x\}$ is non empty. If the subset fX is complete, then S,T and f have a unique point of coincidence $fz \in \overline{B_p(fx_0,r)}$. Also, $p(fz,fz) = 0$.

Chapter 3

Fixed Points of Contractive Mappings in an Ordered Dislocated and Dislocated Quasi Metric Spaces

3.1 Introduction

The theory and some of the de nitions given in this section have been published in [11, 23].

Dislocated metric space (metric-like space) has many applications in the context of logic programming semantics (see [36, 37]). Further useful results can be seen in (see [8, 44, 46, 47, 50, 54, 66, 72, 81]). Furthermore, dislocated quasi metric space (quasi-metric-like space) (see [1, 23, 73, 84, 85]) is a generalization of dislocated metric space and quasi-partial metric space.

From examples and by de nitions given in the rst chapter, it is clear that any partial metric is a d_l-metric whereas a d_l-metric may not be a partial metric. We also remark that for those d_l-metrics which are also partial metrics, we have $B_{d_l}(x;\&)$ $B_p(x;\&)$: Also, for any d_l-metric $B_{d_l}(x;\&)$ B $(x;\&)$: Thus, it is better to nd a xed point in a closed ball de ned by Hitzler in a d_l-metric, because, we restrict ourselves to apply contractive condition on smallest closed ball. In this way, we also weakened the contractive condition.

In Harandi s sence, a sequence fx_ng in the metric-like space $(X;\ )$ converges to a point x 2 X if and only if lim $(x_n;x)$ = $(x;x)$. A sequence $\{x_n\}_{n=0}^{\infty}$ of elements of X is called $_{n!1}$ -Cauchy if the lim $(x_n;x_m)$ exists and is nite. The metric-like space $(X;\ )$ is called $_{n;m!1}$ complete if for each -Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there is some x 2 X, such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m)$$

Romaguera [68] has given the idea of 0-Cauchy Sequence and 0-complete partial metric space. Using his idea, we can observe the following:

(i) Every Cauchy Sequence with respect to Hitzler is a Cauchy Sequence with respect toHarandi.

(ii) Every complete metric space with respect to Harandi is complete with respect to Hitzler.

The following example shows that the converse assertions of (i) and (ii) do not hold.

3.1.1 Example [11]

Let $X = Q^+ \cup \{0\}$ and let $d_l : X \times X \to X$ be de ned by $d_l(x;y) = x + y$: Note that

$\{x_n\} = (1 + \frac{1}{n})^n$ is a Cauchy Sequence with respect to Harandi but it is not a Cauchy Sequence

with respect to Hitzler. Also, every Cauchy Sequence (with respect to Hitzler) in X converges to a point 0in X. Hence, X is complete with respect to Hitzler but X is not complete with respect to Harandi as $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e \notin X$.

3.1.2 De nition [23]

Let $(X;d_q)$ be a dislocated quasi metric space.

(i) A sequence $\{x_n\}$ in $(X;d_q)$ is called left (right) K-Cauchy if $\forall \varepsilon > 0$, $\exists n_0 \in N$, such that $\forall n > m \geq n_0; d_q(x_m;x_n) < \varepsilon$ (respectively $d_q(x_n;x_m) < \varepsilon$): (ii) A sequence $\{x_n\}$ dislocated quasi-converges (for short d_q -converges) to x if $\lim_{n\to\infty} d_q(x_n;x) = \lim_{n\to\infty} d_q(x;x_n) = 0$: In this case x is called a d_q-limit of $\{x_n\}$:

(iii) $(X;d_q)$ is called complete left (right) K-sequentially if every left (right) K-Cauchy sequence in X converges to a point $x \in X$, such that $d_q(x;x) = 0$.

One can easily observe that every complete dislocated quasi metric space is also complete left K-sequentially dislocated quasi metric space but the converse is not true in general.

Now, we discuss the relation between the complete left/right K-sequentially dislocated quasi metric space and 0-complete left/right K-sequentially quasi-partial metric space.

3.1.3 Remark [23]

By comparing the De nition 2.1.1, De nition 2.1.2 with De nition 3.1.1, one can easily observe that if X is 0-complete left/right K-sequentially quasi-partial metric space, then it is also a complete left/right K-sequentially dislocated quasi metric space. But a complete left/right K-sequentially dislocated quasi metric space may not be a 0-complete left/right Ksequentially quasi-partial metric space. So every result which is true for complete left/right Ksequentially dislocated quasi metric spaces, then it will always be true for 0-complete left/right K-sequentially partial metric spaces, but converse does not hold.

[2, 6, 15, 24, 59, 62, 70] gave some xed point theorems in ordered metric spaces. Samet and Vetro [71] generalized the results in ordered metric spaces and introduced the concept of -contractive type mappings and established xed point theorems for such mappings in complete metric spaces.

The existence of xed points of -admissible mappings in complete metric spaces has been studied by several researchers (see [9, 41, 42, 69, 71] and references therein). Now, we introduce the concept of -dominated mappings

3.1.4 De nition

Let $T : X ! X$ and $: X X ! [0;+1)$ be a function. We say that T is -dominated mapping on $A X;$ if

$(x;Tx)$ 1 for all x 2 A. Moreover, if $(x;y)$ 1 and $(y;z)$ 1

implies that $(x;z)$ 1also holds, then we say that T is triangle -dominated mapping$:$

In this chapter, we discuss common xed point results for -dominated mappings in a closed ball in complete dislocated quasi metric space. Su¢ cient conditions for the existence of common xed point for two, three and four mappings in complete dislocated quasi metric space have been obtained. One can easily use this style to prove common xed point results in quasi metric spaces. In section 3.2, we deal with the complete left/right K-sequentially dislocated quasi metric space and prove the existence of common xed points of two, three and four -dominated mappings satisfying a generalized contractive condition. Section 3.3 deals with common xed point results of mappings satisfying - dominated contractive condition

in complete left/right K-sequentially dislocated quasi metric space.

3.2 Common Fixed Point of -Dominated Mappings

Some of the results given in this section have been published in [23]. Some of the results given in this section have been submitted for publication [80].

3.2.1 Theorem

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space. Suppose there exist

a function, $: X X ! [0;+1)$, such that S and T are - dominated mappings on $\overline{B_{d_q}(x_0;r)}$.

Let $x_0;x;y$ 2 $\overline{B_{d_q}(x_0;r)}$, $r> 0$: If there exist some $k;t$, such that $k+2t$ 2 $[0;1)$ and the following

conditions hold for $(x;y)$ 1 and for $x;y$ 2 $\overline{B_{d_q}(x_0;r)}$:

$$d_q(Sx;Ty) \quad kd_q(x;y) + t[d_q(x;Sx) + d_q(y;Ty)]; \tag{3.1}$$

$$d_q(Tx;Sy) \quad kd_q(x;y) + t[d_q(x;Tx) + d_q(y;Sy)] \tag{3.2}$$

and

$$d_q(x_0;Sx_0) \quad (1 \quad)r; \tag{3.3}$$

where $= \frac{k+t}{1-t}$.

If for any sequence fx_ng in $\overline{B_{d_q}(x_0;r)}$, such that $(x_n;x_{n+1})$ 1 for all n 2 N [f0g and

$x_n \to u \in B_{d_q}(x_0;r)$ as $n \to +\infty$ we have $(x_n;u)$ 1 for all $n \in N \cup \{0\}$; then there exists a common

xed point $\hat{w}$ of S and T. Moreover, $d_q(\hat{w};\hat{w}) = 0$.

Proof. Choose a point x_1 in X, such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X, such that

$$x_{2i+1} = Sx_{2i}; \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0;1;2;::::$$

Using the inequality (3.3) and the fact that $= \frac{k+t}{1-t}$, we have

$$d_q(x_0;Sx_0) \quad r.$$

It implies that $x_1 \in \overline{B_{d_q}(x_0;r)}$. Let $x_2; ;x_j \in \overline{B_{d_q}(x_0;r)}$ for some $j \in N$. If $j = 2i+1$, where $i = 0;1;2;::::\frac{j-1}{2}$. As S is -dominated mappings on $\overline{B_{d_q}(x_0;r)}$, then $(x_0;x_1)$ 1. As T is -dominated mappings on $\overline{B_{d_q}(x_0;r)}$, then $(x_1;x_2)$ 1. Continuing in this way we obtain $(x_{2i};x_{2i+1})$ 1 for all $i = 0;1;2;::::\frac{j-1}{2}$. So using the inequalities (3.1) and (3.2), we obtain

$$\begin{aligned} d_q(x_{2i+1};x_{2i+2}) &= d_q(Sx_{2i};Tx_{2i+1}) \quad kd_q(x_{2i};x_{2i+1}) \\ &+t[d_q(x_{2i};Sx_{2i}) + d_q(x_{2i+1};Tx_{2i+1})]; \end{aligned}$$

which implies that

$$d_q(x_{2i+1};x_{2i+2}) \quad d_q(x_{2i};x_{2i+1}) \quad {}^2 d_q(x_{2i-1};x_{2i}) \quad :: \quad {}^{2i+1} d_q(x_0;x_1). \tag{3.4}$$

If $j = 2i+2$, then as $x_1;x_2;:::;x_j \in \overline{B_{d_q}(x_0;r)}$ and $(x_{2i+1};x_{2i+2})$ 1; where $i = 0;1;2;:::;\frac{j-2}{2}$. we obtain

$$d_q(x_{2i+2}; x_{2i+3}) \quad {}^{2i+2} d_q(x_0; x_1). \tag{3.5}$$

Thus, from the inequalities (3.4) and (3.5), we have

$$d_q(x_0, x_1) + .. + \lambda^j d_q(x_0, x_1),$$
$$d_q(x_0, x_1)[1 + .. + \lambda^{j-1} + \lambda^j]$$
$$(1-\lambda) r \frac{(1-\lambda^{j+1})}{1-\lambda} \leq r,$$

$$d_q(x_j; x_{j+1}) \quad {}^j d_q(x_0; x_1) \text{ for some } j \; 2 \; N. \tag{3.6}$$

Now,

$$d_q(x_0; x_{j+1}) \quad d_q(x_0; x_1) + :: + d_q(x_j; x_{j+1})$$
(by 3.6)

$d_q(x_0; x_{j+1})$
gives x_{j+1} 2 $B_{d_q}(x_0; r)$. Hence, x_n 2 $B_{d_q}(x_0; r)$. Also, $(x_n; x_{n+1})$ 1; then

$$d_q(x_n; x_{n+1}) \quad {}^n d_q(x_0; x_1); \text{ for all } n \; 2 \; N. \tag{3.7}$$

So, we have

$$d_q(x_n; x_{n+i}) \quad d_q(x_n, x_{n+1}) + .. + d_q(x_{n+i-1}, x_{n+i})$$
$$: \frac{\lambda^n(1-\lambda^i)}{1-\lambda} d_q(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty$$

Therefore the sequence fx_ng is a left K-Cauchy sequence in $(\overline{B_{d_q}(x_0;r)};d_q):$ As $\overline{B_{d_q}(x_0;r)}$ is

$$\lim_{n!1} d_q(x_n;w\hat{}) = \lim d_q(w;x\hat{}\ _n) = 0: {}^{n}!1 \quad (3.8)$$

Now,

$$d_q(w;S\hat{}\ w\hat{})\ d_q(w;x\hat{}\ _{2n+2}) + d_q(x_{2n+2};Sw\hat{}):$$

On taking limit as n ! 1 and using the fact that $(x_n;w\hat{})$ 1 when $(x_n;x_{n+1})$ 1 and

x_n ! $w;$^ we have

$$d_q(w;S\hat{}\ w\hat{}) \quad \lim_{n!1} [d_q(w;x\hat{}\ _{2n+2}) + kd_q(x_{2n+1};w\hat{})$$

$$+tfd_q(x_{2n+1};x_{2n+2}) + d_q(w;S\hat{}\ w\hat{})g];$$ By

using the inequalities (3.7) and (3.8), we obtain

$$(1\ t)d_q(w;S\hat{}\ w\hat{}) \quad 0$$

and $w\hat{} = Sw:\hat{}$ Similarly, by using,

$$d_q\,(w;T\hat{}\ w\hat{})\ d_q(w;x\hat{}\ _{2n+1}) + d_q(x_{2n+1};Tw\hat{});$$

we can show that $w\hat{} = Tw:\hat{}$ Hence, S and T have a common xed point $w\hat{}$ in $\overline{B_{d_q}(x_0;r)}$. As S closed, it is complete left K-sequentially. Therefore, there exists a point $w\hat{}$ 2

$B_{d_q}(x_0;r)$ with

is - dominated mappings on $B_{d_q}(x_0;r)$ we have $(w; S\hat{w})$ 1 and so $(w;\hat{w})$ 1: Now,

$$d_q(w;\hat{w}) = d_q(Sw;T\hat{w})$$

$$kd_q(w;\hat{w}) + t\{d_q(w;S\hat{w}) + d_q(w;T\hat{w})\}$$

$$(1\ k\ 2t)d_q(w;\hat{w}) \quad 0:$$

This implies that

$$d_q(w;\hat{w}) = 0: \tag{3.9}$$

■

If we take $T = S$ for all $x; y$ 2 X in Theorem 3.2.1, we obtain following result.

3.2.2 Corollary

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space. Suppose there exist

a function, $: X X ! [0;+1)$, such that S is -dominated mappings on $\overline{B_{d_q}(x_0;r)}$. Let

$x_0;x;y$ 2 $\overline{B_{d_q}(x_0;r)}$, $r > 0$: If there exist some $k;t$, such that $k + 2t$ 2 $[0;1)$ and the following

conditions hold:

$$d_q(Sx;Sy)\ kd_q(x;y) + t[d_q(x;Sx) + d_q(y;Sy)];$$

for $(x;y)$ 1 and

$$d_q(x_0;Sx_0) \quad (1 \quad)r;$$

where $= \dfrac{k+t}{1-t}$.

If for any sequence $\{x_n\}$ in $\overline{B_{d_q}(x_0;r)}$, such that $(x_n;x_{n+1})$ 1 for all n 2 N [f0g and $x_n ! u$ 2

$\overline{B_{d_q}(x_0;r)}$ as ~~$n ! +1$ we~~ have $(x_n;u)$ 1 for all n 2 N [f0g, then there exists a

point $\hat{w}$ in $B_{d_q}(x_0;r)$, such that $\hat{w} = S\hat{w}$ and $d_q(w;\hat{w}) = 0$:

3.2.3 Corollary

Let $(X;d)$ be a complete left K-sequentially metric space. Suppose there exists, $: X X !$ $[0;+1)$, such that S and T are -dominated mappings on $\overline{B_{d_q}(x_0;r)}:$ Let

$$(x;y)d(Sx;Ty)\ kd(x;y) + t[d(x;Sx) + d(y;Ty)]$$

holds for all $x;y$ 2 X and $k + 2t$ 2 $[0;1):$

If for any sequence fx_ng in X with $(x_n;x_{n+1})$ 1 for all n 2 N [f0g and x_n ! x as n ! +1, we have $(x_n;x)$ 1 for all n 2 N [f0g, then S and T have a common xed point$:$

3.2.4 Theorem

Adding the following conditions to the hypotheses of Theorem 3.2.1

(i) Let S and T are triangle -dominated mappings on $\overline{B_{d_q}(x_0;r)}$.

(ii) If for any two points $x;y$ in $\overline{B_{d_q}(x_0;r)}$ there exists a point z_0 2 $\overline{B_{d_q}(x_0;r)}$, such that

$(x;z_0)$ 1, $(y;z_0)$ 1$:$

(iii) For all z 2 $\overline{B_{d_q}(x_0;r)}$, such that $(z; Sx_0)$1 implies

$$d_q(x_0;Sx_0) + d_q(z;Tz)\ d_q(x_0;z) + d_q(Sx_0;Tz):$$

then S and T have a unique common xed point $\hat{w}$ and $d_q(\hat{w};\hat{w}) = 0:$

Proof. Let y be another point in $\overline{B_{d_q}(x_0;r)}$, such that $y = Sy = Ty:$ Now,

$$d_q(y;y) = d_q(Sy;Ty)\ kd_q(y;y) + t\text{f}d_q(y;Ty) + d_q(y;Sy)\text{g}$$

$$(1 k 2t)d_q(y;y) \quad 0:$$

This implies that

$$d_q(y;y) = 0: \tag{3.10}$$

Now, if $(w;y\hat{})$ 1, then we have

$$d_q(w;y\hat{}) =$$

$$d_q(Sw;Ty\hat{}) \; kd_q(w;y\hat{}) + t[d_q(w;S\hat{} w) + d_q(y;Ty)]$$

$$(1 k)d_q(w;y\hat{}) \quad 0. \text{ (by 3.9 and 3.10)}$$

This shows that $\hat{w} = y$: Now, if $(w;y\hat{})$ 1, then there exists a point z_0 2 $B_{d_q}(x_0;r)$, such that

$(w;\hat{z}_0)$ 1 and $(y;z_0)$ 1: Choose a point z_1 in X, such that $z_1 = Tz_0$ and $z_2 = Sz_1$.

Continuing this process, we construct a sequence z_n of points in X, such that

$$z_{2i+1} = Tz_{2i}; \text{ and } z_{2i+2} = Sz_{2i+1}, \text{ where } i = 0;1;2;::::$$

As T is -dominated mappings on $B_{d_q}(x_0;r)$, then $(z_0;z_1)$ 1: By assumption $(x_n;w\hat{})$ 1 and $(w;\hat{z}_0)$ 1 implies that $(x_n;z_0)$ 1 for all n 2 N [f0g: Now, we have

$$d_q(Sx_0;Tz_0) \quad kd_q(x_0;z_0) + t[d_q(x_0;x_1) + d_q(z_0;Tz_0)] \; kd_q(x_0;z_0) + t[d_q(x_0;z_0) + d_q(x_1;Tz_0)]; \text{ (by (iii))}$$

$$d_q(Sx_0;Tz_0) \quad d_q(x_0;z_0) \text{ and} \tag{3.11}$$

$$d_q(x_0;z_1)$$

$$d_q(x_0;x_1) + d_q(x_1;z_1)$$

$$(1 \quad)r + d_q(x_0;z_0); \text{ (by 3.3 and 3.11)}$$

$$d_q(x_0;z_1) \quad (1 \quad)r + r \text{ (as } z_0 \text{ 2 } B_{d_q}(x_0;r))$$

$(z_0;z_{j+1})$ 1 implies that $(w;\hat{z}_{j+1})$ 1: Also, $(x_n;w\hat{})$ 1 and $(w;\hat{z}_{j+1})$ 1 implies that

$(x_n; z_{j+1})$ 1: Now, if j is even, then we have

α

$d_q(x_1; Tz_j)$ $kd_q(x_0; z_j) + t[d_q(x_0; x_1) + d_q(z_j; Tz_j)]$

$kd_q(x_0; z_j) + t[d_q(x_0; z_j) + d_q(x_1; Tz_j)]$; (by (iii))

$d_q(x_1; Tz_j)$ $d_q(x_0; z_j)$ r: (as z_j 2 $B_{d_q}(x_0; r)$) (3.12)

implies that z_1 2 $B_{d_q}(x_0; r)$, then $(z_1; z_2)$ 1: As S and T are triangle -dominated mappings on $B_{d_q}(x_0; r)$ and so $(z_0; z_1)$ 1 and $(z_1; z_2)$ 1 implies that $(z_0; z_2)$ 1: Let $z_2; z_3; ::: ; z_j$ 2 $B_{d_q}(x_0; r)$ for some j 2 N, then $(z_j; z_{j+1})$ 1: Now, $(z_0; z_2)$ 1 and $(z_2; z_3)$ 1 implies that $(z_0; z_3)$ 1: Continuing in this way we obtain $(z_0; z_{j+1})$ 1: Now, $(w; \hat{z}_0)$ 1 and Now,

$d_q(x_0; Tz_j)$ $d_q(x_0; x_1) + d_q(x_1; Tz_j)$

$(1 \quad)r + r$; (by 3.12)

$d_q(x_0; z_{j+1})$ r: (3.13)

$d_q(z_j; z_{j+1})$ ${}^j d_q(z_0; z_1)$ for some j 2 N: (3.14)

Now, we have

$d_q(x_2; z_{j+1})$ =

$d_q(Tx_1; Sz_j)$ $kd_q(x_1; z_j) + t[d_q(x_1; Tx_1) + d_q(z_j; Sz_j)]$ $kd_q(x_1; z_j) + t[d_q(x_0; x_1) + d_q(z_{j\ 1}; Tz_{j\ 1})]$, (by 3.7 and 3.14)

$d_q(x_2; z_{j+1})$ $kd_q(x_1; z_j) + t\,[d_q(x_0; z_{j\ 1}) + d_q(x_1; z_j)]$, (by (iii))

$d_q(x_2; z_{j+1})$ $(k + t\,)d_q(x_1; Tz_{j\ 1}) + t\,r$; (as $z_{j\ 1}$ 2 $B_{d_q}(x_0; r)$)

$d_q(x_2; z_{j+1})$ $[(k + t\,) + t\,]r$, (by 3.12, as j 1 is even)

$d_q(x_2; z_{j+1})$ ${}^2 r$: (3.15)

Now,

$$d_q(x_0;z_{j+1}) \quad d_q(x_0;x_1) + d_q(x_1;x_2) + d_q(x_2;z_{j+1})$$

$$d_q(x_0;x_1) + d_q(x_0;x_1) + {}^2r, \text{ (by 3.7 and 3.15)}$$

$$d_q(x_0;z_{j+1}) \quad r: \tag{3.16}$$

Now, if j is odd, then following similar arguments as we have used to prove the inequality (3.6), we have

Therefore, from the inequalities (3.13) and (3.16), z_{j+1} 2 $\overline{B_{d_q}(x_0;r)}$ in both cases*:* Hence,

z_n 2 $\overline{B_{d_q}(x_0;r)}$ for all ^{n} 2 N. Thus, the inequality (3.14) becomes

$$d_q(z_n;z_{n+1}) \quad {}^n d_q(z_0;z_1) \,!\, 0 \text{ as } n \,!\, 1: \tag{3.17}$$

As $(y;z_0)$ 1 and $(z_0;z_{n+1})$1 implies that $(y;z_{n+1})$ 1*:* Also, $(w;\hat{z}_{n+1})$ 1, then for i 2 N; we have

$$d_q(Tw;S\hat{z}_{2i\,1}) \quad kd_q(w;\hat{z}_{2i\,1}) + t[d_q(w;T\hat{w}) + d_q(z_{2i\,1};Sz_{2i\,1})]$$

$$= \quad kd_q(Sw;T\hat{z}_{2i\,2}) + td_q(z_{2i\,1};z_{2i});$$

$$d_q(w;S\hat{z}_{2i\,1}) \quad k^2 d_q(w;\hat{z}_{2i\,2}) + ktd_q(z_{2i\,2};z_{2i\,1}) + td_q(z_{2i\,1};z_{2i})$$

$$\cdots$$

$$k^{2i}d_q(w;\hat{z}_0) + k^{2i\,1}td_q(z_0;z_1) +$$

$$+ktd_q(z_{2i\,2};z_{2i\,1}) + td_q(z_{2i\,1};z_{2i}):$$

On taking limit as i ! 1 and by the inequality (3.17), we have

$$d_q(w;S\hat{z}_{2i\,1}) = 0: \tag{3.18}$$

Similarly, we have

$$d_q(Sz_{2i\,1}, y) \to 0 \text{ as } i \to \infty. \quad (3.19)$$

Now, by using the inequality (3.18) and (3.19), we have

$$d_q(w, \hat{y}) \; d_q(w, S\hat{z}_{2i\,1}) + d_q(Sz_{2i\,1}, y) \to 0 \text{ as } i \to \infty.$$

So, $\hat{w} = y$. Hence, $\hat{w}$ is a unique common fixed point of T and S in $\overline{B_{d_q}(x_0, r)}$. ■

In Theorem 3.2.4, the conditions (i), (ii), (iii) and if for any sequence $\{x_n\}$ in $\overline{B_{d_q}(x_0, r)}$, such that (x_n, x_{n+1}) 1 for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B_{d_q}(x_0, r)}$ as $n \to +\infty$, then (x_n, u) 1 for all $n \in N \cup \{0\}$ are imposed to restrict the conditions (3.1) and (3.2) only for -dominated mappings on $\overline{B_{d_q}(x_0, r)}$ and for those x, y in $\overline{B_{d_q}(x_0, r)}$ for which (x, y) 1. However, the following result relax these restrictions but impose the conditions (3.1) and (3.2) for all elements in $\overline{B_{d_q}(x_0, r)}$.

3.2.5 Theorem

Let (X, d_q) be a complete left K-sequentially dislocated quasi metric space, $x_0 \in \overline{B_{d_q}(x_0, r)}$, $r > 0$ and $S, T : X \to X$ be two mappings. Suppose for $k + 2t \in [0, 1)$, the following conditions hold:

$$d_q(Sx, Ty) \quad k d_q(x, y) + t[d_q(x, Sx) + d_q(y, Ty)]; \; d_q(Tx, Sy)$$
$$k d_q(x, y) + t[d_q(x, Tx) + d_q(y, Sy)];$$

for all x, y in $\overline{B_{d_q}(x_0, r)}$ and

$$d_q(x_0, Sx_0) \quad (1 \quad)r;$$

where $= \dfrac{k+t}{1-t}$, then there exists a unique point $\hat{w}$ in $\overline{B_{d_q}(x_0, r)}$, such that $\hat{w} =$

$S\hat{w} = T\hat{w}$
and $d_q(\hat{w}, \hat{w}) = 0.$ Moreover, S and T have no fixed point other than $\hat{w}.$

Proof. Following similar arguments of Theorem 3.2.1, we can obtain a unique point $\hat{w}$ in $\overline{B_{d_q}(x_0, r)}$, such that $\hat{w} = S\hat{w} = T\hat{w}.$ Let $y = Ty$, then y is the fixed point of T and it may not be a fixed point of $S.$ Now,

$$\begin{aligned} d_q(\hat{w}, y) &= d_q(S\hat{w}, Ty) \leq k d_q(\hat{w}, y) + t[d_q(\hat{w}, S\hat{w}) + d_q(y, Ty)] \\ &\leq \frac{t}{1-k} d_q(y, y). \quad \text{(by 3.9)} \end{aligned}$$

$d_q(w;y^\wedge)$

Similarly,

$$t\, d_q(y;w^\wedge) \quad \frac{}{1\ k} d_q(y;y):$$

then

$$\begin{aligned} &\leq d_q(y, \hat{w}) + d_q(\hat{w}, y) \\ &\leq \frac{t}{1-k} d_q(y, y) + \frac{t}{1-k} d_q(y, y) \\ (1 - \frac{2t}{1-k}) d_q(y, y) &\leq 0, \\ d_q(y, y) &= 0. \end{aligned} \tag{3.20}$$

$d_q(y;y)$

Now,

$$d_q(w;y^\wedge) \quad = d_q(Sw;Ty^\wedge \quad)\ k d_q(w;y^\wedge\ ) + t[d_q(w;^\wedge w^\wedge) + d_q(y;y)]$$

$$(1\ k) d_q(w;y^\wedge\ ) \quad 0: \text{ (by 3.9 and 3.20)}$$

Hence, $\hat{w} = y$. Thus, T has no fixed point other than $\hat{w}.$ Similarly S has no fixed point other than $\hat{w}.$ ■

In Theorem 3.2.4, the conditions (iii) and (3.3) are imposed to restrict the conditions (3.1) and (3.2) only for x, y in $\overline{B_{d_q}(x_0, r)}$. However, the following result relax the conditions (iii) and (3.3) but impose the conditions (3.1) and (3.2) for all elements $x, y \in X$, such that $(x, y) \quad 1.$

3.2.6 Theorem

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space. Suppose there exist

a function, $: X X ! [0;+1)$, such that S and T are triangle -dominated mappings. If

there exist some $k;t$, such that $k + 2t$ 2 $[0;1)$ and the following conditions hold:

$$d_q(Sx;Ty) \quad kd_q(x;y) + t[d_q(x;Sx) + d_q(y;Ty)];$$

$$d_q(Tx;Sy) \quad kd_q(x;y) + t[d_q(x;Tx) + d_q(y;Sy)];$$

for all $x;y$ 2 X, such that $(x;y)$ 1.

If for any sequence fx_ng in X, such that $(x_n;x_{n+1})$ 1 for all n 2 N [f0g and x_n ! u 2 X as n ! +1,

then $(x_n;u)$ 1 for all n 2 N [f0g$:$ Also, for any two points $x;y$ in X there exists a point z_0 2 X,

such that $(x;z_0)$ 1, $(y;z_0)$ 1, then there exists a unique point w^

in X, such that w^ $= Sw$^ $= Tw$^ and $d_q(w;$^ w^$) = 0:$

Now, we apply our Theorem 3.2.4 to obtain unique common xed point of three mappings in closed ball in complete left K-sequentially dislocated quasi metric space.

3.2.7 Theorem

Let $(X;d_q)$ be a dislocated quasi metric space, $S;T;f : X ! X$, such that SX [$TX fX:$ Suppose there

exist a function, $: X X ! [0;+1)$, such that $(fx; Sx)$ 1, $(fx;Tx)$ 1 and $(x;y)$ 1, $(y;z)$ 1 implies that

$(x;z)$ 1 for all $x;y;z$ 2 $X:$ Suppose for $k+2t$ 2 $[0;1)$,

$$d_q (Sx;Ty) \quad kd_q(fx;fy) + t[d_q(fx;Sx) + d_q(fy;Ty)]; \quad (3.21)$$

$$d_q (Tx;Sy) \quad kd_q(fx;fy) + t[d_q(fx;Tx) + d_q(fy;Sy)] \quad (3.22)$$

and for fx_0 2 $B_{dq}(fx_0;r)$

$$d_q(fx_0;Sx_0) \quad (1 \quad)r; \quad (3.23)$$

x_0 2 X, r> 0;B_{d_q}(fx_0;r) fX and for all fx;fy 2 B_{d_q}(fx_0;r); (fx;fy) 1 implies that

where $= \frac{k+t}{1-t}$ and

$$d_q(fx_0;Sx_0) + d_q(fy;Ty) \quad d_q(fx_0;fy) + d_q(Sx_0;Ty); \tag{3.24}$$

for all fy 2 B_{d_q}(fx_0;r), such that (fy; Sx_0) 1:

If for any sequence fx_ng in B_{d_q}(fx_0;r), such that (x_n;x_{n+1}) 1 for all n 2 N [f0g and x_n ! u 2 B_{d_q}(fx_0;r) as n ! +1 we have (x_n;u) 1 for all n 2 N [f0g and for any two points x;y in B_{d_q}(fx_0;r) there exists a point z_0 2 B_{d_q}(fx_0;r), such that (x;z_0) 1, (y;z_0) 1: If the subset fX is complete left K-sequentially and (T;f); (S;f) satis es the condition of weakly compatible pair of functions, then there exists fz 2 B_{d_q}(fx_0;r), such that $S(fz) = T(fz) = f(fz) = fz$. Also, $d_q(fz;fz) = 0$:

Proof. By Lemma 1.4.2, there exists E X, such that $fE = fX$ and f : E ! X is oneto-one. Now, since SX [TX fX; we de ne two mappings g;h : fE ! fE by $g(fx) = Sx$ and $h(fx) = Tx$ respectively. Since f is one-to-one on E, then g;h are well-de ned. As (fx;Sx) 1 implies that (fx;$g(fx)$) 1 and (fx;Tx) 1 implies that (fx;$h(fx)$) 1,

then g and h are -dominated mappings on B_{d_q}(fx_0;r): Now, fx_0 2 B_{d_q}(fx_0;r) fX, then fx_0 2 fX: Let $y_0 = fx_0$; choose a point y_1 in fX, such that $y_1 = g(y_0)$: Also, by the inequality (3.23),

$$d_q(fx_0;g(fx_0)) \quad (1 \quad)r:$$

Then y_1 2 B_{d_q}(fx_0;r): Let $y_2 = h(y_1)$. Continuing this process and having chosen y_n in fX, such that

$$y_{2i+1} = g(y_{2i}) \text{ and } y_{2i+2} = h(y_{2i+1}); \text{ where } i = 0;1;2;:::$$

Following similar arguments of Theorem 3.2.1, y_n 2 $B_{d_q}(fx_0;r)$: Also, by using the inequality (3.23), we obtain

$$d_q(fx_0;g(fx_0)) + d_q(fy;h(fy)) \quad d_q(fx_0;fy) + d_q(g(fx_0);h(fy));$$

for all fy 2 $B_{d_q}(fx_0;r)$, such that $(fy; Sx_0)$ 1. By using the inequalities (3.21) and (3.22),

for $fx;fy$ 2 $B_{d_q}(fx_0;r)$; $(fx;fy)$ 1 implies that

$$d_q\,(g(fx);h(fy)) \quad kd_q(fx;fy) + t[d_q(fx;g(fx)) + d_q(fy;h(fy))];\ d_q$$

$$(h(fx;g(fy)) \quad kd_q(fx;fy) + t[d_q(fx;h(fx)) + d_q(fy;g(fy))]:$$

As fX is a complete left K-sequentially space; all the conditions of Theorem 3.2.1 are satis ed,

we deduce that there exists a unique common xed point fz 2 $B_{d_q}(fx_0;r)$ of g and h: Also,

$d_q(fz;fz) = 0$: The rest of the proof is similar to the proof given in Theorem 2.3.10 and so we leave it. Hence, we obtain a unique common xed point of $S;T$ and f. ■

3.2.8 Corollary

Let $(X;d_q)$ be a dislocated quasi metric space, x_0 2 X, $r> 0$ and $S;T$ and f are self mappings on X, such that SX [TX $fX;B_{d_q}(fx_0;r)$ fX: Suppose for $k + 2t$ 2 $[0;1)$, the following conditions hold:

$$d_q\,(Sx;Ty) \quad kd_q(fx;fy) + t[d_q(fx;Sx) + d_q(fy;Ty)];$$

$$d_q\,(Tx;Sy) \quad kd_q(fx;fy) + t[d_q(fx;Tx) + d_q(fy;Sy)]$$

for all $fx;fy$ 2 $B_{d_q}(fx_0;r)$ and where $= \frac{k+t}{1-t}$: If the subset the condition of weakly compatible pair of functions, then there exists fz 2 $B_{d_q}(fx_0;r)$, such that $S(fz) = T(fz) = f(fz) = fz$. Also, $d_q(fz;fz) = 0$:

Unique common xed point result of four mappings in complete left K-sequentially dislocated quasi metric space in a closed ball is given below which can be proved with the help of Theorem 3.2.4, by using the technique given in Theorem 2.3.13.

3.2.9 Theorem

Let $(X;d_q)$ be a dislocated quasi metric space, x_0 2 X, $r>0$ and $S;T;g$ and f be self mappings on X, such that $SX;TX\ fX = gX$ and $\overline{B_{d_q}(fx_0;r)}\ fX:$ Suppose for $k + 2t$ 2 $[0;1)$, the following conditions hold:

$$d_q(Sx;Ty) \qquad kd_q(fx;gy) + t[d_q(fx;Sx) + d_q(gy;Ty)];$$

$$d_q(Tx;Sy) \qquad kd_q(gx;fy) + t[d_q(fx;Tx) + d_q(gy;Sy)]$$

for all $fx;fy$ 2 $\overline{B_{d_q}(fx_0;r)}$ and where $= \dfrac{k+t}{1-t}:$ If the subset the condition of weakly compatible pair of functions, then there exists fz 2 $B_{d_q}(fx_0;r)$, such that $S(fz) = T(fz) = f(fz) = g(fz) = fz$. Also, $d_q(fz;fz) = 0:$

From Theorem 3.2.1 to Theorem 3.2.6, we derive following important results in preordered complete left K-sequentially dislocated quasi metric space. We de ne the set r by r = f$(x;y)$ 2

$X\ X : x\ y$ or $y\ x$g$:$

3.2.10 Theorem

Let $(X; ;d_q)$ be a preordered complete left K-sequentially dislocated quasi metric space,

$x_0;x;y$ 2 X, $r>0$ and $S;T : X\ !\ X$ be two dominated mappings on $\overline{B_{d_q}(x_0;r)}$. Suppose for

$k + 2t$ 2 $[0;1)$, the following conditions hold:

$$d_q(Sx;Ty) \qquad kd_q(x;y) + t[d_q(x;Sx) + d_q(y;Ty)];$$

$$d_q(Tx;Sy) \qquad kd_q(x;y) + t[d_q(x;Tx) + d_q(y;Sy)]$$

for all $(x;y)$ in $(\overline{B_{d_q}(x_0;r)}\ \overline{B_{d_q}(x_0;r)}) \setminus$ r and

$$d_q(x_0;Sx_0) \quad (1 \quad)r;$$

where $= \frac{k+t}{1-t}$. If for a nonincreasing sequence fx_ng in $\overline{B_{d_q}(x_0;r)};$ fx_ng ! implies that x_n, then there exists a point $\hat{w}$ in $\overline{B_{d_q}(x_0;r)}$, such that $\hat{w} = S\hat{w} = T\hat{w}$ and $d_q(\hat{w};\hat{w}) = 0:$

Also, $\hat{w}$ is unique, if for any $x;y$ 2 $\overline{B_{d_q}(x_0;r)};$ the set $A_{x;y}$ = fz_0 2 $\overline{B_{d_q}(x_0;r)}$: $z_0\ x$ and $z_0\ y$g is non empty and for all z 2 $\overline{B_{d_q}(x_0;r)}$, such that $z\ Sx_0;$ we have

$$d_q(x_0;Sx_0) + d_q(z;Tz)\ d_q(x_0;z) + d_q(Sx_0;Tz)$$

3.2.11 Corollary

Let $(X; ;d_q)$ be a preordered complete left K-sequentially dislocated quasi metric space, x_0 2 X and $S;T : X\ !\ X$ be two dominated mappings. Suppose for $k + 2t$ 2 $[0;1)$, the following conditions hold:

$$d_q(Sx;Ty) \quad kd_q(x;y) + t[d_q(x;Sx) + d_q(y;Ty)];$$
$$d_q(Tx;Sy) \quad kd_q(x;y) + t[d_q(x;Tx) + d_q(y;Sy)]$$

for all $(x;y)$ in r. If for a nonincreasing sequence fx_ng ! implies that x_n, then there exists a point $\hat{w}$ in X, such that $\hat{w} = S\hat{w} = T\hat{w}$ and $d_q(\hat{w};\hat{w}) = 0:$ Also, $\hat{w}$ is unique, if for any $x;y$ 2 $X;$ the set $A_{x;y}$ = fz_0 2 X : $z_0\ x$ and $z_0\ y$g is non empty.

3.2.12 Theorem

Let $(X; ;d_q)$ be a preordered dislocated quasi metric space, x_0 2 X, $r > 0$ and $S;T$ be self mapping and f be a dominated mapping on $\overline{B_{d_q}(fx_0;r)}$, such that SX [$TX\ fX;Tx\ fx;$

$Sx \; fx$ and $B_{d_q}(fx_0;r) \; fX:$ Suppose for $k + 2t$ 2 $[0;1)$, the following conditions hold:

$$d_q(Sx;Ty) \quad kd_q(fx;fy) + t[d_q(fx;Sx) + d_q(fy;Ty)];$$

$$d_q(Tx;Sy) \quad kd_q(fx;fy) + t[d_q(fx;Tx) + d_q(fy;Sy)];$$

for all $(fx;fy)$ 2 $(\overline{B_{d_q}(fx_0;r)} \quad \overline{B_{d_q}(fx_0;r)}) \setminus$ r and

$$d_q(fx_0;Sx_0) \quad (1 \quad)r;$$

where $= \frac{k+t}{1-t}$ and

$$d_q(fx_0;Sx_0) + d_q(fy;Ty) \; d_q(fx_0;fy) + d_q(Sx_0;Ty);$$

for all fy 2 $\overline{B_{d_q}(fx_0;r)}$, such that $fy \; Sx_0:$ If for a nonincreasing sequence fx_ng in $\overline{B_{d_q}(fx_0;r)};$ fx_ng ! implies that x_n and for any two points z and x in $\overline{B_{d_q}(fx_0;r)}$ there exists a point y 2 $\overline{B_{d_q}(fx_0;r)}$, such that $y \; z$ and $y \; x:$ If the subset fX is complete and $(T;f);$ $(S;f)$ satis es the condition of weakly compatible pair of functions, then $S(fz) = T(fz) = f(fz) = fz$. in $\overline{B_{d_q}(fx_0;r)}$. Also, $d_q(fz;fz) = 0:$

3.2.13 Theorem

Let $(X; ;d_q)$ be a preordered dislocated quasi metric space, x_0 2 X, $r > 0$ and $S;T$ be self mapping and f be a dominated mapping on X, such that SX [$TX \; fX$ and $Tx \; fx;$ $Sx \; fx:$ Suppose for $k + 2t$ 2 $[0;1)$, the following conditions hold:

$$d_q(Sx;Ty) \quad kd_q(fx;fy) + t[d_q(fx;Sx) + d_q(fy;Ty)];$$

$$d_q(Tx;Sy) \quad kd_q(fx;fy) + t[d_q(fx;Tx) + d_q(fy;Sy)];$$

for all $(fx;fy)$ 2 r$:$ If for a nonincreasing sequence fx_ng in $X;$ fx_ng ! implies that x_n and for any two points z and x in X there exists a point y 2 X, such that $y \; z$ and $y \; x:$ If the subset fX is complete left K-sequentially and $(T;f);$ $(S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz 2 $\overline{B_{d_q}(fx_0;r)}$, such that $S(fz) = T(fz) = f(fz) = fz$. Also, $d_q(fz;fz) = 0:$

3.2.14 Theorem [23]

Let $(X; ;d_q)$ be an ordered complete left K-sequentially dislocated quasi metric space, S be a self dominated mapping on X and x_0 be an arbitrary point in $\overline{B_{d_q}(x_0;r)}$. Suppose there exists k 2 $[0;1)$ with

$$d_q(Sx;Sy) \quad kd_q(x;y); \text{ for all comparable elements } x;y \text{ in } \overline{B_{d_q}(x_0;r)}$$

and

$$d_q(x_0;Sx_0) \qquad (1\ k)r:$$

If for a nonincreasing sequence fx_ng ! implies that x_n, then there exists a point $\hat{w}$ in $\overline{B_{d_q}(x_0;r)}$, such that $d_q(\hat{w};\hat{w}) = 0$ and $\hat{w} = S\hat{w}:$ Moreover, if for any $x;y$ 2 $\overline{B_{d_q}(x_0;r)};$ the set $A_{x;y}$ = fz 2 $\overline{B_{d_q}(x_0;r)}$: $z\ x$ and $z\ y$g is non empty, then the point $\hat{w}$ is unique.

3.2.15 Theorem [23]

Let $(X; ;d_q)$ be an ordered complete left K-sequentially dislocated quasi metric space, S be a self dominated mapping on X and x_0 be an arbitrary point in $\overline{B_{d_q}(x_0;r)}$. Suppose there exists $k \in [0, \frac{1}{2})$ with

$$d_q(Sx;Sy) \quad k[d_q(x;Sx) + d_q(y;Sy)];$$

for all comparable elements $x;y$ in $\overline{B_{d_q}(x_0;r)}$ and

$$d_q(x_0;Sx_0) \qquad (1 \qquad)r;$$

where $= \frac{k}{1-k}$. If for a nonincreasing sequence $\{x_n\} \to$ implies that x_n, then there exists a point $\hat{w}$ in $\overline{B_{d_q}(x_0,r)}$, such that $d_q(\hat{w},\hat{w}) = 0$ and $\hat{w} = S\hat{w}$. Moreover, if for any $x,y \in \overline{B_{d_q}(x_0,r)}$, the set $A_{x,y} = \{z \in \overline{B_{d_q}(x_0,r)} : z\ x \text{ and } z\ y\}$ is non empty and

$$d_q(x_0,Sx_0) + d_q(z,Sz)\ d_q(x_0,z) + d_q(Sx_0,Sz) \text{ for all } z\ Sx_0.$$

then the point $\hat{w}$ is unique.

3.2.16 Example [23]

Let $X = R^+ \cup \{0\}$ be endowed with usual order and let $d_q : X\ X \to X$ be defined by $d_q(x,y) = \dfrac{x}{2} + y.$ Let $S : X \to X$ be defined by

$$Sx = \begin{cases} \dfrac{x}{7} \text{ if } x \in [0,1] \\ x - \frac{1}{2} \text{ if } x \in (1,\infty). \end{cases}$$

Clearly, S is a dominated mapping, then for $x_0 = 1,\ r = \frac{3}{2},\ \theta = \frac{3}{7},\ \overline{B_{d_q}(x_0,r)} = [0,1]$ and for $k = \frac{3}{10}$,

$$(1-\theta)r = (1-\frac{3}{7})\frac{3}{2} = \frac{6}{7},$$

and

$$d_q(x_0,Sx_0) = d_q(1,S1) = d_q(1,\frac{1}{7}) = \frac{1}{2} + \frac{1}{7} = \frac{9}{14} < \frac{6}{7}.$$

Also, if $x,y \in (1,1)$, then

$$\begin{aligned} 5x + 10y &\geq \frac{9}{2}x + \frac{9}{2}y + \frac{9}{2} \\ &\Rightarrow 5x - \frac{5}{2} + 10y - 5 \geq 3[\frac{3}{2}x + \frac{3}{2}y - 1] \\ &\Rightarrow 10(\frac{x}{2} - \frac{1}{4} + y - \frac{1}{2}) \geq 3[\frac{x}{2} + x - \frac{1}{2} + \frac{y}{2} + y - \frac{1}{2}] \\ &\Rightarrow d_q(Sx,Sy) \geq k[d_q(x,Sx) + d_q(y,Sy)]. \end{aligned}$$

So the contractive condition does not hold on the whole space. Now, if $x,y \in B_{d_q}(x_0,r)$, then

$$
\begin{aligned}
d_q(Sx, Sy) &= \frac{x}{14} + \frac{y}{7} = \frac{1}{7}\{\frac{x}{2} + y\} \\
&\leq \frac{3}{10}\{\frac{x}{2} + \frac{y}{2}\} \leq \frac{3}{10}\{\frac{x}{2} + \frac{x}{7} + \frac{y}{2} + \frac{y}{7}\} \\
&= k[d_q(x, Sx) + d_q(y, Sy)].
\end{aligned}
$$

Also,

$$d_q(x_0;Sx_0) + d_q(z;Sz) = d_q(x_0;z) + d_q(Sx_0;Sz) \text{ for all } z\ Sx_0:$$

Hence, all the conditions of Theorem 3.2.15 are satis ed. Moreover, 0 is equal to $S(0):$

3.3 Common Fixed Point Results Satisfying - Type Contractive Conditions

Some of the results given in this section have been published in [11]. Some of the results given in this section have been submitted for publication [80].

3.3.1 Theorem

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space. Suppose there exist a function, : $X\ X$! $[0;+1)$. Let $r>0$, x_0 2 $\overline{B_{d_q}(x_0;r)}$ and $S;T : X$! X be - dominated mappings on $\overline{B_{d_q}(x_0;r)}$ and 2 . Assume that, for $x;y$ 2 $B_{d_q}(x_0;r)$; $(x;y)$ 1; the following condition holds

$$\max f d_q(Sx;Ty);d_q(Tx;Sy) g \qquad (d_q(x;y)) \tag{3.25}$$

and

$$\sum_{i=0}^{j} \psi^i(d_q(x_0, Sx_0)) \leq r, \text{ for all } j\ 2\ N\ [\ f0g: \tag{3.26}$$

If for any sequence $\{x_n\}$ in $\overline{B_{d_q}(x_0, r)}$, such that $(x_n;x_{n+1})$ 1 for all n 2 N [f0g and x_n ! u 2 $\overline{B_{d_q}(x_0;r)}$ as n ! +1, then $(x_n;u)$ 1 for all n 2 N [f0g; then there exists a common xed point $\hat{w}$ of S and T and $d_q(\hat{w};\hat{w}) = 0:$

Proof. Choose a point x_1 in X, such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X, such that

$$x_{2i+1} = Sx_{2i}; \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0;1;2;::::$$

First, we show that x_n 2 $\overline{B_{d_q}(x_0;r)}$ for all n 2 N. Using (3.26), we have

$$\sum_{i=0}^{j} \psi^i(d_q(x_0, Sx_0)) \leq r \quad \text{for all } j \text{ 2 } N \text{ [f0g}:$$

In particular, it holds for $j = 0$; that is

$$x_1 \text{ 2 } \overline{B_{d_q}(x_0;r)}:$$

is -dominated mappings on $\overline{B_{d_q}(x_0;r)}$, then $(x_0;x_1)$ 1: As T is -dominated mappings on $\overline{B_{d_q}(x_0;r)}$, then $(x_1;x_2)$ 1: Continuing in this way we obtain $(x_{2i};x_{2i+1})$ 1 for all

Let $x_2; ;x_j$ 2 $\overline{B_{d_q}(x_0;r)}$ for some j 2 N. If $j = 2i + 1$, where $i = 0;1;2;:::\frac{j-1}{2}$: As S

$i = 0;1;2;:::\frac{j-1}{2}$: So using (3.25), we obtain

$$\begin{aligned} d_q(x_{2i+1};x_{2i+2}) &= d_q(Sx_{2i};Tx_{2i+1}) \\ & \max f d_q(Sx_{2i};Tx_{2i+1}); d_q(Tx_{2i};Sx_{2i+1}) g \\ & (d_q(x_{2i};x_{2i+1})) \quad {}^2(d_q(x_{2i\ 1};x_{2i})) \\ & {}^{2i+1}(d_q(x_0;x_1)): \end{aligned}$$

Thus, we have

$$d_q(x_{2i+1};x_{2i+2}) \quad {}^{2i+1}(d_q(x_0;x_1)): \tag{3.27}$$

If $j = 2i + 2$, then as $x_1;x_2;:::;x_j$ 2 $\overline{B_{d_q}(x_0;r)}$ where $(i = 0;1;2;::::;\frac{j-2}{2})$: we obtain

$$d_q(x_{2i+2},x_{2i+3}) \leq \psi^{2(i+1)}(d_q(x_0,x_1)). \tag{3.28}$$

Thus, from the inequalities (3.27) and (3.28), we have

$$d_q(x_j,x_{j+1}) \leq \psi^{j}(d_q(x_0,x_1)). \tag{3.29}$$

Now,

$$\begin{aligned} d_q(x_0,x_{j+1}) &= d_q(x_0,x_1) + .. + d_q(x_j,x_{j+1}) \\ &\leq \sum_{i=0}^{j} \psi^i(d_q(x_0,x_1)) \leq r. \end{aligned}$$

Thus, $x_{j+1} \in \overline{B_{d_q}(x_0,r)}.$ Hence, $x_n \in \overline{B_{d_q}(x_0,r)}$ for all $n \in N$. Now, the inequality (3.29) can be written as

$$d_q(x_n,x_{n+1}) \leq \psi^{n}(d_q(x_0,x_1)), \text{ for all } n \in N. \tag{3.30}$$

Fix $\varepsilon > 0$ and let $n(\varepsilon) \in N$, such that $\sum \psi^n(d_q(x_0,x_1)) < \varepsilon.$ Let $n,m \in N$ with $m > n > n(\varepsilon).$

Using the triangle the inequality and the inequality (3.30), we obtain

$$\begin{aligned} d_q(x_n,x_m) &\leq \sum_{k=n}^{m-1} d_q(x_k,x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k(d_q(x_0,x_1)) \\ &\leq \sum_{n \geq n(\varepsilon)} \psi^k(d_q(x_0,x_1)) < \varepsilon. \end{aligned}$$

Thus, we have proved that $\{x_n\}$ is a left K-Cauchy sequence in $(\overline{B_{d_q}(x_0,r)},d_q)$. As $\overline{B_{d_q}(x_0,r)}$ is closed and so it is complete left K-sequentially. Therefore, there exists a point $\hat{w} \in \overline{B_{d_q}(x_0,r)}$, such that $x_n \to \hat{w}.$ Also,

$$\lim_{n\to\infty} d_q(x_n, \hat{w}) = \lim_{n\to\infty} d_q(w, \hat{x}_n) = 0. \quad (3.31)$$

By assumption, we have $\alpha(w, \hat{x}_n) \geq 1$ for all $n \in N \cup \{0\}$. Now, by using (3.25), we get

$$d_q(Sw, \hat{x}_{2i+2}) \leq \max\{d_q(Sw, T\hat{x}_{2i+1}), d_q(Tw, S\hat{x}_{2i+1})\}$$

$$\leq \psi(d_q(w, \hat{x}_{2i+1})) < d_q(w, \hat{x}_{2i+1}).$$

Letting $i \to \infty$ and by the inequality (3.31), we obtain $d_q(Sw, \hat{w}) < 0$. Hence, $S\hat{w} = \hat{w}$. Similarly by using

$$d_q(T\hat{w}, \hat{x}_{2i+1}) \leq \psi(d_q(\hat{w}, \hat{x}_{2i})) < d_q(\hat{w}, \hat{x}_{2i}),$$

we obtain $d_q(T\hat{w}, \hat{w}) = 0$; that is, $T\hat{w} = \hat{w}$. Hence, S and T have a common fixed point in $\overline{B_{d_q}(x_0, r)}$. As S is α- dominated mappings on $\overline{B_{d_q}(x_0, r)}$ we have $\alpha(\hat{w}, S\hat{w}) \geq 1$ and so $\alpha(\hat{w}, \hat{w}) \geq 1$. Now,

$$d_q(\hat{w}, \hat{w}) = \max\{d_q(S\hat{w}, T\hat{w}), d_q(T\hat{w}, S\hat{w})\} \leq \psi(d_q(\hat{w}, \hat{w})).$$

This implies that $d_q(\hat{w}, \hat{w}) = 0$.

■

3.3.2 Example

Let $X = Q^+ \cup \{0\}$ and let $d_q : X \times X \to X$ be the complete left K-sequentially dislocated quasi metric on X defined by,

$$d_q(x, y) = 2x + y \text{ for all } x, y \in X.$$

Let $S, T : X \to X$ be defined by,

$$Sx = \begin{cases} \frac{x}{4} \text{ if } x \in [0,2] \cap X \\ 3x \text{ if } x \in (2,\infty) \cap X \end{cases}$$

and

$$Tx = \begin{cases} \frac{2x}{7} \text{ if } x \in [0,2] \cap X \\ 4x \text{ if } x \in (2,\infty) \cap X. \end{cases}$$

Considering, $x_0 = 1, r = 4,$ then $\overline{B_{d_q}(x_0,r)} = [0,2] \cap X$. Define $\alpha(x,y) = |2x - y + 3|.$ Clearly, S and T are α-dominated mappings on $\overline{B_{d_q}(x_0,r)}$. Let $\psi(t) = \frac{t}{3}.$ Now,

$$d_q(x_0, Sx_0) = d_q(1, S1) = d_q(1, \frac{1}{4}) = \frac{9}{4}.$$

$$\sum_{i=0}^{n} \psi^n(d_q(x_0, Sx_0)) = \frac{9}{4} \sum_{i=0}^{n} \frac{1}{3^n} < (\frac{9}{4})\frac{3}{2} < 4.$$

Now, if $x, y \in (2, \infty) \cap X,$ then we have the following cases.

Case 1. If $\max\{d_q(Sx,Ty), d_q(Tx,Sy)\} = d_q(Sx,Ty)$, then for $x, y \in (2,\infty),$ we have

$$\begin{aligned} d_q(Sx,Ty) &= d_q(3x,4y) = 6x + 4y \\ &> \frac{2x}{3} + \frac{y}{3} = \psi(d_q(x,y)). \end{aligned}$$

Case 2. If $\max\{d_q(Sx,Ty), d_q(Tx,Sy)\} = d_q(Tx,Sy)$,

$$\begin{aligned} d_q(Tx,Sy) &= d_q(4x,3y) = 8x + 3y \\ &> \frac{2x}{3} + \frac{y}{3} = \psi(d_q(x,y)). \end{aligned}$$

So the contractive condition does not hold on the whole space.

Now, if $x, y \in \overline{B_{d_q}(x_0,r)},$ then

Case 3. If $\max\{d_q(Sx,Ty), d_q(Tx,Sy)\} = d_q(Sx,Ty).$

$$\begin{aligned} d_q(Sx,Ty) &= d_q(\frac{x}{4},\frac{2y}{7}) = 2(\frac{x}{4}) + \frac{2y}{7} \\ &\leq 2(\frac{x}{3}) + \frac{y}{3} = \psi(d_q(x,y)). \end{aligned}$$

Case 4. If $\max\{d_q(Sx,Ty), d_q(Tx,Sy)\} = d_q(Tx,Sy)$.

$$\begin{aligned} d_q(Tx,Sy) &= d_q(\frac{2x}{7},\frac{y}{4}) = 2(\frac{2x}{7}) + \frac{y}{4} \\ &\leq 2(\frac{x}{3}) + \frac{y}{3} = \psi(d_q(x,y)). \end{aligned}$$

then the contractive condition holds on $\overline{B_{d_q}(x_0,r)}$. Hence, all the conditions of Theorem 3.3.1 are satisfied and 0 is a common fixed point of S and T.

3.3.3 Corollary

Let (X,d_q) be a complete left K-sequentially dislocated quasi metric space. Suppose there exist a function, $\alpha : X \times X \to [0,+\infty)$. Let $r > 0$, $x_0 \in \overline{B_{d_q}(x_0,r)}$ and $S : X \to X$ be a α-dominated mapping on $\overline{B_{d_q}(x_0,r)}$ and $\psi \in \Psi$. Assume that, for $x,y \in \overline{B_{d_q}(x_0,r)}$, $\alpha(x,y) \geq 1$, the following condition holds

$$d_q(Sx,Sy) \leq \psi(d_q(x,y))$$

and

$$\sum_{i=0}^{j} \psi^i(d_q(x_0,Sx_0)) \leq r, \text{ for all } j \in N \cup \{0\}.$$

If for any sequence $\{x_n\}$ in $\overline{B_{d_q}(x_0,r)}$, such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B_{d_q}(x_0,r)}$ as $n \to +\infty$, then $\alpha(x_n,u) \geq 1$ for all $n \in N \cup \{0\}$, then there exists a fixed point $\hat{w}$ of S and $d_q(\hat{w},\hat{w}) = 0$.

3.3.4 Corollary

Let (X,d_q) be a complete left K-sequentially dislocated quasi metric space. Let $r > 0$, $x_0 \in$

$\overline{B_{d_q}(x_0;r)}$ and $S;T : X ! X$ be dominated mappings on $\overline{B_{d_q}(x_0;r)}$ and 2 . Assume that the following condition holds

$$\max f d_q(Sx;Ty); d_q(Tx;Sy) g \ (d_q(x;y))$$

for all $(x;y)$ in $(\overline{B_{d_q}(x_0;r)}\ \overline{B_{d_q}(x_0;r)}) \setminus$ r and

$$\sum_{i=0}^{j} \psi^i(d_q(x_0, Sx_0)) \leq r, \text{ for all } j \ 2 \ N \ [\ f0g:$$

If for any sequence fx_ng in $\overline{B_{d_q}(x_0;r)}$, such that fx_ng ! implies that x_n, then there exists a point $\hat{w}$ in $\overline{B_{d_q}(x_0;r)}$, such that $\hat{w} = S\hat{w} = T\hat{w}$ and $d_q(\hat{w};\hat{w}) = 0:$

3.3.5 Theorem

Adding condition if $\hat{w}$ is any common xed point in $\overline{B_{d_q}(x_0;r)}$ of S and T, x be any xed point of S or T in $\overline{B_{d_q}(x_0;r)}$, then $(\hat{w};\hat{x})$ 1to the hypotheses of Theorem 3.3.1, then S and T have a unique common xed point $\hat{w}:$

Proof. Assume that y be another xed point of T in $\overline{B_{d_q}(x_0;r)};$ then by assumption, $(\hat{w};\hat{y})$ 1; also,

$$d_q(\hat{w};\hat{y}) = d_q(S\hat{w};T\hat{y}) \ (d_q(\hat{w};\hat{y}))$$

A contradiction to the fact that for each $t > 0$; $(t) < t$: So $\hat{w} = y$: Hence, T has no xed point other than $\hat{w}$: Similarly, S has no xed point other than $\hat{w}$. ■

Now, we apply our Theorem 3.3.5 to obtain unique common xed point of three mappings

on closed ball in complete dislocated quasi d_q-metric space.

3.3.6 Theorem

Let $(X;d_q)$ be a dislocated quasi metric space; $S;T;f : X \to X$, such that $SX \cup TX \subseteq fX$, $r > 0$ and $x_0 \in \overline{B_{d_q}(fx_0;r)}$. Suppose there exist a function, $: X \times X \to [0;+\infty)$, such that $(fx;Sx) \quad 1$, $(fx;Tx) \quad 1$ for all $fx \in \overline{B_{d_q}(fx_0;r)}$: If the following conditions hold for all $fx;fy \in \overline{B_{d_q}(fx_0;r)}$; $(fx;fy) \quad 1$ and 2 ,

$$\max\{d_q(Sx;Ty);d_q(Tx;Sy)\} \quad (d_q(fx;fy)): \tag{3.32}$$

X a n d

$$\sum_{i=0}^{j} {}^{i}(d_q(fx_0;Sx_0)) \quad r; \text{ for all } j \in N \cup \{0\}: \tag{3.33}$$

Suppose that the following conditions hold:

(i) If $\{x_n\}$ is a sequence in $\overline{B_{d_q}(fx_0;r)}$, such that $(x_n;x_{n+1}) \quad 1$ for all n and $x_n \to u \in \overline{B_{d_q}(fx_0;r)}$ as $n \to +\infty$, then $(u;x_n) \quad 1$ for all $n \in N \cup \{0\}$.

(ii) fx, fy be any xed points in $\overline{B_{d_q}(fx_0;r)}$, then $(fx;fy) \quad 1$:

(iii) If the subset fX is complete left K-sequentially and $(T;f)$; $(S;f)$ satis es the condition of weakly compatible pair of functions.

then $S;T$ and f have a unique common xed point fp in $\overline{B_{d_q}(fx_0;r)}$. Moreover $d_q(fp;fp) = 0$:

Proof. By Lemma 1.4.2, there exists $E\ X$, such that $fE = fX$ and $f : E\ !\ X$ is oneto-one.

Now, since SX [$TX\ fX;$ we de ne two mappings $g;h : fE\ !\ fE$ by $g(fx) = Sx$

and $h(fx) = Tx$ respectively. Since f is one-to-one on E, then $g;h$ are well-de ned. As $(fx;Sx)$ 1 implies that $(fx;g(fx))$ 1 and $(fx;Tx)$ 1 implies that $(fx;h(fx))$ 1,

then g and h are -dominated mappings on $\overline{B_{d_q}(fx_0;r)}:$ Now, fx_0 2 $\overline{B_{d_q}(fx_0;r)}$ fX, then fx_0 2 $fX:$ Let $y_0 = fx_0;$ choose a point y_1 in fX, such that $y_1 = g(y_0):$ Also, by the inequality

(3.33).

$$X \quad {}_j\ {}^i(d_q(y_0;gy_0))\ r; \text{ for all } j\ 2\ N\ [\ f0g: \quad {}_{i=0}$$

Then y_1 2 $\overline{B_{d_q}(fx_0;r)}:$ Let $y_2 = h(y_1)$. Continuing this process and having chosen y_n in fX,

such that

$$y_{2i+1} = g(y_{2i}) \text{ and } y_{2i+2} = h(y_{2i+1}); \text{ where } i = 0;1;2;::;$$

Following similar arguments of Theorem 3.3.1, y_n 2 $\overline{B_{d_q}(fx_0;r)}:$ Note that for $fx;\ fy$ 2

$\overline{B_{d_q}(fx_0;r)}$ and $(fx;fy)$ 1, then by using the inequality (3.32), we have

$$\max f d_q\ (g(fx);h(fy));d_q(h(fx);g(fy))g\ (d_q(fx;fy)):$$

As fX is a complete space$;$ all the conditions of Theorem 3.3.5 are satis ed, we deduce that

there exists a unique common xed point fp 2 $\overline{B_{d_q}(fx_0;r)}$ of g and $h:$

The rest of the proof is similar to the proof given in Theorem 2.3.10 and so we leave it.

Hence, we obtain a unique common xed point of $S;T$ and f. ■

Metric version of Theorem 3.3.6 is given below.

3.3.7 Theorem

Let $(X;d)$ be a metric space; $S;T;f : X ! X$, such that SX [TX fX, $r > 0$ and

x_0 2 $\overline{B(fx_0;r)}$. Suppose there exist a function, : $X X$! $[0;+1)$, such that $(fx;Sx)$ 1,

$(fx;Tx)$ 1 for all fx 2 $\overline{B(fx_0;r)}$: If the following conditions hold for all $fx;fy$ 2 $\overline{B(fx_0;r)}$; $(fx;fy)$ 1 and 2 ,

$$d(Sx;Ty) \quad (d(fx;fy)):$$

and

$$\sum_{i=0}^{j} {}^{i}(d(fx_0;Sx_0))\ r;\ \text{for all } j\ 2\ N\ [\ f0g:$$

Suppose that the following conditions hold:

(i) If $fx_n g$ is a sequence in $\overline{B(fx_0;r)}$, such that $(x_n;x_{n+1})$ 1 for all n and $x_n ! u$ 2 $\overline{B(fx_0;r)}$ as $n ! +1$, then $(u;x_n)$ 1 for all n 2 N [f0g.

(ii) fx, fy be any xed points in $B(fx_0;r)$, then $(fx;fy)$ 1:

(iii) If the subset fX is complete left K-sequentially and $(T;f)$; $(S;f)$ satis es the condition of weakly compatible pair of functions.

Then $S;T$ and f have a unique common xed point fp in $\overline{B(fx_0;r)}$.

Now, we obtain the results in [11] as a corollaries of the above results.

3.3.8 Theorem [11]

Let $(X; ;d_l)$ be an ordered complete dislocated metric space, $S;T : X ! X$ be dominated maps and x_0 2 X. Suppose that for k 2 $[0;1)$ and for $S =6 T$, we have

$$d_l(Sx;Ty)\ kd_l(x;y) \text{ for all comparable elements } x;y \text{ in } \overline{B_{d_l}(x_0;r)}$$

and

$$\overline{d_l(x_0;Sx_0)} \quad (1\ k)r:$$

If for a non-increasing sequence fx_ng ! implies that x_n, then there exists a point w^ in $\overline{B_{d_l}(x_0;r)}$, such that $d_l(w;\hat{}\ w\hat{}) = 0$ and $w\hat{} = Sw\hat{} = Tw:\hat{}$ Also, if for any two points $x;y$ in $\overline{B_{d_l}(x_0;r)}$ there exists a point z 2 $B_{d_l}(x_0;r)$, such that $z\ x$ and $z\ y$, then w^ is a unique common xed point in $B_{d_l}(x_0;r):$

3.3.9 Example [11]

Let $X = Q^+$ [f^{0}g be endowed with order $(x_1;y_1)$ $(x_2;y_2)$ if $x_1\ x_2$ and let $S;T : X^2\ !\ X^2$

be de ned by

$$S(x,y)=\begin{cases}(\frac{x}{7},\frac{3y}{11}) \text{ if } x+y\leq 1\\ (x-\frac{1}{3},y-\frac{3}{8}) \text{ if } x+y>1\end{cases}$$

and

$$T(x,y)=\begin{cases}(\frac{4x}{15},\frac{2y}{7}) \text{ if } x+y\leq 1\\ (x-\frac{1}{4},y-\frac{1}{5}) \text{ if } x+y>1\end{cases}:$$

Clearly, S and T are dominated mappings. Let $d_l : X^2\ X^2\ !\ X$ be de ned by $d_l((x_1;y_1);(x_2;y_2))$ $= x_1 + y_1 + x_2 + y_2$, then it is easy to prove that $(X^2;d_l)$ is a complete dislocated metric space.

Let $(x_0,y_0)=(\frac{3}{7},\frac{4}{7}),\ r=2,$ then

$$\overline{B_{d_l}((x_0;y_0);r)} = \text{f}(x;y)\ 2\ X : x + y\ 1\text{g}$$

with $k=\frac{3}{10}\in[0,1),$

$$(1-k)r=(1-\frac{3}{10})2=\frac{7}{5}$$

$$d_l((x_0,y_0),S(x_0,y_0))=\frac{656}{539}<\frac{7}{5}$$

Also, for all comparable elements $(x_1;y_1);(x_2;y_2)$ 2 X^2, such that $x_1 + y_1 > 1$ and $x_2 + y_2 >$ $1;$ we have

$$
\begin{aligned}
d_l(S(x_1,y_1),T(x_2,y_2)) &= x_1 - \frac{1}{3} + y_1 - \frac{3}{8} + x_2 - \frac{1}{4} + y_2 - \frac{1}{5} \\
&\geq \frac{3}{10}\{x_1 + y_1 + x_2 + y_2\} \\
d_l(Sx,Ty) &\geq kd_l[(x_1,y_1),(x_2,y_2)].
\end{aligned}
$$

So the contractive condition does not hold on the whole space$:$ Now, if $(x_1;y_1);(x_2;y_2)$ 2 $\overline{B_{d_l}((x_0;y_0);r)};$ then

$$
\begin{aligned}
d_l(S(x_1,y_1),T(x_2,y_2)) &= \frac{x_1}{7} + \frac{3y_1}{11} + \frac{4x_2}{15} + \frac{2y_2}{7} \\
&\leq \frac{3}{10}\{x_1 + y_1 + x_2 + y_2\} = kd_l[(x_1,y_1),(x_2,y_2)]:
\end{aligned}
$$

Hence, all the conditions of Theorem 3.3.8 are satis ed. Moreover, $(0;0)$ is the common xed point of S and $T:$ Also, note that for any metric d on $X^2;$ the respective condition does not

hold on $\overline{B_{d_l}((x_0;y_0);r)}$ since

$$
\begin{aligned}
d\left(S(\frac{2}{5},\frac{3}{5}),T(\frac{2}{5},\frac{3}{5})\right) &= d\left((\frac{2}{35},\frac{9}{55}),(\frac{8}{75},\frac{6}{35})\right) \\
&> kd\left((\frac{2}{5},\frac{3}{5}),(\frac{2}{5},\frac{3}{5})\right) = 0
\end{aligned}
$$

for any k 2 $[0;1):$

Moreover X^2 is not complete for any metric d on $X^2:$

3.3.10 Remark [11]

If we impose Banach type contractive condition for a pair $S;T : X ! X$ of mappings on a metric space $(X;d)$ that is

$$d(Sx;Ty)\ kd(x;y) \text{ for all } x;y \text{ 2 } X:$$

then it follows that $Sx = Tx;$ for all x 2 X (that is S and T are equal). Therefore the above condition fails to nd common xed points of S and T. However the same condition in dislocated metric space does not assert that $S = T;$ this is seen in Example 3.3.9. Hence, Theorem 3.3.8 cannot be obtained from a metric xed point theorem.

3.3.11 Theorem [11]

Let $(X; ;d_l)$ be a ordered complete dislocated metric space, S be a self dominated mapping on X and x_0 2 X. Suppose there exists k 2 $[0;1)$ with

$$d_l(Sx;Sy)\ kd_l(x;y); \text{ for all comparable elements } x;y \text{ in } \overline{B_{d_l}(x_0;r)}$$

$$\text{and } d_l(x_0;Sx_0)\quad (1\ k)r:$$

If for a non-increasing sequence fx_ng in $B_{d_l}(x_0;r);$ fx_ng ! implies that x_n. Also, if for any $x;y$ 2 $\overline{B_{d_l}(x_0;r)};$ the set $A_{x;y}$ = fz 2 $\overline{B_{d_l}(x_0;r)}$: $z\ x$ and $z\ y$g is non empty, then there exists a unique xed point $\hat{w}$ of S in $\overline{B_{d_l}(x_0;r)}:$ Further $d_l(\hat{w};\hat{w}) = 0:$

3.3.12 Theorem [11]

Let $(X; ;d_l)$ be an ordered complete dislocated metric space, $S;T : X\ !\ X$ be the dominated map and x_0 2 X. Suppose for k 2 $[0;1)$ and for S =6 T, we have

$$d_l(Sx;Ty)\ kd_l(x;y); \text{ for all comparable elements } x;y \text{ in } X:$$

Also, if for a non-increasing sequence fx_ng in $X;$ fx_ng ! implies that x_n and if for any $x;y$ 2 $X;$ the set $A_{x;y}$ = fz 2 X : $z\ x$ and $z\ y$g is non empty, then there exists a unique point $\hat{w}$ in X, such that $\hat{w} = S\hat{w} = T\hat{w}:$ Further $d_l(\hat{w};\hat{w}) = 0:$

3.3.13 Theorem [11]

Let (X,d_l) be a complete dislocated metric space, $S,T : X \to X$ be the self maps and $x_0 \in X$. Suppose for $k \in [0,1)$ and for $S \neq T$, we have

$$d_l(Sx,Ty) \leq k d_l(x,y); \text{ for all elements } x,y \text{ in } \overline{B_{d_l}(x_0,r)}$$

and

$$d_l(x_0,Sx_0) \leq (1-k)r.$$

Then there exists a unique $\hat{w} \in \overline{B_{d_l}(x_0,r)}$, such that $d_l(\hat{w},\hat{w}) = 0$ and $\hat{w} = S\hat{w} = T\hat{w}.$ Further

S and T have no fixed point other than $\hat{w}.$

3.3.14 Theorem [11]

Let $(X,\preceq,d_l)$ be an ordered dislocated metric space and S,T self mappings and f be a dominated mapping on X, such that $SX \cup TX \subseteq fX$, $Tx \preceq fx, Sx \preceq fx$ and $x_0 \in X$. Suppose that for $k \in [0,1)$ and for $S \neq T$, we have

$$d_l(Sx,Ty) \leq k d_l(fx,fy)$$

for all comparable elements $fx,fy \in \overline{B_{d_l}(fx_0,r)} \subseteq fX$ and

$$d_l(fx_0,Tx_0) \leq (1-k)r.$$

If for a non-increasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$ and if for any $x,z \in \overline{B_{d_l}(fx_0,r)}$; the set $A_{x,z} = \{y \in \overline{B_{d_l}(fx_0,r)} : y \preceq z \text{ and } y \preceq x$ for some $z,x \in \overline{B_{d_l}(fx_0,r)}\}$ is non empty that is every pair of elements in $\overline{B_{d_l}(fx_0,r)}$ has a lower bound in $\overline{B_{d_l}(fx_0,r)}$. If the subset fX is complete and $(T,f),(S,f)$ satisfies the condition of weakly compatible pair of functions, then there exists fz in $\overline{B_{d_l}(fx_0;r)}$, such that $S(fz) = T(fz) = f(fz) = fz$. Also $d_l(fz;fz) = 0:$

3.3.15 Theorem [11]

Let $(X;d_l)$ be a dislocated metric space and $S;T;g$ and f be self mappings on X, such that $SX;\ TX\ fX = gX:$ Assume that x_0 2 $X;k$ 2 $[0;1)$ and S =6 T, such that following conditions hold:

$$d_l\,(Sx;Ty)\ kd_l(fx;gy)$$

for all elements $fx;gy$ 2 $\overline{B_{d_l}(fx_0;r)}\ fX$; and

$$d_l(fx_0;Sx_0) \qquad (1\ k)r:$$

If the subset fX is complete, then there exists fz 2 X, such that $d_l(fz;fz) = 0:$ Also if $(T;g);\ (S;f)$ satis es the condition of weakly compatible pair of functions, then there exists fz in

$\overline{B_{d_l}(fx_0;r)}$, such that $S(fz) = T(fz) = f(fz) = g(fz) = fz$.

Chapter 4

Fixed Point Results in Closed Ball for Multivalued Mappings

4.1 Introduction

The theory and the de nitions given in this section have been submitted for publication [78].

Nadler [57], introduced a study of xed point theorems involving multivalued mappings (see also [17, 18]). Asl et al. [14] generalized the notion of -admissible mappings by introducing the concepts of - contractive multifunctions, -admissible mapping and obtained some xed point results for these multifunctions (see also [5, 40, 41]). The aim of this chapter is to establish xed point results for -admissible multivalued mappings on closed ball satisfying generalized - -contractive conditions in complete left K-sequentially dislocated quasi metric space. We derive some new xed point theorems for ordered metric space. The examples have been constructed to demonstrate the novelty of our results. Our results unify, extend and generalize several comparable results in the existing literature. Recently, Lopez [67] introduced the concept of Hausdor fuzzy metric spaces on non empty compact sets. The idea was derived from the concept of Hausdor metric. We also establish a xed point result on closed ball in Hausdor fuzzy metric spaces. We introduce the following de nitions which will be needed in the sequel.

4.1.1 De nition

Let K be a nonempty subset of dislocated quasi metric space X and let x 2 $X:$ An element

y_0 2 K is called a best approximation in K if

$$d_q(x;K) = d_q(x;y_0); \text{ where } d_q(x;K) = \inf_{y2K} d_q(x; y):$$

If each x 2 X has at least one best approximation in $K;$ then K is called a proximinal set.

We denote $CP(X)$ be the set of all closed proximinal subsets of $X:$ Let denote the family

of all nondecreasing functions : $[0;+1)$! $[0;+1)$, such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for all $t>$

$0;$ where ^{n} is the n^{th} iterate of $:$ If 2 $;$ then $(t) < t$ for all $t > 0:$

4.1.2 De nition

Let $(X;d)$ be a metric space, $S : X$! $CP(X)$ be a multivalued mapping and $: X X$!

$[0;+1)$. Let A $X;$ we say that S is -admissible on $A;$ whenever $(x;y)$ 1 implies that

$(Sx; Sy)$ 1$;$ for all $x;y$ 2 $A;$ where $(Sx; Sy)$ = inff $(a;b)$: a 2 $Sx;b$ 2 Syg$:$ If $A = X$, then

we say that S is -admissible on $X:$

4.1.3 De nition

The function $H_{d_q} : CP(X)\ CP(X)$! $X;$ de ned by

$$H_{d_q}(A;B) = \max\{\sup_{a2A} d_q(a;B); \sup_{b2B} d_q(A;b)\}$$

is called dislocated quasi hausdor metric on $CP(X):$

Let X be a nonempty set, then $(X; ;d_l)$ is called a preordered dislocated metric space if d_l is a dislocated metric on X and is a preorder on X. Let $(X; ;d_q)$ be a preordered metric space and $A;B$ X. We say that A B whenever for each a 2 A there exists b 2 B, such that a b. Also,we say that A $_r$ B whenever for each a 2and b 2 B we have a $b:$

4.1.4 Lemma

Let $(X;M;)$ be a fuzzy metric space. Let $(K_0(X);H_M;)$ is a Hausdor fuzzy metric space on $K_0(X)$, then for all $A;B$ 2 $K_0(X)$ and for each a 2 A there exist b_a 2 B satis es $M(a;B;t) = M(a;b_a;t)$, then $H_M(A;B;t)$ $M(a;b_a;t)$. Proof. If $H_M(A;B;t) = \inf M(a;B;t);$ then $H_M(A;B;t)$ $M(a;B;t)$ for each a 2 $A:$ $_{a2A}$

Hence, for each a 2 A there exist b_a 2 B satis es $M(a;B;t) = M(a;b_a;t)$, then $H_M(A;B;t)$

$M(a;b_a;t):$ Now, if $H_M(A;B;t) = \inf M(A;b;t)$ cases, we $_{b2B}$ $_{a2A}$ $\inf M(a;B;t):$ Hence, in both proved the result. ■

4.2 Fixed Point Results for Multivalued Mappings in Dislocated Quasi Metric Spaces

Let $(X;d_q)$ be a dislocated quasi metric space;x_0 2 X and $S : X$! $CP(X)$ be a multivalued mapping on X, then there exist x_1 2 Sx_0, such that $d_q(x_0; Sx_0) = d_q(x_0;x_1):$ Let x_2 2 Sx_1 be, such that $d_q(x_1; Sx_1) = d_q(x_1;x_2):$ Continuing this process, we construct a sequence x_n of points in X, such that x_{n+1} 2 Sx_n and $d_q(x_n; Sx_n) = d_q(x_n;x_{n+1}):$ We denote this iterative sequence by f$XS(x_0)$g$:$

4.2.1 Theorem

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space, $r > 0; x_0$ 2

$\overline{B_{d_q}(x_0;r)}$ and $S : X ! CP(X)$ be a -admissible multifunction on $\overline{B_{d_q}(x_0;r)}$. Assume that for

2 , such that

$$(Sx;Sy)H_{d_q}(Sx;Sy) \qquad (d_q(x;y)) \text{ for all } x;y \ 2 \ \overline{B_{d_q}(x_0;r)} \tag{4.1}$$

and

$$\sum_{i=0}^{n} \psi^i(d_q(x_0, Sx_0)) \leq r \quad \text{for all } n \ 2 \ \mathsf{N} \ [\ f0g: \tag{4.2}$$

If f$XS(x_0)$g is a sequence in $\overline{B_{d_q}(x_0;r)}$ and f$XS(x_0)$g ! x and $(x_n;x_{n+1})$ 1 for $x_n;x_{n+1}$ 2 f$XS(x_0)$g$;n$ 2 N[f0g, then $(x_n;x)$ 1 or $(x;x_n)$ 1 for all n 2 N[f0g. Also, there exist x_1 2 Sx_0, such that $(x_0;x_1)$ 1, then S has a xed point in $\overline{B_{d_q}(x_0;r)}$.

Proof. As x_0 2 $\overline{B_{d_q}(x_0;r)}$; and $S : X ! CP(X)$ be a multivalued mapping on X, then there exist x_1 2 Sx_0, such that $d_q(x_0; Sx_0) = d_q(x_0;x_1)$: If $x_0 = x_1$, then x_0 is a xed point in $\overline{B_{d_q}(x_0;r)}$ of S. Let x_0 =6 x_1: From (4.2), we get

$$d_q(x_0, x_1) \leq \sum_{i=0}^{n} \psi^i(d_q(x_0, x_1)) \leq r.$$

It follows that,

$$x_1 \ 2 \ \overline{B_{d_q}(x_0;r)}:$$

Since $(x_0;x_1)$ 1 and S is -admissible multifunction on $\overline{B_{d_q}(x_0;r)}$ and so $(Sx_0;Sx_1)$ 1. Also, there exist x_2 2 Sx_1, such that $d_q(x_1; Sx_1) = d_q(x_1;x_2)$: If $x_1 = x_2$, then x_1 is a xed point of S in $\overline{B_{d_q}(x_0;r)}$: Let x_1 =6 x_2: Now,

$$d_q(x_1;x_2) \qquad = d_q(x_1;Sx_1) \ H_{d_q}(Sx_0;Sx_1)$$

$$(Sx_0;Sx_1)H_{d_q}(Sx_0;Sx_1):$$

Note that x_2 2 $\overline{B_{d_q}(x_0;r)}$; because

$$d_q(x_0;x_2) \quad d_q(x_0;x_1) + d_q(x_1;x_2)$$

$$d_q(x_0;x_1) + \quad (Sx_0;Sx_1)H_{d_q}(Sx_0;Sx_1)$$

$$d_q(x_0,x_1) + \psi(d_q(x_0,x_1))$$

$$\sum_{i=0}^{n} \psi^i(d_q(x_0,x_1)) \leq r. \quad \text{by (4.1)}$$

As $(Sx_0; Sx_1)$ 1; x_1 2 Sx_0 and x_2 2 Sx_1 and so $(x_1;x_2)$ 1: As S is - admissible

multifunction on $\overline{B_{d_q}(x_0;r)}$. Thus, $(Sx_1; Sx_2)$ 1: Let $x_2; ;x_j$ 2 $\overline{B_{d_q}(x_0;r)}$ for some j 2 N,

such that x_{j+1} 2 Sx_j and $d_q(x_j; Sx_j) = d_q(x_j;x_{j+1})$: As $(Sx_1; Sx_2)$ 1; we have $(x_2;x_3)$ 1;

which further implies $(Sx_2; Sx_3)$ 1: Continuing this process, we have $(Sx_{j\ 1}; Sx_j)$

1. Now,

$$d_q(x_j;x_{j+1}) \quad = \ d_q(x_j;Sx_j)\ H_{d_q}(Sx_{j\ 1};Sx_j)$$

$$(Sx_{j\ 1};Sx_j)H_{d_q}(Sx_{j\ 1};Sx_j)$$

$$(d_q(x_{j\ 1};x_j)) :: {}^j(d_q(x_0;x_1)) \quad (4.3)$$

$$d_q(x_0;x_{j+1}) \quad d_q(x_0;x_1) + :: + d_q(x_j;x_{j+1})\ d_q(x_0;x_1) + :: +$$

$$^j(d_q(x_0;x_1))$$

$$= \sum_{i=0}^{j} {}^i(d_q(x_0;x_1))\ r:$$

Thus, $x_{j+1}\ 2\ B_{d_q}(x_0;r):$ As $(Sx_{j\ 1};Sx_j)\ 1;x_j\ 2\ Sx_{j\ 1};x_{j+1}\ 2\ Sx_j;$ we have $(x_j;x_{j+1})\ 1:$ Also, S is -admissible multifunction on $B_{d_q}(x_0;r);$ therefore $(Sx_j;\ Sx_{j+1})\ 1:$ Hence, by mathematical induction, $x_n\ 2\ B_{d_q}(x_0;r)$ and $(Sx_n;\ Sx_{n+1})\ 1$ for all $n\ 2\ \mathsf{N}$. Now, the inequality (4.3) can be written as

$$d_q(x_n;x_{n+1})\ {}^n(d_q(x_0;x_1));\ \text{for all } n\ 2\ \mathsf{N}:$$

Fix $\&>0$ and let $n(\&)\ 2\ \mathsf{N}$, such that $X\ {}^n(d_q(x_0;x_1))<\&:$ Let $n;m\ 2\ \mathsf{N}$ with $m>n>k(\&);$

then we obtain

$$d_q(x_n;x_m)\qquad \overset{m\ 1}{\underset{k=n}{X}} d_q(x_k;x_{k+1})\ \overset{m\ 1}{\underset{k=n}{X}}\ {}_k(d_q(x_0;x_1))$$

$$\underset{n\ k(\&)}{X}\ {}^k(d_q(x_0;x_1))<\&:$$

Thus, we proved that fx_ng is a Cauchy sequence in $(\overline{B_{d_q}(x_0;r)};d_q)$. As every closed ball in a complete left K-sequentially dislocated quasi metric space is complete left K-sequentially and

so there exists $w\hat{}\ 2\ \overline{B_{d_q}(x_0;r)}$, such that $x_n\ !\ w;\hat{}$ and

$$\lim_{n!1} d_q(x_n;w\hat{}) = \lim_{n!1} d_q(w;x\hat{}_n) = 0:$$

Note that fx_ng is a $fXS(x_0)g$ in $B_{d_q}(x_0;r):$ As $(Sx_n;Sx_{n+1})\ 1$ for all $n\ 2\ \mathsf{N}[f0g;$ we have $(x_{n+1};x_{n+2})\ 1$ for all $n\ 2\ \mathsf{N}[f0g:$ By assumption, we have $(x_n;w\hat{})\ 1$ for all $n\ 2\ \mathsf{N}[f0g:$

Thus, $(Sx_n; S\hat{w}) \quad 1:$ Now,

$$d_q(w;S\hat{}\, \hat{w}) \quad d_q(w;\hat{x}_{n+1}) + d_q(x_{n+1};S\hat{w})$$

$$d_q(w;\hat{x}_{n+1}) + H_{d_q}(Sx_n;S\hat{w})$$

$$d_q(w;\hat{x}_{n+1}) +$$

$$(Sx_n;S\hat{w})H_{d_q}(Sx_n;S\hat{w})$$

$$d_q(w;\hat{x}_{n+1}) + (d_q(x_n;\hat{w})):$$

Letting n ! 1, we obtain $d_q(w; S\hat{}\, \hat{w}) = 0:$ Similarly, if $(w;\hat{x}_n) \quad 1$ for all n 2 N [f0g: Thus,

$(Sw; S\hat{x}_n) \quad 1:$ Now,

$$d_q(Sw;\hat{}\, \hat{w}) \quad (d_q(w;\hat{x}_n)) + d_q(x_{n+1};\hat{w}):$$

We obtain $d_q(Sw;\hat{}\, \hat{w}) = 0:$ Hence, $\hat{w}$ 2 $Sw:\hat{}$ So S has axed point in $B_{d_q}(x_0;r)$. ■

4.2.2 Corollary

Let $(X; ;d_q)$ be a preordered complete left K-sequentially dislocated quasi metric space, S :

X ! $CP(X)$. Suppose there exists 2 ; with

$$H_{d_q}(Sx;Sy) \ (d_q(x;y)); \text{ for all elements } x;y \text{ in } B_{d_q}(x_0;r) \text{ with } x\, y$$

and $\sum_{i=0}^{n} \psi^i(d_q(x_0, Sx_0)) \leq r$ for all n 2 N [f0g

for x_0 2 $B_{d_q}(x_0;r);n$ 2 N, $r> 0:$ If f$XS(x_0)$g is a sequence in $B_{d_q}(x_0;r)$ and f$XS(x_0)$g ! x and

$x_n\, x_{n+1}$ for $x_n;x_{n+1}$ 2 f$XS(x_0)$g; then $x\, x_n$ or $x_n\, x$ for all n 2 N[f0g. Also, there

exist x_1 2 Sx_0, such that $x_0\, x_1$. If $x;y$ 2 $B_{d_q}(x_0;r)$, such that $x\, y$ implies $Sx\, _r\, Sy$, then

there exists a point $\hat{w}$ in $B_{d_q}(x_0;r)$, such that $\hat{w}$ 2 $Sw:\hat{}$

4.2.3 Corollary

Let $(X; ;d_q)$ be a preordered complete left K-sequentially dislocated quasi metric space, S :

X ! $CP(X)$. Suppose there exists k 2 $[0;1)$ with

$$H_{d_q}(Sx;Sy)\ kd_q(x;y); \text{ for all elements } x;y \text{ in } \overline{B_{d_q}(x_0;r)} \text{ with } x\ y$$

and $\sum_{i=0}^{n} k^i d_q(x_0, Sx_0) \leq r$ for all n 2 N [f0g

for x_0 2 $\overline{B_{d_q}(x_0;r)};n$ 2 N[f0g$;r>0$: If f$XS(x_0)$g is a sequence in $\overline{B_{d_q}(x_0;r)}$ and f$XS(x_0)$g !

x and $x_n\ x_{n+1}$ for $x_n;x_{n+1}$ 2 f$XS(x_0)$g$;n$ 2 N[f0g; then $x\ x_n$ or $x_n\ x$ for all n 2 N[f0g.

Also, there exist x_1 2 Sx_0, such that $x_0\ x_1$. If $x;y$ 2 $\overline{B_{d_q}(x_0;r)}$, such that $x\ y$ implies

$Sx\ _r\ Sy$, then there exists a point $\hat{w}$ in $\overline{B_{d_q}(x_0;r)}$, such that $\hat{w}$ 2 $S\hat{w}$.

4.2.4 Corollary

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space and $S : X$! X,

$r>0$ and x_0 be an arbitrary point in $\overline{B_{d_q}(x_0;r)}$. Suppose there exists, : $X\ X$! $[0;+1)$

be a -admissible mapping on $B_{d_q}(x_0;r)$. For 2 , assume that,

$$x;y \ 2\ \overline{B_{d_q}(x_0;r)};\ (x;y) \qquad 1 =)\ d_q(Sx;Sy)\ (d_q(x;y))$$

and $\sum_{i=0}^{j}$ $^i(d_q(x_0;Sx_0))\ r$ for all j 2 N [f0g:

Suppose that the following assertions hold:

(i) $(x_0; Sx_0)$ 1;

(ii) for a Picard sequence $x_{n+1} = Sx_n$ in $B_{d_q}(x_0;r)$, such that $(x_n;x_{n+1})$ 1 for all n 2

N [f0g and x_n ! u 2 $\overline{B_{d_q}(x_0;r)}$ as n ! +1, then $(x_n;u)$ 1 for all n 2 N [f0g.

then there exists a point $\hat{w}$ in $\overline{B_{d_q}(x_0;r)}$, such that $\hat{w} = S\hat{w}$.

4.2.5 Corollary

Let $(X; ;d_q)$ be a preordered complete left K-sequentially dislocated quasi metric space and

let $S : X$! X be nondecreasing mapping and x_0 2 $\overline{B_{d_q}(x_0;r)}$. Suppose that the following assertions hold:

(i) there exists k 2 $[0;1)$, such that $d_q(Sx; Sy)$ $kd_q(x;y)$ for all $x;y$ 2 $\overline{B_{d_q}(x_0;r)}$ with x y;

(ii) $x_0 \preceq Sx_0$ and $\sum_{i=0}^{n} k^i d_q(x_0,\ Sx_0) \leq r$ for all n 2 N [f0g;

(iii) for a Picard sequence $x_{n+1} = Sx_n$ in $\overline{B_{d_q}(x_0;r)}$, such that x_n x_{n+1} for all n 2 N [f0g and x_n ! u 2 $\overline{B_{d_q}(x_0;r)}$ as n ! +1, then x_n u for all n 2 N [f0g.

Then S has a xed point.

4.2.6 Example

Let $X = Q^+$[f0g and let $d_q : X$ X ! X be the complete left K-sequentially dislocated quasi metric on X de ned by,

$$d_q(x,y) = \frac{x}{2} + y \text{ for all } x;y$$

2 X: De ne the multivalued mapping $S : X$! $CP(X)$ by

$$Sx = \begin{cases} [\frac{1}{2}, \frac{2}{3}x] \text{ if } x \in [0,1] \\ [x, x+1] \text{ if } x \in (1,\infty). \end{cases}$$

Considering, $x_0 = 1; r = 4;$ then $\overline{B_{d_q}(x_0;r)} = [0;1] \cap X:$ Now, $d_q(x_0; Sx_0) = d_q(1; S1) = d_q(1, \frac{1}{2}) = 1$. Let $\psi(t) = \frac{t}{3}$ and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x;y \in [0;1] \\ \frac{3}{2} & \text{otherwise:} \end{cases}$$

Now,

$$\alpha(S2, S4)H_{d_q}(S2, S4) = (\frac{3}{2})6 > \psi(d_q(x,y)):$$

So the contractive condition does not hold on the whole space: Clearly

$$\alpha(Sx;Sy)H_{d_q}(Sx;Sy) \le \psi(d_q(x;y)) \text{ for all } x;y \in \overline{B_{d_q}(x_0;r)}:$$

So the contractive condition holds on $\overline{B_{d_q}(x_0;r)}$: Also,

$$\sum_{i=0}^{n} \psi^n(d_q(x_0,x_1)) = \sum_{i=0}^{n} \frac{1}{3^n} < 4 = r.$$

We prove that by Theorem 4.2.1 are satisfied: Moreover, S has a fixed point $\frac{1}{2}$.

4.2.7 Theorem

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space, $r > 0; x_0 \in \overline{B_{d_q}(x_0;r)}$ and $S : X \to CP(X)$ be a α-admissible multifunction on $\overline{B_{d_q}(x_0;r)}$. Assume that for $t \in [0, \frac{1}{2})$, such that

$$\alpha(Sx;Sy)H_{d_q}(Sx;Sy) \le t(d_q(x;Sx) + d_q(y;Sy)) \text{ for all } x;y \in \overline{B_{d_q}(x_0;r)} \quad (4.4)$$

$$\text{and } d_q(x_0;x_1)) \le (1 - \lambda)r \quad (4.5)$$

where $\lambda = \frac{t}{1-t}$. If $\{XS(x_0)\}$ is a sequence in $\overline{B_{d_q}(x_0;r)}$ and $\{XS(x_0)\} \to x$ and $\alpha(x_n;x_{n+1}) \ge 1$ for $x_n;x_{n+1} \in \{XS(x_0)\};$ then $\alpha(x_n;x) \ge 1$ or $\alpha(x;x_n) \ge 1$ for all $n \in \mathbb{N} \cup \{0\}$. Also, there exist $x_1 \in Sx_0$, such that $\alpha(x_0;x_1) \ge 1$, then S has a fixed point in $\overline{B_{d_q}(x_0;r)}$.

Proof. As x_0 2 $B_{d_q}(x_0;r);$ and $S : X\ !\ CP(X)$ be a multivalued mapping on X, then there exist x_1 2 Sx_0, such that $d_q(x_0; Sx_0) = d_q(x_0;x_1):$ If $x_0 = x_1$, then x_0 is a xed point in $\overline{B_{d_q}(x_0;r)}$ of S. Let x_0 =6 $x_1:$ From (4.5), we get

$$d_q(x_0;x_1) \quad (1 \qquad)r<r:$$

It follows that,

$$x_1\ 2\ \overline{B_{d_q}(x_0;r)}:$$

Since $(x_0;x_1)$ 1 and S is -admissible multifunction on $\overline{B_{d_q}(x_0;r)}$ and so $(Sx_0;Sx_1)$ 1. Also, there exist x_2 2 Sx_1, such that $d_q(x_1; Sx_1) = d_q(x_1;x_2):$ If $x_1 = x_2$, then x_1 is a xed point of S in $\overline{B_{d_q}(x_0;r)}:$ Let x_1 =6 $x_2:$ Now, we have

$$\begin{aligned} d_q(x_1;x_2) \quad & H_{d_q}(Sx_0;Sx_1) \quad (Sx_0;Sx_1)H_{d_q}(Sx_0;Sx_1) \\ & t(d_q(x_0;Sx_0) + d_q(x_1;Sx_1)) \\ = \quad & t(d_q(x_0;x_1) + d_q(x_1;x_2)): \end{aligned}$$

Thus,

$$d_q(x_1;x_2)\ d_q(x_0;x_1):$$

Note that x_2 2 $\overline{B_{d_q}(x_0;r)};$ since

$$\begin{aligned} d_q(x_0;x_2) \quad & d_q(x_0;x_1) + d_q(x_1;x_2) \\ & d_q(x_0;x_1) + d_q(x_0;x_1) \\ & (1 + \ )(1 \qquad)r<r: \end{aligned}$$

As $(Sx_0; Sx_1)$ 1; x_1 2 Sx_0 and x_2 2 Sx_1 and so $(x_1;x_2)$ 1: As S is -admissible multifunction on $\overline{B_{d_q}(x_0;r)}$: Thus, $(Sx_1; Sx_2)$ 1: Let $x_2; ;x_j$ 2 $B_{d_q}(x_0;r)$ for some j 2 N, such that x_{j+1} 2 Sx_j and $d_q(x_j; Sx_j) = d_q(x_j;x_{j+1})$: As $(Sx_1; Sx_2)$ 1; we have $(x_2;x_3)$ 1; which further implies $(Sx_2; Sx_3)$ 1: Continuing this process, we have

$(Sx_{j\ 1}; Sx_j)$ 1. Now,

$$d_q(x_j;x_{j+1}) \quad H_{d_q}(Sx_{j\ 1};Sx_j) \quad (Sx_{j\ 1};Sx_j)H_{d_q}(Sx_{j\ 1};Sx_j)$$

$$t(d_q(x_{j\ 1};Sx_{j\ 1}) + d_q(x_j;Sx_j))$$

$$(d_q(x_{j\ 1};x_j)) :: {}^j(d_q(x_0;x_1))$$

Now,

$$d_q(x_0;x_{j+1}) \quad \begin{aligned} & d_q(x_0,x_1)+d_q(x_1,x_2)+..+d_q(x_j,x_{j+1}) \\ & d_q(x_0,x_1)+\theta d_q(x_0,x_1)+..+\theta^j(d_q(x_0,x_1)) \\ & (1-\theta)r\frac{(1-\theta^{j+1})}{(1-\theta)} \leq r. \end{aligned}$$

Thus, x_{j+1} 2 $B_{d_q}(x_0;r)$: As $(Sx_{j\ 1};Sx_j)$ 1;x_j 2 $Sx_{j\ 1};x_{j+1}$ 2 Sx_j; we have $(x_j;x_{j+1})$ 1: Also, S

is -admissible multifunction on $\overline{B_{d_q}(x_0;r)}$; therefore $(Sx_j; Sx_{j+1})$ 1: Hence,

by mathematical induction, x_n 2 $B_{d_q}(x_0;r)$ and $(Sx_n; Sx_{n+1})$ 1 for all n 2 N. Now,

$$d_q(x_n;x_{n+1}) \quad {}^n(d_q(x_0;x_1)); \text{ for all } n \ 2 \ \mathrm{N}:$$

Now,

$$d_q(x_n;x_{n+i}) \quad \begin{aligned} & d_q(x_n,x_{n+1})+..+d_q(x_{n+i-1},x_{n+i}) \\ & :\ \frac{\theta^n(1-\theta^i)}{1-\theta}d_q(x_0,x_1) \longrightarrow 0 \text{ as } n\to\infty \end{aligned}$$

Thus, we proved that fx_ng is a Cauchy sequence in $(\overline{B_{d_q}(x_0;r)};d_q)$. As every closed ball

in a complete left K-sequentially dislocated quasi metric space is complete left K-

sequentially and

so there exists $\hat{w}$ 2 $\overline{B_{d_q}(x_0;r)}$, such that x_n ! $\hat{w}$; and

$\lim d_q(x_n;\hat{w}) = \lim d_q(w;\hat{x}$
$_n) = 0.$ $_{n\to\infty}$ $_{n\to\infty}$

Note that, $\{x_n\}$ is a $\{XS(x_0)\}$ in $B_{d_q}(x_0;r).$ As $(Sx_n; Sx_{n+1})$ 1 for all $n \in \mathbb{N} \cup \{0\};$ we have $(x_{n+1};x_{n+2})$ 1 for all $n \in \mathbb{N} \cup \{0\}.$ By assumption, we have $(x_n;\hat{w})$ $n \in \mathbb{N} \cup \{0\}.$ Thus, 1 for all

$(Sx_n; S\hat{w})$ 1. Now,

$d_q(w;S\hat{w})$ $d_q(w;\hat{x}_{n+1}) + d_q(x_{n+1};S\hat{w})$

$d_q(w;\hat{x}_{n+1}) + H_{d_q}(Sx_n;S\hat{w})$ $d_q(w;\hat{x}_{n+1})$ +
$(Sx_n;S\hat{w})H_{d_q}(Sx_n;S\hat{w})$ $d_q(w;\hat{x}_{n+1})$ +
$t(d_q(x_n;x_{n+1}) + d_q(w;S\hat{w})).$

Letting $n \to \infty$, we obtain $d_q(w; S\hat{w}) = 0.$ Similarly, if $(w;\hat{x}_n)$ 1 for all $n \in \mathbb{N} \cup \{0\};$ we

obtain $d_q(S\hat{w};\hat{w}) = 0.$ Hence, $\hat{w} \in S\hat{w}.$ So S has a xed point in $B_{d_q}(x_0;r)$. ■

4.2.8 Corollary

Let $(X; ;d_q)$ be an ordered complete left K-sequentially dislocated quasi metric space, S : $X \to CP(X)$. Suppose there exists $k \in [0, \frac{1}{2})$ with

$H_{d_q}(Sx;Sy)$ $t(d_q(x;Sx) + d_q(y;Sy))$ for all elements $x;y$ in $B_{d_q}(x_0;r)$ with x y

and

$$d_q(x_0;Sx_0) \quad (1 \quad)r$$

for $x_0 \in B_{d_q}(x_0, r)$, $r > 0$, $\theta = \frac{t}{1\ \ t}$. If $\{XS(x_0)\} \to x$ and x_n x_{n+1} for $x_n;x_{n+1} \in \{XS(x_0)\};$

then x x_n or x_n x for all $n \in \mathbb{N} \cup \{0\}$. If $x;y \in B_{d_q}(x_0;r)$, such that x y implies

Sx $_r$ Sy, then there exists a point $\hat{w}$ in $B_{d_q}(x_0;r)$, such that $\hat{w} \in S\hat{w}.$

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space, $r > 0;$ $x_0 \in$ $B_{d_q}(x_0;r)$ and $S : X \to CP(X)$ be a -admissible multifunction on $B_{d_q}(x_0;r)$. Assume that for 2 , such that

$$(Sx;Sy)H_{d_q}(Sx;Sy)\ (M_q(x;y)) \text{ for all } x;y \ 2\ B_{d_q}(x_0;r) \tag{4.6}$$

where

$$M_q(x;y) = \max f d_q(x;y); d_q(x;Sx); d_q(y;Sy) g$$

and

4.2.9 Theorem

$$\sum_{i=0}^{n} \psi^i(d_q(x_0, Sx_0)) \leq r \quad \text{for all } n\ 2\ \mathbb{N}\ [\ f0g. \tag{4.7}$$

If f$XS(x_0)$g is a sequence in $B_{d_q}(x_0;r)$, such that f$XS(x_0)$g ! x and $(x_n;x_{n+1})$ 1 for $x_n;x_{n+1}$ 2

f$XS(x_0)$g$;n$ 2 N [f0g; then $(x_n;x)$ 1 or $(x;x_n)$ 1 for all n 2 N [f0g. Also,

there exist x_1 2 Sx_0, such that $(x_0;x_1)$ 1, then S has a xed point in $B_{d_q}(x_0;r)$.

Proof. As x_0 2 $B_{d_q}(x_0;r)$; and $S : X ! CP(X)$ be a multivalued mapping on X, then there

exist x_1 2 Sx_0, such that $d_q(x_0; Sx_0) = d_q(x_0;x_1)$: If $x_0 = x_1$, then x_0 is a xed point in

$B_{d_q}(x_0;r)$ of S. Let x_0 =6 x_1: From (4.7), we get

$$d_q(x_0, x_1) \leq \sum_{i=0}^{n} \psi^i(d_q(x_0, x_1)) \leq r.$$

It follows that,

$$x_1 \in \overline{B_{d_q}(x_0;r)}.$$

Since $(x_0;x_1)$ 1 and S is -admissible multifunction on $\overline{B_{d_q}(x_0;r)}$ and so $(Sx_0;Sx_1)$ 1. Also, there exist $x_2 \in Sx_1$, such that $d_q(x_1; Sx_1) = d_q(x_1;x_2).$ If $x_1 = x_2$, then x_1 is a xed point of S in $\overline{B_{d_q}(x_0;r)}.$ Let $x_1 \neq x_2.$ Now,

$$
\begin{aligned}
d_q(x_1;x_2) &= d_q(x_1;Sx_1)\ H_{d_q}(Sx_0;Sx_1) \\
& (Sx_0;Sx_1)H_{d_q}(Sx_0;Sx_1) \\
& (M_q(x_0;x_1)) \quad \text{by (4.6)} \\
& (\max\{d_q(x_0;x_1);d_q(x_0;Sx_0);d_q(x_1;Sx_1)\}) \\
& (\max\{d_q(x_0;x_1);d_q(x_1;x_2)\}).
\end{aligned}
$$

If $\max\{d_q(x_0;x_1);d_q(x_1;x_2)\} = d_q(x_1;x_2);$ then $d_q(x_1;x_2)$ $(d_q(x_1;x_2)).$ This is a contradiction to the fact that $(t) < t$ for all $t > 0.$ Hence, we obtain $\max\{d_q(x_0;x_1);d_q(x_1; Sx_1)\} = d_q(x_0;x_1).$ Now,

$$d_q(x_1;x_2)\ (d_q(x_0;x_1)). \tag{4.8}$$

Note that $x_2 \in \overline{B_{d_q}(x_0;r)};$ because

$$
\begin{aligned}
d_q(x_0;x_2) & \quad d_q(x_0;x_1) + d_q(x_1;x_2) \\
& d_q(x_0, x_1) + \psi(d_q(x_0, x_1)), \quad \text{by (4.8)} \\
& \sum_{i=0}^{n} \psi^i(d_q(x_0, x_1)) \leq r.
\end{aligned}
$$

As $(Sx_0; Sx_1)$ 1; $x_1 \in Sx_0$ and $x_2 \in Sx_1$ and so $(x_1;x_2)$ 1. As S is -admissible multifunction on $B_{d_q}(x_0;r)$. Thus, $(Sx_1; Sx_2)$ 1: Let $x_2; ;x_j$ 2 $B_{d_q}(x_0;r)$ for some j 2 N, such that x_{j+1} 2 Sx_j and $d_q(x_j; Sx_j) = d_q(x_j;x_{j+1})$: As $(Sx_1; Sx_2)$ 1; we have $(x_2;x_3)$ 1; which further implies $(Sx_2; Sx_3)$ 1: Continuing this process, we have

$(Sx_{j\ 1}; Sx_j)$ 1. Now, x_j 2 $Sx_{j\ 1};x_{j+1}$ 2 Sx_j; we have

$$
\begin{aligned}
d_q(x_j;x_{j+1}) &= d_q(x_j;Sx_j)\ H_{d_q}(Sx_{j\ 1};Sx_j)\\
& (Sx_{j\ 1};Sx_j)H_{d_q}(Sx_{j\ 1};Sx_j)\\
& (M_q(x_{j\ 1};x_j))\\
& (\mathrm{maxf}d_q(x_{j\ 1};x_j);d_q(x_{j\ 1};Sx_{j\ 1});d_q(x_j;Sx_j)\mathrm{g})\\
& (\mathrm{maxf}d_q(x_{j\ 1};x_j);d_q(x_j;x_{j+1})):
\end{aligned}
$$

If maxf$d_q(x_{j\ 1};x_j);d_q(x_j;x_{j+1}) = d_q(x_j;x_{j+1})$; then $d_q(x_j;x_{j+1})\ (d_q(x_j;x_{j+1}))$: This is a contradiction to the fact that $(t) < t$ for all $t > 0$: Hence, we obtain maxf$d_q(x_{j\ 1};x_j);d_q(x_j;x_{j+1})\mathrm{g} = d_q(x_{j\ 1};x_j)$:

$$d_q(x_j;x_{j+1})\ :::\ ^j(d_q(x_0;x_1)): \tag{4.9}$$

$$
\begin{aligned}
d_q(x_0;x_{j+1}) &\ d_q(x_0;x_1) + ::: + d_q(x_j;x_{j+1})\\
&\ d_q(x_0;x_1) + ::: + {}^j(d_q(x_0;x_1))\\
&= \sum_{i=0}^{j} {}^i(d_q(x_0;x_1))\ r:
\end{aligned}
$$

Thus, x_{j+1} 2 $\overline{B_{d_q}(x_0;r)}$: As $(Sx_{j\ 1};Sx_j)$ 1;x_j 2 $Sx_{j\ 1};x_{j+1}$ 2 Sx_j; we have $(x_j;x_{j+1})$ 1: Also, S is -admissible multifunction on $\overline{B_{d_q}(x_0;r)}$; therefore $(Sx_j; Sx_{j+1})$ 1: Hence, by mathematical induction, x_n 2 $B_{d_q}(x_0;r)$ and $(Sx_n; Sx_{n+1})$ 1 for all n 2 N. Now, the inequality (4.9) can be written as

$$d_q(x_n;x_{n+1}) \ {}^n(d_q(x_0;x_1)); \text{ for all } n \ 2 \ \mathrm{N}:$$

Fix ″> 0 and let $n(″)$ 2 N, such that X $^n(d_q(x_0;x_1)) <″$: Let $n;m$ 2 N with $m>n>k(″)$; then we obtain

$$\sum_{k=n}^{m\ \ 1} d_q(x_k, x_{k+1}) \le \sum_{k=n}^{m\ \ 1} \psi^k(d_q(x_0, x_1))$$

$$d_q(x_n;x_m) \qquad \sum_{n\ge k(\varepsilon)} \psi^k(d_q(x_0, x_1)) < \varepsilon.$$

Thus, we proved that fx_ng is a Cauchy sequence in $(\overline{B_{d_q}(x_0;r)};d_q)$. As every closed ball in a complete left K-sequentially dislocated quasi metric space is left K-sequentially complete and

so there exists $\hat{w}$ 2 $\overline{B_{d_q}(x_0;r)}$, such that x_n ! $\hat{w}$; and

$$\lim_{n!1} d_q(x_n;\hat{w}) = \lim_{n!1} d_q(\hat{w};x_n) = 0:$$

Note that fx_ng is a f$XS(x_0)$g in $\overline{B_{d_q}(x_0;r)}$: As $(Sx_n;Sx_{n+1})$ 1 for all n 2 N[f0g; we have $(x_{n+1};x_{n+2})$ 1 for all n 2 N[f0g: By assumption, we have $(x_n;\hat{w})$ 1 for all n 2 N[f0g:

Thus, $(Sx_n; S\hat{w})$ 1: Now,

$$d_q(\hat{w};S\hat{w}) \qquad d_q(\hat{w};x_{n+1}) + d_q(x_{n+1};S\hat{w})$$

$$d_q(w;\hat{x}_{n+1}) + H_{d_q}(Sx_n;S\hat{w})\ d_q(w;\hat{x}_{n+1}) +$$

$$(Sx_n;S\hat{w})H_{d_q}(Sx_n;S\hat{w})\ d_q(w;\hat{x}_{n+1}) +$$

$$(\max\{d_q(x_n;\hat{w});d_q(x_n;x_{n+1});d_q(w;S\hat{w})\}).$$

Letting $n \to \infty$, we obtain $d_q(w;S\hat{w}) = 0$; which implies $\hat{w} \in S\hat{w}$. Similarly, if $(w;\hat{x}_n)$ 1 for all $n \in \mathbb{N} \cup \{0\}$. Thus, $(Sw;S\hat{x}_n)$ 1. Now,

$$d_q(S\hat{w};\hat{w})\ (d_q(w;\hat{x}_n)) + d_q(x_{n+1};\hat{w}).$$

We obtain $d_q(S\hat{w};\hat{w}) = 0$. Hence, $\hat{w} \in S\hat{w}$. So S has a fixed point in $\overline{B_{d_q}(x_0;r)}$. ■

4.2.10 Corollary

Let $(X; ;d_q)$ be a preordered complete left K-sequentially dislocated quasi metric space, S : $X \to CP(X)$. Suppose that there exists $\in$; $x_0 \in \overline{B_{d_q}(x_0;r)}$; $r > 0$ with

$H_{d_q}(Sx;Sy)\ (\max\{d_q(x;y);d_q(x;Sx);d_q(y;Sy)\})$; for all elements $x;y$ in $\overline{B_{d_q}(x_0;r)}$ with $x\ y$

and

$$\sum_{i=0}^{n} \psi^i(d_q(x_0, Sx_0)) \leq r$$ for all $n \in \mathbb{N} \cup \{0\}$.

If $\{XS(x_0)\}$ is a sequence in $\overline{B_{d_q}(x_0;r)}$, such that $\{XS(x_0)\} \to x$ and $x_n\ x_{n+1}$ for $x_n;x_{n+1} \in \{XS(x_0)\}$; then $x\ x_n$ or $x_n\ x$ for all $n \in \mathbb{N} \cup \{0\}$. Also, there exist $x_1 \in Sx_0$, such that

$x_0\ x_1$. If $x;y \in \overline{B_{d_q}(x_0;r)}$, such that $x\ y$ implies $Sx\ _r\ Sy$, then there exists a point $\hat{w}$ in $\overline{B_{d_q}(x_0;r)}$, such that $\hat{w} \in S\hat{w}$.

4.2.11 Corollary

Let $(X;d_q)$ be a complete left K-sequentially dislocated quasi metric space and $S : X ! X$,

$r>0$ and x_0 be an arbitrary point in $B_{d_q}(x_0;r)$. Suppose there exists, $: X X ! [0;+1)$

be a -admissible mapping on $B_{d_q}(x_0;r)$. For 2 , assume that,

$$x;y\ 2\ B_{d_q}(x_0;r);\ (x;y) \quad 1 =)\ d_q(Sx;Sy)\ (\max f d_q(x;y);d_q(x;Sx);d_q(y;Sy) g)$$

$$\text{and } X_{i=0}^{j}\ {}^{i}(d_q(x_0;Sx_0))\ r \text{ for all } j\ 2\ \mathbb{N}\ [\ f0g:$$

Suppose that the following assertions hold:

(i) $(x_0; Sx_0) \quad 1;$

(ii) for a Picard sequence $x_{n+1} = Sx_n$ in $B_{d_q}(x_0;r)$, such that $(x_n;x_{n+1}) \quad 1$ for all n 2 N [f0g and $x_n ! u\ 2\ B_{d_q}(x_0;r)$ as $n ! +1$, then $(x_n;u) \quad 1$ for all n 2 N [f0g.

then there exists a point $\hat{w}$ in $B_{d_q}(x_0;r)$, such that $\hat{w} = S\hat{w}$:

4.2.12 Corollary

Let $(X; ;d_q)$ be a preordered complete left K-sequentially dislocated quasi metric space and

let 2 $;x_0$ 2 $B_{d_q}(x_0;r);r>0$ and $S : X ! X$ be a nondecreasing mapping on A. Suppose

that the following assertions hold:

(i) $d_q(Sx; Sy)\ (\max f d_q(x;y);d_q(x; Sx);d_q(y; Sy) g);$ for all elements $x;y$ in $B_{d_q}(x_0;r)$

with $x\ y$;

(ii) $x_0\ Sx_0$ and $P^{j}_{i=0}\ {}^{i}(d_q(x_0; Sx_0))\ r$ for all j 2 N [f0g;

(iii) for a Picard sequence $x_{n+1} = Sx_n$ in $B_{d_q}(x_0;r)$, such that $x_n\ x_{n+1}$ for all n 2 N [f0g and x_n ! u 2 $B_{d_q}(x_0;r)$ as n ! +1, then x_n u for all n 2 N [f0g.

Then S has a xed point.

4.2.13 Example

Let $X = Q^+$[f0g and let $d_q : X$! X be the complete left K-sequentially dislocated quasi metric on X de ned by,

$$d_q(x,y) = \frac{x}{2} + y \text{ for all } x;y$$

2 X: De ne the multivalued mapping $S : X$! $CP(X)$ by

$$Sx = \begin{cases} [\frac{1}{2}, \frac{2}{3}x] \text{ if } x \in [0,1] \\ [x, x+1] \text{ if } x \in (1,\infty). \end{cases}$$

Considering, $x_0 = 1; r = 4$; then $\overline{B_{d_q}(x_0;r)} = [0;1] \setminus X$: Now, $d_q(x_0; Sx_0) = d_q(1; S1) =$ $d_q(1, \frac{1}{2}) = 1$. Let $\psi(t) = \frac{t}{3}$ and

$$(x,y) = \begin{cases} 1 \text{ if } x;y \text{ 2 } [0;1] \\ \frac{3}{2} \text{ otherwise}: \end{cases}$$

8

Now,

$$(S2,S4)H_{d_q}(S2,S4) = (\frac{3}{2})6 > \psi(\max\{d_q(2,4), d_q(2,S2), d_q(4,S4)\}).$$

So the contractive condition does not hold on X: Clearly

$$(Sx;Sy)H_{d_q}(Sx;Sy) \ (\max f d_q(x;y); d_q(x;Tx); d_q(y;Ty) g) \text{ for all } x;y \text{ 2 } \overline{B_{d_q}(x_0;r)}:$$

So the contractive condition holds on $\overline{B_{d_q}(x_0;r)}$: Also,

$$\sum_{i=0}^{n} \psi^n(d_q(x_0,x_1)) = \sum_{i=0}^{n} \frac{1}{3^n} < 4 = r.$$

Hence, all the conditions of Theorem 4.2.9 are satis ed: Moreover, S has a xed point $\frac{1}{2}$.

4.3 Fixed Point Results for Multivalued Mappings in Hausdorff Fuzzy Metric Spaces

4.3.1 Theorem

Let $(X, M, *)$ be a complete fuzzy metric space, where $*$ be a continuous t-norm, defined as $a * a \geq a$ or $a * a = \min\{a, b\}$. Let $(K_0(X), H_M, *)$ is Hausdorff fuzzy metric space on $K_0(X)$. Let $x_0 \in X$ and $S : X \to K_0(X)$ be a multivalued mapping. Assume that for some $k \in (0, 1)$ and $t > 0$, we have

$$H_M(Sx, Sy, kt) \geq M(x, y, t) \text{ for all } x, y \in \overline{B_M}(x_0, r, t) \tag{4.10}$$

and

$$M(x_0, Sx_0, (1-k)t)) \geq 1 - r. \tag{4.11}$$

Then S has a fixed point in $\overline{B_M}(x_0, r, t)$.

Proof. We know that $x_0 \in \overline{B_M}(x_0, r, t).$ We construct a sequence $\{x_n\}$ of points in X as follow. Let $x_1 \in X$ be, such that $x_1 \in Sx_0$ and $M(x_0, Sx_0, t) = M(x_0, x_1, t).$ If $x_0 = x_1$, then x_0 is a fixed point of S. Let $x_0 \neq x_1.$ By Lemma 4.1.4, there exist $x_2 \in Sx_1$ satisfies $M(x_1, Sx_1, t) = M(x_1, x_2, t)$ and

$$M(x_1, x_2, t) \geq H_M(Sx_0, Sx_1, t).$$

If $x_1 = x_2$, then x_1 is a fixed point of $S.$ Let $x_1 \neq x_2.$ For $x_2 \in X$ be, such that $x_2 \in Sx_1$, then by Lemma 4.1.4, there exist $x_3 \in Sx_2$ satisfies $M(x_2, Sx_2, t) = M(x_2, x_3, t)$ and

$$M(x_2, x_3, t) \geq H_M(Sx_1, Sx_2, t).$$

By induction, we have for $x_n \in X; x_{n-1} \neq x_n$ be, such that $x_n \in Sx_{n-1}$, then by Lemma 4.1.4,

$$M(x_n; x_{n+1}; t) \geq H_M(Sx_{n-1}; Sx_n; t). \tag{4.12}$$

First, we will show that $x_n \in \overline{B_M(x_0; r; t)}$. By (4.11), we get

$$M(x_0; x_1; t) > M(x_0; x_1; (1-k)t) = M(x_0; Sx_0; (1-k)t)) \quad 1 - r$$

which shows that $x_1 \in \overline{B_M(x_0; r; t)}$. Now, let $x_2; \ldots; x_j \in \overline{B_M(x_0; r; t)}$

there exist $x_{n+1} \in Sx_n$ satisfies $M(x_n; Sx_n; t) = M(x_n; x_{n+1}; t)$ and

$$
\begin{aligned}
M(x_j; x_{j+1}; t) &\geq H_M(Sx_{j-1}, Sx_j, t) \geq M(x_{j-1}, x_j, \frac{t}{k}) \\
&\geq H_M(Sx_{j-2}, Sx_{j-1}, \frac{t}{k}) \geq M(x_{j-2}, x_{j-1}, \frac{t}{k^2}) \\
&\vdots \\
&\geq M(x_0, x_1, \frac{t}{k^j}).
\end{aligned} \tag{4.13}
$$

Now,

$$
\begin{aligned}
M(x_0; x_{j+1}; t) &> M(x_0; x_{j+1}; (1-k^{j+1})t) \\
&\geq M(x_0; x_1; (1-k)t) * M(x_1; x_2; (1-k)kt) * \cdots * M(x_j; x_{j+1}; (1-k)k^j t) \\
&\geq M(x_0; x_1; (1-k)t) * M(x_0; x_1; (1-k)t) * \cdots * M(x_0; x_1; (1-k)t) \\
&\geq (1-r) * (1-r) * \cdots * (1-r) = 1-r
\end{aligned}
$$

which implies that $x_{j+1} \in \overline{B_M(x_0; r; t)}$. Now, (4.13) can be written as,

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, \frac{t}{k^n}). \tag{4.14}$$

Now, for each $n; m \in \mathbb{N}; n > m$; we have

$M(x_n;x_m;t) > M(x_n;x_m;(1\ k^{m\ n})t)$

$$
\begin{aligned}
&\geq M(x_n, x_{n+1}, (1-k)t) * M(x_{n+1}, x_{n+2}, (1-k)kt) * .. * M(x_{m-1}, x_m, (1-k)k^{m-n-1}t) \\
&\geq M(x_0, x_1, \frac{(1-k)t}{k^n}) * M(x_0, x_1, \frac{(1-k)kt}{k^{n+1}}) * .. * M(x_0, x_1, \frac{(1-k)k^{m-n-1}t}{k^{m-1}}) \\
&= M(x_0, x_1, \frac{(1-k)t}{k^n}) * M(x_0, x_1, \frac{(1-k)t}{k^n}) * .. * M(x_0, x_1, \frac{(1-k)t}{k^n}) \\
&\geq M(x_0, x_1, \frac{(1-k)t}{k^n}). \qquad (4.15)
\end{aligned}
$$

As, $\lim_{t!1} M(x;y;t) = 1$ for all $x;y$ 2 $X:$ In particular

$$M(x_0, x_1, \frac{(1-k)t}{k^n}) = 1 \text{ as } n \to \infty: \qquad (4.16)$$

By using (4.16) in (4.15), we get

$$M(x_n;x_m;t) = 1 \text{ as } n\ !\ 1:$$

Hence, fx_ng is a Cauchy sequence in $\overline{B_M(x_0;r;t)}:$ As every closed ball of a complete fuzzy

metric space is complete. So $\overline{B_M(x_0;r;t)}$ is complete. So there exists $w^\wedge$ 2 $\overline{B_M(x_0;r;t)}$, such

that $x_n\ !\ w^\wedge$ as $n\ !\ 1:$ Now,

$$M(w;S^\wedge w;t^\wedge)\ M(w;x^\wedge\ _n;(1\ k)t)\ M(x_n;Sw;kt^\wedge\ ):$$

By Lemma 4.1.4, we have

$$
\begin{aligned}
M(w;S^\wedge w;t^\wedge\ )\quad & M(w;x^\wedge\ _n;(1\quad k)t)\quad H_M(Sx_{n\ 1};Sw;kt^\wedge\quad) \\
& M(w;x^\wedge\ _n;(1\quad k)t)\quad M(x_{n\ 1};w;t^\wedge\quad). \text{ by } (4.10)
\end{aligned}
$$

Letting $n\ !\ 1;$ we have

$$M(w;S^\wedge w;t^\wedge\ )\quad 1\quad 1 = 1$$

which implies that $w^\wedge$ 2 $Sw:^\wedge$ ■

4.3.2 Corollary

Let $(X;M;\ )$ be a complete fuzzy metric space, where be a continuous t-norm, de ned as

$a\ a\ a$ or $a\ a = \min\{a;b\}$. Let $x_0 \in X$ and $S : X \to X$ be a self mapping. Assume that for some $k \in (0;1)$ and $t > 0$, we have

$$M(Sx;Sy;kt)\ M(x;y;t) \text{ for all } x;y \in \overline{B_M(x_0;r;t)}$$

and

$$M(x_0;Sx_0;(1\ k)t)) \quad 1\ r.$$

Then S has a fixed point in $\overline{B_M(x_0;r;t)}$.

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