

Metric Fixed Point Results for Nonlinear Mappings and Some Related Applications



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2026**

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE
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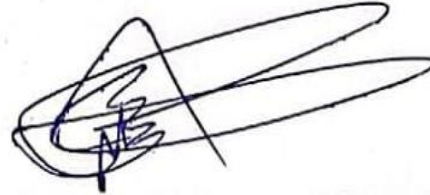
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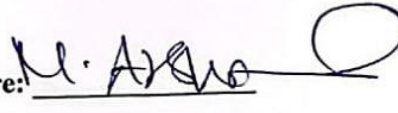
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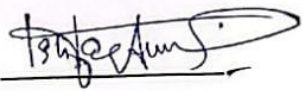
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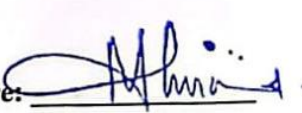
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DEDICATED TO....

*“My Parents, Wife, Son, Daughters, Brothers,
Sisters, Teachers and Friends for supporting
and encouraging me”.*

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List of Publications

The list of the research articles deduced from the work presented in this thesis, published in international journals (ESCI & SCI) is given below

1. **M. Mudhesh**, A. Hussain, M. Arshad, H. Alsulami, A Contemporary Approach of Integral Khan-Type Multivalued Contractions with Generalized Dynamic Process and an Application, **Mathematics**, 11(20), (2023), 1--18.
2. **M. Mudhesh**, H. A. Hammad, E. Ameer, A. Ali, Fixed Point Results under New Contractive Conditions on Closed Balls, **Applied Mathematics & Information Sciences**, 16(4), (2022), 1-9.
3. **M. Mudhesh**, H. A. Hammad, E. Ameer, M. Arshad, F. Jarad, Novel Results on Fixed-Point Methodologies for Hybrid Contraction Mappings in M_b -Metric Spaces with an Application, **AIMS Mathematics**, 8(1), (2022), 1530–1549.
4. **M. Mudhesh**, A. Hussain, M. Arshad, H. Al-Sulami, A. Ali, New Techniques on Fixed Point Theorems for Symmetric Contraction Mappings with its Application, **AIMS Mathematics**, 8(4), (2023), 9118–9145.
5. **M. Mudhesh**, M. Arshad, A. Hussain, A. H. Albargi, E. Ameer, The Solvability of Fourth-Order Differential Equations under Almost Hardy–Rogers (ψ, ϕ) -Type Multivalued Contractions in M_ϕ -Metric Space, **International Journal of Mathematics and Mathematical Sciences**, V. 2024, 6620429, (2024), 1-14.
6. **M. Mudhesh**, H. A. Hammad, H. Alsamir, M. Arshad, E. Ameer, Recent Fixed-Point Results for θ -Contraction Mappings in Rectangular M-Metric Spaces with Supportive Application, **Journal of Mathematics**, Volume 2021, Article ID 5564248, (2021), 1-9.

7. **M. Mudhesh**, M. Arshad, A. Hussain, H. Alsulami, The Solution of Nonlinear Matrix Equations Using Suzuki Multivalued Contractions in Rectangular M-metric Spaces, **International Journal of Mathematics and Mathematical Sciences**, 2025, 590804, (2025), 1-17.
8. **M. Mudhesh**, M. Arshad, A. Hussain, H. Alsulami, The Existence of Solutions for Second-Order Differential Equations under Almost Fisher-Type Multivalued F-Contractions in Metric Spaces, **European Journal of Pure and Applied Mathematics**, 18(4), 6441, (2025), 1-21.

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Preface

The research outcomes presented in this dissertation revolve around the broad domain of Metric Fixed Point Theory (abbreviated as M.F.P.T), a fundamental area of nonlinear analysis with numerous applications in mathematical modeling, optimization, and theoretical computer science. It has been dedicated for application in diverse fields like biology, chemistry, economics, engineering, game theory, and physics. In particular, this work explores significant generalizations of Banach's Contraction Principle (BCP), extending its classical results to three distinct types of metric spaces (M.Ss).

The dissertation introduces and develops the notion of semi α_* -admissibility with respect to η , and employs this framework to study fixed points (F.Ps) of Ćirić-type rational contractions on closed ball of complete metric space (C-M.S). Moreover, F.P results for multivalued mappings are derived under the assumptions of both integral Khan (ψ, ϕ) -type and integral θ -type multivalued contractions with respect to (w.r.t) generalized dynamic processes (G.D.Ps) within C-M.Ss. Furthermore, this research establishes F.P theorems for multivalued mappings of almost Fisher-type $\mathcal{F}$ -contractions endowed with a Γ -transitive binary relation in M.Ss. Details of these results can be found in Chapter 2.

Additionally, this study formulates the concept of η -cyclic (α_*, β_*) -admissibility and investigates criteria for the existence of common fixed points (C.F.Ps) for multivalued Geraghty $\mathcal{F}$ -contractive mappings. We also analyze and extend results on self-mappings associated with symmetric α - β - η - ψ -contractions and almost Hardy-Rogers-type multivalued (ψ, ϕ) -contractions in M_b -metric spaces (Mb-M.Ss), as discussed in Chapter 3.

We present a study on the existence of unique fixed points (U.F.Ps) for single-valued θ -contractions and Suzuki-rational multivalued (ψ, ϕ) -contractions on rectangular M -metric spaces (R.M-M.Ss) (see Chapter 4).

In essence, this thesis offers a comprehensive and comparative investigation into the existence of F.Ps and C.F.Ps for both single-valued and multivalued generalized contractions across three categories of M.Ss: (1) Standard metric space (M.S) (2) M_b -metric space (Mb-M.S) (3) Rectangular M -metric space (R.M-M.S).

The results presented herein not only enrich the theoretical landscape of M.F.P.T, but also

lay the groundwork for further examples and applications in nonlinear analysis and related mathematical fields.

This dissertation is made up of four chapters:

Chapter 1 consists of generic introduction of F.P theory, major concepts, notations and some of the well-known outcomes.

Chapter 2 is arranged as: Sections 2.1 we utilize the notion of integral Khan (ψ, ϕ) -type and integral θ -type multivalued contractions with G.D.P to prove novel C.F.P results in M.Ss with two supported examples. In section 2.2 an application to solve a nonlinear fractional differential equation (NFDE) is provided. These results have been published in **Mathematics**, **2023 (11)**, **4318**, **18 pages**.

Section 2.3 deals with some new C.F.P theorems concerning Ćirić-Type rational contractions on closed ball of M.Ss. These results have been published in **Applied Mathematics & Information Sciences**, **16(4)**, **2022**, **555-564**.

In section 2.4 we discuss the existence of F.Ps of almost Fisher-type multivalued $\mathcal{F}_{\mathfrak{R}}$ -contraction endowed with a Γ -transitive binary relation $\mathfrak{R}$ with two supported examples in M.Ss. These results have been published in **European Journal of Pure and Applied Mathematics**, **18(4)**, **2025**, **1-21**.

Chapter 3 is divided into six sections. In first three sections, we put forward the notion of η -cyclic (α_*, β_*) -Geraghty F -contractive for multivalued maps by combining the results of Wardowski cyclic contraction and admissible multivalued Geraghty contraction with establishing C.F.Ps for such contraction in Mb-M.Ss. Some examples and an application of ordinary differential equation (ODE) are shown. These results have been published in **AIMS Mathematics**, **8(1)**, **2022**, **1530-1549**.

In sections 4 and 5, novel techniques on F.P theorem for self-mappings of four types of symmetric contractions are discussed on Mb-M.S and on closed ball of Mb-M.S with an application to fractional-order differential equation (FODE). These results have been published in **AIMS Mathematics**, **4(8)**, **2023**, **9118-9145**.

Section 6 presents the concept of almost Hardy-Rogars (ψ, ϕ) -contractions with discussion of some related C.F.P results for multivalued mappings in Mb-M.S. Examples are furnished to confirm the practical relevance of the obtained findings. The outcomes here have been published

in **International Journal of Mathematics and Mathematical Sciences**, Vol. 2024, 14 pages. <https://doi.org/10.1155/2024/6620429>

Chapter 4 aims to study three results: Section 4.1 presents the motif of almost Suzuki-Rational (ψ, ϕ) -type multivalued contractions and corresponding C.F.P theorems on a C-R.M-M.S with supported examples. These results have been published in **International Journal of Mathematics and Mathematical Sciences**, 2025, 590804, (2025), 1-17.

In Sections 4.2 and 4.3 we study θ -contraction mappings and establish related F.P theorems for single-valued maps in a C-R.M-M.S. An example and an application are outlined to substantiate the theory. The findings here have been published in **Journal of Mathematics** 2021, Article ID 5564248, 9 pages.

Additional to abbreviations above and throughout this thesis, we use some more abbreviated notations as follows:

$d(\xi_1, \xi_2) = d _{\xi_2}^{\xi_1};$	$\hat{H}_d(A, B) = \hat{H}_d _B^A;$	$\alpha(\xi_1, \xi_2) = \alpha _{\xi_2}^{\xi_1};$
$CB(\Omega) = CB ^\Omega;$	for all $\xi = \forall_\xi;$	there exists $\xi = \exists_\xi;$
Integral Equation=IE;	Green Function=G.F;	boundary value problem=BVP.

Chapter 1

Introduction and Preliminaries

Fixed point theory (F.P.T) constitutes a foundational and prominent area within functional analysis and topology. As a core discipline, it is deeply rooted in the structure of C-M.Ss. Its scope extends beyond pure mathematics, offering powerful tools to address complex problems in engineering, computer science, and economics. The origins of this theory trace back to 1922 when Banach [27] introduced the fundamental contraction principle, which states that the inequality $d|_{\Gamma\xi_2}^{\xi_1} \leq c d|_{\xi_2}^{\xi_1}$, $c \in [0, 1)$ defines a contraction $\forall \xi_1, \xi_2$ in a C-M.S $(\Omega, d|)$, whereas the map $\Gamma : \Omega \rightarrow \Omega$ admits a U.F.P $\xi^* \in \Omega$, so that $\Gamma\xi^* = \xi^*$. Since then, a wide array of generalizations has emerged, significantly broadening the scope of F.P results for contractive-type maps over C-M.Ss. Nadler [89] laid early groundwork by extending the BCP to multivalued mappings. Samet et al. [101] introduced the frameworks of single-valued α -admissible and α - ψ -contractive maps, further enriching the theory. Mohammadi et al. [74] contributed additional results for α - ψ -Ćirić generalized multifunctions (see [26, 61]), while Shoaib [105] advanced the notion of semi α^* -admissible multivalued maps, proving F.P theorems within closed balls of left (or right) K-sequentially complete dislocated quasi M.Ss [98, 106]. Further contributions include generalized multivalued Khan-type (ψ, ϕ) -contractions and θ -Khan-type multivalued contractions [114], along with the JS-contraction introduced by Jleli and Samet [53], formulated via the inequality $\theta \left(d|_{\Gamma\xi_2}^{\xi_1} \right) \leq \left[\theta \left(d|_{\xi_2}^{\xi_1} \right) \right]^k$, $k \in [0, 1)$ and $\theta \in \Phi$ is called θ -contraction (*JS-contraction*) $\forall \xi_1, \xi_2$ in a C-G.M.S $(\Omega, d|)$ see also [4, 12, 14, 21, 37, 44, 47, 52, 94, 111]. Cyclic (α, β) -admissibility and cyclic (α_*, β_*) -admissibility mappings were presented in [11, 19], further discussed in [16, 17]. Klim and Wardowski [60] defined the concept of a D.P and

obtained F.P outcomes for a specific case of nonlinear $\mathcal{F}$ -contractions in C-M.Ss. Hussain et al. [45] explored G.D.Ps leading to coincidence and C.F.P results for generalized (f, L) -almost $\mathcal{F}$ -contractions. Jaggi [51] introduced rational expressions in F.P theory, marking a significant milestone. Karapinar [56] further enriched the field by introducing interpolative-type contractions, linking it with interpolation theory. Karapinar and Fulga [57] proposed a hybrid contraction combining Jaggi and interpolative-type contractions. In 1980, Fisher [34] explored new results that generalized the BCP by employing a new rational inequality $\forall_{\xi_1, \xi_2 \in \Omega} \exists_{\lambda_1, \lambda_2 \in [0, \infty)}$ so that $d_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \lambda_1 d_{\xi_2}^{\xi_1} + \lambda_2 \frac{d_{\Gamma\xi_1}^{\xi_1} \cdot d_{\Gamma\xi_2}^{\xi_2}}{1 + d_{\xi_2}^{\xi_1}}$, broadening fixed point theory and inspiring further research on generalized contractive mappings. Wardowski [112] presented a new family of mappings known as $\mathcal{F}$ -contraction, offering a compelling generalization of Banach's principle. Subsequent work by Wardowski et al. [113] introduced the notion of $\mathcal{F}$ -weak contraction, and Secolean [102], proved related F.P theorems within iterated function systems. Piri et al. [93] extended these findings by applying some weaker conditions on self-maps. Cosentino and Vetro [31] established some F.P results of Hardy-Rogers-type in C-M.Ss and C-ordered M.Ss. Hussain and Salimi [44] defined α -GF-contractions w.r.t general function families G , producing Wardowski results. Acar and Altun [2] further extended multivalued $\mathcal{F}$ -contractions by introducing δ -distance concepts and established F.P results in C-M.Ss. Recently, Numerous studies on $\mathcal{F}$ -contractions have been published (e.g., ([3, 9, 31, 70, 92, 95, 104])). In 1973, Geraghty [35] provided a vital generalization of BCP. Asl et al. [39] introduced the notion of α^* - ψ -Geraghty contractive multifunctions. Alizadeh et al. [11] expanded results on cyclic (α, β) -admissible mappings, and several F.P results under this idea were proved (see [30, 63]). In 2016, Liu et al. [66] defined (ψ, ϕ) -type and (ψ, ϕ) -Suzuki-type contractions, establishing relevant F.P theorems. Xiao et al. [114] introduced generalized multivalued Khan-type (ψ, ϕ) -contractions with accompanying F.P results. Newly, the notion of fractional symmetric α - η -contractions were discussed in [36, 42, 43], with applications for solving FODEs. The analysis of four variants of the stated contractions is carried out, leading to F.P findings in $\mathcal{F}$ -M.Ss. Hardy and Rogers [38] earlier developed F.P theorems based on Banach's formulation, which were later generalized across multiple settings. The concept of relation-theoretic contractions was introduced by Alam and Imdad [5] in M.Ss equipped with arbitrary binary relations see [6, 7, 110, 115]. These contributions were later broadened to set-valued mappings by several

researchers [20, 40, 68, 90].

In contrast, FDEs occupy a central position within contemporary mathematical analysis, as they encompass a broad class of models whose diverse solution structures have been rigorously investigated across numerous studies. In this regard, the methodology of F.P theory provides a foundational apparatus for the qualitative and quantitative exploration of NFDEs. Beyond this, it furnishes an effective scheme for resolving initial value problems encountered in disciplines such as dynamic optimization and applied mechanics. Hence, this theoretical framework proves to be remarkably influential in validating existence and uniqueness results for an extensive spectrum of problems governed by nonlinear interdependencies.

Parallel to these theoretical developments, M.S popularizations have played an integral role in functional analysis. One of them is an M-M.S introduced by Asadi et al. [23], further explored in [72, 75, 76, 77]. Mlaiki et al. [71] introduced Mb-M.S. In 2018, Özgür et al. [91] proposed R.M-M.S, extending the framework of this field with proving impressive F.Ps under the advanced BCP, while Patle et al. [96] explored Pompeiu-Hausdorff distance in the setting of M-M.Ss, establishing Nadler and Kannan type F.P results. In this chapter, we touch on some basic concepts and terms that are related to our research and given without proof.

1.1 Some essential concepts

Definition 1.1.1. Let $\Omega \neq \emptyset$ and $\Gamma, \mathfrak{S} : \Omega \longrightarrow \Omega$, then $\xi \in \Omega$ is called

- (i) F.P of Γ if $\Gamma\xi = \xi$.
- (ii) C.F.P of the pair $(\Gamma, \mathfrak{S})$ if $\Gamma\xi = \mathfrak{S}\xi = \xi$.
- (iii) Coincidence point of the pair $(\Gamma, \mathfrak{S})$ if $\Gamma\xi = \mathfrak{S}\xi$.

Definition 1.1.2. [28] Let Ψ symbolize the set of comparison functions $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, so that

- (ψ_1) ψ materializes $\forall \xi_1, \xi_2 > 0, \xi_1 < \xi_2 \implies \psi(\xi_1) \leq \psi(\xi_2)$;
- (ψ_2) $\lim_{\kappa \rightarrow \infty} \psi^\kappa(\xi_1) = 0$, where ψ^κ is κ^{th} iterative of ψ . Patently, if $\psi \in \Psi$ then $\psi(\xi_1) < \xi_1, \forall \xi_1 > 0$.

Example 1.1.3. [28] For all $t > 0$ and $0 < a < 1$, $\psi(t) = at \in \Psi$.

Definition 1.1.4. [66] Let Φ symbolize the set of functions $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, so that

(ϕ_1) ϕ is nondecreasing; $\forall \xi_1, \xi_2 > 0, \xi_1 < \xi_2 \implies \phi(\xi_1) \leq \phi(\xi_2)$;

(ϕ_2) for each $\{\xi_\kappa\} \subset (0, \infty)$, $\lim_{\kappa \rightarrow \infty} \phi(\xi_\kappa) = 0 \Leftrightarrow \lim_{\kappa \rightarrow \infty} \xi_\kappa = 0$;

(ϕ_3) ϕ is continuous.

Example 1.1.5. The following $\phi_1, \phi_2, \phi_3 \in \Phi$, $\forall \varsigma > 0$

$$\phi_1(\varsigma) = \varsigma;$$

$$\phi_2(\varsigma) = \ln \theta(\varsigma), \theta \in \Theta;$$

$$\phi_3(\varsigma) = e^{\mathcal{F}(\varsigma)}, \mathcal{F} \in \Delta_w.$$

Lemma 1.1.6. [66] Let $\phi : (0, \infty) \longrightarrow (0, \infty)$ achieves (ϕ_1), (ϕ_3) with $\inf_{\varsigma \in \mathbb{R}^+} \phi(\varsigma) = 0$ and a sequence $\{\varsigma_\kappa\} \subset (0, \infty)$. Thus, $\lim_{\kappa \rightarrow \infty} \phi(\varsigma_\kappa) = 0$ iff $\lim_{\kappa \rightarrow \infty} \varsigma_\kappa = 0$.

Definition 1.1.4 was extended in [86] by adding (ϕ_4) as follows:

$$(\phi_4) \phi \left(\sum_{\iota=1}^{\kappa} A_{\iota} \right) \leq \sum_{\iota=1}^{\kappa} \phi(A_{\iota}), A_{\iota} \in \mathbb{R}^+, \iota = 1, 2, \dots, \kappa.$$

Example 1.1.7. [86] The following ϕ_1 and $\phi_2 \in \Phi$, $\forall \varsigma \in (0, \infty)$

$$(1) \phi_1(\varsigma) = a\varsigma, a > 0;$$

$$(2) \phi_2(\varsigma) = |\varsigma|.$$

Definition 1.1.4 was modified in [78] by adding (ϕ_5) as follows:

$$(\phi_5) \phi(\inf G) = \inf \phi(G) \quad \forall G \subset (0, \infty) \text{ with } \inf(G) > 0.$$

Let Φ^* denotes the set of functions $\phi : (0, \infty) \rightarrow (0, \infty)$ satisfy (ϕ_1) – (ϕ_3) and (ϕ_5).

The forthcoming outcomes have been rigorously addressed in earlier works, (see [48, 62, 64, 73, 87]):

Let Ψ_b denotes the class of all non-decreasing functions $\psi : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ so that $\sum_{\kappa=1}^{\infty} \psi^\kappa(\varsigma) < \infty$, $\forall \varsigma > 0$, whereas ψ^κ is the κ^{th} iterate of ψ .

Lemma 1.1.8. Let $\psi \in \Psi_b$, thus:

$$(\psi_{b1}) (\psi^\kappa(\varsigma))_{\kappa \in \mathbb{N}} \rightarrow 0 \text{ as } \kappa \rightarrow \infty, \forall \varsigma \in (0, \infty);$$

$$(\psi_{b2}) \psi(\varsigma) < \varsigma, \forall \varsigma > 0;$$

$$(\psi_{b3}) \psi(\varsigma) = 0 \text{ iff } \varsigma = 0.$$

Let Ψ_s , $s \geq 1$ denotes the class of all non-decreasing functions $\psi : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ so that

$$(\psi_{s1}) \sum_{n=1}^{\infty} s^n \psi^n(\varsigma) < \infty, \forall \varsigma > 0;$$

$$(\psi_{s2}) s\psi(\varsigma) < \varsigma, \forall \varsigma > 0;$$

$$(\psi_{s3}) s^{\kappa+1} \psi^{\kappa+1}(\varsigma) < s^\kappa s \psi^\kappa \psi(\varsigma) < s^\kappa \psi^\kappa(\varsigma), \text{ where } \psi^\kappa \text{ is the } \kappa^{th} \text{ iterate of } \psi.$$

Definition 1.1.9. [106] Let $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multi-valued mapping, $\alpha : \Omega \times \Omega \rightarrow [0, \infty)$ be a function and $\emptyset \neq \beta \subset \Omega$. So, Γ is termed semi α_* -admissible on β , if $\alpha|_{\xi_2}^{\xi_1} \geq 1 \implies \alpha_*|_{\Gamma\xi_2}^{\Gamma\xi_1} \geq 1 \ \forall_{\xi_1, \xi_2 \in \beta}$, where

$$\alpha_*|_{\Gamma\xi_2}^{\Gamma\xi_1} = \inf \{ \alpha|_b^a : a \in \Gamma\xi_1, b \in \Gamma\xi_2 \}.$$

It should be noted that if $\beta = \Omega$, then we say that Γ is an α_* -admissible on Ω .

Definition 1.1.10. [54] Let $(\Omega, d|)$ be an M.S and $\Gamma, \mathfrak{S} : \Omega \rightarrow \Omega$ are compatible iff $\lim_{i \rightarrow \infty} d|_{\mathfrak{S}\Gamma\xi_i}^{\Gamma\mathfrak{S}\xi_i} = 0$, if $\{\xi_i\}$ is a sequence in Ω so that, $\lim_{i \rightarrow \infty} \Gamma\xi_i = \lim_{i \rightarrow \infty} \mathfrak{S}\xi_i = r$, $r \in \Omega$.

Lèmma 1.1.11. Let $(\Omega, d|)$ be an M.S. If $\exists \{\xi_i\}, \{\mu_i\}$ so that

$$\lim_{i \rightarrow \infty} d|_{\mu_i}^{\xi_i} = 0, \lim_{i \rightarrow \infty} \xi_i = r, r \in \Omega. \text{ Thus, } \lim_{i \rightarrow \infty} \mu_i = r.$$

If $(\xi_1, \xi_2) \in \mathfrak{R}$, then we express it by $\xi_1 \mathfrak{R} \xi_2$ and it is said that “ ξ_1 is related to ξ_2 ”. Here, we take $\mathfrak{R}$ as a binary relation on a nonempty subset Ω and $(\Omega, d|)$ as an M.S equipped with a binary relation $\mathfrak{R}$.

Definition 1.1.12. [5] A binary relation $\mathfrak{R}$ on $\Omega \neq \emptyset$ is a subset of $\Omega \times \Omega$. For $\xi_1, \xi_2 \in \Omega$, we say that ξ_1 is related to ξ_2 ($\xi_1 \mathfrak{R} \xi_2$) if and only if $(\xi_1, \xi_2) \in \mathfrak{R}$.

Definition 1.1.13. [5, 59] Let $\Gamma : \Omega \longrightarrow \Omega$. So, $\mathfrak{R}$ on Ω is regarded as Γ -closed if for $\xi_1, \xi_2 \in \Omega$,

$$(\xi_1, \xi_2) \in \mathfrak{R} \Rightarrow (\Gamma\xi_1, \Gamma\xi_2) \in \mathfrak{R}.$$

Definition 1.1.14. [5] Let $\mathfrak{R}$ be a binary relation on $\Omega \neq \emptyset$. Then, a sequence $\{\xi_\kappa\} \subset \Omega$ is called $\mathfrak{R}$ -preserving, if

$$(\xi_\kappa, \xi_{\kappa+1}) \in \mathfrak{R}, \ \forall_{\kappa \in \mathbb{N}}.$$

Definition 1.1.15. [7] Let $\Gamma : \Omega \longrightarrow \Omega$. Thus, $\mathfrak{R}$ is regarded as Γ -transitive, if for $\xi_1, \xi_2, \xi_3 \in \Omega$,

$$(\Gamma\xi_1, \Gamma\xi_2), (\Gamma\xi_2, \Gamma\xi_3) \in \mathfrak{R} \Rightarrow (\Gamma\xi_1, \Gamma\xi_3) \in \mathfrak{R}.$$

Definition 1.1.16. [100] An M.S $(\Omega, d|)$ is $\mathfrak{R}$ -regular if for every $\{\xi_\kappa\} \subseteq \Omega$,

$$(\xi_\kappa, \xi_{\kappa+1}) \in \mathfrak{R} \text{ and } \xi_\kappa \rightarrow \xi \in \Omega \implies (\xi_\kappa, \xi) \in \mathfrak{R} \ \forall_{\kappa \in \mathbb{N}}.$$

Definition 1.1.17. [6] Let $\mathfrak{R}$ be a binary relation on an M.S $(\Omega, d|)$ and $\xi^* \in \Omega$. A map $\Gamma : \Omega \longrightarrow \Omega$ is $\mathfrak{R}$ -continuous at ξ^* if for any sequence $\{\xi_\kappa\} \xrightarrow{d|} \xi^*$, we have $\Gamma\xi_\kappa \xrightarrow{d|} \Gamma\xi^*$. Γ is $\mathfrak{R}$ -continuous if it is $\mathfrak{R}$ -continuous at each point of Ω .

Definition 1.1.18. [90] Let $(\Omega_p, d|)$ be a partial M.S with a binary relation $\mathfrak{R}$ and Γ a multivalued-maps on Ω_p . Then,

- $\mathfrak{R}$ is called Γ -closed, if for any $\xi_1, \xi_2 \in \Omega_p$, $(\xi_1, \xi_2) \in \mathfrak{R} \Rightarrow (a, b) \in \mathfrak{R}$ for some $a \in \Gamma\xi_1$ and $b \in \Gamma\xi_2$.

- Γ is called $\mathfrak{R}$ -continuous at $\xi^* \in \Omega_p$, if for any $\mathfrak{R}$ -preserving sequence $\{\xi_\kappa\} \subset \Omega_p$ with $\{\xi_\kappa\} \xrightarrow{d|} \xi^*$, we have $\Gamma\xi_\kappa \xrightarrow{\hat{H}_d} \Gamma\xi^*$, Γ is termed $\mathfrak{R}$ -continuous if it is $\mathfrak{R}$ -continuous at each point of Ω .

- $\mathfrak{R}$ is called Γ -transitive, if for any $\xi_1, \xi_2, \xi_3 \in \Omega_p$, $a \in \Gamma\xi_1$, $b \in \Gamma\xi_2$ and $c \in \Gamma\xi_3$, we have

$$(a, b) \in \mathfrak{R}, (b, c) \in \mathfrak{R} \Rightarrow (a, c) \in \mathfrak{R}.$$

Definition 1.1.19. [20] Let $\Omega \neq \emptyset$ and $\Gamma : \Omega \rightarrow CP|^\Omega$. A binary relation $\mathfrak{R}$ on Ω is called Γ -transitive, if for any $\xi_1, \xi_2, \xi_3 \in \Omega_p$, $a \in \Gamma\xi_1$, $b \in \Gamma\xi_2$ and $c \in \Gamma\xi_3$, we have

$$(a, b) \in \mathfrak{R}, (b, c) \in \mathfrak{R} \Rightarrow (a, c) \in \mathfrak{R}.$$

Definition 1.1.20. [107] Let $(\Omega, d|)$ be a M.S with a binary relation $\mathfrak{R}$ and $\Gamma : \Omega \rightarrow CP|^\Omega$. Then, $\mathfrak{R}$ is called Γ - $d|$ -closed, if

$$(\xi_1, \xi_2) \in \mathfrak{R}, r \in \Gamma\xi_1, s \in \Gamma\xi_2, d|_s^r \leq d|_{\xi_2}^{\xi_1} \Rightarrow (r, s) \in \mathfrak{R}.$$

Definition 1.1.21. Let $\Omega \neq \emptyset$ and $2|^\Omega$ denotes the collection of all non-empty subset of Ω . Thus, $\Gamma : \Omega \longrightarrow 2|^\Omega$ is regarded as a set-valued map. A point $\xi \in \Omega$ is regarded as:

- (i) F.P of Γ if $\xi \in \Gamma\xi$.
- (ii) C.F.P of the pair $(\Gamma, \Im)$ if $\xi \in \Gamma\xi \cap \Im\xi$.

Let $(\Omega, d|)$ be an M.S, $CB|^\Omega$ be a family of all bounded and closed subsets of Ω and $K|^\Omega$ the family of all compact subsets of Ω .

Definition 1.1.22. [89] Let $\hat{H}_d$ be the Pompeiu-Hausdorff metric $\hat{H}_d : CB|^\Omega \times CB|^\Omega \rightarrow [0, \infty)$

induced by $d|$ and defined as

$$\hat{H}_d|_{\mathbb{B}}^{\mathbb{A}} = \max \left\{ \sup_{\xi_1 \in \mathbb{A}} D|_{\mathbb{B}}^{\xi_1}, \sup_{\xi_2 \in \mathbb{B}} D|_{\xi_2}^{\mathbb{A}} \right\}, \quad \forall_{\mathbb{A}, \mathbb{B} \in CB|^\Omega},$$

where

$$D|_{\mathbb{B}}^{\xi_1} = \inf_{\xi_2 \in \mathbb{B}} d|_{\xi_2}^{\xi_1}, \quad \forall_{\xi_1 \in \Omega} \text{ and } \forall_{\mathbb{A}, \mathbb{B} \in CB|^\Omega}.$$

Additionally, $(CB|^\Omega, \hat{H}_d)$ is a Pompeiu-Hausdorff M.S. So, $a \in \Omega$ is regarded as an F.P of a multivalued-map $\Gamma : \Omega \rightarrow CB|^\Omega$ if $a \in \Gamma a$.

Lemmæ 1.1.23. [89]. If $\mathbb{A}, \mathbb{B} \in CB|^\Omega$, then, $D|_{\mathbb{B}}^{\xi} \leq \hat{H}|_{\mathbb{B}}^{\mathbb{A}}, \quad \forall_{\xi \in \mathbb{A}}$.

Lemmæ 1.1.24. [89] If $\mathbb{A}, \mathbb{B} \in CB|^\Omega$. For $\check{N} > 0$, $\bar{a} \in \mathbb{A} \exists_{\check{a} \in \mathbb{B}}$ so that, $d|_{\check{a}}^{\bar{a}} \leq \hat{H}|_{\mathbb{B}}^{\mathbb{A}} + \check{N}$.

Definition 1.1.25. [11]. Let $\Omega \neq \emptyset$, $\alpha, \beta : \Omega \rightarrow [0, \infty)$ be functions. a mapping $\Gamma : \Omega \rightarrow \Omega$ is regarded as cyclic (α, β) -admissible if for some $\xi \in \Omega$,

$$\alpha(\xi) \geq 1 \implies \beta(\Gamma\xi) \geq 1,$$

and

$$\beta(\xi) \geq 1 \implies \alpha(\Gamma\xi) \geq 1.$$

Definition 1.1.26. [19]. Let $\Omega \neq \emptyset$, $\alpha, \beta : \Omega \rightarrow [0, \infty)$ be functions and $\mathbb{A}, \mathbb{B} \subset \Omega$. a mapping $\Gamma : \Omega \rightarrow CB|^\Omega$ is termed cyclic (α, β) -admissible if for some $\xi \in \Omega$,

$$\alpha(\xi) \geq 1 \implies \beta_*(\Gamma\xi) \geq 1,$$

and

$$\beta(\xi) \geq 1 \implies \alpha_*(\Gamma\xi) \geq 1,$$

where $\beta_*(\mathbb{A}) = \inf_{a \in \mathbb{A}} \beta(a)$ and $\alpha_*(\mathbb{B}) = \inf_{b \in \mathbb{B}} \alpha(b)$.

Denote by $\aleph|^\Omega$ a collection of all non-empty subsets of Ω , by $K|^\Omega$ a collection of all non-empty closed subsets of Ω .

Definition 1.1.27. [60] Let $\Gamma : \Omega \rightarrow \mathbb{N}^\Omega$ and $\xi_0 \in \Omega$ be arbitrary and fixed. Define

$$\mathcal{L}|_{\xi_0}^\Gamma = \left\{ \{\xi_\kappa\}_{\kappa \in \mathbb{N} \cup \{0\}} \subset \Omega : \xi_{\kappa+1} \in \Gamma \xi_{\kappa-1}, \forall \kappa \in \mathbb{N} \right\},$$

each element of $\mathcal{L}|_{\xi_0}^\Gamma$ is called a D.P of Γ starting with ξ_0 . If Γ is a single-valued mapping, then the D.P is uniquely defined and is of the form $(\Gamma^\kappa \xi_0)$, see [24, 25].

Definition 1.1.28. [45] A G.D.P of mappings $\mathfrak{I} : \Omega \rightarrow \Omega$ and $\Gamma : \Omega \rightarrow CB|^\Omega$ is defined as:

$$\mathcal{L}|_{\xi_0}^{\mathfrak{I}, \Gamma} = \left\{ \{\xi_\kappa\}_{\kappa \in \mathbb{N} \cup \{0\}} : \xi_{\kappa+1} = \mathfrak{I} \xi_\kappa \in \Gamma \xi_{\kappa-1}, \forall \kappa \in \mathbb{N} \right\},$$

where $\xi_0 \in \Omega$ is termed as a start point. Briefly, $\mathcal{L}|_{\xi_0}^{\mathfrak{I}, \Gamma}$ stands for $\mathfrak{I} \xi_\kappa$. The sequence $\{\xi_\kappa\}$ for which (ξ_κ) is regarded as the $\mathfrak{I}$ iterate sequence of Γ , starts in ξ_0 .

Definition 1.1.29. [50] Let $\Omega \neq \emptyset$ and $G = (V_G, E_G)$ be a graph so that $V_G = \Omega$, and let $\Gamma : \Omega \rightarrow CB|^\Omega$. Γ is told to be edge preserving if for each $u \in \Gamma \xi_1$ and $v \in \Gamma \xi_2$, if $(\xi_1, \xi_2) \in E_G$, then $(u, v) \in E_G$.

Lemma 1.1.30. [67] Let $\{\xi_\kappa\}_{\kappa \in \mathbb{N}}$ be a non-negative sequence such that $\lim_{\kappa \rightarrow \infty} \xi_\kappa = \xi$. Then,

$$\lim_{\kappa \rightarrow \infty} \int_0^{\xi_\kappa} \varphi(r) dr = \int_0^\xi \varphi(r) dr.$$

Lemma 1.1.31. [67] Let $\{\xi_\kappa\}_{\kappa \in \mathbb{N}}$ be a non-negative sequence. Then,

$$\lim_{\kappa \rightarrow \infty} \int_0^{\xi_\kappa} \varphi(r) dr = 0 \iff \lim_{\kappa \rightarrow \infty} \xi_\kappa = 0.$$

Lemma 1.1.32. [66] Let $\phi : (0, \infty) \rightarrow (0, \infty)$ be a continuous nondecreasing map, $\inf_{\xi \in (0, \infty)} \phi(\xi) = 0$. Let $\{\xi_\kappa\} \subset (0, \infty)$, so that $\lim_{\kappa \rightarrow \infty} \phi(\xi_\kappa) = 0 \iff \lim_{\kappa \rightarrow \infty} \xi_\kappa = 0$.

Remark 1.1.33. [32, 37] If θ is right continuous and satisfies $(\Theta 1)$, then

$$\theta(\inf G) = \inf \theta(G), \forall G \subset (0, \infty) \text{ with } \inf(G) > 0.$$

1.2 Some related contractions

Theorem 1.2.1. Let $(\Omega, d|)$ be a $C - M.S$ and $\Gamma : \Omega \longrightarrow \Omega$ be a contraction-mapping, if $\exists_{h \in [0,1)}$, such that

$$d|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq h d|_{\xi_2}^{\xi_1}, \forall_{\xi_1, \xi_2 \in \Omega}.$$

Then, Γ has a U.F.P.

Theorem 1.2.2 [38] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \longrightarrow \Omega$ be a self-mapping satisfying the condition $\forall_{\xi_1, \xi_2 \in \Omega}$,

$$\frac{1}{2} d|_{\Gamma\xi_1}^{\xi_1} < d|_{\xi_2}^{\xi_1} \implies d|_{\Gamma\xi_2}^{\Gamma\xi_1} < a d|_{\xi_2}^{\xi_1} + b d|_{\Gamma\xi_1}^{\xi_1} + c d|_{\Gamma\xi_2}^{\xi_2} + d d|_{\Gamma\xi_2}^{\xi_1} + f d|_{\Gamma\xi_1}^{\xi_2},$$

where $a, b, c, d, f \geq 0$ with $a + b + c + d + f < 1$.

Then, Γ has a U.F.P.

Theorem 1.2.3 [35] Let $(\Omega, d|)$ be a $C - M.S$ and $\Gamma : \Omega \longrightarrow \Omega$ be a self-map satisfies the condition If $\exists_{\varphi \in \Pi}$, we have

$$d|_{\Gamma\xi_2}^{\Gamma\xi_1} < \left(\varphi \left(d|_{\xi_2}^{\xi_1} \right) \right) d|_{\xi_2}^{\xi_1}, \forall_{\xi_1, \xi_2 \in \Omega},$$

where Π is the set of all functions $\varphi : [0, \infty) \rightarrow (0, 1)$ satisfying $\lim_{n \rightarrow \infty} t_n = 0$ whenever $\lim_{n \rightarrow \infty} \varphi(t_n) = 1$.

Then, Γ has a U.F.P.

Theorem 1.2.4 [29] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \longrightarrow \Omega$ be a mapping such that

$$\int_0^{d|_{\Gamma\xi_2}^{\Gamma\xi_1}} \mu(r) dr \leq h \int_0^{d|_{\xi_2}^{\xi_1}} \mu(r) dr, \forall_{\xi_1, \xi_2 \in \Omega},$$

$h \in (0, 1)$ and $\mu \in \Upsilon$, where Υ is the class of all functions $\mu : [0, \infty) \rightarrow [0, \infty)$ with finite integral on every compact subset of $[0, \infty)$ and $\int_0^\varepsilon \mu(r) dr > 0$ for each $\varepsilon > 0$. Then, Γ has a U.F.P $a \in \Omega$ so that $\forall_{\xi \in \Omega}, \lim_{\kappa \rightarrow \infty} \Gamma^\kappa \xi = a$.

Definition 1.2.5 [51] Let $(\Omega, d|)$ be a M.S and $\Gamma : \Omega \longrightarrow \Omega$ is called a Jaggi contraction map

if there exists $\lambda_1, \lambda_2 \in [0, 1)$ with $\lambda_1 + \lambda_2 < 1$, such that

$$d|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \lambda_1 d|_{\xi_2}^{\xi_1} + \lambda_2 \frac{d|_{\Gamma\xi_1}^{\xi_1} d|_{\Gamma\xi_2}^{\xi_2}}{d|_{\xi_2}^{\xi_1}}, \quad \forall \xi_1, \xi_2 \in \Omega.$$

Theorem 1.2.6 [51] Let $(\Omega, d|)$ be a C-MS and $\Gamma : \Omega \longrightarrow \Omega$ be a Jaggi contraction map.

Then, Γ has a U.F.P in Ω .

Definition 1.2.7 [56] Let $(\Omega, d|)$ be a M.S. We say that the self-mapping $\Gamma : \Omega \longrightarrow \Omega$ is an interpolative Kannan type contraction, if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, such that

$$d|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \lambda \left[d|_{\Gamma\xi_1}^{\xi_1} \right]^\alpha \cdot \left[d|_{\Gamma\xi_2}^{\xi_2} \right]^{1-\alpha}, \quad \forall \xi_1, \xi_2 \in \Omega \text{ with } \xi_1 \neq \Gamma\xi_1.$$

Theorem 1.2.8 [56] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \longrightarrow \Omega$ be an interpolative Kannan type contraction. Then, Γ has a U.F.P in Ω .

Definition 1.2.9 [57] A selfmap Γ on $(\Omega, d|)$ is regarded as a Jaggi-type hybrid contraction if $\exists \psi \in \Psi$ so that

$$d|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \psi \left(M|_{\xi_2}^{\xi_1} \right),$$

where $s \geq 0$ and $\lambda_i \geq 0$, $i = 1, 2, \dots$ with $\lambda_1 + \lambda_2 = 1$ and

$$M|_{\xi_2}^{\xi_1} = \begin{cases} \left[\lambda_1 \left(\frac{d|_{\Gamma\xi_1}^{\xi_1} d|_{\Gamma\xi_2}^{\xi_2}}{d|_{\xi_2}^{\xi_1}} \right)^s + \lambda_2 \left(d|_{\xi_2}^{\xi_1} \right)^s \right]^{\frac{1}{s}}, & \text{for } s > 0, \quad \xi_1, \xi_2 \in \Omega, \quad \xi_1 \neq \xi_2; \\ \left(d|_{\Gamma\xi_1}^{\xi_1} \right)^{\lambda_1} \left(d|_{\Gamma\xi_2}^{\xi_2} \right)^{\lambda_2}, & \text{for } s = 0, \quad \xi_1, \xi_2 \in \Omega \setminus \mathbb{F}_\Gamma|^\Omega, \end{cases}$$

where $\mathbb{F}_\Gamma|^\Omega = \{\xi \in \Omega : \Gamma\xi = \xi\}$.

Theorem 1.2.10 [57] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \longrightarrow \Omega$ be a continuous Jaggi type hybrid contraction. Then, Γ has a F.P in Ω . Moreover, for any $\xi_0 \in \Omega$, the sequence $\{\Gamma\xi_0\}$ converges to ξ .

Suzuki [109] proved a generalized BCP expressed as:

Theorem 1.2.11 [109] Let $(\Omega, d|)$ be a C-M.S, and $\Gamma : \Omega \longrightarrow \Omega$ be a self-map. Assume that $\forall \xi_1, \xi_2 \in \Omega$ with $\xi_1 \neq \xi_2$,

$$\frac{1}{2} d|_{\Gamma\xi_1}^{\xi_1} < d|_{\xi_2}^{\xi_1} \implies d|_{\Gamma\xi_2}^{\Gamma\xi_1} < d|_{\xi_2}^{\xi_1}.$$

Then, Γ has a U.F.P.

Theorem 1.2.12 [66] Let $(\Omega, d|)$ be a C-M.S. A map $\Gamma : \Omega \rightarrow \Omega$ is told to be an (ψ, ϕ) -type Suzuki contraction, if there exist $\phi \in \Phi$ and $\psi \in \Psi$ so that $\forall_{\xi_1, \xi_2 \in \Omega}$ with $\Gamma\xi_1 \neq \Gamma\xi_2$, we have

$$\frac{1}{2} d|_{\Gamma\xi_1}^{\xi_1} < d|_{\xi_2}^{\xi_1} \implies \phi \left(d|_{\Gamma\xi_2}^{\Gamma\xi_1} \right) \leq \psi \left[\phi \left(U|_{\xi_2}^{\xi_1} \right) \right],$$

where

$$U|_{\xi_2}^{\xi_1} = \max \left\{ d|_{\xi_2}^{\xi_1}, d|_{\Gamma\xi_1}^{\xi_1}, d|_{\Gamma\xi_2}^{\xi_2}, \frac{d|_{\Gamma\xi_2}^{\xi_1}}{2}, d|_{\Gamma\xi_1}^{\xi_2} \right\}.$$

Then, Γ has a U.F.P.

Definition 1.2.13 [112] Let $(\Omega, d|)$ be a M.S. A mapping $\Gamma : \Omega \rightarrow \Omega$ is regarded as an $\mathcal{F}$ -contraction if $\exists_{\tau \in \mathbb{R}^+}$ so that $\forall_{\xi_1, \xi_2 \in \Omega}$,

$$d|_{\Gamma\xi_2}^{\Gamma\xi_1} > 0 \implies \tau + \mathcal{F} \left(d|_{\Gamma\xi_2}^{\Gamma\xi_1} \right) \leq \mathcal{F} \left(d|_{\xi_2}^{\xi_1} \right), \quad (1.1)$$

where Δ_w is the class of functions $\mathcal{F} : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following three assumptions:

- (F1) $\mathcal{F}$ is strictly increasing, i.e. $\forall_{\varrho_1, \varrho_2 \in \mathbb{R}^+}$ so that $\varrho_1 < \varrho_2$, $\mathcal{F}(\varrho_1) < \mathcal{F}(\varrho_2)$;
- (F2) For $\{\Upsilon_\kappa\}_{\kappa=1}^\infty \subseteq \mathbb{R}^+$, $\lim_{\kappa \rightarrow \infty} \Upsilon_\kappa = 0 \iff \lim_{\kappa \rightarrow \infty} \mathcal{F}(\Upsilon_\kappa) = -\infty$;
- (F3) $\exists \hat{h} \in (0, 1)$ such that $\lim_{\kappa \rightarrow 0^+} \Upsilon^{\hat{h}} \mathcal{F}(\Upsilon_\kappa) = 0$.

This Definition 1.2.13 has been generalized in [33] by adding the condition below:

- (F4) for any sequence $\{\Upsilon_\kappa\}_{\kappa=1}^\infty \subset \mathbb{R}^+$, so that

$$\tau + \mathcal{F}(s\Upsilon_\kappa) \leq \mathcal{F}(\Upsilon_{\kappa-1}), \quad s \geq 1,$$

$\forall_{\kappa \in \mathbb{N}}$ and some $\tau > 0$, then

$$\tau + \mathcal{F}(s^\kappa \Upsilon_\kappa) \leq \mathcal{F}(s^{\kappa-1} \Upsilon_{\kappa-1}).$$

Here, Δ_s denotes the set of all functions $\mathcal{F}$ fulfilling (F1) - (F4).

Exæmple 1.2.14 [112] Let $\mathcal{F} : (0, \infty) \rightarrow \mathbb{R}$ be defined as $\mathcal{F}(k) = \ln k$. Then, $\mathcal{F} \in \Delta_s$. Notice

that condition (1.1) reduces to the following

$$d|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq e^{-\tau} d|_{\xi_2}^{\xi_1}, \quad \forall_{\xi_1, \xi_2 \in \Omega}, \quad \Gamma\xi_1 \neq \Gamma\xi_2.$$

It is clear that for $\xi_1, \xi_2 \in \Omega$ such that $\Gamma\xi_1 = \Gamma\xi_2$ the inequality $d|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq e^{-\tau} d|_{\xi_2}^{\xi_1}$, is true, i.e. Γ is a BC map.

Exæmple 1.2.15. [112] Let $\mathcal{F}(k) = \ln k + k$, $k \in (0, \infty)$, then $\mathcal{F} \in \Delta_s$ and (1.1) can be reduced to

$$\frac{d|_{\Gamma\xi_2}^{\Gamma\xi_1}}{d|_{\xi_2}^{\xi_1}} e^{[d|_{\Gamma\xi_2}^{\Gamma\xi_1} - d|_{\xi_2}^{\xi_1}]} \leq e^{-\tau}, \quad \forall_{\xi_1, \xi_2 \in \Omega}, \quad \Gamma\xi_1 \neq \Gamma\xi_2.$$

Remark 1.2.16. From (F1) and Definition 1.2.13, one can readily deduce that each $\mathcal{F}$ -contraction is necessarily continuous.

Theorem 1.2.17. [112] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow \Omega$ be an $\mathcal{F}$ -contractive. If $\exists_{\tau > 0}$, $\mathcal{F} \in \Delta_s$ and $\forall_{\xi_1, \xi_2 \in \Omega}$, then Γ has a U.F.P $\xi^* \in \Omega$.

$\mathcal{F}$ -contraction was modified in [93] by replacing (F3) by (F3') $\mathcal{F}$ is continuous.

Let Δ^* denotes the set of maps $\mathcal{F} : (0, \infty) \rightarrow \mathbb{R}$ satisfy (F1) – (F3').

Theorem 1.2.18. Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow \Omega$ be an $\mathcal{F}$ -contractive. If $\exists_{\tau > 0}$, $\mathcal{F} \in \Delta^*$ and $\forall_{\xi_1, \xi_2 \in \Omega}$, then Γ has a U.F.P $\xi^* \in \Omega$.

Example 1.2.19. The following $\mathcal{F}_i \in \Delta^*$ $\forall_{k \in \mathbb{R}^+}$, $i = 1, 2, 3$:

$$(1) \mathcal{F}_1(k) = \ln k \quad (2) \mathcal{F}_2(k) = k - \frac{1}{k} \quad (3) \mathcal{F}_3(t) = \frac{e^t}{1 - e^{2t}}.$$

Remark 1.2.20. [13] If $\mathcal{F}$ is right continuous and satisfies (F2), then

$$\mathcal{F}(\inf A) = \inf \mathcal{F}(A) \quad \forall_{A \subset (0, \infty)} \text{ with } \inf(A) > 0.$$

Definition 1.2.21. [89] A mapping $\Gamma : \Omega \rightarrow CB|^\Omega$ is set-valued contraction if $\exists_{\lambda \in [0, 1)}$, such that,

$$\hat{H}_d|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \lambda d|_{\xi_2}^{\xi_1}, \quad \forall_{\xi_1, \xi_2 \in \Omega}.$$

Theorem 1.2.22. [89] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow CB|^\Omega$ a set-valued contraction. Thus, Γ admit a U.F.P $\xi^* \in \Omega$ so that $\xi^* \in \Gamma\xi^*$.

The notion of multi-valued $\mathcal{F}$ -contractive maps were discussed in [3].

Definition 1.2.23. [3] Let $(\Omega, d|)$ be a M.S and $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multivalued-mapping.

Then, Γ is regarded as a generalized multi-valued $\mathcal{F}$ -contraction if $\exists_{\mathcal{F} \in \Delta^*, \tau \in \mathbb{R}^+}$ so that

$$\forall_{\xi_1, \xi_2 \in \Omega}, \quad \hat{H}_d \Big|_{\Gamma \xi_2}^{\Gamma \xi_1} > 0 \implies \tau + \mathcal{F}(\hat{H}_d \Big|_{\Gamma \xi_2}^{\Gamma \xi_1}) \leq \mathcal{F}(\mathcal{G}_1 \Big|_{\xi_2}^{\xi_1}),$$

where

$$\mathcal{G}_1 \Big|_{\xi_2}^{\xi_1} = \max \left\{ d \Big|_{\xi_2}^{\xi_1}, D \Big|_{\Gamma \xi_1}^{\xi_1}, D \Big|_{\Gamma \xi_2}^{\xi_2}, \frac{1}{2} [D \Big|_{\Gamma \xi_2}^{\xi_1} + D \Big|_{\Gamma \xi_1}^{\xi_2}] \right\}.$$

Theorem 1.2.24. [3] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow K|^\Omega$ be a generalized multi-valued $\mathcal{F}$ -contraction. If Γ or $\mathcal{F}$ is continuous, then Γ has a F.P in Ω .

Theorem 1.2.25. [104] Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow CL|^\Omega$ a multivalued-mapping. presume that $\exists_{\mathcal{F} \in \Delta^*, \tau \in \mathbb{R}^+}$ so that

$$2\tau + \mathcal{F} \left(\hat{H}_d \Big|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \mathcal{F} \left(a_1 d \Big|_{\xi_2}^{\xi_1} + a_2 D \Big|_{\Gamma \xi_1}^{\xi_1} + a_3 D \Big|_{\Gamma \xi_2}^{\xi_2} + a_4 D \Big|_{\Gamma \xi_2}^{\xi_1} + a_5 D \Big|_{\Gamma \xi_1}^{\xi_2} \right),$$

$\forall_{\xi_1, \xi_2 \in \Omega}$, with $\Gamma \xi_1 \neq \Gamma \xi_2$, where $a_1, a_2, a_3, a_4, a_5 \geq 0$, $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$. Then Γ has a F.P.

Jleli and Samet [53] have established the following results:

Consider $\Theta = \{\theta : (0, \infty) \rightarrow (1, \infty) : \theta \text{ achieves } \Theta_1, \Theta_2 \text{ and } \Theta_3\}$. where,

(Θ_1) θ is nondecreasing;

(Θ_2) for each $\{\partial_\kappa\} \subset (0, \infty)$,

$$\lim_{\kappa \rightarrow \infty} \theta(\partial_\kappa) = 1 \Leftrightarrow \lim_{\kappa \rightarrow \infty} \partial_\kappa = 0^+;$$

(Θ_3) $\exists_{0 < q < 1}$ and $\exists_{\ell \in (0, \infty]}$ so that $\lim_{\partial \rightarrow 0^+} \frac{\theta(\partial) - 1}{\partial^q} = \ell$.

Theorem 1.2.26. [53] Let $(\Omega, d|)$ be a G.M.S and $\Gamma : \Omega \rightarrow \Omega$ be a map, if $\exists_{\mathbb{k} \in (0, 1)}$, $\theta \in \Theta$ so that,

$$\forall_{\xi_1, \xi_2 \in \Omega}, \quad d \Big|_{\Gamma \xi_2}^{\Gamma \xi_1} > 0 \implies \theta \left(d \Big|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \left[\theta \left(d \Big|_{\xi_2}^{\xi_1} \right) \right]^\mathbb{k}. \quad (1.2)$$

Then, Γ has a U.F.P.

Example 1.2.27. Let $\theta_1, \theta_2 : (0, \infty) \rightarrow (1, \infty)$ be defined by,

$$\theta_1(r) = e^{\sqrt{r}}, \quad \theta_2(r) = e^{\sqrt{re^r}}, \quad \forall r \in (0, \infty).$$

Thus, θ_1 & $\theta_2 \in \Theta$.

In line with [4], let $\Theta^* = \{\theta : (0, \infty) \rightarrow (1, \infty) : \theta \text{ achieves } (\Theta_1), (\Theta_2) \text{ and } (\Theta_4)\}$, where (Θ_4) θ is continuous on $(0, \infty)$.

In line with [66], Ξ is the class of functions $\theta : (0, \infty) \rightarrow (1, \infty)$ achieving (Θ_1) , (Θ_5) & (Θ_4) , whereas

$$(\Theta_5): \inf_{r \in (0, \infty)} \theta(r) = 1.$$

Jaleli et al.[52] introduced extension of [53] as follows:

Theorem 1.2.28. [52] Let $(\Omega, d|)$ be a G.M.S and $\Gamma : \Omega \rightarrow \Omega$ be a map, if $\exists_{k \in (0,1)}$ and $\exists_{\theta \in \Theta \cup \Theta^*}$ so that,

$$\xi_1, \xi_2 \in \Omega, \quad d|_{\Gamma\xi_2}^{\Gamma\xi_1} > 0 \Rightarrow \theta \left(d|_{\Gamma\xi_2}^{\Gamma\xi_1} \right) \leq \left[\theta \left(M|_{\xi_2}^{\xi_1} \right) \right]^k, \quad (1.2)$$

where

$$M|_{\xi_2}^{\xi_1} = \max \left\{ d|_{\xi_2}^{\xi_1}, d|_{\Gamma\xi_1}^{\xi_1}, d|_{\Gamma\xi_2}^{\xi_2} \right\}.$$

Then, Γ has a U.F.P.

θ -contractive for multivalued-mappings were obtained in [37].

Definition 1.2.29. [37] Let $(\Omega, d|)$ be a M.S, $\Gamma : \Omega \rightarrow CB|^\Omega$ and $\theta \in \Theta$. Thus, Γ is a multi-valued θ -contractive if $\exists_{k \in [0,1)}$ so that $\forall_{\xi_1, \xi_2 \in \Omega}$ with $\hat{H}_d|_{\Gamma\xi_2}^{\Gamma\xi_1} > 0$,

$$\theta \left(\hat{H}_d|_{\Gamma\xi_2}^{\Gamma\xi_1} \right) \leq \left[\theta \left(d|_{\xi_2}^{\xi_1} \right) \right]^k.$$

Definition 1.2.30. [114] Let $(\Omega, d|)$ be an M.S and $\Gamma : \Omega \rightarrow CB|^\Omega$ is a generalized multi-valued Khan-type (ψ, ϕ) -contractive map, if $\exists_{\phi \in \Phi, \psi \in \Psi}$ so that $\forall_{\xi_1, \xi_2 \in \Omega}$, we have

(1) If $\max \left\{ d|_{\Gamma\xi_2}^{\xi_1}, d|_{\Gamma\xi_1}^{\xi_2} \right\} \neq 0$, then $\Gamma\xi_1 \neq \Gamma\xi_2$ and

$$\phi \left(\hat{H}_d|_{\Gamma\xi_2}^{\Gamma\xi_1} \right) \leq \psi \left[\phi \left(M|_{\xi_2}^{\xi_1} \right) \right],$$

where

$$M|_{\xi_2}^{\xi_1} = \frac{d|_{\Gamma\xi_1}^{\xi_1} d|_{\Gamma\xi_2}^{\xi_1} + d|_{\Gamma\xi_2}^{\xi_2} d|_{\Gamma\xi_1}^{\xi_2}}{\max \left\{ d|_{\Gamma\xi_2}^{\xi_1}, d|_{\Gamma\xi_1}^{\xi_2} \right\}}.$$

(2) If $\max \left\{ d|_{\Gamma\xi_2}^{\xi_1}, d|_{\Gamma\xi_1}^{\xi_2} \right\} = 0$, then $\Gamma\xi_1 = \Gamma\xi_2$.

Theorem 1.2.31. Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow CB|^\Omega$ be a generalized multi-valued Khan-type (ψ, ϕ) -contractive map. Then, Γ yields an F.P $\xi^* \in \Omega$.

Theorem 1.2.32. [71] Let (Ω_b, m_b) be an C-Mb-M.S with $s \geq 1$ and $\Gamma : \Omega \rightarrow \Omega$ be a map. If $\exists_{k \in [0,1)}$ so that

$$m_b|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq k m_b|_{\xi_2}^{\xi_1} \quad \forall_{\xi_1, \xi_2 \in \Omega_b}.$$

Then, Γ has an F.P $\xi^* \in \Omega$.

Theorem 1.2.33. [91] Let (Ω_r, m_r) be a C-R.M-M.S and $\Gamma : \Omega \rightarrow \Omega$ be a mapping. If $\exists_{k \in [0,1)}$ so that

$$m_r|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq k m_r|_{\xi_2}^{\xi_1} \quad \forall_{\xi_1, \xi_2 \in \Omega_r}.$$

Then, Γ has an F.P $\xi^* \in \Omega$, where $m_r|_{\xi^*}^{\xi^*} = 0$.

Definition 1.2.34. [43] Let $(\Omega_b, d_{\mathcal{F}})$ be an $\mathcal{F}$ -M.S, $\Gamma : \Omega \rightarrow \Omega$ be a map and $\alpha, \eta : \Omega_b \rightarrow [0, \infty)$ be two functions. Γ is regarded as a symmetric fractional α - η -contraction of type-I, if $\exists_{a,b,c \in (0,1), \lambda \in [0,1)}$ so that $\forall_{\xi_1, \xi_2 \in \Omega \setminus \text{Fix}(\Gamma)}$, whenever $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$, we have

$$d_{\mathcal{F}}|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \psi \left[\lambda(\mathbb{H}_1|_{\xi_2}^{\xi_1}) \right],$$

where

$$\begin{aligned} \mathbb{H}_1|_{\xi_2}^{\xi_1} &= d_{\mathcal{F}}|_{\xi_2}^{\xi_1} \cdot \left[d_{\mathcal{F}}|_{\Gamma\xi_1}^{\xi_1} \right]^{\frac{1}{(a-b)(a-c)}} \cdot \left[d_{\mathcal{F}}|_{\Gamma\xi_2}^{\xi_2} \right]^{\frac{1}{(a-b)(a-c)}} \\ &\quad \cdot \left[d_{\mathcal{F}}|_{\Gamma\xi_1}^{\xi_1} + d_{\mathcal{F}}|_{\Gamma\xi_2}^{\xi_2} \right]^{\frac{1}{(b-a)(b-c)}} \cdot \left[d_{\mathcal{F}}|_{\Gamma\xi_2}^{\xi_1} + d_{\mathcal{F}}|_{\Gamma\xi_1}^{\xi_2} \right]^{\frac{1}{(c-a)(c-b)}}. \end{aligned}$$

Theorem 1.2.35. [43] Let $(\Omega, d_{\mathcal{F}})$ be a C- $\mathcal{F}$ -M.S and Γ be a symmetric fractional α - η -contraction type-I satisfies the following statements:

- (i) Γ is an α -admissible map w.r.t η ;
- (ii) $\exists_{\xi_0 \in \Omega}$ so that $\alpha|_{\Gamma\xi_0}^{\xi_0} \geq \eta|_{\Gamma\xi_0}^{\xi_0}$;

(iii) Γ is α - η -continuous.

Then, Γ has an F.P $\xi^* \in \Omega$.

Definition 1.2.36. [8] A self-map Γ on a graphical b-M.S $(\Omega, d|)$ with $s \geq 1$ is regarded as a Fisher-type graph contraction for H_G on $(\Omega, d|)$ if H_G is graph preserving and if $\exists_{\lambda_1, \lambda_2 > 0}$ with $\lambda_1 + \lambda_2 < \frac{1}{s}$ so that $\forall_{\zeta_1, \zeta_2 \in \Omega}$ with $(\zeta_1, \zeta_2) \in H_G$, we have

$$d|_{\Gamma \xi_2}^{\Gamma \xi_1} \leq \lambda_1 d|_{\xi_2}^{\xi_1} + \lambda_2 \frac{d|_{\Gamma \xi_1}^{\xi_1} \cdot d|_{\Gamma \xi_2}^{\xi_2}}{1 + d|_{\xi_2}^{\xi_1}}.$$

1.3 Some generalized spaces

In this section, we deal with two spaces, that are, Mb-M.S, and R.M-M.S, which are generalizations of M.S and modify the classical distance function to allow for broader applications in F.P theory and functional analysis.

1.3.1 Mb-metric spaces

►Mlaiki et al. [71] generalized the notion of Mb-M.S as follows:

Definition 1.3.1.1. Let $\Omega_b \neq \emptyset$ and a function $m_b : \Omega_b \times \Omega_b \rightarrow \mathbb{R}_0^+$ is an Mb-M if $\forall_{\xi_1, \xi_2, \xi_3 \in \Omega_b}$,

$$(M_{bi}) \quad m_b|_{\xi_1}^{\xi_1} = m_b|_{\xi_2}^{\xi_2} = m_b|_{\xi_2}^{\xi_1} \text{ iff } \xi_1 = \xi_2;$$

$$(M_{bii}) \quad m_b|_{\xi_1, \xi_2} \leq m_b|_{\xi_2}^{\xi_1};$$

$$(M_{biii}) \quad m_b|_{\xi_2}^{\xi_1} = m_b|_{\xi_1}^{\xi_2};$$

$$(M_{biv}) \quad \text{There exists a coefficient } s \geq 1 \text{ so that,}$$

$$m_b|_{\xi_3}^{\xi_1} - m_b|_{\xi_1, \xi_3} \leq s \left[\left(m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} \right) + \left(m_b|_{\xi_3}^{\xi_2} - m_b|_{\xi_2, \xi_3} \right) \right] - m_b|_{\xi_2}^{\xi_2}.$$

The pair (Ω_b, m_b) is said to be an Mb-M.S.

Note that, $m_b|_{\xi_1, \xi_2}$ and $M_b|_{\xi_1, \xi_2}$ are defined in [71] as follows

$$m_b|_{\xi_1, \xi_2} = \min \left\{ m_b|_{\xi_1}^{\xi_1}, m_b|_{\xi_2}^{\xi_2} \right\};$$

and

$$M_b|_{\xi_1, \xi_2} = \max \left\{ m_b|_{\xi_1}^{\xi_1}, m_b|_{\xi_2}^{\xi_2} \right\}.$$

Example 1.3.1.2. Let $\Omega_b = [0, \infty)$ and $p > 1$. Define $m_b : \Omega_b \times \Omega_b \rightarrow [0, \infty)$ by

$$m_b|_{\xi_2}^{\xi_1} = (\max\{\xi_1, \xi_2\})^p + |\xi_1 - \xi_2|^p, \quad \forall_{\xi_1, \xi_2 \in \Omega_b}.$$

Then (Ω_b, m_b) is an Mb-MS with $s = 2^p$ and not M-M.S.

Definition 1.3.1.3. Let (Ω_b, m_b) be an Mb-M.S. Then

- A sequence $\{\xi_\kappa\}$ in Ω_b converge to a point ξ iff

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\xi}^{\xi_\kappa} - m_b|_{\xi_\kappa, \xi} \right) = 0.$$

- A sequence $\{\xi_\kappa\}$ in Ω_b is called m_b -Cauchy iff

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(m_b|_{\xi_\varkappa}^{\xi_\kappa} - m_b|_{\xi_\kappa, \xi_\varkappa} \right) \text{ and } \lim_{\kappa, \varkappa \rightarrow \infty} \left(M_b|_{\xi_\kappa, \xi_\varkappa} - m_b|_{\xi_\kappa, \xi_\varkappa} \right),$$

exist and finite.

- An Mb-M.S is a m_b -cømplete if every m_b -Cauchy sequence cønverges to ζ so that

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\xi}^{\xi_\kappa} - m_b|_{\xi_\kappa, \xi} \right) = 0 \text{ and } \lim_{\kappa \rightarrow \infty} \left(M_b|_{\xi_\kappa, \xi} - m_b|_{\xi_\kappa, \xi} \right) = 0.$$

Example 1.3.1.4. [99] Let $\Omega_b = [0, 1]$ and $m_b : \Omega_b \times \Omega_b \rightarrow [0, \infty)$ be defined by

$$m_b|_{\xi_2}^{\xi_1} = \left(\frac{\xi_1 + \xi_2}{2} \right)^2, \quad \forall_{\xi_1, \xi_2 \in \Omega_b}.$$

Then (Ω_b, m_b) is an Mb-M.S with $s = 2$.

1.3.2 Rectangular M-metric space

►Özgür et al. [91] provided the notion of R.M-M.S as follows:

Definition 1.3.2.1. Let $\Omega_r \neq \emptyset$ and $m_r : \Omega_r \times \Omega_r \rightarrow [0, \infty)$ be a mapping satisfies $\forall_{\xi_1, \xi_2 \in \Omega_r}$,

- (i) $m_r|_{\xi_1}^{\xi_1} = m_r|_{\xi_2}^{\xi_2} = m_r|_{\xi_2}^{\xi_1}$ iff $\xi_1 = \xi_2$;
- (ii) $m_r|_{\xi_1, \xi_2} \leq m_r|_{\xi_2}^{\xi_1}$;
- (iii) $m_r|_{\xi_2}^{\xi_1} = m_r|_{\xi_1}^{\xi_2}$

(iv) $\forall_{\xi_3, \xi_4 \in \Omega_r \setminus \{\xi_1, \xi_2\}}$, we have

$$m_r|_{\xi_2}^{\xi_1} - m_r|_{\xi_1, \xi_2} \leq \left(m_r|_{\xi_3}^{\xi_1} - m_r|_{\xi_1, \xi_3} \right) + \left(m_r|_{\xi_4}^{\xi_3} - m_r|_{\xi_3, \xi_4} \right) + \left(m_r|_{\xi_2}^{\xi_4} - m_r|_{\xi_4, \xi_2} \right).$$

The pair (Ω_r, m_r) is said to be an R.M-M.S.

Remark 1.3.2.2. Let (Ω_r, m_r) be a R.M-M.S. Clearly, we have

- (1) $0 \leq M_r|_{\xi_1, \xi_2} + m_r|_{\xi_1, \xi_2} = m_r|_{\xi_1}^{\xi_1} + m_r|_{\xi_2}^{\xi_2}$;
- (2) $0 \leq M_r|_{\xi_1, \xi_2} - m_r|_{\xi_1, \xi_2} = \left| m_r|_{\xi_1}^{\xi_1} - m_r|_{\xi_2}^{\xi_2} \right| \forall_{\xi_1, \xi_2 \in \Omega_r}$;
- (3) $M_r|_{\xi_1, \xi_2} - m_r|_{\xi_1, \xi_2} \leq \left(M_r|_{\zeta_1, \zeta_3} - m_r|_{\zeta_1, \zeta_3} \right) + \left(M_r|_{\zeta_3, \zeta_4} - m_r|_{\zeta_3, \zeta_4} \right) + \left(M_r|_{\zeta_4, \zeta_2} - m_r|_{\zeta_4, \zeta_2} \right).$

Example 1.3.2.3. Take $\mathbb{C}$ to be the set of complex-numbers, $\Omega_{r\varphi} = \{z \in \mathbb{C} : \arg|z| = \varphi\} \cup \{0\}$ for a fixed φ , $0 \leq \varphi < 2\pi$, and define $m_r : \Omega_{r\varphi} \times \Omega_{r\varphi} \longrightarrow [0, \infty)$ by $m_r|_{\xi_2}^{\xi_1} = \frac{|\xi_1| + |\xi_2|}{2} \forall_{\xi_1, \xi_2 \in \Omega_{r\varphi}}$. Then $(\Omega_{r\varphi}, m_r)$ is a R.M-M.S.

Definition 1.3.2.4. Let (Ω_r, m_r) be a R.M-M.S. Then, we have

- (1) $\{\xi_\kappa\} \subset \Omega_r$ converges to ξ iff

$$\lim_{\kappa \rightarrow \infty} \left(m_r|_{\xi}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi} \right) = 0.$$

- (2) $\{\xi_\kappa\} \subset \Omega_r$ is m_r -Cauchy iff

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(m_r|_{\xi_\varkappa}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right) \text{ and } \lim_{\kappa, \varkappa \rightarrow \infty} \left(M_r|_{\xi_\kappa, \xi_\varkappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right),$$

exist and finite.

- (3) A R.M-M.S is regarded as m_r -complete if every m_r -Cauchy $\{\xi_\kappa\}$ converges to ξ so that

$$\lim_{\kappa \rightarrow \infty} \left(m_r|_{\xi}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi} \right) = 0 \text{ and } \lim_{\kappa \rightarrow \infty} \left(M_r|_{\xi_\kappa, \xi} - m_r|_{\xi_\kappa, \xi} \right) = 0.$$

Lemma 1.3.2.5. presume that $\xi_\kappa \rightarrow \xi$ and $\varrho_\kappa \rightarrow \varrho$ as $\kappa \rightarrow \infty$ in a R.M-M.S (Ω_r, m_r) . Then

$$\lim_{\kappa \rightarrow \infty} \left(m_r|_{\varrho}^{\xi_\kappa} - m_r|_{\xi_\kappa, \varrho_\kappa} \right) = m_r|_{\varrho}^{\xi} - m_r|_{\xi, \varrho}.$$

Lemma 1.3.2.6. presume that $\xi_\kappa \rightarrow \xi$ as $\kappa \rightarrow \infty$ in a R.M-M.S (Ω_r, m_r) . Then

$$\lim_{n \rightarrow \infty} \left(m_r|_{\varrho}^{\xi_n} - m_r|_{\xi_n, \varrho} \right) = m_r|_{\varrho}^{\xi} - m_r|_{\xi, \varrho} \quad \forall \varrho \in \Omega_r.$$

Lemma 1.3.2.7. presume that $\xi_\kappa \rightarrow \xi$ and $\varrho_\kappa \rightarrow \varrho$ as $\kappa \rightarrow \infty$ in a R.M-M.S (Ω_r, m_r) . Then $m_r|_{\varrho}^{\xi} = m_r|_{\xi, \varrho}$. Further if $m_r|_{\xi}^{\xi} = m_r|_{\varrho}^{\varrho}$, then $\xi = \varrho$.

Lemma 1.3.2.8. Let $\{\xi_\kappa\}$ be a sequence in a R.M-M.S (Ω_r, m_r) and $r \in [0, 1)$ such that $m_r|_{\xi_\kappa}^{\xi_{\kappa+1}} \leq r m_r|_{\xi_{\kappa-1}}^{\xi_\kappa} \quad \forall \kappa \in \mathbb{N}$. Then,

$$(A) \lim_{\kappa \rightarrow \infty} m_r|_{\xi_{\kappa-1}}^{\xi_\kappa} = 0;$$

$$(B) \lim_{\kappa \rightarrow \infty} m_r|_{\xi_\kappa}^{\xi_\kappa} = 0;$$

$$(C) \lim_{\kappa, \kappa' \rightarrow \infty} m_r|_{\xi_\kappa, \xi_{\kappa'}} = 0;$$

(D) $\{\xi_\kappa\}$ is an m_r -Cauchy sequence.

Lemma 1.3.2.9. Let (Ω_r, m_r) be a R.M-M.S and Γ be a self-mapping on Ω_r . if $\exists_{k \in [0, 1)}$ so that $m_r|_{\Gamma \varrho}^{\Gamma \xi} \leq k m_r|_{\varrho}^{\xi} \quad \forall \xi, \varrho \in \Omega_r$ and $\{\xi_\kappa\}_{\kappa \geq 0}$ defined by $\xi_{\kappa+1} = \Gamma \xi_\kappa$, if $\xi_\kappa \rightarrow \xi$ as $\kappa \rightarrow \infty$, then $\Gamma \xi_\kappa \rightarrow \Gamma \xi$ as $\kappa \rightarrow \infty$.

Definition 1.3.2.10. Let m_r be a R.M-M.S on Ω_r , $\forall \xi_1 \in \Omega_r$ and $\varepsilon > 0$. The open-ball with center ξ_1 and radius ε is

$$B_r|_{\varepsilon}^{\xi_1} = \left\{ \xi_2 \in \Omega_r : m_r|_{\xi_2}^{\xi_1} - m_r|_{\xi_1, \xi_2} < \varepsilon \right\}.$$

Likewise, the closed-ball with the center ξ_1 and radius ε is

$$\overline{B_r|_{\varepsilon}^{\xi_1}} = \left\{ \xi_2 \in \Omega_r : m_r|_{\xi_2}^{\xi_1} - m_r|_{\xi_1, \xi_2} \leq \varepsilon \right\}.$$

Lemma 1.3.2.11. Let m_r be a R.M-M.S on Ω_r . The collection of all open balls, $\beta_b = \left\{ B_b|_{\varepsilon}^{\xi_1} \right\}_{\xi_1 \in \Omega_b}^{\varepsilon > 0}$, forms a basis on Ω_r .

Chapter 2

Fixed Point Results for Multivalued Mappings in Metric Spaces

Berinde [28] introduced the concept of a comparison function satisfying the conditions (ψ_1) and (ψ_2) from Definition 1.1.2, Liu et al. [66] presented the notions of (ψ, ϕ) -contraction-type and (ψ, ϕ) -Suzuki contraction-type and established new F.P theorem for such mappings. Xiao et al. [114] developed a new F.P results for multivalued (ψ, ϕ) -Khan contraction-type. Latif et al. [64] proposed another-type of comparison function and ensured F.P outcomes for generalized (α, ψ) -type-Meir-Keeler contractions. Subsequently, Samet et al. [101], Ali and Kamran [10], Hussain et al. [46], and Alsulami et al. [15] further extended these ideas. The findings in this chapter build upon the results in Shoaib et al. [106], Alsulami et al. [15], Xiao et al. [114], Wardowski [112], Fisher [34], Karapinar and Fulga [57]. The chapter is outlined below:

In sections 2.1 and 2.2, we examine the link between (ψ, ϕ) -integral Khan contraction-type and θ -contraction w.r.t G.D.Ps. We establish F.P outcomes for multivalued-maps that achieve both contractions along with a discussion on the solution of NFDE. In section 2.3, we derive new C.F.P theorems involving Ćirić-Type rational $(\alpha_*, \eta_*, \psi, \phi)$ -contractions on closed ball of M.S, and introduce the notion of semi α_* -admissible mapping w.r.t η . In section 2.4, we explore almost Fisher-type multivalued $\mathcal{F}_{\mathfrak{R}}$ -contraction endowed with a Γ -transitive binary relation $\mathfrak{R}$, and establish related F.P theorems in M.Ss.

The results given in this chapter have been published in [80, 82, 84].

2.1 Fixed points of (ψ, ϕ) -integral Khan-type contractive mappings with generalized dynamic process

In this section, we derive F.Ps for (ψ, ϕ) -integral Khan-type and θ -integral multi-valued contractive mappings w.r.t. G.D.P $\mathcal{L}|_{\xi_0}^{f, \Gamma}$ in C-M.Ss.

Definition 2.1.1. Let $f : \Omega \rightarrow \Omega$ be a map on a M.S $(\Omega, d|)$ and $\Gamma : \Omega \rightarrow CB|^\Omega$ is termed an (ψ, ϕ) -integral Khan-type multi-valued contractive mapping w.r.t. G.D.P $\mathcal{L}|_{\xi_0}^{f, \Gamma}$, $\xi_0 \in \Omega$, if

$\exists \phi \in \Phi, \psi \in \Psi$ and $\mu \in \Upsilon$ such that $\forall_{\xi_\kappa, \xi_{\kappa+1} \in \mathcal{L}|_{\xi_0}^{f, \Gamma}, \kappa \in \mathbb{N}}$.

(i) If $\max \left\{ D|_{\Gamma \xi_{\kappa+1}}^{f \xi_\kappa}, D|_{\Gamma \xi_\kappa}^{f \xi_{\kappa+1}} \right\} \neq 0$, then $D|_{\Gamma \xi_{\kappa+1}}^{f \xi_\kappa} \neq 0$ and

$$d|_{f \xi_{\kappa+1}}^{f \xi_\kappa} > 0 \Rightarrow \phi \left(\int_0^{d|_{f \xi_{\kappa+1}}^{f \xi_\kappa}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{M|_{\xi_\kappa}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right], \quad (2.1)$$

where

$$M|_{\xi_\kappa}^{\xi_{\kappa-1}} = \frac{D|_{\Gamma \xi_{\kappa-1}}^{f \xi_{\kappa-1}} \cdot D|_{\Gamma \xi_\kappa}^{f \xi_{\kappa-1}} + D|_{\Gamma \xi_\kappa}^{f \xi_\kappa} \cdot D|_{\Gamma \xi_{\kappa-1}}^{f \xi_\kappa}}{\max \left\{ D|_{\Gamma \xi_\kappa}^{f \xi_{\kappa-1}}, D|_{\Gamma \xi_{\kappa-1}}^{f \xi_\kappa} \right\}}. \quad (2.2)$$

(ii) If $\max \left\{ D|_{\Gamma \xi_{\kappa+1}}^{f \xi_\kappa}, D|_{\Gamma \xi_\kappa}^{f \xi_{\kappa+1}} \right\} = 0$, then $D|_{\Gamma \xi_{\kappa+1}}^{f \xi_\kappa} = 0$.

Lemma 2.1.2. Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow \Omega$ be a self-map. Thus, the next propositions are equivalent:

(i) Γ is a θ -integral contractive map with $\theta \in \Theta$;

(ii) Γ is an (ψ, ϕ) -integral contractive map with $\psi \in \Psi$ and $\phi \in \Phi$.

Proof. (i) $\implies$ (ii) Let Γ be an θ -integral contractive map, then $\exists_{\theta \in \Theta, \mu \in \Upsilon, k \in (0,1)}$, such

that $\forall_{\xi_\kappa, \xi_{\kappa+1} \in \Omega}$, we have

$$d|_{\Gamma \xi_{\kappa+1}}^{\Gamma \xi_\kappa} \neq 0 \implies \theta \left(\int_0^{d|_{\Gamma \xi_{\kappa+1}}^{\Gamma \xi_\kappa}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{d|_{\xi_{\kappa+1}}^{\xi_\kappa}} \mu(r) dr \right) \right]^k.$$

Using $\phi(r) = \ln \theta(r)$ and $\psi(r) = kr$, then it is immediate to verify that Γ is an (ψ, ϕ) -integral

contractive map and

$$\begin{aligned}
& \theta \left(\int_0^{d_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_{\kappa}}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{d_{\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \right]^k \\
& \Leftrightarrow \ln \theta \left(\int_0^{d_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_{\kappa}}} \mu(r) dr \right) \leq k \ln \left[\theta \left(\int_0^{d_{\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \right] \\
& \Leftrightarrow \phi \left(\int_0^{d_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_{\kappa}}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{d_{\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \right].
\end{aligned}$$

Hence, Γ is an (ψ, ϕ) -integral contractive map.

(ii) $\Rightarrow$ (i) Let Γ be an (ψ, ϕ) -integral contractive map, then $\exists \psi \in \Psi, \phi \in \Phi, \mu \in \Upsilon$, such that $\forall_{\xi_{\kappa}, \xi_{\kappa+1} \in \Omega}$, we have

$$d_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_{\kappa}} \neq 0 \Rightarrow \phi \left(\int_0^{d_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_{\kappa}}} \varphi(r) dr \right) \leq \psi \left[\phi \left(\int_0^{d_{\xi_{\kappa+1}}^{\xi_{\kappa}}} \varphi(r) dr \right) \right].$$

Using $\theta(r) = e^{\phi(r)}$ and $\psi(r) = kr$, then it is immediate to verify that Γ is a θ -integral contractive map and

$$\begin{aligned}
& \phi \left(\int_0^{d_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_{\kappa}}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{d_{\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \right] \\
& \Leftrightarrow \theta \left(\int_0^{d_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_{\kappa}}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{d_{\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \right]^k.
\end{aligned}$$

Hence, Γ is a θ -integral contractive map.

Lemma 2.1.3. Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multi-valued map. Then, the next propositions are equivalent:

(i) Γ is a θ -integral multi-valued contractive map with $\theta \in \Theta$;

(ii) Γ is an (ψ, ϕ) -integral multi-valued contractive map with $\psi \in \Psi$ and $\phi \in \Phi$. ■

Proof. We follow the similar way in the proof of Lemma 2.1.2. ■

Theorem 2.1.4. Presume that $(\Omega, d|)$ is a C-M.S and $f : \Omega \rightarrow \Omega$ is a non-linear self-map. Let $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multivalued-mapping satisfying (2.1) $\forall_{\xi_\kappa, \xi_{\kappa+1} \in \mathcal{L}|_{\xi_0}^{f, \Gamma}, \kappa \in \mathbb{N}}$. Then, Γ & f possess a C.F.P ξ^* so that $\lim_{\kappa \rightarrow \infty} f\xi_\kappa = \xi^* \in \Gamma\xi^*$.

Proof. As $\xi_0 \in \mathcal{L}|_{\xi_0}^{\Gamma, f}$, if $\exists_{\kappa_0 \in \mathbb{N}}$ so that $\xi_{\kappa_0} = \xi_{\kappa_0+1}$, then the C.F.P of Γ and f is done. We form a sequence according to next G.D.P:

$$\mathcal{L}|_{\xi_0}^{\Gamma, f} = \left\{ (\xi_\kappa)_{\kappa \in \mathbb{N} \cup \{0\}} : \xi_{\kappa+1} = f\xi_\kappa \in \Gamma\xi_{\kappa-1}, \forall_{\kappa \in \mathbb{N}} \right\}.$$

Now, we take $\xi_\kappa \neq \xi_{\kappa+1}$, then $d|_{f\xi_{\kappa+1}}^{f\xi_\kappa} = d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} > 0 \forall_{\kappa \in \mathbb{N}}$. Utilising (2.1), we get

$$\begin{aligned} \phi \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_\kappa}} \mu(r) dr \right) &\leq \psi \left[\phi \left(\int_0^{M|_{\xi_\kappa}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right] \\ &= \psi \left[\phi \left(\int_0^{\frac{D|_{\Gamma\xi_{\kappa-1}}^{f\xi_{\kappa-1}} \cdot D|_{\Gamma\xi_\kappa}^{f\xi_{\kappa-1}} + D|_{\Gamma\xi_\kappa}^{f\xi_\kappa} \cdot D|_{\Gamma\xi_{\kappa-1}}^{f\xi_\kappa}}{\max\{D|_{\Gamma\xi_\kappa}^{f\xi_{\kappa-1}}, D|_{\Gamma\xi_{\kappa-1}}^{f\xi_\kappa}\}}} \mu(r) dr \right) \right] \\ &= \psi \left[\phi \left(\int_0^{d|_{f\xi_\kappa}^{f\xi_{\kappa-1}}} \mu(r) dr \right) \right] < \phi \left(\int_0^{d|_{f\xi_\kappa}^{f\xi_{\kappa-1}}} \mu(r) dr \right). \end{aligned} \tag{2.3}$$

From (2.3) and (ϕ_1) , we get

$$\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_\kappa}} \mu(r) dr < \int_0^{d|_{f\xi_\kappa}^{f\xi_{\kappa-1}}} \mu(r) dr.$$

Consequently, the sequence $\left\{ \int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_\kappa}} \mu(r) dr \right\}$ is decreasing and bounded from below. There-

fore, $\exists \varrho \geq 0$ such that $\lim_{\kappa \rightarrow \infty} \int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr = \varrho$. Assume that $\varrho > 0$. Thus, from (2.3), we have

$$\begin{aligned}
\phi \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr \right) &\leq \psi \left[\phi \left(\int_0^{d|_{f\xi_{\kappa}}^{f\xi_{\kappa-1}}} \mu(r) dr \right) \right] \\
&\leq \psi^2 \left[\phi \left(\int_0^{d|_{f\xi_{\kappa-1}}^{f\xi_{\kappa-2}}} \mu(r) dr \right) \right] \\
&\vdots \\
&\leq \psi^{\kappa} \left[\phi \left(\int_0^{d|_{f\xi_1}^{f\xi_0}} \mu(r) dr \right) \right].
\end{aligned} \tag{2.4}$$

Taking $\kappa \rightarrow \infty$ in (2.4), and as $\psi \in \Psi$, we have

$$0 \leq \lim_{\kappa \rightarrow \infty} \phi \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr \right) \leq \lim_{\kappa \rightarrow \infty} \psi^{\kappa} \left[\phi \left(\int_0^{d|_{f\xi_1}^{f\xi_0}} \mu(r) dr \right) \right] = 0,$$

hence

$$\lim_{\kappa \rightarrow \infty} \phi \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr \right) = 0.$$

From (ϕ_2) , we get

$$\lim_{\kappa \rightarrow \infty} \int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr = 0.$$

From Lemma 1.1.31, we get

$$\lim_{\kappa \rightarrow \infty} d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}} = \lim_{\kappa \rightarrow \infty} d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} = 0. \tag{2.5}$$

Now, we verify that $\{\xi_\kappa\}$ is a Cauchy sequence in $(\Omega, d|)$. Assume on the contrary that $\{\xi_\kappa\}$ is not Cauchy, then $\exists_{\epsilon>0}$ so that $\forall_{p \in \mathbb{N}}, \exists_{\kappa(p), \varkappa(p) \in \mathbb{N}}$ with $\kappa(p) > \varkappa(p) > p$ attaining

$$d|_{f_{\xi_{\varkappa(p)}}}^{f_{\xi_{\kappa(p)}}} \geq \epsilon. \quad (2.6)$$

Let $\varkappa(p)$ to be the least integer exceeding $\kappa(p)$ and attaining (2.6), such that

$$d|_{f_{\xi_{\varkappa(p)-1}}}^{f_{\xi_{\kappa(p)}}} < \epsilon, \quad \forall_{p \in \mathbb{N}}. \quad (2.7)$$

Using the triangle inequality in (2.6), we get

$$\epsilon \leq d|_{f_{\xi_{\varkappa(p)}}}^{f_{\xi_{\kappa(p)}}} \leq d|_{f_{\xi_{\varkappa(p)+1}}}^{f_{\xi_{\kappa(p)}}} + d|_{f_{\xi_{\varkappa(p)}}}^{f_{\xi_{\varkappa(p)+1}}}. \quad (2.8)$$

Letting $p \rightarrow \infty$ in (2.8), using (2.5) and (2.7), we obtain

$$\lim_{p \rightarrow \infty} d|_{f_{\xi_{\varkappa(p)+1}}}^{f_{\xi_{\kappa(p)}}} \geq \epsilon. \quad (2.9)$$

Thus, $\exists_{p_1 \in \mathbb{N}}$, such that

$$d|_{f_{\xi_{\varkappa(p)+1}}}^{f_{\xi_{\kappa(p)}}} > \frac{\epsilon}{2}, \quad \forall_{p > p_1}.$$

This implies that

$$\max \left\{ d|_{f_{\xi_{\varkappa(p)+1}}}^{f_{\xi_{\kappa(p)}}}, d|_{f_{\xi_{\varkappa(p)}}}^{f_{\xi_{\kappa(p)+1}}} \right\} > \frac{\epsilon}{2}, \quad \forall_{p > p_1}. \quad (2.10)$$

From (2.6) and applying triangle inequality again, we have

$$\epsilon \leq d|_{f_{\xi_{\varkappa(p)}}}^{f_{\xi_{\kappa(p)}}} \leq d|_{f_{\xi_{\kappa(p)+1}}}^{f_{\xi_{\kappa(p)}}} + d|_{f_{\xi_{\varkappa(p)+1}}}^{f_{\xi_{\kappa(p)+1}}} + d|_{f_{\xi_{\varkappa(p)}}}^{f_{\xi_{\varkappa(p)+1}}}.$$

Taking $p \rightarrow \infty$ and using (2.5), we get

$$\lim_{p \rightarrow \infty} d|_{f_{\xi_{\varkappa(p)+1}}}^{f_{\xi_{\kappa(p)+1}}} \geq \epsilon.$$

Thus, $\exists_{p_2 \in \mathbb{N}}$, such that

$$d|_{f_{\xi_{\varkappa(p)+1}}}^{f_{\xi_{\kappa(p)+1}}} > \frac{\epsilon}{2}, \quad \forall_{p > p_2}. \quad (2.11)$$

Now, from (2.11), $(\phi 1)$ and (2.1), we have

$$\begin{aligned} \phi \left(\int_0^{\frac{\epsilon}{2}} \mu(r) dr \right) &\leq \phi \left(\int_0^{d|_{f\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{M|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)+1}}} \mu(r) dr \right) \right] \\ &< \phi \left(\int_0^{\frac{D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} + D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}}{\max \left\{ D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}, D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \right\}}} \mu(r) dr \right). \end{aligned} \quad (2.12)$$

On the other hand, from (2.10), we find that

$$0 \leq \frac{D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} + D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}}{\max \left\{ D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}, D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \right\}} \leq D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} + D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}.$$

Let $p \rightarrow \infty$ above and using (2.5), we deduce that

$$\lim_{p \rightarrow \infty} \frac{D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} + D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}}{\max \left\{ D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}, D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \right\}} = 0.$$

So, $\exists_{p_3 \in \mathbb{N}}$, such that

$$\frac{D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} + D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}}{\max \left\{ D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}, D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \right\}} < \frac{\epsilon}{2}, \quad \forall p > p_3. \quad (2.13)$$

Fusing (2.12) and (2.13), we get

$$\phi \left(\int_0^{\frac{\epsilon}{2}} \mu(r) dr \right) < \phi \left(\int_0^{\frac{D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} + D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \cdot D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}}{\max \left\{ D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}}, D|_{\Gamma\xi_{\kappa(p)+1}}^{f\xi_{\kappa(p)+1}} \right\}}} \mu(r) dr \right) < \phi \left(\int_0^{\frac{\epsilon}{2}} \mu(r) dr \right),$$

$\forall p > \max\{p_1, p_2, p_3\}$, a contradiction. Thus, $\{\xi_\kappa\}$ is a Cauchy in a C-M.S $(\Omega, d|)$. So, $\exists \xi^* \in \Omega$ so that $\lim_{\kappa \rightarrow \infty} f\xi_\kappa = \xi^*$, that is $\lim_{\kappa \rightarrow \infty} d|_{\xi^*}^{\xi_\kappa} = 0$. By virtue of (2.1), we infer that

$$\begin{aligned}
\phi \left(\int_0^{d|_{f\xi^*}^{\xi^*}} \mu(r) dr \right) &= \lim_{\kappa \rightarrow \infty} \phi \left(\int_0^{d|_{f\xi^*}^{f\xi_{\kappa+1}}} \mu(r) dr \right) \leq \lim_{\kappa \rightarrow \infty} \psi \left[\phi \left(\int_0^{M|_{\xi^*}^{\xi_\kappa}} \mu(r) dr \right) \right] \\
&= \lim_{\kappa \rightarrow \infty} \psi \left[\phi \left(\int_0^{\frac{D|_{\Gamma\xi_\kappa}^{f\xi_\kappa} \cdot D|_{\Gamma\xi^*}^{f\xi_\kappa} + D|_{\Gamma\xi^*}^{f\xi^*} \cdot D|_{\Gamma\xi_\kappa}^{f\xi^*}}{\max\{D|_{\Gamma\xi^*}^{f\xi_\kappa}, D|_{\Gamma\xi_\kappa}^{f\xi^*}\}}} \mu(r) dr \right) \right] \\
&< \lim_{\kappa \rightarrow \infty} \phi \left(\int_0^{\frac{d|_{f\xi_{\kappa+1}}^{f\xi_\kappa} \cdot d|_{f\xi^*}^{f\xi_\kappa} + d|_{f\xi^*}^{f\xi^*} \cdot d|_{f\xi_{\kappa+1}}^{f\xi^*}}{\max\{d|_{f\xi^*}^{f\xi_\kappa}, d|_{f\xi_{\kappa+1}}^{f\xi^*}\}}} \mu(r) dr \right),
\end{aligned} \tag{2.14}$$

which together with (ϕ_1) and (ϕ_3) yield that $\int_0^{d|_{f\xi^*}^{\xi^*}} \mu(r) dr = 0$. According to Lemma 1.1.31, we deduce that $d|_{f\xi^*}^{\xi^*} = 0$, that is $\xi^* = f\xi^* \in \Gamma\xi^*$. Therefore, f and Γ have a C.F.P ξ^* . ■

Example 2.1.5. Let $\Omega = [0, \infty)$ with the usual metric $d|_{\xi_2}^{\xi_1} = |\xi_1 - \xi_2|$. Then, $(\Omega, d|)$ is a C-M.S. Consider the maps $f : \Omega \rightarrow \Omega$ by $f\xi = \frac{\xi}{4}$ and $\Gamma : \Omega \rightarrow CB|^\Omega$ by $\Gamma\xi = \left[0, \frac{4\xi}{3}\right]$. The sequence $\{\xi_\kappa\}$ can be defined as $\xi_\kappa = \xi_0 l^{\kappa-1} \forall \kappa \in \mathbb{N}$ with $\xi_0 = 4$ and $l = \frac{1}{4}$, so that $\xi_1 = 4$ and

$$\begin{aligned}
\xi_2 &= 1 = f\xi_1 \in \Gamma\xi_0 = \left[0, \frac{4}{3}\right], \quad \xi_3 = \frac{1}{4} = f\xi_2 \in \Gamma\xi_1 = \left[0, \frac{4}{3}\right]; \\
\xi_4 &= \frac{1}{16} = f\xi_3 \in \Gamma\xi_2 = \left[0, \frac{1}{3}\right], \quad \xi_5 = \frac{1}{64} = f\xi_4 \in \Gamma\xi_3 = \left[0, \frac{1}{12}\right].
\end{aligned}$$

Proceeding in this aproach, we get

$$\mathcal{L}|_4^{\Gamma, f} = \left\{1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \dots\right\},$$

is a G.D.P of f & Γ having a start point $\xi_0 = 4$. Define $\mu(t) = \frac{16^t}{3} \in \Upsilon$, $\phi(r) = 4r \in \Phi$ and $\psi(r) = \frac{2r}{\ln 16} \in \Psi$. $\forall_{\xi_\kappa, \xi_{\kappa+1} \in \mathcal{L}_4^{\Gamma, f}}$ and $d|_{f\xi_{\kappa+1}}^{f\xi_\kappa} > 0$, we have

$$\begin{aligned} \phi \left(\int_0^{d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}} \mu(r) dr \right) &= \phi \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_\kappa}} \mu(r) dr \right) = \phi \left(\int_0^{\frac{1}{4}|\xi_\kappa - \xi_{\kappa+1}|} \mu(r) dr \right) = 4 \int_0^{\frac{1}{4}|\xi_\kappa - \xi_{\kappa+1}|} \frac{16^r}{3} dr \\ &= 4 \left(\frac{16^{\frac{1}{4}|\xi_\kappa - \xi_{\kappa+1}|}}{3 \ln 16} - \frac{1}{3 \ln 16} \right) \leq \frac{2}{\ln 16} \left[4 \left(\frac{16^{\frac{1}{4}|\xi_\kappa - \xi_{\kappa+1}|}}{3 \ln 16} - \frac{1}{3 \ln 16} \right) \right] \\ &= \frac{2}{\ln 16} \left[4 \left(\int_0^{d|_{\xi_{\kappa+1}}^{\xi_\kappa}} \mu(r) dr \right) \right] \leq \psi \left[\phi \left(\int_0^{M|_{\xi_\kappa}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right]. \end{aligned}$$

We can select any two arbitrary points $\xi_1 = 1$, $\xi_2 = \frac{1}{4} \in \mathcal{L}_4^{\Gamma, f}$ to obtain the following:

$$\phi \left(\int_0^{d|_{f\xi_2}^{f\xi_1}} \mu(r) dr \right) = \phi \left(\int_0^{d|_{f\frac{1}{4}}^{f1}} \mu(r) dr \right) = 4 \left(\frac{16^{\frac{3}{16}}}{3 \ln 16} - \frac{1}{3 \ln 16} \right) \approx 0.33.$$

Therefore, we have

$$M|_{\frac{1}{4}}^1 = \frac{D|_{\Gamma 1}^{f1} \cdot D|_{\Gamma \frac{1}{4}}^{f1} + D|_{\Gamma \frac{1}{4}}^{f\frac{1}{4}} \cdot D|_{\Gamma 1}^{f\frac{1}{4}}}{\max \left\{ D|_{\Gamma \frac{1}{4}}^{f1}, D|_{\Gamma 1}^{f\frac{1}{4}} \right\}} = \frac{17}{64},$$

so that

$$\psi \left[\phi \left(\int_0^{\frac{17}{64}} \frac{16^r}{3} dr \right) \right] = \frac{2}{\ln 16} \left[4 \left(\frac{16^{\frac{17}{64}}}{3 \ln 16} - \frac{1}{3 \ln 16} \right) \right] \approx 0.38.$$

Therefore, we conclude that

$$\phi \left(\int_0^{d|_{f\frac{1}{4}}^{f1}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{M|_{\frac{1}{4}}^1} \mu(r) dr \right) \right].$$

Hence, the condition of Theorem 2.1.4 is achieved, so that f & Γ have a C.F.P $0 = f0 \in \Gamma 0$.

Otherwise, we get

$$\phi \left(\int_0^{d_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}} \mu(r) dr \right) > \psi \left[\phi \left(\int_0^{M_{\xi_{\kappa}}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right].$$

For instance, if $\xi_1 = 0$ and $\xi_2 = 2$, then we obtain

$$\phi \left(\int_0^{d_{f\xi_2}^{f\xi_1}} \mu(r) dr \right) = \phi \left(\int_0^{d_{f2}^{f0}} \mu(r) dr \right) = 4 \left(\frac{16^{\frac{1}{2}}}{3 \ln 16} - \frac{1}{3 \ln 16} \right) \approx 1.44.$$

Therefore, we have

$$M_2^0 = \frac{d_{\Gamma 0}^{f0} \cdot d_{\Gamma 2}^{f0} + d_{\Gamma 2}^{f2} \cdot d_{\Gamma 0}^{f2}}{\max \left\{ d_{\Gamma 2}^{f0}, d_{\Gamma 0}^{f2} \right\}} = \frac{1}{2},$$

so that

$$\psi \left[\phi \left(\int_0^{\frac{1}{2}} \frac{16^r}{3} ds \right) \right] = \frac{2}{\ln 16} \left[4 \left(\frac{16^{\frac{1}{2}}}{3 \ln 16} - \frac{1}{3 \ln 16} \right) \right] \approx 1.04.$$

Therefore, we conclude that

$$\phi \left(\int_0^{d_{f2}^{f0}} \mu(r) dr \right) > \psi \left[\phi \left(\int_0^{M_2^0} \mu(r) dr \right) \right].$$

That is, the premise of Theorem 2.1.4 is not attained for some $\xi_{\kappa}, \xi_{\kappa+1} \in \Omega$.

Definition 2.1.6. Let $f : \Omega \rightarrow \Omega$ be a map on a M.S $(\Omega, d|)$ and $\Gamma : \Omega \rightarrow CB|\Omega$ is termed an θ -integral Khan-type multi-valued contractive map w.r.t. the G.D.P $\mathcal{L}_{\xi_0}^{\Gamma, f}$, $\xi_0 \in \Omega$, if

$\exists_{k \in (0,1), \theta \in \Theta}$ and $\mu \in \Upsilon$ so that $\forall_{\xi_{\kappa}, \xi_{\kappa+1} \in \mathcal{L}_{\xi_0}^{\Gamma, f}, \kappa \in \mathbb{N}}$.

(i) If $\max \left\{ D_{\Gamma \xi_{\kappa+1}}^{f\xi_{\kappa}}, D_{\Gamma \xi_{\kappa}}^{f\xi_{\kappa+1}} \right\} \neq 0$, then $D_{\Gamma \xi_{\kappa+1}}^{f\xi_{\kappa}} \neq 0$ and

$$d_{f\xi_{\kappa+1}}^{f\xi_{\kappa}} > 0 \Rightarrow \theta \left(\int_0^{d_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{M_{\xi_{\kappa}}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right]^k, \quad (2.15)$$

where $M_{\xi_{\kappa}}^{\xi_{\kappa-1}}$ is defined as in (2.2)

(ii) If $\max \left\{ D|_{\Gamma\xi_{\kappa+1}}^{f\xi_{\kappa}}, D|_{\Gamma\xi_{\kappa}}^{f\xi_{\kappa+1}} \right\} = 0$, then $D|_{\Gamma\xi_{\kappa+1}}^{f\xi_{\kappa}} = 0$.

Theorem 2.1.7. Presume that $(\Omega, d|)$ is a C-M.S and $f : \Omega \rightarrow \Omega$ is a non-linear self-map. Let $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multivalued-mapping achieving (2.15). Then, Γ & f possess a C.F.P ξ^* so that $\lim_{\kappa \rightarrow \infty} f\xi_{\kappa} = \xi^* \in \Gamma\xi^*$.

Proof. As $\xi_0 \in \mathcal{L}|_{\xi_0}^{\Gamma, f}$, if there is $\kappa_0 \in \mathbb{N}$ so that $\xi_{\kappa_0} = \xi_{\kappa_0+1}$, then the existence of a F.P is done. We form a sequence according to next G.D.P:

$$\mathcal{L}|_{\xi_0}^{\Gamma, f} = \left\{ (\xi_{\kappa})_{\kappa \in \mathbb{N} \cup \{0\}} : \xi_{\kappa+1} = f\xi_{\kappa} \in \Gamma\xi_{\kappa-1}, \forall \kappa \in \mathbb{N} \right\}.$$

Now, we take $\xi_{\kappa} \neq \xi_{\kappa+1}$, then $d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}} = d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} > 0 \forall \kappa \in \mathbb{N}$. Utilising (2.15), we have

$$\theta \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{d|_{f\xi_{\kappa}}^{f\xi_{\kappa-1}}} \mu(r) dr \right) \right]^k. \quad (2.16)$$

Applying $\ln$ to each side of (2.16), referring to Lemma 2.1.3 with taking $\psi(r) = kr \in \Psi$ and $\phi(r) = \ln \theta(r) \in \Phi$, leads to

$$\phi \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{d|_{f\xi_{\kappa}}^{f\xi_{\kappa-1}}} \mu(r) dr \right) \right].$$

Henceforth, by tracing the line of proof employed in the Theorem 2.1.4, we deduct that f & Γ possess a C.F.P ξ^* .

■

Example 2.1.8. Let $\Omega = [0, \infty)$ and $d|_{\xi_2}^{\xi_1} = |\xi_1 - \xi_2|$, then $(\Omega, d|)$ is a C-M.S. Consider $\theta(r) = e^{\sqrt{r}} \in \Theta$ and the maps $f : \Omega \rightarrow \Omega$ by $f\xi = \frac{\xi}{2}$ and $\Gamma : \Omega \rightarrow CB|^\Omega$ by $\Gamma\xi = \left[0, \frac{2\xi}{3}\right]$. The sequence $\{\xi_{\kappa}\}$ can be defined as $\xi_{\kappa} = \xi_0 l^{\kappa-1} \forall \kappa \in \mathbb{N}$ with $\xi_0 = 2$ and $l = \frac{1}{2}$ so that $\xi_1 = 2$ and

$$\begin{aligned} \xi_2 &= 1 = f\xi_1 \in \Gamma\xi_0 = \left[0, \frac{4}{3}\right], \quad \xi_3 = \frac{1}{2} = f\xi_2 \in \Gamma\xi_1 = \left[0, \frac{4}{3}\right]; \\ \xi_4 &= \frac{1}{4} = f\xi_3 \in \Gamma\xi_2 = \left[0, \frac{2}{3}\right], \quad \xi_5 = \frac{1}{8} = f\xi_4 \in \Gamma\xi_3 = \left[0, \frac{1}{3}\right]. \end{aligned}$$

Proceeding in this approach, we get

$$\mathcal{L}|_2^{\Gamma, f} = \left\{ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\},$$

is a G.D.P of f and Γ having start point $\xi_0 = 2$. Let $\mu(t) = \frac{1}{3}$. For $\xi_\kappa, \xi_{\kappa+1} \in \mathcal{L}|_4^{f, \Gamma}$ and $d|_{f\xi_{\kappa+1}}^{f\xi_\kappa} > 0$, we have

$$\begin{aligned} \theta \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_\kappa}} \mu(r) dr \right) &= \theta \left(\int_0^{\frac{1}{2}|\xi_\kappa - \xi_{\kappa+1}|} \frac{1}{3} dr \right) = \theta \left(\frac{|\xi_\kappa - \xi_{\kappa+1}|}{6} \right) \\ &= e^{\frac{1}{\sqrt{2}} \sqrt{\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_\kappa}} \mu(r) dr}} \leq \left(e^{\sqrt{\int_0^{M|_{\xi_\kappa}^{\xi_{\kappa-1}}} \mu(r) dr}} \right)^{\frac{1}{\sqrt{2}}} \\ &= \left[\theta \left(\int_0^{M|_{\xi_\kappa}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right]^k. \end{aligned}$$

We can select any arbitrary points $\xi_1 = 1$ and $\xi_2 = \frac{1}{2}$ of $\mathcal{L}|_2^{\Gamma, f}$ to obtain the following

$$\theta \left(\int_0^{d|_{f\xi_2}^{f\xi_1}} \mu(r) dr \right) = \theta \left(\int_0^{d|_{f\frac{1}{2}}^{f1}} \frac{1}{3} dr \right) = \theta \left(\int_0^{\frac{1}{4}} \frac{1}{3} dr \right) = 1.33.$$

Thereafter, we have

$$M|_{\frac{1}{2}}^1 = \frac{d|_{\Gamma 1}^{f1} \cdot d|_{\Gamma \frac{1}{2}}^{f1} + d|_{\Gamma \frac{1}{2}}^{f\frac{1}{2}} \cdot d|_{\Gamma 1}^{f\frac{1}{2}}}{\max \left\{ d|_{\Gamma \frac{1}{2}}^{f1}, d|_{\Gamma 1}^{f\frac{1}{2}} \right\}} = \frac{5}{8}.$$

So that for $k = \frac{1}{\sqrt{2}}$, we get

$$\left[\theta \left(\int_0^{M|_{\frac{1}{2}}^{\frac{1}{2}}} \mu(r) dr \right) \right]^k = \left[\theta \left(\int_0^{\frac{5}{8}} \frac{1}{3} dr \right) \right]^k = \left[\theta \left(\frac{5}{24} \right) \right]^{\frac{1}{\sqrt{2}}} \approx 1.38.$$

Therefore, we conclude

$$\theta \left(\int_0^{d|_{f\frac{1}{2}}^{f\frac{1}{2}}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{M|_{\frac{1}{2}}^{\frac{1}{2}}} \mu(r) dr \right) \right]^k.$$

Hence, the required condition of Theorem 2.1.7 is satisfied, and $0 = f0 \in \Gamma 0$ is a C.F.P of f and Γ .

Otherwise, we have

$$\theta \left(\int_0^{d|_{f\xi_{\kappa+1}}^{f\xi_{\kappa}}} \mu(r) dr \right) > \left[\theta \left(\int_0^{M|_{\xi_{\kappa}}^{\xi_{\kappa}-1}} \mu(r) dr \right) \right]^k.$$

For instance, if $\xi_1 = 0$ and $\xi_2 = 2$, then we obtain

$$\theta \left(\int_0^{d|_{f\xi_2}^{f\xi_1}} \mu(r) dr \right) = \left[\theta \left(\int_0^{d|_{f2}^{f0}} \mu(r) dr \right) \right]^k = \left[\theta \left(\int_0^{d|_1^0} \frac{1}{3} dr \right) \right]^k = e^{\frac{1}{\sqrt{3}}} \approx 1.8.$$

Therefore, we have

$$M|_2^0 = \frac{d|_{\Gamma 0}^{f0} \cdot d|_{\Gamma 2}^{f0} + d|_{\Gamma 2}^{f2} \cdot d|_{\Gamma 0}^{f2}}{\max \left\{ d|_{\Gamma 2}^{f0}, d|_{\Gamma 0}^{f2} \right\}} = 1.$$

So that for $k = \frac{1}{\sqrt{2}}$, we get

$$\begin{aligned} \left[\theta \left(\int_0^{M|_2^0} \mu(r) dr \right) \right]^k &= \left[\theta \left(\int_0^1 \frac{1}{3} dr \right) \right]^k \\ &= e^{\frac{1}{\sqrt{6}}} \approx 1.5. \end{aligned}$$

Therefore, we conclude

$$\theta \left(\int_0^{d|_{f_2}^{f_0}} \mu(r) dr \right) > \left[\theta \left(\int_0^{M|_2^0} \mu(r) dr \right) \right]^k.$$

That is, the premise of Theorem 2.1.7 is not satisfied for some $\xi_\kappa, \xi_{\kappa+1} \in \Omega$.

Corollary 2.1.9. Let $\Gamma : \Omega \rightarrow CB|\Omega$ be a multivalued-mapping on a M.S $(\Omega, d|)$. Then, Γ is termed an (ψ, ϕ) -integral Khan-type contractive map w.r.t. D.P $\mathcal{L}|_{\xi_0}^\Gamma$, $\xi_0 \in \Omega$, if $\exists_{\phi \in \Phi, \psi \in \Psi, \mu \in \Upsilon}$ so that $\forall_{\xi_\kappa, \xi_{\kappa+1} \in \mathcal{L}|_{\xi_0}^\Gamma, \kappa \in \mathbb{N}}$.

(i) If $\max \left\{ D|_{\Gamma\xi_{\kappa+1}}^{\xi_\kappa}, D|_{\Gamma\xi_\kappa}^{\xi_{\kappa+1}} \right\} \neq 0$, then $\Gamma\xi_\kappa \neq \Gamma\xi_{\kappa+1}$ and

$$\hat{H}|_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_\kappa} > 0 \Rightarrow \phi \left(\int_0^{\hat{H}|_{\Gamma\xi_{\kappa+1}}^{\Gamma\xi_\kappa}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{M|_{\xi_\kappa}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right], \quad (2.17)$$

where

$$M|_{\xi_\kappa}^{\xi_{\kappa-1}} = \frac{D|_{\Gamma\xi_{\kappa-1}}^{\xi_{\kappa-1}} \cdot D|_{\Gamma\xi_\kappa}^{\xi_{\kappa-1}} + D|_{\Gamma\xi_\kappa}^{\xi_\kappa} \cdot D|_{\Gamma\xi_{\kappa-1}}^{\xi_\kappa}}{\max \left\{ D|_{\Gamma\xi_\kappa}^{\xi_{\kappa-1}}, D|_{\Gamma\xi_{\kappa-1}}^{\xi_\kappa} \right\}}. \quad (2.18)$$

(ii) If $\max \left\{ D|_{\Gamma\xi_{\kappa+1}}^{\xi_\kappa}, D|_{\Gamma\xi_\kappa}^{\xi_{\kappa+1}} \right\} = 0$, then $\Gamma\xi_\kappa = \Gamma\xi_{\kappa+1}$.

Hence, Γ possesses an F.P ξ^* such that $\xi^* \in \lim_{\kappa \rightarrow \infty} \Gamma\xi_\kappa = \Gamma\xi^*$.

Corollary 2.2.10. Let $\Gamma : \Omega \rightarrow \Omega$ be a self-map on a M.S $(\Omega, d|)$. Then, Γ is termed an (ψ, ϕ) -integral Khan-type contractive map w.r.t. D.P $\mathcal{L}|_{\xi_0}^\Gamma$, $\xi_0 \in \Omega$, if $\exists_{\phi \in \Phi, \psi \in \Psi}$ and $\mu \in \Upsilon$ such that $\forall_{\xi_\kappa, \xi_{\kappa+1} \in \mathcal{L}|_{\xi_0}^\Gamma, \kappa \in \mathbb{N}}$.

(i) If $\max \left\{ d|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}, d|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa+1}} \right\} \neq 0$, then $\Gamma\xi_{\kappa} \neq \Gamma\xi_{\kappa+1}$ and

$$d|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}} > 0 \Rightarrow \phi \left(\int_0^{d|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{M|_{\xi_{\kappa}}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right], \quad (2.19)$$

where $M|_{\xi_{\kappa}}^{\xi_{\kappa-1}}$ is defined as in (2.18).

(ii) If $\max \left\{ d|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}, d|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa+1}} \right\} = 0$, then $\Gamma\xi_{\kappa} = \Gamma\xi_{\kappa+1}$.

Thus, Γ possesses an F.P ξ^* such that $\xi^* = \lim_{j \rightarrow \infty} \Gamma\xi_{\kappa} = \Gamma\xi^*$.

Corollary 2.1.11. Let $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multivalued-map on a M.S $(\Omega, d|)$. Then Γ is termed a θ -integral contractive map w.r.t. $D.P \mathcal{L}_{\xi_0}^\Gamma$, $\xi_0 \in \Omega$, if $\exists_{k \in (0,1), \theta \in \Theta}$ and $\mu \in \Upsilon$ such that $\forall_{\xi_{\kappa}, \xi_{\kappa+1} \in \mathcal{L}_{\xi_0}^\Gamma, \kappa \in \mathbb{N}}$

(i) If $\max \left\{ D|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}, D|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa+1}} \right\} \neq 0$, then $\Gamma\xi_{\kappa} \neq \Gamma\xi_{\kappa+1}$ and

$$\hat{H}|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}} > 0 \Rightarrow \theta \left(\int_0^{\hat{H}|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{M|_{\xi_{\kappa}}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right]^k, \quad (2.20)$$

where $M|_{\xi_{\kappa}}^{\xi_{\kappa-1}}$ is defined as in (2.18).

(ii) If $\max \left\{ D|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}, D|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa+1}} \right\} = 0$, then $\Gamma\xi_{\kappa} = \Gamma\xi_{\kappa+1}$.

Thus, Γ possesses an F.P ξ^* such that $\xi^* \in \lim_{\kappa \rightarrow \infty} \Gamma\xi_{\kappa} = \Gamma\xi^*$.

Corollary 2.1.12. Let $\Gamma : \Omega \rightarrow \Omega$ be a mapping on a M.S $(\Omega, d|)$. Then, Γ is termed an integral Khan-type θ -contraction w.r.t. $D.P \mathcal{L}_{\xi_0}^\Gamma$, if $\exists_{k \in (0,1), \theta \in \Theta}$ and $\mu \in \Upsilon$ such that $\forall_{\xi_{\kappa}, \xi_{\kappa+1} \in \mathcal{L}_{\xi_0}^\Gamma, \kappa \in \mathbb{N}}$

(i) If $\max \left\{ D|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}, D|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa+1}} \right\} \neq 0$, then $\Gamma\xi_{\kappa} \neq \Gamma\xi_{\kappa+1}$ and

$$d|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}} > 0 \Rightarrow \theta \left(\int_0^{d|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}} \mu(r) dr \right) \leq \left[\theta \left(\int_0^{M|_{\xi_{\kappa}}^{\xi_{\kappa-1}}} \mu(r) dr \right) \right]^k, \quad (2.21)$$

where $M|_{\xi_j}^{\xi_{j-1}}$ is defined as in (2.18).

(ii) If $\max \left\{ D|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa}}, D|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa+1}} \right\} = 0$, then $\Gamma\xi_{\kappa} = \Gamma\xi_{\kappa+1}$.

Thus, Γ possesses an F.P ξ^* such that $\xi^* = \lim_{\kappa \rightarrow \infty} \Gamma \xi_\kappa = \Gamma \xi^*$.

Corollary 2.1.13. Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow \Omega$ be a self-map. Thus, the next propositions are equivalent:

(i) Γ is a θ -contractive map with $\theta \in \Theta$;

(ii) Γ is an (ψ, ϕ) -contractive map with $\psi \in \Psi$ and $\phi \in \Phi$.

Corollary 2.1.14. Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multivalued mapping. Thus, the next propositions are equivalent:

(i) Γ is a multi-valued θ -contractive map with $\theta \in \Theta$;

(ii) Γ is a multi-valued (ψ, ϕ) -contractive map with $\psi \in \Psi$ and $\phi \in \Phi$.

2.2 Application to nonlinear fractional diferential equations

In this part, we apply the F.P technique to find a solution of NFDE using Corollary 2.1.10. In particular, inspired from [36, 42, 43], we recall the definition of Caputo derivative of function $\xi : [0, \infty) \rightarrow \mathbb{R}$ order $\beta > 0$, which is Considered as following:

$$D^\beta (\xi (r)) = \frac{1}{\Upsilon(n-\beta)} \int_0^r (r-k)^{n-\beta-1} \xi^{(n)}(k) dk \quad (n-1 < \beta < n, \quad n = [\beta] + 1),$$

where $[\beta]$ means the integer term of $\beta \in \mathbb{R}^+$, and Υ is a γ -function.

Hence, observe the next NFDE:

$$\begin{cases} D^\beta (\xi (r)) + \Xi|_{\xi(r)}^r = 0, & r \in I = [0, 1], \quad \beta < 1 \\ \xi(0) = \xi(1) = 0, \end{cases}, \quad (2.22)$$

where the function $\Xi : I \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$ is continuous. So, the NFDE (2.22) is equalized to the next IE $\forall_{r \in I}$:

$$\xi(r) = \int_0^1 \mathcal{U}|_k^r \Xi|_{\xi(k)}^k dk. \quad (2.23)$$

Where, $\mathcal{U} : I^2 \rightarrow \mathbb{R}$ is a G.F defined as

$$\mathcal{U}|_k^r = \begin{cases} (r(1-k))^{\alpha-1} - (r-k)^{\alpha-1}, & 0 \leq k \leq r \leq 1 \\ \frac{(r(1-k))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq r \leq k \leq 1 \end{cases}.$$

Define $d : \Omega^2 \rightarrow [0, \infty)$ such that $\Omega = C|_{\mathbb{R}}^I$ is a space of continuous functions on I by

$$d|_{\xi_2}^{\xi_1} = \|\xi_1(r) - \xi_2(r)\|_{\infty} = \sup_{r \in I} |\xi_1(r) - \xi_2(r)|, \forall \xi_1, \xi_2 \in \Omega.$$

Distinctly, $(\Omega, d|)$ is a C-M.S.

Theorem 2.2.1. Under the two postulates below, NFDE (2.22) has at least one solution $\xi^* \in \Omega$ iff Γ has an F.P.

(♣1) $\exists_{\partial \geq 1}$ so that $\forall_{q_1, q_2 \in \mathbb{R}}$, we have

$$\left| \Xi|_{q_1}^r - \Xi|_{q_2}^r \right| \leq \frac{e^{-\partial}}{\partial} M|_{q_2}^{q_1} \quad \forall_{r \in I},$$

where

$$M|_{q_2}^{q_1} = \frac{|q_1 - \Gamma q_1| \cdot |q_1 - \Gamma q_2| + |q_2 - \Gamma q_2| \cdot |q_2 - \Gamma q_1|}{\max\{|q_1 - \Gamma q_2|, |q_2 - \Gamma q_1|\}}.$$

(♣2) $\exists_{\partial \geq 1}$ so that $\max_{r, k \in I} \int_0^1 \mathcal{U}|_k^r dk \leq \frac{1}{\partial}$.

Proof. Define the operator $\Gamma : \Omega \longrightarrow \Omega$ by

$$\Gamma \xi(r) = \int_0^1 \mathcal{U}|_k^r \Xi|_{\xi(k)}^k dk \quad \forall_{r \in I}. \quad (2.24)$$

Evidently, the solvability of the IE (2.23) is equivalized to find a F.P of Γ . From above propo-

sitions, $\forall_{\xi_1, \xi_2 \in \Omega, r \in I}$, we get

$$\begin{aligned}
|\Gamma \xi_1(r) - \Gamma \xi_2(r)| &= \left| \int_0^1 \mathcal{U}|_k^r \Xi|_{\xi_1(k)}^k dk - \int_0^1 \mathcal{U}|_k^r \Xi|_{\xi_2(k)}^k dk \right| \\
&\leq \int_0^1 \left[\left| \mathcal{U}|_k^r \Xi|_{\xi_1(k)}^k - \mathcal{U}|_k^r \Xi|_{\xi_2(k)}^k \right| \right] ds \\
&\leq \int_0^1 \mathcal{U}|_k^r \frac{e^{-\partial}}{\partial} M|_{\xi_2}^{\xi_1} dk \\
&= \left[\frac{e^{-\partial}}{\partial} M|_{\xi_2}^{\xi_1} \right] \left[\int_0^1 \mathcal{U}|_k^r dk \right].
\end{aligned}$$

Consequently, we have

$$\max_{t \in I} |\Gamma \xi_1(r) - \Gamma \xi_2(r)| \leq \left[\frac{e^{-\partial}}{\partial^2} M|_{\xi_2}^{\xi_1} \right] \left(\max_{r, k \in I} \int_0^1 \mathcal{U}|_k^r dk \right),$$

implying that

$$d|_{\Gamma \xi_2}^{\Gamma \xi_1} \leq \frac{1}{\partial^2 e^\partial} M|_{\xi_2}^{\xi_1}.$$

Choosing $\psi(r) = \frac{1}{\partial^2 e^\partial} r \in \Psi$ and $\phi(r) = r \in \Phi$ to obtain

$$\phi \left(d|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \psi \left(\phi \left[M|_{\xi_2}^{\xi_1} \right] \right).$$

Furthermore, by a contractive condition in Corollary 2.1.10 with setting $\mu(r) = 1 \forall_{r \in \mathbb{R}}$, we get

$$\phi \left(\int_0^{d|_{\Gamma \xi_{j+1}}^{\Gamma \xi_j}} \mu(r) dr \right) \leq \psi \left[\phi \left(\int_0^{M|_{\xi_j}^{\xi_{j-1}}} \mu(r) dr \right) \right].$$

Hence, the hypotheses of Theorem 2.2.1 are realized and Γ possesses an F.P which is a solution of (2.22). ■

Example 2.2.2. Consider the next IE

$$\xi(r) = \frac{e^{-\partial}}{12\partial} \int_0^1 (r(1-k))^{\alpha-1} - (r-k)^{\alpha-1} \xi(k) dk, \quad 0 \leq k \leq r \leq 1. \quad (2.25)$$

Let $\Gamma : \Omega \rightarrow \Omega$ be defined by

$$\Gamma\xi(r) = \frac{e^{-\partial}}{12\partial} \int_0^1 (r(1-k))^{\alpha-1} - (r-k)^{\alpha-1} \xi(k) dk, \quad 0 \leq k \leq r \leq 1.$$

By specifying $\mathcal{U}|_k^r = \frac{1}{12} (r(1-k))^{\alpha-1} - (r-k)^{\alpha-1}$ and $\Xi|_{\xi(k)}^k = \frac{e^{-\partial}}{\partial} \xi(k)$ in Theorem 2.2.1, we obtain that the IE (2.25) has at least one solution in Ω . Now, we have

$$\begin{aligned} \left| \Xi|_{\xi_1(k)}^k - \Xi|_{\xi_2(k)}^k \right| &= \left| \frac{e^{-\partial}}{\partial} \xi_1(k) - \frac{e^{-\partial}}{\partial} \xi_2(k) \right| \\ &\leq \left| \frac{2e^{-\partial}}{\partial} (\xi_1(k) - \xi_2(k)) \right| \\ &= \frac{2e^{-\partial}}{\partial} d|_{\xi_2}^{\xi_1} \\ &\leq \frac{2e^{-\partial}}{\partial} M|_{\xi_2}^{\xi_1}. \end{aligned}$$

For $\partial = 2$, $\alpha = 2$ and $0 \leq k \leq r \leq 1$, we get

$$\max_{r,k \in I} \int_0^1 \mathcal{U}|_k^r dk = \max_{r,k \in I} \int_0^1 \frac{1}{12} \left[(r(1-k))^{\alpha-1} - (r-k)^{\alpha-1} \right] dk \leq \frac{1}{2}.$$

Hence, the hypotheses of Theorem 2.2.1 are satisfied, and Γ possesses an F.P which is a solution of (2.25).

2.3 Common fixed point theorems for Ćirić-type contraction on closed ball

Let (Ω, d) be a M.S, $\xi_0 \in \Omega$ and $\Gamma, \mathfrak{I} : \Omega \rightarrow CB|^\Omega$ be multivalued mappings. Then there exist $\xi_1 \in \Gamma\xi_0$ such that $D|_{\Gamma\xi_0}^{\xi_0} = d|_{\xi_1}^{\xi_0}$. Let $\xi_2 \in \mathfrak{I}\xi_1$ be such that $D|_{\mathfrak{I}\xi_1}^{\xi_1} = d|_{\xi_2}^{\xi_1}$. Continuing this manner, we build a sequence $\{\xi_\kappa\}$ in Ω such that $\xi_{\kappa+1} \in \Gamma\xi_\kappa \Rightarrow D|_{\Gamma\xi_\kappa}^{\xi_\kappa} = d|_{\xi_{\kappa+1}}^{\xi_\kappa}$ and $\xi_{\kappa+2} \in \mathfrak{I}\xi_{\kappa+1} \Rightarrow D|_{\mathfrak{I}\xi_{\kappa+1}}^{\xi_{\kappa+1}} = d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}$. We say that $\{\Gamma\mathfrak{I}(\xi_\kappa)\}$ is a sequence in Ω generated by ξ_0 .

Definition 2.3.1 Let $\Gamma, \mathfrak{I} : \Omega \rightarrow CB|^\Omega$ be two multivalued mappings and $\alpha : \Omega \times \Omega \rightarrow [0, +\infty)$. Let $A \subseteq \Omega$. The pair $(\Gamma, \mathfrak{I})$ is regarded as a semi α_* -admissible on A , whenever $\alpha|_{\xi_2}^{\xi_1} \geq 1$ implies

that $\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \geq 1$ and $\alpha_*|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \geq 1$, $\forall_{\xi_1, \xi_2 \in A}$, where

$$\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} = \inf\{\alpha|_b^a : a \in \Gamma\xi_1, b \in \mathfrak{S}\xi_2\}.$$

If $A = \Omega$, then we say that $(\Gamma, \mathfrak{S})$ is a pair of α_* -admissible on Ω .

Definition 2.3.2 Let $\Gamma : \Omega \rightarrow CB|^\Omega$ be a multivalued mapping and $\alpha, \eta : \Omega \times \Omega \rightarrow [0, +\infty)$. Let $A \subseteq \Omega$. Γ is regarded as a semi α_* -admissible w.r.t η on A , whenever $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$ implies that $\alpha_*|_{\Gamma\xi_2}^{\Gamma\xi_1} \geq \eta_*|_{\Gamma\xi_2}^{\Gamma\xi_1}$, $\forall_{\xi_1, \xi_2 \in A}$, where

$$\alpha_*|_{\Gamma\xi_2}^{\Gamma\xi_1} = \inf\{\alpha|_b^a : a \in \Gamma\xi_1, b \in \Gamma\xi_2\},$$

and

$$\eta_*|_{\Gamma\xi_2}^{\Gamma\xi_1} = \sup\{\eta|_b^a : a \in \Gamma\xi_1, b \in \Gamma\xi_2\}.$$

If $A = \Omega$, then we say that Γ is α_* -admissible w.r.t η on Ω .

Definition 2.3.3 Let $\Gamma, \mathfrak{S} : \Omega \rightarrow CB|^\Omega$ be two mappings and $\alpha, \eta : \Omega \times \Omega \rightarrow [0, +\infty)$. Let $A \subseteq \Omega$. We say that $(\Gamma, \mathfrak{S})$ is a pair of semi α_* -admissible w.r.t η on A , whenever $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$ implies that $\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \geq \eta_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1}$ and $\alpha_*|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \geq \eta_*|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1}$, $\forall_{\xi_1, \xi_2 \in A}$, where

$$\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} = \inf\{\alpha|_b^a : a \in \Gamma\xi_1, b \in \mathfrak{S}\xi_2\},$$

and

$$\eta_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} = \sup\{\eta|_b^a : a \in \Gamma\xi_1, b \in \mathfrak{S}\xi_2\}.$$

Again, if $A = \Omega$, then we say that $(\Gamma, \mathfrak{S})$ is a pair of α_* -admissible w.r.t η on Ω .

Definition 2.3.4. Let $(\Omega, d|)$ be a M.S, $\alpha, \eta : \Omega \times \Omega \rightarrow [0, +\infty)$ be two functions and $\Gamma, \mathfrak{S} : \Omega \rightarrow CB|^\Omega$ be two multivalued mappings. The pair $(\Gamma, \mathfrak{S})$ is called Ćirić-type rational $(\alpha_*, \eta_*, \psi, \phi)$ -contraction, if there exists $\phi \in \Theta$ and $\psi \in \Psi$ such that $\forall_{\xi_1, \xi_2 \in \{\Gamma\mathfrak{S}(\xi_n)\}}$,

$$\hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} > 0 \Rightarrow \phi \left(\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \right) \leq \psi \left[\phi \left(M|_{\xi_2}^{\xi_1} \right) \right], \quad (2.26)$$

where

$$M|_{\xi_2}^{\xi_1} = \max \left\{ d|_{\xi_2}^{\xi_1}, D|_{\Gamma\xi_1}^{\xi_1}, D|_{\Im\xi_2}^{\xi_2}, \frac{D|_{\Gamma\xi_1}^{\xi_1} \cdot D|_{\Im\xi_2}^{\xi_2}}{1 + d|_{\xi_2}^{\xi_1}} \right\}. \quad (2.27)$$

Theorem 2.3.5. Let $(\Omega, d|)$ be a C-M.S, $\alpha, \eta : \Omega \times \Omega \rightarrow [0, +\infty)$ be two functions. Assume that $\Gamma, \Im : \Omega \rightarrow CB|^\Omega$ are a pair of semi α_* -admissible w.r.t η satisfying (2.26) on a closed ball $\overline{B_d|_r^{\xi_0}}$, for $r > 0$ and $\xi_0 \in \overline{B_d|_r^{\xi_0}}$. Suppose that $\{\Gamma\Im(\xi_\kappa)\}$ is a sequence in Ω generated by ξ_0 . Then $\{\Gamma\Im(\xi_\kappa)\} \rightarrow z \in \overline{B_d|_r^{\xi_0}}$, and

$$\phi \left(d|_{\xi_1}^{\xi_0} \right) \leq \sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] \leq r, \quad r > 0. \quad (2.28)$$

Moreover, if $\forall \xi_1, \xi_2 \in \left(\overline{B_d|_r^{\xi_0}} \cap \{\Gamma\Im(\xi_n)\} \right) \cup \{z\}$ the contractive condition (2.26) holds .

Then, Γ and $\Im$ have a C.F.P in $\overline{B_d|_r^{\xi_0}}$.

Proof. Since $\xi_0 \in \overline{B_d|_r^{\xi_0}}$, and $\Gamma, \Im : \Omega \rightarrow CB|^\Omega$ are two multivalued mappings, then there is $\xi_1 \in \Gamma\xi_0$ so that $D|_{\Gamma\xi_0}^{\xi_0} = d|_{\xi_1}^{\xi_0}$. If $\xi_0 = \xi_1$, then ξ_0 is an F.P of Γ in $\overline{B_d|_r^{\xi_0}}$. Let $\xi_0 \neq \xi_1$. From (2.28), we get,

$$\phi \left(d|_{\xi_1}^{\xi_0} \right) \leq \sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] \leq r, \quad r > 0.$$

It follows that $\xi_1 \in \overline{B_d|_r^{\xi_0}}$. As $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$ and $(\Gamma, \Im)$ is a pair of semi α_* -admissible w.r.t. η on $\overline{B_d|_r^{\xi_0}}$, so $\alpha_*|_{\Im\xi_1}^{\Gamma\xi_0} \geq \eta_*|_{\Im\xi_1}^{\Gamma\xi_0}$. As $\alpha_*|_{\Im\xi_1}^{\Gamma\xi_0} \geq \eta_*|_{\Im\xi_1}^{\Gamma\xi_0}$, $\xi_1 \in \Gamma\xi_0$ and $\xi_2 \in \Im\xi_1$, so $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$. Let $\xi_2, \dots, \xi_i \in \overline{B_d|_r^{\xi_0}}$ for some $i \in \mathbb{N}$. As $(\Gamma, \Im)$ is a pair of semi α_* -admissible w.r.t. η on $\overline{B_d|_r^{\xi_0}}$, we have

$$\alpha_*|_{\Gamma\xi_2}^{\Im\xi_1} \geq \eta_*|_{\Gamma\xi_2}^{\Im\xi_1},$$

implying that $\alpha|_{\xi_3}^{\xi_2} \geq \eta|_{\xi_3}^{\xi_2}$, which further implies

$$\alpha_*|_{\Im\xi_3}^{\Gamma\xi_2} \geq \eta_*|_{\Im\xi_3}^{\Gamma\xi_2}.$$

Continuing in this manner and if $i = 2 + 1, \dots, i = 1, 2, \dots, \frac{i-1}{2}$, we have

$$\alpha_*|_{\Im\xi_{2+1}}^{\Gamma\xi_2} \geq \eta_*|_{\Im\xi_{2+1}}^{\Gamma\xi_2},$$

which leadss to

$$\alpha|_{\xi_{2+2}}^{\xi_{2+1}} \geq \eta|_{\xi_{2+2}}^{\xi_{2+1}}.$$

Now, we write

$$\begin{aligned} \phi\left(d|_{\xi_{2+2}}^{\xi_{2+1}}\right) &\leq \phi\left(\hat{H}_d|_{\Im\xi_{2+1}}^{\Gamma\xi_2}\right) \leq \phi\left(\alpha_*|_{\Im\xi_{2+1}}^{\Gamma\xi_2} \hat{H}_d|_{\Im\xi_{2+1}}^{\Gamma\xi_2}\right) \leq \psi\left[\phi\left(M|_{\xi_{2+1}}^{\xi_2}\right)\right] \\ &= \psi\left[\phi\left(\max\left\{d|_{\xi_{2+1}}^{\xi_2}, D|_{\Gamma\xi_2}^{\xi_2}, D|_{\Im\xi_{2+1}}^{\xi_{2+1}}, \frac{D|_{\Gamma\xi_2}^{\xi_2} \cdot D|_{\Im\xi_{2+1}}^{\xi_{2+1}}}{1 + \mathcal{L}_{\Im\xi_{2+1}}^{\xi_{2j}}}\right\}\right)\right] \\ &= \psi\left[\phi\left(\max\left\{d|_{\xi_{2+1}}^{\xi_2}, d|_{\xi_{2+2}}^{\xi_{2+1}}, \frac{d|_{\xi_{2+1}}^{\xi_2} \cdot d|_{\xi_{2+2}}^{\xi_{2+1}}}{1 + d|_{\xi_{2+1}}^{\xi_2}}\right\}\right)\right]. \end{aligned}$$

If $M|_{\xi_{2+1}}^{\xi_2} = d|_{\xi_{2+2}}^{\xi_{2+1}}$, then

$$\phi\left(d|_{\xi_{2+2}}^{\xi_{2+1}}\right) \leq \psi\left[\phi\left(d|_{\xi_{2+2}}^{\xi_{2+1}}\right)\right].$$

Using (ϕ_1) and sicethe $\psi \in \Psi$, we get

$$d|_{\xi_{2+2}}^{\xi_{2+1}} < d|_{\xi_{2+2}}^{\xi_{2+1}},$$

which is an inconsistency as $d|_{\xi_{2+2}}^{\xi_{2+1}} \geq 0$.

Similarly if $M|_{\xi_{2+1}}^{\xi_2} = \frac{d|_{\xi_{2+1}}^{\xi_2} \cdot d|_{\xi_{2+2}}^{\xi_{2+1}}}{1 + d|_{\xi_{2+1}}^{\xi_2}}$, we obtain an inconsistency. Therefore, $M|_{\xi_{2+1}}^{\xi_2} = d|_{\xi_{2+1}}^{\xi_2}$, which implies that

$$\begin{aligned} \phi\left(d|_{\xi_{2+2}}^{\xi_{2+1}}\right) &\leq \psi\left[\phi\left(d|_{\xi_{2+1}}^{\xi_2}\right)\right] \\ &\leq \psi\left[\phi\left(\alpha_*|_{\Gamma\xi_2}^{\Im\xi_{2-1}} \hat{H}|_{\Gamma\xi_2}^{\Im\xi_{2-1}}\right)\right] \\ &\leq \psi^2\left[\phi\left(d|_{\xi_2}^{\xi_{2-1}}\right)\right] \\ &\leq \dots \leq \psi^{2+1}\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right]. \end{aligned}$$

Following that,

$$\phi\left(d|_{\xi_{2+2}}^{\xi_{2+1}}\right) \leq \psi^{2+1}\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right]. \quad (2.29)$$

Utilizing triangle inequality, (ϕ_4) , (2.28) and (2.29), we obtain

$$\begin{aligned}
\phi\left(d|_{\xi_{2+1}}^{\xi_0}\right) &\leq \phi\left(d|_{\xi_1}^{\xi_0} + \dots + d|_{\xi_{2+1}}^{\xi_2} + d|_{\xi_{2+2}}^{\xi_{2+1}}\right) \\
&\leq \phi\left(d|_{\xi_1}^{\xi_0}\right) + \dots + \phi\left(d|_{\xi_{2+1}}^{\xi_2}\right) + \phi\left(d|_{\xi_{2+2}}^{\xi_{2+1}}\right) \\
&\leq \phi\left(d|_{\xi_1}^{\xi_0}\right) + \dots + \psi^2\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right] + \psi^{2+1}\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right] \\
&\leq \sum_{i=0}^{2+1} \psi^i\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right] \leq r.
\end{aligned}$$

Thus, $\xi_{2+1} \in \overline{B|_r^{\xi_0}}$. Therefore, by induction, $\xi_\kappa \in \overline{B|_r^{\xi_0}}$ and $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa} \forall \kappa \in \mathbb{N}$. As $\Gamma, \mathfrak{S}$ are semi α_* -admissible w.r.t. η on $\overline{B|_r^{\xi_0}}$, then $\alpha_*|_{\mathfrak{S}\xi_{\kappa+1}}^{\Gamma\xi_\kappa} \geq \eta_*|_{\mathfrak{S}\xi_{\kappa+1}}^{\Gamma\xi_\kappa} \forall \kappa \in \mathbb{N} \cup \{0\}$. Now, inequality (2.29) can be written as

$$\phi\left(d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}\right) \leq \psi^{\kappa+1}\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right]. \quad (2.30)$$

Passing $\kappa \rightarrow \infty$ in (2.30), we get

$$0 \leq \lim_{\kappa \rightarrow \infty} \phi\left(d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}\right) \leq \lim_{\kappa \rightarrow \infty} \psi^{\kappa+1}\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right] = 0,$$

hence

$$\lim_{\kappa \rightarrow \infty} \phi\left(d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}\right) = 0.$$

From (ϕ_2) , we get

$$\lim_{\kappa \rightarrow \infty} d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} = 0. \quad (2.31)$$

To prove that $\{\xi_\kappa\}$ is a Cauchy in $(\overline{B|_r^{\xi_0}}, d|)$. Let $\kappa, \varkappa \in \mathbb{N}$ with $\varkappa > \kappa > p$. Then, we get

$$\begin{aligned}
\phi\left(d|_{\xi_\varkappa}^{\xi_\kappa}\right) &\leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi_\kappa} + d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} + \dots + d|_{\xi_\varkappa}^{\xi_{\varkappa-1}}\right) \\
&\leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi_\kappa}\right) + \phi\left(d|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}\right) + \dots + \phi\left(d|_{\xi_\varkappa}^{\xi_{\varkappa-1}}\right) \\
&\leq \psi^\kappa\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right] + \psi^{\kappa+1}\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right] + \dots + \psi^{\varkappa-1}\left[\phi\left(d|_{\xi_1}^{\xi_0}\right)\right].
\end{aligned} \quad (2.32)$$

Letting $\kappa, \varkappa \rightarrow \infty$ in (2.32), we get

$$\lim_{\kappa, \varkappa \rightarrow \infty} \phi\left(d|_{\xi_\varkappa}^{\xi_\kappa}\right) = 0.$$

Applying (ϕ_2) , we get

$$\lim_{\kappa, \mathcal{K} \rightarrow \infty} d|_{\xi_{\mathcal{K}}}^{\xi_{\kappa}} = 0. \quad (2.33)$$

Since every closed ball in a C-M.S is also complete, so there is $\xi^* \in \overline{B|_r^{\xi_0}}$ so that $\xi_{\kappa} \rightarrow \xi^*$, and

$$\lim_{n \rightarrow \infty} d|_{\xi^*}^{\xi_n} = 0. \quad (2.34)$$

Hence, $\{\Gamma\mathfrak{I}(\xi_{\kappa})\}$ is a sequence in $\overline{B|_r^{\xi_0}}$ generated by ξ_0 and $\{\Gamma\mathfrak{I}(\xi_{\kappa})\} \rightarrow \xi^* \in \overline{B|_r^{\xi_0}}$. So, for $\xi_{\kappa}, \xi_{\kappa+1} \in \{\Gamma\mathfrak{I}(\xi_{\kappa})\}$, one can write

$$\alpha|_{\xi_{\kappa+1}}^{\xi_{\kappa}} \geq \eta|_{\xi_{\kappa+1}}^{\xi_{\kappa}}, \quad \forall \kappa \geq 0.$$

Due to

$$\alpha_*|_{\Gamma\xi_{\kappa+1}}^{\mathfrak{I}\xi_{\kappa}} \geq \eta_*|_{\Gamma\xi_{\kappa+1}}^{\mathfrak{I}\xi_{\kappa}}, \quad \forall \kappa \geq 0.$$

Thus, we get

$$\alpha|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} \geq \eta|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}, \quad \forall \kappa \geq 0.$$

From our assumption, we have

$$\alpha|_{\xi^*}^{\xi_{\kappa}} \geq \eta|_{\xi^*}^{\xi_{\kappa}}, \quad \forall \kappa \geq 0.$$

Hence

$$\alpha_*|_{\Gamma\xi^*}^{\mathfrak{I}\xi_{\kappa}} \geq \eta_*|_{\Gamma\xi^*}^{\mathfrak{I}\xi_{\kappa}}.$$

Now, to claim that $\xi^* \in \Gamma\xi^*$, Suppose that $d|_{\Gamma\xi^*}^{\xi^*} > 0$, then, we have

$$\begin{aligned} \phi\left(D|_{\Gamma\xi^*}^{\xi^*}\right) &\leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*} + D|_{\Gamma\xi^*}^{\xi_{\kappa+1}}\right) \leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \phi\left(D|_{\Gamma\xi^*}^{\xi_{\kappa+1}}\right) \\ &\leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \phi\left(\alpha_*|_{\Gamma\xi^*}^{\mathfrak{I}\xi_{\kappa}} \hat{H}_d|_{\Gamma\xi^*}^{\mathfrak{I}\xi_{\kappa}}\right) \leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \psi\left[\phi\left(M|_{\xi^*}^{\xi_{\kappa}}\right)\right] \\ &= \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \psi\left[\phi\left(\max\left\{d|_{\xi_{\kappa}}^{\xi^*}, D|_{\mathfrak{I}\xi_{\kappa}}^{\xi_{\kappa}}, D|_{\Gamma\xi^*}^{\xi^*}, \frac{D|_{\mathfrak{I}\xi_{\kappa}}^{\xi_{\kappa}} \cdot D|_{\Gamma\xi^*}^{\xi^*}}{1 + d|_{\xi_{\kappa}}^{\xi^*}}\right\}\right)\right] \\ &= \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \psi\left[\phi\left(\max\left\{d|_{\xi_{\kappa}}^{\xi^*}, d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}, D|_{\Gamma\xi^*}^{\xi^*}, \frac{d|_{\xi_{\kappa+1}}^{\xi_{\kappa}} \cdot D|_{\Gamma\xi^*}^{\xi^*}}{1 + d|_{\xi_{\kappa}}^{\xi^*}}\right\}\right)\right]. \end{aligned}$$

Letting $\kappa \rightarrow \infty$ in the previous inequality, using (ϕ_2) , as $\psi \in \Psi$ and (2.34), we get

$$D|_{\Gamma\zeta^*}^{\xi^*} < D|_{\Gamma\zeta^*}^{\xi^*},$$

a contradiction. Thus, $D|_{\Gamma\zeta^*}^{\xi^*} = 0$ and $\xi^* \in \Gamma\zeta^*$.

Similarly, we can obtain that $D|_{\zeta^*}^{\mathfrak{S}\xi^*} = 0$. Hence $\xi^* \in \mathfrak{S}\xi^*$. So, Γ and $\mathfrak{S}$ have a C.F.P. ■

The following outcome shows that our results are valid in the context of partially ordered M.Ss (for short P.O.M.Ss).

Via this space, let $A, B \subseteq \Omega$. If $\forall a \in A, \exists b \in B$ so that $a \preceq b$ and $\Gamma a \preceq_r \mathfrak{S}b$, then we say that $A \preceq B$ and $\Gamma A \preceq_r \mathfrak{S}B$, respectively.

Theorem 2.3.6. Let $(\Omega, \preceq, d|)$ be a C-P.O.M.S, $\alpha, \eta : \Omega \times \Omega \rightarrow [0, \infty)$ be two functions and $\Gamma, \mathfrak{S} : \Omega \rightarrow CB|^\Omega$ be two non-decreasing semi α_* -admissible maps w.r.t η . Suppose there exist $\phi \in \Phi$ and $\psi \in \Psi$ so that

$$\hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} > 0 \Rightarrow \phi \left(\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \right) \leq \psi \left[\phi \left(M|_{\xi_2}^{\xi_1} \right) \right], \quad (2.35)$$

where, $M|_{\xi_2}^{\xi_1}$ is defined as in (2.27) for all $\xi_1, \xi_2 \in \overline{B|_r^{\xi_0}} \cap \{\Gamma\mathfrak{S}(\xi_\kappa)\}$, $r > 0$ and

$$\sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] \leq r, \quad r > 0.$$

with $\xi_1 \preceq \xi_2$, $\Gamma\xi_1 \preceq_r \mathfrak{S}\xi_2$ and $\sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] \leq r$. Then, $\{\Gamma\mathfrak{S}(\xi_\kappa)\}$ is a sequence in $\overline{B|_r^{\xi_0}}$, $\xi_\kappa \preceq \xi_{\kappa+1}$ and $\{\Gamma\mathfrak{S}(\xi_\kappa)\} \rightarrow \xi^* \in \overline{B|_r^{\xi_0}}$. Moreover, if $\xi^* \preceq \xi_\kappa$ or $\xi_\kappa \preceq \xi^*$ and inequality (2.35) holds for all $\xi_1, \xi_2 \in \left(\overline{B|_r^{\xi_0}} \cap \{\Gamma\mathfrak{S}(\xi_\kappa)\} \right) \cup \{\xi^*\}$. Then, ξ^* is a C.F.P of Γ and $\mathfrak{S}$ in $\overline{B|_r^{\xi_0}}$.

Proof. Let $\xi_0 \in \overline{B|_r^{\xi_0}}$ so that $\xi_0 \preceq \Gamma\xi_0$. Define a sequence $\{\Gamma\mathfrak{S}(\xi_\kappa)\}_{\kappa \in \mathbb{N}}$ by letting $\xi_1 \in \Gamma\xi_0$ so that $\xi_0 \preceq \xi_1$ and $\xi_2 \in \mathfrak{S}\xi_1$ so that $\xi_1 \preceq \xi_2$. As $\Gamma, \mathfrak{S}$ are non-decreasing, we have $\xi_3 \in \Gamma\xi_2$ such that $\xi_2 \preceq \xi_3$. Continuing the same manner, we get a sequence $\{\Gamma\mathfrak{S}(\xi_\kappa)\}_{\kappa \in \mathbb{N}} \subseteq \overline{B|_r^{\xi_0}}$ generated by ξ_0 such that

$$\xi_{2\kappa+1} \in \Gamma\xi_{2\kappa} \text{ and } \xi_{2\kappa+2} \in \mathfrak{S}\xi_{2\kappa+1}, \Rightarrow \xi_{2\kappa} \preceq \xi_{2\kappa+1} \text{ and } \xi_{2\kappa+1} \preceq \xi_{2\kappa+2}, \quad \forall \kappa \geq 0.$$

Following that

$$\xi_0 \preceq \xi_1 \preceq \xi_2 \preceq \dots \preceq \xi_\kappa \preceq \xi_{\kappa+1} \preceq \dots$$

As the pair $(\Gamma, \mathfrak{I})$ is semi α_* -admissible w.r.t. η , we have $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa} \quad \forall \kappa \geq 0$. Tracing the line of proof employed in the Theorem 2.3.5, we deduce that

$$\lim_{n \rightarrow \infty} d|_{\xi^*}^{\xi_\kappa} = 0. \quad (2.36)$$

Hence, $\{\Gamma\mathfrak{I}(\xi_\kappa)\}$ is a sequence in $\overline{B|_r^{\xi_0}}$ generated by ξ_0 and $\{\Gamma\mathfrak{I}(\xi_\kappa)\} \rightarrow \xi^* \in \overline{B|_r^{\xi_0}}$. Also, for $\xi_\kappa, \xi_{\kappa+1} \in \{\Gamma\mathfrak{I}(\xi_\kappa)\}$, we get

$$\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa}, \quad \forall \kappa \geq 0.$$

As $\alpha_*|_{\mathfrak{I}\xi_{\kappa+1}}^{\Gamma\xi_\kappa} \geq \eta_*|_{\mathfrak{I}\xi_{\kappa+1}}^{\Gamma\xi_\kappa} \quad \forall \kappa \geq 0$, we have $\alpha|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} \geq \eta|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}$. By assumption, we get $\alpha|_{\xi^*}^{\xi_\kappa} \geq \eta|_{\xi^*}^{\xi_\kappa}, \quad \forall \kappa \geq 0$. Thus $\alpha_*|_{\mathfrak{I}\xi^*}^{\Gamma\xi_\kappa} \geq \eta_*|_{\mathfrak{I}\xi^*}^{\Gamma\xi_\kappa}$. Now to prove $\xi^* \in \Gamma\xi^*$, let $d|_{\Gamma\xi^*}^{\xi^*} > 0$. Then, one gets

$$\begin{aligned} \phi\left(d|_{\Gamma\xi^*}^{\xi^*}\right) &\leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*} + D|_{\Gamma\xi^*}^{\xi_{\kappa+1}}\right) \leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \phi\left(D|_{\Gamma\xi^*}^{\xi_{\kappa+1}}\right) \\ &\leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \phi\left(\alpha_*|_{\Gamma\xi^*}^{\mathfrak{I}\xi_\kappa} \hat{H}_d|_{\Gamma\xi^*}^{\mathfrak{I}\xi_\kappa}\right) \leq \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \psi\left[\phi\left(M|_{\xi^*}^{\xi_\kappa}\right)\right] \\ &= \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \psi\left[\phi\left(\max\left\{d|_{\xi^*}^{\xi_\kappa}, D|_{\mathfrak{I}\xi_\kappa}^{\xi_\kappa}, D|_{\Gamma\xi^*}^{\xi^*}, \frac{D|_{\mathfrak{I}\xi_\kappa}^{\xi_\kappa} \cdot D|_{\Gamma\xi^*}^{\xi^*}}{1 + d|_{\xi^*}^{\xi_\kappa}}\right\}\right)\right] \\ &= \phi\left(d|_{\xi_{\kappa+1}}^{\xi^*}\right) + \psi\left[\phi\left(\max\left\{d|_{\xi^*}^{\xi_\kappa}, d|_{\xi_{\kappa+1}}^{\xi_\kappa}, D|_{\Gamma\xi^*}^{\xi^*}, \frac{d|_{\xi_{\kappa+1}}^{\xi_\kappa} \cdot D|_{\Gamma\xi^*}^{\xi^*}}{1 + d|_{\xi^*}^{\xi_\kappa}}\right\}\right)\right]. \end{aligned}$$

Letting $\kappa \rightarrow \infty$ above, using (ϕ_2) , as $\psi \in \Psi$, and from (2.36), we get

$$D|_{\Gamma\xi^*}^{\xi^*} < D|_{\Gamma\xi^*}^{\xi^*}.$$

A contradiction. Thus, $D|_{\Gamma\xi^*}^{\xi^*} = 0$ and $\xi^* \in \Gamma\xi^*$.

Analogously, we can obtain that $D|_{\mathfrak{I}\xi^*}^{\xi^*} = 0$ and $\xi^* \in \mathfrak{I}\xi^*$. So, Γ and $\mathfrak{I}$ have a C.F.P. ■

Definition 2.3.7. Let $\Gamma : \Omega \rightarrow \Omega$ be a self-mapping and $\alpha, \eta : \Omega \times \Omega \rightarrow [0, +\infty)$ be two functions, then the mapping f is termed semi α -admissible w.r.t. η , if for some $\xi_1, \xi_2 \in \mathbb{A} \subseteq \Omega$, $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1} \implies \alpha|_{\Gamma\xi_2}^{\Gamma\xi_1} \geq \eta|_{\Gamma\xi_2}^{\Gamma\xi_1}$. If $\mathbb{A} = \Omega$, then Γ is told to be α -admissible w.r.t. η .

Corollary 2.3.8. Let $(\Omega, d|)$ be a C-M.S and $\Gamma : \Omega \rightarrow \Omega$, $r > 0$ and ξ_0 be an arbitrary

point in $\overline{B|_r^{\xi_0}}$, for $r > 0$. Let $\{\xi_\kappa\}$ be a Picard sequence in Ω with initial guess ξ_0 . Let $\alpha, \eta : \Omega \times \Omega \rightarrow [0, +\infty)$ be semi α -admissible mappings w.r.t. η on $\overline{B|_r^{\xi_0}}$ with $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$. Assume that $\exists_{\phi \in \Theta, \psi \in \Psi}$ so that $\forall_{\xi_1, \xi_2 \in \overline{B|_r^{\xi_0}}}$, $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$, implies

$$\phi \left(d|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \psi \left[\phi \left(M|_{\xi_2}^{\xi_1} \right) \right], \quad (2.37)$$

where

$$M|_{\xi_2}^{\xi_1} = \max \left\{ d|_{\xi_2}^{\xi_1}, d|_{\Gamma \xi_1}^{\xi_1}, d|_{\Gamma \xi_2}^{\xi_2}, \frac{d|_{\Gamma \xi_1}^{\xi_1} \cdot d|_{\Gamma \xi_2}^{\xi_2}}{1 + d|_{\xi_2}^{\xi_1}} \right\}, \quad (2.38)$$

and $\sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] \leq r$, $r > 0$. Then $\{\xi_\kappa\}$ is a sequence in $\overline{B|_r^{\xi_0}}$, $\xi_n \rightarrow \xi^* \in \overline{B|_r^{\xi_0}}$ and $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa} \forall_{\kappa \geq 0}$. Also, if $\alpha|_{\xi^*}^{\xi_\kappa} \geq \eta|_{\xi^*}^{\xi_\kappa} \forall_{\kappa \geq 0}$, and the inequality (2.37) holds for all $\xi_1, \xi_2 \in \left(\overline{B|_r^{\xi_0}} \cap \{\Omega \Gamma(\xi_\kappa)\} \right) \cup \{\xi^*\}$. Then, ξ^* is a F.P of Γ in $\overline{B|_r^{\xi_0}}$.

Bear in mind that if $(\Omega, \preceq)$ is a partially ordered set and $\Gamma : \Omega \rightarrow \Omega$ is a mapping so that for $\xi_1, \xi_2 \in \Omega$, with $\xi_1 \preceq \xi_2$ implies $\Gamma \xi_1 \preceq \Gamma \xi_2$, then Γ is said to be non-decreasing.

Corollary 2.3.9. Let $(\Omega, d|)$ be a C-P.O.M.S and $\Gamma : \Omega \rightarrow \Omega$ be a non-decreasing mapping. Assume that ξ_0 be an arbitrary point in $\overline{B|_r^{\xi_0}}$, $\{\xi_\kappa\}$ be a Picard sequence in Ω with initial guess ξ_0 and $\xi_0 \preceq \xi_1$. Presume that there are $\phi \in \Theta$ and $\psi \in \Psi$ so that

$$\phi \left(d|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \psi \left[\phi \left(M|_{\xi_2}^{\xi_1} \right) \right], \quad (2.39)$$

where $M|_{\xi_2}^{\xi_1}$ is defined as in (2.38) $\forall \xi_1, \xi_2 \in \overline{B|_r^{\xi_0}} \cap \{\Omega \Gamma(\xi_\kappa)\}$ with $\xi_1 \preceq \xi_2$ and

$$\sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] \leq r, \quad r > 0.$$

Then $\{\xi_\kappa\}$ is a sequence in $\overline{B|_r^{\xi_0}}$, $\xi_\kappa \preceq \xi_{\kappa+1}$ and $\{\xi_\kappa\} \rightarrow \xi^* \in \overline{B|_r^{\xi_0}}$. Also, if $\xi^* \preceq \xi_\kappa$ or $\xi_\kappa \preceq \xi^*$ and the inequality (2.39) holds for each $\xi_1, \xi_2 \in \left(\overline{B|_r^{\xi_0}} \cap \{\Omega \Gamma(\xi_\kappa)\} \right) \cup \{\xi^*\}$. Then ξ^* is a F.P of Γ in $\overline{B|_r^{\xi_0}}$.

Example 2.3.10. Let $\Omega = [0, \infty)$ with metric $d|_{\xi_2}^{\xi_1} = |\xi_1 - \xi_2|$. Then, $(\Omega, d|)$ is a C-M.S.

Define $\Gamma, \mathfrak{I} : \Omega \rightarrow CB|^\Omega$ by

$$\Gamma \xi_1 = \begin{cases} \left\{ \frac{3\xi_1}{e^3}, \frac{\xi_1}{e^3} \right\} & \text{if } \xi_1 \in [0, 1], \\ \{1, \xi_1 + 4\} & \text{if } \xi_1 \in (1, \infty). \end{cases} \quad \text{and } \mathfrak{I} \xi_1 = \begin{cases} \left\{ \frac{3\xi_1}{e^4}, \frac{\xi_1}{e^4} \right\} & \text{if } \xi_1 \in [0, 1], \\ \{0, \xi_1 + 5\} & \text{if } \xi_1 \in (1, \infty). \end{cases}$$

Consider, $\xi_0 = 1$, $r = 10$. Then, $\overline{B|_r^{\xi_0}} = [0, 11]$ and $D|_{\Gamma \xi_0}^{\xi_0} = D|_{\Gamma 1}^1 = d|_{\frac{1}{e^3}}^1 = 1 - \frac{1}{e^3}$. Hence, we obtain a sequence $\{\Gamma \mathfrak{I}(\xi_n)\} = \{1, \frac{1}{e^3}, \frac{1}{e^7}, \frac{1}{e^{10}}, \frac{1}{e^{14}}, \dots\}$ in Ω generated by ξ_0 . Let $\phi(t) = 2t$ and $\psi(t) = \frac{2}{e}t$. Define the functions,

$$\alpha|_{\xi_2}^{\xi_1} = \begin{cases} 2, & \text{if } \xi_1, \xi_2 \in [0, 1]; \\ \frac{5}{4}, & \text{otherwise.} \end{cases} \quad \text{and } \eta|_{\xi_2}^{\xi_1} = \begin{cases} 1, & \text{if } \xi_1, \xi_2 \in [0, 1]; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}.$$

Now, for $\xi_1 = 4$ and $\xi_2 = 6$, we get

$$\phi \left(\alpha_*|_{\Gamma 6}^{\mathfrak{I} 4}(\cdot) \hat{H}_d|_{\Gamma 6}^{\mathfrak{I} 4} \right) = 2 \left(\frac{5}{4} \right) 10 > \frac{2}{e} \left[2 \max \left\{ 2, 4, 5, \frac{20}{3}, \frac{20}{9} \right\} \right] = \frac{80}{3e} = 9.8.$$

Hence, the condition (2.26) does not hold $\forall_{\xi_1, \xi_2 \in \Omega}$ and $\forall_{\xi_1, \xi_2 \in \overline{B|_r^{\xi_0}}}$. Now, $\forall \xi_1, \xi_2 \in \overline{B|_r^{\xi_0}} \cap \{\Gamma \mathfrak{I}(\xi_n)\}$, we have

$$\begin{aligned} \alpha_*|_{\Gamma \xi_2}^{\mathfrak{I} \xi_1} \hat{H}_d|_{\Gamma \xi_2}^{\mathfrak{I} \xi_1} &= 2 \left[\max \left\{ \sup_{a \in \mathfrak{I} \xi_1} d|_{\Gamma \xi_2}^a, \sup_{b \in \Gamma \xi_2} d|_b^{\mathfrak{I} \xi_1} \right\} \right] \\ &= 2 \left[\max \left\{ \sup_{a \in \mathfrak{I} \xi_1} d|_{\left[\frac{3\xi_2}{e^4}, \frac{\xi_2}{e^4} \right]}^a, \sup_{b \in \Gamma \xi_2} d|_b^{\left[\frac{3\xi_1}{e^3}, \frac{\xi_1}{e^3} \right]} \right\} \right] \\ &= 2 \left[\max \left\{ d|_{\left[\frac{3\xi_2}{e^4}, \frac{\xi_2}{e^4} \right]}^{\frac{3\xi_1}{e^3}}, d|_{\frac{3\xi_2}{e^4}}^{\left[\frac{3\xi_1}{e^3}, \frac{\xi_1}{e^3} \right]} \right\} \right] = 2 \left[\max \left\{ d|_{\frac{\xi_2}{e^4}}^{\frac{3\xi_1}{e^3}}, d|_{\frac{3\xi_2}{e^4}}^{\frac{\xi_1}{e^3}} \right\} \right] \\ &\leq \frac{1}{e} \left[2 \max \left\{ |\xi_1 - \xi_2|, \left| \xi_1 - \frac{\xi_1}{e^3} \right|, \left| \xi_2 - \frac{\xi_2}{e^4} \right|, \frac{\left| \xi_1 - \frac{\xi_1}{e^3} \right| \left| \xi_2 - \frac{\xi_2}{e^4} \right|}{1 + |\xi_1 - \xi_2|} \right\} \right], \end{aligned}$$

implying that

$$2 \alpha_*|_{\Gamma \xi_2}^{\mathfrak{I} \xi_1} \hat{H}_d|_{\Gamma \xi_2}^{\mathfrak{I} \xi_1} \leq \frac{2}{e} \left[2 M|_{\xi_2}^{\xi_1} \right],$$

which yields that

$$\Lambda \left(\alpha_*|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \hat{H}_d|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \right) \leq \psi \left[\phi \left(M|_{\xi_2}^{\xi_1} \right) \right].$$

Therefore, the condition (2.26) holds on $\overline{B|_r^{\xi_0}} \cap \{\Gamma\mathfrak{S}(\xi_\kappa)\}$. Also, $\forall \kappa \geq 0$, we obtain

$$\begin{aligned} \phi \left(d|_{\xi_1}^{\xi_0} \right) &\leq \sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] = \sum_{i=0}^{\kappa} \psi^i \left[\phi \left(1 - \frac{1}{e^3} \right) \right] \\ &= 2 \left(1 - \frac{1}{e^3} \right) \sum_{i=0}^{\kappa} \left(\frac{2}{e} \right)^i \leq 10 = r. \end{aligned}$$

Hence, all requirements of Theorem 2.3.5 are fulfilled. Moreover, $\{\Gamma\mathfrak{S}(\xi_\kappa)\}$ is a sequence in $\overline{B|_r^{\xi_0}}$, $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa}$ and $\{\Gamma\mathfrak{S}(\xi_\kappa)\} \rightarrow 0 \in \overline{B|_r^{\xi_0}}$. Also, $\alpha|_0^{\xi_\kappa} \geq \eta|_0^{\xi_\kappa}$ or $\alpha|_{\xi_\kappa}^0 \geq \eta|_{\xi_\kappa}^0 \forall \kappa \geq 0$. Further, $\mathfrak{S}$ and Γ have a C.F.P 0.

Now, we apply Theorem 2.3.5 in graph theory

Theorem 2.3.11. Let $(\Omega, d|)$ be a C-M.S endowed with a graph G , $r > 0$, $\xi_0 \in \overline{B|_r^{\xi_0}}$, $\Gamma, \mathfrak{S} : \Omega \rightarrow CB|^\Omega$, $\alpha, \eta : \Omega \times \Omega \rightarrow [0, \infty)$ and let $\{\Gamma\mathfrak{S}(\xi_\kappa)\}$ be a sequence in Ω generated by ξ_0 with $(\xi_0, \xi_1) \in E_G$. Assume that:

- (i) the pair $(\Gamma, \mathfrak{S})$ is edge preserving,
- (ii) for all $\xi_1, \xi_2 \in \overline{B|_r^{\xi_0}} \cap \{\Gamma\mathfrak{S}(\xi_\kappa)\}$ and $(\xi_1, \xi_2) \in E_G$, $\exists \phi \in \Theta$, $\psi \in \Psi$ so that

$$\hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} > 0 \Rightarrow \phi \left(\hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \right) \leq \psi \left[\phi \left(M|_{\xi_2}^{\xi_1} \right) \right], \quad (2.40)$$

where, $M|_{\xi_2}^{\xi_1}$ is defined as in (2.27)

- (iii) $\exists_{\xi_0 \in \overline{B|_r^{\xi_0}}}$ so that

$$\phi \left(D|_{\Gamma\xi_0}^{\xi_0} \right) \leq \sum_{i=0}^{\kappa} \psi^i \left[\phi \left(d|_{\xi_1}^{\xi_0} \right) \right] \leq r, \quad r > 0.$$

Then, $\{\Gamma\mathfrak{S}(\xi_\kappa)\}$ is a sequence in $\overline{B|_r^{\xi_0}}$, $(\xi_\kappa, \xi_{\kappa+1}) \in E_G$ and $\{\Gamma\mathfrak{S}(\xi_\kappa)\} \rightarrow \xi^*$. Also, if $(\xi_\kappa, \xi^*) \in E_G$ or $(\xi^*, \xi_\kappa) \in E_G \forall \kappa \geq 0$ and inequality (3.1) holds for all $\xi_1, \xi_2 \in \left(\overline{B|_r^{\xi_0}} \cap \{\Gamma\mathfrak{S}(\xi_\kappa)\} \right) \cup \{\xi^*\}$. Then, ξ^* is a C.F.P of Γ and $\mathfrak{S}$ in $\overline{B|_r^{\xi_0}}$.

Proof. Define, $\alpha, \eta : \Omega \times \Omega \rightarrow [0, +\infty)$ by

$$\alpha|_{\xi_2}^{\xi_1} = \eta|_{\xi_2}^{\xi_1} = \begin{cases} 1, & \text{if } (\xi_1, \xi_2) \in E_G \\ 0, & \text{otherwise} \end{cases}$$

As $\{\Gamma\mathfrak{S}(\xi_\kappa)\}$ is a sequence in Ω generated by ξ_0 with $(\xi_0, \xi_1) \in E_G$, we have $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0} \geq 1$. Let $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1} = 1$, then $(\xi_1, \xi_2) \in E_G$. From (i), we get $(u, v) \in E_G \forall u \in \Gamma\xi_1$ and $\forall v \in \mathfrak{S}\xi_2$. This implies that $\alpha|_v^u \geq \eta|_v^u = 1 \forall u \in \Gamma\xi_1$ and $\forall v \in \mathfrak{S}\xi_2$. It follows that $\inf\{\alpha|_v^u : u \in \Gamma\xi_1, v \in \mathfrak{S}\xi_2\} \geq \sup\{\eta|_v^u : u \in \Gamma\xi_1, v \in \mathfrak{S}\xi_2\} = 1$. Thus, $(\Gamma, \mathfrak{S})$ is a pair of semi α_* -admissible w.r.t. η on $\overline{B|_r^{\xi_0}}$. Furthermore, if $(\xi_1, \xi_2) \in E_G$, then $\alpha|_{\xi_2}^{\xi_1} = \eta|_{\xi_2}^{\xi_1} = 1$ and hence, $\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} = \eta_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} = 1$.

Now, condition (ii) can be written as

$$\phi\left(\alpha_*|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1} \hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1}\right) = \phi\left(\hat{H}_d|_{\mathfrak{S}\xi_2}^{\Gamma\xi_1}\right) \leq \psi\left[\phi\left(M|_{\xi_2}^{\xi_1}\right)\right],$$

$\forall \xi_1, \xi_2 \in \left(\overline{B|_r^{\xi_0}} \cap \{\Gamma\mathfrak{S}(\xi_n)\}\right)$. Condition (iii) leads to all assumptions of Theorem 2.3.5. Now, we have $\{\Gamma\mathfrak{S}(\xi_\kappa)\}$ is a sequence in $\overline{B|_r^{\xi_0}}$, $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa}$, that is, $(\xi_\kappa, \xi_{\kappa+1}) \in E_G$ and $\{\Gamma\mathfrak{S}(\xi_\kappa)\} \rightarrow \xi^* \in \overline{B|_r^{\xi_0}}$. Further, if $(\xi_\kappa, \xi^*) \in E_G$ or $(\xi^*, \xi_\kappa) \in E_G \forall \kappa \geq 0$ and inequality (2.40) holds for all $\xi_1, \xi_2 \in \left(\overline{B|_r^{\xi_0}} \cap \{\Gamma\mathfrak{S}(\xi_\kappa)\}\right) \cup \{\xi^*\}$. Then, we have $\alpha|_{\xi^*}^{\xi_\kappa} \geq \eta|_{\xi^*}^{\xi_\kappa}$ or $\alpha|_{\xi_\kappa}^{\xi^*} \geq \eta|_{\xi_\kappa}^{\xi^*} \forall \kappa \geq 0$. Thus, the existence of C.F.P $\xi^* \in \overline{B|_r^{\xi_0}}$ of Γ and $\mathfrak{S}$ immediately follows by Theorem 2.3.5.

■

2.4 Fixed point theorems for almost Fisher-type $\mathcal{F}_{\mathfrak{R}}$ -contraction with binary relation

In this part, we study existence of F.Ps for an almost Fisher-type multivalued $\mathcal{F}_{\mathfrak{R}}$ -contraction mappings into a C-M.S endowed with a Γ -transitive binary relation $\mathfrak{R}$.

Definition 2.4.1. Let $(\Omega, d|)$ be a M.S. A map $\Gamma : \Omega \rightarrow CB|^\Omega$ is called an almost Fisher-type multivalued $\mathcal{F}_{\mathfrak{R}}$ -contraction endowed with a Γ -transitive binary relation $\mathfrak{R}$, If $\exists_{\tau \in \mathbb{R}^+, \mathcal{F}_{\mathfrak{R}} \in \Delta_w}$ and $\exists_{\lambda_1, \lambda_2, L \geq 0}$ with

$\lambda_1 + \lambda_2 \leq 1$ such that $\forall_{\xi_1, \xi_2 \in \mathfrak{R}^*}$, $\mathfrak{R}^* = \{(\xi_1, \xi_2) \in \mathfrak{R} : \xi_1, \xi_2 \in \Omega \setminus \text{Fix}(\Gamma)\}$, we have

$$\tau + \mathcal{F}_{\mathfrak{R}} \left(\hat{H}_d \Big|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \mathcal{F}_{\mathfrak{R}} \left(M \Big|_{\xi_2}^{\xi_1} \right) + L N \Big|_{\xi_2}^{\xi_1}, \quad (2.41)$$

where

$$M \Big|_{\xi_2}^{\xi_1} = \begin{cases} \left[\lambda_1 \left(\frac{D \Big|_{\Gamma \xi_1}^{\xi_1} \cdot D \Big|_{\Gamma \xi_2}^{\xi_2}}{1 + d \Big|_{\xi_2}^{\xi_1}} \right)^{\beta} + \lambda_2 \left(d \Big|_{\xi_2}^{\xi_1} \right)^{\beta} \right]^{\frac{1}{\beta}} & \text{if } \beta > 0; \\ \left(D \Big|_{\Gamma \xi_1}^{\xi_1} \right)^{\lambda_1} \left(D \Big|_{\Gamma \xi_2}^{\xi_2} \right)^{\lambda_2} & \text{if } \beta = 0, \end{cases} \quad (2.42)$$

and

$$N \Big|_{\xi_2}^{\xi_1} = \min \left\{ D \Big|_{\Gamma \xi_1}^{\xi_1}, D \Big|_{\Gamma \xi_2}^{\xi_2}, D \Big|_{\Gamma \xi_2}^{\xi_1}, D \Big|_{\Gamma \xi_1}^{\xi_2} \right\}. \quad (2.43)$$

Theorem 2.4.2. Let $(\Omega, d|)$ be a C-M.S, and let $\Gamma : \Omega \rightarrow CB|^\Omega$ be an almost Fisher-type multivalued $\mathcal{F}_{\mathfrak{R}}$ -contraction mapping endowed with a Γ -transitive binary relation $\mathfrak{R}$. Assume that

$$(T1) \quad \Omega|_{\mathfrak{R}}^\Gamma = \{\xi_1 \in \Omega : (\xi_1, \xi_2) \in \mathfrak{R} \text{ for some } \xi_2 \in \Gamma \xi_1\} \neq \emptyset;$$

$$(T2) \quad \mathfrak{R} \text{ is } \Gamma\text{-closed};$$

$$(T3) \quad \Gamma \text{ is } \mathfrak{R}\text{-continuous or } (\Omega, d|) \text{ is } \mathfrak{R}\text{-regular space.}$$

Then, Γ has a F.P $\xi^* \in \Omega$.

Proof. From (T1) $\exists_{\xi_0 \in \Omega}$ such that $(\xi_0, \Gamma \xi_0) \in \mathfrak{R}$ and from (T2), we have $(\Gamma \xi_0, \Gamma^2 \xi_0) \in \mathfrak{R}$. Since $\Gamma \xi_0 \neq \emptyset$ and closed $\exists_{\xi_1 \in \Omega}$ such that $\xi_1 \in \Gamma \xi_0 \subset \Omega$ such that $(\xi_1, \Gamma \xi_1) \in \mathfrak{R}$ and from (T2), we have $(\Gamma^2 \xi_0, \Gamma^3 \xi_0) \in \mathfrak{R}$. Continuing in this way, we construct a sequence $\{\xi_\kappa\}$ by $\xi_\kappa \in \Gamma \xi_{\kappa-1} = \Gamma^\kappa \xi_0$, $\forall_{\kappa \in \mathbb{N}_0}$. If $\xi_\kappa \in \Gamma \xi_\kappa$ for some $\kappa \in \mathbb{N}_0$ then ξ_κ becomes a F.P of Γ and the proof is finished, so we presume that $\xi_\kappa \notin \Gamma \xi_\kappa$, $\forall_{\kappa \in \mathbb{N}_0}$ then $D \Big|_{\Gamma \xi_\kappa}^{\xi_\kappa} > 0$ and from Lemma 1.1.23, we get

$$0 < D \Big|_{\Gamma \xi_\kappa}^{\xi_\kappa} \leq \hat{H}_d \Big|_{\Gamma \xi_{\kappa+1}}^{\Gamma \xi_\kappa}, \quad \forall_{\kappa \in \mathbb{N}_0}. \quad (2.44)$$

Since $(\xi_0, \Gamma \xi_0) \in \mathfrak{R}$ and from (T2), we conclude that $(\Gamma^\kappa \xi_0, \Gamma^{\kappa+1} \xi_0) \in \mathfrak{R}$, $\forall_{\kappa \in \mathbb{N}_0}$. That is

$$(\xi_\kappa, \xi_{\kappa+1}) \in \mathfrak{R}, \quad \forall_{\kappa \in \mathbb{N}_0}. \quad (2.45)$$

Now, from (2.44) and (2.45), we have $(\xi_{\kappa-1}, \xi_\kappa) \in \mathfrak{R}^* \quad \forall_{\kappa \in \mathbb{N}_0}$. Utilizing (F_1) in (2.44) and

applying Remark 1.2.20 together with (2.41), we get

$$\begin{aligned}\mathcal{F}_{\mathfrak{R}}\left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}\right) &= \mathcal{F}_{\mathfrak{R}}\left(D|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa}}\right) \leq \mathcal{F}_{\mathfrak{R}}\left(\hat{H}_d|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa-1}}\right) \\ &\leq \mathcal{F}_{\mathfrak{R}}\left(M|_{\xi_{\kappa}}^{\xi_{\kappa-1}}\right) + L N|_{\xi_{\kappa}}^{\xi_{\kappa-1}} - \tau,\end{aligned}\tag{2.46}$$

where if $\beta > 0$, then

$$\begin{aligned}M|_{\xi_{\kappa}}^{\xi_{\kappa-1}} &= \left[\lambda_1 \left(\frac{D|_{\Gamma\xi_{\kappa-1}}^{\xi_{\kappa-1}} \cdot D|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa}}}{1 + d|_{\xi_{\kappa}}^{\xi_{\kappa-1}}} \right)^{\beta} + \lambda_2 \left(d|_{\xi_{\kappa}}^{\xi_{\kappa-1}} \right)^{\beta} \right]^{\frac{1}{\beta}} \\ &\leq \left[\lambda_1 \left(\frac{d|_{\xi_{\kappa}}^{\xi_{\kappa-1}} \cdot d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}}{d|_{\xi_{\kappa}}^{\xi_{\kappa-1}}} \right)^{\beta} + \lambda_2 \left(d|_{\xi_{\kappa}}^{\xi_{\kappa-1}} \right)^{\beta} \right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1 \left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}} \right)^{\beta} + \lambda_2 \left(d|_{\xi_{\kappa}}^{\xi_{\kappa-1}} \right)^{\beta} \right]^{\frac{1}{\beta}}.\end{aligned}$$

Assume that $d|_{\xi_{\kappa}}^{\xi_{\kappa-1}} \leq d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}$, then we get

$$\begin{aligned}M|_{\xi_{\kappa}}^{\xi_{\kappa-1}} &\leq \left[\lambda_1 \left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}} \right)^{\beta} + \lambda_2 \left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}} \right)^{\beta} \right]^{\frac{1}{\beta}} \\ &= \left[(\lambda_1 + \lambda_2) \left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}} \right)^{\beta} \right]^{\frac{1}{\beta}} \\ &< d|_{\xi_{\kappa+1}}^{\xi_{\kappa}},\end{aligned}\tag{2.47}$$

and

$$N|_{\xi_{\kappa}}^{\xi_{\kappa-1}} = \min \left\{ d|_{\xi_{\kappa}}^{\xi_{\kappa-1}}, d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}, d|_{\xi_{\kappa+1}}^{\xi_{\kappa-1}}, d|_{\xi_{\kappa}}^{\xi_{\kappa}} \right\} = 0.$$

Then, from (2.46), we have

$$\begin{aligned}\mathcal{F}_{\mathfrak{R}}\left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}\right) &\leq \mathcal{F}_{\mathfrak{R}}\left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}\right) - \tau \\ &< \mathcal{F}_{\mathfrak{R}}\left(d|_{\xi_{\kappa+1}}^{\xi_{\kappa}}\right).\end{aligned}$$

A contradiction. Therefore, $d|_{\xi_\kappa}^{\xi_{\kappa-1}} > d|_{\xi_{\kappa+1}}^{\xi_\kappa}$, which implies that

$$\begin{aligned} M|_{\xi_\kappa}^{\xi_{\kappa-1}} &\leq \left[\lambda_1 \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^\beta + \lambda_2 \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^\beta \right]^{\frac{1}{\beta}} \\ &= \left[(\lambda_1 + \lambda_2) \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^\beta \right]^{\frac{1}{\beta}} \\ &< d|_{\xi_\kappa}^{\xi_{\kappa-1}}. \end{aligned}$$

Then, from (2.46), we have

$$\begin{aligned} \mathcal{F}_{\mathfrak{R}} \left(d|_{\xi_{\kappa+1}}^{\xi_\kappa} \right) &\leq \mathcal{F}_{\mathfrak{R}} \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) - \tau \\ &< \mathcal{F}_{\mathfrak{R}} \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right). \end{aligned} \tag{2.48}$$

Thereafter, the iterative sequence $\left\{ d|_{\xi_{\kappa+1}}^{\xi_\kappa} \right\}$ is decreasing and convergent. Through induction, we obtain

$$\mathcal{F}_{\mathfrak{R}} \left(d|_{\xi_{\kappa+1}}^{\xi_\kappa} \right) \leq \mathcal{F}_{\mathfrak{R}} \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) - \tau < \dots < \mathcal{F}_{\mathfrak{R}} \left(d|_{\xi_1}^{\xi_0} \right) - \kappa\tau.$$

Taking limit as $\kappa \rightarrow \infty$ above, we get

$$\lim_{\kappa \rightarrow \infty} \mathcal{F}_{\mathfrak{R}} \left(d|_{\xi_{\kappa+1}}^{\xi_\kappa} \right) = -\infty.$$

Applying (F_2) , we deduct

$$\lim_{\kappa \rightarrow \infty} d|_{\xi_{\kappa+1}}^{\xi_\kappa} = 0. \tag{2.49}$$

We claim that $\{\xi_\kappa\}$ is a Cauchy, by supposing on the contrary that it is not. Then $\exists_{\varepsilon>0}$ and subsequences $\{\xi_{l_\kappa}\}$ and $\{\xi_{q_\kappa}\}$ so that for $l_\kappa > q_\kappa > \kappa$, we have

$$d|_{\xi_{q_\kappa}}^{\xi_{l_\kappa}} \geq \varepsilon \text{ and } d|_{\xi_{q_\kappa}}^{\xi_{l_\kappa-1}} < \varepsilon, \quad \forall_{\kappa \in \mathbb{N}}. \tag{2.50}$$

By triangle inequality, we have

$$\varepsilon \leq d|_{\xi_{q_\kappa}}^{\xi_{l_\kappa}} \leq d|_{\xi_{l_\kappa-1}}^{\xi_{l_\kappa}} + d|_{\xi_{q_\kappa}}^{\xi_{l_\kappa-1}}.$$

Taking limit as $\kappa \rightarrow \infty$ and using (2.49) and (2.50), we get

$$\lim_{\kappa \rightarrow \infty} d|_{\xi_{q\kappa}}^{\xi_{l\kappa}} = \varepsilon. \quad (2.51)$$

Using triangle inequality again, we get

$$d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa}} \leq d|_{\xi_{q\kappa}}^{\xi_{l\kappa}} + d|_{\xi_{q\kappa+1}}^{\xi_{q\kappa}}.$$

Taking limit as $\kappa \rightarrow \infty$ above, using (2.49) and (2.51), we get

$$\lim_{\kappa \rightarrow \infty} d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa}} \leq \varepsilon. \quad (2.52)$$

Similarly, we have

$$\varepsilon \leq d|_{\xi_{q\kappa}}^{\xi_{l\kappa}} \leq d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa}} + d|_{\xi_{q\kappa}}^{\xi_{q\kappa+1}}.$$

Taking limit as $\kappa \rightarrow \infty$ above, using (2.49) and (2.51), we get

$$\lim_{n \rightarrow \infty} d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa}} \geq \varepsilon. \quad (2.53)$$

Therefore, from (2.52) and (2.53), we get

$$\lim_{\kappa \rightarrow \infty} d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa}} = \varepsilon.$$

By similar way, we conclude

$$\lim_{\kappa \rightarrow \infty} d|_{\xi_{l\kappa+1}}^{\xi_{q\kappa}} = \varepsilon.$$

Next, we claim that

$$d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa+1}} > 0, \quad \forall \kappa \in \mathbb{N}. \quad (2.54)$$

Arguing by contradiction, $\exists \kappa \in \mathbb{N}$, such that

$$d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa+1}} = 0. \quad (2.55)$$

Using triangle inequality, we have

$$\begin{aligned}\varepsilon &\leq d|_{\xi_{q\kappa}}^{\xi_{l\kappa}} \leq d|_{\xi_{l\kappa}+1}^{\xi_{l\kappa}} + d|_{\xi_{q\kappa}}^{\xi_{l\kappa}+1} \\ &\leq d|_{\xi_{l\kappa}+1}^{\xi_{l\kappa}} + d|_{\xi_{q\kappa}+1}^{\xi_{l\kappa}+1} + d|_{\xi_{q\kappa}}^{\xi_{q\kappa}+1}.\end{aligned}$$

Taking limit as $\kappa \rightarrow \infty$ above, from (2.49) and (2.55), we get a contradiction. Hence (2.54) hold true.

Now, we have

$$0 < d|_{\xi_{q\kappa}+1}^{\xi_{l\kappa}+1} = D|_{\Gamma\xi_{q\kappa}}^{\xi_{l\kappa}+1} \leq \hat{H}_d|_{\Gamma\xi_{q\kappa}}^{\Gamma\xi_{l\kappa}}. \quad (2.56)$$

Since $\{\xi_\kappa\}$ is $\mathfrak{R}$ -preserving sequence, then by Γ -transitivity of $\mathfrak{R}$, we have $(\xi_{l\kappa}, \xi_{q\kappa}) \in \mathfrak{R}$ and from (2.56), we have $(\xi_{l\kappa}, \xi_{q\kappa}) \in \mathfrak{R}^*$. Utilizing (F_1) in (2.56) and applying (2.41), we obtain

$$\begin{aligned}\mathcal{F}_{\mathfrak{R}}\left(d|_{\xi_{q\kappa}+1}^{\xi_{l\kappa}+1}\right) &\leq \mathcal{F}_{\mathfrak{R}}\left(\hat{H}_d|_{\Gamma\xi_{q\kappa}}^{\Gamma\xi_{l\kappa}}\right) \\ &\leq \mathcal{F}_{\mathfrak{R}}\left(M|_{\xi_{q\kappa}}^{\xi_{l\kappa}}\right) + L N|_{\xi_{q\kappa}}^{\xi_{l\kappa}} - \tau,\end{aligned} \quad (2.57)$$

where

$$\begin{aligned}M|_{\xi_{q\kappa}}^{\xi_{l\kappa}} &= \left[\lambda_1 \left(\frac{D|_{\Gamma\xi_{l\kappa}}^{\xi_{l\kappa}} \cdot D|_{\Gamma\xi_{q\kappa}}^{\xi_{q\kappa}}}{1 + d|_{\xi_{q\kappa}}^{\xi_{l\kappa}}} \right)^\beta + \lambda_2 \left(d|_{\xi_{q\kappa}}^{\xi_{l\kappa}} \right)^\beta \right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1 \left(\frac{d|_{\xi_{l\kappa}+1}^{\xi_{l\kappa}} \cdot d|_{\xi_{q\kappa}+1}^{\xi_{q\kappa}}}{1 + d|_{\xi_{q\kappa}}^{\xi_{l\kappa}}} \right)^\beta + \lambda_2 \left(d|_{\xi_{q\kappa}}^{\xi_{l\kappa}} \right)^\beta \right]^{\frac{1}{\beta}},\end{aligned}$$

and

$$\begin{aligned}N|_{\xi_{q\kappa}}^{\xi_{l\kappa}} &= \min \left\{ D|_{\Gamma x_{l\kappa}}^{\xi_{l\kappa}}, D|_{\Gamma\xi_{q\kappa}}^{\xi_{q\kappa}}, D|_{\Gamma\xi_{q\kappa}}^{\xi_{l\kappa}}, D|_{\Gamma\xi_{l\kappa}}^{\xi_{q\kappa}} \right\} \\ &= \min \left\{ d|_{\xi_{l\kappa}+1}^{\xi_{l\kappa}}, d|_{\xi_{q\kappa}+1}^{\xi_{q\kappa}}, d|_{\xi_{q\kappa}+1}^{\xi_{l\kappa}}, D|_{\Gamma\xi_{l\kappa}+1}^{\xi_{q\kappa}} \right\}.\end{aligned}$$

As $\mathcal{F}_{\mathfrak{R}}$ is continuous, then taking limit as $\kappa \rightarrow \infty$ in (2.57), we get

$$\mathcal{F}_{\mathfrak{R}}(\varepsilon) \leq \mathcal{F}_{\mathfrak{R}}\left(\sqrt[\beta]{\lambda_2 \varepsilon}\right) + L(0) - \tau < \mathcal{F}_{\mathfrak{R}}(\varepsilon),$$

which gives a contradiction. So, $\{\xi_\kappa\}$ is $\mathfrak{R}$ -Cauchy sequence in a C-M.S such that $\exists_{\xi^* \in \Omega}$ implies that $\lim_{\kappa \rightarrow \infty} \xi_\kappa = \xi^*$. To proof that $\xi^* \in \Gamma^{\xi^*}$, we claim that $D|_{\Gamma^{\xi^*}}^{\xi^*} > 0$ and by using (2.41), Lemma 1.1.23 and (T3), we write

$$\begin{aligned} \mathcal{F}_{\mathfrak{R}} \left(D|_{\Gamma^{\xi^*}}^{\xi^*} \right) &\leq \lim_{\kappa \rightarrow \infty} \mathcal{F}_{\mathfrak{R}} \left(D|_{\Gamma^{\xi_\kappa}}^{\xi_\kappa} \right) \leq \lim_{\kappa \rightarrow \infty} \mathcal{F}_{\mathfrak{R}} \left(\hat{H}_d|_{\Gamma^{\xi_\kappa}}^{\Gamma^{\xi_{\kappa-1}}} \right) \\ &\leq \lim_{\kappa \rightarrow \infty} \mathcal{F}_{\mathfrak{R}} \left(M|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) + \lim_{\kappa \rightarrow \infty} N|_{\xi_\kappa}^{\xi_{\kappa-1}} - \tau, \end{aligned} \quad (2.58)$$

where

$$\begin{aligned} M|_{\xi_\kappa}^{\xi_{\kappa-1}} &= \left[\lambda_1 \left(\frac{D|_{\Gamma^{\xi_{\kappa-1}}}^{\xi_{\kappa-1}} \cdot D|_{\Gamma^{\xi_\kappa}}^{\xi_\kappa}}{1 + d|_{\xi_\kappa}^{\xi_{\kappa-1}}} \right)^\beta + \lambda_2 \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^\beta \right]^{\frac{1}{\beta}} \\ &= \left[\lambda_1 \left(\frac{d|_{\xi_\kappa}^{\xi_{\kappa-1}} \cdot D|_{\Gamma^{\xi_\kappa}}^{\xi_\kappa}}{1 + d|_{\xi_\kappa}^{\xi_{\kappa-1}}} \right)^\beta + \lambda_2 \left(D|_{\xi_\kappa}^{\Gamma^{\xi_{\kappa-2}}} \right)^\beta \right]^{\frac{1}{\beta}}, \end{aligned}$$

and

$$\begin{aligned} N|_{\xi_\kappa}^{\xi_{\kappa-1}} &= \min \left\{ D|_{\Gamma^{\xi_{\kappa-1}}}^{\xi_{\kappa-1}}, D|_{\Gamma^{\xi_\kappa}}^{\xi_\kappa}, D|_{\Gamma^{\xi_\kappa}}^{\xi_{\kappa-1}}, D|_{\Gamma^{\xi_{\kappa-1}}}^{\xi_\kappa} \right\} \\ &= \min \left\{ d|_{\xi_\kappa}^{\xi_{\kappa-1}}, D|_{\Gamma^{\xi_\kappa}}^{\xi_\kappa}, D|_{\Gamma^{\xi_\kappa}}^{\xi_{\kappa-1}}, d|_{\xi_\kappa}^{\xi_\kappa} \right\}. \end{aligned}$$

Taking limit as $\kappa \rightarrow \infty$ in (2.58) with continuity of $\mathcal{F}_{\mathfrak{R}}$, we obtain

$$\begin{aligned} \mathcal{F}_{\mathfrak{R}} \left(D|_{\Gamma^{\xi^*}}^{\xi^*} \right) &\leq \mathcal{F}_{\mathfrak{R}} \left(\sqrt[\beta]{\lambda_2} D|_{\Gamma^{\xi^*}}^{\xi^*} \right) + L(0) - \tau \\ &< \mathcal{F}_{\mathfrak{R}} \left(D|_{\Gamma^{\xi^*}}^{\xi^*} \right). \end{aligned}$$

A contradiction, thence $D|_{\Gamma^{\xi^*}}^{\xi^*} = 0$ and $\xi^* \in \Gamma^{\xi^*}$ is a F.P of Γ .

Now, if $\beta = 0$, then

$$\begin{aligned} M|_{\xi_\kappa}^{\xi_{\kappa-1}} &= \left(D|_{\Gamma^{\xi_{\kappa-1}}}^{\xi_{\kappa-1}} \right)^{\lambda_1} \left(D|_{\Gamma^{\xi_\kappa}}^{\xi_\kappa} \right)^{\lambda_2} \\ &= \left(d|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^{\lambda_1} \left(d|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\lambda_2}. \end{aligned} \quad (2.59)$$

Assume that $d|_{\xi_\kappa}^{\xi_{\kappa-1}} \leq d|_{\xi_{\kappa+1}}^{\xi_\kappa}$, then (2.59) becomes

$$\begin{aligned} M|_{\xi_\kappa}^{\xi_{\kappa-1}} &\leq \left(d|_{\xi_{\kappa+1}}^{\xi_\kappa}\right)^{\lambda_1} \left(d|_{\xi_{\kappa+1}}^{\xi_\kappa}\right)^{\lambda_2} \\ &= \left(d|_{\xi_{\kappa+1}}^{\xi_\kappa}\right)^{\lambda_1+\lambda_2} \\ &\leq d|_{\xi_{\kappa+1}}^{\xi_\kappa}. \end{aligned} \quad (2.60)$$

Thereafter, from (2.46) and (2.60), we get an inconsistency. Thus, $d|_{\xi_\kappa}^{\xi_{\kappa-1}} > d|_{\xi_{\kappa+1}}^{\xi_\kappa}$ and (2.59) becomes

$$M|_{\xi_\kappa}^{\xi_{\kappa-1}} < d|_{\xi_\kappa}^{\xi_{\kappa-1}} \quad (2.61)$$

Applying (2.61) in (2.46) and following the same steps from (2.48) till (2.57), we get

$$\begin{aligned} M|_{\xi_{q\kappa}}^{\xi_{l\kappa}} &= \left(D|_{\Gamma\xi_{l\kappa}}^{\xi_{l\kappa}}\right)^{\lambda_1} \left(D|_{\Gamma\xi_{q\kappa}}^{\xi_{q\kappa}}\right)^{\lambda_2} \\ &= \left(d|_{\xi_{l\kappa+1}}^{\xi_{l\kappa}}\right)^{\lambda_1} \left(d|_{\xi_{q\kappa+1}}^{\xi_{q\kappa}}\right)^{\lambda_2}, \end{aligned}$$

and

$$\begin{aligned} N|_{\xi_{q\kappa}}^{\xi_{l\kappa}} &= \min \left\{ D|_{\Gamma\xi_{l\kappa}}^{\xi_{l\kappa}}, D|_{\Gamma\xi_{q\kappa}}^{\xi_{q\kappa}}, D|_{\Gamma\xi_{q\kappa}}^{\xi_{l\kappa}}, D|_{\Gamma\xi_{l\kappa}}^{\xi_{q\kappa}} \right\} \\ &= \min \left\{ d|_{\xi_{l\kappa+1}}^{\xi_{l\kappa}}, d|_{\xi_{q\kappa+1}}^{\xi_{q\kappa}}, d|_{\xi_{q\kappa+1}}^{\xi_{l\kappa}}, d|_{\xi_{l\kappa+1}}^{\xi_{q\kappa}} \right\}. \end{aligned}$$

As $\mathcal{F}_{\mathfrak{R}}$ is continuous, then taking limit as $m \rightarrow \infty$ in (2.57), we get

$$\mathcal{F}_{\mathfrak{R}}(\varepsilon) < \mathcal{F}_{\mathfrak{R}}(0) + L(0) - \tau,$$

which gives a contradiction again. So, $\{\xi_\kappa\}$ is $\mathfrak{R}$ -Cauchy sequence in a C-M.S such that $\exists_{\xi^* \in \Omega}$ implies that $\lim_{\kappa \rightarrow \infty} \xi_\kappa = \xi^*$. To proof that $\xi^* \in \Gamma\xi^*$, we claim that $D|_{\Gamma\xi^*}^{\xi^*} > 0$ and by using (2.41), Lemma 1.1.23 and (T3), we write

$$\begin{aligned} \mathcal{F}_{\mathfrak{R}}\left(D|_{\Gamma\xi^*}^{\xi^*}\right) &\leq \lim_{\kappa \rightarrow \infty} \mathcal{F}_{\mathfrak{R}}\left(D|_{\Gamma\xi^*}^{\xi_\kappa}\right) \leq \lim_{\kappa \rightarrow \infty} \mathcal{F}_{\mathfrak{R}}\left(\hat{H}_d|_{\Gamma\xi^*}^{\Gamma\xi_{\kappa-1}}\right) \\ &\leq \lim_{\kappa \rightarrow \infty} \mathcal{F}_{\mathfrak{R}}\left(M|_{\xi^*}^{\xi_{\kappa-1}}\right) + \lim_{\kappa \rightarrow \infty} L N|_{\xi^*}^{\xi_{\kappa-1}} - \tau, \end{aligned} \quad (2.62)$$

where

$$\begin{aligned} M|_{\xi^*}^{\xi_{\kappa-1}} &= \left(D|_{\Gamma_{\xi_{\kappa-1}}}^{\xi_{\kappa-1}} \right)^{\lambda_1} \left(D|_{\Gamma_{\xi^*}}^{\xi^*} \right)^{\lambda_2} \\ &= \left(d|_{\xi_{\kappa}}^{\xi_{\kappa-1}} \right)^{\lambda_1} \left(D|_{\Gamma_{\xi^*}}^{\xi^*} \right)^{\lambda_2}, \end{aligned}$$

and

$$\begin{aligned} N|_{\xi^*}^{\xi_{\kappa-1}} &= \min \left\{ D|_{\Gamma_{\xi_{\kappa-1}}}^{\xi_{\kappa-1}}, D|_{\Gamma_{\xi^*}}^{\xi^*}, D|_{\Gamma_{\xi^*}}^{\xi_{\kappa-1}}, D|_{\Gamma_{\xi_{\kappa-1}}}^{\xi^*} \right\} \\ &= \min \left\{ d|_{\xi_{\kappa}}^{\xi_{\kappa-1}}, D|_{\Gamma_{\xi^*}}^{\xi^*}, D|_{\Gamma_{\xi^*}}^{\xi_{\kappa-1}}, d|_{\xi_{\kappa}}^{\xi^*} \right\}. \end{aligned}$$

Taking limit as $\kappa \rightarrow \infty$ in (2.62) with continuity of $\mathcal{F}_{\mathfrak{R}}$, leads to

$$\mathcal{F}_{\mathfrak{R}} \left(D|_{\Gamma_{\xi^*}}^{\xi^*} \right) \leq \mathcal{F}_{\mathfrak{R}}(0) + L(0) - \tau,$$

this implies that $D|_{\Gamma_{\xi^*}}^{\xi^*} = 0$ and $\xi^* \in \Gamma_{\xi^*}$ is a F.P of Γ . ■

Example 2.4.3. Let $\Omega = [0, \infty)$ equipped with a usual metric. Take a sequence $\{\vartheta_{\kappa}\} \subset \Omega$ by $\vartheta_{\kappa} = \frac{\kappa^2(\kappa+1)^2}{4} \forall_{\kappa \geq 1}$. Set $\mathfrak{R} = \{(\vartheta_{\kappa}, \vartheta_{\kappa}), (\vartheta_{\kappa}, \vartheta_{\kappa+1}), (\vartheta_{\kappa}, \vartheta_{\kappa+2}) : \kappa = 1, 2, \dots\}$, and $\Gamma : \Omega \rightarrow CB|^{\Omega}$ by

$$\Gamma\xi = \begin{cases} \{3\xi\} & \text{if } 0 \leq \xi \leq \vartheta_1 \\ \{0, \vartheta_1\} & \text{if } \vartheta_1 \leq \xi \leq \vartheta_2 \\ \left\{ \vartheta_{\kappa-1} + \left(\frac{\vartheta_{\kappa} - \vartheta_{\kappa-1}}{\vartheta_{\kappa+1} - \vartheta_{\kappa}} \right) (\xi - \vartheta_{\kappa}) \right\} & \text{if } \vartheta_{\kappa} \leq \xi \leq \vartheta_{\kappa+1}, \kappa = 2, \dots \end{cases}.$$

Then, $(\Omega, d|)$ is a C-M.S, $\Omega|_{\mathfrak{R}}^{\Gamma} \neq \emptyset$ as $\vartheta_1 = 1 \in \Omega$, $1 \in \Gamma\vartheta_1 = \Gamma 1 = \{0, 1\}$ and $(1, 1) \in \mathfrak{R}$, also $\mathfrak{R}$ is Γ -transitive and Γ is $\mathfrak{R}$ -continuous or $(\Omega, d|)$ is $\mathfrak{R}$ -regular space. Now, for $\kappa = 2, \dots$, we have

$$\begin{aligned} \hat{H}_d|_{\Gamma\vartheta_{\kappa+1}}^{\Gamma\vartheta_{\kappa}} &= \max \left\{ \sup_{r \in \Gamma\vartheta_{\kappa}} D|_{\Gamma\vartheta_{\kappa+1}}^r, \sup_{s \in \Gamma\vartheta_{\kappa+1}} D|_s^{\Gamma\vartheta_{\kappa}} \right\} \\ &= \max \left\{ d|_{\vartheta_{\kappa}}^{\vartheta_{\kappa-1}}, d|_{\vartheta_{\kappa}}^{\vartheta_{\kappa-1}} \right\} = d|_{\vartheta_{\kappa-1}}^{\vartheta_{\kappa}}. \end{aligned}$$

From (2.42) and (2.43) with $\beta \geq 0$, we get $M|_{\vartheta_{\kappa+1}}^{\vartheta_{\kappa}} = d|_{\vartheta_{\kappa}}^{\vartheta_{\kappa+1}}$ and $N|_{\vartheta_{\kappa+1}}^{\vartheta_{\kappa}} = 0$.

Applying (2.41), we get

$$\tau + \mathcal{F}_{\mathfrak{R}} \left(\hat{H}_d \Big|_{\Gamma^{\vartheta_{\kappa+1}}}^{\Gamma^{\vartheta_{\kappa}}} \right) = \tau + \mathcal{F}_{\mathfrak{R}} \left(d|_{\vartheta_{\kappa-1}}^{\vartheta_{\kappa}} \right) = \tau + \ln \left(d|_{\vartheta_{\kappa-1}}^{\vartheta_{\kappa}} \right), \quad (2.63)$$

and

$$\mathcal{F}_{\mathfrak{R}} \left(M|_{\vartheta_{\kappa+1}}^{\vartheta_{\kappa}} \right) + L N|_{\vartheta_{\kappa+1}}^{\vartheta_{\kappa}} \leq \mathcal{F}_{\mathfrak{R}} \left(2 d|_{\vartheta_{\kappa}}^{\vartheta_{\kappa+1}} \right) = \ln \left(2 d|_{\vartheta_{\kappa}}^{\vartheta_{\kappa+1}} \right). \quad (2.64)$$

From (2.63) and (2.64), we deduce that

$$\tau + \ln \left(d|_{\vartheta_{\kappa-1}}^{\vartheta_{\kappa}} \right) \leq \ln \left(d|_{\vartheta_{\kappa}}^{\vartheta_{\kappa+1}} \right),$$

implies that

$$\tau \leq \ln \left(\frac{d|_{\vartheta_{\kappa}}^{\vartheta_{\kappa+1}}}{d|_{\vartheta_{\kappa-1}}^{\vartheta_{\kappa}}} \right) = \ln \left(\frac{|\vartheta_{\kappa+1} - \vartheta_{\kappa}|}{|\vartheta_{\kappa} - \vartheta_{\kappa-1}|} \right) = f(\kappa). \quad (2.65)$$

In the line of Table 1 and Figure 1, the Eq (2.65) holds for $0 < \tau \leq 1.21$ and $\forall_{\kappa \in [2, 51]}$. So, the contraction (2.41) is satisfied $\forall_{\vartheta_1, \vartheta_1 \in \Omega}$ such that $(\vartheta_1, \vartheta_1) \in \mathfrak{R}^*$. Then, Γ has at least one F.P.

Table 1: Iteration and $f(\kappa)$

Iter	$f(\kappa)$	Iter	$f(\kappa)$	Iter	$f(\kappa)$	Iter	$f(\kappa)$	Iter	$f(\kappa)$
$\kappa = 2$	1.21	$\kappa = 12$	0.24	$\kappa = 22$	0.133	$\kappa = 32$	0.092	$\kappa = 42$	0.071
$\kappa = 3$	0.86	$\kappa = 13$	0.22	$\kappa = 23$	0.127	$\kappa = 33$	0.090	$\kappa = 43$	0.069
$\kappa = 4$	0.66	$\kappa = 14$	0.20	$\kappa = 24$	0.122	$\kappa = 34$	0.086	$\kappa = 44$	0.067
$\kappa = 5$	0.55	$\kappa = 15$	0.19	$\kappa = 25$	0.117	$\kappa = 35$	0.085	$\kappa = 45$	0.066
$\kappa = 6$	0.46	$\kappa = 16$	0.18	$\kappa = 26$	0.113	$\kappa = 36$	0.082	$\kappa = 46$	0.065
$\kappa = 7$	0.40	$\kappa = 17$	0.17	$\kappa = 27$	0.109	$\kappa = 37$	0.080	$\kappa = 47$	0.063
$\kappa = 8$	0.35	$\kappa = 18$	0.16	$\kappa = 28$	0.105	$\kappa = 38$	0.078	$\kappa = 48$	0.062
$\kappa = 9$	0.31	$\kappa = 19$	0.153	$\kappa = 29$	0.102	$\kappa = 39$	0.076	$\kappa = 49$	0.061
$\kappa = 10$	0.28	$\kappa = 20$	0.146	$\kappa = 30$	0.098	$\kappa = 40$	0.074	$\kappa = 50$	0.059
$\kappa = 11$	0.26	$\kappa = 21$	0.139	$\kappa = 31$	0.095	$\kappa = 41$	0.072	$\kappa = 51$	0.058

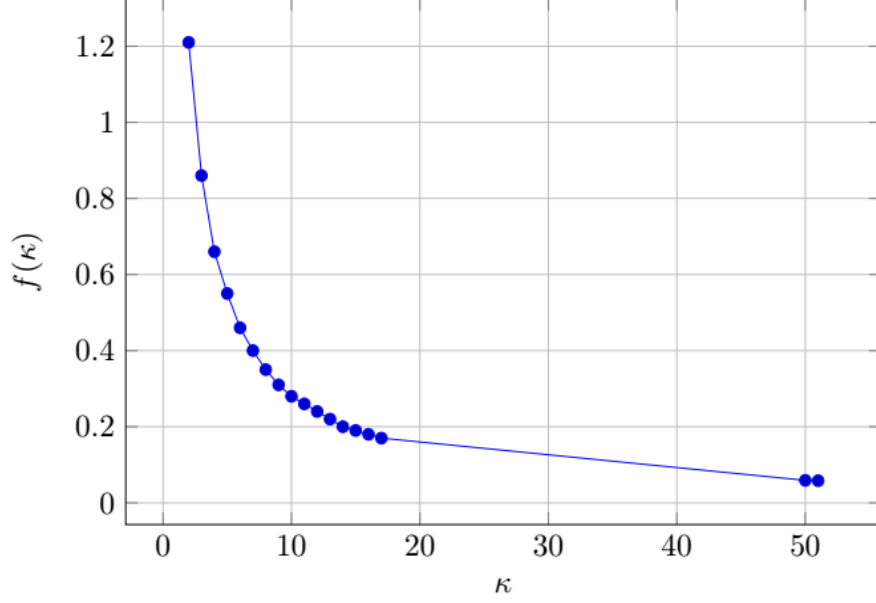


Figure 1: Behavior of $f(\kappa)$ for $\kappa \in [2, 51]$.

Example 2.4.4 Let $\Omega = \{1, 2, 3, 4\}$ and $d| : \Omega \times \Omega \rightarrow [0, \infty)$ be a metric on Ω such that $d|_2^1 = 4$, $d|_3^1 = 6$, $d|_4^1 = 3$, $d|_3^2 = 4$, $d|_4^2 = 3$, $d|_4^3 = 5$, $d|_{\xi_2}^{\xi_1} = d|_{\xi_1}^{\xi_2}$ and $d|_{\xi_1}^{\xi_1} = 0$, $\forall \xi_1, \xi_2 \in \Omega$. We define a binary relation on Ω as $\mathfrak{R} = \left\{ \begin{array}{l} (1, 1), (1, 2), (1, 3), (1, 4), (4, 1), (3, 4), \\ (2, 1), (2, 2), (2, 4), (3, 1), (4, 3), (4, 4), \end{array} \right\}$. Consider a map $\Gamma : \Omega \rightarrow CB|^\Omega$ as

$$\Gamma\xi = \begin{cases} \{3, 4\}, & \xi \in \{1, 4\}; \\ \{3\}, & \xi = 2; \\ \{4\}, & \xi = 3. \end{cases}$$

Then, $\Omega|_{\mathfrak{R}}^\Gamma \neq \emptyset$ as $4 \in \Omega$, $4 \in \Gamma 3 = \{4\}$ and $(4, 3) \in \mathfrak{R}$. Also, it is easy to check that $\mathfrak{R}$ is Γ -closed, Γ -transitive and Γ is $\mathfrak{R}$ -continuous or $(\Omega, d|)$ is $\mathfrak{R}$ -regular space. Now, $\forall \lambda_1 = \frac{1}{5}$, $\lambda_2 = \frac{3}{5}$, $\beta = 2$, $L = \frac{1}{5}$ and $(\xi_1, \xi_2) \in \mathfrak{R}^* = \{(1, 1), (1, 2), (1, 3), (3, 1), (2, 2), (2, 1)\}$, we have for $(\xi_1, \xi_2) = (1, 2)$

$$\begin{aligned} \hat{H}_d|_{\Gamma 2}^{\Gamma 1} &= \max \left\{ \sup_{a \in \Gamma 1} D|_{\Gamma 2}^a, \sup_{b \in \Gamma 2} D|_b^{\Gamma 1} \right\} \\ &= \max \left\{ d|_3^4, d|_3^3 \right\} = \max \{5, 0\} = 5, \end{aligned}$$

where

$$\begin{aligned}
M|_2^1 &= \left[\lambda_1 \left(\frac{D|_{\Gamma_1}^1 \cdot D|_{\Gamma_2}^2}{1 + d|_2^1} \right)^\beta + \lambda_2 \left(d|_2^1 \right)^\beta \right]^{\frac{1}{\beta}} \\
&= \left[\lambda_1 \left(\frac{d|_3^1 d|_3^2}{1 + d|_2^1} \right)^\beta + \lambda_2 \left(d|_2^1 \right)^\beta \right]^{\frac{1}{\beta}} \\
&= \left[\lambda_1 \left(\frac{6 \times 4}{1 + 4} \right)^\beta + \lambda_2 (4)^\beta \right]^{\frac{1}{\beta}} \\
&\leq \left[\frac{1}{3} \left(\frac{24}{5} \right)^2 + \frac{2}{3} (4)^2 \right]^{\frac{1}{2}} \\
&= \left(\frac{86}{75} \right)^{\frac{1}{2}} \times 4 = 4.283300908,
\end{aligned}$$

and

$$\begin{aligned}
N|_2^1 &= \min \left\{ D|_{\Gamma_1}^1, D|_{\Gamma_2}^2, D|_{\Gamma_2}^1, D|_{\Gamma_1}^2 \right\} \\
&= \min \left\{ d|_3^1, d|_3^2, d|_3^1, d|_3^2 \right\} \\
&= \min \{6, 4, 6, 4\} = 4.
\end{aligned}$$

For $\beta = 0$, we get

$$M|_2^1 = \left(D|_{\Gamma_1}^1 \right)^{\frac{1}{5}} \cdot \left(D|_{\Gamma_2}^2 \right)^{\frac{3}{5}} = \left(d|_3^1 \right)^{\frac{1}{5}} \left(d|_3^2 \right)^{\frac{3}{5}} = 6^{\frac{1}{5}} \times 4^{\frac{3}{5}} \approx 3.3.$$

Therefore, the contraction condition is satisfied.

For $(\xi_1, \xi_2) = (1, 3)$, we have $\hat{H}|_{\Gamma_3}^{\Gamma_1} = 5$, $M|_3^1 \approx 5.488$ and $N|_3^1 = 0$. Therefore

$$\tau + \mathcal{F}_{\mathfrak{R}}(5) \leq \mathcal{F}_{\mathfrak{R}}(5.488).$$

For $(\xi_1, \xi_2) = (1, 1)$, we have $\hat{H}|_{\Gamma_1}^{\Gamma_1} = 5$, $M|_1^1 = 12\sqrt{3}$ and $N|_1^1 = 6$. Therefore

$$\tau + \mathcal{F}_{\mathfrak{R}}(5) \leq \mathcal{F}_{\mathfrak{R}}(12\sqrt{3}) + \frac{6}{5}.$$

For $(\xi_1, \xi_2) = (2, 2)$, we have $\hat{H}|_{\Gamma_1}^{\Gamma_1} = 0$, $M|_1^1 = \frac{16}{\sqrt{3}}$ and $N|_1^1 = 4$. Therefore

$$\tau + \mathcal{F}_{\mathfrak{R}}(0) \leq \mathcal{F}_{\mathfrak{R}}\left(\frac{16}{\sqrt{3}}\right) + \frac{4}{5}.$$

Hence, $\forall_{\xi_1, \xi_2 \in \mathfrak{R}^*}$, we find that (2.41) is achieved and all conditions of Theorem 2.4.2 are satisfied so that Γ has a F.P 4 $\in \Omega$.

Corollary 2.4.5. Let $(\Omega, d|)$ be a C-M.S. A self-map $\Gamma : \Omega \rightarrow \Omega$ is called an almost Fisher-type $\mathcal{F}_{\mathfrak{R}}$ -contraction endowed with a Γ -transitive binary relation $\mathfrak{R}$ satisfies conditions of Theorem 2.4.2. If $\exists_{\tau \in \mathbb{R}^+, \mathcal{F}_{\mathfrak{R}} \in \Delta_w}$ and $\exists_{\lambda_1, \lambda_2, L \geq 0}$ with $\lambda_1 + \lambda_2 \leq 1$ such that $\forall_{\xi_1, \xi_2 \in \mathfrak{R}^*}$, $\mathfrak{R}^* = \{(\xi_1, \xi_2) \in \mathfrak{R} : \xi_1, \xi_2 \in \Omega \setminus \text{Fix}(\Gamma)\}$, we have

$$\tau + \mathcal{F}_{\mathfrak{R}}\left(d|_{\Gamma\xi_2}^{\Gamma\xi_1}\right) \leq \mathcal{F}_{\mathfrak{R}}\left(M|_{\xi_2}^{\xi_1}\right) + L N|_{\xi_2}^{\xi_1},$$

where $M|_{\xi_2}^{\xi_1}$ and $N|_{\xi_2}^{\xi_1}$ are defined as in (2.42) and (2.43) respectively. Then, Γ has an F.P.

Corollary 2.4.6. Let $(\Omega, d|)$ be a C-M.S. A self-map $\Gamma : \Omega \rightarrow \Omega$ is called a Fisher-type $\mathcal{F}_{\mathfrak{R}}$ -contraction endowed with a Γ -transitive binary relation $\mathfrak{R}$ satisfies conditions of Theorem 2.4.2. If $\exists_{\tau \in \mathbb{R}^+, \mathcal{F}_{\mathfrak{R}} \in \Delta_w}$ and $\exists_{\lambda_1, \lambda_2 \geq 0}$ with $\lambda_1 + \lambda_2 \leq 1$ such that $\forall_{\xi_1, \xi_2 \in \mathfrak{R}^*}$, $\mathfrak{R}^* = \{(\xi_1, \xi_2) \in \mathfrak{R} : \xi_1, \xi_2 \in \Omega \setminus \text{Fix}(\Gamma)\}$, we have

$$\tau + \mathcal{F}_{\mathfrak{R}}\left(d|_{\Gamma\xi_2}^{\Gamma\xi_1}\right) \leq \mathcal{F}_{\mathfrak{R}}\left(M|_{\xi_2}^{\xi_1}\right),$$

where $M|_{\xi_2}^{\xi_1}$ is defined as in (2.42). Then, Γ has an F.P.

Corollary 2.4.7. Let $(\Omega, d|)$ be a C-M.S. A map $\Gamma : \Omega \rightarrow CB|^\Omega$ is called an almost Fisher-type multivalued $\mathcal{F}_{\mathfrak{R}}$ -contraction endowed with a Γ -transitive binary relation $\mathfrak{R}$ satisfies conditions of Theorem 2.4.2. If $\exists_{\tau \in \mathbb{R}^+, \mathcal{F}_{\mathfrak{R}} \in \Delta_w}$, $\exists_{\lambda, L \geq 0}$ and $\lambda + L \leq 1$ such that $\forall_{\xi_1, \xi_2 \in \mathfrak{R}^*}$, $\mathfrak{R}^* = \{(\xi_1, \xi_2) \in \mathfrak{R} : \xi_1, \xi_2 \in \Omega \setminus \text{Fix}(\Gamma)\}$, we have

$$\tau + \mathcal{F}_{\mathfrak{R}}\left(\hat{H}_d|_{\Gamma\xi_2}^{\Gamma\xi_1}\right) \leq \mathcal{F}_{\mathfrak{R}}\left(\lambda d|_{\xi_2}^{\xi_1}\right) + L d|_{\xi_2}^{\xi_1}.$$

Then, Γ has an F.P.

Chapter 3

Fixed Point Results for Hybrid Contractions in M_b -Metric Spaces

Mlaiki et al. [71] extended the idea of Mb-M.S and established notable F.Ps under an advanced BCP. Alizadeh et al. [11] proposed the formulation of cyclic (α, β) -admissible mapping which was subsequently generalized by Isik et al. [49], Padhan et al. [92], Sezen [103] and Ameer et al. [19]. Geraghty [35] contributed a valuable extension of BCP and derived F.P theorems for single-valued mappings. The idea of symmetrical fractional contractive maps were developed in [36, 42, 43], where four distinct classes of such contractions were analyzed, resulting in significant F.P findings. Hardy and Rogers [38] investigated the BCP and formulated a series of impactful F.P theorems.

This chapter comprises six sections. In first three sections, we explore the motif of η -cyclic (α_*, β_*) -admissible type F -contraction multivalued mappings with establishing a C.F.P results for such mappings, along with relevant consequences and practical application within the Mb-M.S. Sections 4 and 5 introduce four categories of symmetric fractional contractions for single-valued mappings within Mb-MS and closed ball therein, with application to FODE. Finally, section 6 is devoted to the analysis of almost Hardy-Rogers multivalued contractions, where we establish several C.F.P outcomes in Mb-M.S.

The results given in this chapter have been published in [78, 83, 85].

3.1 Common fixed points for (α_*, β_*) -Geraghty multivalued $\mathcal{F}$ -contractions

We propose the definitions of η -cyclic (α_*, β_*) -admissible mappings and so improve the hybrid multivalued $\mathcal{F}$ -contraction-type of (α_*, β_*) -Geraghty and establish some new C.F.P theorems in Mb-M.S. Moreover, consistent with the results in [23, 96], we introduce the followig Lemmas to be useful in the future study:

Let (Ω_b, m_b) be an Mb-M.S and $CB_b|_{\Omega_b}$ be a family of nonempty, closed and bounded subsets of Ω_b . For $\aleph, \mathbb{Q} \in CB_b|_{\Omega_b}$, define

$$\hat{H}_b|_{\mathbb{Q}}^{\aleph} = \max \left\{ \delta_b|_{\mathbb{Q}}^{\aleph}, \delta_b|_{\aleph}^{\mathbb{Q}} \right\},$$

where $\delta_b|_{\mathbb{Q}}^{\aleph} = \sup \left\{ m_b|_{\mathbb{Q}}^p : p \in \aleph \right\}$ and $\delta_b|_{\aleph}^{\mathbb{Q}} = \sup \left\{ m_b|_q^{\aleph} : q \in \mathbb{Q} \right\}$.

Let $\bar{\aleph}$ denotes the closure of $\aleph$ with respect to m_b . Then $\aleph$ is closed in (Ω_b, m_b) iff $\aleph = \bar{\aleph}$.

Lemma 3.1.1. Let $\aleph \neq \emptyset$ in an Mb-M.S (Ω_b, m_b) , then

$$p \in \bar{\aleph} \iff m_b|_{\aleph}^p = \sup_{\lambda \in \aleph} m_b|_{p, \lambda}.$$

Proof. Let $p \in \bar{\aleph}$

$$\begin{aligned} &\Leftrightarrow B_b|_{\epsilon}^p \cap \aleph \neq \emptyset, \text{ for every open ball } B_b \text{ in } \Omega_b \text{ and } \epsilon > 0 \\ &\Leftrightarrow m_b|_{\lambda}^p < m_b|_{p, \lambda} + \epsilon, \text{ for some } \lambda \in \aleph \\ &\Leftrightarrow m_b|_{\lambda}^p - m_b|_{p, \lambda} < \epsilon \\ &\Leftrightarrow \inf \left\{ m_b|_{\lambda}^p - m_b|_{p, \lambda} : \lambda \in \aleph \right\} = 0 \\ &\Leftrightarrow \inf \left\{ m_b|_{\lambda}^p : \lambda \in \aleph \right\} = \sup \left\{ m_b|_{p, \lambda} : \lambda \in \aleph \right\} \\ &\Leftrightarrow m_b|_{\lambda}^p = \sup_{\lambda \in \aleph} m_b|_{p, \lambda}. \end{aligned}$$

Lemma 3.1.2. Let $\aleph, \mathbb{Q}, \mathfrak{R} \in CB_b|_{\Omega_b}$, then ■

$$(a) \delta_b|_{\aleph}^{\aleph} = \sup_{p \in \aleph} \left\{ \sup_{q \in \aleph} m_b|_{p, q} \right\};$$

(b) for $s \geq 1$, we have

$$\delta_b|_{\mathbb{Q}}^{\aleph} - \sup_{p \in \aleph} \sup_{q \in \mathbb{Q}} m_b|_{p,q} \leq s \left[\delta_b|_{\aleph}^{\aleph} - \inf_{p \in \aleph} \inf_{r \in \aleph} m_b|_{p,r} + \delta_b|_{\mathbb{Q}}^{\aleph} - \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \right] - \inf_{r \in \aleph} m_b|_r^r.$$

Proof.

(a) Since $\aleph \in CB_b|_{\Omega_b}$ and $\bar{\aleph} = \aleph$. Then from Lemma 3.1.1, $m_b|_{\lambda}^p = \sup_{\lambda \in \aleph} m_b|_{p,\lambda}$. Thus,

$$\delta_b|_{\aleph}^{\aleph} = \sup_{p \in \aleph} \{ m_b|_{\aleph}^p \} = \sup_{p \in \aleph} \left\{ \sup_{q \in \aleph} m_b|_{p,q} \right\}.$$

(b) For any $p \in \aleph$, $q \in \mathbb{Q}$ and $r \in \aleph$, $\exists_{s \geq 1}$ such that

$$m_b|_q^p - m_b|_{p,q} \leq s \left[m_b|_r^p - m_b|_{p,r} + m_b|_q^r - m_b|_{r,q} \right] - m_b|_r^r. \quad (3.1)$$

We can write (3.1) as

$$m_b|_q^p - m_b|_{p,q} + s m_b|_{p,r} + s m_b|_{r,q} \leq s \left[m_b|_r^p + m_b|_q^r \right] - m_b|_r^r.$$

Since q is arbitrary element in $\mathbb{Q}$ and $m_b|_{\mathbb{Q}}^r \leq \delta_b|_{\mathbb{Q}}^{\aleph}$, we have

$$\begin{aligned} m_b|_{\mathbb{Q}}^p - \sup_{q \in \mathbb{Q}} m_b|_{p,q} + s m_b|_{p,r} + s \inf_{q \in \mathbb{Q}} m_b|_{r,q} &\leq s \left[m_b|_r^p + m_b|_{\mathbb{Q}}^r \right] - m_b|_r^r \\ &\leq s \left[m_b|_r^p + \delta_b|_{\mathbb{Q}}^{\aleph} \right] - m_b|_r^r. \end{aligned}$$

Since r is arbitrary element in $\aleph$, we have

$$m_b|_{\mathbb{Q}}^p - \sup_{q \in \mathbb{Q}} m_b|_{p,q} + s \inf_{r \in \aleph} m_b|_{p,r} + s \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \leq s \left[m_b|_{\aleph}^p + \delta_b|_{\mathbb{Q}}^{\aleph} \right] - \inf_{r \in \aleph} m_b|_r^r.$$

As p is arbitrary element in $\aleph$, we have

$$\delta_b|_{\mathbb{Q}}^{\aleph} - \sup_{p \in \aleph} \sup_{q \in \mathbb{Q}} m_b|_{p,q} + s \inf_{p \in \aleph} \inf_{r \in \aleph} m_b|_{p,r} + s \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \leq s \left[\delta_b|_{\aleph}^{\aleph} + \delta_b|_{\mathbb{Q}}^{\aleph} \right] - \inf_{r \in \aleph} m_b|_r^r.$$

Therefore, we deduce that

$$\delta_b|_{\mathbb{Q}}^{\aleph} - \supsup_{p \in \aleph, q \in \mathbb{Q}} m_b|_{p,q} \leq s \left[\delta_b|_{\aleph}^{\aleph} - \inf_{p \in \aleph} \inf_{r \in \aleph} m_b|_{p,r} + \delta_b|_{\mathbb{Q}}^{\aleph} - \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \right] - \inf_{r \in \aleph} m_b|_r^r.$$

■

Lemma 3.1.3. Let $\aleph, \mathbb{Q}, \aleph \in CB_b|_{\Omega_b}$, then

- (1) $\hat{H}_b|_{\aleph}^{\aleph} = \delta_b|_{\aleph}^{\aleph} = \sup_{p \in \aleph} \left\{ \sup_{q \in \aleph} m_b|_{p,q} \right\};$
- (2) $\hat{H}_b|_{\mathbb{Q}}^{\aleph} = \hat{H}_b|_{\aleph}^{\mathbb{Q}};$
- (3) for $s \geq 1$, we have

$$\begin{aligned} & \left(\hat{H}_b|_{\mathbb{Q}}^{\aleph} - \supsup_{p \in \aleph, q \in \mathbb{Q}} m_b|_{p,q} \right) \\ & \leq s \left[\left(\hat{H}_b|_{\aleph}^{\aleph} - \inf_{p \in \aleph} \inf_{r \in \aleph} m_b|_{p,r} \right) + \left(\hat{H}_b|_{\mathbb{Q}}^{\aleph} - \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \right) \right] - \inf_{r \in \aleph} m_b|_r^r. \end{aligned}$$

Proof.

- (1) It is straightforward from (a) of Lemma 3.1.2.
- (2) It directly follows from (M_{biii}) of Definition 1.3.1.1.
- (3) Using Definition of $\hat{H}_b|_{\mathbb{Q}}^{\aleph}$ and (b) of Lemma 3.1.2, we get

$$\begin{aligned} \hat{H}_b|_{\mathbb{Q}}^{\aleph} &= \max \left\{ \delta_b|_{\mathbb{Q}}^{\aleph}, \delta_b|_{\aleph}^{\mathbb{Q}} \right\} \\ &\leq \max \left\{ s \left[\delta_b|_{\aleph}^{\aleph} - \inf_{p \in \aleph} \inf_{r \in \aleph} m_b|_{p,r} + \delta_b|_{\mathbb{Q}}^{\aleph} - \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \right] - \inf_{r \in \aleph} m_b|_r^r + \supsup_{p \in \aleph, q \in \mathbb{Q}} m_b|_{p,q}, \right. \\ &\quad \left. s \left[\delta_b|_{\aleph}^{\mathbb{Q}} - \inf_{q \in \mathbb{Q}} \inf_{r \in \aleph} m_b|_{q,r} + \delta_b|_{\mathbb{Q}}^{\aleph} - \inf_{r \in \aleph} \inf_{p \in \aleph} m_b|_{r,p} \right] - \inf_{r \in \aleph} m_b|_r^r + \supsup_{q \in \mathbb{Q}, p \in \aleph} m_b|_{q,p} \right\} \\ &\leq \max \left\{ s \left[\delta_b|_{\aleph}^{\aleph}, \delta_b|_{\aleph}^{\mathbb{Q}} \right] \right\} + \max \left\{ s \left[\delta_b|_{\mathbb{Q}}^{\aleph}, \delta_b|_{\aleph}^{\mathbb{Q}} \right] \right\} - s \inf_{p \in \aleph} \inf_{r \in \aleph} m_b|_{p,r} - s \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \\ &\quad - \inf_{r \in \aleph} m_b|_r^r + \supsup_{p \in \aleph, q \in \mathbb{Q}} m_b|_{p,q}. \end{aligned}$$

That is

$$\hat{H}_b|_{\mathbb{Q}}^{\aleph} - \supsup_{p \in \aleph, q \in \mathbb{Q}} m_b|_{p,q} \leq s \left[\hat{H}_b|_{\aleph}^{\aleph} - \inf_{p \in \aleph} \inf_{r \in \aleph} m_b|_{p,r} + \hat{H}_b|_{\mathbb{Q}}^{\aleph} - \inf_{r \in \aleph} \inf_{q \in \mathbb{Q}} m_b|_{r,q} \right] - \inf_{r \in \aleph} m_b|_r^r$$

■

Lemma 3.1.4. Let $\aleph, \mathbb{Q} \in CB_b|_{\Omega_b}$ and $h > 1$, then $\forall_{p \in \aleph}, \exists_{q \in \mathbb{Q}}$ so that

- (i) $m_b|_q^p \leq h \hat{H}_b|_{\mathbb{Q}}^{\aleph}$;
- (ii) $m_b|_q^p \leq \hat{H}_b|_{\mathbb{Q}}^{\aleph} + h$.

Proof. (i) Suppose that $\exists_{p \in \aleph}$ such that $m_b|_q^p > h \hat{H}_b|_{\mathbb{Q}}^{\aleph}, \forall_{q \in \mathbb{Q}}$. This implies that $\inf \left\{ m|_q^p, q \in \mathbb{Q} \right\} \geq h \hat{H}_b|_{\mathbb{Q}}^{\aleph}$, that is $m|_{\mathbb{Q}}^p \geq h \hat{H}_b|_{\mathbb{Q}}^{\aleph}$.

Note that

$$\hat{H}_b|_{\mathbb{Q}}^{\aleph} \geq \delta_b|_{\mathbb{Q}}^{\aleph} = \sup \left\{ m|_{\mathbb{Q}}^{\lambda}, \lambda \in \aleph \right\} \geq m|_{\mathbb{Q}}^p \geq h \hat{H}_b|_{\mathbb{Q}}^{\aleph}.$$

That is a contradiction. Thus, $m_b|_q^p \leq h \hat{H}_b|_{\mathbb{Q}}^{\aleph}$.

(ii) Suppose that $\exists_{p \in \aleph}$ such that $m_b|_q^p > \hat{H}_b|_{\mathbb{Q}}^{\aleph} + h, \forall_{q \in \mathbb{Q}}$. This implies that $\inf \left\{ m|_q^p, q \in \mathbb{Q} \right\} \geq \hat{H}_b|_{\mathbb{Q}}^{\aleph} + h$, that is $m|_{\mathbb{Q}}^p \geq \hat{H}_b|_{\mathbb{Q}}^{\aleph} + h$.

Note that

$$\hat{H}_b|_{\mathbb{Q}}^{\aleph} \geq \delta_b|_{\mathbb{Q}}^{\aleph} = \sup \left\{ m|_{\mathbb{Q}}^{\lambda}, \lambda \in \aleph \right\} \geq m|_{\mathbb{Q}}^p \geq \hat{H}_b|_{\mathbb{Q}}^{\aleph} + h.$$

That is a contradiction. Thus, $m_b|_q^p \leq \hat{H}_b|_{\mathbb{Q}}^{\aleph} + h$. ■

Remark 3.1.5. For all ξ_1, ξ_2, ξ_3 in an Mb-M.S (Ω_b, m_b) , then

- (1) $M_b|_{\xi_1, \xi_2} + m_b|_{\xi_1, \xi_2} = m_b|_{\xi_1}^{\xi_1} + m_b|_{\xi_2}^{\xi_2}$;
- (2) $M_b|_{\xi_1, \xi_2} - m_b|_{\xi_1, \xi_2} = \left| m_b|_{\xi_1}^{\xi_1} + m_b|_{\xi_2}^{\xi_2} \right|$;
- (3) For $s \geq 1$, we have

$$M_b|_{\xi_1, \xi_2} - m_b|_{\xi_1, \xi_2} \leq s \left[\left(M_b|_{\xi_1, \xi_3} - m_b|_{\xi_1, \xi_3} \right) + \left(M_b|_{\xi_3, \xi_2} - m_b|_{\xi_3, \xi_2} \right) \right].$$

Notice that: If $s = 1$, then we get Remark 1.1 in [23].

Definition 3.1.6. Let $\Omega_b \neq \emptyset$, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be three functions and $\mathbb{A}, \mathbb{B} \subseteq \Omega_b$. A mapping $\Gamma : \Omega_b \rightarrow CB_b|_{\Omega_b}$ is regared as η -cyclic (α_*, β_*) -admissible if for some $\xi \in \Omega_b$,

$$\alpha(\xi) \geq \eta(\xi) \Rightarrow \beta_*(\Gamma\xi) \geq \eta_*(\Gamma\xi),$$

and

$$\beta(\xi) \geq \eta(\xi) \Rightarrow \alpha_*(\Gamma\xi) \geq \eta_*(\Gamma\xi),$$

where $\beta_*(A) = \inf_{a \in A} \beta(a)$ and $\alpha_*(B) = \inf_{b \in B} \alpha(b)$.

Definition 3.1.7. Let $\Omega_b \neq \emptyset$, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be three functions and $A, B \subseteq \Omega_b$. The two mappings $\mathfrak{S}, \Gamma : \Omega_b \rightarrow CB_b|^{|\Omega_b|}$ are regarded as η -cyclic (α_*, β_*) -admissible if for some $\xi \in \Omega_b$,

$$\alpha(\xi) \geq \eta(\xi) \Rightarrow \beta_*(\mathfrak{S}\xi) \geq \eta_*(\mathfrak{S}\xi),$$

and

$$\beta(\xi) \geq \eta(\xi) \Rightarrow \alpha_*(\Gamma\xi) \geq \eta_*(\Gamma\xi).$$

Notice that:

- If $\eta = \eta_* = 1$ and $\mathfrak{S} = \Gamma$, then we get Definition 2.2 in [19].
- Definition 3.1.7 reduces to Definition 3.1.6 if we put $\mathfrak{S} = \Gamma$.

Example 3.1.8. Let $\Omega_b = [0, \infty)$. Define the mappings $\mathfrak{S}, \Gamma : \Omega_b \rightarrow CB|^{|\Omega_b|}$ and $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ by $\mathfrak{S}\xi = \{3\xi\}$, $\Gamma\xi = \{\xi^2\}$, $\eta(\xi) = \xi \ \forall \xi \in \Omega_b$,

$$\alpha(\xi) = \begin{cases} e^{3\xi^2} & \text{if } \xi > 0 \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(\xi) = \begin{cases} 5^{2\xi} & \text{if } \xi > 0 \\ 1 & \text{otherwise} \end{cases}.$$

Then, $\forall \xi > 0$, we get

$$\alpha(\xi) = e^{3\xi^2} \geq \xi = \eta(\xi) \implies \beta_*(\mathfrak{S}\xi) = \beta_*(3\xi) = 5^{6\xi} \geq 3\xi = \eta_*(\mathfrak{S}\xi).$$

Similarly

$$\beta(\xi) = 5^{2\xi} \geq \xi = \eta(\xi) \implies \alpha_*(\Gamma\xi) = \alpha_*(\xi^2) = e^{3\xi^4} \geq \xi^2 = \eta_*(\Gamma\xi).$$

Otherwise, for $\xi = 0$ the conditions are satisfied and the pair $(\mathfrak{S}, \Gamma)$ is η -cyclic (α_*, β_*) -admissible mappings.

Definition 3.1.9. Let (Ω_b, m_b) be an Mb-M.S, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be functions. Two multi-valued maps $\mathfrak{S}, \Gamma : \Omega_b \rightarrow CB_b|^{|\Omega_b|}$ are regarded as generalized (α_*, β_*) -Geraghty $\mathcal{F}$ -contractive

maps if $\exists_{\varphi \in \Pi}$ and $\exists_{\mathcal{F} \in \Delta_s}$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b}$, and $\tau \in \mathbb{R}_+$ with $\hat{H}_b \Big|_{\Gamma \xi_2}^{\mathfrak{S} \xi_1} > 0$,

$$\begin{aligned} \beta_* (\mathfrak{S} \xi_1) \alpha_* (\Gamma \xi_2) &\geq \eta_* (\mathfrak{S} \xi_1) \eta_* (\Gamma \xi_2) \\ \implies \tau + \mathcal{F} \left(s \hat{H}_b \Big|_{\Gamma \xi_2}^{\mathfrak{S} \xi_1} \right) &\leq \mathcal{F} \left(\varphi \left(M_b \Big|_{\xi_2}^{\xi_1} \right) \cdot M_b \Big|_{\xi_2}^{\xi_1} \right), \end{aligned} \quad (3.2)$$

where

$$M_b \Big|_{\xi_2}^{\xi_1} = \max \left\{ m_b \Big|_{\xi_2}^{\xi_1}, m_b \Big|_{\mathfrak{S} \xi_1}^{\xi_1}, m_b \Big|_{\Gamma \xi_2}^{\xi_2}, \frac{m_b \Big|_{\mathfrak{S} \xi_1}^{\xi_1} \cdot m_b \Big|_{\Gamma \xi_2}^{\xi_2}}{s + m_b \Big|_{\xi_2}^{\xi_1}} \right\}. \quad (3.3)$$

Theorem 3.1.10. Let (Ω_b, m_b) be a C-Mb-M.S, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be three functions, and $\mathfrak{S}, \Gamma : \Omega_b \rightarrow CB_b \Big|_{\Omega_b}^{\Omega_b}$ be multivalued maps satisfy the conditions below:

- (1) The pair $(\mathfrak{S}, \Gamma)$ is generalized (α_*, β_*) -Geraghty $\mathcal{F}$ -contraction;
- (2) The pair $(\mathfrak{S}, \Gamma)$ is η -cyclic (α_*, β_*) -admissible;
- (3) Either $\exists_{\xi_0 \in \Omega_b}$ such that $\alpha_*(\Gamma \xi_0) \geq \eta_*(\Gamma \xi_0)$ or $\exists_{\sigma_0 \in \Omega_b}$ such that $\beta_*(\mathfrak{S} \sigma_0) \geq \eta_*(\mathfrak{S} \sigma_0)$.

Then, $\mathfrak{S}$ and Γ have a C.F.P $\xi^* \in \Omega_b$.

Proof. Let $\xi_0 \in \Omega_b$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$, by axiom (2) $\exists_{\xi_1 \in \mathfrak{S} \xi_0}$ and $\exists_{\xi_2 \in \Gamma \xi_1}$, such that

$$\alpha(\xi_0) \geq \eta(\xi_0) \Rightarrow \beta(\xi_1) \geq \beta_*(\mathfrak{S} \xi_0) \geq \eta_*(\mathfrak{S} \xi_0),$$

and

$$\alpha(\xi_2) \geq \alpha_*(\Gamma \xi_1) \geq \eta_*(\Gamma \xi_1).$$

Therefore

$$\alpha_*(\Gamma \xi_1) \beta_*(\mathfrak{S} \xi_0) \geq \eta_*(\Gamma \xi_1) \eta_*(\mathfrak{S} \xi_0).$$

From Remark 1.2.20, we get

$$\mathcal{F} \left(s m_b \Big|_{\Gamma \xi_1}^{\xi_1} \right) = \inf_{\sigma \in \Gamma \xi_1} \mathcal{F} \left(s m_b \Big|_{\sigma}^{\xi_1} \right).$$

Thus, $\exists_{\sigma = \xi_2 \in \Gamma \xi_1}$, such that

$$\begin{aligned} \mathcal{F} \left(s m_b \Big|_{\xi_2}^{\xi_1} \right) &\leq \mathcal{F} \left(s H_b \Big|_{\Gamma \xi_1}^{\mathfrak{S} \xi_0} \right) \\ &\leq \mathcal{F} \left(\varphi \left(M_b \Big|_{\xi_1}^{\xi_0} \right) M_b \Big|_{\xi_1}^{\xi_0} \right) - \tau, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned}
M_b|_{\xi_1}^{\xi_0} &= \max \left\{ m_b|_{\xi_1}^{\xi_0}, m_b|_{\mathfrak{S}\xi_0}^{\xi_0}, m_b|_{\Gamma\xi_1}^{\xi_1}, \frac{m_b|_{\mathfrak{S}\xi_0}^{\xi_0} \cdot m_b|_{\Gamma\xi_1}^{\xi_1}}{s + m_b|_{\xi_1}^{\xi_0}} \right\} \\
&\leq \max \left\{ m_b|_{\xi_1}^{\xi_0}, m_b|_{\xi_2}^{\xi_1}, \frac{m_b|_{\xi_1}^{\xi_0} \cdot m_b|_{\xi_2}^{\xi_1}}{m_b|_{\xi_1}^{\xi_0}} \right\} \\
&= \max \left\{ m_b|_{\xi_1}^{\xi_0}, m_b|_{\xi_2}^{\xi_1} \right\}.
\end{aligned}$$

If $M_b|_{\xi_1}^{\xi_0} \leq m_b|_{\xi_2}^{\xi_1}$, then from (3.4), we can write

$$\begin{aligned}
\mathcal{F}(s m_b|_{\xi_2}^{\xi_1}) &\leq \mathcal{F}(\varphi(m_b|_{\xi_2}^{\xi_1}) \cdot m_b|_{\xi_2}^{\xi_1}) - \tau \\
&< \mathcal{F}(m_b|_{\xi_2}^{\xi_1}).
\end{aligned}$$

Applying (F1), we have

$$s m_b|_{\xi_2}^{\xi_1} < m_b|_{\xi_2}^{\xi_1},$$

a contradiction. If $M_b|_{\xi_1}^{\xi_0} \leq m_b|_{\xi_1}^{\xi_0}$, then by (3.4), we get

$$\begin{aligned}
\mathcal{F}(s m_b|_{\xi_2}^{\xi_1}) &\leq \mathcal{F}(\varphi(m_b|_{\xi_1}^{\xi_0}) m_b|_{\xi_1}^{\xi_0}) - \tau \\
&< \mathcal{F}(m_b|_{\xi_1}^{\xi_0}).
\end{aligned}$$

From (F1) again, we get

$$s m_b|_{\xi_2}^{\xi_1} < m_b|_{\xi_1}^{\xi_0}.$$

Analogous to (3.4), there is $\xi_3 \in \mathfrak{S}\xi_2$ such that

$$\begin{aligned}
\mathcal{F}(s m_b|_{\xi_3}^{\xi_2}) &\leq \mathcal{F}(\varphi(M_b|_{\xi_2}^{\xi_1}) \cdot M_b|_{\xi_2}^{\xi_1}) - \tau \\
&\quad \mathcal{F}(\varphi(m_b|_{\xi_2}^{\xi_1}) \cdot m_b|_{\xi_2}^{\xi_1}) - \tau.
\end{aligned}$$

Continuing with the same manners, we construct a sequence $\{\xi_\kappa\}$ in Ω_b so that $\xi_{2\kappa+1} \in \mathfrak{S}\xi_{2\kappa}$

and $\xi_{2\kappa+2} \in \Gamma \xi_{2\kappa+1}$. As the pair $(\mathfrak{S}, \Gamma)$ is η -cyclic (α_*, β_*) -admissible, we have

$$\beta_*(\mathfrak{S}\xi_{2\kappa})\alpha_*(\Gamma\xi_{2\kappa+1}) \geq \eta_*(\mathfrak{S}\xi_{2\kappa})\eta_*(\Gamma\xi_{2\kappa+1}) \quad \forall \kappa \geq 0.$$

Subsequently by (3.2), we have

$$\begin{aligned} \tau + \mathcal{F}\left(s m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}}\right) &\leq \tau + \mathcal{F}\left(s H_b|_{\Gamma\xi_{2\kappa+1}}^{\mathfrak{S}\xi_{2\kappa}}\right) \\ &\leq \mathcal{F}\left(\varphi\left(M_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right) \cdot M_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right) \\ &\leq \mathcal{F}\left(\varphi\left(m_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right) \cdot m_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right) \\ &\leq \mathcal{F}\left(m_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right). \end{aligned} \tag{3.5}$$

Therefore, (3.5) implies that

$$\tau + \mathcal{F}\left(s m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}}\right) \leq \mathcal{F}\left(m_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right).$$

Let $\rho_{2\kappa+1} = m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}}$ and $\mu_{2\kappa+1} = s^{2\kappa+1}\rho_{2\kappa+1} \quad \forall \kappa \geq 0$, then we write

$$\tau + \mathcal{F}\left(s\rho_{2\kappa+1}\right) \leq \mathcal{F}\left(\rho_{2\kappa}\right), \quad \forall \kappa \geq 0.$$

By (F4), we get

$$\tau + \mathcal{F}\left(\mu_{2\kappa+1}\right) \leq \mathcal{F}\left(\mu_{2\kappa}\right), \quad \forall \kappa \geq 0. \tag{3.6}$$

From (3.6) again, we infer that

$$\mathcal{F}\left(\mu_{2\kappa+1}\right) \leq \mathcal{F}\left(\mu_{2\kappa}\right) - \tau \leq \dots \leq \mathcal{F}\left(\mu_0\right) - (2\kappa+1)\tau, \quad \forall \kappa \geq 0. \tag{3.7}$$

Litting $\kappa \rightarrow \infty$ in (3.7), we have

$$\lim_{\kappa \rightarrow \infty} \mathcal{F}\left(\mu_{2\kappa+1}\right) = -\infty,$$

which implies from (F2) that

$$\lim_{\kappa \rightarrow \infty} \mu_{2\kappa+1} = 0.$$

From (F3), $\exists_{\hat{h} \in (0,1)}$ for which

$$\lim_{\kappa \rightarrow \infty} (\mu_{2\kappa+1})^{\hat{h}} \mathcal{F}(\mu_{2\kappa+1}) = 0.$$

Using (3.7), we have

$$\begin{aligned} (\mu_{2\kappa+1})^{\hat{h}} \mathcal{F}(\mu_{2\kappa+1}) - (\mu_{2\kappa+1})^{\hat{h}} \mathcal{F}(\mu_0) &\leq \left[\begin{aligned} &(\mu_{2\kappa+1})^{\hat{h}} (\mathcal{F}(\mu_0) - (2\kappa+1)\tau) \\ &- (\mu_{2\kappa+1})^{\hat{h}} \mathcal{F}(\mu_0) \end{aligned} \right] \\ &= -(\mu_{2\kappa+1})^{\hat{h}} (2\kappa+1)\tau \\ &\leq 0, \quad \forall_{\kappa \geq 0}. \end{aligned}$$

Taking limit as $\kappa \rightarrow \infty$ and because of $\tau > 0$, we get

$$\lim_{\kappa \rightarrow \infty} (\mu_{2\kappa+1})^{\hat{h}} (2\kappa+1) = 0.$$

Thus, $\exists_{\kappa_1 \in \mathbb{N}}$ such that

$$(\mu_{2\kappa+1})^{\hat{h}} (2\kappa+1) \leq 1 \quad \forall_{\kappa \geq \kappa_1} \implies \mu_{2\kappa+1} \leq \frac{1}{(2\kappa+1)^{\frac{1}{\hat{h}}}} \quad \forall_{\kappa \geq \kappa_1}.$$

Henceforth, the series $\sum_{\kappa} \mu_{2\kappa+1}$ is convergent.

Now, we verify that $\{\xi_{\kappa}\}$ is an m_b -Cauchy in Ω_b . Using (M_{biv}) , we get

$$\begin{aligned} &m_b|_{\xi_{2\kappa+3}}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+3}} \\ &\leq s \left[m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} + m_b|_{\xi_{2\kappa+3}}^{\xi_{2\kappa+2}} - m_b|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}} \right] - m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+2}} \\ &\leq s \left[m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} + m_b|_{\xi_{2\kappa+3}}^{\xi_{2\kappa+2}} - m_b|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}} \right] \\ &\leq s m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} + s^2 m_b|_{\xi_{2\kappa+3}}^{\xi_{2\kappa+2}}. \end{aligned}$$

Similarly

$$\begin{aligned}
& m_b|_{\xi_{2\kappa+4}}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+4}} \\
& \leq s \left[m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} + m_b|_{\xi_{2\kappa+4}}^{\xi_{2\kappa+2}} - m_b|_{\xi_{2\kappa+2}, \xi_{2\kappa+4}} \right] - m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+2}} \\
& \leq s m_b|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} + s^2 m_b|_{\xi_{2\kappa+3}}^{\xi_{2\kappa+2}} + s^3 m_b|_{\xi_{2\kappa+4}}^{\xi_{2\kappa+3}}.
\end{aligned}$$

In general, $\forall q > p > \kappa_1$ with $p = 2\kappa + 1$, we obtain

$$m_b|_{\xi_q}^{\xi_p} - m_b|_{\xi_p, \xi_q} \leq \sum_{i=p}^{q-1} s^{i-p+1} m_b|_{\xi_{i+1}}^{\xi_i} \leq \sum_{i=p}^{q-1} s^i m_b|_{\xi_{i+1}}^{\xi_i} \leq \sum_{i=p}^{\infty} \mu_i.$$

The convergence of $\sum_{i=p}^{\infty} \mu_i$ leads to

$$\lim_{p, q \rightarrow \infty} \left(m_b|_{\xi_q}^{\xi_p} - m_b|_{\xi_p, \xi_q} \right) = 0.$$

By the same way and from Remark 3.1.5, we obtain

$$\begin{aligned}
& M_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+4}} - m_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+4}} \\
& \leq s \left(M_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} - m_b|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} \right) + s^2 \left(M_b|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}} - m_b|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}} \right) \\
& \quad + s^3 \left(M_b|_{\xi_{2\kappa+3}, \xi_{2\kappa+4}} - m_b|_{\xi_{2\kappa+3}, \xi_{2\kappa+4}} \right).
\end{aligned}$$

In general, $\forall q > p > \kappa_1$ with $p = 2\kappa + 1$, we obtain

$$\begin{aligned}
M_b|_{\xi_p, \xi_q} - m_b|_{\xi_p, \xi_q} & \leq \sum_{i=p}^{q-1} s^{i-p+1} \left(M_b|_{\xi_i, \xi_{i+1}} - m_b|_{\xi_i, \xi_{i+1}} \right) \leq \sum_{i=p}^{q-1} s^{i-p+1} M_b|_{\xi_i, \xi_{i+1}} \\
& \leq \sum_{i=p}^{q-1} s^{i-p+1} m_b|_{\xi_i}^{\xi_i} \leq \sum_{i=p}^{q-1} s^{i-p+1} m_b|_{\xi_{i+1}}^{\xi_i} \leq \sum_{i=p}^{q-1} s^i m_b|_{\xi_{i+1}}^{\xi_i} \leq \sum_{i=p}^{\infty} \mu_i.
\end{aligned}$$

The convergence of $\sum_{i=p}^{\infty} \mu_i$ leads to

$$\lim_{p, q \rightarrow \infty} \left(M_b|_{\xi_p, \xi_q} - m_b|_{\xi_p, \xi_q} \right) = 0.$$

Therefore, $\{\xi_\kappa\}$ is an m_b -Cauchy in Ω_b . Because Ω_b is m_b -complete, $\exists_{\xi^* \in \Omega_b}$ such that $\xi_\kappa \longrightarrow \xi^*$ as $\kappa \longrightarrow \infty$, implies $\xi_{2\kappa+1} \rightarrow \xi^*$ and $\xi_{2\kappa+2} \rightarrow \xi^*$ as $\kappa \rightarrow \infty$. Thus, we have

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\xi^*}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \xi^*} \right) = 0. \quad (3.8)$$

Since $\lim_{\kappa \rightarrow \infty} m_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa+1}} = 0$, then from (3.8), we get

$$\lim_{\kappa \rightarrow \infty} m_b|_{\xi^*}^{\xi_{2\kappa+1}} = 0. \quad (3.9)$$

From (3.2), (3.5) and (3.9), we conclude that

$$\lim_{\kappa \rightarrow \infty} \hat{H}_b|_{\Gamma \xi^*}^{\mathfrak{S} \xi_{2\kappa}} = 0. \quad (3.10)$$

Since $\xi_{2\kappa+1} \in \mathfrak{S} \xi_{2\kappa}$ and

$$m_b|_{\Gamma \xi^*}^{\xi_{2\kappa+1}} \leq \hat{H}_b|_{\Gamma \xi^*}^{\mathfrak{S} \xi_{2\kappa}}.$$

So, taking limit as $\kappa \rightarrow \infty$ above, we get

$$\lim_{\kappa \rightarrow \infty} m_b|_{\Gamma \xi^*}^{\xi_{2\kappa+1}} = 0. \quad (3.11)$$

By (M_{bii}) , we write

$$m_b|_{\xi_{2\kappa+1}, \Gamma \xi^*} \leq m_b|_{\Gamma \xi^*}^{\xi_{2\kappa+1}},$$

that is

$$\lim_{\kappa \rightarrow \infty} m_b|_{\xi_{2\kappa+1}, \Gamma \xi^*} = 0. \quad (3.12)$$

Now, utilizing (M_{biv}) , we get

$$\begin{aligned} & m_b|_{\Gamma \xi^*}^{\xi^*} - \sup_{\gamma \in \Gamma \xi^*} m_b|_{\xi^*, \gamma} \\ & \leq m_b|_{\Gamma \xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma \xi^*} \\ & \leq s \left[m_b|_{\xi_{2\kappa+1}}^{\xi^*} - m_b|_{\xi^*, \xi_{2\kappa+1}} + m_b|_{\Gamma \xi^*}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \Gamma \xi^*} \right] \\ & \quad - m_b|_{\xi_{2\kappa+1}}^{\xi_{2\kappa+1}}. \end{aligned} \quad (3.13)$$

Letting $\kappa \rightarrow \infty$ in (3.13), from (3.9), (3.11) and (3.12), we deduce that

$$m_b|_{\Gamma\xi^*}^{\xi^*} \leq \sup_{\gamma \in \Gamma\xi^*} m_b|_{\xi^*,\gamma}. \quad (3.14)$$

Using (M_{bii}) , $m_b|_{\xi^*,\gamma} \leq m_b|_{\gamma}^{\xi^*} \quad \forall \gamma \in \Gamma\xi^*$ implies that

$$m_b|_{\xi^*,\gamma} - m_b|_{\gamma}^{\xi^*} \leq 0.$$

Thus

$$\sup \left\{ m_b|_{\xi^*,\gamma} - m_b|_{\gamma}^{\xi^*} : \gamma \in \Gamma\xi^* \right\} \leq 0.$$

Implying that

$$\sup_{\gamma \in \Gamma\xi^*} m_b|_{\xi^*,\gamma} - \inf_{\gamma \in \Gamma\xi^*} m_b|_{\gamma}^{\xi^*} \leq 0.$$

Therefore

$$\sup_{\gamma \in \Gamma\xi^*} m_b|_{\xi^*,\gamma} \leq m_b|_{\Gamma\xi^*}^{\xi^*}. \quad (3.15)$$

From (3.14) and (3.15), we have

$$m_b|_{\Gamma\xi^*}^{\xi^*} = \sup_{\gamma \in \Gamma\xi^*} m_b|_{\xi^*,\gamma}.$$

Thence, by Lemma 3.1.1, we get $\xi^* \in \overline{\Gamma\xi^*} = \Gamma\xi^*$. Similarly, we can easily conclude that $\xi^* \in \mathfrak{I}\xi^*$. Hence, ξ^* is a C.F.P of $\mathfrak{I}$ and Γ .

■

Remark 3.1.11. Theorem 3.1.10, stil valid if we consider the following:

◆ If we put $s = 1$ in Definition 3.1.9, then generalized (α_*, β_*) -Geraghty $\mathcal{F}$ -contraction mappings takes the form $H_m|_{\Gamma\xi_2}^{\mathfrak{I}\xi_1} > 0$,

$$\begin{aligned} \alpha_*(\mathfrak{I}\xi_1)\beta_*(\Gamma\xi_2) &\geq \eta_*(\mathfrak{I}\xi_1)\eta_*(\Gamma\xi_2) \\ \implies \tau + \mathcal{F}\left(H_m|_{\Gamma\xi_2}^{\mathfrak{I}\xi_1}\right) &\leq \mathcal{F}\left(\varphi\left(M_m|_{\xi_2}^{\xi_1}\right) \cdot M_m|_{\xi_2}^{\xi_1}\right), \end{aligned}$$

where $\varphi \in \Pi$, $\mathcal{F} \in \Delta_{\mathcal{F}}$, $\tau \in R_+$ and

$$M_m|_{\xi_2}^{\xi_1} = \max \left\{ m|_{\xi_2}^{\xi_1}, m|_{\mathfrak{S}\xi_1}^{\xi_1}, m|_{\Gamma\xi_2}^{\xi_2}, \frac{m|_{\mathfrak{S}\xi_1}^{\xi_1} \cdot m|_{\Gamma\xi_2}^{\xi_2}}{1 + m|_{\xi_2}^{\xi_1}} \right\}.$$

Moreover, under the same conditions of Theorem 3.1.10 $\mathfrak{S}$ and Γ have a C.F.P in a C-M-M.S.

◆ If we take $M_b|_{\xi_2}^{\xi_1} = m_b|_{\xi_2}^{\xi_1}$ in Definition 3.1.9, then $\mathfrak{S}$ and Γ have a C.F.P in a C-Mb-M.S, provided that the stipulations (1) – (3) of Theorem 3.1.10 hold.

◆ If we consider $\mathfrak{S} = \Gamma$ in Definition 3.1.9, then the results is given immidiately by tracing the line of proof employed in the Theorem 3.1.10.

Example 3.1.12. Let $\Omega_b = [0, \infty)$ and $m_b : \Omega_b \times \Omega_b \longrightarrow [0, \infty)$ defined by $m_b|_{\xi_2}^{\xi_1} = \max \{ \xi_1, \xi_2 \}^p + |\xi_1 - \xi_2|^p$, $\forall \xi_1, \xi_2 \in \Omega_b$. Then, (Ω_b, m_b) is an Mb-M.S with $p > 1$ and $s = 2^p$, which is not an M-M.S. Since if $\xi_1 = 5$, $\xi_2 = 1$ and $\xi_3 = 4$, we obtain $m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2}^{\xi_1} > m_b|_{\xi_3}^{\xi_1} - m_b|_{\xi_1, \xi_3}^{\xi_1} - m_b|_{\xi_2}^{\xi_3} - m_b|_{\xi_3, \xi_2}^{\xi_3}$. Define $\mathfrak{S}, \Gamma : \Omega_b \rightarrow CB|_{\Omega_b}$ by

$$\mathfrak{S}\xi = \begin{cases} \left\{ \frac{\xi}{64} \right\}, & \text{if } \xi \in (0, 1] \\ \left\{ 0, \frac{1}{16} \right\} & \text{otherwise} \end{cases} \quad \text{and} \quad \Gamma\xi = \begin{cases} \left\{ 0, \frac{\xi}{48} \right\} & \text{if } \xi \in (0, 1] \\ \{0\} & \text{otherwise} \end{cases}.$$

Also, we define $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ by $\eta(\xi) = \xi + 1 \ \forall \xi \in \Omega_b$,

$$\alpha(\xi) = \begin{cases} 3e^{2\xi^2} & \text{if } \xi > 0 \\ 3 & \xi = 0 \end{cases} \quad \text{and} \quad \beta(\xi) = \begin{cases} 5\xi + 1 & \xi > 0 \\ 1 & \xi = 0 \end{cases}.$$

For $\xi \in (0, 1]$, we have

$$\alpha(\xi) \geq \eta(\xi) \Rightarrow \beta_*(\mathfrak{S}\xi) = \beta_*\left(\left\{\frac{\xi}{64}\right\}\right) = \frac{5\xi}{64} + 1 \geq \frac{\xi}{64} + 1 = \eta_*\left(\left\{\frac{\xi}{64}\right\}\right) = \eta_*(\mathfrak{S}\xi).$$

If $\Gamma\xi = 0$, then

$$\beta(\xi) \geq \eta(\xi) \Rightarrow \alpha_*(\Gamma\xi) = \alpha_*(\{0\}) = 3 \geq 1 = \eta_*(\{0\}) = \eta_*(\Gamma\xi).$$

If $\Gamma\xi = \frac{\xi}{48}$, then

$$\beta(\xi) \geq \eta(\xi) \Rightarrow \alpha_*(\Gamma\xi) = \alpha_*\left(\left\{\frac{\xi}{48}\right\}\right) = 3e^{2\left(\frac{\xi}{48}\right)^2} \geq \frac{\xi}{48} + 1 = \eta_*\left(\left\{\frac{\xi}{48}\right\}\right) = \eta_*(\Gamma\xi).$$

Hence, $(\mathfrak{S}, \Gamma)$ is η -cyclic (α_*, β_*) -admissible maps. Define $\varphi(t) = \frac{1}{6^p} \in \Pi$ for $\xi_1, \xi_2 \in (0, 1]$, we write

$$\begin{aligned} \mathcal{F}\left(s \hat{H}_b \Big|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1}\right) &= \mathcal{F}\left(2^p \left(\max \left\{ \sup_{a \in \mathfrak{S}\xi_1} m_b \Big|_{\Gamma\xi_2}^a, \sup_{b \in \Gamma\xi_2} m_b \Big|_b^{\mathfrak{S}\xi_1} \right\}\right)\right) \\ &= \mathcal{F}\left(2^p \max \left(m_b \Big|_{\left\{0, \frac{\xi_2}{48}\right\}}^{\frac{\xi_1}{64}}, m_b \Big|_{\frac{\xi_2}{48}}^{\left\{\frac{\xi_1}{64}\right\}} \right)\right) \\ &= \mathcal{F}\left(2^p \max \left(m_b \Big|_0^{\frac{\xi_1}{64}}, m_b \Big|_{\frac{\xi_2}{48}}^{\frac{\xi_1}{64}} \right)\right) \\ &= \mathcal{F}\left(2^p m_b \Big|_{\frac{\xi_2}{48}}^{\frac{\xi_1}{64}}\right) = \mathcal{F}\left(\frac{2^p}{16^p} m_b \Big|_{\frac{\xi_2}{3}}^{\frac{\xi_1}{4}}\right) \\ &= \mathcal{F}\left(\frac{1}{2^{3p}} m_b \Big|_{\frac{\xi_2}{3}}^{\frac{\xi_1}{4}}\right) \leq \mathcal{F}\left(\frac{1}{6^p} m_b \Big|_{\xi_2}^{\xi_1}\right). \end{aligned}$$

By taking $\mathcal{F}(\xi_1) = \ln \xi_1$, we get

$$\ln \left(s \hat{H}_b \Big|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \right) \leq \ln \left(\frac{1}{6^p} m_b \Big|_{\xi_2}^{\xi_1} \right) = -p \ln(6) + \ln \left(m_b \Big|_{\xi_2}^{\xi_1} \right),$$

implying that

$$\ln \left(s \hat{H}_b \Big|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \right) \leq \ln \left(m_b \Big|_{\xi_2}^{\xi_1} \right) - \tau.$$

Since $m_b \Big|_{\xi_2}^{\xi_1} \leq M_b \Big|_{\xi_2}^{\xi_1}$, and $\varphi \in \Pi$, we obtain

$$\mathcal{F}\left(s \hat{H}_b \Big|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1}\right) \leq F\left(\varphi\left(M_b \Big|_{\xi_2}^{\xi_1}\right)\right) - \tau \leq F\left(\varphi\left(M_b \Big|_{\xi_2}^{\xi_1}\right) \cdot M_b \Big|_{\xi_2}^{\xi_1}\right) - \tau.$$

Similarly, for each $\xi_1, \xi_2 \in \Omega_b$ and $\tau = p \ln(6)$, $p > 1$ that satisfy the above inequality. Hence, all conditions of Theorem 3.1.10 are satisfied and $\xi^* = 0$ is a C.F.P of $\mathfrak{S}$ and Γ .

3.2 Some consequences

This part is reduction of previous part by considering that $\mathfrak{S}$ and Γ are single-valued maps.

Definition 3.2.1. Let $\Omega_b \neq \emptyset$, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be three functions. The map $\Gamma : \Omega_b \rightarrow \Omega_b$ is regarded as η -cyclic (α, β) -admissible if for some $\xi \in \Omega_b$,

$$\alpha(\xi) \geq \eta(\xi) \Rightarrow \beta(\Gamma\xi) \geq \eta(\Gamma\xi),$$

and

$$\beta(\xi) \geq \eta(\xi) \Rightarrow \alpha(\Gamma\xi) \geq \eta(\Gamma\xi).$$

Definition 3.2.2. Let $\Omega_b \neq \emptyset$, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be three functions. The maps $\mathfrak{S}, \Gamma : \Omega_b \rightarrow \Omega_b$ are regarded as η -cyclic (α, β) -admissible if for some $\xi \in \Omega_b$,

$$\alpha(\varsigma) \geq \eta(\xi) \Rightarrow \beta(\mathfrak{S}\xi) \geq \eta(\mathfrak{S}\xi),$$

and

$$\beta(\xi) \geq \eta(\xi) \Rightarrow \alpha(\Gamma\xi) \geq \eta(\Gamma\xi).$$

The followig results can be virified in the similar way as in Theorem 3.1.10.

Corollary 3.2.3. Let (Ω_b, m_b) be a C-Mb-M.S and $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be three functions. $\Gamma : \Omega_b \rightarrow \Omega_b$ is regarded as a generalized (α, β) -Geraghty $\mathcal{F}$ -contractive map if $\exists_{\varphi \in \Pi}$ and $\exists_{\mathcal{F} \in \Delta_s}$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b}$, $s \geq 1$ and $\tau \in \mathbb{R}_+$,

$$\begin{aligned} \alpha(\xi_1)\beta(\xi_2) &\geq \eta(\xi_1)\eta(\xi_2) \\ \implies \tau + \mathcal{F}\left(s m_b|_{\Gamma\xi_2}^{\Gamma\xi_1}\right) &\leq \mathcal{F}\left(\varphi\left(M_b|_{\xi_2}^{\xi_1}\right) M_b|_{\xi_2}^{\xi_1}\right), \end{aligned}$$

where $M_b|_{\varsigma_2}^{\varsigma_1}$ is defined as in (3.3) with taking $\mathfrak{S} = \Gamma$. Assume also that the followig hypotheses hold:

(i) Γ is an η -cyclic (α, β) -admissible;

(ii) $\exists_{\xi_0 \in \Omega_b}$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$ or $\beta(\xi_0) \geq \eta(\xi_0)$.

Then, Γ has a F.P $\xi^* \in \Omega_b$.

Corollary 3.2.4. Let (Ω_b, m_b) be a C-Mb-M.S and $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be functions. $\mathfrak{S}, \Gamma : \Omega_b \rightarrow \Omega_b$ are regarded as generalized (α, β) -Geraghty $\mathcal{F}$ -contractive-type maps if $\exists_{\varphi \in \Pi}$ and $\exists_{\mathcal{F} \in \Delta_s}$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b}$, $s \geq 1$ and $\tau \in \mathbb{R}_+$,

$$\begin{aligned} \alpha(\xi_1)\beta(\xi_2) &\geq \eta(\xi_1)\eta(\xi_2) \\ \implies \tau + \mathcal{F}\left(s m_b|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1}\right) &\leq \mathcal{F}\left(\varphi\left(M_b|_{\xi_2}^{\xi_1}\right) \cdot M_b|_{\xi_2}^{\xi_1}\right), \end{aligned}$$

where $M_b|_{\xi_2}^{\xi_1}$ is defined as in (3.3). Assume also that the followig hypotheses hold:

(i) $(\mathfrak{S}, \Gamma)$ is a pair of η -cyclic (α, β) -admissible;

(ii) $\exists_{\xi_0 \in \Omega_b}$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$ or $\exists_{\sigma_0 \in \Omega_b}$ such that $\beta(\sigma_0) \geq \eta(\sigma_0)$.

Then, $\mathfrak{S}$ and Γ have a C.F.P $\xi^* \in \Omega_b$.

Setting $\alpha(\xi_1) = \beta(\xi_2) = \eta(\xi_1) = \eta(\xi_2) = 1$ in Corollary 3.2.4, we have the following result:

Corollary 3.2.5. Let (Ω_b, m_b) be a C-Mb-M.S. The maps $\mathfrak{S}, \Gamma : \Omega_b \rightarrow \Omega_b$ are regarded as generalized Geraghty $\mathcal{F}$ -contractive maps if $\exists_{\varphi \in \Pi}$ and $\exists_{\mathcal{F} \in \Delta_s}$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b}$, $s \geq 1$ and $\tau \in \mathbb{R}_+$,

$$\tau + \mathcal{F}\left(s m_b|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1}\right) \leq \mathcal{F}\left(\varphi\left(M_b|_{\xi_2}^{\xi_1}\right) M_b|_{\xi_2}^{\xi_1}\right),$$

where $M_b|_{\xi_2}^{\xi_1}$ is defined as in (3.3). Then, $\mathfrak{S}$ and Γ have a C.F.P $\xi^* \in \Omega_b$.

3.3 Application to ordinary differential equations

Consider the next pair of ODEs:

$$\left\{ \begin{array}{l} -\frac{d^2\xi_1}{dr^2} = f|_{\xi_1(r)}^r, \quad r \in I = [0, 1] \\ \xi_1(0) = \xi_1(1) = 0. \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} -\frac{d^2\xi_2}{dr^2} = K|_{\xi_2(r)}^r, \quad r \in I = [0, 1] \\ \xi_2(0) = \xi_2(1) = 0. \end{array} \right., \quad (3.16)$$

where $f, K : I \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions and the pair of ODEs (3.16) are equalized

to the next IEs:

$$\xi_1(r) = \int_0^1 G|_k^r f|_{\xi_1(k)}^k dk \text{ and } \xi_2(r) = \int_0^1 G|_k^r K|_{\xi_1(k)}^k dk. \quad (3.17)$$

where $G : I^2 \rightarrow (-\infty, \infty)$ is a G.F defined as

$$G|_k^r = \begin{cases} r(1-k), & 0 \leq r \leq k \leq 1; \\ k(1-r), & 0 \leq k \leq r \leq 1. \end{cases}$$

Define $d : \Omega_b^2 \rightarrow [0, \infty)$ such that $\Omega = C|_{\mathbb{R}}^I$ is a space of continuous function on I by

$$m_b|_{\xi_2}^{\xi_1} = \max_{r \in I} \left(\left| \frac{\xi_1(r) + \xi_2(r)}{2} \right| \right)^2, \forall_{\xi_1, \xi_2 \in \Omega_b}.$$

Obviously, (Ω_b, m_b) is a C-Mb-M.S with a constant $s = 2$.

Theorem 3.3.1 The system (3.16) has at least one solution under the stepulations below.

(♠1) There is a function $\omega : \mathbb{R} \rightarrow (0, 1)$ such that $\forall_{\vartheta_1, \vartheta_2 \in \mathbb{R}}$, we have

$$|f|_{\vartheta_1}^r + K|_{\vartheta_2}^r| \leq \sqrt{\omega(r) M_b|_{\vartheta_2}^{\vartheta_1}} \forall_{r \in I},$$

where

$$M_b|_{\vartheta_2}^{\vartheta_1} = \max \left\{ \left| \frac{\vartheta_1 + \vartheta_2}{2} \right|^2, \left| \frac{\vartheta_1 + \Im \vartheta_1}{2} \right|^2, \left| \frac{\vartheta_2 + \Gamma \vartheta_2}{2} \right|^2, \frac{\left| \frac{\vartheta_2 + \Gamma \vartheta_2}{2} \right|^2 \cdot \left| \frac{\vartheta_1 + \Im \vartheta_1}{2} \right|^2}{s + \left| \frac{\vartheta_1 + \vartheta_2}{2} \right|^2} \right\}.$$

(♠2) There is $s \geq 1$ so that $\max_{r, k \in I} \int_0^1 G|_k^r dk \leq \sqrt{\frac{12e^{-\tau}}{7s}}$ for some $\tau > 0$.

Proof. define the operators $\Im, \Gamma : \Omega_b \rightarrow \Omega_b$ by

$$\Im \xi_1(r) = \int_0^1 G|_k^r f|_{\xi_1(k)}^k dk \text{ and } \Gamma \xi_2(r) = \int_0^1 G|_k^r K|_{\xi_2(k)}^k dk, \forall_{r \in I}.$$

Distinctly, the solvability of the IEs (3.17) is equalized to find a C.F.P of $\Im$ and Γ . Let $\xi_1, \xi_2 \in \Omega_b$,

by our assumption $\forall_{r \in I}$, we get

$$\begin{aligned}
[|\mathfrak{S}\xi_1(r) + \Gamma\xi_2(r)|]^2 &= \left[\left| \int_0^1 G|_k^r f|_{\xi_1(k)}^k ds + \int_0^1 G|_k^r K|_{\xi_2(k)}^k dk \right| \right]^2 \\
&\leq \left[\int_0^1 G|_k^r \left| f|_{\xi_1(k)}^k + K|_{\xi_2(k)}^k \right| dk \right]^2 \\
&\leq \left[\int_0^1 G|_k^r \sqrt{\omega(r) M_b|_{\xi_2}^{\xi_1}} dk \right]^2 \\
&= \left[\sqrt{\omega(r) M_b|_{\xi_2}^{\xi_1}} \right]^2 \left[\int_0^1 G|_k^r dk \right]^2 \\
&\leq \left[\omega(r) M_b|_{\xi_2}^{\xi_1} \right] \left[\sqrt{\frac{12e^{-\tau}}{7s}} \right]^2 \\
&= \omega(r) \frac{12e^{-\tau}}{7s} M_b|_{\xi_2}^{\xi_1}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\max_{r \in I} [|\mathfrak{S}\xi_1(r) + \Gamma\xi_2(r)|]^2 &\leq \left[\omega(r) M_b|_{\xi_2}^{\xi_1} \right] \left[\max_{t, s \in I} \int_0^1 G|_k^r dk \right]^2 \\
&\leq \left[\omega(r) M_b|_{\xi_2}^{\xi_1} \right] \frac{12e^{-\tau}}{7s},
\end{aligned}$$

implying that

$$s m_b|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \leq \frac{3\omega(r)}{7} e^{-\tau} M_b|_{\xi_2}^{\xi_1} \leq e^{-\tau} \varphi \left(M_b|_{\xi_2}^{\xi_1} \right) M_b|_{\xi_2}^{\xi_1},$$

which implies that

$$\tau + \ln \left(s m_b|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \right) \leq \ln \left[\varphi \left(M_b|_{\xi_2}^{\xi_1} \right) M_b|_{\xi_2}^{\xi_1} \right],$$

where $\mathcal{F}(\xi) = \ln \xi$ and $\varphi(r) = \frac{3\omega(r)}{7} \forall_{r \in I}$. Thus, all stipulations of Corollary 3.2.5 are held.

Therefore, $\mathfrak{S}$ and Γ have a C.F.P, that is a solution to the ODEs (3.16). ■

3.4 Symmetric fractional contractions and related fixed point theorems

Herein, we introduce four types of symmetric fractional $(\alpha, \beta, \eta, \psi)$ -contractions for self-mappings and obtain related F.P results in Mb-M.S and the same results is discussing on closed ball of

Mb-M.S. Consistent with [88, 91], we introduce the following results to be useful in our study.

Definition 3.4.1. Let (Ω_b, m_b) be an Mb-M.S. The open ball of Ω_b with center ξ_1 and radius ε is

$$B_b|_{\varepsilon}^{\xi_1} = \left\{ \xi_2 \in \Omega_b : m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} < \varepsilon, \forall \xi_1 \in \Omega_b \text{ and } \forall \varepsilon > 0 \right\}.$$

Notice that we have $\xi_1 \in B_b|_{\varepsilon}^{\xi_1} \forall \varepsilon > 0$. Indeed, we get

$$m_b|_{\xi_1}^{\xi_1} - m_b|_{\xi_1, \xi_1} = m_b|_{\xi_1}^{\xi_1} - m_b|_{\xi_1}^{\xi_1} = 0 < \varepsilon.$$

Analogously, the closed ball of Ω_b is

$$\overline{B_b|_{\varepsilon}^{\xi_1}} = \left\{ \xi_2 \in \Omega_b : m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} \leq \varepsilon, \forall \xi_1 \in \Omega_b \text{ and } \forall \varepsilon > 0 \right\}.$$

Lemma 3.4.2. Let (Ω_b, m_b) be an Mb-M.S and $\beta_b = \left\{ B_b|_{\varepsilon}^{\xi_1} \right\}_{\xi_1 \in \Omega_b}^{\varepsilon > 0}$ be a collection of open balls of Ω_b , then β_b forms a basis on Ω_b .

Proof. Let $\xi_2 \in B_b|_{\varepsilon}^{\xi_1}$, we get

$$m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} < \varepsilon, \forall \xi_1 \in \Omega_b, \varepsilon > 0. \quad (3.18)$$

Take $\delta = \varepsilon - m_b|_{\xi_2}^{\xi_1} + m_b|_{\xi_1, \xi_2} \implies \delta > 0$. Now, we show that $B_b|_{\delta}^{\xi_2} \subset B_b|_{\varepsilon}^{\xi_1}$. Let $\xi_3 \in B_b|_{\delta}^{\xi_2}$. Then, we obtain

$$m_b|_{\xi_3}^{\xi_2} - m_b|_{\xi_2, \xi_3} < \delta. \quad (3.19)$$

From (M_{biii}) , we get

$$m_b|_{\xi_3}^{\xi_1} - m_b|_{\xi_1, \xi_3} \leq s \left[m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} + m_b|_{\xi_3}^{\xi_2} - m_b|_{\xi_2, \xi_3} \right] - m_b|_{\xi_2}^{\xi_2}.$$

Utilizing (3.18) and (3.19), we concluded that

$$\delta = \frac{1}{s} \left(\varepsilon + m_b|_{\xi_2}^{\xi_2} \right) - \left(m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} \right). \quad (3.20)$$

Again from (M_{biii}) and using (3.18) - (3.20), we get

$$\begin{aligned}
m_b|_{\xi_3}^{\xi_1} - m_b|_{\xi_1, \xi_3} &\leq s \left[m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} + m_b|_{\xi_3}^{\xi_2} - m_b|_{\xi_2, \xi_3} \right] - m_b|_{\xi_2}^{\xi_2} \\
&< s \left[m_b|_{\xi_2}^{\xi_1} - m_b|_{\xi_1, \xi_2} + \delta \right] - m_b|_{\xi_2}^{\xi_2} \\
&< s \left[\frac{1}{s} \left(\varepsilon + m_b|_{\xi_2}^{\xi_2} \right) \right] - m_b|_{\xi_2}^{\xi_2} \\
&= \varepsilon.
\end{aligned}$$

Hence, $\xi_3 \in B_b|_{\varepsilon}^{\xi_1}$, that is $B_b|_{\delta}^{\xi_2} \subset B_b|_{\varepsilon}^{\xi_1}$. Therefore β_b satisfies the conditions of basis. ■

This segment introduces F.P results for symmetric fractional α - β - η - ψ -contraction type-I.

Definition 3.4.3. Let (Ω_b, m_b) be an Mb-M.S, $\Gamma : \Omega_b \rightarrow \Omega_b$ be a selfmapping, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be functions and $\psi \in \Psi$. Γ is termed a symmetric fractional α - β - η - ψ -contraction of pattern-I, if $\exists_{s \geq 1}$, $\exists_{a, b, c \in (0, 1)}$ and $\lambda = \left(s m_b|_{\xi_1, \xi_2} \right)^{\frac{-1}{(c-a)(c-b)}} \in [0, \infty)$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b \setminus Fix(\Gamma)}$, whenever $\alpha(\xi_1)\beta(\xi_2) \geq \eta(\xi_1)\eta(\xi_2)$, we have

$$s^2 m_b|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \psi \left[\lambda(\mathbb{H}_1|_{\xi_2}^{\xi_1}) \right], \quad (3.21)$$

where

$$\begin{aligned}
\mathbb{H}_1|_{\xi_2}^{\xi_1} &= m_b|_{\xi_2}^{\xi_1} \cdot \left[m_b|_{\Gamma\xi_1}^{\xi_1} \right]^{\frac{1}{(a-b)(a-c)}} \cdot \left[m_b|_{\Gamma\xi_2}^{\xi_2} \right]^{\frac{1}{(a-b)(a-c)}} \\
&\quad \cdot \left[m_b|_{\Gamma\xi_1}^{\xi_1} + m_b|_{\Gamma\xi_2}^{\xi_2} \right]^{\frac{1}{(b-a)(b-c)}} \cdot \left[m_b|_{\Gamma\xi_2}^{\xi_1} + m_b|_{\Gamma\xi_1}^{\xi_2} \right]^{\frac{1}{(c-a)(c-b)}}.
\end{aligned}$$

Example 3.4.4. Let $\Omega_b = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$ and $m_b : \Omega_b \times \Omega_b \rightarrow [0, \infty)$ be defined by $m_b|_{\xi_2}^{\xi_1} = \left(\frac{\xi_1 + \xi_2}{2} \right)^2$. Then (Ω_b, m_b) is a C-Mb-M.S with $s = 2$. Define $\Gamma : \Omega_b \rightarrow \Omega_b$ by

$$T0 = T\frac{1}{3} = T\frac{2}{3} = T1 = 0, \quad T\frac{1}{2} = \frac{1}{2},$$

and $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ by

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 1, & \text{if } \xi \in \Omega_b \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(\xi) = \begin{cases} \frac{1}{2}, & \text{if } \xi \in \Omega_b \\ 0, & \text{otherwise} \end{cases}.$$

Let $\psi(t) = \frac{3}{4}t$. If $\xi_1, \xi_2 \in \Omega_b$. Clearly $\alpha(\xi_1)\beta(\xi_2) \geq \eta(\xi_1)\eta(\xi_2)$, such that

$$\begin{aligned}
s^2 m_b|_{\Gamma \frac{1}{3}}^{\frac{2}{3}} &= 0 \leq \psi[\lambda m_b|_{\Gamma \frac{1}{3}}^{\frac{2}{3}} \cdot \left(m_b|_{\Gamma \frac{1}{3}}^{\frac{1}{3}}\right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma \frac{2}{3}}^{\frac{2}{3}}\right)^{\frac{1}{(a-b)(a-c)}} \\
&\quad \cdot \left(m_b|_{\Gamma \frac{1}{3}}^{\frac{1}{3}} + m_b|_{\Gamma \frac{2}{3}}^{\frac{2}{3}}\right)^{\frac{1}{(b-a)(b-c)}} \cdot \left(m_b|_{\Gamma \frac{2}{3}}^{\frac{1}{3}} + m_b|_{\Gamma \frac{1}{3}}^{\frac{2}{3}}\right)^{\frac{1}{(c-a)(c-b)}}] \\
&= \psi \left[\frac{\lambda}{4} \left(\frac{1}{36}\right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(\frac{1}{9}\right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(\frac{1}{36} + \frac{1}{9}\right)^{\frac{1}{(b-a)(b-c)}} \cdot \right. \\
&\quad \left. \left(\frac{1}{36} + \frac{1}{9}\right)^{\frac{1}{(c-a)(c-b)}} \right] \\
&= \psi \left[\frac{\lambda}{4} \left(\frac{1}{36} \times \frac{4}{36}\right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(\frac{5}{36}\right)^{\frac{1}{(b-a)(b-c)}} \cdot \left(\frac{5}{36}\right)^{\frac{1}{(c-a)(c-b)}} \right] \\
&\leq \psi \left[\frac{\lambda}{4} \left(\frac{5}{36}\right)^{\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}} \right] \\
&= \frac{3}{16}\lambda \in [0, \infty).
\end{aligned}$$

By Choosing any value of $\lambda \in [0, \infty)$ and $a, b, c \in (0, 1)$, we find that (3.21) holds $\forall_{\xi_1, \xi_2 \in \Omega_b \setminus Fix(\Gamma)}$. Thus, Γ has two F.Ps of 0 and $\frac{1}{2}$.

Theorem 3.4.5. Let (Ω_b, m_b) be a C-Mb-M.S with coefficient $s \geq 1$ and Γ be a mapping satisfies (3.21) along with assertions below:

- (i) Γ is an η -Ćyclic (α, β) -admissible;
- (ii) Either $\exists_{\xi_0 \in \Omega_b}$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$ or $\exists_{\sigma_0 \in \Omega_b}$ such that $\beta(\sigma_0) \geq \eta(\sigma_0)$;
- (iii) Γ is continuous.

Then, Γ has an F.P $\xi^* \in \Omega_b$.

Proof. Let ξ_0 in Ω_b such that $\alpha(\xi_0) \geq \eta(\xi_0)$ and $\beta(\xi_0) \geq \eta(\xi_0)$. Define an iterative $\{\xi_\kappa\}$ in Ω_b by $\xi_\kappa = \Gamma \xi_{\kappa-1} \forall_{\kappa \in \mathbb{N}}$. If $\exists_{\kappa_0 \in \mathbb{N}}$ for which $\Gamma \xi_{\kappa_0} = \xi_{\kappa_0}$, then ξ_{κ_0} is an F.P of Γ and the proof is finished. Presume that $m_b|_{\Gamma \xi_{\kappa_0}}^{\xi_{\kappa_0}} > 0$, from axiom (i) $\exists_{\xi_1 \in \Omega_b}$ so that

$$\alpha(\xi_0) \geq \eta(\xi_0) \implies \beta(\xi_1) = \beta(\Gamma \xi_0) \geq \eta(\Gamma \xi_0) = \eta(\xi_1),$$

and

$$\beta(\xi_0) \geq \eta(\xi_0) \implies \alpha(\xi_1) = \alpha(\Gamma \xi_0) \geq \eta(\Gamma \xi_0) = \eta(\xi_1).$$

By continuing this approach, we get

$$\alpha(\xi_\kappa) \geq \eta(\xi_\kappa) \implies \beta(\xi_{\kappa+1}) \geq \eta(\xi_{\kappa+1}).$$

Similarly

$$\beta(\xi_\kappa) \geq \eta(\xi_\kappa) \implies \alpha(\xi_{\kappa+1}) \geq \eta(\xi_{\kappa+1}).$$

Hence $\forall_{\kappa \in \mathbb{N}}$, we have

$$\alpha(\xi_\kappa) \beta(\xi_{\kappa+1}) \geq \eta(\xi_\kappa) \eta(\xi_{\kappa+1}). \quad (3.22)$$

If $\xi_{\kappa+1} = \xi_\kappa$ for some $\kappa \in \mathbb{N}$, then $\xi_\kappa = \xi^*$, and the proof is finished. Thus, we Presume that $\forall_{\kappa \in \mathbb{N}}$, $\xi_\kappa \neq \xi_{\kappa+1}$ accompanied by $m_b|_{\Gamma\xi_\kappa}^{\xi_{\kappa-1}} = m_b|_{\Gamma\xi_\kappa}^{\xi_\kappa} > 0$. From (3.21) and $\forall_{\kappa \in \mathbb{N}}$, we have

$$\begin{aligned} m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} &\leq s^2 m_b|_{\Gamma\xi_\kappa}^{\xi_{\kappa-1}} \\ &\leq \psi \left[\lambda(\Xi_1|_{\xi_\kappa}^{\xi_{\kappa-1}}) \right], \end{aligned} \quad (3.23)$$

where

$$\begin{aligned} \Xi_1|_{\xi_\kappa}^{\xi_{\kappa-1}} &= \left[\begin{aligned} &m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \cdot (m_b|_{\Gamma\xi_{\kappa-1}}^{\xi_{\kappa-1}})^{\frac{1}{(a-b)(a-c)}} \cdot (m_b|_{\Gamma\xi_\kappa}^{\xi_\kappa})^{\frac{1}{(a-b)(a-c)}} \cdot \\ &\left(m_b|_{\Gamma\xi_{\kappa-1}}^{\xi_{\kappa-1}} + m_b|_{\Gamma\xi_\kappa}^{\xi_\kappa} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \\ &\left(m_b|_{\Gamma\xi_\kappa}^{\xi_{\kappa-1}} + m_b|_{\Gamma\xi_{\kappa-1}}^{\xi_\kappa} \right)^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right] \\ &= \left[\begin{aligned} &m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \cdot (m_b|_{\xi_\kappa}^{\xi_{\kappa-1}})^{\frac{1}{(a-b)(a-c)}} \cdot (m_b|_{\xi_{\kappa+1}}^{\xi_\kappa})^{\frac{1}{(a-b)(a-c)}} \cdot \\ &\left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} + m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \\ &\left(m_b|_{\xi_{\kappa+1}}^{\xi_{\kappa-1}} + m_b|_{\xi_n}^{\xi_n} \right)^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right] \\ &\leq \left[\begin{aligned} &(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}})^{1+\frac{1}{(a-b)(a-c)}} \cdot (m_b|_{\xi_{\kappa+1}}^{\xi_\kappa})^{\frac{1}{(a-b)(a-c)}} \cdot \\ &\left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} + m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \\ &\left[s \left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} - m_b|_{\xi_{\kappa-1}, \xi_\kappa} + m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} - m_b|_{\xi_\kappa, \xi_{\kappa+1}} \right) + m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right]^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right] \\ &\leq \left[\begin{aligned} &\left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^{1+\frac{1}{(a-b)(a-c)}} \cdot \left(m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \\ &\left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} + m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \\ &\left[s \left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} + m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right) + m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right]^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right] \end{aligned} \quad (3.24)$$

$$\begin{aligned}
&\leq \left[\begin{aligned} &\left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^{1+\frac{1}{(a-b)(a-c)}} \cdot \left(m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \\ &\left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} + m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \\ &\left[s \left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \cdot m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right) \cdot m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right]^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right] \\
&= \left[\begin{aligned} &s^{\frac{1}{(c-a)(c-b)}} \left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^{1+\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}} \cdot \\ &\left(m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}} \cdot \left(m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right)^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right] \\
&= \left[\begin{aligned} &s^{\frac{1}{(c-a)(c-b)}} \left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right)^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}} \cdot \\ &\left(m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}} \cdot \\ &\left(m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right)^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right] m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \\
&= \left(s m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right)^{\frac{1}{(c-a)(c-b)}} m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}.
\end{aligned}$$

Now, from (3.23) and (3.24), we obtain that

$$\begin{aligned}
m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} &\leq s^2 m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \\
&\leq \Upsilon \left(\lambda \left[s m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right]^{\frac{1}{(c-a)(c-b)}} m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) \\
&= \Upsilon \left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) < m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}.
\end{aligned} \tag{3.25}$$

Henceafter (3.25) implies that $\left\{ m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right\}$ is decreasing and $\exists \varrho \geq 0$ such that $\lim_{\kappa \rightarrow \infty} m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} = \varrho$. Presume that $\varrho > 0$. From (3.25), we write

$$\begin{aligned}
m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} &\leq s^2 m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \leq \psi \left[m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right] \\
&\leq \psi^2 \left[m_b|_{\xi_{\kappa-1}}^{\xi_{\kappa-2}} \right] \leq \psi^3 \left[m_b|_{\xi_{\kappa-1}}^{\xi_{\kappa-3}} \right] \\
&\leq \dots \leq \psi^\kappa \left[m_b|_{\xi_1}^{\xi_0} \right].
\end{aligned} \tag{3.26}$$

Taking $\kappa \rightarrow \infty$ in (3.26) and as $\psi \in \Psi$, we have

$$0 \leq \lim_{\kappa \rightarrow +\infty} m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \leq \lim_{\kappa \rightarrow +\infty} \psi_\kappa \left[m_b|_{\xi_1}^{\xi_0} \right] = 0.$$

Leading to

$$\lim_{\kappa \rightarrow +\infty} m_b|_{\xi_{\kappa+1}}^{\xi_{\kappa}} = 0. \quad (3.27)$$

Now, we verify that $\{\xi_{\kappa}\}$ is an M_b -Cauchy in (Ω_b, m_b) . From (M_{bii}) , we have

$$0 \leq m_b|_{\xi_{\kappa}, \xi_{\kappa+1}} \leq m_b|_{\xi_{\kappa+1}}^{\xi_{\kappa}} \quad \forall \kappa \in \mathbb{N}.$$

Since from (3.27), we have

$$\lim_{\kappa \rightarrow \infty} m_b|_{\xi_{\kappa}, \xi_{\kappa+1}} = 0, \quad (3.28)$$

which denotes to

$$\lim_{\kappa \rightarrow +\infty} m_b|_{\xi_{\kappa}}^{\xi_{\kappa}} = 0 \text{ or } \lim_{\kappa \rightarrow +\infty} m_b|_{\xi_{\kappa+1}}^{\xi_{\kappa+1}} = 0. \quad (3.29)$$

Therefore

$$\lim_{\kappa, \kappa \rightarrow \infty} m_b|_{\xi_{\kappa}, \xi_{\kappa}} = \lim_{\kappa, \kappa \rightarrow \infty} \min \left\{ m_b|_{\xi_{\kappa}}^{\xi_{\kappa}}, m_b|_{\xi_{\kappa}}^{\xi_{\kappa}} \right\} = 0. \quad (3.30)$$

Hence

$$\lim_{\kappa, \kappa \rightarrow \infty} \left(M_b|_{\xi_{\kappa}, \xi_{\kappa}} - m_b|_{\xi_{\kappa}, \xi_{\kappa}} \right) = \lim_{\kappa, \kappa \rightarrow \infty} \left| m_b|_{\xi_{\kappa}}^{\xi_{\kappa}} - m_b|_{\xi_{\kappa}}^{\xi_{\kappa}} \right| = 0.$$

Next, we verify that $\lim_{\kappa, \kappa \rightarrow \infty} \left(m_b|_{\xi_{\kappa}}^{\xi_{\kappa}} - m_b|_{\xi_{\kappa}, \xi_{\kappa}} \right) = 0$. Suppose on the contrary

$$\lim_{\kappa, \kappa \rightarrow \infty} \left(m_b|_{\xi_{\kappa}}^{\xi_{\kappa}} - m_b|_{\xi_{\kappa}, \xi_{\kappa}} \right) \neq 0.$$

Then, $\exists \epsilon_{>0}$ and subsequence $\{\mathcal{L}_k\} \subset \mathbb{N}$ such that

$$m_b|_{\xi_{n_k}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{n_k}} \geq \epsilon. \quad (3.31)$$

Let $\mathcal{L}_k$ represent the least quantity for which (3.31) holds, so that

$$m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_{k-1}}} - m_b|_{\xi_{\mathcal{L}_{k-1}}, \xi_{\kappa_k}} < \epsilon. \quad (3.32)$$

By (M_{biv}) in (3.31) and using (3.32), we get

$$\begin{aligned}
\epsilon &\leq m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\kappa_k}} \\
&\leq s \left[\left(m_b|_{\xi_{\mathcal{L}_{k-1}}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\mathcal{L}_{k-1}}} \right) + \left(m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_{k-1}}} - m_b|_{\xi_{\mathcal{L}_{k-1}}, \xi_{\kappa_k}} \right) \right] \\
&\quad - m_b|_{\xi_{\mathcal{L}_{k-1}}}^{\xi_{\mathcal{L}_{k-1}}} \\
&\leq s\epsilon + s \left[m_b|_{\xi_{\mathcal{L}_{k-1}}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\mathcal{L}_{k-1}}} \right] - m_b|_{\xi_{\mathcal{L}_{k-1}}}^{\xi_{\mathcal{L}_{k-1}}}.
\end{aligned} \tag{3.33}$$

Letting $k \rightarrow \infty$ in (3.33), using (3.27), (3.28) and (3.29), we get

$$\epsilon \leq \lim_{k \rightarrow \infty} \left(m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\kappa_k}} \right) \leq s\epsilon. \tag{3.34}$$

Applying (3.30) in (3.34), we have

$$\epsilon \leq \lim_{k \rightarrow \infty} m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} \leq s\epsilon. \tag{3.35}$$

Similarly from (M_{biv}) and (3.31), we obtain

$$\begin{aligned}
\epsilon &\leq m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\kappa_k}} \\
&\leq s \left[\left(m_b|_{\xi_{\mathcal{L}_{k+1}}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\mathcal{L}_{k+1}}} \right) + \left(m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_{k+1}}} - m_b|_{\xi_{\mathcal{L}_{k+1}}, \xi_{\kappa_k}} \right) \right] \\
&\quad - m_b|_{\xi_{\mathcal{L}_{k+1}}}^{\xi_{\mathcal{L}_{k+1}}} \\
&\leq s \left[\begin{aligned} &\left(m_b|_{\xi_{\mathcal{L}_{k+1}}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\mathcal{L}_{k+1}}} \right) \\ &+ s \left[\left(m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_{k+1}}} - m_b|_{\xi_{\mathcal{L}_{k+1}}, \xi_{\kappa_k+1}} \right) + \left(m_b|_{\xi_{\kappa_k}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\kappa_k+1}, \xi_{\kappa_k}} \right) \right] \\ &- m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k+1}} \end{aligned} \right] \\
&\quad - m_b|_{\xi_{\mathcal{L}_{k+1}}}^{\xi_{\mathcal{L}_{k+1}}} \\
&= s \left(m_b|_{\xi_{\mathcal{L}_{k+1}}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\mathcal{L}_{k+1}}} \right) + s^2 \left(m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_{k+1}}} - m_b|_{\xi_{\mathcal{L}_{k+1}}, \xi_{\kappa_k+1}} \right) \\
&\quad + s^2 \left(m_b|_{\xi_{\kappa_k}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\kappa_k+1}, \xi_{\kappa_k}} \right) - s m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\mathcal{L}_{k+1}}}^{\xi_{\mathcal{L}_{k+1}}}.
\end{aligned} \tag{3.36}$$

Similar to (3.33), we find that

$$\begin{aligned}
\epsilon &\leq m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}, \xi_{\kappa_k+1}} \\
&\leq s \left[m_b|_{\xi_{\mathcal{L}_k}}^{\xi_{\mathcal{L}_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}, \xi_{\mathcal{L}_k}} \right] + s^2 \left[m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\kappa_k}} \right] \\
&\quad + s^2 \left[m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k}} - m_b|_{\xi_{\kappa_k}, \xi_{\kappa_k+1}} \right] - s m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\mathcal{L}_k}}^{\xi_{\mathcal{L}_k}}.
\end{aligned} \tag{3.37}$$

Utilizing (3.36) and (3.37), then

$$\begin{aligned}
\epsilon &\leq m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\kappa_k}} \\
&\leq s \left(m_b|_{\xi_{\mathcal{L}_k+1}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\mathcal{L}_k+1}} \right) + s^2 \left(m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}, \xi_{\kappa_k+1}} \right) \\
&\quad + s^2 \left(m_b|_{\xi_{\kappa_k}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\kappa_k+1}, \xi_{\kappa_k}} \right) - s m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}}^{\xi_{\mathcal{L}_k+1}} \\
&\leq s \left(m_b|_{\xi_{\mathcal{L}_k+1}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\mathcal{L}_k+1}} \right) + s^2 \left(\begin{aligned} &s \left[m_b|_{\xi_{\mathcal{L}_k}}^{\xi_{\mathcal{L}_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}, \xi_{\mathcal{L}_k}} \right] \\ &+ s^2 \left[m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} - m_b|_{\xi_{\mathcal{L}_k}, \xi_{\kappa_k}} \right] \\ &+ s^2 \left[m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k}} - m_b|_{\xi_{\kappa_k}, \xi_{\kappa_k+1}} \right] \\ &- s m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\mathcal{L}_k}}^{\xi_{\mathcal{L}_k}} \end{aligned} \right) \\
&\quad + s^2 \left(m_b|_{\xi_{\kappa_k}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\kappa_k+1}, \xi_{\kappa_k}} \right) - s m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}}^{\xi_{\mathcal{L}_k+1}}.
\end{aligned} \tag{3.38}$$

Inserting limit $k \rightarrow \infty$ in (3.38), and utilizing (3.27), (3.28), (3.29) and (3.34), we obtain

$$\epsilon \leq \lim_{k \rightarrow \infty} s^2 \left(m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}, \xi_{\kappa_k+1}} \right) \leq s^5 \epsilon.$$

Therefore

$$\frac{\epsilon}{s^2} \leq \lim_{k \rightarrow \infty} \left(m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k+1}} - m_b|_{\xi_{\mathcal{L}_k+1}, \xi_{\kappa_k+1}} \right) \leq s^3 \epsilon. \tag{3.39}$$

From (3.30), we have

$$\frac{\epsilon}{s^2} \leq \lim_{k \rightarrow \infty} m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k+1}} \leq s^3 \epsilon. \tag{3.40}$$

Now, from (3.21), we obtain

$$\begin{aligned} s^2 m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k+1}} &= s^2 m_b|_{\Gamma\xi_{\kappa_k}}^{\Gamma\xi_{\mathcal{L}_k}} \\ &\leq \psi \left[\lambda(\mathbb{F}_1|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}}) \right], \end{aligned}$$

where

$$\begin{aligned} \mathbb{F}_1|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} &= m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} \cdot \left(m_b|_{\Gamma\xi_{\mathcal{L}_k}}^{\xi_{\mathcal{L}_k}} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma\xi_{\kappa_k}}^{\xi_{\kappa_k}} \right)^{\frac{1}{(a-b)(a-c)}} \\ &\quad \cdot \left(m_b|_{\Gamma\xi_{\mathcal{L}_k}}^{\xi_{\mathcal{L}_k}} + m_b|_{\Gamma\xi_{\kappa_k}}^{\xi_{\kappa_k}} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \left(m_b|_{\Gamma\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} + m_b|_{\Gamma\xi_{\mathcal{L}_k}}^{\xi_{\kappa_k}} \right)^{\frac{1}{(c-a)(c-b)}} \\ &= m_b|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} \cdot \left(m_b|_{\xi_{\mathcal{L}_k+1}}^{\xi_{\mathcal{L}_k}} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k}} \right)^{\frac{1}{(a-b)(a-c)}} \\ &\quad \cdot \left(m_b|_{\xi_{\mathcal{L}_k+1}}^{\xi_{\mathcal{L}_k}} + m_b|_{\xi_{\kappa_k+1}}^{\xi_{\kappa_k}} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \left(m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k}} + m_b|_{\xi_{\mathcal{L}_k+1}}^{\xi_{\kappa_k}} \right)^{\frac{1}{(c-a)(c-b)}}. \end{aligned}$$

Inserting limit $k \rightarrow \infty$ in the above equation with utilizing (3.27) and (3.28), we get

$$0 \leq \lim_{k \rightarrow \infty} \mathbb{F}_1|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} \leq 0 \implies \lim_{k \rightarrow \infty} \mathbb{F}_1|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} = 0. \quad (3.41)$$

Thence, it follows from (3.40), (3.41) and (ψ_{b3}) of Lemma 1.1.8 that

$$\begin{aligned} \epsilon &= s^2 \left(\frac{\epsilon}{s^2} \right) \leq s^2 \lim_{k \rightarrow \infty} m_b|_{\xi_{\kappa_k+1}}^{\xi_{\mathcal{L}_k+1}} = s^2 \lim_{k \rightarrow \infty} m_b|_{\Gamma\xi_{\kappa_k}}^{\Gamma\xi_{\mathcal{L}_k}} \\ &\leq \psi \left[\lambda \lim_{k \rightarrow \infty} \mathbb{F}_1|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} \right] < \psi \left[\lim_{k \rightarrow \infty} \mathbb{F}_1|_{\xi_{\kappa_k}}^{\xi_{\mathcal{L}_k}} \right] \\ &= \psi[0] = 0. \end{aligned}$$

Hence, we conclude that $\epsilon < 0$, that is a discrepancy. Thus $\lim_{\kappa, \kappa \rightarrow \infty} \left(m_b|_{\xi_{\kappa}}^{\xi_{\mathcal{L}_\kappa}} - m_b|_{\xi_{\kappa}, \xi_{\kappa}} \right) = 0$, therefore $\{\xi_{\kappa}\}$ is an M_b -Cauchy in Ω_b . Because Ω_b is complete, $\exists$ some $\xi^* \in \Omega_b$ so that $\xi_{\kappa} \rightarrow \xi^*$ as $\kappa \rightarrow \infty$. Since Γ is continuous, then $\lim_{\kappa \rightarrow \infty} \Gamma\xi_{\kappa} = \Gamma\xi^*$. So, we have

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \xi^*} \right) = 0. \text{ and } \lim_{\kappa \rightarrow \infty} \left(M_b|_{\xi_{\kappa+1}, \xi^*} - m_b|_{\xi_{\kappa+1}, \xi^*} \right) = 0. \quad (3.42)$$

Since from (3.29) and (3.42), we get

$$\begin{aligned}
\lim_{\kappa \rightarrow \infty} \left(m_b|_{\xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \xi^*} \right) &= 0 \\
&= \lim_{\kappa \rightarrow \infty} m_b|_{\xi^*}^{\xi_{\kappa+1}} \\
&= \lim_{\kappa \rightarrow \infty} m_b|_{\xi^*}^{\Gamma \xi_{\kappa}} \\
&= m_b|_{\xi^*}^{\Gamma \xi^*}.
\end{aligned} \tag{3.43}$$

That is $\Gamma \xi^* = \xi^*$ and ξ^* is an F.P of Γ . ■

Theorem 3.4.6. Let (Ω_b, m_b) be a C-Mb-M.S and $\Gamma : \Omega_b \rightarrow \Omega_b$ be a mapping satisfies (3.21) along with affirmations below:

- (i) Γ is an η -cyclic (α, β) -admissible;
- (ii) Either $\exists_{\xi_0 \in \Omega_b}$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$ or $\exists_{\sigma_0 \in \Omega_b}$ such that $\beta(\sigma_0) \geq \eta(\sigma_0)$;
- (iii) if $\{\xi_{\kappa}\}$ is a sequence in Ω_b such that $\xi_{\kappa} \rightarrow \xi^*$ as $\kappa \rightarrow \infty$, and $\beta(\xi_{\kappa}) \geq \eta(\xi_{\kappa}) \forall_{\kappa \in \mathbb{N}}$, then $\beta(\xi^*) \geq \eta(\xi^*)$.

Then, Γ has an F.P ξ^* in Ω_b .

Proof. By tracing the line of proof employed in the Theorem 3.4.6, we acquire $\beta(\xi^*) \geq \eta(\xi^*)$. Now, we show that $m_b|_{\xi^*}^{\Gamma \xi^*} = 0$. Since $\xi_{\kappa} \rightarrow \xi^*$ as $n \rightarrow \infty$, from (M_{biv}) , we have

$$\begin{aligned}
0 &\leq \left| \left(m_b|_{\Gamma \xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \Gamma \xi^*} \right) - \left(m_b|_{\Gamma \xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma \xi^*} \right) \right| \\
&\leq \left| s \left(\left(m_b|_{\xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \xi^*} \right) + \left(m_b|_{\Gamma \xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma \xi^*} \right) \right) - m_b|_{\xi^*}^{\xi^*} \right. \\
&\quad \left. - \left(s \left(\left(m_b|_{\xi^*}^{\xi^*} - m_b|_{\xi^*, \xi^*} \right) + \left(m_b|_{\Gamma \xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma \xi^*} \right) \right) - m_b|_{\xi^*}^{\xi^*} \right) \right|.
\end{aligned} \tag{3.44}$$

So, taking limit as $\kappa \rightarrow \infty$ in (3.44) with using (3.29) and (3.42), we get

$$0 \leq \lim_{\kappa \rightarrow \infty} \left| \left(m_b|_{\Gamma \xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \Gamma \xi^*} \right) - \left(m_b|_{\Gamma \xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma \xi^*} \right) \right| \leq 0,$$

this implies that

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\Gamma \xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \Gamma \xi^*} \right) = m_b|_{\Gamma \xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma \xi^*} = m_b|_{\Gamma \xi^*}^{\xi^*}. \tag{3.45}$$

Now, from (3.23) and (3.45), we have

$$\begin{aligned}
m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1},\Gamma\xi^*} &\leq s^2 m_b|_{\Gamma\xi^*}^{\xi_{\kappa}} - m_b|_{\Gamma\xi_{\kappa},\Gamma\xi^*} \leq \psi \left[\lambda \left(\mathbb{F}_1|_{\xi^*}^{\xi_{\kappa}} \right) \right] \\
&\leq \psi \left(\begin{aligned} &\lambda m_b|_{\xi_{\kappa}}^{\xi^*} \cdot \left(m_b|_{\xi^*}^{\Gamma\xi^*} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa}} \right)^{\frac{1}{(a-b)(a-c)}} \\ &\cdot \left(m_b|_{\xi^*}^{\Gamma\xi^*} + m_b|_{\Gamma\xi_{\kappa}}^{\xi_{\kappa}} \right)^{\frac{1}{(b-a)(b-c)}} \\ &\cdot \left(m_b|_{\xi^*}^{\Gamma\xi_{\kappa}} + m_b|_{\Gamma\xi^*}^{\xi_{\kappa}} \right)^{\frac{1}{(c-a)(c-b)}} \end{aligned} \right).
\end{aligned} \tag{3.46}$$

Letting $\kappa \rightarrow \infty$ in (3.46) and since $\psi \in \Psi$, we get

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1},\Gamma\xi^*} \right) = 0. \tag{3.47}$$

Therefore, from (3.45) and (3.47), we get $m_b|_{\Gamma\xi^*}^{\xi^*} = 0$ and ξ^* is an F.P of Γ . ■

Example 3.4.7. Let $\Omega_b = [0, 1]$ and $m_b : \Omega_b \times \Omega_b \rightarrow \mathbb{R}^+$ defined by

$$m_b|_{\xi_2}^{\xi_1} = \left(\frac{\xi_1 + \xi_2}{2} \right)^2.$$

Then, (Ω_b, m_b) is an Mb-M.S with $s = 2$. Define $\Gamma : \Omega_b \rightarrow \Omega_b$ by

$$\Gamma\xi = \begin{cases} \frac{\xi^2}{9}, & \text{if } \xi \in (0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Describe the functions $\alpha, \beta, \eta : \Omega_b \times \Omega_b \rightarrow (0, 1]$ by

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 2, & \text{if } \xi \in (0, 1] \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(\xi) = \begin{cases} 1, & \text{if } \xi \in (0, 1] \\ 0, & \text{otherwise} \end{cases}.$$

Clearly for $\xi_1, \xi_2 \in (0, 1]$, $\alpha(\xi_1) = 2 \geq 1 = \eta(\xi_1) \implies \beta(\Gamma\xi_1) = 2 \geq 1 = \eta(\Gamma\xi_1)$, and $\beta(\xi_1) = 2 \geq 1 = \eta(\xi_1) \implies \alpha(\Gamma\xi_1) \geq \eta(\Gamma\xi_1)$. So, Γ is η -cyclic (α, β) -admissible, as long as

$\alpha(\xi_1)\beta(\xi_2) \geq \eta(\xi_1)\eta(\xi_2)$, such that

$$\begin{aligned}
s^2 m_b|_{\Gamma_{\xi_2}^{\xi_1}} &= 4 m_b|_{\frac{\xi_1^2}{9}} = 4 \left(\frac{\xi_1^2 + \xi_2^2}{18} \right)^2 \leq 4 \left(\frac{\xi_1 + \xi_2}{18} \right)^2 = \frac{4}{9} \left[\frac{1}{9} \left(\frac{\xi_1 + \xi_2}{2} \right)^2 \right] \\
&\leq \frac{4}{9} \left[\frac{1}{9} \left(\left(\frac{\xi_1 + \xi_2}{2} \right)^2 \cdot \left(\frac{3\xi_1}{4} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(\frac{3\xi_2}{4} \right)^{\frac{1}{(a-b)(a-c)}} \right. \right. \\
&\quad \left. \left. \cdot \left(\left(\frac{3\xi_1}{4} \right)^2 + \left(\frac{3\xi_2}{4} \right)^2 \right)^{\frac{1}{(b-a)(b-c)}} \cdot \left(\left(\frac{2\xi_1 + \xi_2}{4} \right)^2 + \left(\frac{\xi_1 + 2\xi_2}{4} \right)^2 \right)^{\frac{1}{(c-a)(c-b)}} \right) \right] \\
&= \psi[\lambda(\mathbb{H}_1(\xi_1, \xi_2))].
\end{aligned}$$

That is attained by taking $\psi(r) = \frac{4}{9}r$, $\lambda = \frac{1}{9} \in [0, 1)$ and $a, b, c \in (0, 1)$, $\forall \xi_1, \xi_2 \in \Omega_b \setminus Fix(\Gamma)$.

Otherwise, for $\xi_1 = \xi_2 = 0$, So, Γ is η -cyclic (α, β) -admissible, as long as $\alpha(\xi_1)\beta(\xi_2) \geq \eta(\xi_1)\eta(\xi_2)$, and

$$s^2 m_b|_{\Gamma_{\xi_2}^{\xi_1}} = 0 \leq \psi[\lambda(\mathbb{H}_1|_{\xi_2}^{\xi_1})].$$

Therefore, all affirmations of Theorems 3.4.5 and 3.4.6 are satisfied. Hence Γ has an F.P $\xi^* = 0 \in \Omega_b$. (Notice that 9 is an another F.P of Γ , but $9 \notin \Omega_b$).

This segment introduces F.P results for symmetric fractional α - β - η - ψ -contraction type-II:

Definition 3.4.8. Let (Ω_b, m_b) be an Mb-M.S, $\Gamma : \Omega_b \rightarrow \Omega_b$ be a selfmapping, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be functions and $\psi \in \Psi$. Γ is termed a symmetric fractional α - β - η - ψ -contraction pattern-II provided that $\exists_{s \geq 1}$, $\exists_{a, b, c \in (0, 1)}$ and $\lambda = \left(s m_b|_{\xi_1, \xi_2} \right)^{\frac{c}{(c-a)(c-b)}} \in [0, \infty)$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b \setminus Fix(\Gamma)}$, whenever $\alpha(\xi_1)\beta(\xi_2) \geq \eta(\xi_1)\eta(\xi_2)$, we have

$$s^2 m_b|_{\Gamma_{\xi_2}^{\xi_1}} \leq \psi[\lambda(\mathbb{H}_2|_{\xi_2}^{\xi_1})], \quad (3.48)$$

where

$$\mathbb{H}_2|_{\xi_2}^{\xi_1} = \left\{ \begin{aligned} &m_b|_{\xi_2}^{\xi_1} \cdot \left(m_b|_{\Gamma_{\xi_1}^{\xi_1}} \right)^{\frac{a}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma_{\xi_2}^{\xi_2}} \right)^{\frac{a}{(a-b)(a-c)}} \\ &\cdot \left(m_b|_{\Gamma_{\xi_1}^{\xi_1}} + m_b|_{\Gamma_{\xi_2}^{\xi_2}} \right)^{\frac{b}{(b-a)(b-c)}} \cdot \left(m_b|_{\Gamma_{\xi_2}^{\xi_1}} + m_b|_{\Gamma_{\xi_1}^{\xi_2}} \right)^{\frac{c}{(c-a)(c-b)}} \end{aligned} \right\}.$$

Theorem 3.4.9. Let (Ω_b, m_b) be a C-Mb-M.S and Γ be a mapping satisfies (3.48) along with the affirmations of Theorem 3.4.5. Then, Γ has a F.P in Ω_b .

Proof. By tracing the line of proof employed in the Theorem 3.4.5, we find that $\xi^* \in \Omega_b$ is an F.P of Γ . ■

Theorem 3.4.10. Let (Ω_b, m_b) be a C-Mb-M.S and Γ be a mapping satisfies (3.48) along with the affirmations of Theorem 3.4.6. Then, Γ has an F.P in Ω_b .

Proof. By tracing the line of proof employed in the Theorem 3.4.6, we find that $\xi^* \in \Omega_b$ is an F.P of Γ . ■

This segment introduces F.P results for symmetric fractional α - β - η - ψ -contraction type-III:

Definition 3.4.11. Let (Ω_b, m_b) be an Mb-M.S, $\Gamma : \Omega_b \rightarrow \Omega_b$ be a selfmapping, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be functions and $\psi \in \Psi$. Γ is termed a symmetric fractional α - β - η - ψ -contraction pattern-III, if $\exists_{s \geq 1}$, $\exists_{a, b, c \in (0, 1)}$ and $\lambda = \left(s m|_{\xi_1, \xi_2} \right)^{\frac{-c^2}{(c-a)(c-b)}} \in [0, \infty)$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b \setminus Fix(\Gamma)}$, whenever $\alpha(\xi_1)\beta(\xi_2) \geq \eta(\xi_1)\eta(\xi_2)$, we have

$$s^2 m_b|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq \Upsilon \left[\lambda(\mathbb{H}_3|_{\xi_2}^{\xi_1}) \right], \quad (3.49)$$

where

$$\mathbb{H}_3|_{\xi_2}^{\xi_1} = \max \left\{ \begin{array}{l} m_b|_{\xi_2}^{\xi_1}, \left(m_b|_{\Gamma\xi_1}^{\xi_1} \right)^{\frac{a^2}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma\xi_2}^{\xi_2} \right)^{\frac{a^2}{(a-b)(a-c)}} \\ \cdot \left(m_b|_{\Gamma\xi_1}^{\xi_1} + m_b|_{\Gamma\xi_2}^{\xi_2} \right)^{\frac{b^2}{(b-a)(b-c)}} \cdot \left(m_b|_{\Gamma\xi_2}^{\xi_1} + m_b|_{\Gamma\xi_1}^{\xi_2} \right)^{\frac{c^2}{(c-a)(c-b)}} \end{array} \right\}.$$

Theorem 3.4.12. Let (Ω_b, m_b) be a C-Mb-M.S and Γ be a mapping satisfies (3.49) along with the affirmations of theorem 3.4.5. Then, Γ has an F.P in Ω_b .

Proof. Let $\xi_0 \in \Omega_b$ such that $\alpha(\xi_0) \geq \eta(\xi_0)$ and $\beta(\xi_0) \geq \eta(\xi_0)$. $\forall_{\kappa \in \mathbb{N}}$, we construct an iteration $\{\xi_\kappa\}_{\kappa=1}^\infty$ so that $\xi_1 = \Gamma\xi_0$, $\xi_2 = \Gamma\xi_1 = \Gamma^2\xi_0$. By continuing this approach, we get $\xi_{\kappa+1} = \Gamma\xi_\kappa = \Gamma^{\kappa+1}\xi_0$. From (i), we obtain that

$$\alpha(\xi_\kappa)\beta(\xi_{\kappa+1}) \geq \eta(\xi_\kappa)\eta(\xi_{\kappa+1}), \quad \forall_{\kappa \in \mathbb{N}}. \quad (3.50)$$

If $\xi_{\kappa+1} = \xi_\kappa$ for some $\kappa \in \mathbb{N}$, then $\xi_\kappa = \xi^*$ is a F.P of Γ . So, we presume that $\xi_\kappa \neq \xi_{\kappa+1}$ in conjunction with

$$m_b|_{\Gamma\xi_\kappa}^{\Gamma\xi_{\kappa-1}} = m_b|_{\Gamma\xi_\kappa}^{\xi_\kappa} > 0, \quad \forall_{\kappa \in \mathbb{N}}.$$

From (3.49), we own

$$m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \leq s^2 m_b|_{\Gamma\xi_\kappa}^{\Gamma\xi_{\kappa-1}} \leq \psi \left[\lambda \left(\mathbb{H}_3|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) \right]. \quad (3.51)$$

Followig the same steps in (3.24), we deduce that

$$\mathbb{E}_3|_{\xi_\kappa}^{\xi_{\kappa-1}} = \left\{ m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}, \left(s m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right)^{\frac{c^2}{(c-a)(c-b)}} m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right\}.$$

Now, if

$$\mathbb{E}_3|_{\xi_\kappa}^{\xi_{\kappa-1}} = \left(s m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right)^{\frac{c^2}{(c-a)(c-b)}} m_b|_{\xi_{\kappa+1}}^{\xi_\kappa}. \quad (3.52)$$

Then, from (3.51) and (3.52), we get

$$\begin{aligned} m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} &\leq \psi \left[\lambda \left(s m_b|_{\xi_{\kappa-1}, \xi_{\kappa+1}} \right)^{\frac{c^2}{(c-a)(c-b)}} m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right] \\ &= \psi \left[m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right] < m_b|_{\xi_{\kappa+1}}^{\xi_\kappa}, \end{aligned}$$

a contradiction, thus

$$\mathbb{E}_3|_{\xi_\kappa}^{\xi_{\kappa-1}} = m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}. \quad (3.53)$$

From (3.51) and (3.53), we deduce that

$$\begin{aligned} m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} &\leq s^2 m_b|_{\Gamma \xi_\kappa}^{\Gamma \xi_{\kappa-1}} \\ &\leq \Upsilon \left[\lambda \left(\mathbb{E}_3|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) \right] < m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}. \end{aligned} \quad (3.54)$$

By tracing the line of proof employed in the Theorem 3.4.5, we find that $\xi^* \in \Omega_b$ is an F.P of Γ .

■

Theorem 3.4.13. Let (Ω_b, m_b) be a C-Mb-M.S and Γ be a mapping satisfies (3.49) along with the assertions of Theorem 3.4.6. Then, Γ possesses an F.P in Ω_b .

Proof. By tracing the line of proof employed in the Theorem 3.4.6, we find that $\xi^* \in \Omega_b$ is an F.P of Γ .

■

Example 3.4.14. Let $\Omega_b = [0, 1]$ and $m_b : \Omega_b \times \Omega_b \rightarrow \mathbb{R}^+$ defined by

$$m_b|_{\xi_2}^{\xi_1} = \left(\frac{\xi_1 + \xi_2}{2} \right)^2 \quad \forall_{\xi_1, \xi_2 \in \Omega_b}.$$

Then, (Ω_b, m_b) is an Mb-M.S with $s = 2$. Define $\Gamma : \Omega_b \rightarrow \Omega_b$ by

$$\Gamma \xi = \begin{cases} \frac{1}{15}, & \text{if } \xi \in [0, 1); \\ 1, & \text{if } \xi = 1. \end{cases}$$

Describe the functions $\alpha, \beta, \eta : \Omega_b \times \Omega_b \rightarrow [0, \infty)$ by

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 2, & \text{if } \xi \in (0, 1] \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(\xi) = \begin{cases} 1, & \text{if } \xi \in (0, 1] \\ 0, & \text{otherwise} \end{cases}.$$

Clearly, Γ is η -cyclic (α, β) -admissible mapping. Moreover if $\{\xi_\kappa\}$ is an iterative sequence in Ω_b so that $\xi_\kappa \rightarrow \xi^*$ as $\kappa \rightarrow \infty$ and $\beta(\xi_\kappa) \geq \eta(\xi_\kappa)$. Then, $\beta(\xi^*) \geq \eta(\xi^*)$ whenever $\alpha(\xi_1) \beta(\xi_2) \geq \eta(\xi_1) \eta(\xi_2)$, and for $\xi_1 \in [0, 1), \xi_2 = 1$, we have

$$\begin{aligned} s^2 m_b|_{\Gamma \xi_2}^{\Gamma \xi_1} &= 4 m_b|_{\Gamma 1}^{\Gamma \xi_1} = 4 \left(\frac{\frac{1}{15} + 1}{2} \right)^2 \leq 4 \left(\frac{8}{15} \right)^2 \\ &= 4 \left(\frac{8}{3 \times 5} \right)^2 = \frac{4}{5} \left[\frac{64}{5} \left(\frac{1}{3} \right)^2 \right] \\ &\leq \psi[\lambda(\mathbb{F}_3(\xi, 1))]. \end{aligned}$$

That is attained by taking $\psi(r) = \frac{4}{5}r$, $\lambda = \frac{64}{5} \in [0, \infty)$ and $a, b, c \in (0, 1)$. Therefore, all affirmations of Theorems 3.4.12 and 3.4.13 are satisfied. Hence, Γ has two F.Ps $\xi^* = 1, \frac{1}{5} \in \Omega_b$.

This segment introduces F.P results for symmetric fractional α - β - η - ψ -contraction type-IV:
Definition 3.4.15. Let (Ω_b, m_b) be an Mb-M.S with a self-map $\Gamma : \Omega_b \rightarrow \Omega_b$, $\alpha, \beta, \eta : \Omega_b \rightarrow [0, \infty)$ be functions and $\psi \in \Psi$. Γ is termed a symmetric fractional α - β - η - ψ -contraction pattern-IV, if $\exists_{s \geq 1}, \exists_{a, b, c \in (0, 1)}$ and $\lambda = \left(s m_b|_{\xi_1, \xi_2} \right)^{\frac{-a^3}{(c-a)(c-b)}} \in [0, 1)$ with $(a + b + c < 1)$ such that $\forall_{\xi_1, \xi_2 \in \Omega_b \setminus Fix(\Gamma)}$, whenever $\alpha(\xi_1) \beta(\xi_2) \geq \eta(\xi_1) \eta(\xi_2)$, we have

$$s^2 m_b|_{\Gamma \xi_2}^{\Gamma \xi_1} \leq \psi \left[\lambda(\mathbb{F}_4|_{\xi_2}^{\xi_1}) \right], \quad (3.55)$$

where

$$\mathbb{F}_4|_{\xi_2}^{\xi_1} = \left\{ \begin{aligned} &\left(m_b|_{\xi_2}^{\xi_1} \right)^{\frac{a^3}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma \xi_1}^{\xi_1} \right)^{\frac{a^3}{(a-b)(a-c)}} \\ &\cdot \left(m_b|_{\Gamma \xi_1}^{\xi_1} + m_b|_{\Gamma \xi_2}^{\xi_2} \right)^{\frac{b^3}{(b-a)(b-c)}} \cdot \left(m_b|_{\Gamma \xi_2}^{\xi_1} + m_b|_{\Gamma \xi_1}^{\xi_2} \right)^{\frac{c^3}{(c-a)(c-b)}} \end{aligned} \right\}.$$

Theorem 3.4.16. Let (Ω_b, m_b) be a C-Mb-M.S and Γ be a mapping satisfies (3.55) along with the affirmations of Theorem 3.4.5.

Then, Γ possesses an F.P in Ω_b .

Proof. Let $\xi_0 \in \Omega_b$ so that $\alpha(\xi_0) \geq \eta(\xi_0)$ and $\beta(\xi_0) \geq \eta(\xi_0)$. We build a sequence $\{\xi_\kappa\}_{\kappa=1}^\infty$ as $\xi_1 = \Gamma\xi_0$, $\xi_2 = \Gamma\xi_1 = \Gamma^2\xi_0$. Proceeding in this manner, we get $\xi_{\kappa+1} = \Gamma\xi_\kappa = \Gamma^{n+1}\xi_0$, $\forall \kappa \in \mathbb{N}$. From (i) we can conclude that

$$\alpha(\xi_\kappa) \beta(\xi_{\kappa+1}) \geq \eta(\xi_\kappa) \eta(\xi_{\kappa+1}), \quad \forall \kappa \in \mathbb{N}. \quad (3.56)$$

If $\xi_{\kappa+1} = \xi_\kappa$ for some $\kappa \in \mathbb{N}$, then $\xi_\kappa = \xi^*$ is an F.P of Γ . So, we presume that $\xi_\kappa \neq \xi_{\kappa+1}$ in conjunction with

$$m_b|_{\Gamma\xi_\kappa}^{\Gamma\xi_{\kappa-1}} = m_b|_{\Gamma\xi_\kappa}^{\xi_\kappa}, \quad \forall \kappa \in \mathbb{N}.$$

Now, from (3.55), we have

$$m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \leq s^2 m_b|_{\Gamma\xi_\kappa}^{\Gamma\xi_{\kappa-1}} \leq \psi \left[\lambda \left(\mathbb{H}_4|_{\xi_\kappa}^{\xi_{\kappa-1}} \right) \right],$$

where by the same steps in (3.24), we deduce that

$$\mathbb{H}_4|_{\xi_\kappa}^{\xi_{\kappa-1}} \leq \left(m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \cdot m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right)^{a+b+c} \leq \max \left\{ m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}, m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right\}.$$

If $\max \left\{ m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}, m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right\} = m_b|_{\xi_{\kappa+1}}^{\xi_\kappa}$, then

$$m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \leq \psi \left[m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \right] < m_b|_{\xi_{\kappa+1}}^{\xi_\kappa},$$

a contradiction, thus

$$m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} \leq \psi \left[m_b|_{\xi_\kappa}^{\xi_{\kappa-1}} \right] < m_b|_{\xi_\kappa}^{\xi_{\kappa-1}}. \quad (3.57)$$

By tracing the line of proof employed in the Theorem 3.4.5, we find that $\xi^* \in \Omega_b$ is an F.P of Γ . ■

Theorem 3.4.17. Let (Ω_b, m_b) be a C-Mb-M.S, Γ be a mapping satisfies (3.55) along with the affirmations of theorem 3.4.6.

Then, Γ possesses an F.P in Ω_b .

Proof. By tracing the line of proof employed in the Theorem 3.4.6, we find that $\xi^* \in \Omega_b$ is an F.P of Γ . ■

Example 3.4.18. Let $\Omega_b = [0, \infty)$, $p \geq 1$ and $m_b : \Omega_b \times \Omega_b \rightarrow \mathbb{R}^+$ defined by

$$m_b|_{\xi_2}^{\xi_1} = \max \{ \xi_1, \xi_2 \}^p + |\xi_1 - \xi_2|^p, \quad \forall \xi_1, \xi_2 \in \Omega_b.$$

Then, (Ω_b, m_b) is an Mb-M.S with $s = 2^p$. Define $\Gamma : \Omega_b \rightarrow \Omega_b$ by

$$\Gamma \xi = \begin{cases} \frac{\xi+0.5}{128}, & \text{if } \xi \in (0, 1]; \\ 0, & \text{otherwise.} \end{cases}$$

Describe the functions $\alpha, \beta, \eta : \Omega_b \times \Omega_b \rightarrow [0, \infty)$ as

$$\alpha(\xi) = \beta(\xi) = \begin{cases} 2, & \text{if } \xi \in (0, 1]; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } \eta(\xi) = \begin{cases} 1, & \text{if } \xi \in (0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\forall \xi_1, \xi_2 \in (0, 1]$, Γ is η -cyclic (α, β) -admissible mapping, whenever $\alpha(\xi_1) \beta(\xi_2) \geq \eta(\xi_1) \eta(\xi_2)$, we have

$$\begin{aligned} s^2 m_b|_{\Gamma \xi_2}^{\Gamma \xi_1} &= 2^{2p} m_b|_{\frac{\xi_2+0.5}{125}}^{\frac{\xi_1+0.5}{125}} \leq 2^{2p} m_b|_{\frac{\xi_2}{64}}^{\frac{\xi_1}{64}} \\ &= 2^{2p} \left(\max \left\{ \frac{\xi_1}{64}, \frac{\xi_2}{64} \right\}^p + \left| \frac{\xi_1}{64} - \frac{\xi_2}{64} \right|^p \right) \\ &= \frac{2^{2p}}{64^p} (\max \{ \xi_1, \xi_2 \}^p + |\xi_1 - \xi_2|^p) \\ &= \frac{1}{2^{4p}} m_b|_{\xi_2}^{\xi_1} \leq \frac{1}{2^{2p}} \left[\frac{1}{2^p} \left(\mathbb{E}_4|_{\xi_2}^{\xi_1} \right) \right] \\ &\leq \psi \left[\lambda \left(\mathbb{E}_4|_{\xi_2}^{\xi_1} \right) \right]. \end{aligned}$$

That is attained by taking $\psi(t) = \frac{1}{2^{2p}}t$, $\lambda = \frac{1}{2^p} \in [0, \infty)$ and $a, b, c \in (0, 1)$.

Otherwise, it is obvious that Γ is η -cyclic (α, β) -admissible, whenever $\alpha(\xi_1) \beta(\xi_2) \geq \eta(\xi_1) \eta(\xi_2)$, we have

$$s^2 m_b|_{\Gamma \xi_2}^{\Gamma \xi_1} = 0 \leq \psi \left[\lambda \left(\mathbb{E}_4|_{\xi_2}^{\xi_1} \right) \right].$$

Thus, all affirmations of Theorem 3.4.16 are satisfied. Hence, Γ has two F.Ps 0 and $\frac{1}{254}$ in Ω_b .

Taking $\eta(\xi_1) = \eta(\xi_2) = 1$, in Theorems 3.4.5, 3.4.6, 3.4.9 and 3.4.10, we derive the next corollaries.

Corollary 3.4.19. Let (Ω_b, m_b) be a C-Mb-M.S and Γ be a symmetric fractional α - β - ψ -contraction-type-I fulfilling affirmations below:

- (i) Γ is a cyclic (α, β) -admissible;
- (ii) Either $\exists_{\xi_0 \in \Omega_b}$ such that $\alpha(\xi_0) \geq 1$ or $\exists_{\sigma_0 \in \Omega_b}$ such that $\beta(\sigma_0) \geq 1$;
- (iii) Γ is continuous.

Then, Γ has an F.P in Ω_b .

Corollary 3.4.20. Let (Ω_b, m_b) be a C-Mb-M.S and Γ be a symmetric fractional α - β - ψ -contraction-type-I fulfilling affirmations below:

- (i) Γ is a cyclic (α, β) -admissible;
- (ii) Either $\exists_{\xi_0 \in \Omega_b}$ such that $\alpha(\xi_0) \geq 1$ or $\exists_{\sigma_0 \in \Omega_b}$ such that $\beta(\sigma_0) \geq 1$;
- (iii) if $\{\xi_\kappa\}$ is an iterative sequence in Ω_b so that $\xi_\kappa \rightarrow \xi^*$ as $\kappa \rightarrow \infty$, and $\beta(\xi_\kappa) \geq 1 \forall \kappa \in \mathbb{N}$, then $\beta(\xi^*) \geq 1$.

Then, Γ has an F.P $\xi^* \in \Omega_b$.

Analogously, we can drive corollaries for symmetric fractional contraction-type-II, III and IV.

Now, we derive F.P outcomes for symmetric fractional contractions on closed-ball of Mb-M.S.

Theorem 3.4.21. Let (Ω_b, m_b) be a C-Mb-M.S, ξ_0 be any point in a closed-ball $\overline{B_b|_\varepsilon^{\xi_0}}$, $\alpha, \eta : \Omega_b \times \Omega_b \rightarrow [0, \infty)$ be semi α -admissible mappings w.r.t. η on $\overline{B_b|_\varepsilon^{\xi_0}}$ with $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$ and $\psi \in \Psi$. Let $\Gamma : \Omega_b \rightarrow \Omega_b$ be a continuous semi α -admissible to satisfy (3.21), $\forall_{\xi_1, \xi_2 \in \overline{B_b|_\varepsilon^{\xi_0}} \subseteq \Omega_b \setminus Fix(\Gamma)}$, $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$. Moreover,

$$m_b|_{\xi_1}^{\xi_0} - m_b|_{\xi_0, \xi_1} \leq \sum_{\iota=0}^{\kappa} s^{\iota+1} \psi^\iota \left[m_b|_{\xi_1}^{\xi_0} \right] \leq \varepsilon, \text{ for } \varepsilon > 0. \quad (3.58)$$

Then, Γ has an F.P in $\overline{B_b|_\varepsilon^{\xi_0}} \subseteq \Omega_b$.

Proof. Since $\xi_0 \in \overline{B_b|_\varepsilon^{\xi_0}}$, $\exists_{\xi_1 \in \Omega_b}$ such that $\xi_1 = \Gamma \xi_0$ and $\exists_{\xi_2 \in \Omega_b}$ such that $\xi_2 = \Gamma \xi_1$. By continuing this approach, we construct an iterative $\{\xi_\kappa\}$ in Ω_b so that, $\xi_\kappa = \Gamma \xi_{\kappa-1}$. As $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$ and Γ is semi α -admissible mapping w.r.t η , we have $\alpha|_{\Gamma \xi_1}^{\Gamma \xi_0} \geq \eta|_{\Gamma \xi_1}^{\Gamma \xi_0}$ from which we deduce that $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$, which also implies that $\alpha|_{\Gamma \xi_2}^{\Gamma \xi_1} \geq \eta|_{\Gamma \xi_2}^{\Gamma \xi_1}$. Continuing in this way, we obtain that $\alpha|_{\Gamma \xi_\kappa}^{\Gamma \xi_{\kappa-1}} \geq \eta|_{\Gamma \xi_\kappa}^{\Gamma \xi_{\kappa-1}}$, this leads to $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa} \forall \kappa \in \mathbb{N}$.

Now, we show that $\xi_\kappa \in \overline{B_b|_\varepsilon^{\xi_0}} \forall \kappa \in \mathbb{N}$. Utilizing inequality (3.58), we have

$$m_b|_{\xi_1}^{\xi_0} - m_b|_{\xi_0, \xi_1} \leq \sum_{\iota=0}^{\kappa} s^{\iota+1} \psi^\iota \left[m_b|_{\xi_1}^{\xi_0} \right] \leq \varepsilon, \text{ for } \varepsilon > 0.$$

That is $\xi_1 \in \overline{B_b|_\varepsilon^{\xi_0}}$. Let $\xi_2, \xi_3, \dots, \xi_j \in \overline{B_b|_\varepsilon^{\xi_0}}$ for some $j \in \mathbb{N}$, we write

$$m_b|_{\xi_{j+1}}^{\xi_j} \leq s^2 m_b|_{\Gamma \xi_j}^{\Gamma \xi_{j-1}} \leq \psi \left[\lambda(\mathbb{H}_1|_{\xi_j}^{\xi_{j-1}}) \right]. \quad (3.59)$$

Thereafter, by tracing the line employed in (3.24), we get

$$\mathbb{H}_1|_{\xi_j}^{\xi_{j-1}} \leq \left(s m_b|_{\xi_{j-1}, \xi_{j+1}} \right)^{\frac{1}{(c-a)(c-b)}} m_b|_{\xi_j}^{\xi_{j-1}}. \quad (3.60)$$

From (3.59), (3.60) and similar to (3.23) and (3.24), we conclude that

$$m_b|_{\xi_{j+1}}^{\xi_j} < \psi^j \left[m_b|_{\xi_1}^{\xi_0} \right], \quad \forall j \in \mathbb{N}. \quad (3.61)$$

Using (M_{biv}) and (3.61), we have

$$\begin{aligned} m_b|_{\xi_{j+1}}^{\xi_0} - m_b|_{\xi_0, \xi_{j+1}} &\leq s \left[\left(m_b|_{\xi_1}^{\xi_0} - m_b|_{\xi_0, \xi_1} \right) + \left(m_b|_{\xi_{j+1}}^{\xi_1} - m_b|_{\xi_1, \xi_{j+1}} \right) \right] - m_b|_{\xi_1}^{\xi_1} \\ &\leq s \left[m_b|_{\xi_1}^{\xi_0} + s m_b|_{\xi_2}^{\xi_1} + s^2 m_b|_{\xi_3}^{\xi_2} + \dots + s^j m_b|_{\xi_{j+1}}^{\xi_j} \right] \\ &= s m_b|_{\xi_1}^{\xi_0} + s^2 m_b|_{\xi_2}^{\xi_1} + s^3 m_b|_{\xi_3}^{\xi_2} + \dots + s^{j+1} m_b|_{\xi_{j+1}}^{\xi_j} \\ &< s m_b|_{\xi_1}^{\xi_0} + s^2 \psi \left(m_b|_{\xi_1}^{\xi_0} \right) + \dots + s^{j+1} \psi^j \left(m_b|_{\xi_1}^{\xi_0} \right) \\ &< \sum_{\iota=0}^j s^{\iota+1} \psi^\iota \left(m_b|_{\xi_1}^{\xi_0} \right) < \varepsilon. \end{aligned}$$

That is $\xi_{j+1} \in \overline{B_b|_\varepsilon^{\xi_0}}$. Hence, by induction, we get $\xi_\kappa \in \overline{B_b|_\varepsilon^{\xi_0}} \forall \kappa \in \mathbb{N}$, therefore $\{\xi_\kappa\}$ is a sequence in $\overline{B_b|_\varepsilon^{\xi_0}}$. As Γ is simi α -admissible w.r.t η on $\overline{B_b|_\varepsilon^{\xi_0}}$, so $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa}$. Also, inequality (3.61) can be written as

$$m_b|_{\xi_{\kappa+1}}^{\xi_\kappa} < \psi^\kappa \left[m_b|_{\xi_1}^{\xi_0} \right], \quad \forall \kappa \in \mathbb{N}. \quad (3.62)$$

As $\sum_{\iota=1}^{\infty} s^\iota \psi^\iota(r) < \infty$, thus for some $k \in \mathbb{N}$ the series $\sum_{\iota=1}^{\infty} s^\iota \psi^\iota \left[\psi^{k-1} \left(m_b|_{\xi_1}^{\xi_0} \right) \right]$ converges. Fix

$\varepsilon > 0$, then $\exists_{k(\varepsilon) \in \mathbb{N}}$, so that

$$\sum_{\iota=1}^{\infty} s^{\iota} \psi^{\iota} \left[\psi^{\kappa-1} \left(m_b|_{\xi_1}^{\xi_0} \right) \right] < \varepsilon. \quad (3.63)$$

Let $\kappa, \varkappa \in \mathbb{N}$ with $\kappa > \varkappa > k(\varepsilon)$ and from (M_{biv}) , (3.62) and (3.63), we get

$$\begin{aligned} m_b|_{\xi_{\varkappa}}^{\xi_{\kappa}} - m_b|_{\xi_{\kappa}, \xi_{\varkappa}} &\leq \sum_{\iota=\kappa}^{\varkappa-1} s^{\iota-\kappa+1} m_b|_{\xi_{\iota+1}}^{\xi_{\iota}} \\ &\leq \sum_{\iota=\kappa}^{\varkappa-1} s^{\iota-\kappa+1} \psi^{\iota} \left[m_b|_{\xi_{\iota+1}}^{\xi_{\iota}} \right] \\ &< \sum_{\iota=\kappa}^{\varkappa-\kappa} s^{\iota-\kappa} \psi^{i-\kappa} \left[\psi^{\kappa-1} \left(m_b|_{\xi_1}^{\xi_0} \right) \right] \\ &< \sum_{\iota=1}^{\infty} s^{\iota} \psi^{\iota} \left[\psi^{k(\varepsilon)-1} \left(m_b|_{\xi_1}^{\xi_0} \right) \right] \\ &< \varepsilon. \end{aligned}$$

The convergence of the series $\sum_{\iota=1}^{\infty} s^{\iota} \psi^{\iota} \left[\psi^{k-1} \left(m_b|_{\xi_1}^{\xi_0} \right) \right]$ leads to

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(m_b|_{\xi_{\varkappa}}^{\xi_{\kappa}} - m_b|_{\xi_{\kappa}, \xi_{\varkappa}} \right) = 0.$$

By the same way, we can show that

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(M_b|_{\xi_{\kappa}, \xi_{\varkappa}} - m_b|_{\xi_{\kappa}, \xi_{\varkappa}} \right) = 0.$$

Therefore, $\{\xi_{\kappa}\}$ is an m_b -Cauchy sequence in $\overline{B_b|_{\varepsilon}^{\xi_0}}$. So, $\exists_{\xi^* \in \overline{B_b|_{\varepsilon}^{\xi_0}}}$ so that $\xi_{\kappa} \rightarrow \xi^*$ as $\kappa \rightarrow \infty$.

By continuity of Γ then $\lim_{\kappa \rightarrow \infty} \Gamma \xi_{\kappa} = \Gamma \xi^*$ and

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\xi^*}^{\xi_{\kappa}} - m_b|_{\xi_{\kappa}, \xi^*} \right) = 0. \quad (3.64)$$

Next, we show that $\Gamma\xi^* = \xi^*$. Suppose that $m_b|_{\Gamma\xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma\xi^*} > 0$, then by (M_{biv}) , we have

$$\begin{aligned}
m_b|_{\Gamma\xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma\xi^*} &\leq s \left[\left(m_b|_{\xi_{\kappa+1}}^{\xi^*} - m_b|_{\xi^*, \xi_{\kappa+1}} \right) + \left(m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \Gamma\xi^*} \right) \right] \\
&\quad - m_b|_{\xi_{\kappa+1}}^{\xi_{\kappa+1}} \\
&\leq s \left(m_b|_{\xi_{\kappa+1}}^{\xi^*} - m_b|_{\xi^*, \xi_{\kappa+1}} \right) + s \left(m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} - m_b|_{\xi_{\kappa+1}, \Gamma\xi^*} \right) \\
&= s \left(m_b|_{\xi_{\kappa+1}}^{\xi^*} - m_b|_{\xi^*, \xi_{\kappa+1}} \right) + \frac{1}{s} \left(s^2 m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} \right) - s m_b|_{\xi_{\kappa+1}, \Gamma\xi^*} \\
&\leq s \left(m_b|_{\xi_{\kappa+1}}^{\xi^*} - m_b|_{\xi^*, \xi_{\kappa+1}} \right) + \frac{1}{s} \psi \left[\lambda \left(m_b|_{\xi_{\kappa+1}}^{\xi_{\kappa+1}} \right) \right] - s m_b|_{\xi_{\kappa+1}, \Gamma\xi^*} \\
&= s \left(m_b|_{\xi_{\kappa+1}}^{\xi^*} - m_b|_{\xi^*, \xi_{\kappa+1}} \right) + \\
&\quad \frac{1}{s} \psi \left[\lambda \begin{bmatrix} m_b|_{\xi^*}^{\xi_{\kappa+1}} \cdot \left(m_b|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa+1}} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \\ \left(m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \\ \left(m_b|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa+1}} + m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \\ \left(m_b|_{\Gamma\xi^*}^{\xi_{\kappa+1}} + m_b|_{\Gamma\xi_{\kappa+1}}^{\xi_{\kappa+1}} \right)^{\frac{1}{(c-a)(c-b)}} \end{bmatrix} \right] - s m_b|_{\xi_{\kappa+1}, \Gamma\xi^*}.
\end{aligned} \tag{3.65}$$

By taking $\kappa \rightarrow \infty$ in (3.65) and applying (3.64), we get

$$\begin{aligned}
m_b|_{\Gamma\xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma\xi^*} &\leq \frac{1}{s} \psi \left[\lambda \begin{bmatrix} m_b|_{\Gamma\xi^*}^{\xi^*} \cdot \left(m_b|_{\xi^*}^{\xi^*} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma\xi^*}^{\xi^*} \right)^{\frac{1}{(a-b)(a-c)}} \cdot \\ \left(m_b|_{\xi^*}^{\xi^*} + m_b|_{\Gamma\xi^*}^{\xi^*} \right)^{\frac{1}{(b-a)(b-c)}} \cdot \\ \left(m_b|_{\Gamma\xi^*}^{\xi^*} + s m_b|_{\xi^*}^{\xi^*} + m_b|_{\xi^*, \Gamma\xi^*} \right)^{\frac{1}{(c-a)(c-b)}} \end{bmatrix} \right] - s m_b|_{\xi^*, \Gamma\xi^*} \\
&= \frac{1}{s^2} s \psi \left[\lambda \begin{bmatrix} m_b|_{\Gamma\xi^*}^{\xi^*} \cdot \left(m_b|_{\xi^*}^{\xi^*} \right)^{\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}} \cdot \\ \left(m_b|_{\Gamma\xi^*}^{\xi^*} \right)^{\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}} \cdot \\ \left(s m_b|_{\xi^*, \Gamma\xi^*} \right)^{\frac{1}{(c-a)(c-b)}} \end{bmatrix} \right] - s m_b|_{\xi^*, \Gamma\xi^*} \\
&< \frac{1}{s^2} m_b|_{\Gamma\xi^*}^{\xi^*} - s m_b|_{\xi^*, \Gamma\xi^*} \text{ by } (\psi 3) \\
&< m_b|_{\Gamma\xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma\xi^*},
\end{aligned}$$

a contradiction. Therefore $m_b|_{\Gamma\xi^*}^{\xi^*} - m_b|_{\xi^*, \Gamma\xi^*} = 0$ implies that $m_b|_{\Gamma\xi^*}^{\xi^*} = m_b|_{\xi^*, \Gamma\xi^*}$. So, that is $\Gamma\xi^* = \xi^*$ and $\xi^* \in \overline{B_b|_{\xi^*}^{\xi_0}}$ is an F.P of Γ . ■

Example 3.4.22. Let $\Omega_b = [0, \infty)$, $p \geq 1$ and $m_b : \Omega_b \times \Omega_b \rightarrow \mathbb{R}^+$ defined by

$$m_b|_{\xi_2}^{\xi_1} = \max \{ \xi_1, \xi_2 \}^p + |\xi_1 - \xi_2|^p, \quad \forall \xi_1, \xi_2 \in \Omega_b.$$

Then, (Ω_b, m_b) is an Mb-M.S with $s = 2^p$. Define $\Gamma : \Omega_b \rightarrow \Omega_b$ by

$$\Gamma \xi = \begin{cases} \frac{\xi}{e^5}, & \text{if } \xi \in [0, 10]; \\ 4\xi - 45, & \text{if } \xi \in (10, \infty). \end{cases}$$

Describe $\alpha, \eta : \Omega_b \times \Omega_b \rightarrow [0, \infty)$ as

$$\alpha|_{\xi_2}^{\xi_1} = \begin{cases} 2, & \text{if } \xi \in [0, 10]; \\ 0, & \text{if } \xi \in (10, \infty), \end{cases} \quad \text{and} \quad \eta|_{\xi_2}^{\xi_1} = \begin{cases} 1, & \text{if } \xi \in [0, 10]; \\ 0, & \text{if } \xi \in (10, \infty). \end{cases}$$

Consider $\xi_0 = 1$ and $\varepsilon = 180$, then $\overline{B_b|_{\varepsilon}^{\xi_0}} = [0, 10]$ and $m_b|_{\Gamma \xi_0}^{\xi_0} = m_b|_{\Gamma 1}^1 = m_b|_{\frac{1}{e^5}}^1 = 1 + \left(1 - \frac{1}{e^5}\right)^p$. Therefore $\alpha|_{\Gamma 1}^1 = 2 \geq 1 = \eta|_{\Gamma 1}^1$. Now, $\forall \xi_1, \xi_2 \in [0, 10]$, the semi α -admissibility of Γ w.r.t. η is held, whenever $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$, we have

$$\begin{aligned} s^2 m_b|_{\Gamma \xi_2}^{\Gamma \xi_1} &= 2^{2p} m_b|_{\frac{\xi_2}{e^5}}^{\frac{\xi_1}{e^5}} = 2^{2p} \left(\max \left\{ \frac{\xi_1}{e^5}, \frac{\xi_2}{e^5} \right\}^p + \left| \frac{\xi_1}{e^5} - \frac{\xi_2}{e^5} \right|^p \right) \\ &= \frac{2^{2p}}{e^{5p}} (\max \{ \xi_1, \xi_2 \}^p + |\xi_1 - \xi_2|^p) \\ &\leq \frac{2^p}{e^{5p}} \left(2^{2p} m_b|_{\xi_2}^{\xi_1} \right) \leq \psi \left[\lambda \left(\mathbb{H}_1|_{\xi_2}^{\xi_1} \right) \right]. \end{aligned}$$

That is attained by taking $\psi(t) = \frac{2^p}{e^{5p}} t$, $\lambda = 2^{2p} \in [0, \infty)$ and $a, b, c \in (0, 1)$. Now, $\forall \kappa \geq 0, p > 1$, we have

$$\begin{aligned} m_b|_{\xi_1}^{\xi_0} - m_b|_{\xi_0, \xi_1} &= \sum_{i=0}^{\kappa} s^{i+1} \psi^i \left[m_b|_{\xi_1}^{\xi_0} \right] \leq \sum_{i=0}^{\kappa} s^{i+1} \psi^i \left[1 + \left(1 - \frac{1}{e^5} \right)^p \right] \\ &= 2^2 \left[1 + \left(1 - \frac{1}{e^5} \right)^p \right] \sum_{i=0}^{\kappa} \left(\frac{2^{2p}}{e^{5p}} \right)^i \\ &\leq 180 = \varepsilon. \end{aligned}$$

Note that $20, 21 \in \Omega_b$ and for $p = 2$, we get $\alpha|_{21}^{20} \geq \eta|_{21}^{20}$ and

$$s^2 m_b|_{\Gamma_{21}}^{\Gamma_{20}} > \Upsilon \left[\lambda \left(\mathbb{F}_1|_{21}^{20} \right) \right],$$

yields that (3.21) is not attained. Thence, all affirmations of Theorem 3.4.21 are satisfied, and Γ possesses an F.P $\xi^* = 0 \in \overline{B_b|_{\varepsilon}^{\xi_0}}$. (Note that $15 \in \Omega_b$ is an another F.P of Γ , but $15 \notin \overline{B_b|_{\varepsilon}^{\xi_0}}$).

3.5 An application to a fractional-order differential equation

In this section, The outcomes given by Theorem 3.4.9 are applying to show the existence of a solution to FODEs.

Consider the function $\mathcal{L} : (0, 1) \rightarrow \mathbb{R}$. The conformable fractional-order derivative of order κ of $\mathcal{L}$ at $r > 0$ is defined in [1]:

$$D^\kappa \mathcal{L}(r) = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{L}(r + \varepsilon r^{1-\kappa}) - \mathcal{L}(r)}{\varepsilon}. \quad (3.66)$$

The conformable fractional integral related to (3.66) is defined in [1, 58] as given below:

$$I_0^\alpha \mathcal{L}(r) = \int_0^r k^{\kappa-1} \mathcal{L}(k) dk. \quad (3.67)$$

The BVP of a FODE can be formulated as:

$$\left\{ \begin{array}{l} D_r^\kappa \xi(r) = \lambda \mathcal{L}|_{\xi(r)}^r, \quad r \in (0, 1), \quad \kappa \in (1, 2) \\ \xi(0) = 0, \quad \xi(1) = \int_0^1 \xi(k) dk \end{array} \right. . \quad (3.68)$$

The IE related with BVP (3.68) can be expressed as follows:

$$\xi(r) = \lambda \int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi(k)}^k dk, \quad (3.69)$$

where, $\mathcal{U}_k^r$ is a G.F given by

$$\mathcal{U}_k^r = \begin{cases} -2rk^\kappa + k^{\kappa-1}, & 0 \leq k \leq r \leq 1 \\ -2rk^\kappa, & 0 \leq r \leq k \leq 1 \end{cases}, \quad (3.70)$$

and $\int_0^1 \xi(k)dk$ is the Riemann integrable of ξ w.r.t k and $\mathcal{L} : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Consider

$$\xi(r) = c_0 + c_1 r + \lambda \int_0^1 k^{\kappa-1} \mathcal{L}_{|\xi(k)}^k dk, \quad (3.71)$$

when $\xi(0) = 0$, then $c_0 = 0$, $\xi(1) = c_1 + \lambda \int_0^1 k^{\alpha-1} \mathcal{L}_{|\xi(k)}^k dk$, and from the condition $\xi(1) = \int_0^1 \xi(k)dk$, we have

$$\begin{aligned} \int_0^1 \xi(k)dk &= \int_0^1 c_1 k dk + \lambda \int_0^1 \int_0^k v^{\kappa-1} \mathcal{L}_{|\xi(k)}^k dv dk = \frac{1}{2}c_1 + \lambda \int_0^1 \int_v^1 v^{\kappa-1} \mathcal{L}_{|\xi(k)}^k dv dk \\ &= \frac{1}{2}c_1 + \lambda \int_0^1 (1-v) v^{\kappa-1} \mathcal{L}_{|\xi(k)}^k dv dk \\ &= \frac{1}{2}c_1 + \lambda \int_0^1 (k^{\kappa-1} - k^\kappa) \mathcal{L}_{|\xi(k)}^k dk, \end{aligned}$$

this implies that

$$\frac{1}{2}c_1 = \lambda \int_0^1 k^{\kappa-1} \mathcal{L}_{|\xi(k)}^k dk + \lambda \int_0^1 (k^{\kappa-1} - k^\kappa) \mathcal{L}_{|\xi(k)}^k dk = -\lambda \int_0^1 k^\kappa \mathcal{L}(k, \xi(k)) dk.$$

Hence

$$c_1 = -2\lambda \int_0^1 k^\kappa \mathcal{L}_{|\xi(k)}^k dk.$$

It follows from (3.71), c_0 , and c_1 that

$$\begin{aligned}
\xi(r) &= -2\lambda r \int_0^1 k^\kappa \mathcal{L}|_{\xi(k)}^k dk + \lambda \int_0^r k^{\kappa-1} \mathcal{L}|_{\xi(k)}^k dk \\
&= -2\lambda r \int_0^r k^\kappa \mathcal{L}|_{\xi(k)}^k dk - 2\lambda t \int_t^1 k^\kappa \mathcal{L}|_{\xi(k)}^k dk + \lambda \int_0^r k^{\kappa-1} \mathcal{L}|_{\xi(k)}^k dk \\
&= \lambda \int_0^r (-2rk^\kappa + k^{\kappa-1}) \mathcal{L}|_{\xi(k)}^k dk + \lambda \int_t^1 (-2rk^\kappa) \mathcal{L}|_{\xi(k)}^k dk \\
&= \lambda \int_0^r \mathcal{U}|_k^r \mathcal{L}|_{\xi(k)}^k dk.
\end{aligned}$$

Let $Q^\kappa(I)$ be the space of continuous functions on I , where $I = [0, 1]$, $\kappa > 0$. Let $m_b|_{\xi_2}^{\xi_1} = \left\| \frac{\xi_1 + \xi_2}{2} \right\|_\infty^2 = \max_{r \in I} \left(\left| \frac{\xi_1(r) + \xi_2(r)}{2} \right| \right)^2 \forall_{\xi_1, \xi_2 \in Q^\kappa(I)}$. Then $(Q^\kappa(I), m_b)$ is a C-Mb-M.S with a constant $\delta = 2$.

Theorem 3.5.1. The BVP (3.68) has at least one solution $\xi^* \in Q^\kappa(I)$, under the affirmations below:

(1) $\exists \varpi : \mathbb{R} \rightarrow (0, 1)$, $\rho, \sigma, \varrho : \mathbb{R} \rightarrow \mathbb{R}$ are four functions and $\delta \geq 1$ so that $\forall_{r \in I, \xi_1, \xi_2 \in \mathbb{R}}$ with $\rho(\xi_1)\sigma(\xi_2) \geq \varrho(\xi_1)\varrho(\xi_2)$, we have

$$\left| \mathcal{L}|_{\xi_1(k)}^k + \mathcal{L}|_{\xi_2(k)}^k \right| \leq \sqrt{\frac{4\varpi(r)}{10\delta^2}} \mathbb{H}|_{\xi_2(k)}^{\xi_1(k)},$$

where

$$\begin{aligned}
\mathbb{H}|_{\xi_2(k)}^{\xi_1(k)} &= \left| \frac{\xi_1(k) + \xi_2(k)}{2} \right|^2 \cdot \left| \frac{\xi_1(k) + \Gamma\xi_1(k)}{2} \right|^{\frac{2a}{(a-b)(a-c)}} \cdot \left| \frac{\xi_2(k) + \Gamma\xi_2(k)}{2} \right|^{\frac{2a}{(a-b)(a-c)}} \\
&\quad \left[\left| \frac{\xi_1(k) + \Gamma\xi_1(k)}{2} \right|^2 + \left| \frac{\xi_2(k) + \Gamma\xi_2(k)}{2} \right|^2 \right]^{\frac{b}{(b-a)(b-c)}} \\
&\quad \left[\left| \frac{\xi_1(k) + \Gamma\xi_2(k)}{2} \right|^2 + \left| \frac{\xi_2(k) + \Gamma\xi_1(k)}{2} \right|^2 \right]^{\frac{c}{(c-a)(c-b)}}, \quad \forall_{a, b, c \in (0, 1)};
\end{aligned}$$

(2) $\exists_{\xi_1 \in Q^\kappa(I)}$ such that $\forall_{r \in I}$, we get

$$\rho(\xi_1(r)) \sigma \left(\int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi_1(k)}^k dk \right) \geq \varrho(\xi_1(r)) \varrho \left(\int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi_1(k)}^k dk \right);$$

(3) $\forall_{r \in I}$ and $\forall_{\xi_1, \xi_2 \in Q^\kappa(I)}$, $\exists_{\xi_3, \xi_4 \in C^\alpha(I)}$, such that

$$\rho(\xi_1(r)) \geq \varrho(\xi_1(r)) \implies \sigma \left(\int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi_3(k)}^k dk \right) \geq \varrho \left(\int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi_3(k)}^k dk \right),$$

and

$$\sigma(\xi_2(r)) \geq \varrho(\xi_2(r)) \implies \rho \left(\int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi_4(k)}^k dk \right) \geq \varrho \left(\int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi_4(k)}^k dk \right);$$

(4) For any cluster point ξ of $\{\xi_\kappa\}$ in $Q^\kappa(I)$ with $\rho(\xi_\kappa) \sigma(\xi_{\kappa+1}) \geq \varrho(\xi_\kappa) \varrho(\xi_{\kappa+1})$ so that $\lim_{n \rightarrow \infty} \inf \rho(\xi_\kappa) \sigma(\xi) \geq \lim_{n \rightarrow \infty} \inf \varrho(\xi_\kappa) \varrho(\xi)$.

Proof. It is known that $\xi^* \in Q^\kappa(I)$ is a solution of (3.68) iff $\xi^* \in Q^\kappa(I)$ is a solution of the FODE:

$$\xi(r) = \lambda \int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi(k)}^k dk, \quad \forall_{\lambda, r \in I}.$$

Define a map $\Gamma : Q^\kappa(I) \rightarrow Q^\kappa(I)$ by

$$\Gamma \xi(r) = \lambda \int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi(k)}^k dk, \quad \forall_{\lambda, r \in I}.$$

Then, the problem (3.68) is equivalent to find $\xi^* \in Q^\kappa(I)$ that is a F.P of Γ . Let $\xi_1, \xi_2 \in Q^\kappa(I)$, such that $\rho(\xi_1(r)) \sigma(\xi_2(r)) \geq 0, \forall_{r \in I}$. Using the stipulation (1), we get

$$\begin{aligned} [|\Gamma \xi_1(r) + \Gamma \xi_2(r)|]^2 &= \left[|\lambda| \left| \int_0^1 \mathcal{U}|_k^r \mathcal{L}|_{\xi_1(k)}^k dk + \mathcal{U}|_k^r \mathcal{L}|_{\xi_2(k)}^k dk \right| \right]^2 \\ &\leq |\lambda| \int_0^1 \mathcal{U}|_k^r \left| \mathcal{L}|_{\xi_1(k)}^k + \mathcal{L}|_{\xi_2(k)}^k \right| dk \\ &\leq \left[|\lambda| \int_0^1 \mathcal{U}|_k^r dk \sqrt{\frac{4\varpi(r)}{10\delta^2}} \mathbb{H}_{|\xi_2(k)}^{\xi_1(k)} \right]^2 \\ &= \left[|\lambda| \int_0^1 \mathcal{U}|_k^r dk \right]^2 \frac{4\varpi(r)}{10\delta^2} \mathbb{H}_{|\xi_2(k)}^{\xi_1(k)}. \end{aligned}$$

For $r \in I$, we get

$$\begin{aligned} \max_{r \in I} [|\Gamma \xi_1(r) + \Gamma \xi_2(r)|]^2 &= \max_{r \in I} \left[|\lambda| \int_0^1 \mathcal{U}_k^r dk \right]^2 \frac{4\varpi(r)}{10\delta^2} \cdot \\ &\quad \left| \frac{\xi_1(k) + \xi_2(k)}{2} \right|^2 \cdot \left| \frac{\xi_1(k) + \Gamma \xi_1(k)}{2} \right|^{\frac{2a}{(a-b)(a-c)}} \cdot \left| \frac{\xi_2(k) + \Gamma \xi_2(k)}{2} \right|^{\frac{2a}{(a-b)(a-c)}} \cdot \\ &\quad \left[\left| \frac{\xi_1(k) + \Gamma \xi_1(k)}{2} \right|^2 + \left| \frac{\xi_2(k) + \Gamma \xi_2(k)}{2} \right|^2 \right]^{\frac{b}{(b-a)(b-c)}} \cdot \\ &\quad \left[\left| \frac{\xi_1(k) + \Gamma \xi_2(k)}{2} \right|^2 + \left| \frac{\xi_2(k) + \Gamma \xi_1(k)}{2} \right|^2 \right]^{\frac{c}{(c-a)(c-b)}}, \end{aligned}$$

this implies that

$$\begin{aligned} [||\Gamma \xi_1(r) + \Gamma \xi_2(r)||_\infty]^2 &\leq \frac{4\varpi(r)}{10\delta^2} \left\| \frac{\xi_1(k) + \xi_2(k)}{2} \right\|_\infty^2 \cdot \\ &\quad \left\| \frac{\xi_1(k) + \Gamma \xi_1(k)}{2} \right\|_\infty^{\frac{2a}{(a-b)(a-c)}} \cdot \left\| \frac{\xi_2(k) + \Gamma \xi_2(k)}{2} \right\|_\infty^{\frac{2a}{(a-b)(a-c)}} \cdot \\ &\quad \left[\left\| \frac{\xi_1(k) + \Gamma \xi_1(k)}{2} \right\|_\infty^2 + \left\| \frac{\xi_2(k) + \Gamma \xi_2(k)}{2} \right\|_\infty^2 \right]^{\frac{b}{(b-a)(b-c)}} \cdot \\ &\quad \left[\left\| \frac{\xi_1(k) + \Gamma \xi_2(k)}{2} \right\|_\infty^2 + \left\| \frac{\xi_2(k) + \Gamma \xi_1(k)}{2} \right\|_\infty^2 \right]^{\frac{c}{(c-a)(c-b)}}. \end{aligned}$$

Thus,

$$\delta^2 m_b|_{\Gamma \xi_2}^{\Gamma \xi_1} \leq \psi \left[\lambda \mathbb{F}_2|_{\xi_2}^{\xi_1} \right],$$

where, $\psi(r) = \frac{\omega(r)}{5}r$, $\lambda = \frac{1}{2}$ and

$$\mathbb{F}_2|_{\xi_2}^{\xi_1} = \left\{ \begin{aligned} &m_b|_{\xi_2}^{\xi_1} \cdot \left(m_b|_{\Gamma \xi_1}^{\xi_1} \right)^{\frac{a}{(a-b)(a-c)}} \cdot \left(m_b|_{\Gamma \xi_2}^{\xi_2} \right)^{\frac{a}{(a-b)(a-c)}} \cdot \\ &\left(m_b|_{\Gamma \xi_1}^{\xi_1} + m_b|_{\Gamma \xi_2}^{\xi_2} \right)^{\frac{b}{(b-a)(b-c)}} \cdot \left(m_b|_{\Gamma \xi_2}^{\xi_1} + m_b|_{\Gamma \xi_1}^{\xi_2} \right)^{\frac{c}{(c-a)(c-b)}} \end{aligned} \right\}.$$

$\forall_{\xi_1, \xi_2 \in Q^\kappa(I), r \in I}$ such that $\rho(\xi_1(r))\sigma(\xi_2(r)) \geq \varrho(\xi_1(r))\varrho(\xi_2(r))$. We define $\alpha, \beta, \eta : Q^\kappa(I) \rightarrow$

$[0, \infty)$ assuming $\alpha(\xi_1) = \beta(\xi_1)$ by

$$\alpha(\xi_1) = \left\{ \begin{array}{ll} 2 & \text{if } \rho(\xi_1(r)) \sigma(\xi_2(r)) \geq 0, \\ 0 & \text{otherwise} \end{array} \right\} \text{ and } \eta(\xi_1) = \left\{ \begin{array}{ll} \frac{1}{4} & \text{if } \varrho(\xi_1(r)) \varrho(\xi_2(r)) \geq 0, \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then, $\forall_{\xi_1, \xi_2 \in Q^\kappa(I)}$, $\alpha(\xi_1) \geq \eta(\xi_1)$ and $\beta(\xi_2) \geq \eta(\xi_2)$. If $\alpha(\xi_1) \geq \eta(\xi_1)$ and $\beta(\xi_2) \geq \eta(\xi_2)$ $\forall_{\xi_1, \xi_2 \in Q^\kappa(I)}$, then $\rho(\xi_1(r)) \geq \varrho(\xi_1(r))$ and $\sigma(\xi_2(r)) \geq \varrho(\xi_2(r))$. From (3), we have $\rho(\Gamma\xi_1(r)) \geq \varrho(\Gamma\xi_1(r))$ and $\sigma(\Gamma\xi_2(r)) \geq \varrho(\Gamma\xi_2(r))$, and so $\alpha(\Gamma\xi_1) \geq \eta(\Gamma\xi_1)$ and $\beta(\Gamma\xi_2) \geq \eta(\Gamma\xi_2)$. Thus, Γ is η -cyclic α -admissible mapping. From (2) there subsist $\xi_1 \in Q^\kappa(I)$ parallel to $\alpha(\xi_1) \beta(\xi_2) \geq \eta(\xi_1) \eta(\xi_2)$. By (4), for any cluster point ξ of $\{\xi_\kappa\}$ in $Q^\kappa(I)$ with $\rho(\xi_\kappa) \sigma(\xi_{\kappa+1}) \geq \varrho(\xi_\kappa) \varrho(\xi_{\kappa+1})$ implies $\alpha(\xi_\kappa) \beta(\xi_{\kappa+1}) \geq \eta(\xi_\kappa) \eta(\xi_{\kappa+1})$ and $\lim_{\kappa \rightarrow \infty} \inf \alpha(\xi_\kappa) \beta(\xi_{\kappa+1}) \geq \lim_{\kappa \rightarrow \infty} \inf \eta(\xi_\kappa) \eta(\xi_{\kappa+1})$. So, all affirmations of Theorem 3.4.9 are satisfied, and then Γ has an F.P ξ^* in $Q^\kappa(I)$, that is a solvability of BVP (3.66). ■

3.6 Common fixed points of almost Hardy-Rogers (ψ, ϕ) -type multivalued contractions

This section contains C.F.P outcomes for almost Hardi-Rogers (ψ, ϕ) -type and θ -type multivalued contractions in C-Mb-M.S.

Definition 3.6.1. Let (Ω_b, m_b) be an Mb-M.S and $\Gamma, \mathfrak{S} : \Omega_b \rightarrow CB_b|_{\Omega_b}$ are told to be almost Hardy-Rogers (ψ, ϕ) -type multivalued contraction maps, if $\exists_{\phi \in \Phi^*, \psi \in \Psi}$ and $\exists_{s \geq 1, \alpha \in [0,1]}$ such that $\forall_{\zeta_1, \zeta_2 \in \Omega_b}$, $H_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} > 0$, implies,

$$\phi \left(s^3 H_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} \right) \leq \psi \left(\phi \left[\alpha U_b|_{\zeta_2}^{\zeta_1} + L m_b|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \right] \right), \quad (3.72)$$

where

$$\begin{aligned} U_b|_{\zeta_2}^{\zeta_1} &= a m_b|_{\zeta_2}^{\zeta_1} + b m_b|_{\Gamma(\zeta_1)}^{\zeta_1} + c m_b|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} + \\ & d \frac{m_b|_{\Gamma(\zeta_2)}^{\zeta_2} \left[s + m_b|_{\Gamma(\zeta_1)}^{\zeta_1} \right]}{s + m_b|_{\zeta_2}^{\zeta_1}} + e \frac{m_b|_{\Gamma(\zeta_1)}^{\zeta_1} m_b|_{\Gamma(\zeta_2)}^{\zeta_2}}{s + m_b|_{\zeta_2}^{\zeta_1}}, \end{aligned} \quad (3.73)$$

and $a, b, c, d, e \geq 0$ with $\alpha a < \frac{1}{s^3}$ and $0 \leq \alpha a + \alpha b + \alpha c + \alpha d + \alpha e + L < 1$.

Theorem 3.6.2. Let (Ω_b, m_b) be a C-Mb-M.S and the maps $\Gamma, \mathfrak{S} : \Omega_b \rightarrow CB_b|_{\Omega_b}$ be an almost Hardy-Rogers (ψ, ϕ) -type multivalued contraction maps. If ψ, Γ and $\mathfrak{S}$ are continuous.

Then, Γ and $\mathfrak{S}$ have a C.F.P.

Proof. Let $\zeta_0 \in \Omega_b$ be an arbitrary point. Since $\Gamma(\zeta) \neq \emptyset \forall \zeta \in \Omega_b$, we take $\zeta_1 \in \Gamma(\zeta_0)$ with $\zeta_1 \notin \mathfrak{S}(\zeta_1)$ and $m_b|_{\Gamma(\zeta_0)}^{\zeta_0} > 0$. Then, $\exists \zeta_2 \in \Omega_b$ such that $\zeta_2 \in \mathfrak{S}(\zeta_1)$ with $\zeta_2 \notin \Gamma(\zeta_2)$. Since $\Gamma(\zeta_2)$ is closed and $m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} > 0$, we have

$$0 < m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \leq H_b|_{\mathfrak{S}(\zeta_1)}^{\Gamma(\zeta_0)}.$$

From (ϕ_1) , we get

$$\phi\left(m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1}\right) \leq \phi\left(H_b|_{\mathfrak{S}(\zeta_1)}^{\Gamma(\zeta_0)}\right). \quad (3.74)$$

By (ϕ_5) , $\exists \zeta_2 \in \mathfrak{S}(\zeta_1)$ such that $\phi\left(m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1}\right) = \phi\left(m_b|_{\zeta_2}^{\zeta_1}\right)$. Thence, from (3.72) and (3.74), we have

$$0 \leq \phi\left(m_b|_{\zeta_2}^{\zeta_1}\right) \leq \phi\left(s^3 H_b|_{\mathfrak{S}(\zeta_1)}^{\Gamma(\zeta_0)}\right) \leq \psi\left(\phi\left[\alpha U_b|_{\zeta_1}^{\zeta_0} + L m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1}\right]\right). \quad (3.75)$$

From (M_{biv}) and (3.73), we have

$$\begin{aligned} U_b|_{\zeta_1}^{\zeta_0} &= a m_b|_{\zeta_1}^{\zeta_0} + b m_b|_{\Gamma(\zeta_0)}^{\zeta_0} + c m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} + \\ &\quad d \frac{m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \left[s + m_b|_{\Gamma(\zeta_0)}^{\zeta_0}\right]}{s + m_b|_{\zeta_1}^{\zeta_0}} + e \frac{m_b|_{\Gamma(\zeta_0)}^{\zeta_0} m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1}}{s + m_b|_{\zeta_1}^{\zeta_0}} \\ &\leq a m_b|_{\zeta_1}^{\zeta_0} + b m_b|_{\zeta_1}^{\zeta_0} + c m_b|_{\zeta_2}^{\zeta_1} + d m_b|_{\zeta_2}^{\zeta_1} + e m_b|_{\zeta_2}^{\zeta_1} \\ &\leq (a + b) m_b|_{\zeta_1}^{\zeta_0} + (c + d + e) m_b|_{\zeta_2}^{\zeta_1}. \end{aligned}$$

From (3.75), (M_{bii}) and since $\psi \in \Psi$, we get

$$\begin{aligned} \phi\left(m_b|_{\zeta_2}^{\zeta_1}\right) &\leq \psi\left(\phi\left[\begin{array}{c} (\alpha a + \alpha b) m_b|_{\zeta_1}^{\zeta_0} + (\alpha c + \alpha d + \alpha e) m_b|_{\zeta_2}^{\zeta_1} \\ + L m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \end{array}\right]\right) \\ &\leq \psi\left(\phi\left[(\alpha a + \alpha b) m_b|_{\zeta_1}^{\zeta_0} + (\alpha c + \alpha d + \alpha e + L) m_b|_{\zeta_2}^{\zeta_1}\right]\right). \end{aligned}$$

Since $\psi \in \Psi$ and from (ϕ_1) , we get

$$m_b|_{\zeta_2}^{\zeta_1} \leq (\alpha a + \alpha b) m_b|_{\zeta_1}^{\zeta_0} + (\alpha c + \alpha d + \alpha e + L) m_b|_{\zeta_2}^{\zeta_1}.$$

Since $\alpha a + \alpha b + \alpha c + \alpha d + \alpha e + L < 1$, we deduce that $1 - (\alpha c + \alpha d + \alpha e + L) > \alpha a + \alpha b > 0$.

Thus

$$\begin{aligned} m_b|_{\zeta_2}^{\zeta_1} &\leq \frac{(\alpha a + \alpha b)}{1 - (\alpha c + \alpha d + \alpha e + L)} m_b|_{\zeta_1}^{\zeta_0} \\ &< m_b|_{\zeta_1}^{\zeta_0}. \end{aligned}$$

Again, from (ϕ_1) and since $\psi \in \Psi$, we have

$$\phi \left(m_b|_{\zeta_2}^{\zeta_1} \right) \leq \psi \left(\phi \left(m_b|_{\zeta_1}^{\zeta_0} \right) \right).$$

By continuing this approach, we construct an iterative sequence $\{\zeta_\kappa\}$ in Ω_b by $\zeta_{2\kappa+1} \in \Gamma(\zeta_{2\kappa})$, $\zeta_{2\kappa+2} \in \mathfrak{F}(\zeta_{2\kappa+1})$ and $\zeta_{2\kappa+1} \neq \zeta_{2\kappa}$, $\forall \kappa \in \mathbb{N} \cup \{0\}$, we find that

$$\phi \left(m_b|_{\zeta_{2\kappa+2}}^{\zeta_{2\kappa+1}} \right) \leq \psi \left(\phi \left(m_b|_{\zeta_{2\kappa+1}}^{\zeta_{2\kappa}} \right) \right) < \phi \left(m_b|_{\zeta_{2\kappa+1}}^{\zeta_{2\kappa}} \right). \quad (3.76)$$

The inequality (3.76) yields that $\left\{ m_b|_{\zeta_{2\kappa+1}}^{\zeta_{2\kappa}} \right\}$ is a decreasing sequence. From (3.76), we get

$$\begin{aligned} \phi \left(m_b|_{\zeta_{2\kappa+1}}^{\zeta_{2\kappa}} \right) &\leq \psi \left(\phi \left(m_b|_{\zeta_{2\kappa}}^{\zeta_{2\kappa-1}} \right) \right) \leq \psi^2 \left(\phi \left(m_b|_{\zeta_{2\kappa-1}}^{\zeta_{2\kappa-2}} \right) \right) \\ &\leq \dots \leq \psi^{2\kappa} \left(\phi \left(m_b|_{\zeta_1}^{\zeta_0} \right) \right). \end{aligned} \quad (3.77)$$

Similarly, one gets

$$\begin{aligned} \phi \left(m_b|_{\zeta_{2\kappa+2}}^{\zeta_{2\kappa+1}} \right) &\leq \psi \left(\phi \left(m_b|_{\zeta_{2\kappa+1}}^{\zeta_{2\kappa}} \right) \right) \leq \psi^2 \left(\phi \left(m_b|_{\zeta_{2\kappa}}^{\zeta_{2\kappa-1}} \right) \right) \\ &\leq \dots \leq \psi^{2\kappa+1} \left(\phi \left(m_b|_{\zeta_1}^{\zeta_0} \right) \right). \end{aligned} \quad (3.78)$$

Letting $\kappa \longrightarrow \infty$ in (3.77) and (3.78), we get

$$0 \leq \lim_{\kappa \longrightarrow \infty} \phi \left(m_b|_{\zeta_{\kappa+1}}^{\zeta_\kappa} \right) \leq \lim_{\kappa \longrightarrow \infty} \psi^\kappa \left(\phi \left(m_b|_{\zeta_1}^{\zeta_0} \right) \right) = 0,$$

which implies that

$$\lim_{\kappa \rightarrow \infty} \phi \left(m_b|_{\zeta_{\kappa+1}}^{\zeta_{\kappa}} \right) = 0.$$

Applying (ϕ_2) and Lemma 1.1.32, we have

$$\lim_{\kappa \rightarrow \infty} m_b|_{\zeta_{\kappa+1}}^{\zeta_{\kappa}} = 0. \quad (3.79)$$

We prove that $\{\zeta_{\kappa}\}$ is an M_b -Cauchy in (Ω_b, m_b) . From (M_{bii}) and $\forall_{\kappa \in \mathbb{N}}$, we have

$$0 \leq m_b|_{\zeta_{\kappa}, \zeta_{\kappa+1}} \leq m_b|_{\zeta_{\kappa+1}}^{\zeta_{\kappa}}. \quad (3.80)$$

As of (3.79) and letting $\kappa \rightarrow \infty$ in (3.80), we obtain

$$\lim_{\kappa \rightarrow \infty} m_b|_{\zeta_{\kappa}, \zeta_{\kappa+1}} = 0, \quad (3.81)$$

where (3.81) denotes that

$$\lim_{\kappa \rightarrow \infty} m_b|_{\zeta_{\kappa}}^{\zeta_{\kappa}} = 0 \text{ or } \lim_{\kappa \rightarrow \infty} m_b|_{\zeta_{\kappa+1}}^{\zeta_{\kappa+1}} = 0. \quad (3.82)$$

Therefore

$$\lim_{\kappa, \kappa \rightarrow \infty} m_b|_{\zeta_{\kappa}, \zeta_{\kappa}} = \lim_{\kappa, \kappa \rightarrow \infty} \min \left\{ m_b|_{\zeta_{\kappa}}^{\zeta_{\kappa}}, m_b|_{\zeta_{\kappa}}^{\zeta_{\kappa}} \right\} = 0. \quad (3.83)$$

From (3.82) and (3.83), we get

$$\lim_{\kappa, \kappa \rightarrow \infty} \left(M_b|_{\zeta_{\kappa}, \zeta_{\kappa}} - m_b|_{\zeta_{\kappa}, \zeta_{\kappa}} \right) = \lim_{\kappa, \kappa \rightarrow \infty} \left| m_b|_{\zeta_{\kappa}}^{\zeta_{\kappa}} - m_b|_{\zeta_{\kappa}}^{\zeta_{\kappa}} \right| = 0. \quad (3.84)$$

Next, we shall prove that $\lim_{\kappa, \kappa \rightarrow \infty} \left(m_b|_{\zeta_{\kappa}}^{\zeta_{\kappa}} - m_b|_{\zeta_{\kappa}, \zeta_{\kappa}} \right) = 0$. We argue by contradiction that

$$\lim_{\kappa, \kappa \rightarrow \infty} \left(m_b|_{\zeta_{\kappa}}^{\zeta_{\kappa}} - m_b|_{\zeta_{\kappa}, \zeta_{\kappa}} \right) \neq 0,$$

Consider $\varepsilon > 0$ and subsequences $\{\hat{h}_{\kappa}\}_{\kappa=1}^{\infty}$ and $\{\hat{j}_{\kappa}\}_{\kappa=1}^{\infty}$ such that $\forall_{\kappa \in \mathbb{N}}, \hat{h}_{\kappa} > \hat{j}_{\kappa} > \kappa$ with

$$m_b|_{\zeta_{\hat{j}(\kappa)}}^{\zeta_{\hat{h}(\kappa)}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{\hat{j}(\kappa)}} \geq \varepsilon. \quad (3.85)$$

Let $\hat{h}(\kappa) \in \mathbb{N}$ be the smallest satisfies (3.85), so that

$$m_b|_{\zeta_{j(\kappa)}^{\zeta_{\hat{h}(\kappa)}-1}} - m_b|_{\zeta_{\hat{h}(\kappa)-1}, \zeta_{j(\kappa)}} < \varepsilon. \quad (3.86)$$

Therefore, by (M_{biv}) in (3.85), we get

$$\begin{aligned} \varepsilon &\leq m_b|_{\zeta_{j(\kappa)}^{\zeta_{\hat{h}(\kappa)}}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{j(\kappa)}} \\ &\leq s \left[m_b|_{\zeta_{\hat{h}(\kappa)-1}^{\zeta_{\hat{h}(\kappa)}}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{\hat{h}(\kappa)-1}} + m_b|_{\zeta_{j(\kappa)}^{\zeta_{\hat{h}(\kappa)}-1}} - m_b|_{\zeta_{\hat{h}(\kappa)-1}, \zeta_{j(\kappa)}} \right] \\ &\quad - m_b|_{\zeta_{\hat{h}(\kappa)-1}^{\zeta_{\hat{h}(\kappa)}-1}}. \end{aligned} \quad (3.87)$$

Letting $\kappa \rightarrow \infty$ in (3.87), using (3.79) and (3.86), we obtain

$$\varepsilon \leq \lim_{\kappa \rightarrow \infty} \left[m_b|_{\zeta_{j(\kappa)}^{\zeta_{\hat{h}(\kappa)}}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{j(\kappa)}} \right] \leq s\varepsilon. \quad (3.88)$$

Again, by (M_{biv}) in (3.85), we get

$$\begin{aligned} \varepsilon &\leq m_b|_{\zeta_{j(\kappa)}^{\zeta_{\hat{h}(\kappa)}}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{j(\kappa)}} \\ &\leq s \left[m_b|_{\Gamma(\zeta_{j(\kappa)})}^{\zeta_{\hat{h}(\kappa)}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{\Gamma(j(\kappa))}} + m_b|_{\zeta_{j(\kappa)}}^{\Gamma(\zeta_{j(\kappa)})} - m_b|_{\Gamma(\zeta_{j(\kappa)}), \zeta_{j(\kappa)}} \right] \\ &\quad - m_b|_{\Gamma(\zeta_{j(\kappa)})}^{\Gamma(\zeta_{j(\kappa)})} \\ &\leq s \left[m_b|_{\zeta_{j(\kappa)+1}^{\zeta_{\hat{h}(\kappa)}}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{j(\kappa)+1}} + m_b|_{\zeta_{j(\kappa)}}^{\zeta_{j(\kappa)+1}} - m_b|_{\zeta_{j(\kappa)+1}, \zeta_{j(\kappa)}} \right]. \end{aligned} \quad (3.89)$$

Letting $\kappa \rightarrow \infty$ in (3.89), we get

$$\lim_{\kappa \rightarrow \infty} \left[m_b|_{\Gamma(\zeta_{j(\kappa)})}^{\zeta_{\hat{h}(\kappa)}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \Gamma(\zeta_{j(\kappa)})} \right] \geq \frac{\varepsilon}{s}. \quad (3.90)$$

Applying (M_{biv}) in (3.90) together with (3.79) and (3.88), we have

$$\frac{\varepsilon}{s} \leq \lim_{\kappa \rightarrow \infty} \left[m_b|_{\zeta_{j(\kappa)+1}^{\zeta_{\hat{h}(\kappa)}}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{j(\kappa)+1}} \right] \leq s^2\varepsilon. \quad (3.91)$$

Using (M_{biv}) twice in (3.91), we have

$$\begin{aligned}
\frac{\varepsilon}{s} &\leq m_b|_{\zeta_{j(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{j(\kappa)+1}} \\
&\leq s \left[m_b|_{\Gamma(\zeta_{\hat{h}(\kappa)})}^{\zeta_{\hat{h}(\kappa)}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \Gamma(\zeta_{\hat{h}(\kappa)})} + m_b|_{\zeta_{j(\kappa)+1}}^{\Gamma(\zeta_{\hat{h}(\kappa)})} - m_b|_{\Gamma(\zeta_{\hat{h}(\kappa)}), \zeta_{j(\kappa)+1}} \right] \\
&\quad - m_b|_{\zeta_{\hat{h}(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)+1}} \\
&\leq s \left[s \left[\hat{H}_b|_{\Im(\zeta_{j(\kappa)+1})}^{\Gamma(\zeta_{\hat{h}(\kappa)})} - m_b|_{\Gamma(\zeta_{\hat{h}(\kappa)}), \Im(\zeta_{j(\kappa)+1})} + m_b|_{\zeta_{j(\kappa)+1}}^{\Im(\zeta_{j(\kappa)+1})} \right] \right. \\
&\quad \left. - m_b|_{\zeta_{j(\kappa)+1}}^{\zeta_{j(\kappa)+1}} \right] \\
&\quad - m_b|_{\zeta_{\hat{h}(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)+1}} \\
&\leq s m_b|_{\Gamma(\zeta_{\hat{h}(\kappa)})}^{\zeta_{\hat{h}(\kappa)}} + s^2 \hat{H}_b|_{\Im(\zeta_{j(\kappa)+1})}^{\Gamma(\zeta_{\hat{h}(\kappa)})} + s^2 m_b|_{\zeta_{j(\kappa)+1}}^{\Im(\zeta_{j(\kappa)+1})}.
\end{aligned} \tag{3.92}$$

Taking limit as $\kappa \rightarrow \infty$ in (3.92) and applying (3.79) together with (3.91),

$$\lim_{\kappa \rightarrow \infty} \hat{H}_b|_{\Im(\zeta_{j(\kappa)+1})}^{\Gamma(\zeta_{\hat{h}(\kappa)})} \geq \frac{\varepsilon}{s^3}. \tag{3.93}$$

From (3.72), (3.79), (3.91) and (3.93), using (ϕ_2) , (ϕ_3) and since $\psi \in \Psi$, we get

$$\begin{aligned}
\phi(\varepsilon) &= \phi\left(s^3 \frac{\varepsilon}{s^3}\right) \leq \phi\left(\lim_{\kappa \rightarrow \infty} s^3 \hat{H}_b|_{\Im(\zeta_{j(\kappa)+1})}^{\Gamma(\zeta_{\hat{h}(\kappa)})}\right) \\
&\leq \lim_{\kappa \rightarrow \infty} \psi\left(\phi\left[\alpha U_b|_{\zeta_{j(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}} + L m_b|_{\Im(\zeta_{j(\kappa)+1})}^{\zeta_{j(\kappa)+1}}\right]\right) \\
&\leq \lim_{\kappa \rightarrow \infty} \psi\left(\phi\left(\left(\alpha a s \left(m_b|_{\zeta_{\hat{h}(\kappa)}}^{\zeta_{\hat{h}(\kappa)}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{\hat{h}(\kappa)}} + m_b|_{\zeta_{j(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}} - m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{j(\kappa)+1}}\right) \right. \right. \\
&\quad \left. \left. + \alpha a m_b|_{\zeta_{\hat{h}(\kappa)}, \zeta_{\hat{h}(\kappa)}} + \alpha b m_b|_{\zeta_{\hat{h}(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}} + \alpha c m_b|_{\zeta_{j(\kappa)+2}}^{\zeta_{j(\kappa)+1}} \right. \right. \\
&\quad \left. \left. + d \frac{m_b|_{\zeta_{j(\kappa)+2}}^{\zeta_{j(\kappa)+1}} \left[s + m_b|_{\zeta_{\hat{h}(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}}\right]}{s + m_b|_{\zeta_{j(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}}} + e \frac{m_b|_{\zeta_{\hat{h}(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}} m_b|_{\zeta_{j(\kappa)+2}}^{\zeta_{j(\kappa)+1}}}{s + m_b|_{\zeta_{j(\kappa)+1}}^{\zeta_{\hat{h}(\kappa)}}} + L m_b|_{\zeta_{j(\kappa)+2}}^{\zeta_{j(\kappa)+1}} \right)\right) \\
&= \psi\left(\phi\left(\alpha a s^3 \varepsilon\right)\right) \\
&< \phi(\varepsilon),
\end{aligned}$$

a discrepancy. Therefore, $\lim_{\kappa, \kappa \rightarrow \infty} \left(m_b|_{\xi_\kappa}^{\xi_\kappa} - m_b|_{\xi_\kappa, \xi_\kappa}\right) = 0$ and $\{\zeta_\kappa\}$ is m_b -Cauchy sequence in Ω_b . Since (Ω_b, m_b) is a C-Mb-M.S, by Lemma 1.1.32, the sequence $\{\zeta_\kappa\}$ converges to $\zeta^* \in$

(Ω_b, m_b) . Again, by Lemma 1.1.32, $\exists_{\zeta^* \in \Omega_b}$ such that $\zeta_\kappa \rightarrow \zeta^*$ as $\kappa \rightarrow \infty \implies \zeta_{2\kappa+1} \rightarrow \zeta^*$ and $\zeta_{2\kappa+2} \rightarrow \zeta^*$ as $\kappa \rightarrow \infty$. Thus, we have

$$\lim_{\kappa \rightarrow \infty} \left(m_b|_{\zeta^*}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \zeta^*} \right) = 0. \text{ and } \lim_{\kappa \rightarrow \infty} \left(M_b|_{\xi_{2\kappa+1}, \zeta^*} - m_b|_{\xi_{2\kappa+1}, \zeta^*} \right) = 0. \quad (3.94)$$

Next, we prove that $\zeta^* \in \Gamma(\zeta^*) \cap \mathfrak{S}(\zeta^*)$. Since Γ and $\mathfrak{S}$ are continuous with using (3.80) and (3.94), we get

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} \left(m_b|_{\zeta^*}^{\xi_{2\kappa+1}} - m_b|_{\xi_{2\kappa+1}, \zeta^*} \right) &= 0 \\ &= \lim_{\kappa \rightarrow \infty} m_b|_{\zeta^*}^{\xi_{2\kappa+1}} \\ &= \lim_{\kappa \rightarrow \infty} m_b|_{\zeta^*}^{\Gamma \xi_{2\kappa}} \\ &= m_b|_{\zeta^*}^{\Gamma \xi^*}. \end{aligned}$$

That is $\zeta^* \in \Gamma(\zeta^*)$ and ζ^* is an F.P of Γ .

Similarly,

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} \left(m_b|_{\zeta^*}^{\xi_{2\kappa+2}} - m_b|_{\xi_{2\kappa+2}, \zeta^*} \right) &= 0 \\ &= \lim_{\kappa \rightarrow \infty} m_b|_{\zeta^*}^{\xi_{2\kappa+2}} \\ &= \lim_{\kappa \rightarrow \infty} m_b|_{\zeta^*}^{\mathfrak{S} \xi_{2\kappa+1}} \\ &= m_b|_{\zeta^*}^{\mathfrak{S} \xi_{2\kappa+1}}. \end{aligned}$$

Thence, $\zeta^* \in \mathfrak{S}(\zeta^*)$ and ζ^* is an F.P of $\mathfrak{S}$. Due to the limit uniqueness, we obtain that $\zeta^* \in \Gamma(\zeta^*) \cap \mathfrak{S}(\zeta^*)$. Thus, Γ and $\mathfrak{S}$ have a C.F.P $\zeta^* \in \Omega_b$. ■

Example 3.6.3. Let $\Omega_b = [0, 1]$. Define $m_b : \Omega_b \times \Omega_b \rightarrow [0, +\infty)$ by $m_b|_{\eta}^{\zeta} = \left(\frac{\zeta + \eta}{2} \right)^2$, $\forall_{\zeta, \eta \in \Omega_b}$. Clearly, (Ω_b, m_b) is an Mb-M.S with $S = 2$. Let $\phi(r) = r \in \Phi^*$ and $\psi(r) = \frac{2}{25}r \in \Psi$, $\forall_{r > 0}$. Acquaint $\Gamma, \mathfrak{S} : \Omega_b \longrightarrow CB_b|_{\Omega_b}$ by

$$\Gamma(\zeta) = \begin{cases} \left\{ \frac{\zeta}{30}, \frac{\zeta}{15} \right\}, & \text{if } \zeta \in (0, 1] \\ \{0\}, & \text{otherwise.} \end{cases} \quad \text{and } \mathfrak{S}(\zeta) = \begin{cases} \left\{ \frac{\zeta}{40}, \frac{3\zeta}{40} \right\}, & \text{if } \zeta \in (0, 1] \\ \left\{ 0, \frac{1}{10} \right\}, & \text{otherwise} \end{cases}.$$

Now, we have

$$\begin{aligned}
\phi \left(s^3 H_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} \right) &= \phi \left[2^3 \max \left\{ \sup_{a \in \Gamma(\zeta_1)} m_b|_{\mathfrak{S}(\zeta_2)}^a, \sup_{b \in \mathfrak{S}(\zeta_2)} m_b|_b^{\Gamma(\zeta_1)} \right\} \right] \\
&= \phi \left[2^3 \max \left\{ m_b|_{\left\{ \frac{\zeta_2}{40}, \frac{3\zeta_2}{40} \right\}}^{\frac{\zeta_1}{15}}, m_b|_{\frac{3\zeta_2}{40}}^{\left\{ \frac{\zeta_1}{30}, \frac{\zeta_1}{15} \right\}} \right\} \right] \\
&= \frac{8}{100} \phi \left[\max \left\{ m_b|_{\frac{\zeta_2}{4}}^{\frac{2\zeta_1}{3}}, m_b|_{\frac{3\zeta_2}{4}}^{\frac{\zeta_1}{3}} \right\} \right] \\
&= \frac{2}{25} \phi \left[\max \left\{ \left(\frac{2\zeta_1}{6} + \frac{\zeta_2}{8} \right)^2, \left(\frac{\zeta_1}{6} + \frac{3\zeta_2}{8} \right)^2 \right\} \right] \\
&\leq \psi \left(\phi \left[\alpha U_b|_{\zeta_2}^{\zeta_1} + L m_b|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \right] \right),
\end{aligned}$$

where, $\alpha = \frac{1}{2}$, $a = \frac{1}{6}$, $b = d = \frac{1}{3}$, $c = e = 0$, $L = \frac{1}{4}$ and $U_b|_{\zeta_2}^{\zeta_1}$ is defined as in (3.73). Thereafter, all condations of Theorem 3.6.2 are satisfied so that, $0 \in \Omega_b$ is a C.F.P of Γ and $\mathfrak{S}$.

Definition 3.6.4. Let (Ω_b, m_b) be an Mb-M.S and $\Gamma, \mathfrak{S} : \Omega_b \rightarrow CB_b|_{\Omega_b}^{\Omega_b}$ are termed an almost Hardy-Rogers θ -type contraction mappings, if there exist $\theta \in \Theta$, $\alpha \in [0, 1]$ and $k \in (0, 1)$ so that $\forall_{\zeta_1, \zeta_2 \in \Omega_b}$ and $H_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} > 0$,

$$\theta \left(s^3 H_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} \right) \leq \left[\theta \left(\alpha U_b|_{\zeta_2}^{\zeta_1} + L m_b|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \right) \right]^k, \quad (3.95)$$

where, $U_b|_{\zeta_2}^{\zeta_1}$ is defined as in (3.73) and $a, b, c, d, e \geq 0$ with $\alpha a < \frac{1}{s^3}$ and $0 \leq \alpha a + \alpha b + \alpha c + \alpha d + \alpha e + L < 1$.

Theorem 3.6.5. Let (Ω_b, m_b) be a C-Mb-M.S and $\Gamma, \mathfrak{S} : \Omega_b \rightarrow CB_b|_{\Omega_b}^{\Omega_b}$ be almost Hardy-Rogers θ -type contraction mappings. If ψ , Γ and $\mathfrak{S}$ are continuous. Then, Γ and $\mathfrak{S}$ have a C.F.P.

Proof. Let $\zeta_0 \in \Omega_b$ be an arbitrary point and $\zeta_1 \in \Gamma(\zeta_0)$, $\exists_{\zeta_2 \in \Omega_b}$ such that $\zeta_2 \in \mathfrak{S}(\zeta_1)$. Assume that $m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} > 0$, as

$$0 < m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \leq H_b|_{\mathfrak{S}(\zeta_1)}^{\Gamma(\zeta_0)}.$$

From Remark 1.1.33, we have

$$\theta \left(m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \right) = \inf_{\gamma \in \mathfrak{S}(\zeta_1)} \theta \left(m_b|_{\gamma}^{\zeta_1} \right).$$

Thus, there is $\gamma = \zeta_2 \in \mathfrak{S}(\zeta_1)$ and from (3.95), we have

$$\theta \left(m_b|_{\zeta_2}^{\zeta_1} \right) \leq \theta \left(s^3 H_b|_{\mathfrak{S}(\zeta_1)}^{\Gamma(\zeta_0)} \right) \leq \left[\theta \left(\alpha U_b|_{\zeta_1}^{\zeta_0} + L m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \right) \right]^k. \quad (3.96)$$

Inserting $\ln$ in both sides of (3.96), we obtain

$$\ln \theta \left(m_b|_{\zeta_2}^{\zeta_1} \right) \leq k \ln \theta \left(\alpha U_b|_{\zeta_1}^{\zeta_0} + L m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \right). \quad (3.97)$$

From Corollary 2.2.14 with taking $\psi(r) = kr \in \Psi$ and $\phi(r) = \ln \theta(r) \in \Phi$ in (3.97), leads to

$$\phi \left(m_b|_{\zeta_2}^{\zeta_1} \right) \leq \psi \left[\phi \left(\alpha U_b|_{\zeta_1}^{\zeta_0} + L m_b|_{\mathfrak{S}(\zeta_1)}^{\zeta_1} \right) \right].$$

By tracing the line of proof employed in the Theorem 3.6.2, we find that Γ and $\mathfrak{S}$ have a C.F.P.

Example 3.6.6. From Example 3.6.3. Let $\theta(r) = e^{\sqrt{r}} \in \Theta$ and $\Gamma, \mathfrak{S} : \Omega_b \longrightarrow CB_b|_{\Omega_b}$ by

$$\Gamma(\zeta) = \begin{cases} \left\{ \frac{\zeta}{12\kappa}, \frac{\zeta}{6\kappa} \right\}, & \text{if } \zeta \in (0, 1], \kappa \in \mathbb{N} \\ \{0\}, & \text{otherwise.} \end{cases} \quad \text{and } \mathfrak{S}(\zeta) = \begin{cases} \left\{ \frac{\zeta}{16\kappa}, \frac{3\zeta}{16\kappa} \right\}, & \text{if } \zeta \in (0, 1], \kappa \in \mathbb{N} \\ \left\{ 0, \frac{1}{10\kappa} \right\}, & \text{otherwise} \end{cases}.$$

Now, $\forall_{\zeta \in (0,1]}$ and $\forall_{\kappa \in \mathbb{N}}$, we have

$$\begin{aligned} \theta \left(s^3 H_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} \right) &= \theta \left[2^3 \max \left\{ \sup_{a \in \Gamma(\zeta_1)} m_b|_{\mathfrak{S}(\zeta_2)}^a, \sup_{b \in \mathfrak{S}(\zeta_2)} m_b|_b^{\Gamma(\zeta_1)} \right\} \right] \\ &= \theta \left[2^3 \max \left\{ m_b|_{\left\{ \frac{\zeta_2}{16\kappa}, \frac{3\zeta_2}{16\kappa} \right\}}^{\frac{\zeta_1}{6\kappa}}, m_b|_{\frac{3\zeta_2}{16\kappa}}^{\left\{ \frac{\zeta_1}{12\kappa}, \frac{\zeta_1}{6\kappa} \right\}} \right\} \right] \\ &= \theta \left[\frac{8}{16\kappa^2} \max \left\{ m_b|_{\frac{\zeta_2}{4}}^{\frac{2\zeta_1}{3}}, m_b|_{\frac{3\zeta_2}{4}}^{\frac{\zeta_1}{3}} \right\} \right] \\ &= \theta \left[\frac{1}{2\kappa^2} \max \left\{ \left(\frac{2\zeta_1}{6} + \frac{\zeta_2}{8} \right)^2, \left(\frac{\zeta_1}{6} + \frac{3\zeta_2}{8} \right)^2 \right\} \right] \\ &\leq \theta \left[\frac{1}{2\kappa^2} \left(\alpha U_b|_{\zeta_2}^{\zeta_1} + L m_b|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \right) \right] \\ &= \left(e^{\sqrt{\alpha U_b|_{\zeta_2}^{\zeta_1} + L m_b|_{\mathfrak{S}(\zeta_2)}^{\zeta_2}}} \right)^{\frac{1}{\sqrt{2\kappa}}} = \left[\theta \left(\alpha U_b|_{\zeta_2}^{\zeta_1} + L m_b|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \right) \right]^k, \end{aligned}$$

where $k = \frac{1}{\sqrt{2\kappa}} \in (0, 1)$, $\alpha = \frac{1}{2}$, $a = \frac{1}{6}$, $b = d = \frac{1}{3}$, $c = e = 0$, $L = \frac{1}{4}$ and $U_b|_{\zeta_2}^{\zeta_1}$ is clear in

(3.73). Thus, all condations of Theorem 3.6.2 hold and $0 \in \Omega_b$ is a C.F.P of Γ and $\mathfrak{S}$.

Corollary 3.6.7. Let (Ω_b, m_b) be a C-M-M.S and $\Gamma, \mathfrak{S} : \Omega_b \rightarrow \Omega_b$ be almost Hardy-Rogers (ψ, ϕ) -type contraction maps, if there exist $\phi \in \Phi^*$, $\psi \in \Psi$, $L \geq 0$ and $\alpha \in [0, 1]$ such that $\forall_{\zeta_1, \zeta_2 \in \Omega_b}$, $m_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} > 0$, implies,

$$\phi \left(m|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} \right) \leq \psi \left(\phi \left[\alpha U_m|_{\zeta_2}^{\zeta_1} + L m|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \right] \right),$$

where

$$\begin{aligned} U_m|_{\zeta_2}^{\zeta_1} &= a m|_{\zeta_2}^{\zeta_1} + b m|_{\Gamma(\zeta_1)}^{\zeta_1} + c m|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} + \\ &\quad d \frac{m|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \left[1 + m|_{\Gamma(\zeta_1)}^{\zeta_1} \right]}{1 + m|_{\zeta_2}^{\zeta_1}} + e \frac{m|_{\Gamma(\zeta_1)}^{\zeta_1} m|_{\mathfrak{S}(\zeta_2)}^{\zeta_2}}{1 + m|_{\zeta_2}^{\zeta_1}}, \end{aligned} \quad (3.98)$$

and $a, b, c, d, e \geq 0$ with $0 \leq \alpha a + \alpha b + \alpha c + \alpha d + \alpha e + L < 1$. If ψ, Γ and $\mathfrak{S}$ are continuous, then $\mathfrak{S}$ and Γ have a C.F.P

Corollary 3.6.8. Let (Ω_b, m_b) be a C-M-M.S and $\Gamma, \mathfrak{S} : \Omega_b \rightarrow \Omega_b$ be almost Hardy-Rogers θ -type contraction maps, if there exist $\theta \in \Theta$, $k \in (0, 1)$, $L \geq 0$ and $\alpha \in [0, 1]$ such that $\forall_{\zeta_1, \zeta_2 \in \Omega_b}$, $m_b|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} > 0$, implies,

$$\theta \left(m|_{\mathfrak{S}(\zeta_2)}^{\Gamma(\zeta_1)} \right) \leq \left[\theta \left(\alpha U_m|_{\zeta_2}^{\zeta_1} + L m|_{\mathfrak{S}(\zeta_2)}^{\zeta_2} \right) \right]^k,$$

where $U_m|_{\zeta_2}^{\zeta_1}$ is defined in (3.98). If Γ and $\mathfrak{S}$ are continuous, then $\mathfrak{S}$ and Γ have a C.F.P

Remark 3.6.9. Definitions 3.6.1 and 3.6.4 are stil valid if we consider the following: ■

◆ If we take $M_b|_{\zeta_2}^{\zeta_1} = m_b|_{\zeta_2}^{\zeta_1}$ and $L = 0$, then $\mathfrak{S}$ and Γ have a C.F.P in a C-Mb-M.S, provided that the conditions of Theorems 3.6.2 and 3.6.5 hold.

◆ If $L = 0$, then $\mathfrak{S}$ and Γ have a C.F.P in a C-Mb-M.S, provided that the conditions of Theorems 3.6.2 and 3.6.5 hold.

◆ If we consider $\mathfrak{S} = \Gamma$, then the results are given immidiatly in the same steps as the proof of Theorems 3.6.2 and 3.6.5.

◆ If we consider $\Gamma, \mathfrak{S}$ as single-valued maps, then the results are given immidiatly in the same steps as the proof of Theorems 3.6.2 and 3.6.5.

Chapter 4

Fixed Point Results for Contraction Mappings in Rectangular M-Metric Spaces

The concept of R.M-M.S was introduced by Özgür et al. [91] as an extension of M-M.S. Suzuki [108, 109] developed the BCP [27] and proved new F.P results. His contribution has been widely extended in various directions, (see, for instance [18, 22, 41, 66, 69]). Jleli et al. [52, 53] introduced the notion of θ -contraction, which was later modified by Liet al. [65] into θ -quasi-contraction, leading to several F.P findings. Altun et al. [12] explored F.P theorems for a new category of θ -contractive mappings. The outcomes of Jleli et al. were further generalized by several researchers (see, e.g., [4, 44, 47, 94]).

The objective of this chapter is to analyze the idea of almost Suzuki-Rational (ψ, ϕ) -type multivalued contraction and to establish related C.F.P theorems within R.M-M.S and closed ball therein, which are generalization of almost (ψ, ϕ) -contractions self-mappings. Moreover, we present F.P theorems for θ -contraction type self-mappings in the same framework. As an application, the obtained theorems are utilized to show the existence of solution to NIEs.

These results have been published in [79, 81].

4.1 Common fixed points of almost Suzuki-Rational (ψ, ϕ) -type multivalued contractions

Next results generalize some C.F.P theorems for Suzuki-Rational (ψ, ϕ) -type multivalued contraction into R.M-M.S and on closed ball of R.M-M.S. We introduce the following results to be consistent with the similar results in [23, 96] and to be useful in our study.

Definition 4.1.1. Let (Ω_r, m_r) be an R.M-M.S and $\Delta \subset \Omega_b$. Then, Δ is called bounded if $\forall_{g \in \Delta}$ there exists an open ball $B_r|_{\varepsilon}^{\varsigma_1}$ of center $\varsigma_1 \in \Omega_b$ and radius $\varepsilon > 0$, so that $g \in B_r|_{\varepsilon}^{\varsigma_1}$, that is $m_r|_g^{\varsigma_1} - m_r|_{\varsigma_1, g} < \varepsilon$.

Let (Ω_r, m_r) be an R.M-M.S and $CB_r|_{\Omega_r}^{\Omega_r}$ be the family of non-empty closed and bounded subsets of Ω_r . For $\aleph, \mathbb{Q} \in CB_r|_{\Omega_r}^{\Omega_r}$, define

$$H_r|_{\mathbb{Q}}^{\aleph} = \max \left\{ \delta_r|_{\mathbb{Q}}^{\aleph}, \delta_r|_{\aleph}^{\mathbb{Q}} \right\},$$

where $\delta_r|_{\mathbb{Q}}^{\aleph} = \sup \left\{ m_r|_{\mathbb{Q}}^p : p \in \aleph \right\}$ and $\delta_r|_{\aleph}^{\mathbb{Q}} = \sup \left\{ m_r|_q^{\aleph} : q \in \mathbb{Q} \right\}$.

Let $\overline{\aleph}$ denotes the closure of $\aleph$ w.r.t m_r . Then $\aleph$ is closed in (Ω_r, m_r) iff $\aleph = \overline{\aleph}$.

Lemma 4.1.2. Let $\aleph \neq \emptyset$ in an Mb-M.S (Ω_r, m_r) , then

$$p \in \overline{\aleph} \iff m_r|_{\aleph}^p = \sup_{\lambda \in \aleph} m_r|_{p, \lambda}.$$

Proof. We follow the similar way in the proof of Lemma 3.1.1.

■

Lemma 4.1.3. Let $\aleph, \mathbb{Q}, \aleph, \mathbb{U} \in CB_r|_{\Omega_r}^{\Omega_r}$, then

- (a) $\delta_r|_{\aleph}^{\aleph} = \sup_{p \in \aleph} \left\{ \sup_{q \in \aleph} m_r|_{p, q} \right\};$
- (b) for $a \in \aleph, b \in \mathbb{Q}, c \in \aleph$ and $d \in \mathbb{U}$, we have

$$\left(\delta_r|_{\mathbb{Q}}^{\aleph} - \sup_{a \in \aleph} \sup_{b \in \mathbb{Q}} m_r|_{a, b} \right) \leq \left[\begin{array}{c} \left(\delta_r|_{\aleph}^{\aleph} - \inf_{a \in \aleph} \inf_{c \in \aleph} m_r|_{a, c} \right) + \left(\delta_r|_{\mathbb{U}}^{\aleph} - \inf_{c \in \aleph} \inf_{d \in \mathbb{U}} m_r|_{c, d} \right) + \\ \left(\delta_r|_{\mathbb{Q}}^{\mathbb{U}} - \inf_{d \in \mathbb{U}} \inf_{b \in \mathbb{Q}} m_r|_{d, b} \right) \end{array} \right].$$

Proof. We follow the similar way in the proof of Lemma 3.1.2. ■

Lemma 4.1.4. Let $\aleph, \mathbb{Q}, \aleph \in CB_r|_{\Omega_r}$, then

- (1) $\hat{H}_r|_{\aleph}^{\aleph} = \delta_r|_{\aleph}^{\aleph} = \sup_{p \in \aleph} \left\{ \sup_{q \in \aleph} m_r|_{p,q} \right\}$; (2) $\hat{H}_r|_{\mathbb{Q}}^{\aleph} = \hat{H}_r|_{\aleph}^{\mathbb{Q}}$;
(3) for $a \in \aleph$, $b \in \mathbb{Q}$, $c \in \aleph$ and $d \in \mathbb{U}$, we have

$$\left(\hat{H}_r|_{\mathbb{Q}}^{\aleph} - \sup_{a \in \aleph} \sup_{b \in \mathbb{Q}} m_r|_{a,b} \right) \leq \left[\begin{aligned} & \left(\hat{H}_r|_{\aleph}^{\aleph} - \inf_{a \in \aleph} \inf_{c \in \aleph} m_r|_{a,c} \right) + \left(\hat{H}_r|_{\mathbb{U}}^{\aleph} - \inf_{c \in \aleph} \inf_{d \in \mathbb{U}} m_r|_{c,d} \right) \\ & + \left(\hat{H}_r|_{\mathbb{Q}}^{\mathbb{U}} - \inf_{d \in \mathbb{U}} \inf_{b \in \mathbb{Q}} m_r|_{d,b} \right) \end{aligned} \right].$$

Proof. We follow the similar way in the proof of Lemma 3.1.3. ■

Lemma 3.1.4. Let $\aleph, \mathbb{Q} \in CB_r|_{\Omega_r}$ and $h > 1$, then $\forall_{p \in \aleph}, \exists_{q \in \mathbb{Q}}$ so that

- (i) $m_r|_q^p \leq h \hat{H}_r|_{\mathbb{Q}}^{\aleph}$;
(ii) $m_r|_q^p \leq \hat{H}_r|_{\mathbb{Q}}^{\aleph} + h$.

Proof. We follow the similar way in the proof of Lemma 3.1.4. ■

Definition 4.1.6. Let (Ω_r, m_r) be a R.M-M.S and $\alpha, \eta : \Omega_r \times \Omega_r \rightarrow [0, \infty)$ be two functions. The mappings $\Gamma, \aleph : \Omega_r \rightarrow CB_r|_{\Omega_r}$ are termed (ψ, ϕ) -Suzuki Rational-type multivalued contractions, if $\exists_{\phi \in \Phi, \psi \in \Psi}$ and $\exists_{\gamma \in (0,1), L \geq 0}$, with $\gamma + L \leq 1$ such that $\forall_{\xi_1, \xi_2 \in \Omega_r}$ with $\Gamma \xi_1 \neq \aleph \xi_2$.

$$\begin{aligned} \frac{1}{2} \min \left\{ m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\aleph \xi_2}^{\xi_2} \right\} &< m_r|_{\xi_2}^{\xi_1} \Rightarrow \\ \phi \left(\hat{H}_r|_{\aleph \xi_2}^{\Gamma \xi_1} \right) &\leq \psi \left[\phi \left(\gamma M_r|_{\xi_2}^{\xi_1} + L N_r|_{\xi_2}^{\xi_1} \right) \right], \end{aligned} \quad (4.1)$$

where

$$M_r|_{\xi_2}^{\xi_1} = \max \left\{ m_r|_{\xi_2}^{\xi_1}, m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\aleph \xi_2}^{\xi_2}, \frac{m_r|_{\aleph \xi_2}^{\xi_2} \left[1 + m_r|_{\Gamma \xi_1}^{\xi_1} \right]}{1 + m_r|_{\xi_2}^{\xi_1}} \right\}, \quad (4.2)$$

and

$$N_r|_{\xi_2}^{\xi_1} = \min \left\{ m_r|_{\xi_2}^{\xi_1}, m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\aleph \xi_2}^{\xi_2}, \frac{m_r|_{\Gamma \xi_1}^{\xi_1} \cdot m_r|_{\aleph \xi_2}^{\xi_2}}{1 + m_r|_{\xi_2}^{\xi_1}} \right\}. \quad (4.3)$$

Theorem 4.1.7. Let (Ω_r, m_r) be a C-R.M-M.S, $\alpha, \eta : \Omega_r \times \Omega_r \rightarrow [0, \infty)$ be given functions and let $\Gamma, \aleph : \Omega_r \rightarrow CB_r|_{\Omega_r}$ be a pair of α_* -admissible mappings w.r.t. η satisfying (4.1). If $\psi, \aleph$ and Γ are continuous. Then, $\aleph$ and Γ have a C.F.P.

Proof. Let $\xi_0 \in \Omega_r$ be an arbitrary point. Since $\Gamma\xi_0 \neq \emptyset \forall \xi \in \Omega_r$, we select $\xi_1 \in \Gamma\xi_0$ and $m_r|_{\Gamma\xi_0}^{\xi_0} > 0$. If $\xi_1 \in \mathfrak{S}\xi_1$, then ξ_1 is a F.P of $\mathfrak{S}$. Now, consider $\xi_1 \notin \mathfrak{S}\xi_1$. Since $\mathfrak{S}\xi_1$ is closed and $m_r|_{\mathfrak{S}\xi_1}^{\xi_1} > 0$, then $\frac{1}{2} \min \left\{ m_r|_{\Gamma\xi_0}^{x_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} < m_r|_{\xi_1}^{\xi_0}$, we have $m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \leq \hat{H}_r|_{\mathfrak{S}\xi_1}^{\xi_0}$. Utilizing (ϕ_1) , we get $\phi \left(m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right) \leq \phi \left(\hat{H}_r|_{\mathfrak{S}\xi_1}^{\xi_0} \right)$. Since $(\Gamma, \mathfrak{S})$ is a pair of α_* -admissible w.r.t. η , then $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$ implies that $\alpha_*|_{\mathfrak{S}\xi_1}^{\Gamma\xi_0} \geq \eta_*|_{\mathfrak{S}\xi_1}^{\Gamma\xi_0}$. Applying (4.1), we can write

$$\begin{aligned}
\phi \left(m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right) &\leq \phi \left(\hat{H}_r|_{\mathfrak{S}\xi_1}^{\Gamma\xi_0} \right) \leq \psi \left[\phi \left(\gamma M_r|_{\xi_1}^{\xi_0} + L N_r|_{\xi_1}^{\xi_0} \right) \right] \\
&= \psi \left[\phi \left(\gamma \max \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\Gamma\xi_0}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1}, \frac{m_r|_{\mathfrak{S}\xi_1}^{\xi_1} [1 + m_r|_{\Gamma\xi_0}^{\xi_0}]}{1 + m_r|_{\xi_1}^{\xi_0}} \right\} \right. \right. \\
&\quad \left. \left. + L \min \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\Gamma\xi_0}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1}, \frac{m_r|_{\Gamma\xi_0}^{\xi_0} m_r|_{\mathfrak{S}\xi_1}^{\xi_1}}{1 + m_r|_{\xi_1}^{\xi_0}} \right\} \right) \right] \\
&= \psi \left[\phi \left(\gamma \max \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1}, \frac{m_r|_{\mathfrak{S}\xi_1}^{\xi_1} [1 + m_r|_{\xi_1}^{\xi_0}]}{1 + m_r|_{\xi_1}^{\xi_0}} \right\} \right. \right. \\
&\quad \left. \left. + L \min \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1}, \frac{m_r|_{\xi_1}^{\xi_0} m_r|_{\mathfrak{S}\xi_1}^{\xi_1}}{1 + m_r|_{\xi_1}^{\xi_0}} \right\} \right) \right] \\
&\leq \psi \left[\phi \left(\gamma \max \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} + L \min \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} \right) \right] \\
&< \phi \left(\gamma \max \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} + L \min \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} \right).
\end{aligned} \tag{4.4}$$

If $\max \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} = m_r|_{\mathfrak{S}\xi_1}^{\xi_1}$, then $\min \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} = m_r|_{\xi_1}^{\xi_0}$, from (4.4) and (ϕ_1) , we get

$$m_r|_{\mathfrak{S}\xi_1}^{\xi_1} < \gamma m_r|_{\mathfrak{S}\xi_1}^{\xi_1} + L m_r|_{\xi_1}^{\xi_0}.$$

Since $\gamma + L \leq 1$, then $1 - \gamma \geq L > 0$, implying that

$$m_r|_{\mathfrak{S}\xi_1}^{\xi_1} < \frac{L}{1 - \gamma} m_r|_{\xi_1}^{\xi_0} < m_r|_{\xi_1}^{\xi_0}. \tag{4.5}$$

Similarly, if $\max \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} = m_r|_{\xi_1}^{\xi_0}$, then $\min \left\{ m_r|_{\xi_1}^{\xi_0}, m_r|_{\mathfrak{S}\xi_1}^{\xi_1} \right\} = m_r|_{\mathfrak{S}\xi_1}^{\xi_1}$, from (4.4) and (ϕ_1) , we get

$$m_r|_{\mathfrak{S}\xi_1}^{\xi_1} < \gamma m_r|_{\xi_1}^{\xi_0} + L m_r|_{\mathfrak{S}\xi_1}^{\xi_1}.$$

Since $\gamma + L \leq 1$, then $1 - L \geq \gamma > 0$, implying that

$$m_r|_{\mathfrak{S}\xi_1}^{\xi_1} < \frac{\gamma}{1-L} m_r|_{\xi_1}^{\xi_0} < m_r|_{\xi_1}^{\xi_0}. \quad (4.6)$$

Since $\mathfrak{S}\xi_1 \neq \emptyset$ is compact $\forall \xi_1 \in \Omega_r$, $\exists \xi_2 \in \mathfrak{S}\xi_1$. If $\xi_2 \in \Gamma\xi_2$, then ξ_2 is a F.P of Γ and the proof is done. Consider $\xi_2 \notin \Gamma\xi_2$ such that $m_r|_{\xi_2}^{\xi_1} = m_r|_{\mathfrak{S}\xi_1}^{\xi_1}$ from (4.5) or (4.6), we get

$$m_r|_{\xi_2}^{\xi_1} < m_r|_{\xi_1}^{\xi_0}. \quad (4.7)$$

Therefore, $\frac{1}{2} \min \left\{ m_r|_{\mathfrak{S}\xi_1}^{\xi_1}, m_r|_{\Gamma\xi_2}^{\xi_2} \right\} < m_r|_{\xi_2}^{\xi_1}$, and $m_r|_{\Gamma\xi_2}^{\xi_2} \leq H_r|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1}$. Since $(\Gamma, \mathfrak{S})$ is a pair of α_* -admissible w.r.t. η , then $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$ implies that $\alpha_*|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \geq \eta_*|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1}$. From (4.1), we have

$$\phi \left(m_r|_{\Gamma\xi_2}^{\xi_2} \right) \leq \phi \left(H_r|_{\Gamma\xi_2}^{\mathfrak{S}\xi_1} \right) \leq \psi \left[\phi \left(\gamma M_r|_{\xi_2}^{\xi_1} + L N_r|_{\xi_2}^{\xi_1} \right) \right].$$

By similar way in (4.4), (4.6) and (4.7), we obtain

$$m_r|_{\xi_3}^{\xi_2} < m_r|_{\xi_2}^{\xi_1}.$$

Because of $\Gamma\xi_2 \neq \emptyset$ and is compact $\forall \xi_2 \in \Omega_r$, $\exists \xi_3 \in \Gamma\xi_2$ and if $\xi_3 \in \mathfrak{S}\xi_3$, then ξ_3 is a F.P of $\mathfrak{S}$, consider $\xi_3 \notin \mathfrak{S}\xi_3$ such that $m_r|_{\xi_4}^{\xi_3} = m_r|_{\mathfrak{S}\xi_3}^{\xi_3}$. Therefore, $\frac{1}{2} \min \left\{ m_r|_{\Gamma\xi_2}^{\xi_2}, m_r|_{\mathfrak{S}\xi_3}^{\xi_3} \right\} < m_r|_{\xi_3}^{\xi_2}$, we have $m_r|_{\mathfrak{S}\xi_3}^{\xi_3} \leq H_r|_{\mathfrak{S}\xi_3}^{\Gamma\xi_2}$ and from (4.1), we get

$$\phi \left(m_r|_{\mathfrak{S}\xi_3}^{\xi_3} \right) \leq \phi \left(H_r|_{\mathfrak{S}\xi_3}^{\Gamma\xi_2} \right) \leq \psi \left[\phi \left(\gamma M_r|_{\xi_3}^{\xi_2} + L N_r|_{\xi_3}^{\xi_2} \right) \right] < \phi \left(m_r|_{\xi_3}^{\xi_2} \right),$$

implying that

$$m_r|_{\xi_4}^{\xi_3} < m_r|_{\xi_3}^{\xi_2}.$$

Continuing in the same vein, we find that $\forall \kappa \geq 0$, $\exists \xi_{2\kappa+1} \in \Gamma\xi_{2\kappa}$, $\xi_{2\kappa+2} \in \mathfrak{S}\xi_{2\kappa+1}$, so that

$$\frac{1}{2} \min \left\{ m_r|_{\Gamma\xi_{2\kappa}}^{\xi_{2\kappa}}, m_r|_{\mathfrak{S}\xi_{2\kappa+1}}^{\xi_{2\kappa+1}} \right\} < m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}},$$

and $\alpha|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} \geq \eta|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} \implies \alpha_*|_{\Gamma\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} \geq \eta_*|_{\Gamma\xi_{2\kappa+2}}^{\xi_{2\kappa+1}}$. Hence from (4.1), we have

$$\begin{aligned} \phi\left(m_r|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}}\right) &\leq \phi\left(\hat{H}_r|_{\mathfrak{S}\xi_{2\kappa+1}}^{\Gamma\xi_{2\kappa}}\right) \leq \psi\left[\phi\left(\gamma M_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}} + L N_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right)\right] \\ &\leq \psi\left[\phi\left(m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right)\right] < \phi\left(m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right). \end{aligned} \quad (4.8)$$

The inequality (4.8) confirms that $\left\{m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right\}$ is decreasing and $\exists_{\sigma \geq 0}$ such that $\lim_{\kappa \rightarrow \infty} m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}} = \sigma$. Presume that $\sigma > 0$. From (4.1), using (4.4) and (4.8), we can write

$$\begin{aligned} \phi\left(m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right) &\leq \phi\left(\hat{H}_r|_{\mathfrak{S}\xi_{2\kappa-1}}^{\Gamma\xi_{2\kappa}}\right) \leq \psi\left[\phi\left(\gamma M_r|_{\xi_{2\kappa-1}}^{\xi_{2\kappa}} + L N_r|_{\xi_{2\kappa-1}}^{\xi_{2\kappa}}\right)\right] \\ &\leq \psi\left[\phi\left(m_r|_{\xi_{2\kappa-1}}^{\xi_{2\kappa}}\right)\right] \leq \psi^2\left[\phi\left(m_r|_{\xi_{2\kappa-2}}^{\xi_{2\kappa-1}}\right)\right] \\ &\leq \dots \leq \psi^{2\kappa}\left[\phi\left(m_r|_{\xi_1}^{\xi_0}\right)\right]. \end{aligned} \quad (4.9)$$

Letting $\kappa \rightarrow \infty$ in (4.9), and as $\psi \in \Psi$, we obtain

$$0 \leq \lim_{\kappa \rightarrow \infty} \phi\left(m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right) \leq \lim_{\kappa \rightarrow \infty} \psi^{2\kappa}\left[\phi\left(m_r|_{\xi_1}^{\xi_0}\right)\right] = 0,$$

implying that

$$\lim_{\kappa \rightarrow \infty} \phi\left(m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}}\right) = 0.$$

From (ϕ_2) , we get

$$\lim_{\kappa \rightarrow \infty} m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa}} = 0. \quad (4.10)$$

Next, we verify that $\{\xi_\kappa\}$ is an m_r -Cauchy in (Ω_r, m_r) . Recollect that, from (M_{rii}) and $\forall_{\kappa \geq 0}$, we have

$$0 \leq m_r|_{\xi_\kappa, \xi_{\kappa+1}}^{\xi_\kappa} \leq m_r|_{\xi_{\kappa+1}}^{\xi_\kappa}. \quad (4.11)$$

Let $\kappa \rightarrow \infty$ in (4.11) and using (4.10), we get

$$\lim_{\kappa \rightarrow \infty} m_r|_{\xi_\kappa, \xi_{\kappa+1}}^{\xi_\kappa} = 0, \quad (4.12)$$

leading to

$$\lim_{\kappa \rightarrow \infty} m_r|_{\xi_\kappa}^{\xi_\kappa} = 0 \text{ or } \lim_{\kappa \rightarrow \infty} m_r|_{\xi_{\kappa+1}}^{\xi_{\kappa+1}} = 0. \quad (4.13)$$

Therefore

$$\lim_{\kappa, \varkappa \rightarrow \infty} m_r|_{\xi_\kappa, \xi_\varkappa} = \lim_{\kappa, \varkappa \rightarrow \infty} \min \left\{ m_r|_{\xi_\kappa}^{\xi_\kappa}, m_r|_{\xi_\varkappa}^{\xi_\varkappa} \right\} = 0. \quad (4.14)$$

Thence, using (4.14), we get

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(M_r|_{\xi_\kappa, \xi_\varkappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right) = \lim_{\kappa, \varkappa \rightarrow \infty} \left| m_r|_{\xi_\kappa}^{\xi_\kappa} - m_r|_{\xi_\varkappa}^{\xi_\varkappa} \right| = 0.$$

Next, we manifest that $\lim_{\kappa, \varkappa \rightarrow \infty} \left(m_r|_{\xi_\varkappa}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right) = 0$. Suppose on the contrary that

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(m_r|_{\xi_\varkappa}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right) \neq 0.$$

Then, $\exists \epsilon > 0$ and subsequences $\{\xi_{\kappa(p)}\}$ and $\{\xi_{\varkappa(p)}\}$ of $\{\xi_\kappa\}$ such that $\forall p \in \mathbb{N}$, $\exists \kappa(p), \varkappa(p) \in \mathbb{N}$ with $\varkappa(p) > \kappa(p) > p$ attaining

$$m_r|_{\xi_{\varkappa(p)}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\varkappa(p)}} \geq \epsilon. \quad (4.15)$$

Let $\kappa(p)$ be the least exceeding $\varkappa(p)$ and satisfying (4.15) so that

$$m_r|_{\xi_{\varkappa(p)}}^{\xi_{\kappa(p)}-1} - m_r|_{\xi_{\kappa(p)-1}, \xi_{\varkappa(p)}} < \epsilon, \quad \forall p \in \mathbb{N}. \quad (4.16)$$

Utilizing (M_{riv}) in (4.15), we get

$$\begin{aligned} \epsilon &\leq m_r|_{\xi_{\varkappa(p)}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\varkappa(p)}} \\ &\leq m_r|_{\xi_{\kappa(p)}-2}^{\xi_{\varkappa(p)}} - m_r|_{\xi_{\varkappa(p)}, \xi_{\kappa(p)}-2} + \\ &\quad m_r|_{\xi_{\kappa(p)}-1}^{\xi_{\kappa(p)}-2} - m_r|_{\xi_{\kappa(p)}-2, \xi_{\kappa(p)}-1} + m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)}-1} - m_r|_{\xi_{\kappa(p)}-1, \xi_{\kappa(p)}} \\ &\leq \epsilon + m_r|_{\xi_{\kappa(p)}-1}^{\xi_{\kappa(p)}-2} + m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)}-1}. \end{aligned} \quad (4.17)$$

Let $p \rightarrow \infty$ in (4.17), using (4.10) and (4.16), we obtain

$$\lim_{p \rightarrow \infty} \left(m_r|_{\xi_{\varkappa(p)}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\varkappa(p)}} \right) = \epsilon. \quad (4.18)$$

By (M_{riv}) again, we have

$$\begin{aligned}
m_r|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\kappa(p)+1}} &\leq m_r|_{\xi_{\kappa(p)-1}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\kappa(p)-1}} \\
&\quad + m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)-1}} - m_r|_{\xi_{\kappa(p)-1}, \xi_{\kappa(p)}} \\
&\quad + m_r|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\kappa(p)+1}} \\
&\leq m_r|_{\xi_{\kappa(p)-1}}^{\xi_{\kappa(p)}} + m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)-1}} - m_r|_{\xi_{\kappa(p)-1}, \xi_{\kappa(p)}} \\
&\quad + m_r|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)}}.
\end{aligned} \tag{4.19}$$

Letting $p \rightarrow \infty$ in (4.19), utilising (4.10) and (4.16), we obtain

$$\lim_{p \rightarrow \infty} \left(m_r|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\kappa(p)+1}} \right) \leq \epsilon. \tag{4.20}$$

Similarly, by (M_{riv}) , we have

$$\begin{aligned}
&m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)-1}} - m_r|_{\xi_{\kappa(p)-1}, \xi_{\kappa(p)}} \\
&\leq m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)-1}} - m_r|_{\xi_{\kappa(p)-1}, \xi_{\kappa(p)}} + m_r|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\kappa(p)+1}} \\
&\quad + m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)+1}} - m_r|_{\xi_{\kappa(p)+1}, \xi_{\kappa(p)}}. \\
&\leq m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)-1}} + m_r|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \xi_{\kappa(p)+1}} + m_r|_{\xi_{\kappa(p)+1}}^{\xi_{\kappa(p)+1}}.
\end{aligned} \tag{4.21}$$

Letting $p \rightarrow \infty$ in (4.21), using (4.10) and (4.16), we get

$$\lim_{p \rightarrow \infty} \left(m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)-1}} - m_r|_{\xi_{\kappa(p)-1}, \xi_{\kappa(p)}} \right) \leq \epsilon. \tag{4.22}$$

From (4.10) and (4.18), we can choose $\kappa_0 \geq 1$ such that $\forall \kappa \geq \kappa_0$

$$\begin{aligned}
&\frac{1}{2} \min \left\{ m_r|_{\Gamma \xi_{\kappa(p)}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \Gamma \xi_{\kappa(p)}} , m_r|_{\Im \xi_{\kappa(p)}}^{\xi_{\kappa(p)}} - m_r|_{\xi_{\kappa(p)}, \Im \xi_{\kappa(p)}} \right\} \\
&\leq \frac{1}{2} \min \left\{ m_r|_{\Gamma \xi_{\kappa(p)}}^{\xi_{\kappa(p)}} , m_r|_{\Im \xi_{\kappa(p)}}^{\xi_{\kappa(p)}} \right\} < m_r|_{\xi_{\kappa(p)}}^{\xi_{\kappa(p)}}.
\end{aligned}$$

Now, utilizing (M_{riv}) , (4.10) and (4.16), we get

$$\begin{aligned}
& \lim_{p \rightarrow \infty} \left(\gamma M_r|_{\xi_{\mathcal{X}(p)}^{\xi_{\kappa(p)}-1}} + L N_r|_{\xi_{\mathcal{X}(p)}^{\xi_{\kappa(p)}-1}} \right) \\
&= \lim_{p \rightarrow \infty} \left(\gamma \max \left\{ \begin{array}{l} m_r|_{\xi_{\mathcal{X}(p)}^{\xi_{\kappa(p)}-1}}, M_r|_{\xi_{\kappa(p)}^{\xi_{\kappa(p)}-1}}, m_r|_{\xi_{\mathcal{X}(p)+1}^{\xi_{\mathcal{X}(p)}}}, \\ \frac{m_r|_{\xi_{\kappa(p)}^{\xi_{\kappa(p)}-1}} [1 + m_r|_{\xi_{\mathcal{X}(p)+1}^{\xi_{\mathcal{X}(p)}}}]}{1 + m_r|_{\xi_{\mathcal{X}(p)}^{\xi_{\kappa(p)}-1}}} \end{array} \right\} \right. \\
&\quad \left. + L \min \left\{ \begin{array}{l} m_r|_{\xi_{\mathcal{X}(p)}^{\xi_{\kappa(p)}-1}}, m_r|_{\xi_{\kappa(p)}^{\xi_{\kappa(p)}-1}}, m_r|_{\xi_{\mathcal{X}(p)+1}^{\xi_{\mathcal{X}(p)}}}, \\ \frac{m_r|_{\xi_{\kappa(p)}^{\xi_{\kappa(p)}-1}} \cdot m_r|_{\xi_{\mathcal{X}(p)+1}^{\xi_{\mathcal{X}(p)}}}}{1 + m_r|_{\xi_{\mathcal{X}(p)}^{\xi_{\kappa(p)}-1}}} \end{array} \right\} \right) \\
&\leq \gamma \max \{0 + \epsilon + 0, 0, 0, 0\} + L \min \{0 + \epsilon + 0, 0, 0, 0\} \\
&= \gamma \epsilon.
\end{aligned} \tag{4.23}$$

Thence, from (2.26), (2.1), (ϕ_1) and since $\psi \in \Psi$, one concludes

$$\begin{aligned}
\phi(\epsilon) &= \lim_{p \rightarrow \infty} \phi \left(m_r|_{\xi_{m(p)+1}^{\xi_{\kappa(p)}}} \right) \leq \lim_{p \rightarrow \infty} \phi \left(\hat{H}_r|_{\mathfrak{S}\xi_{m(p)}}^{\Gamma\xi_{\kappa(p)}-1} \right) \\
&\leq \psi \left[\phi \left(\lim_{p \rightarrow \infty} \gamma M_r|_{\xi_{m(p)}^{\xi_{\kappa(p)}-1}} + L N_r|_{\xi_{m(p)}^{\xi_{\kappa(p)}-1}} \right) \right] \\
&\leq \psi [\phi(\gamma\epsilon)] < \phi(\epsilon),
\end{aligned}$$

a contradiction. Therefore, $\lim_{\kappa, \mathcal{X} \rightarrow \infty} (m_r|_{\xi_{\mathcal{X}}^{\xi_{\kappa}}} - m_r|_{\xi_{\kappa}, \xi_{\mathcal{X}}}) = 0$ and hence $\{\xi_{\kappa}\}$ is an m_r -Cauchy in a C-R.M-M.S (Δ, m_r) , so $\exists \xi^* \in \Omega_r$ such that $\xi_{\kappa} \rightarrow \xi^*$ as $\kappa \rightarrow \infty$, $\implies \xi_{2\kappa+1} \rightarrow \xi^*$ and $\xi_{2\kappa+2} \rightarrow \xi^*$ as $\kappa \rightarrow \infty$. By virtue of (4.1), we infer that

$$\lim_{\kappa \rightarrow \infty} \left(m_r|_{\xi^*}^{\xi_{2\kappa+1}} - m_r|_{\xi_{2\kappa+1}, \xi^*} \right) = 0 \text{ and } \lim_{\kappa \rightarrow \infty} \left(M_r|_{\xi_{2\kappa+1}, \xi^*} - m_r|_{\xi_{2\kappa+1}, \xi^*} \right) = 0. \tag{4.24}$$

Next, we prove that $\xi^* \in \mathfrak{S}\xi^*$. Since from (4.13) and by (4.24), we get

$$\lim_{\kappa \rightarrow \infty} m_r|_{\xi^*}^{\xi_{2\kappa+1}} = 0. \tag{4.25}$$

It follows from (4.1), (4.8) and (4.25) that

$$\lim_{\kappa \rightarrow \infty} \hat{H}_r|_{\mathfrak{S}\xi^*}^{\Gamma\xi_{2\kappa}} = 0. \tag{4.26}$$

Since $\xi_{2\kappa+1} \in \Gamma\xi_{2\kappa}$, then

$$m_r|_{\mathfrak{S}\xi^*}^{\xi_{2\kappa+1}} \leq \hat{H}_r|_{\mathfrak{S}\xi^*}^{\Gamma\xi_{2\kappa}}.$$

Letting $\kappa \rightarrow \infty$ in above inequality, we obtain that

$$\lim_{\kappa \rightarrow \infty} m_r|_{\mathfrak{S}\xi^*}^{\xi_{2\kappa+1}} = 0. \quad (4.27)$$

From (M_{rii}) , one can write

$$m_r|_{\xi_{2\kappa+1}, \mathfrak{S}\xi^*} \leq m_r|_{\mathfrak{S}\xi^*}^{\xi_{2\kappa+1}},$$

which yields that

$$\lim_{\kappa \rightarrow \infty} m_r|_{\xi_{2\kappa+1}, \mathfrak{S}\xi^*} = 0. \quad (4.28)$$

Now, utilizing (M_{riv}) , we have

$$\begin{aligned} & m_r|_{\mathfrak{S}\xi^*}^{\xi^*} - \sup_{y \in \mathfrak{S}x^*} m_r|_{\xi^*, y} \\ & \leq m_r|_{\mathfrak{S}\xi^*}^{\xi^*} - m_r|_{\xi^*, \mathfrak{S}x^*} \\ & \leq m_r|_{\xi_{2\kappa+1}}^{\xi^*} - m_r|_{\xi^*, \xi_{2\kappa+1}} + m_r|_{\Gamma\xi_{2\kappa}}^{\xi_{2\kappa+1}} - m_r|_{\xi_{2\kappa+1}, \Gamma x_{2\kappa}} \\ & \quad + \hat{H}_r|_{\mathfrak{S}\xi^*}^{\Gamma\xi_{2\kappa}} - m_r|_{\Gamma\xi_{2\kappa}, \mathfrak{S}\xi^*} \\ & \leq m_r|_{\xi_{2\kappa+1}}^{\xi^*} + m_r|_{\xi_{2\kappa+1}}^{\xi_{2\kappa+1}} + \hat{H}_r|_{\mathfrak{S}\xi^*}^{\Gamma\xi_{2\kappa}}. \end{aligned} \quad (4.29)$$

Letting $\kappa \rightarrow \infty$ in (4.29) and from (4.24), (4.25), (4.26) and (4.27), we infer that

$$m_r|_{\mathfrak{S}\xi^*}^{\xi^*} \leq \sup_{y \in \mathfrak{S}\xi^*} m_r|_{\xi^*, y}. \quad (4.30)$$

Using $(M_{rii}) \forall_{y \in \mathfrak{S}\xi^*}$, we get

$$m_r|_{\xi^*, y} \leq m_r|_y^{\xi^*},$$

yields

$$m_r|_{\xi^*, y} - m_r|_y^{\xi^*} \leq 0.$$

Thus

$$\sup \left\{ m_r|_{\xi^*, y} - m_r|_y^{\xi^*} : y \in \mathfrak{S}\xi^* \right\} \leq 0,$$

this implies that

$$\sup_{y \in \Im \xi^*} m_r|_{\xi^*, y} - \sup_{y \in \Im \xi^*} m_r|_y^{\xi^*} \leq 0.$$

Therefore

$$\sup_{y \in \Im \xi^*} m_r|_{\xi^*, y} \leq m_r|_{\Im \xi^*}^{\xi^*}. \quad (4.31)$$

Now, from (4.30) and (4.31), we obtain

$$m_r|_{\Im \xi^*}^{\xi^*} = \sup_{y \in \Im \xi^*} m_r|_{\xi^*, y}.$$

Thence, by Lemma 4.1.2, we get $\xi^* \in \overline{\Im \xi^*} = \Im \xi^*$. Similarly, we can easily conclude that $\xi^* \in \overline{\Gamma \xi^*} = \Gamma \xi^*$. Hence, $\Im$ and Γ have a C.F.P. ■

Example 4.1.8. According to Example 1.3.2.3, we take $\theta \in \{0, \frac{3\pi}{2}\}$ and the R.M-M.S becomes $(\Omega_{r\{0, \frac{3\pi}{2}\}}, m_r)$ so that

$$\Omega_{r\{0, \frac{3\pi}{2}\}} = [0, \infty) \cup (V_i = \{z = \kappa i : \kappa < 0\}),$$

and $m_r|_{z_2}^{z_1} = \frac{|z_1| + |z_2|}{2} \forall_{z_1, z_2 \in \Omega_{r\{0, \frac{3\pi}{2}\}}}$. Consider the two mappings $\Gamma, \Im : \Omega_{r\{0, \frac{3\pi}{2}\}} \rightarrow CB|_{\Omega_{r\{0, \frac{3\pi}{2}\}}}$ by

$$\Gamma z = \begin{cases} \left\{ \frac{3z}{8}, \frac{z}{8} \right\} & z \in V_i; \\ \left\{ \frac{z}{5} \right\} & z \in [0, \infty), \end{cases} \quad \text{and} \quad \Im z = \begin{cases} \left\{ \frac{3z}{16} \right\} & z \in V_i; \\ \left\{ \frac{z}{15} \right\} & z \in [0, \infty). \end{cases}$$

Define $\alpha, \eta : \Omega_{r\{0, \frac{3\pi}{2}\}} \times \Omega_{r\{0, \frac{3\pi}{2}\}} \rightarrow [0, \infty)$ by $\alpha|_{z_2}^{z_1} = 2$ and $\eta|_{z_2}^{z_1} = 1, \forall_{z_1, z_2 \in \Omega_{r\{0, \frac{3\pi}{2}\}}}$. Then, $\Gamma, \Im$ are α_* -admissible maps w.r.t η . Now, $\forall_{z_1, z_2 \in V_i}$, we have $\frac{1}{2} \min \{m_r|_{\Gamma z_1}^{z_1}, m_r|_{\Im z_2}^{z_2}\} < m_r|_{z_2}^{z_1}$. Define $\psi(r) = \frac{99}{100}r \in \Psi$, $\phi(r) = 2r \in \Phi$, $\gamma = \frac{1}{3} \in (0, 1)$ and $L = \frac{2}{3} > 0$, so that

$$\begin{aligned} \phi \left(\hat{H}_r \Big|_{\Im z_2}^{\Gamma z_1} \right) &= 2 \left(\max \left\{ \sup_{a \in \Gamma z_1} m_r|_{\Im z_2}^a, \sup_{b \in \Im z_2} m_r|_b^{\Gamma z_1} \right\} \right) = 2 \left(\max \left\{ m_r|_{\left\{ \frac{3z_2}{16} \right\}}^{\frac{z_1}{8}}, m_r|_{\frac{3z_2}{16}}^{\left\{ \frac{3z_1}{8}, \frac{z_1}{8} \right\}} \right\} \right) \\ &= 2 \left(\frac{3|z_1| + \left| \frac{z_2}{2} \right|}{8} \right) = \frac{3}{8} \left[2 \left(\frac{1|z_1| + \left| \frac{z_2}{2} \right|}{3} + \frac{2|z_1| + \left| \frac{z_2}{2} \right|}{2} \right) \right] \\ &\leq \frac{99}{100} \left[2 \left(\frac{1|z_1| + |z_2|}{3} + \frac{2|z_1| + |z_2|}{2} \right) \right] \\ &= \psi \left[\phi \left(\gamma m_r|_{z_2}^{z_1} + L m_r|_{z_2}^{z_1} \right) \right] \leq \psi \left[\phi \left(\gamma M_r|_{z_2}^{z_1} + L N_r|_{z_2}^{z_1} \right) \right]. \end{aligned}$$

Similarly, $\forall_{z_1, z_2 \in [0, \infty)}$ (4.1) satisfies. So, the hypotheses of Theorem 4.1.7 hold, and 0 is a C.F.P of Γ and $\mathfrak{S}$.

Theorem 4.1.9. Presume that (Ω_r, m_r) is a C-R.M-M.S, $\xi_0 \in \overline{B|_\epsilon^{\xi_0}} \subseteq \Omega_r$ be an arbitrary, $\alpha, \eta : \Omega_r \times \Omega_r \rightarrow [0, \infty)$ be given functions with $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$, $\phi \in \Phi$ and $\psi \in \Psi$. Let $\Gamma, \mathfrak{S} : \Omega_r \rightarrow CB_r|_{\Omega_r}$ be continuous semi α_* -admissible multivalued maps w.r.t η satisfying (4.1) $\forall_{\xi_1, \xi_2 \in \overline{B|_\epsilon^{\xi_0}}}$. Moreover,

$$\phi \left(m_r|_{\xi_1}^{\xi_0} - m_r|_{\xi_0, \xi_1} \right) \leq \sum_{i=0}^{\kappa} \psi^i \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right] \leq \epsilon, \quad \forall_{\epsilon > 0}. \quad (4.32)$$

Then, Γ and $\mathfrak{S}$ have a C.F.P in $\overline{B|_\epsilon^{\xi_0}} \subseteq \Omega_r$.

Proof. Let $\xi_0 \in \overline{B|_\epsilon^{\xi_0}}$, $\exists_{\xi_1 \in \Omega_r}$ such that $\xi_1 \in \Gamma \xi_0$ and $\exists_{\xi_2 \in \Omega_r}$ such that $\xi_2 \in \mathfrak{S} \xi_1$. By continuing this approach, we construct an iterative $\{\xi_\kappa\}$ so that, $\xi_{2\kappa+1} \in \Gamma \xi_{2\kappa}$ and $\xi_{2\kappa+2} \in \mathfrak{S} \xi_{2\kappa+1}$. As $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$ and $(\Gamma, \mathfrak{S})$ is a pair of semi α_* -admissible multivalued maps w.r.t η , then $\alpha_*|_{\mathfrak{S} \xi_1}^{\Gamma \xi_0} \geq \eta_*|_{\mathfrak{S} \xi_1}^{\Gamma \xi_0}$ which implies that $\alpha|_{\xi_2}^{\xi_1} \geq \eta|_{\xi_2}^{\xi_1}$ from which we deduce that $\alpha_*|_{\Gamma \xi_2}^{\mathfrak{S} \xi_1} \geq \eta_*|_{\Gamma \xi_2}^{\mathfrak{S} \xi_1}$. Continuing in this way and if $\iota = 2\kappa + 1$, $\kappa = 1, 2, \dots, \frac{\iota-1}{2}$, we obtain $\alpha_*|_{\mathfrak{S} \xi_{2\kappa+1}}^{\Gamma \xi_{2\kappa}} \geq \eta_*|_{\mathfrak{S} \xi_{2\kappa+1}}^{\Gamma \xi_{2\kappa}}$, which leads to $\alpha_*|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} \geq \eta_*|_{\xi_{2\kappa+2}}^{\xi_{2\kappa+1}} \quad \forall_{\kappa \geq 0}$. Now, we evince that $\xi_{2\kappa} \in \overline{B|_\epsilon^{\xi_0}}$. From (4.32), we have

$$\phi \left(m_r|_{\xi_1}^{\xi_0} - m_r|_{\xi_0, \xi_1} \right) \leq \sum_{i=0}^{\kappa} \psi^i \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right] \leq \epsilon, \quad \forall_{\epsilon > 0}.$$

That is $\xi_1 \in \overline{B|_\epsilon^{\xi_0}}$. Let $\xi_2, \xi_3, \dots, \xi_{2i+1} \in \overline{B|_\epsilon^{\xi_0}}$, for some $i \geq 0$. Now, from (4.1), we can write

$$\phi \left(m_r|_{\xi_{2i+2}}^{\xi_{2i+1}} \right) \leq \phi \left(\hat{H}_r|_{\mathfrak{S} \xi_{2i+1}}^{\Gamma \xi_{2i}} \right) \leq \psi \left[\phi \left(m_r|_{\xi_{2i+1}}^{\xi_{2i}} \right) \right].$$

Similar to steps (4.4)-(4.9), we deduce that

$$\phi \left(m_r|_{\xi_{2i+2}}^{\xi_{2i+1}} \right) \leq \psi^{2i+1} \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right] \quad \forall_{i \geq 0}. \quad (4.33)$$

Using (M_{riv}) , (ϕ_4) , (4.32) and (4.33), we obtain

$$\begin{aligned}
\phi \left(m_r|_{\xi_{2i+1}}^{\xi_0} - m_r|_{\xi_0, \xi_{2i+1}} \right) &\leq \phi \left(m_r|_{\xi_1}^{\xi_0} - m_r|_{\xi_0, \xi_1} + m_r|_{\xi_2}^{\xi_1} - m_r|_{\xi_1, \xi_2} + \dots \right. \\
&\quad \left. + m_r|_{\xi_{2i+1}}^{\xi_{2i}} - m_r|_{\xi_{2i}, \xi_{2i+1}} \right) \\
&\leq \phi \left(m_r|_{\xi_1}^{\xi_0} + m_r|_{\xi_2}^{\xi_1} + \dots + m_r|_{\xi_{2i+1}}^{\xi_{2i}} \right) \\
&\leq \phi \left(m_r|_{\xi_1}^{\xi_0} + m_r|_{\xi_2}^{\xi_1} + \dots + m_r|_{\xi_{2i+1}}^{\xi_{2i}} \right) \\
&\leq \phi \left(m_r|_{\xi_1}^{\xi_0} \right) + \phi \left(m_r|_{\xi_2}^{\xi_1} \right) + \dots + \phi \left(m_r|_{\xi_{2i+1}}^{\xi_{2i}} \right) \\
&\leq \phi \left(m_r|_{\xi_1}^{\xi_0} \right) + \psi \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right] + \dots + \psi^{2i} \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right] \\
&\leq \sum_{i=0}^{2i} \psi^i \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right] \leq \epsilon.
\end{aligned}$$

Thence, $\xi_{2i+1} \in \overline{B|_\epsilon^{\xi_0}}$ by induction $\xi_\kappa \in \overline{B|_\epsilon^{\xi_0}}$, therefore $\{\xi_\kappa\}$ is a sequence in $\overline{B|_\epsilon^{\xi_0}}$ and $\alpha|_{\xi_{\kappa+1}}^{\xi_\kappa} \geq \eta|_{\xi_{\kappa+1}}^{\xi_\kappa} \forall \kappa \geq 0$. As Γ and $\mathfrak{S}$ are simi α_* -admissible maps w.r.t η on $\overline{B|_\epsilon^{\xi_0}}$, then $\alpha_*|_{\mathfrak{S}_{\kappa+1}}^{\Gamma_{\xi_\kappa}} \geq \eta_*|_{\mathfrak{S}_{\kappa+1}}^{\Gamma_{\xi_\kappa}}$. Now, inequality (4.33) can be written as

$$\phi \left(m_r|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} \right) \leq \psi^{\kappa+1} \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right]. \quad (4.34)$$

Passing $\kappa \rightarrow \infty$ in (4.34), we get

$$\lim_{\kappa \rightarrow \infty} \phi \left(m_r|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} \right) = 0.$$

From (ϕ_2) , we have

$$\lim_{\kappa \rightarrow \infty} m_r|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} = 0. \quad (4.35)$$

Now, we evidence that $\{\xi_\kappa\}$ is an m_r -Cauchy sequence in $\left(\overline{B|_\epsilon^{\xi_0}}, m_r \right)$. Let $\varkappa, \kappa \in \mathbb{N}$ with

$\varkappa > \kappa > p$. Then

$$\begin{aligned}
\phi\left(m_r|_{\xi_\varkappa}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_\varkappa}\right) &\leq \phi\left(\begin{aligned} &m_r|_{\xi_{\kappa+1}}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_{\kappa+1}} \\ &+ m_r|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} - m_r|_{\xi_{\kappa+1}, \xi_{\kappa+2}} + \dots \\ &+ m_r|_{\xi_\varkappa}^{\xi_{\varkappa-1}} - m_r|_{\xi_{\varkappa-1}, \xi_\varkappa} \end{aligned}\right) \\
&\leq \phi\left(m_r|_{\xi_{\kappa+1}}^{\xi_\kappa} + m_r|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}} + \dots + m_r|_{\xi_\varkappa}^{\xi_{\varkappa-1}}\right) \\
&\leq \phi\left(m_r|_{\xi_{\kappa+1}}^{\xi_\kappa}\right) + \phi\left(m_r|_{\xi_{\kappa+2}}^{\xi_{\kappa+1}}\right) + \dots + \phi\left(m_r|_{\xi_\varkappa}^{\xi_{\varkappa-1}}\right) \\
&\leq \psi^\kappa\left[\phi\left(m_r|_{\xi_1}^{\xi_0}\right)\right] + \psi^{\kappa+1}\left[\phi\left(m_r|_{\xi_1}^{\xi_0}\right)\right] + \dots \\
&\quad + \psi^{\varkappa-1}\left[\phi\left(m_r|_{\xi_1}^{\xi_0}\right)\right].
\end{aligned} \tag{4.36}$$

Taking limit as $\kappa, \varkappa \rightarrow \infty$ in (4.36), we get

$$\lim_{\kappa, \varkappa \rightarrow \infty} \phi\left(m_r|_{\xi_\varkappa}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_\varkappa}\right) = 0.$$

Applying the condition (ϕ_2) , we have

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(m_r|_{\xi_\varkappa}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_\varkappa}\right) = 0.$$

By the same way and from Remark 1.3.2.2, we obtain

$$\begin{aligned}
\phi\left(M_r|_{\xi_{2\kappa+1}, \xi_{2\kappa+4}} - m_r|_{\xi_{2\kappa+1}, \xi_{2\kappa+4}}\right) &\leq \phi\left(\begin{aligned} &M_r|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} - m_r|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} \\ &+ M_r|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}} - m_r|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}} \\ &+ M_r|_{\xi_{2\kappa+3}, \xi_{2\kappa+4}} - m_r|_{\xi_{2\kappa+3}, \xi_{2\kappa+4}} \end{aligned}\right) \\
&\leq \phi\left(M_r|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}} - m_r|_{\xi_{2\kappa+1}, \xi_{2\kappa+2}}\right) \\
&\quad + \phi\left(M_r|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}} - m_r|_{\xi_{2\kappa+2}, \xi_{2\kappa+3}}\right) \\
&\quad + \phi\left(M_r|_{\xi_{2\kappa+3}, \xi_{2\kappa+4}} - m_r|_{\xi_{2\kappa+3}, \xi_{2\kappa+4}}\right).
\end{aligned}$$

In general, $\forall_{\kappa, \varkappa, p \in \mathbb{N}}$ with $\varkappa > \kappa > p$. Then

$$\begin{aligned}
\phi \left(M_r|_{\xi_\kappa, \xi_\varkappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right) &\leq \phi \left(\begin{array}{c} M_r|_{\xi_\kappa, \xi_{\kappa+1}} - m_r|_{\xi_\kappa, \xi_{\kappa+1}} + \\ M_r|_{\xi_{\kappa+1}, \xi_{\kappa+2}} - m_r|_{\xi_{\kappa+1}, \xi_{\kappa+2}} + \dots + \\ M_r|_{\xi_{\varkappa-1}, \xi_\varkappa} - m_r|_{\xi_{\varkappa-1}, \xi_\varkappa} \end{array} \right) \\
&= \phi \left(\sum_{i=\kappa}^{\varkappa-1} \left(M_r|_{\xi_i, \xi_{i+1}} - m_r|_{\xi_i, \xi_{i+1}} \right) \right) \\
&\leq \sum_{i=0}^{\varkappa-1} \phi \left(M_r|_{\xi_i, \xi_{i+1}} - m_r|_{\xi_i, \xi_{i+1}} \right) \\
&\leq \sum_{i=0}^{\varkappa-1} \phi \left(m_r|_{\xi_{i+1}}^{\xi_i} \right) \\
&\leq \sum_{i=0}^{m-1} \psi^i \left[\phi \left(m_r|_{\xi_1}^{\xi_0} \right) \right] \leq \epsilon,
\end{aligned}$$

this leads to

$$\lim_{\kappa, \varkappa \rightarrow \infty} \phi \left(M_r|_{\xi_\kappa, \xi_\varkappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right) = 0.$$

Applying (ϕ_2) again, we get

$$\lim_{\kappa, \varkappa \rightarrow \infty} \left(M_r|_{\xi_\kappa, \xi_\varkappa} - m_r|_{\xi_\kappa, \xi_\varkappa} \right) = 0.$$

Therefore, $\{\xi_\kappa\}$ is an m_r -Cauchy in $\overline{B_r|_{\xi}^{\xi_0}}$. So, $\exists \xi^* \in \overline{B_r|_{\xi}^{\xi_0}}$ such that $\xi_\kappa \rightarrow \xi^*$ as $\kappa \rightarrow \infty$, $\implies \xi_{2\kappa+1} \rightarrow \xi^*$ and $\xi_{2\kappa+2} \rightarrow \xi^*$ as $\kappa \rightarrow \infty$. Since Γ and $\mathfrak{S}$ are continuous, and

$$\lim_{n \rightarrow \infty} \left(m_r|_{\xi^*}^{\xi_{2\kappa+1}} - m_r|_{\xi_{2\kappa+1}, \xi^*} \right) = 0 \text{ and } \lim_{\kappa \rightarrow \infty} \left(M_r|_{\xi_{2\kappa+1}, \xi^*} - m_r|_{\xi_{2\kappa+1}, \xi^*} \right) = 0. \quad (4.37)$$

Next, we prove that $\xi^* \in \mathfrak{S}\xi^* \cap \Gamma\xi^*$. Pressume that $m_r|_{\Gamma\xi^*}^{\xi^*} - m_r|_{\xi^*, \Gamma\xi^*} > 0$. So, by (M_{riv}) , we have

$$\phi \left(m_r|_{\Gamma\xi^*}^{\xi^*} - m_r|_{\xi^*, \Gamma\xi^*} \right) \leq \phi \left(\begin{array}{c} m_r|_{\xi_\kappa}^{\xi^*} - m_r|_{\xi^*, \xi_\kappa} + m_r|_{\xi_{\kappa+1}}^{\xi_\kappa} - m_r|_{\xi_\kappa, \xi_{\kappa+1}} \\ + m_r|_{\Gamma\xi^*}^{\xi_{\kappa+1}} - m_r|_{\xi_{\kappa+1}, \Gamma\xi^*} \end{array} \right). \quad (4.38)$$

Applying (ϕ_1) in (4.38), we get

$$m_r|_{\Gamma\xi^*}^{\xi^*} - m_r|_{\xi^*,\Gamma\xi^*} < m_r|_{\xi_\kappa}^{\xi^*} - m_r|_{\xi^*,\xi_\kappa} + m_r|_{\xi_{\kappa+1}}^{\xi_\kappa} + m_r|_{\Gamma\xi^*}^{\xi_{\kappa+1}} - m_r|_{\xi_{\kappa+1},\Gamma\xi^*}. \quad (4.39)$$

Taking limit as $\kappa \rightarrow \infty$ in (4.39), we have

$$m_r|_{\Gamma\xi^*}^{\xi^*} - m_r|_{\xi^*,\Gamma\xi^*} < m_r|_{\Gamma\xi^*}^{\xi^*} - m_r|_{\xi^*,\Gamma\xi^*},$$

a contradiction. Therefore, $m_r|_{\Gamma\xi^*}^{\xi^*} - m_r|_{\xi^*,\Gamma\xi^*} = 0$, and hence $\xi^* \in \Gamma\xi^*$. Similarly, we can evidence that $\xi^* \in \Im\xi^*$. Thus, $\xi^* \in \Im\xi^* \cap \Gamma\xi^*$, that is Γ and $\Im$ has a C.F.P $\xi^* \in \overline{B_r|_{\epsilon}^{\xi^0}}$. ■

Example 4.1.10. From Example 1.3.2.3, we consider $\theta \in \{0, \frac{\pi}{2}, \pi\}$. Then, the R.M-M.S becomes $(\Omega_r\{0, \frac{\pi}{2}, \pi\}, m_r)$ so that

$$\Omega_r\{0, \frac{\pi}{2}, \pi\} = \mathbb{R} \cup (V_i = \{z = \kappa i : \kappa > 0\}),$$

and $m_r|_{z_2}^{z_1} = \frac{|z_1|+|z_2|}{2} \forall_{z_1, z_2 \in \Omega_r\{0, \frac{\pi}{2}, \pi\}}$. Define $\Gamma, \Im : \Omega_r\{0, \frac{\pi}{2}, \pi\} \rightarrow CB_r|_{\Omega_r\{0, \frac{\pi}{2}, \pi\}}$ by

$$\Gamma z = \begin{cases} \{\frac{z}{120}, \frac{z}{40}\} & z \in (0, 20i] \cup [0, 20]; \\ \{1, 4z - 63\} & \text{otherwise,} \end{cases} \quad \text{and} \quad \Im z = \begin{cases} \{\frac{z}{30}, \frac{z}{60}\} & z \in (0, 20i] \cup [0, 20]; \\ \{0, 6z - 105\} & \text{otherwise.} \end{cases}$$

Consider $z_0 = 0$, $\epsilon = 10$. Then, $\overline{B_r|_{\epsilon}^{z_0}} = (0, 20i] \cup [0, 20]$ and $m_r|_{z_1}^{z_0} = m_r|_{\Im z_0}^0 = 0$, therefore, $\forall_{n \geq 0}$, $\epsilon > 0$ and $\forall_{z_0, z_1 \in \overline{B_r|_{\epsilon}^{z_0}}}$, $\frac{1}{2} \min \{m_r|_{\Gamma z_0}^{z_0}, m_r|_{\Im z_1}^{z_1}\} < m_r|_{z_1}^{z_0}$, we have

$$\phi \left(m_r|_{z_1}^{z_0} - m_r|_{z_0, z_1} \right) \leq \sum_{i=0}^{\kappa} \psi^i [\phi (m_r|_{z_1}^{z_0})] \leq \epsilon.$$

Define $\alpha, \eta : \Omega_r\{0, \frac{\pi}{2}, \pi\} \times \Omega_r\{0, \frac{\pi}{2}, \pi\} \rightarrow [0, \infty)$ by

$$\alpha|_{z_2}^{z_1} = \begin{cases} \frac{7}{2}z & z \in V_i; \\ 2 & z \in \mathbb{R}, \end{cases} \quad \text{and} \quad \eta|_{z_2}^{z_1} = \begin{cases} 2z & z \in V_i; \\ 1 & z \in \mathbb{R}. \end{cases}$$

Thus, $\alpha|_{z_2}^{z_1} \geq \eta|_{z_2}^{z_1}$ implies that $\alpha_*|_{\Im z_2}^{\Gamma z_1} \geq \eta_*|_{\Im z_2}^{\Gamma z_1} \forall_{z_1, z_2 \in \Omega_r\{0, \frac{\pi}{2}, \pi\}}$. Now, let $\psi(r) = \frac{399}{400}r$ and

$\phi(r) = 2r$. Then, for $z_1 = 30$ and $z_2 = 31$, we have

$$\phi \left(\hat{H}_r \Big|_{\mathfrak{S}31}^{\Gamma 30} \right) = 187 > 37.99 = \psi \left[\phi \left(\gamma M_r|_{31}^{30} + L N_r|_{31}^{30} \right) \right].$$

So, the condition (4.1) does not hold for some $z_1, z_2 \in \Omega_r \setminus \{0, \frac{\pi}{2}, \pi\}$. On the other hand, $\forall_{z_1, z_2 \in \overline{B_r|_\epsilon^{z_0}}}$ with considering $\gamma = \frac{1}{3} \in (0, 1)$ and $L = \frac{2}{3} > 0$, we have

$$\begin{aligned} \phi \left(\hat{H}_r \Big|_{\mathfrak{S}z_2}^{\Gamma z_1} \right) &= 2 \max \left\{ \sup_{a \in \Gamma z_1} m_r|_{\left\{ \frac{z_2}{30}, \frac{z_2}{60} \right\}}^a, \sup_{b \in \mathfrak{S}z_2} m_r|_{\left\{ \frac{z_1}{120}, \frac{z_1}{40} \right\}}^b \right\} \\ &= 2 \max \left\{ m_r|_{\left\{ \frac{z_2}{30}, \frac{z_2}{60} \right\}}^{\frac{z_1}{40}}, m_r|_{\frac{z_2}{30}}^{\left\{ \frac{z_1}{120}, \frac{z_1}{40} \right\}} \right\} \\ &= 2 \max \left\{ m_r|_{\frac{z_2}{60}}^{\frac{z_1}{40}}, m_r|_{\frac{z_2}{30}}^{\frac{z_1}{120}} \right\} \\ &= \frac{2}{20} \max \left\{ m_r|_{\frac{z_2}{3}}^{\frac{z_1}{2}}, m_r|_{\frac{2z_2}{3}}^{\frac{z_1}{6}} \right\} \\ &\leq \frac{399}{400} \left[2 \left(\frac{1}{3} \frac{|z_1| + |z_2|}{2} + \frac{2}{3} \frac{|z_1| + |z_2|}{2} \right) \right] \\ &= \psi \left[\phi \left(\gamma m_r|_{z_2}^{z_1} + L m_r|_{z_2}^{z_1} \right) \right] \\ &\leq \psi \left[\phi \left(\gamma M_r|_{z_2}^{z_1} + L N_r|_{z_2}^{z_1} \right) \right]. \end{aligned}$$

Hence, all hypotheses of Theorem 4.1.9 are satisfied, and $0 \in \overline{B_r|_\epsilon^{z_0}}$ is a C.F.P of Γ and $\mathfrak{S}$. (Note that $21 \in \Omega_r \setminus \{0, \frac{\pi}{2}, \pi\}$ is an another C.F.P of Γ and $\mathfrak{S}$, but $21 \notin \overline{B_r|_\epsilon^{z_0}}$).

Corollary 4.1.11. Let (Ω_r, m_r) be a C-R.M-M.S and the pair $\Gamma, \mathfrak{S} : \Omega_r \rightarrow \Omega_r$ is said to be (ψ, ϕ) -Suzuki Rational-type contractive maps, if $\exists_{\phi \in \Phi, \psi \in \Psi}$ and $\exists_{\gamma \in (0,1), L \geq 0}$, with $\gamma + L \leq 1$ such that $\forall_{\xi_1, \xi_2 \in \Omega_r}$ with $\Gamma \xi_1 \neq \mathfrak{S} \xi_2$.

$$\begin{aligned} \frac{1}{2} \min \left\{ m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\mathfrak{S} \xi_2}^{\xi_2} \right\} &< m_r|_{\xi_2}^{\xi_1} \Rightarrow \\ \phi \left(m_r|_{\mathfrak{S} \xi_2}^{\Gamma \xi_1} \right) &\leq \psi \left[\phi \left(\gamma M_r|_{\xi_2}^{\xi_1} + L N_r|_{\xi_2}^{\xi_2} \right) \right], \end{aligned} \tag{4.40}$$

where $M_r|_{\xi_2}^{\xi_1}$ and $N_r|_{\xi_2}^{\xi_2}$ are defined as in (4.2) and (4.3) respectively.

Hence, Γ and $\mathfrak{S}$ possess a C.F.P $\xi^* \in \Omega_r$.

Corollary 4.1.12. Let (Ω_r, m_r) be a C-R.M-M.S, and the map $\Gamma : \Omega_r \rightarrow \Omega_r$ is termed (ψ, ϕ) -Suzuki Rational-type contraction, if $\exists_{\phi \in \Phi, \psi \in \Psi}$ and $\exists_{\gamma \in (0,1), L \geq 0}$, with $\gamma + L \leq 1$ such that

$\forall_{\xi_1, \xi_2 \in \Omega_r}$ with $\Gamma \xi_1 \neq \Gamma \xi_2$.

$$\begin{aligned} \frac{1}{2} \min \left\{ m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\Gamma \xi_2}^{\xi_2} \right\} &< m_r|_{\xi_2}^{\xi_1} \Rightarrow \\ \phi \left(m_r|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) &\leq \psi \left[\phi \left(\gamma M_r|_{\xi_2}^{\xi_1} + L N_r|_{\xi_2}^{\xi_2} \right) \right], \end{aligned} \quad (4.41)$$

where

$$M_r|_{\xi_2}^{\xi_1} = \max \left\{ m_r|_{\xi_2}^{\xi_1}, m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\Gamma \xi_2}^{\xi_2}, \frac{m_r|_{\Gamma \xi_2}^{\xi_2} \left[1 + m_r|_{\Gamma \xi_1}^{\xi_1} \right]}{1 + m_r(x, y)} \right\}, \quad (4.42)$$

and

$$N_r|_{\xi_2}^{\xi_1} = \min \left\{ m_r|_{\xi_2}^{\xi_1}, m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\Gamma \xi_2}^{\xi_2}, \frac{m_r|_{\Gamma \xi_1}^{\xi_1} \cdot m_r|_{\Gamma \xi_2}^{\xi_2}}{1 + m_r|_{\xi_2}^{\xi_1}} \right\}. \quad (4.43)$$

Hence, Γ possesses an F.P in Ω_r .

Corollary 4.1.13. Let (Ω_r, m_r) be a C-R.M-M.S, $\xi_0 \in \overline{B_r|_{\epsilon}^{\xi_0}} \subseteq \Omega_r$ be an arbitrary, $\alpha, \eta : \Omega_r \times \Omega_r \rightarrow [0, \infty)$ be functions with $\alpha|_{\xi_1}^{\xi_0} \geq \eta|_{\xi_1}^{\xi_0}$. Assume $\exists_{\phi \in \Phi, \psi \in \Psi}$ and $\exists_{\gamma \in (0,1), L \geq 0}$ with $\gamma + L \leq 1$. Let $\Gamma, \mathfrak{F} : \Omega_r \rightarrow \Omega_r$ be continuous semi α -admissible maps w.r.t η satisfying (4.40) and (4.32), $\forall_{\xi_0, \xi_1 \in \overline{B_r|_{\epsilon}^{\xi_0}}}$. Then, Γ and $\mathfrak{F}$ have a C.F.P in $\overline{B_r|_{\epsilon}^{\xi_0}} \subseteq \Omega_r$.

4.2 Fixed point theorems for single-valued θ -contraction mappings

Here, we initiate generalized θ -contraction and prove related F.P results in R.M-M.Ss.

Theorem 4.2.1. Let (Ω_r, m_r) be a C-R.M-M.S, and $\Gamma : \Omega_r \rightarrow \Omega_r$ be a mapping. Suppose that $\exists_{\theta \in \Theta}$ and $\exists_{k \in (0,1)}$ such that $\forall_{\xi_1, \xi_2 \in \Omega_r}$,

$$m_r|_{\Gamma \xi_2}^{\Gamma \xi_1} \neq 0 \implies \theta \left(m_r|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \left[\theta \left(N_r|_{\xi_2}^{\xi_1} \right) \right]^k, \quad (4.44)$$

where

$$N_r|_{\xi_2}^{\xi_1} = \max \left\{ m_r|_{\xi_2}^{\xi_1}, m_r|_{\Gamma \xi_1}^{\xi_1}, m_r|_{\Gamma \xi_2}^{\xi_2} \right\}. \quad (4.45)$$

Then, Γ has a U.F.P.

Proof. Let $\xi_0 \in \Omega_r$ be an arbitrary and define a sequence $\{\xi_\kappa\}_{\kappa \in \mathbb{N}}$ by letting $\xi_1 = \Gamma \xi_0$,

$\xi_2 = \Gamma\xi_1$, $\xi_3 = \Gamma\xi_2$, ..., $\xi_{\kappa+1} = \Gamma\xi_\kappa = \Gamma^{\kappa+1}\xi_0$. If for some $q \in \mathbb{N}$, we have $\Gamma^q\xi = \Gamma^{q+1}\xi$, then $\Gamma^q\xi$ is a F.P of Γ . So, we suppose that $m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} > 0$, $\forall \kappa \in \mathbb{N}$. Now, from (4.44) we get

$$\theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \leq \left[\theta \left(N_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi} \right) \right]^k, \quad (4.46)$$

where

$$\begin{aligned} N_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi} &= \max \left\{ m_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi}, m_r|_{\Gamma^{\kappa-1}\xi}^{\Gamma^{\kappa-1}\xi}, m_r|_{\Gamma^{\kappa}\xi}^{\Gamma^\kappa\xi} \right\} \\ &= \max \left\{ m_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi}, m_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi}, m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right\} \\ &= \max \left\{ m_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi}, m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right\}. \end{aligned} \quad (4.47)$$

If $N_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi} = m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi}$, then from (4.46), we write

$$\theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \leq \left[\theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \right]^k. \quad (4.48)$$

Taking $\ln$ in both sides of (4.48)

$$\ln \left[\theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \right] \leq k \ln \left[\theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \right],$$

a contradiction because $k \in (0, 1)$. Therefore, we get

$$N_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi} = m_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi}.$$

It follows from (4.46) that

$$\theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \leq \left[\theta \left(m_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi} \right) \right]^k \leq \left[\theta \left(m_r|_{\Gamma^{\kappa-1}\xi}^{\Gamma^{\kappa-2}\xi} \right) \right]^{k^2} \leq \dots \leq \left[\theta \left(m_r|_{\Gamma\xi}^\xi \right) \right]^{k^\kappa}.$$

That leads to

$$1 \leq \theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \leq \left[\theta \left(m_r|_{\Gamma\xi}^\xi \right) \right]^{k^\kappa}, \quad \forall \kappa \in \mathbb{N}. \quad (4.49)$$

Letting $\kappa \rightarrow \infty$ in (4.49), we infer that

$$\theta \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right) \rightarrow 1 \text{ as } \kappa \rightarrow \infty, \quad (4.50)$$

which implies from (Θ2) that $\lim_{\kappa \rightarrow \infty} m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} = 0$. Now, from condition (Θ3), $\exists_{r \in (0,1), l \in (0,\infty]}$ such that

$$\lim_{\kappa \rightarrow \infty} \frac{\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) - 1}{\left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r} = l.$$

Presume that $l < \infty$. Let $B = \frac{l}{2} > 0$. Invoking the definition of the limit $\exists_{\kappa_0 \in \mathbb{N}}$

$$\left| \frac{\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) - 1}{\left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r} - l \right| \leq B, \quad \forall_{\kappa \geq \kappa_0}.$$

This leads to

$$\frac{\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) - 1}{\left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r} \geq l - B = B, \quad \forall_{\kappa \geq \kappa_0}.$$

Thus

$$n \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r \leq A\kappa \left[\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) - 1 \right], \quad \forall_{\kappa \geq \kappa_0},$$

where $A = \frac{1}{B}$. Presume now $l = \infty$ and let $B > 0$ be chosen. Invoking the definition of the limit, $\exists_{\kappa_0 \in \mathbb{N}}$ such that

$$\frac{\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) - 1}{\left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r} \geq B, \quad \forall_{\kappa \geq \kappa_0}.$$

This implies that

$$n \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r \leq A\kappa [\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) - 1] \quad \forall_{\kappa \geq \kappa_0},$$

where $A = \frac{1}{B}$. Thus, in all cases, $\exists_{A>0}$ and $\exists_{\kappa_0 \in \mathbb{N}}$ such that

$$n \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r \leq A\kappa \left[\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) - 1 \right] \quad \forall_{\kappa \geq \kappa_0}.$$

Using (4.49), we get

$$\kappa \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right)^r \leq A\kappa \left(\left[\theta \left(m_r|_{\Gamma^{\kappa\xi}_{\kappa+1\xi}} \right) \right]^{k^\kappa} - 1 \right) \quad \forall_{\kappa \geq \kappa_0}.$$

Taking $\kappa \rightarrow \infty$ above, we infer that

$$\lim_{\kappa \rightarrow \infty} \kappa \left(m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \right)^r = 0.$$

Thus, $\exists_{\kappa_1 \in \mathbb{N}}$ so that

$$m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^\kappa\xi} \leq \frac{1}{\kappa^{1/r}} \quad \forall_{\kappa \geq \kappa_1}. \quad (4.51)$$

Now, we discuss two steps.

Step 1. When $\Gamma^\kappa\xi = \Gamma^\varkappa\xi$ for some integers $\kappa \neq \varkappa$ with $\varkappa > \kappa$, we get $\Gamma^{\varkappa-\kappa}(\Gamma^\kappa\xi) = \Gamma^\kappa\xi$. choose $\mu = \Gamma^\kappa\xi$ and $p = m - n$. Then, $\Gamma^p\mu = \mu$, and μ is a periodic point of Γ . By (4.44) and (4.49), we obtain

$$\theta \left(m_r|_{\Gamma_\mu}^\mu \right) = \theta \left(m_r|_{\Gamma^{p+1}\mu}^{\Gamma^p\mu} \right) \leq \left[\theta \left(m_r|_{\Gamma_\mu}^\mu \right) \right]^{k^p}.$$

If $p \rightarrow \infty$, we get $m_r|_{\Gamma_\mu}^\mu = 0$. On the other hand, we write

$$\theta \left(m_r|_{\Gamma_\mu}^\mu \right) = \theta \left(m_r|_{\Gamma^p\mu}^{\Gamma^p\mu} \right) \leq \left[\theta \left(m_r|_{\Gamma^{p-1}\mu}^{\Gamma^{p-1}\mu} \right) \right]^k \leq \dots \leq \left[\theta \left(m_r|_{\Gamma_\mu}^\mu \right) \right]^{k^{p-1}} \leq \left[\theta \left(m_r|_{\Gamma_\mu}^\mu \right) \right]^{k^p},$$

which leads to

$$\ln \theta \left(m_r|_{\Gamma_\mu}^\mu \right) \leq k^{p-1} \ln \left[\theta \left(m_r|_{\Gamma_\mu}^\mu \right) \right] \leq k^p \ln \left[\theta \left(m_r|_{\Gamma_\mu}^\mu \right) \right],$$

implying that

$$\theta \left(m_r|_{\Gamma_\mu}^\mu \right) = \theta \left(m_r|_{\Gamma_\mu}^{\Gamma_\mu} \right).$$

That yields $m_r|_{\Gamma_\mu}^\mu = m_r|_{\Gamma_\mu}^{\Gamma_\mu} = 0$. Hence $\Gamma\mu = \mu$, that is μ is a F.P of Γ .

Step 2. When $\Gamma^\kappa\xi \neq \Gamma^\varkappa\xi$ for every $\kappa, \varkappa \in \mathbb{N}$ with $\kappa \neq \varkappa$, we prove that Γ has a periodic point. suppose on the contrary, utilizing (4.44) and (4.45), we obtain

$$\theta \left(m_r|_{\Gamma^{\kappa+2}\xi}^{\Gamma^\kappa\xi} \right) \leq \left[\theta \left(N_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^{\kappa-1}\xi} \right) \right]^k, \quad (4.52)$$

where

$$N_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^{\kappa-1}\xi} = \max \left\{ m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^{\kappa-1}\xi}, m_r|_{\Gamma^\kappa\xi}^{\Gamma^{\kappa-1}\xi}, m_r|_{\Gamma^{\kappa+2}\xi}^{\Gamma^{\kappa+1}\xi} \right\}. \quad (4.53)$$

From $(\Theta 1)$, $(\Theta 4)$, (4.52) and (4.53), we can write

$$\theta \left(m_r |_{\Gamma^{\kappa} \xi}^{\Gamma^{\kappa} \xi} \right) \leq \left[\max \left\{ \theta \left(m_r |_{\Gamma^{\kappa+1} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa+1} \xi} \right) \right\} \right]^k. \quad (4.54)$$

Let $\kappa \in I \subset \mathbb{N}$, so that

$$u_{\kappa} = \max \left\{ \theta \left(m_r |_{\Gamma^{\kappa+1} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa+1} \xi} \right) \right\} = \theta \left(m_r |_{\Gamma^{\kappa+1} \xi}^{\Gamma^{\kappa-1} \xi} \right).$$

If $|I| < \infty$, $\exists_{N \in \mathbb{N}}$ such that $\forall_{\kappa \geq N}$,

$$\max \left\{ \theta \left(m_r |_{\Gamma^{\kappa+1} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa+1} \xi} \right) \right\} = \max \left\{ \theta \left(m_r |_{\Gamma^{\kappa} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa+1} \xi} \right) \right\}.$$

In this case, from (4.54), we infer that

$$1 \leq \theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa} \xi} \right) \leq \left[\max \left\{ \theta \left(m_r |_{\Gamma^{\kappa} \xi}^{\Gamma^{\kappa-1} \xi} \right), \theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa+1} \xi} \right) \right\} \right]^k \forall_{\kappa \geq N}.$$

Let $\kappa \rightarrow \infty$ above and using (4.50), we get

$$\theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa} \xi} \right) \rightarrow 1 \text{ as } \kappa \rightarrow \infty.$$

If $|I| = \infty$, we find $\{\tilde{u}_{\kappa}\} \subset \{u_{\kappa}\}$, such that

$$\tilde{u}_{\kappa} = \theta \left(m_r |_{\Gamma^{\kappa+1} \xi}^{\Gamma^{\kappa-1} \xi} \right) \text{ for } \kappa \text{ sufficiently large.}$$

From (4.54), we get

$$1 \leq \theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa} \xi} \right) \leq \left[\theta \left(m_r |_{\Gamma^{\kappa+1} \xi}^{\Gamma^{\kappa-1} \xi} \right) \right]^k \leq \left[\theta \left(m_r |_{\Gamma^{\kappa} \xi}^{\Gamma^{\kappa-2} \xi} \right) \right]^{k^2} \leq \dots \leq \left[\theta \left(m_r |_{\Gamma^2 \xi}^{\xi} \right) \right]^{k^{\kappa}},$$

for κ sufficiently large. Let $\kappa \rightarrow \infty$ above, we obtain

$$\theta \left(m_r |_{\Gamma^{\kappa+2} \xi}^{\Gamma^{\kappa} \xi} \right) \rightarrow 1 \text{ as } \kappa \rightarrow \infty. \quad (4.55)$$

Then, in all cases, (4.55) holds. Using (4.55) and (Θ2), we have

$$\lim_{n \rightarrow \infty} m_r|_{\Gamma^{\kappa+2}\xi}^{\Gamma^{\kappa}\xi} = 0.$$

Similarly, from (Θ3), $\exists_{\kappa_2 \in \mathbb{N}}$ such that

$$m_r|_{\Gamma^{n+2}\xi}^{\Gamma^n\xi} \leq \frac{1}{n^{1/r}} \quad \forall_{n \geq n_2}. \quad (4.56)$$

Let $\acute{N} = \max\{\kappa_0, \kappa_1\}$. we consider the following:

If $\varkappa > 2$ is odd, then we consider $\varkappa = 2q + 1$, $q \geq 1$, Using (4.51) $\forall_{\kappa \geq \acute{N}}$, we obtain

$$\begin{aligned} m_r|_{\Gamma^{\kappa+\varkappa}\xi}^{\Gamma^{\kappa}\xi} - m_r|_{\Gamma^{\kappa}\xi, \Gamma^{\kappa+\varkappa}\xi} &\leq m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^{\kappa}\xi} - m_r|_{\Gamma^{\kappa}\xi, \Gamma^{\kappa+1}\xi} \\ &\quad + m_r|_{\Gamma^{\kappa+2}\xi}^{\Gamma^{\kappa+1}\xi} - m_r|_{\Gamma^{\kappa+1}\xi, \Gamma^{\kappa+2}\xi} \\ &\quad + \dots + m_r|_{\Gamma^{\kappa+2q+1}\xi}^{\Gamma^{\kappa+2q}\xi} - m_r|_{\Gamma^{\kappa+2q}\xi, \Gamma^{\kappa+2q+1}\xi} \\ &\leq m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^{\kappa}\xi} + m_r|_{\Gamma^{\kappa+2}\xi}^{\Gamma^{\kappa+1}\xi} + \dots + m_r|_{\Gamma^{\kappa+2q+1}\xi}^{\Gamma^{\kappa+2q}\xi} \\ &\leq \frac{1}{\kappa^{1/r}} + \frac{1}{(\kappa+1)^{1/r}} + \dots + \frac{1}{(\kappa+2q)^{1/r}} \\ &\leq \sum_{i=\kappa}^{\infty} \frac{1}{i^{1/r}}. \end{aligned} \quad (4.57)$$

Now, if $\varkappa > 2$ is even, then we write $\varkappa = 2q$, $q \geq 2$, Using (4.51) and (4.56), $\forall_{\kappa \geq \acute{N}}$, we get

$$\begin{aligned} m_r|_{\Gamma^{\kappa+\varkappa}\xi}^{\Gamma^{\kappa}\xi} - m_r|_{\Gamma^{\kappa}\xi, \Gamma^{\kappa+\varkappa}\xi} &\leq m_r|_{\Gamma^{\kappa+2}\xi}^{\Gamma^{\kappa}\xi} - m_r|_{\Gamma^{\kappa}\xi, \Gamma^{\kappa+2}\xi} \\ &\quad + m_r|_{\Gamma^{\kappa+3}\xi}^{\Gamma^{\kappa+2}\xi} - m_r|_{\Gamma^{\kappa+2}\xi, \Gamma^{\kappa+3}\xi} \\ &\quad + \dots + m_r|_{\Gamma^{\kappa+2q}\xi}^{\Gamma^{\kappa+2q-1}\xi} - m_r|_{\Gamma^{\kappa+2q-1}\xi, \Gamma^{\kappa+2q}\xi} \\ &\leq m_r|_{\Gamma^{\kappa+2}\xi}^{\Gamma^{\kappa}\xi} + m_r|_{\Gamma^{\kappa+3}\xi}^{\Gamma^{\kappa+2}\xi} + \dots + m_r|_{\Gamma^{\kappa+2q}\xi}^{\Gamma^{\kappa+2q-1}\xi} \\ &\leq \frac{1}{\kappa^{1/r}} + \frac{1}{(\kappa+2)^{1/r}} + \dots + \frac{1}{(\kappa+2q-1)^{1/r}} \\ &\leq \sum_{i=\kappa}^{\infty} \frac{1}{i^{1/r}}. \end{aligned} \quad (4.58)$$

Combining (4.57) and (4.58), we have

$$m_r|_{\Gamma^{\kappa+\varkappa}\xi}^{\Gamma^{\kappa}\xi} - m_r|_{\Gamma^{\kappa}\xi, \Gamma^{\kappa+\varkappa}\xi} \leq \sum_{i=\kappa}^{\infty} \frac{1}{i^{1/r}} \quad \forall_{\kappa \geq N, \varkappa \in \mathbb{N}}.$$

Since the series $\sum_{i=\kappa}^{\infty} \frac{1}{i^{1/r}}$ is convergent, we find that $m_r|_{\Gamma^{\kappa+\varkappa}\xi}^{\Gamma^{\kappa}\xi} - m_r|_{\Gamma^{\kappa}\xi, \Gamma^{\kappa+\varkappa}\xi}$ is also convergent as $\kappa, \varkappa \rightarrow \infty$, which implies that

$$m_r|_{\Gamma^{\kappa+\varkappa}\xi}^{\Gamma^{\kappa}\xi} - m_r|_{\Gamma^{\kappa}\xi, \Gamma^{\kappa+\varkappa}\xi} \leq m_r|_{\Gamma^{\kappa+\varkappa}\xi}^{\Gamma^{\kappa}\xi}.$$

Thus, the sequence $\{\Gamma^{\kappa}\xi\}$ is m_r -Cauchy in m_r -C-R.M-M.S (Ω_r, m_r) . Hence, there is some $u \in \Omega_r$ such that $\Gamma^{\kappa}\xi \rightarrow u$ as $\kappa \rightarrow \infty$. With no affecting generality, we can suppose that $\Gamma^{\kappa}\xi \neq u \quad \forall \kappa$. Assume that $m_r|_{\Gamma u}^u > 0$, Using (4.44), we get

$$\theta\left(m_r|_{\Gamma u}^{\Gamma^{\kappa+1}\xi}\right) \leq \left[\theta\left(N_r|_u^{\Gamma^{\kappa}\xi}\right)\right]^k \quad \forall_{\kappa \in \mathbb{N}},$$

where

$$N_r|_u^{\Gamma^{\kappa}\xi} = \max \left\{ m_r|_u^{\Gamma^{\kappa}\xi}, m_r|_{\Gamma^{\kappa+1}\xi}^{\Gamma^{\kappa}\xi}, m_r|_{\Gamma u}^u \right\}.$$

Taking $\kappa \rightarrow \infty$ above, using $(\Theta 4)$, Lemma's 1.3.2.5, 1.3.2.6 and 1.3.2.7, we have

$$\theta(m_r|_{\Gamma u}^u) \leq [\theta(m_r|_{\Gamma u}^u)]^k < \theta(m_r|_{\Gamma u}^u),$$

a contradiction. Hence, $\Gamma u = u$, which again contradicts the assumption that Γ has no periodic point. Thence, Γ has a periodic point u of period q . Presume that $Fix\{\Gamma\} = \emptyset$. Then, we obtain

$$m_r|_{\Gamma u}^u > 0, \quad q > 1.$$

Using (4.44), we obtain $\theta(m_r|_{\Gamma u}^u) = \theta\left(m_r|_{\Gamma^{q+1}u}^{\Gamma^q u}\right) \leq [\theta(m_r|_{\Gamma u}^u)]^{k^q} < \theta(m_r|_{\Gamma u}^u)$, a discrepancy. Thus, $Fix\{\Gamma\} \neq \emptyset$, that is, Γ has more than one F.P. Now, assume that $u, v \in \Omega_r$ are two F.Ps of Γ such that $m_r|_v^u = m_r|_{\Gamma v}^{\Gamma u} > 0$. Utilizing (4.44), we get

$$\theta(m_r|_v^u) = \theta\left(m_r|_{\Gamma v}^{\Gamma u}\right) \leq [\theta(m_r|_v^u)]^k < \theta(m_r|_v^u),$$

a contradiction. Then, Γ has a U.F.P. ■

Remark 4.2.2. Theorem 4.2.1 is still valid if we replace condition (4.44) with one of the following stipulations, $\forall_{\xi_1, \xi_2, u \in \Omega_r}$ with $m_r|_u^u = 0$:

(▼1) There is $0 < k < 1$ so that

$$m_r|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq k \max \left\{ m_r|_{\xi_2}^{\xi_1}, m_r|_{\Gamma\xi_1}^{\xi_1}, m_r|_{\Gamma\xi_2}^{\xi_2} \right\}.$$

(▼2) There is $0 \leq k < \frac{1}{2}$ so that

$$m_r|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq k \left(m_r|_{\Gamma\xi_1}^{\xi_1} + m_r|_{\Gamma\xi_2}^{\xi_2} \right).$$

(▼3) There is $0 \leq k < \left(\frac{\sqrt{3}-1}{2} \right)$ so that

$$m_r|_{\Gamma\xi_2}^{\Gamma\xi_1} \leq k \left(m_r|_{\Gamma\xi_2}^{\xi_1} + m_r|_{\Gamma\xi_1}^{\xi_2} \right).$$

(▼4) There are $\theta \in \Theta$ and $k \in (0, 1)$ so that

$$m_r|_{\Gamma\xi_2}^{\Gamma\xi_1} \neq 0 \implies \theta \left(m_r|_{\Gamma\xi_2}^{\Gamma\xi_1} \right) \leq \left[\theta \left(m_r|_{\xi_2}^{\xi_1} \right) \right]^k.$$

We note that Θ contains a large class of functions. For instance

$$\theta(r) = 2 - \frac{2}{\pi} \arctan \left(\frac{1}{r^\alpha} \right), \quad 0 < \alpha < 1, \quad r > 0.$$

Corollary 4.2.3. Let (Ω_r, m_r) be a C-R.M-M.S, and $\Gamma : \Omega_r \rightarrow \Omega_r$ be a self-mapping. If $\exists_{\theta \in \Theta}$ and $\exists_{\alpha, k \in (0,1)}$ such that $\forall_{\xi_1, \xi_2 \in \Omega_r}$ and $\Gamma\xi_1 \neq \Gamma\xi_2$

$$2 - \frac{2}{\pi} \arctan \left(\frac{1}{\left[m_r|_{\Gamma\xi_2}^{\Gamma\xi_1} \right]^\alpha} \right) \leq \left[2 - \frac{2}{\pi} \arctan \left(\frac{1}{\left[N_r|_{\xi_2}^{\xi_1} \right]^\alpha} \right) \right]^k,$$

where $N_r|_{\xi_2}^{\xi_1}$ is defined as in (4.45). Then, Γ has a U.F.P.

Example 4.2.4. Let $\Omega_{r\varphi} = \{0\} \cup \{z \in \mathbb{C} : \arg(z) = \varphi\}$ for a fixed φ , $0 \leq \varphi < 2\pi$. Define $m_r : \Omega_r \times \Omega_r \longrightarrow [0, \infty)$ by $m_r|_{\xi_2}^{\xi_1} = \frac{|\xi_1|+|\xi_2|}{2} \forall_{\xi_1, \xi_2 \in \Omega_{r\varphi}}$. Then $(\Omega_{r\varphi}, m_r)$ is a R.M-M.S. We take $\varphi = 0, \frac{\pi}{2}$ and the R.M-M.S becomes $(\Omega_{r0, \frac{\pi}{2}}, m_r)$ such that $\Omega_{r0, \frac{\pi}{2}} = [0, \infty) \cup (A_i = \{z = \kappa i : 1 \leq \kappa < \infty\})$ and $m_r|_{z_2}^{z_1} = \frac{|z_1|+|z_2|}{2} \forall_{z_1, z_2 \in \Omega_{r0, \frac{\pi}{2}}}$. Let $\theta(r) = e^{\sqrt{r}} \in \Theta$. Define $\Gamma : \Omega_{r0, \frac{\pi}{2}} \rightarrow \Omega_{r0, \frac{\pi}{2}}$ by

$$\Gamma z = \begin{cases} \frac{z}{2+\kappa} : & z \in A_i, 1 \leq \kappa < \infty; \\ \frac{z}{2}, & z \in [0, 1]; \\ \frac{z}{2^\kappa} & z \in (1, \infty), 2 \leq \kappa < \infty. \end{cases}$$

Now, for $z \in A_i$, $1 \leq \kappa < \infty$, we have

$$\begin{aligned} \theta \left(m_r|_{\Gamma z_2}^{\Gamma z_1} \right) &= e^{\sqrt{m_r|_{\Gamma z_2}^{\Gamma z_1}}} = \theta \left(\frac{\left| \frac{z_1}{2+\kappa} \right| + \left| \frac{z_2}{2+\kappa} \right|}{2} \right) = \theta \left(\frac{1}{2+\kappa} m_r|_{z_2}^{z_1} \right) \\ &= e^{\sqrt{\frac{1}{2+\kappa}} \sqrt{m_r|_{z_2}^{z_1}}} = \left(e^{\sqrt{m_r|_{z_2}^{z_1}}} \right)^{\sqrt{\frac{1}{2+\kappa}}} \leq \left(e^{\sqrt{N_r|_{z_2}^{z_1}}} \right)^{\sqrt{\frac{1}{2+\kappa}}} = [\theta(N_r|_{z_2}^{z_1})]^k. \end{aligned}$$

Then, Γ satisfies the contraction (4.44) with $0 < k = \sqrt{\frac{1}{2+\kappa}} < 1$.

For $z \in [0, 1]$, we have

$$\begin{aligned} \theta \left(m_r|_{\Gamma z_2}^{\Gamma z_1} \right) &= \theta \left(\frac{\left| \frac{z_1}{2} \right| + \left| \frac{z_2}{2} \right|}{2} \right) = \theta \left(\frac{1}{2} m_r|_{z_2}^{z_1} \right) = e^{\frac{1}{2} \sqrt{m_r|_{z_2}^{z_1}}} \\ &= \left(e^{\sqrt{m_r|_{z_2}^{z_1}}} \right)^{\frac{1}{2}} \leq \left(e^{\sqrt{N_r|_{z_2}^{z_1}}} \right)^{\frac{1}{2}} = [\theta(N_r|_{z_2}^{z_1})]^k. \end{aligned}$$

Similarly, for $z \in (1, \infty)$, $2 \leq \kappa < \infty$, we get

$$\begin{aligned} \theta \left(m_r|_{\Gamma z_2}^{\Gamma z_1} \right) &= \theta \left(\frac{\left| \frac{z_1}{2^\kappa} \right| + \left| \frac{z_2}{2^\kappa} \right|}{2} \right) = \theta \left(\frac{1}{2^\kappa} m_r|_{z_2}^{z_1} \right) = e^{\frac{1}{2^\kappa} \sqrt{m_r|_{z_2}^{z_1}}} \\ &= \left(e^{\sqrt{m_r|_{z_2}^{z_1}}} \right)^{\frac{1}{2^\kappa}} \leq \left(e^{\sqrt{N_r|_{z_2}^{z_1}}} \right)^{\frac{1}{2^\kappa}} = [\theta(N_r|_{z_2}^{z_1})]^k, \end{aligned}$$

where $k = \frac{1}{2^\kappa} \in (0, 1)$. Hence, the conditions of Theorem 4.2.1 are satisfied and $z = 0$ is a U.F.P of Γ .

4.3 Application for existence a solution to a non-linear integral equation

In this section, we apply Theorem 4.2.1 to consider the following NIE:

$$\xi(r) = \int_0^t \mathcal{L}|_k^r \varpi|_{\xi(k)}^k dk, \quad r \in [0, D], \quad D > 0, \quad (4.59)$$

where $\mathcal{L} : [0, D] \times [0, D] \rightarrow \mathbb{R}$ and $\varpi : [0, D] \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

Define $\Gamma : \Omega_r \rightarrow \Omega_r$ by:

$$\Gamma \xi(r) = \int_0^t \mathcal{L}|_k^r \varpi|_{\xi(k)}^k dk. \quad (4.60)$$

Let $\Omega_r = C([0, D], \mathbb{R})$ be the space of all continuous functions on $[0, D]$ to $\mathbb{R}$ with

$$m_r|_{\xi_2}^{\xi_1} = \frac{\|\xi_1\| + \|\xi_2\|}{2} = \max_{r \in [0, D]} \left(\frac{|\xi_1(r)| + |\xi_2(r)|}{2} \right), \quad \forall \xi_1, \xi_2 \in \Omega_r.$$

Obviously, (Ω_r, m_r) is a C-R.M-M.S.

Theorem 4.3.1. Via follownig assumptions, the problem (4.59) has a unique solution in Ω_r .

(▲1) $\mathcal{L} : [0, D] \times [0, D] \rightarrow \mathbb{R}$ is a continuous such that $\max_{r, k \in [0, D]} |\mathcal{L}|_k^r| \leq 1$;

(▲2) $\varpi : [0, D] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous such that $\exists \lambda \geq 1$ and

$$\left| \varpi|_{\xi_1(r)}^r \right| + \left| \varpi|_{\xi_2(r)}^r \right| < \lambda^2 e^{-\lambda} P_r|_{\xi_2}^{\xi_1}, \quad \forall \xi_1, \xi_2 \in \Omega_r, \quad r \in [0, D]$$

where $P_r|_{\xi_2}^{\xi_1} = \max \left\{ \frac{|\xi_1| + |\xi_2|}{2}, \frac{|\xi_1| + |\Gamma \xi_1|}{2}, \frac{|\xi_2| + |\Gamma \xi_2|}{2} \right\}$;

(▲3) $\forall \xi_1, \xi_2 \in \Omega_r$, $\|\xi_1\|_\lambda = \max_{r \in [0, D]} |\xi_1(r)| e^{-\lambda r}$ and

$$m_{r\lambda}|_{\xi_2}^{\xi_1} = \frac{\|\xi_1\|_\lambda + \|\xi_2\|_\lambda}{\lambda} = \max_{r \in [0, D]} \left(\frac{|\xi_1(r)| + |\xi_2(r)|}{\lambda} \right) e^{-\lambda r}.$$

Then, $(\Omega_r, m_{r\lambda})$ is a C-R.M-M.S with $\lambda = 2$;

(▲4) $\forall \xi_1, \xi_2 \in \Omega_r$, we define $\theta \in \Theta$ by $\theta(r) = e^{\sqrt{r}}$, $r > 0$.

Proof. Define the operator $\Gamma : \Omega_r \rightarrow \Omega_r$ as in (4.60), we get

$$\begin{aligned}
|\Gamma \xi_1(r)| + |\Gamma \xi_2(r)| &= \left| \int_0^r \mathcal{L}|_k^r \varpi|_{\xi_1(k)}^k dk \right| + \left| \int_0^r \mathcal{L}|_k^r \varpi|_{\xi_2(k)}^k dk \right| \\
&\leq \int_0^r |\mathcal{L}|_k^r \left(\left| \varpi|_{\xi_1(k)}^k \right| + \left| \varpi|_{\xi_2(k)}^k \right| \right) dk \\
&\leq \int_0^r \left(\left| \varpi|_{\xi_1(k)}^k \right| + \left| \varpi|_{\xi_2(k)}^k \right| \right) dk = \int_0^r \lambda^2 e^{-\lambda} P_r|_{\xi_2(k)}^{\xi_1(k)} dk \\
&= \lambda^2 e^{-\lambda} \int_0^r e^{r\lambda} \max \left\{ \frac{|\xi_1(k)| + |\xi_2(k)|}{\lambda} e^{-r\lambda}, \frac{|\xi_1(k)| + |\Gamma \xi_1(k)|}{\lambda} e^{-r\lambda}, \frac{|\xi_2(k)| + |\Gamma \xi_2(k)|}{\lambda} e^{-r\lambda} \right\} dk \\
&\leq \lambda^2 e^{-\lambda} \int_0^r e^{r\lambda} \max \left\{ m_{r\lambda}|_{\xi_2}^{\xi_1}, m_{r\lambda}|_{\Gamma \xi_1}^{\xi_1}, m_{r\lambda}|_{\Gamma \xi_2}^{\xi_2} \right\} dk \\
&= \lambda^2 e^{-\lambda} N_r|_{\xi_2}^{\xi_1} \int_0^t e^{r\lambda} dk \\
&= \lambda^2 e^{-\lambda} N_r|_{\xi_2}^{\xi_1} \frac{1}{\lambda} (e^{\lambda r} - 1) \\
&= \lambda e^{-\lambda} N_r|_{\xi_2}^{\xi_1} e^{\lambda r} - \lambda e^{-\lambda} N_r|_{\xi_2}^{\xi_1} \\
&\leq \lambda e^{-\lambda} N_r|_{\xi_2}^{\xi_1} e^{\lambda r}.
\end{aligned}$$

Consequently, we have

$$\frac{|\Gamma \xi_1(k)| + |\Gamma \xi_2(k)|}{\lambda} e^{-\lambda r} \leq e^{-\lambda} N_r|_{\xi_2}^{\xi_1}.$$

This implies that

$$m_{r\lambda}|_{\xi_2}^{\xi_1} = \max_{r \in [0, D]} \frac{|\Gamma \xi_1(k)| + |\Gamma \xi_2(k)|}{\lambda} e^{-\lambda r} \leq e^{-\lambda} N_r|_{\xi_2}^{\xi_1}.$$

From (▲4), we get

$$e^{\sqrt{m_{r\lambda}|_{\xi_2}^{\xi_1}}} \leq e^{\sqrt{e^{-\lambda} N_r|_{\xi_2}^{\xi_1}}} = \left(e^{\sqrt{N_r|_{\xi_2}^{\xi_1}}} \right)^{e^{-\frac{\lambda}{2}}},$$

yielding that

$$\theta \left(m_{r\lambda}|_{\Gamma \xi_2}^{\Gamma \xi_1} \right) \leq \left[\theta \left(N_r|_{\xi_2}^{\xi_1} \right) \right]^K, \quad \forall_{\xi_1, \xi_2 \in \Omega_b},$$

where $K = e^{-\frac{\lambda}{2}} \in (0, 1)$ with $\lambda = 2$. Thus, all the conditions of Theorem 4.2.1 are satisfied and Γ have a U.F.P, that is a unique solution to NIE (4.59).

■

Conclusion

In this dissertation, we introduced the concept of semi α_* -admissibility w.r.t η and investigated F.Ps of Ćirić-Type rational contractions, we also proposed the concept of a G.D.P as a novel framework to establish F.P findings for both (ψ, ϕ) -integral Khan-type contraction and θ -integral contraction. Furthermore, we examined F.Ps of almost Fisher-type multivalued $\mathcal{F}_{\mathfrak{R}}$ -contractions within C-M.Ss endowed with a Γ -transitive binary relation $\mathfrak{R}$. These findings are supported by illustrative examples and an application for solving NFDEs. The extensions developed in this section yield various multivalued F.P outcomes, considerably deepening our knowledge within this domain of analysis. These contributions extended the results reported in [106, 15, 114, 112, 51, 57], as well as in other works cited in the literature. In the second part of the dissertation, we combined Wardowski's cyclic contraction with admissible multivalued Geraghty contractions to introduce the notion of η -Cyclic (α_*, β_*) -admissible contraction maps and established C.F.Ps for such contraction in C-Mb-M.Ss. We also proved F.P results for self-mappings of symmetric α - β - η - ψ -contraction and almost Hardy-Rogars-type multivalued (ψ, ϕ) -contractions in C-Mb-M.Ss. These theorems were then applied to examine the existence of solutions to pairs of ODEs and FODEs. These outcomes extended the results reported in [35, 11, 38, 19, 42, 43, 36], along with other relevant references in the literature.

In the third part, we explored the concepts of θ -contraction self-maps and almost Suzuki-Rational (ψ, ϕ) -contraction multivalued maps, and established several F.P and C.F.P results for these mappings in C-R.M-M.Ss. An application provided for solving a NIE. These outcomes serve as extensions of the work presented in [109, 66, 52, 53] among other documented studies.

Overall, this dissertation makes significant contributions to the field of F.P.T by providing new insights and methodologies that enhance both the theoretical understanding and practical application of these concepts in broader domain of M.F.P.T.

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