Existence and Stability Results for Fractional Differential Equations with Non-local Boundary Conditions

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Differential equations
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Boundary conditions

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A dissertation submitted in

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Dedicated

to

My Parents

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Preface

In fractional calculus (FC) we study differentiation and integration of non-integer order. FC has played crucial role in different fields like mechanics, electricity, economics, physics, biology, biophysics, chemistry, control theory, aerodynamics, blood flow phenomena, signal and image processing etc. It also plays vital role in the modeling of different number of phenomena, particularly in the modeling of complex media, memory dependent phenomena and many more. FC has emerged as an efficient technique in the analysis of dynamical systems. Leibniz, in the seventeenth century introduced the notation of derivative $\frac{d^n y}{du^n}$, where n is non-negative integers. Later on in 1695, first time L'Hopital, raised a question to Leibniz. what will be derivative if $n = \frac{1}{2}$? In his reply, Leibniz wrote to L'Hopital
"*This is an apparent paradox from which, one day, useful consequences will be drawn*".
After that Euler in 1730 wrote, when $G = G(\mathrm{u})$, G is function of u and the ratio $d^n(G(\mathrm{u}))$ to $d^n\mathrm{u}$ can be expressed algebraically, where n is positive integer. He also raised question. what happen if n is taken fraction ?. After few year Lagrange (1772) and Laplace (1812) worked on fractional derivatives but first time in 1819 Lacroix, developed fractional derivative

$$\frac{d^{\frac{1}{2}}G}{d\mathrm{u}^{\frac{1}{2}}} = \frac{2\sqrt{\mathrm{u}}}{\sqrt{\pi}}.$$

Later some other researchers such as Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Holmgren (1865), Letnikov (1868), Laurent (1884), Nekrassov (1888), Krug (1890), Hadamard (1892), Heaviside (1892), Pincherle (1902), Weyl (1919), Marchaud (1927), Davis (1924), Zygmund (1935), Love (1938), Kober (1940), Widder (1941). Riesz (1949). Feller (1952), Oldham and Spanier (1974), Samko and Kilbas (1993). Podlubny (1999) and Trujillo and Srivastava (2006) etc., worked in this field and developed some useful fractional derivatives. Most common fractional derivatives in fractional calculus are Riemann-Liouville (R-L). Hadamard. Caputo, Weyl and Grunwald-Letnikov (G-L) etc.

In this thesis, six chapters are presented. Chapter 1 presents some basic definitions and results which are related to the special functions (Gamma function, Beta function and M-L function), fractional derivatives and integrals (Caputo, R-L, CF, AB). The statements of some famous fixed point theorems and basic results are given, which provide the base for existence and uniqueness results in FDEs. Also literature review and research methodology are discussed in this chapter.

In Chapter 2, we discuss non-linear fractional boundary value problem (BVP) of order $\sigma \in (1, 2]$ involving AB-Caputo derivative with non-separated boundary conditions. We find existence, uniqueness and stability of solution in this chapter. For existence results, we use Krasnoselskii's fixed point theorem. We obtain unique solution via Banach contraction principle. The criteria for Hyers-Ulam stability is presented to get stable solution. For validity of results, an example is provided. This work is published in journal **Fractals, Vol. 29, No. 5 (2021) 2140016.**

In Chapter 3, we investigate non-linear fractional BVP of order $\sigma \in (2, 3]$ involving AB-Caputo derivative with integral type boundary conditions. Existence, uniqueness and stability of solution are derived in this chapter. Existence of the solution is obtained via Krasnoselskii's and Schauder fixed point theorems. To get unique solution, we apply Banach contraction principle. To discuss the stability of given AB-Caputo fractional BVP, the criteria for Hyers-Ulam stability is used. In the end, example is provided for the validity of results. This work is submitted in the journal **Chaos, Solitons & Fractals.**

In Chapter 4, we discuss multi-term AB-Caputo fractional BVP of order $\sigma \in (0, 1)$ involving non-local boundary conditions with different cases ($\delta_1^2 - 4\delta_0\delta_2 = 0$, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$). We find existence results via Krasnoselskii's fixed point theorems. We obtain unique solution via Banach contraction principle for all cases ($\delta_1^2 - 4\delta_0\delta_2 = 0$, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$). In the last, three examples are given, for validity of our work. The work is published in journal **Fractals, Vol. 31, No. 2 (2023) 2340024.**

In Chapter 5, we present non-linear coupled FDEs with non-local boundary conditions of order $\sigma \in (1, 2]$ involving AB-Caputo fractional derivative. Existence of the solution of given AB-Caputo fractional coupled BVP is obtained via Krasnoselskii's fixed point theorem. We obtain unique solution via Banach contraction principle. To discuss the stability of AB-Caputo fractional coupled BVP, the criteria for Hyers-Ulam stability is used. Examples are also given to validate the results which are given in this chapter. This work is published in journal **Fractals, Vol. 31, No. 2 (2023) 2340023.**

In Chapter 6, we explore AB-piecewise fractional differential system (FDS) of order $\sigma \in (0,1)$ with initial conditions. Existence results of AB-piecewise FDS is derived by using Schauder fixed point theorem. To get unique solution, we apply Banach contraction principle. To analyze the stability of the solution of AB-piecewise FDS, the Hyers-Ulam stability is discussed. Numerical scheme is presented, which is based on Euler's formula to obtain the approximate solution In the last, example is also provided in which we find unique and stable solution. Further we calculate approximate solution with the help of Euler's formula. This work is submitted in **Alexandria Engineering Journal.**

Table of Contents

1. Preliminaries

2. Existence and Stability Results for FDEs Involving Non-Separated Boundary Conditions

3. Existence and Stability Results of FDEs Involving Integral Type Boundary Conditions

4. Existence Results for Multi-term AB-Caputo Fractional BVP

5. Existence and Stability Results of Coupled System

6. Mathematical Analysis of Dynamical Systems Involving AB-Piecewise Fractional Derivative

Chapter 1

Preliminaries

In this chapter, six sections are presented. In Section 1, some special function such as Gamma function, Beta function and M-L function are discussed. Section 2 contains some basic definitions and results which are related to different fractional derivatives and fractional integrals. In Section 3, Fredholm and Volterra integral equations with different kinds are given. In Section 4, the statement of some famous fixed point theorems are given. In Section 5, literature review is presented. In Section 6, research methodology is provided.

1.1 Some Special Function

This section presents Gamma function, Beta function, relation between Beta and Gamma functions, M-L function and generalized M-L function.

Definition 1.1. [1] The Gamma function is defined as

$$\Gamma(\mathrm{u}) = \int\limits_{0}^{\infty} \ell^{\mathrm{u}-1} e^{-\ell} d\ell,$$

where $0 < \mathrm{u} < \infty$.

Definition 1.2. [1] The Beta function is defined as

$$\beta(\mathrm{u}, \vartheta) = \int\limits_{0}^{1} \ell^{\mathrm{u}-1} (1-\ell)^{\vartheta-1} d\ell, \quad \mathrm{u}, \vartheta > 0.$$

The relation between Beta and Gamma functions is given as

$$\beta(\mathrm{u},\vartheta)=\frac{\Gamma(\mathrm{u})\Gamma(\vartheta)}{\Gamma(\mathrm{u}+\vartheta)},\ \mathrm{u},\vartheta>0.$$

Definition 1.3. [2] M-L function is generalization of the exponential which was introduced by Mittag-Leffler in 1903 defined as

$$E_{\sigma}(\mathrm{u})=\sum_{\xi=0}^{\infty}\frac{\mathrm{u}^{\xi}}{\Gamma(\sigma\xi+1)},\ \Re(\sigma)>0,\ \mathrm{u}\in\mathbb{C}. \tag{1.1}$$

After that Wiman [3] in 1905 generalized (1.1), as:

$$E_{\sigma,\gamma}(\mathrm{u})=\sum_{\xi=0}^{\infty}\frac{\mathrm{u}^{\xi}}{\Gamma(\sigma\xi+\gamma)},\ \Re(\sigma)>0,\ \Re(\gamma)>0,\ \mathrm{u}\in\mathbb{C}.$$

Few year later Prabhakar [4], in 1971 presented generalized M-L function and defined as

$$E_{\sigma,\gamma}^{\hbar}(\mathrm{u})=\sum_{\xi=0}^{\infty}\frac{(\hbar)_{\mathrm{p}}\mathrm{u}^{\xi}}{\mathrm{p}!\Gamma(\sigma\xi+\gamma)},\ \Re(\sigma)>0,\ \Re(\gamma)>0,\ \mathrm{u}\in\mathbb{C},$$

where $(\kappa)_{\mathrm{p}}$ is Pochhammer symbol and represented as

$$(\hbar)_{\mathrm{p}}=\hbar(\hbar+1)(\hbar+2)..........(\hbar+\mathrm{p}-1)\text{ with }(\hbar)_0=1,\ \mathrm{p}\in\mathbb{N}.$$

1.2 Fractional Calculus

This section presents some basic definitions, lemmas and propositions of various fractional derivatives and fractional integrals such as Caputo, R-L, CF-Caputo, CF-Riemann Liouville (CF-R–L), AB-Caputo, AB-Riemann–Liouville (AB-R L) and AB-Piecewise derivatives are presented.

Definition 1.4. [5] Suppose a finite interval $\Omega=[a,b]$ $(-\infty<a<b<\infty)$ on real axis $\mathbb{R}$, then left and right R-L fractional derivatives are defined as

$$\left({}^{RL}{}_{a}D^{\sigma}G\right)(\ell)=\frac{1}{\Gamma(\mathrm{n}-\sigma)}\left(\frac{d}{d\ell}\right)^{\mathrm{n}}\int_{a}^{\ell}\left(\frac{G(\varsigma)}{(\ell-\varsigma)^{\sigma-\mathrm{n}+1}}\right)d\varsigma,\ \Re(\sigma)>0,\ \ell>a,$$

and

$$\left({}^{RL}D_{b}^{\sigma}G\right)(\ell)=\frac{1}{\Gamma(\mathfrak{n}-\sigma)}\left(-\frac{d}{d\ell}\right)^{\mathfrak{n}}\int\limits_{\ell}^{b}\left(\frac{G(\varsigma)}{(\varsigma-\ell)^{\sigma-\mathfrak{n}+1}}\right)d\varsigma,\quad \Re(\sigma)>0,\ \ell<b.$$

where if $\sigma\in\mathbb{N}_0$ then $\mathfrak{n}=\sigma$ and if $\sigma\notin\mathbb{N}_0$ then $\mathfrak{n}=[\Re(\sigma)]+1$.

Definition 1.5. [5] The left and right R-L fractional integrals are defined as

$$\left({}_{a}^{RL}I^{\sigma}G\right)(\ell)=\frac{1}{\Gamma(\sigma)}\int\limits_{a}^{\ell}\left(\frac{G(\varsigma)}{(\ell-\varsigma)^{1-\sigma}}\right)d\varsigma,\quad \Re(\sigma)>0,\ \ell>a,$$

and

$$\left({}^{RL}I_{b}^{\sigma}G\right)(\ell)=\frac{1}{\Gamma(\sigma)}\int\limits_{\ell}^{b}\left(\frac{G(\varsigma)}{(\varsigma-\ell)^{1-\sigma}}\right)d\varsigma,\quad \Re(\sigma)>0,\ \ell<b.$$

Definition 1.6. [5] Suppose a finite interval $\Omega=[a,b]$ $(-\infty<a<b<\infty)$ on real axis $\mathbb{R}$, then left and right Caputo fractional derivatives are defined as

$$\left({}_{a}^{C}D^{\sigma}G\right)(\ell)=\left[{}^{RL}D_{a^+}^{\sigma}\left(G(\varsigma)-\sum_{\kappa=0}^{\mathfrak{n}-1}\frac{G^{\kappa}(a)}{\kappa!}(\varsigma-a)^{\kappa}\right)\right](\varsigma),$$

and

$$\left({}^{C}D_{b}^{\sigma}G\right)(\ell)=\left[{}^{RL}D_{b^-}^{\sigma}\left(G(\varsigma)-\sum_{\kappa=0}^{\mathfrak{n}-1}\frac{G^{\kappa}(b)}{\kappa!}(b-\varsigma)^{\kappa}\right)\right](\varsigma),$$

where if $\sigma\in\mathbb{N}_0$ then $\mathfrak{n}=\sigma$ and if $\sigma\notin\mathbb{N}_0$ then $\mathfrak{n}=[\Re(\sigma)]+1$.

Particularly, when $0<\Re(\sigma)<1$ and $G(\ell)\in AC[a,b]$ (Absolutely continuous function) then we have

$$\left({}_{a}^{C}D^{\sigma}G\right)(\ell)=\frac{1}{\Gamma(1-\sigma)}\int\limits_{a}^{\ell}\left(\frac{G'(\varsigma)}{(\ell-\varsigma)^{\sigma}}\right)d\varsigma. \tag{1.2}$$

and

$$\left({}^{C}D_{b}^{\sigma}G\right)(\ell)=-\frac{1}{\Gamma(1-\sigma)}\int\limits_{\ell}^{b}\left(\frac{G'(\varsigma)}{(\varsigma-\ell)^{\sigma}}\right)d\varsigma. \tag{1.3}$$

Now by replacing $(\ell-\varsigma)^{-\sigma}$ with $\exp\left(-\sigma\frac{\ell}{1-\sigma}\right)$ and $\Gamma(1-\sigma)$ with $\frac{1-\sigma}{M(\sigma)}$ in (1.2) and (1.3) then we get the following definition.

Definition 1.7. [6, 7] Suppose $G \in H^1(a,b)$ and $\sigma \in (0,1)$ then left and right CF-Caputo fractional derivatives are defined as

$$\left({}^{CFC}{}_{a}D^{\sigma}G\right)(\ell) = \frac{M(\sigma)}{1-\sigma}\int_{a}^{\ell}\left(G'(\varsigma)\exp\left[-\sigma\frac{(\ell-\varsigma)}{1-\sigma}\right]\right)d\varsigma$$

and

$$\left({}^{CFC}D_{b}^{\sigma}G\right)(\ell) = -\frac{M(\sigma)}{1-\sigma}\int_{\ell}^{b}\left(G'(\varsigma)\exp\left[-\sigma\frac{(\varsigma-\ell)}{1-\sigma}\right]\right)d\varsigma,$$

where $M(\sigma) = 1-\sigma+\frac{\sigma}{\Gamma(\sigma)}$ is a normalization function and satisfying $M(0) = M(1) = 1$.

Definition 1.8. [7] Suppose $G \in H^1(a,b)$, $b > a$ and $\sigma \in (0,1)$ then left and right CF-R L fractional derivatives are defined as

$$\left({}^{CFR}{}_{a}D^{\sigma}G\right)(\ell) = \frac{M(\sigma)}{1-\sigma}\frac{d}{d\ell}\int_{a}^{\ell}\left(G(\varsigma)\exp\left[-\sigma\frac{(\ell-\varsigma)}{1-\sigma}\right]\right)d\varsigma$$

and

$$\left({}^{CFR}D_{b}^{\sigma}G\right)(\ell) = -\frac{M(\sigma)}{1-\sigma}\frac{d}{d\ell}\int_{\ell}^{b}\left(G(\varsigma)\exp\left[-\sigma\frac{(\varsigma-\ell)}{1-\sigma}\right]\right)d\varsigma.$$

Definition 1.9. [7] The CF-fractional integrals are defined as

$$\left({}^{CF}{}_{a}I^{\sigma}G\right)(\ell) = \frac{1-\sigma}{M(\sigma)}G(\ell) + \frac{\sigma}{B(\sigma)}\int_{a}^{\ell}G(\varsigma)\,d\varsigma$$

and

$$\left({}^{CF}I_{b}^{\sigma}G\right)(\ell) = \frac{1-\sigma}{M(\sigma)}G(\ell) + \frac{\sigma}{B(\sigma)}\int_{\ell}^{b}G(\varsigma)\,d\varsigma.$$

Next, we define left and right AB-Caputo and AB-Riemann-Liouville (AB-R L) fractional derivatives using generalized Mittag-Leffler.

Definition 1.10. [8, 9] Let $G \in H^1(a,b)$, $b > a$ and $\sigma \in (0,1)$ then the left and right AB-Caputo fractional derivatives are defined as

$$ {}^{ABC}_{\ \ a}D^{\sigma}(G(\ell)) = \frac{AB(\sigma)}{1-\sigma}\int_{a}^{\ell} G'(\varsigma)E_{\sigma}\left[-\sigma\frac{(\ell-\varsigma)^{\sigma}}{1-\sigma}\right]d\varsigma $$

and

$$ {}^{ABC}D_{b}^{\sigma}(G(\ell)) = -\frac{AB(\sigma)}{1-\sigma}\int_{\ell}^{b} G'(\varsigma)E_{\sigma}\left[-\sigma\frac{(\varsigma-\ell)^{\sigma}}{1-\sigma}\right]d\varsigma. $$

where $M(\sigma) = 1 - \sigma + \frac{\sigma}{\Gamma(\sigma)}$ is a normalization function and satisfying $M(0) = M(1) = 1$.

Definition 1.11. [8, 9] Let $G \in H^{1}(a, b)$, $b > a$ and $\sigma \in (0, 1)$ then the left and right new AB-R–L fractional derivatives are defined as

$$ {}^{ABR}_{\ \ a}D^{\sigma}(G(\ell)) = \frac{AB(\sigma)}{1-\sigma}\frac{d}{d\ell}\int_{a}^{\ell} G(\varsigma)E_{\sigma}\left[-\sigma\frac{(\ell-\varsigma)^{\sigma}}{1-\sigma}\right]d\varsigma $$

and

$$ {}^{ABR}D_{b}^{\sigma}(G(\ell)) = -\frac{AB(\sigma)}{1-\sigma}\frac{d}{d\ell}\int_{\ell}^{b} G(\varsigma)E_{\sigma}\left[-\sigma\frac{(\varsigma-\ell)^{\sigma}}{1-\sigma}\right]d\varsigma. $$

Definition 1.12. [8, 9] Now we define left and right AB- fractional integrals as,

$$ {}^{AB}_{\ a}I^{\sigma}(G(\ell)) = \frac{1-\sigma}{AB(\sigma)}G(\ell) + \frac{\sigma}{AB(\sigma)}\ {}_{a}I^{\sigma}(G(\ell)) $$

and

$$ {}^{AB}I_{b}^{\sigma}(G(\ell)) = \frac{1-\sigma}{AB(\sigma)}G(\ell) + \frac{\sigma}{AB(\sigma)}\ I_{b}^{\sigma}(G(\ell)). $$

Lemma 1.1. [9] For $0 < \sigma < 1$, we have

$$ ({}^{AB}_{\ a}I^{\sigma}\,{}^{ABR}_{\ \ a}D^{\sigma})(G(\ell)) = G(\ell) $$

and

$$ ({}^{AB}I_{b}^{\sigma}\ {}^{ABR}D_{b}^{\sigma})(G(\ell)) = G(\ell). $$

Lemma 1.2. [10] For $0 < \sigma < 1$, we have

$$ ({}^{AB}_{\ a}I^{\sigma}\,{}^{ABC}_{\ \ a}D^{\sigma})(G(\ell)) = G(\ell) - G(a) $$

and

$$({}^{AB}I_b^{\sigma}\ {}^{ABC}D_b^{\sigma})(G(\ell)) = G(\ell) - G(b).$$

Proposition 1.1. [11] For $G(\ell)$ defined on $[a,b]$ and $\sigma \in (\kappa, \kappa+1]$ for some $\kappa \in \mathbb{N}_0$, we have

1. $({}^{ABR}_{\quad a}D^{\sigma}\,{}^{AB}_{\ a}I^{\sigma})(G(\ell)) = G(\ell)$.
2. $({}^{AB}_{\ a}I^{\sigma}\,{}^{ABR}_{\quad a}D^{\sigma})(G(\ell)) = G(\ell) - \sum\limits_{\xi=0}^{\kappa-1}\frac{G^{\xi}(a)}{\xi!}(\ell-a)^{\xi}$.
3. $({}^{AB}_{\ a}I^{\sigma}\,{}^{ABC}_{\quad a}D^{\sigma})(G(\ell)) = G(\ell) - \sum\limits_{\xi=0}^{\kappa}\frac{G^{\xi}(a)}{\xi!}(\ell-a)^{\xi}$.

Example 1.1. [12] In the following example, we find AB-Caputo derivative of $\sin \ell$. Consider

$$\begin{aligned}
{}^{ABC}_{\quad 0}D^{\frac{1}{2}}(\sin(\ell)) &= \frac{AB\left(\frac{1}{2}\right)}{1-\frac{1}{2}}\int_0^{\ell}\cos\varsigma E_{\frac{1}{2}}\left(\frac{-\frac{1}{2}(\ell-\varsigma)^{\frac{1}{2}}}{1-\frac{1}{2}}\right)d\varsigma \\
&= 2AB\left(\frac{1}{2}\right)\int_0^{\ell}\cos\varsigma E_{\frac{1}{2}}\left(-(\ell-\varsigma)^{\frac{1}{2}}\right)d\varsigma, \\
&= 2\int_0^{\ell}\cos\varsigma\ E_{\frac{1}{2}}\left(-(\ell-\varsigma)^{\frac{1}{2}}\right)d\varsigma \\
&= \frac{2}{\Gamma\left(\frac{\hbar}{2}+1\right)}\int_0^{\ell}\cos\varsigma\ \sum_{\hbar=0}^{\infty}(-1)^{\hbar}(\ell-\varsigma)^{\frac{\hbar}{2}}d\varsigma \\
&= \frac{2}{\Gamma\left(\frac{\hbar}{2}+1\right)}\sum_{\hbar=0}^{\infty}(-1)^{\hbar}\left[\int_0^{\ell}\left\{1-\frac{\ell^2}{2!}+\frac{\ell^4}{4!}-\ldots\ldots\right\}(\ell-\varsigma)^{\frac{\hbar}{2}}d\varsigma\right] \\
&= \frac{2}{\Gamma\left(\frac{\hbar}{2}+1\right)}\sum_{\hbar=0}^{\infty}(-1)^{\hbar}\left[\int_0^{\ell}\left\{(\ell-\varsigma)^{\frac{\hbar}{2}}-\frac{\ell^2}{2!}(\ell-\varsigma)^{\frac{\hbar}{2}}+\frac{\ell^4}{4!}(\ell-\varsigma)^{\frac{\hbar}{2}}-\ldots\ldots\right\}d\varsigma\right] \\
&= 2\sum_{\hbar=1}^{\infty}(-1)^{(\hbar-1)}(\ell)^{2\hbar-1}E_{\frac{1}{2},2\hbar}(-\ell)^{\frac{1}{2}}.
\end{aligned}$$

Where we take $\sigma = \frac{1}{2}$, and $AB\left(\frac{1}{2}\right) = 1$. Similarly we find AB-Caputo derivative of $\cos \ell$ and $(\exp(\ell))$,

$${}^{ABC}_{\quad 0}D^{\frac{1}{2}}(\cos(\ell)) = 2\sum_{\hbar=1}^{\infty}(-1)^{(\hbar)}(\ell)^{2\hbar}E_{\frac{1}{2},2\hbar+1}(-\ell)^{\frac{1}{2}},\ \ {}^{ABC}_{\quad 0}D^{\frac{1}{2}}(\exp\ell) = 2\sum_{\hbar=1}^{\infty}\ell^{\hbar}E_{\frac{1}{2},\hbar+1}(-\ell)^{\frac{1}{2}}.$$

Definition 1.13. [13] Suppose G is differentiable function then the AB-piecewise fractional derivative having classical and M-L kernel is defined as

$$ {}^{PAB}{}_{0}D_{\ell}^{\sigma}(G(\ell)) = \begin{cases} G^{/}(\ell) \text{ if } \quad 0 \leq \ell \leq \ell_1, \\ {}^{ABC}{}_{0}D_{\ell}^{\sigma}(G(\ell)) \text{ if } \ \ell_1 \leq \ell \leq T, \end{cases} $$

which means ${}^{PAB}{}_{0}D_{\ell}^{\sigma}(G(\ell))$ is a classical derivative if $0 \leq \ell \leq \ell_1$ and AB-fractional derivative if $\ell_1 \leq \ell \leq T$.

Definition 1.14. [13] Suppose G is continuous function then the AB-piecewise fractional integral having classical and M-L kernel is defined as

$$ {}^{PAB}{}_{0}I_{\ell}^{\sigma}(G(\ell)) = \begin{cases} \int\limits_{0}^{\ell} G(\varsigma)\, d\varsigma \text{ if } \ 0 \leq \ell \leq \ell_1, \\ \frac{1-\sigma}{AB(\sigma)}G(\ell) + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell} G(\varsigma)(\ell-\varsigma)^{\sigma-1}d\varsigma \quad \text{if } \ \ell_1 \leq \ell \leq T, \end{cases} $$

which means ${}^{PAB}{}_{0}I_{\ell}^{\sigma}(G(\ell))$ is a classical integral if $0 \leq \ell \leq \ell_1$ and AB-fractional integral if $\ell_1 \leq \ell \leq T$.

1.3 Fredholm and Volterra Integral Equations

This section contains Fredholm and Volterra integral equations. In integral equation, unknown determined function $\mathfrak{u}(\ell)$ appears under the integral sign and we write integral equation in the typical form as follows

$$ \mathfrak{u}(\ell) = G(\ell) + \int\limits_{h(\varsigma)}^{m(\varsigma)} Q(\ell,\varsigma)\mathfrak{u}(\varsigma)d\varsigma. $$

where $h(\varsigma)$ and $m(\varsigma)$ are limits of integration, $G(\ell)$ is known function and $Q(\ell,\varsigma)$ represents the kernel.

Definition 1.15. [14] In Fredholm integral equation, limits of integration are constant $(h(\varsigma) = a,\ m(\varsigma) = b)$ and given as

$$\Phi(\ell)\mathrm{u}(\ell) = G(\ell) + \lambda\int_a^b Q(\ell,\varsigma)\mathrm{u}(\varsigma)d\varsigma,$$

where λ is parameter and $a \leq \ell, \varsigma \leq b$.

For first kind Fredholm integral equation, we take $\Phi(\ell) = 0$, so

$$G(\ell) + \lambda\int_a^b Q(\ell,\varsigma)\mathrm{u}(\varsigma)d\varsigma = 0.$$

For second kind Fredholm integral equation, we take $\Phi(\ell) = 1$, so

$$\mathrm{u}(\ell) = G(\ell) + \lambda\int_a^b Q(\ell,\varsigma)\mathrm{u}(\varsigma)d\varsigma.$$

Definition 1.16. [14] In Volterra integral equation, limits of integration are functions of ℓ, given as

$$\Phi(\ell)\mathrm{u}(\ell) = G(\ell) + \lambda\int_a^\ell Q(\ell,\varsigma)\mathrm{u}(\varsigma)d\varsigma,$$

where $\ell \in [a, b]$ and when we take $\varsigma > \ell$ then $Q(\ell, \varsigma)$ vanishes.

For first kind Volterra integral equation, we take $\Phi(\ell) = 0$, so

$$G(\ell) + \lambda\int_a^\ell Q(\ell,\varsigma)\mathrm{u}(\varsigma)d\varsigma = 0.$$

For second kind Volterra integral equation, we take $\Phi(\ell) = 1$, so

$$\mathrm{u}(\ell) = G(\ell) + \lambda\int_a^\ell Q(\ell,\varsigma)\mathrm{u}(\varsigma)d\varsigma.$$

1.4 Fixed Points Results

In this section, statements of some famous fixed point theorems and results are presented, which are main tools for uniqueness and existence results in FDEs. Banach Contraction theorem is used to find unique fixed point. Schauder fixed point theorem and Krasnoselskii's fixed point theorem are used to find atleast one fixed point. For compactness, we use Arzela-Ascoli theorem.

Theorem 1.1. [15] (**Banach Contraction Principle**) Suppose $X = (X, d)$ is complete metric space and $\mathcal{F} : X \to X$ is contraction i.e there exists $\delta \in (0, 1)$ such that

$$d(\mathcal{F}\mathrm{u}, \mathcal{F}\vartheta) \leq \delta d(\mathrm{u}, \vartheta),$$

for all $\mathrm{u}, \vartheta \in X$. Then $\mathcal{F}$ has a unique fixed point.

Theorem 1.2. [15] (**Schauder fixed point theorem**) Suppose Ω is bounded, convex, closed and nonempty subset of Banach space X. If the mapping $\mathcal{F}$ from Ω into Ω, is continuous such that $\mathcal{F}\Omega \subset X$, $\mathcal{F}\Omega$ is relatively compact so $\mathcal{F}$ has atleast one fixed point in Ω.

Theorem 1.3. [15] (**Krasnoselskii's fixed point theorem**) Suppose Ω is bounded, convex, closed and nonempty subset of Banach space X. Assume $\mathcal{F}_1, \mathcal{F}_2$ are two operators such that

(i). $\mathcal{F}_1\mathrm{u}_1 + \mathcal{F}_2\mathrm{u}_2 \in X$, whenever $\mathrm{u}_1, \mathrm{u}_2 \in X$.

(ii). $\mathcal{F}_2$ is a contraction.

(iii). $\mathcal{F}_1$ is continuous and compact.

Then there exist $\mathrm{u}_3 \in X$ such that $\mathrm{u}_3 = \mathcal{F}_1\mathrm{u}_3 + \mathcal{F}_2\mathrm{u}_3$.

Lemma 1.3. [15] (**Arzela-Ascoli theorem**) Suppose $J = [a, b] = (-\infty < a < b < \infty)$ is finite interval on $\mathbb{R}$. A subset Ω in $C(J, \mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on J.

1.5 Literature Review

FC helps to solve numerous problems including special functions of mathematical analysis as well as their generalizations and extensions in one and more than one variables. In FC, various type of fractional derivatives are introduced. Some results and applications related to FC can be seen in [5, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

In 2015, Caputo and Fabrizio (CF) in [6] defined a fractional derivative involving exponential kernel. It consists of the temporal and the spatial variables representations. The temporal representation is related with time variables, in this case where real powers are involved in the solution of fractional derivative will be changed into integer powers with some simplifications. For this framework, the Laplace transform is suitable to solve. The spatial variables representation involves non-local fractional derivatives. For this framework, Fourier transform is convenient. After that in 2016, Atangana and Baleanu (AB) [8], have modified CF and present a new type fractional derivative having kernel involving generalized M-L function. This type of derivative is said to be AB-fractional derivative and used in the of both sense R-L and Caputo derivative. Fractional integration and differentiation with M-L kernel have been considered the most important and powerful tool in mathematics, which enable us to reproduce the cross effect into mathematical models describing real world problems. Some hidden aspects of non-local dynamical systems can easily be studied if we blend classical fractional derivatives with AB-derivative. Some properties and applications related to CF and AB derivatives can be seen in [7, 9, 10, 11, 29, 30, 31, 32, 33, 34].

Existence and uniqueness are two main aspects in the theory of fractional differential equations (FDEs). To obtain existence and uniqueness results of FDEs involving various initial and boundary conditions like periodic, anti-periodic, non-separated, integral, multi-points and multi-strip etc., different fixed point theorems are used. Schaefer, Schauder, Krasnoselskii's, and Leray-Schauder fixed point theorems are used to get atleast one solution of FDEs. It guarantees that solution of FDEs with in function spaces like the space of continuous function, Banach spaces, spaces with differentiable functions, Lebesgue integrable function space, Sobolev spaces and many more. To obtain unique solution of FDEs, famous Banach contraction principle is used. It means no other than one solution for same initial and boundary condition exists. Many researchers worked on existence and uniqueness theory of FDEs involving different boundary conditions, which can be seen in [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46].

Stability analysis is another key characteristic of FDEs. In recent times, stability of the solutions of FDEs has gained much importance as compare to existence and uniqueness. A stable solution provides important information under the given domain but on the hand unstable solution may not give the required information. First time Ulam [47], raised the question about stability of additive mappings. Later Hyers [48], developed a method to obtain the criteria for the stability of these mappings. Rassias [49], explained the method to determine the conditions for both linear and nonlinear mappings. In [50], Rus presented stability theory of differential equations of four different types such as Hyers-Ulam (HU) stability, HU-Rassias stability, generalized HU stability and generalized HU-Rassias stability. Some results which are related to the stability of the solution of FDEs are given in [51, 52, 53, 54, 55, 56, 57, 58].

1.6 Research Methodology

To find solution of linear and non-linear FDEs and fractional differential systems (FDSs) with non-local boundary conditions, AB-Caputo fractional derivative and AB-Piecewise fractional derivative are used. Schauder and Krasnoselskii's fixed point theorems are applied to get existence of the solution of FDEs and FDSs. To obtain unique solution of FDEs and FDSs, classical Banach contraction principle is used. To investigate the stability of given fractional BVPs with non-local boundary conditions, the HU stability is discussed. The analysis of approximate and exact solutions with errors of nonlinear integral equations is also elaborated with graphs by using various numerical technique.

Chapter 2

Existence and Stability Results for FDEs Involving Non-Separated Boundary Conditions

2.1 Introduction

We discuss FDEs involving non-separated boundary condition in this chapter. Non-separated BVP are those problem in which boundary conditions can not be expressed in term of separate boundary points. Non-separated BVPs required some special technique to be get solved. These boundary conditions are used in various BVPs of differential equations (DE), partial differential equations (PDEs) and FDEs. For example, Ahmad [59], presented the following FDE

$$\left({}^{C}D^{\sigma}\mathrm{u}\right)(\ell) = G(\ell, \mathrm{u}(\ell)), \ \ell \in [0, \mathsf{T}], \ \mathsf{T} > 0,$$

with non-separated boundary conditions

$$\mathrm{u}(0) = \xi_1 \mathrm{u}(\mathsf{T}) + \mathrm{p}_1, \ \mathrm{u}^{'}(0) = \xi_2 \mathrm{u}^{'}(\mathsf{T}) + \mathrm{p}_2.$$

Recently some researchers develop some new results in existence and stability theory of FDEs with non-separated boundary conditions involving various fractional operators. They used different fixed point theorems for existence results of FDEs and presented numerous conditions for stability analysis. For instance, Zhou and Liu [60], presented existence results of given BVP under the assumptions of Pettis integrability for the class of BVP fractional differential inclusions. Their analysis relied on Monch fixed point theorem, which they used the measures of weak noncompactness. After that Samei et al. [61], investigated nonlinear q-integro multi-term FDEs with non-separated boundary conditions including Caputo fractional derivative. They used multivalued map and derived existence results of given BVP. For this, they applied Covitz–Nadler and Leray–Schauder fixed point theorems to obtain solutions of given inclusion problems. They also discussed different results of convex and nonconvex multifunctions. Later Ibnelazyz et al. [62], explored uniqueness and existence results of integro-differential equations. They used norm to develop the existence results via Krasnoselskii's fixed point theorem. They obtained uniqueness via Banach contraction principle. In same year, Treanbucha and Sudsutad [63], discussed stability, uniqueness and existence results for impulsive FDEs involving Caputo proportional derivative. They obtain stability results of given problem by using Ulam stability. For existence results, they applied Schaefer fixed point theorem. They applied Banach contraction principle for uniqueness of the solution. Some other related results for FDEs involving non-separated boundary conditions are discussed with the help of different fixed point theorems can be seen in [64, 65, 66, 67, 68, 69].

For motivation of above work, we discuss the following non-linear AB-Caputo fractional BVP

$$\left({}^{ABC}{}_{0}D^{\sigma}\mathrm{u}\right)(\ell) = G(\ell, \mathrm{u}(\ell)), \tag{2.1}$$

with non-separated boundary conditions

$$\begin{cases} \mathrm{u}(0) = \xi_1 \mathrm{u}(\mathrm{T}) + \mathrm{p}_1 \int\limits_0^{\mathrm{T}} E(\varsigma, \mathrm{u}(\varsigma))d\varsigma, \\ \\ \mathrm{u}'(0) = \xi_2 \mathrm{u}'(\mathrm{T}) + \mathrm{p}_2 \int\limits_0^{\mathrm{T}} \eta(\varsigma, \mathrm{u}(\varsigma))d\varsigma. \end{cases} \tag{2.2}$$

Where $G, E, \eta : [0, \mathrm{T}] \times \mathbb{R} \to \mathbb{R}$ are all continuous functions and $1 < \sigma \leq 2$, $\ell \in [0, \mathrm{T}]$, $\mathrm{T} > 0$, $\xi_1, \xi_2, \mathrm{p}_1, \mathrm{p}_2 > 0$.

We assume the following assumptions in this chapter, which are useful in our next discussion.

$$\begin{cases} (i) \quad |G(\ell,\mathfrak{u}(\ell)) - G(\ell,\vartheta(\ell))| \le \mathfrak{L}_1 \left|\mathfrak{u}(\ell) - \vartheta(\ell)\right|. \\ (ii) \ |E(\ell,\mathfrak{u}(\ell)) - E(\ell,\vartheta(\ell))| \le \mathfrak{L}_2 \left|\mathfrak{u}(\ell) - \vartheta(\ell)\right|. \\ (iii) \ |\eta(\ell,\mathfrak{u}(\ell)) - \eta(\ell,\vartheta(\ell))| \le \mathfrak{L}_3 \left|\mathfrak{u}(\ell) - \vartheta(\ell)\right|. \end{cases} \tag{H1}$$

$$\begin{cases} (i) \ |G(\ell,\mathfrak{u}(\ell))| \le \mathfrak{q}_1(\ell) + c_1 \left|\mathfrak{u}(\ell)\right|^{\hbar_1}. \\ (ii) \ |E(\ell,\mathfrak{u}(\ell))| \le \mathfrak{q}_2(\ell) + c_2 \left|\mathfrak{u}(\ell)\right|^{\hbar_2}. \\ (iii) \ |\eta(\ell,\mathfrak{u}(\ell))| \le \mathfrak{q}_3(\ell) + c_3 \left|\mathfrak{u}(\ell)\right|^{\hbar_3}. \end{cases} \tag{H2}$$

$$\mathfrak{S}_1 = \left\{ \begin{array}{c} \frac{\mathfrak{L}_1}{AB(\sigma-1)}[\mathsf{T}\{|\mathfrak{z}_1\xi_1| + 1)\} \\ +\frac{1}{\Gamma(\sigma+1)}\{(\mathsf{T})^{\sigma}(|\xi_2\mathfrak{z}_2(1+\xi_1)|\sigma + |\mathfrak{z}_1\xi_1| + 1)\}] \\ +\mathfrak{L}_2\left|\mathfrak{z}_1\mathfrak{p}_1\right|\mathsf{T} + \mathfrak{L}_3\left|\mathfrak{p}_2\mathfrak{z}_2(1+\xi_1)\right|\mathsf{T}^2 \end{array} \right\} < 1. \tag{H3}$$

$$\mathfrak{S}_2 = \left\{ \begin{array}{c} \frac{\mathfrak{L}_1}{AB(\sigma-1)}[\{\mathsf{T}(|\xi_2\mathfrak{z}_2(1+\xi_1)| + |\mathfrak{z}_1\xi_1| + 1)\} \\ +\frac{1}{\Gamma(\sigma+1)}\{(\mathsf{T})^{\sigma}(|\xi_2\mathfrak{z}_2(1+\xi_1)|\sigma + |\mathfrak{z}_1\xi_1| + 1)\}] \\ +\mathfrak{L}_2\left|\mathfrak{z}_1\mathfrak{p}_1\right|\mathsf{T} + \mathfrak{L}_3\left|\mathfrak{p}_2\mathfrak{z}_2(1+\xi_1)\right|\mathsf{T}^2 \end{array} \right\} < 1. \tag{H4}$$

Where $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3 \ge 0$, $c_1, c_2, c_3 \ge 0$, $0 < \hbar_1, \hbar_2, \hbar_3 < 1$ and also $\mathfrak{q}_i \in C([0,\mathsf{T}], \mathbb{R}^+)$, $i = 1,2,3$.

2.2 Existence Results

This section presents existence of the solution of AB–Caputo fractional BVP (2.1)-(2.2) using Theorem 1.3. First we derive our main result for linear AB–Caputo fractional BVP

$$\left({}^{ABC}_{\ \ 0}D^{\sigma}\mathfrak{u}\right)(\ell) = Q(\ell), \quad 1 < \sigma \le 2, \ \ell \in [0,\mathsf{T}], \ \mathsf{T} > 0, \tag{2.3}$$

with

$$\begin{cases} \mathfrak{u}(0) = \xi_1\mathfrak{u}(\mathsf{T}) + \mathfrak{p}_1\int\limits_0^{\mathsf{T}}\Phi(\varsigma)d\varsigma, \\ \mathfrak{u}'(0) = \xi_2\mathfrak{u}'(\mathsf{T}) + \mathfrak{p}_2\int\limits_0^{\mathsf{T}}\varpi(\varsigma)d\varsigma. \end{cases} \tag{2.4}$$

Lemma 2.1. Suppose $Q, \Phi, \varpi, : [0, \mathsf{T}] \to \mathbb{R}$ are continuous functions then the solution of AB-Caputo fractional BVP (2.3) – (2.4) is given as

$$\mathfrak{u}(\ell) = \delta(\ell) + \int_0^{\mathsf{T}} \Psi(\ell, \varsigma) Q(\varsigma)\, d\varsigma + \mathfrak{z}_1 \mathfrak{p}_1 \int_0^{\mathsf{T}} \Phi(\varsigma) d\varsigma + \mathfrak{z}_2 \mathfrak{p}_2 (\xi_1 \mathsf{T} + (1 - \xi_1)\ell) \int_0^{\mathsf{T}} \varpi(\varsigma) d\varsigma. \tag{2.5}$$

where

$$\mathfrak{z}_1 = \frac{1}{(1-\xi_1)}, \ \mathfrak{z}_2 = \frac{1}{(1-\xi_1)(1-\xi_2)} \text{ with } \xi_1, \ \xi_2 \neq 1.$$

$$\delta(\ell) = (2-\sigma)\xi_2 \left(\mathfrak{z}_2 \frac{(\xi_1 \mathsf{T} + (1-\xi_1)\ell)}{AB(\sigma - 1)} \right) G(\mathsf{T}) \tag{2.6}$$

and

$$\Psi(\ell, \varsigma) = \begin{cases} (\sigma-1)^2 \xi_2 \left(\mathfrak{z}_2 \frac{(\xi_1 \mathsf{T} + (1-\xi_1)\ell)}{AB(\sigma-1)\Gamma(\sigma)} \right) (\mathsf{T} - \varsigma)^{\sigma-2} + \left(\mathfrak{z}_1 \frac{(\xi_1(\sigma-1))}{AB(\sigma-1)\Gamma(\sigma)} \right) (\mathsf{T} - \varsigma)^{\sigma-1} \\ + \left(\frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)} \right) (\ell - \varsigma)^{\sigma-1} + \left(\mathfrak{z}_1 \frac{(\xi_1(2-\sigma))}{AB(\sigma-1)} \right) + \left(\frac{(2-\sigma)}{AB(\sigma-1)} \right), \ 0 \leq \varsigma \leq \ell, \\ \\ (\sigma-1)^2 \xi_2 \left(\mathfrak{z}_2 \frac{(\xi_1 \mathsf{T} + (1-\xi_1)\ell)}{AB(\sigma-1)\Gamma(\sigma)} \right) (\mathsf{T} - \varsigma)^{\sigma-2} + \left(\mathfrak{z}_1 \frac{(\xi_1(\sigma-1))}{AB(\sigma-1)\Gamma(\sigma)} \right) (\mathsf{T} - \varsigma)^{\sigma-1} \\ + \left(\mathfrak{z}_1 \frac{(\xi_1(2-\sigma))}{AB(\sigma-1)} \right), \ \ell < \varsigma \leq \mathsf{T}. \end{cases} \tag{2.7}$$

Proof. We have

$$\left({}^{ABC}_{\ \ 0}D^{\sigma} \mathfrak{u} \right)(\ell) = Q(\ell), \quad 1 < \sigma \leq 2, \ 0 \leq \ell \leq \mathsf{T}.$$

Taking ${}^{AB}I^{\sigma}$ on both sides and using Proposition 1.1, we get

$$\mathfrak{u}(\ell) = c_1 + c_2 \ell + \frac{(2-\sigma)}{AB(\sigma-1)} \int_0^{\ell} Q(\varsigma) d\varsigma + \frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)} \int_0^{\ell} (\ell - \varsigma)^{\sigma-1} Q(\varsigma) d\varsigma. \tag{2.8}$$

Using the boundary conditions (2.4) with $Q(0) = 0$, we obtain

$$\begin{aligned} c_1 &= \frac{\xi_1 \xi_2}{(1-\xi_1)(1-\xi_2)} \mathsf{T} \frac{(2-\sigma)}{AB(\sigma-1)} Q(\mathsf{T}) \\ &+ \frac{\xi_1 \xi_2}{(1-\xi_1)(1-\xi_2)} \mathsf{T} \frac{(\sigma-1)^2}{AB(\sigma-1)\Gamma(\sigma)} \int_0^{\mathsf{T}} (\mathsf{T} - \varsigma)^{\sigma-2} Q(\varsigma)\, d\varsigma \end{aligned}$$

$$
\begin{aligned}
&+\frac{\xi_1\mathfrak{p}_2}{(1-\xi_1)(1-\xi_2)}\mathsf{T}\int_0^{\mathsf{T}}\varpi(\varsigma)d\varsigma+\frac{\xi_1}{(1-\xi_1)}\frac{(2-\sigma)}{AB\,(\sigma-1)}\int_0^{\mathsf{T}}Q\,(\varsigma)\,d\varsigma \\
&+\frac{\xi_1}{(1-\xi_1)}\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-1}Q\,(\varsigma)\,d\varsigma+\frac{\mathfrak{p}_1}{(1-\xi_1)}\int_0^{\mathsf{T}}\Phi(\varsigma)d\varsigma
\end{aligned}
$$

and

$$
\begin{aligned}
c_2 &= \frac{\xi_2}{(1-\xi_2)}\frac{(2-\sigma)}{AB\,(\sigma-1)}Q\,(\mathsf{T}) \\
&\quad+\frac{\xi_2}{(1-\xi_2)}\frac{(\sigma-1)^2}{AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-2}Q\,(\varsigma)\,d\varsigma+\frac{\mathfrak{p}_2}{(1-\xi_2)}\int_0^{\mathsf{T}}\varpi(\varsigma)d\varsigma.
\end{aligned}
$$

Putting the values of c_1 and c_2 in equation (2.8), we obtain

$$
\begin{aligned}
\mathfrak{u}(\ell) &= \left(\frac{\xi_1\xi_2\mathsf{T}(2-\sigma)+\xi_2(1-\xi_1)\ell(2-\sigma)}{(1-\xi_1)(1-\xi_2)AB\,(\sigma-1)}\right)Q\,(\mathsf{T}) \\
&\quad+\left(\frac{\xi_1\xi_2\mathsf{T}(\sigma-1)^2+\xi_2(1-\xi_1)\ell(\sigma-1)^2}{(1-\xi_1)(1-\xi_2)AB\,(\sigma-1)\,\Gamma(\sigma)}\right)\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-2}Q\,(\varsigma)\,d\varsigma \\
&\quad+\frac{\xi_1(\sigma-1)}{(1-\xi_1)AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-1}Q\,(\varsigma)\,d\varsigma \\
&\quad+\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma \\
&\quad+\frac{\xi_1(2-\sigma)}{(1-\xi_1)AB\,(\sigma-1)}\int_0^{\mathsf{T}}Q\,(\varsigma)\,d\varsigma+\frac{(2-\sigma)}{AB\,(\sigma-1)}\int_0^{\ell}Q(\varsigma)d\varsigma \\
&\quad+\frac{\mathfrak{p}_1}{(1-\xi_1)}\int_0^{\mathsf{T}}\Phi(\varsigma)d\varsigma+\left(\frac{\xi_1\mathfrak{p}_2\mathsf{T}+\mathfrak{p}_2(1-\xi_1)\ell}{(1-\xi_1)(1-\xi_2)}\right)\int_0^{\mathsf{T}}\varpi(\varsigma)d\varsigma.
\end{aligned}
$$

After simplifications and replacing the values of $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\Psi\,(\ell,\varsigma)$ and $\delta(\ell)$, we get the required solution. ■

Consider a Banach space $X=C([0,\mathsf{T}],\mathbb{R})$ having norm

$$
\|\mathfrak{u}\|=\sup_{\ell\in[0,\mathsf{T}]}|\mathfrak{u}(\ell)|\,.
$$

Now we transfer AB-Caputo fractional BVP (2.1)-(2.2) into fixed point problem, given as

$$\mathrm{u} = \mathcal{F}\mathrm{u}, \tag{2.9}$$

where $\mathcal{F} : X \to X$ is given by

$$\begin{aligned}
(\mathcal{F}\mathrm{u})(\ell) \;=\;& (2-\sigma)\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathrm{T}+(1-\xi_1)\ell}{AB(\sigma-1)}\right)G(\mathrm{T},\mathrm{u}(\mathrm{T}))\\
&+(\sigma-1)^2\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathrm{T}+(1-\xi_1)\ell}{AB(\sigma-1)\Gamma(\sigma)}\right)\int\limits_0^{\mathrm{T}}(\mathrm{T}-\varsigma)^{\sigma-2}G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma\\
&+\mathfrak{z}_1\frac{\xi_1(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int\limits_0^{\mathrm{T}}(\mathrm{T}-\varsigma)^{\sigma-1}G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma\\
&+\frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int\limits_0^{\ell}(\ell-\varsigma)^{\sigma-1}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma\\
&+\mathfrak{z}_1\frac{\xi_1(2-\sigma)}{AB(\sigma-1)}\int\limits_0^{\mathrm{T}}G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma+\frac{(2-\sigma)}{AB(\sigma-1)}\int\limits_0^{\ell}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma\\
&+\mathfrak{z}_1\mathfrak{p}_1\int\limits_0^{\mathrm{T}}E(\varsigma,\mathrm{u}(\varsigma))d\varsigma+\mathfrak{p}_2\mathfrak{z}_2(\xi_1\mathrm{T}+(1-\xi_1)\ell)\int\limits_0^{\mathrm{T}}\eta(\varsigma,\mathrm{u}(\varsigma))d\varsigma.
\end{aligned}\tag{2.10}$$

Remark 2.1. To prove the existence results of AB-Caputo BVP (2.1)-(2.2), we show that the function $\mathcal{F}$ defined in (2.10) has a fixed point.

Theorem 2.1. Suppose all assumptions of $(H_1)-(H3)$ hold. Then AB-Caputo fractional BVP (2.1)-(2.2) has atleast one solution for all $[0,\mathrm{T}]$.

Proof. Consider

$$\mathfrak{B}_{r_1}=\left\{\mathrm{u}\in X:\|\mathrm{u}\|\le r_1>\max\begin{bmatrix}3\left(\|\mathfrak{q}_1\|+c_1\|r_1\|^{h_1}\right)\Theta_1,\\ 3\left(\|\mathfrak{q}_2\|+c_2\|r_1\|^{h_2}\right)\Theta_2,\\ 3\|\mathfrak{q}_3\|+c_3\|r_1\|^{h_3}\Theta_3\end{bmatrix}\right\},$$

where

$$\begin{cases} \Theta_1 = \frac{1}{AB(\sigma-1)}[\{\mathsf{T}(|\xi_2\mathfrak{z}_2(1+\xi_1)|+|\mathfrak{z}_1\xi_1|+1)\} \\ \quad +\frac{1}{\Gamma(\sigma+1)}\{\mathsf{T}^{\sigma}(|\xi_2\mathfrak{z}_2(1+\xi_1)|\sigma+|\mathfrak{z}_1\xi_1|+1)\}], \\ \Theta_2 = |\mathfrak{z}_1\mathfrak{p}_1|\mathsf{T}, \\ \Theta_3 = |\mathfrak{p}_2\mathfrak{z}_2|(1+\xi_1)\mathsf{T}^2. \end{cases} \tag{2.11}$$

Now we define operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathfrak{B}_{r_1}$ as

$$\begin{aligned} (\mathcal{F}_1\mathfrak{u})(\ell) &= (2-\sigma)\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell}{AB(\sigma-1)}\right)G(\mathsf{T},\mathfrak{u}(\mathsf{T})), \\ (\mathcal{F}_2\mathfrak{u})(\ell) &= (\sigma-1)^2\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell}{AB(\sigma-1)\Gamma(\sigma)}\right)\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-2}G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma \\ &\quad +\mathfrak{z}_1\frac{\xi_1(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma \\ &\quad +\frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma \\ &\quad +\mathfrak{z}_1\frac{\xi_1(2-\sigma)}{AB(\sigma-1)}\int_0^{\mathsf{T}}G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma+\frac{(2-\sigma)}{AB(\sigma-1)}\int_0^{\ell}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma \\ &\quad +\mathfrak{z}_1\mathfrak{p}_1\int_0^{\mathsf{T}}E(\varsigma,\mathfrak{u}(\varsigma))d\varsigma+\mathfrak{p}_2\mathfrak{z}_2(\xi_1\mathsf{T}+(1-\xi_1)\ell)\int_0^{\mathsf{T}}\eta(\varsigma,\mathfrak{u}(\varsigma))d\varsigma. \end{aligned}$$

Step 1. In this step, we prove $\mathcal{F}_1\mathfrak{u}+\mathcal{F}_2\vartheta\in\mathfrak{B}_{r_1}$. For this we take $\mathfrak{u},\vartheta\in\mathfrak{B}_{r_1}$ and consider

$$\begin{aligned} |\mathcal{F}_1\{\mathfrak{u}(\ell)\}+\mathcal{F}_2\{\vartheta(\ell)\}| &= \Bigg|(2-\sigma)\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell}{AB(\sigma-1)}\right)G(\mathsf{T},\mathfrak{u}(\mathsf{T})) \\ &\quad +(\sigma-1)^2\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell}{AB(\sigma-1)\Gamma(\sigma)}\right)\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-2}G(\varsigma,\vartheta(\varsigma))\,d\varsigma \\ &\quad +\mathfrak{z}_1\frac{\xi_1(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-1}G(\varsigma,\vartheta(\varsigma))\,d\varsigma \\ &\quad +\frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}G(\varsigma,\vartheta(\varsigma))d\varsigma \end{aligned}$$

$$
\begin{aligned}
&+\mathfrak{z}_1\frac{\xi_1(2-\sigma)}{AB\,(\sigma-1)}\int_0^{\mathsf{T}} G\left(\varsigma,\vartheta(\varsigma)\right)d\varsigma+\frac{(2-\sigma)}{AB\,(\sigma-1)}\int_0^{\ell} G(\varsigma,\vartheta(\varsigma))d\varsigma\\
&+\mathfrak{z}_1\mathfrak{p}_1\int_0^{\mathsf{T}} E\left(\varsigma,\vartheta(\varsigma)\right)d\varsigma+\mathfrak{p}_2\mathfrak{z}_2\left(\xi_1\mathsf{T}+(1-\xi_1)\ell\right)\int_0^{\mathsf{T}}\eta\left(\varsigma,\vartheta(\varsigma)\right)d\varsigma\Bigg|\\
\le\;&\left|(2-\sigma)\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell}{AB\,(\sigma-1)}\right)\right|\left|G\left(\mathsf{T},\mathfrak{u}(\mathsf{T})\right)\right|\\
&+\left|(\sigma-1)^2\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right)\right|\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-2}\left|G\left(\varsigma,\vartheta(\varsigma)\right)\right|d\varsigma\\
&+\left|\mathfrak{z}_1\frac{\xi_1(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right|\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-1}\left|G\left(\varsigma,\vartheta(\varsigma)\right)\right|d\varsigma\\
&+\left|\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right|\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}\left|G(\varsigma,\vartheta(\varsigma))\right|d\varsigma\\
&+\left|\mathfrak{z}_1\frac{\xi_1(2-\sigma)}{AB\,(\sigma-1)}\right|\int_0^{\mathsf{T}}\left|G\left(\varsigma,\vartheta(\varsigma)\right)\right|d\varsigma+\frac{(2-\sigma)}{AB\,(\sigma-1)}\int_0^{\ell}\left|G(\varsigma,\vartheta(\varsigma))\right|d\varsigma\\
&+\left|\mathfrak{z}_1\mathfrak{p}_1\right|\int_0^{\mathsf{T}}\left|E\left(\varsigma,\vartheta(\varsigma)\right)\right|d\varsigma+\left|\mathfrak{p}_2\mathfrak{z}_2\left(\xi_1\mathsf{T}+(1-\xi_1)\ell\right)\right|\int_0^{\mathsf{T}}\left|\eta\left(\varsigma,\vartheta(\varsigma)\right)\right|d\varsigma,
\end{aligned}
$$

taking $\sup\limits_{\ell\in[0,\mathsf{T}]}$ on both sides

$$
\begin{aligned}
\left\|\mathcal{F}_1\mathfrak{u}+\mathcal{F}_2\vartheta\right\| &\le \left(\|\mathfrak{q}_1\|+c_1\|r_1\|^{h_1}\right)\Theta_1+\left(\|\mathfrak{q}_2\|+c_2\|r_1\|^{h_2}\right)\Theta_2\\
&\quad+\left(\|\mathfrak{q}_3\|+c_3\|r_1\|^{h_3}\right)\Theta_3\\
&\le \frac{r_1}{3}+\frac{r_1}{3}+\frac{r_1}{3}\le r_1.
\end{aligned}
$$

Hence $\mathcal{F}_1\mathfrak{u}+\mathcal{F}_2\vartheta\in\mathfrak{B}_{r_1}$.

Step 2. In this step, we show a mapping $\mathcal{F}_2$ is contraction. For this, consider

$$
\begin{aligned}
\left|\mathcal{F}_2\{\mathfrak{u}(\ell)\}-\mathcal{F}_2\{\vartheta(\ell)\}\right| \le\;& \left|(\sigma-1)^2\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right)\right|\\
&\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-2}\left|G\left(\varsigma,\mathfrak{u}(\varsigma)\right)-G\left(\varsigma,\vartheta(\varsigma)\right)\right|d\varsigma
\end{aligned}
$$

$$
\begin{aligned}
& +\left|\mathfrak{z}_1 \frac{\xi_1(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right| \int_0^{\mathsf{T}} (\mathsf{T}-\varsigma)^{\sigma-1} \left|G\left(\varsigma, \mathfrak{u}(\varsigma)\right) - G\left(\varsigma, \vartheta(\varsigma)\right)\right| d\varsigma \\
& +\left|\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right| \int_0^{\ell} (\ell-\varsigma)^{\sigma-1} \left|G\left(\varsigma, \mathfrak{u}(\varsigma)\right) - G\left(\varsigma, \vartheta(\varsigma)\right)\right| d\varsigma \\
& +\left|\mathfrak{z}_1 \frac{\xi_1(2-\sigma)}{AB\,(\sigma-1)}\right| \int_0^{\mathsf{T}} \left|G\left(\varsigma, \mathfrak{u}(\varsigma)\right) - G\left(\varsigma, \vartheta(\varsigma)\right)\right| d\varsigma \\
& +\left|\frac{(2-\sigma)}{AB\,(\sigma-1)}\right| \int_0^{\ell} \left|G\left(\varsigma, \mathfrak{u}(\varsigma)\right) - G\left(\varsigma, \vartheta(\varsigma)\right)\right| d\varsigma \\
& +\left|\mathfrak{z}_1\mathfrak{p}_1\right| \int_0^{\mathsf{T}} \left|E\left(\varsigma, \mathfrak{u}(\varsigma)\right) - E\left(\varsigma, \vartheta(\varsigma)\right)\right| d\varsigma \\
& +\left|\mathfrak{p}_2\mathfrak{z}_2\left(\xi_1\mathsf{T} + (1-\xi_1)\ell\right)\right| \int_0^{\mathsf{T}} \left|\eta\left(\varsigma, \mathfrak{u}(\varsigma)\right) - \eta\left(\varsigma, \vartheta(\varsigma)\right)\right| d\varsigma \\
\leq\; & \frac{\mathfrak{L}_1}{AB\,(\sigma-1)}[\mathsf{T}\{|\mathfrak{z}_1\xi_1| + 1)\} + \frac{1}{\Gamma(\sigma+1)} \\
& \{(\mathsf{T})^{\sigma}(|\xi_2\mathfrak{z}_2(1+\xi_1)|\,\sigma + |\mathfrak{z}_1\xi_1| + 1)\}]\,|\mathfrak{u}(\ell) - \vartheta(\ell)| \\
& +\mathfrak{L}_2\,|\mathfrak{z}_1\mathfrak{p}_1|\,\mathsf{T}\,|\mathfrak{u}(\ell) - \vartheta(\ell)| \\
& +\mathfrak{L}_3\,|\mathfrak{p}_2\mathfrak{z}_2(1+\xi_1)|\,\mathsf{T}^2\,|\mathfrak{u}(\ell) - \vartheta(\ell)| \\
=\; & \mathfrak{S}_1\,|\mathfrak{u}(\ell) - \vartheta(\ell)|\,,
\end{aligned}
$$

taking $\sup\limits_{\ell\in[0,\mathsf{T}]}$ on both sides, we get

$$\|\mathcal{F}_2\mathfrak{u} - \mathcal{F}_2\vartheta\| \leq \mathfrak{S}_1\,\|\mathfrak{u} - \vartheta\|\,.$$

This implies $\mathcal{F}_2$ is contraction mapping.

Step 3. In this step, we prove $\mathcal{F}_1$ is continuous and compact operator. The continuity of G implies continuous of $\mathcal{F}_1$. Since

$$\|\mathcal{F}_1\mathfrak{u}\| \leq \frac{\|\mathfrak{q}_1\|\,\|\xi_2\mathfrak{z}_2(1+\xi_1)\|\,\mathsf{T}}{AB\,(\sigma-1)} \leq r_1.$$

Therefore $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is uniformly bounded for all $[0,\mathsf{T}]$.

Next, we prove $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is equicontinuous. For this, consider

$$\begin{aligned}
&|(\mathcal{F}_1\mathfrak{u})(\ell_1) - (\mathcal{F}_1\mathfrak{u})(\ell_2)| \\
= &\left| \begin{array}{l} (2-\sigma)\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell_1}{AB(\sigma-1)}\right)G\left(\mathsf{T},\mathfrak{u}(\mathsf{T})\right) \\ -(2-\sigma)\xi_2\mathfrak{z}_2\left(\frac{\xi_1\mathsf{T}+(1-\xi_1)\ell_2}{AB(\sigma-1)}\right)G\left(\mathsf{T},\mathfrak{u}(\mathsf{T})\right) \end{array} \right| \\
= &\left| \frac{(2-\sigma)\xi_2\mathfrak{z}_2}{AB\,(\sigma-1)}\left\{(1-\xi_1)\ell_1 - (1-\xi_1)\ell_2\right\}G\left(\mathsf{T},\mathfrak{u}(\mathsf{T})\right)\right| \longrightarrow 0.
\end{aligned}$$

when $\ell_2 \longrightarrow \ell_1$. Thus $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is equicontinuous for all $\ell \in [0, \mathsf{T}]$. So by using Arzela-Ascoli theorem (Lemma 1.3), the set $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is relatively compact. Since $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is relatively compact so $\overline{\mathcal{F}_1(\mathfrak{B}_{r_1})}$ is compact. As $\mathcal{F}_1(\mathfrak{B}_{r_1}) \subset \overline{\mathcal{F}_1(\mathfrak{B}_{r_1})} \subset X$, which implies $\mathcal{F}_1$ is compact. Hence all conditions of Theorem 1.3, are verified. Therefore given AB-Caputo fractional BVP (2.1)-(2.2) has atleast one solution on $[0, \mathsf{T}]$. ■

2.3 Uniqueness

This section provides unique solution of AB-Caputo fractional BVP (2.1)-(2.2). For this purpose, we use Banach contraction principle (Theorem 1.1).

Theorem 2.2. Suppose all conditions of $(H1)$ and $(H4)$ hold. Then AB-Caputo fractional BVP (2.1) – (2.2), has a unique solution .

Proof. Consider

$$\mathfrak{B}_{r_2} = \left\{\mathfrak{u} \in X : \|\mathfrak{u}\| \leq r_2 \geq \frac{\mathfrak{S}_3}{(1-\mathfrak{S}_2)} \text{ with } (1-\mathfrak{S}_2) \neq 0\right\}.$$

where

$$\mathfrak{S}_3 = \mathfrak{R}_1\Theta_1 + \mathfrak{R}_2\Theta_2 + \mathfrak{R}_3\Theta_3,$$

and Θ_i, $i = 1, 2, 3$ are defined in (2.11).

We set

$$\sup_{\ell\in[0,\mathsf{T}]}|G(\ell,0)| = \mathfrak{R}_1, \ \sup_{\ell\in[0,\mathsf{T}]}|E(\ell,0)| = \mathfrak{R}_2, \ \sup_{\ell\in[0,\mathsf{T}]}|\eta(\ell,0)| = \mathfrak{R}_3,$$

where $\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3 \geq 0$.

Now consider

$$\begin{aligned}
|G(\ell,\mathfrak{u}(\ell))| &= |G(\ell,\mathfrak{u}(\ell))-G(\ell,0)+G(\ell,0)| \\
&\le |G(\ell,\mathfrak{u}(\ell))-G(\ell,0)|+|G(\ell,0)| \\
&\le \mathfrak{L}_1\,\|\mathfrak{u}\|+\mathfrak{R}_1=\mathfrak{L}_1 r_2+\mathfrak{R}_1.
\end{aligned}$$

Similarly, we consider

$$\begin{aligned}
|E(\ell,\mathfrak{u}(\ell))| &\le \mathfrak{L}_2 r_2+\mathfrak{R}_2. \\
|\eta(\ell,\mathfrak{u}(\ell))| &\le \mathfrak{L}_3 r_2+\mathfrak{R}_3.
\end{aligned}$$

Step 1. First we show that $\mathcal{F}\left(\mathfrak{B}_{r_2}\right)\subset\mathfrak{B}_{r_2}$. For this we consider

$$\begin{aligned}
|(\mathcal{F}\mathfrak{u})(\ell)| \le\ & (2-\sigma)\,|\xi_2\mathfrak{Z}_2(\xi_1\mathsf{T}+(1-\xi_1)\ell)|\,\frac{1}{AB\,(\sigma-1)}\,|G\,(\mathsf{T},\mathfrak{u}(\mathsf{T}))| \\
&+(\sigma-1)^2\,|\xi_2\mathfrak{Z}_2(\xi_1\mathsf{T}+(1-\xi_1)\ell)|\,\frac{1}{AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-2}\,|G\,(\varsigma,\mathfrak{u}(\varsigma))|\,d\varsigma \\
&+|\mathfrak{Z}_1\xi_1|\,\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\mathsf{T}}(\mathsf{T}-\varsigma)^{\sigma-1}\,|G\,(\varsigma,\mathfrak{u}(\varsigma))|\,d\varsigma \\
&+\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}\,|G\,(\varsigma,\mathfrak{u}(\varsigma))|\,d\varsigma \\
&+|\mathfrak{Z}_1\xi_1|\,\frac{(2-\sigma)}{AB\,(\sigma-1)}\int_0^{\mathsf{T}}|G\,(\varsigma,\mathfrak{u}(\varsigma))|\,d\varsigma+\frac{(2-\sigma)}{AB\,(\sigma-1)}\int_0^{\ell}|G\,(\varsigma,\mathfrak{u}(\varsigma))|\,d\varsigma \\
&+|\mathfrak{Z}_1\mathfrak{p}_1|\int_0^{\mathsf{T}}|E\,(\varsigma,\mathfrak{u}(\varsigma))|\,d\varsigma+|\mathfrak{p}_2\mathfrak{Z}_2\,(\xi_1\mathsf{T}+(1-\xi_1)\ell)|\int_0^{\mathsf{T}}|\eta\,(\varsigma,\mathfrak{u}(\varsigma))|\,d\varsigma.
\end{aligned}$$

taking $\sup\limits_{\ell\in[0,\mathsf{T}]}$ on both sides

$$\begin{aligned}
\|\mathcal{F}\mathfrak{u}\| \le\ & \frac{\mathfrak{L}_1 r_2}{AB\,(\sigma-1)}[\{\mathsf{T}(|\xi_2\mathfrak{Z}_2(1+\xi_1)|+|\mathfrak{Z}_1\xi_1|+1)\} \\
&+\frac{1}{\Gamma(\sigma+1)}\{(\mathsf{T})^{\sigma}(|\xi_2\mathfrak{Z}_2(1+\xi_1)|\,\sigma+|\mathfrak{Z}_1\xi_1|+1)\}]
\end{aligned}$$

$$
\begin{aligned}
& +\mathfrak{L}_2 \left|\mathfrak{Z}_1\mathfrak{p}_1\right| r_2\mathsf{T} + \mathfrak{L}_3 \left|\mathfrak{p}_2\mathfrak{Z}_2(1+\xi_1)\right| \mathsf{T}^2 r_2 \\
& +\frac{\mathfrak{N}_1}{AB\,(\sigma-1)}\left[\{\mathsf{T}(\left|\xi_2\mathfrak{Z}_2(1+\xi_1)\right| + \left|\mathfrak{Z}_1\xi_1\right| + 1)\}\right. \\
& +\frac{1}{\Gamma(\sigma+1)}\left.\{(\mathsf{T})^{\sigma}(\left|\xi_2\mathfrak{Z}_2(1+\xi_1)\right|\sigma + \left|\mathfrak{Z}_1\xi_1\right| + 1)\}\right] \\
& +\mathfrak{N}_2 \left|\mathfrak{Z}_1\mathfrak{p}_1\right| \mathsf{T} + \mathfrak{N}_3 \left|\mathfrak{p}_2\mathfrak{Z}_2(1+\xi_1)\right| \mathsf{T}^2 \\
= \ & \mathfrak{S}_2 r_2 + \mathfrak{S}_3 \leq r_2.
\end{aligned}
$$

This shows that $\mathcal{F}\,(\mathfrak{B}_{r_2}) \subset \mathfrak{B}_{r_2}$.

Step 2. In this step, we prove a mapping $\mathcal{F}$ is contraction. For this consider $\mathfrak{u}, \vartheta \in \mathfrak{B}_{r_2}$, we have

$$
\begin{aligned}
\left|(\mathcal{F}\mathfrak{u})(\ell) - (\mathcal{F}\vartheta)(\ell)\right| \leq \ & \{(2-\sigma)\left|\xi_2\mathfrak{Z}_2(\xi_1\mathsf{T} + (1-\xi_1)\ell)\right| \\
& \frac{1}{AB\,(\sigma-1)}\left|G\,(\mathsf{T},\mathfrak{u}(\mathsf{T})) - G\,(\mathsf{T},\vartheta(\mathsf{T}))\right|\} \\
& +\{(\sigma-1)^2\left|\xi_2\mathfrak{Z}_2(\xi_1\mathsf{T} + (1-\xi_1)\ell)\right| \frac{1}{AB\,(\sigma-1)\,\Gamma(\sigma)} \\
& \int_0^{\mathsf{T}} (\mathsf{T}-\varsigma)^{\sigma-2}\left|G\,(\varsigma,\mathfrak{u}(\varsigma)) - G\,(\varsigma,\vartheta(\varsigma))\right| d\varsigma\} \\
& +\left|\mathfrak{Z}_1\xi_1\right| \frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)} \int_0^{\mathsf{T}} (\mathsf{T}-\varsigma)^{\sigma-1}\left|G\,(\varsigma,\mathfrak{u}(\varsigma)) - G\,(\varsigma,\vartheta(\varsigma))\right| d\varsigma \\
& +\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)} \int_0^{\ell} (\ell-\varsigma)^{\sigma-1}\left|G\,(\varsigma,\mathfrak{u}(\varsigma)) - G\,(\varsigma,\vartheta(\varsigma))\right| d\varsigma \\
& +\left|\mathfrak{Z}_1\xi_1\right| \frac{(2-\sigma)}{AB\,(\sigma-1)} \int_0^{\mathsf{T}} \left|G\,(\varsigma,\mathfrak{u}(\varsigma)) - G\,(\varsigma,\vartheta(\varsigma))\right| d\varsigma \\
& +\frac{(2-\sigma)}{AB\,(\sigma-1)} \int_0^{\ell} \left|G\,(\varsigma,\mathfrak{u}(\varsigma)) - G\,(\varsigma,\vartheta(\varsigma))\right| d\varsigma \\
& +\left|\mathfrak{Z}_1\mathfrak{p}_1\right| \int_0^{\mathsf{T}} \left|E\,(\varsigma,\mathfrak{u}(\varsigma)) - E\,(\varsigma,\vartheta(\varsigma))\right| d\varsigma \\
& +\left|\mathfrak{p}_2\mathfrak{Z}_2\,(\xi_1\mathsf{T} + (1-\xi_1)\ell)\right| \int_0^{\mathsf{T}} \left|\eta\,(\varsigma,\mathfrak{u}(\varsigma)) - \eta\,(\varsigma,\vartheta(\varsigma))\right| d\varsigma,
\end{aligned}
$$

taking $\sup\limits_{\ell\in[0,\mathsf{T}]}$ on both sides

$$\begin{aligned}\left\|(\digamma\mathfrak{u})(\ell)-(\digamma\vartheta)(\ell)\right\| &\leq \|\mathfrak{u}-\vartheta\|\left\{\frac{\mathfrak{L}_1}{AB\,(\sigma-1)}\left[\{\mathsf{T}(|\xi_2\mathfrak{z}_2(1+\xi_1)|+|\mathfrak{z}_1\xi_1|+1)\}\right.\right.\\ &\left.+\frac{1}{\Gamma(\sigma+1)}\{(\mathsf{T})^{\sigma}(|\xi_2\mathfrak{z}_2(1+\xi_1)|\sigma+|\mathfrak{z}_1\xi_1|+1)\}\right]\\ &\left.+\mathfrak{L}_2\,|\mathfrak{z}_1\mathfrak{p}_1|\,\mathsf{T}+\mathfrak{L}_3\,|\mathfrak{p}_2\mathfrak{z}_2(1+\xi_1)|\,\mathsf{T}^2\right\}\\ &= \mathfrak{S}_2\,\|\mathfrak{u}-\vartheta\|.\end{aligned}$$

As $\mathfrak{S}_2<1$, so $\digamma$ is contraction mapping. Hence from Theorem 1.1, we get a unique solution of AB-Caputo fractional BVP (2.1)-(2.2). ■

2.4 Hyers-Ulam Stability

To investigate the stability of given AB-Caputo fractional BVP (2.1)-(2.2), we use HU stability in this section.

Remark 2.2. From Lemma 2.1, the solution $\mathfrak{u}(\ell)\in X$ of AB-Caputo fractional BVP (2.1)-(2.2) is given as;

$$\begin{aligned}\mathfrak{u}(\ell) &= \delta(\ell)+\int\limits_0^{\mathsf{T}}\Psi(\ell,\varsigma)\,G(\varsigma)\,d\varsigma+\mathfrak{z}_1\mathfrak{p}_1\int\limits_0^{\mathsf{T}}E(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\\ &+\mathfrak{z}_2\mathfrak{p}_2\,(\xi_1\mathsf{T}+(1-\xi_1)\ell)\int\limits_0^{\mathsf{T}}\eta(\varsigma,\mathfrak{u}(\varsigma))d\varsigma.\end{aligned}$$

where $\delta(\ell)$ and $\Psi(\ell,\varsigma)$ are defined in (2.6) and (2.7).

Note that

$$\begin{aligned}|\Psi(\ell,\varsigma)| &\leq \left|(\sigma-1)^2\xi_2\left(\mathfrak{z}_2\frac{(\xi_1\mathsf{T}+(1-\xi_1)\ell)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right)(\mathsf{T}-\varsigma)^{\sigma-2}\right|\\ &+\left|\left(\mathfrak{z}_1\frac{(\xi_1(\sigma-1))}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right)(\mathsf{T}-\varsigma)^{\sigma-1}\right|+\left|\left(\frac{(\sigma-1)}{AB\,(\sigma-1)\,\Gamma(\sigma)}\right)(\ell-\varsigma)^{\sigma-1}\right|\\ &+\left|\left(\mathfrak{z}_1\frac{(\xi_1(2-\sigma))}{AB\,(\sigma-1)}\right)\right|+\left|\left(\frac{(2-\sigma)}{AB\,(\sigma-1)}\right)\right|\\ &= \varrho.\end{aligned}$$

Definition 2.1. The AB-Caputo fractional BVP (2.1)-(2.2) is HU stable if there exists a constant $\lambda > 0$ such that for every $\epsilon > 0$ and each solution $\mathfrak{u}(\ell) \in X$, satisfying

$$\left| \left({}_{0}^{ABC}D^{\sigma}\mathfrak{u} \right)(\ell) - G(\ell, \mathfrak{u}(\ell)) \right| \leq \epsilon, \tag{2.12}$$

for all $\ell \in [0, \mathsf{T}]$. Then there exists $\vartheta(\ell) \in X$ a solution of given AB-Caputo fractional BVP (2.1)-(2.2), such that

$$\left| \mathfrak{u}(\ell) - \vartheta(\ell) \right| \leq \lambda\epsilon, \text{ for all } \ell \in [0, \mathsf{T}]. \tag{2.13}$$

Remark 2.3. $\mathfrak{u}(\ell) \in X$ is a solution of inequality (2.12) if and only if there exists a function $\tau(\ell) \in X$, such that

(i) $|\tau(\ell)| \leq \epsilon$, for all $\ell \in [0, \mathsf{T}]$.

(ii) $\left({}^{ABC}{}_{0}D^{\sigma}\mathfrak{u} \right)(\ell) = G(\ell, \mathfrak{u}(\ell)) + \tau(\ell)$, for all $\ell \in [0, \mathsf{T}]$.

Theorem 2.3. Suppose $G, E, \eta : [0, \mathsf{T}] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and satisfying all conditions of (H_1). If

$$1 - \left(\varrho \mathsf{T} \mathfrak{L}_1 - \Theta_2 \mathfrak{L}_2 - \Theta_3 \mathfrak{L}_3 \right) \neq 0,$$

then AB-Caputo fractional BVP (2.1)-(2.2) is HU stable.

Proof. Suppose $\mathfrak{u}(\ell) \in X$ is solution of (2.12), then from Remark 2.3, we have

$$\left({}_{0}^{ABC}D^{\sigma}\mathfrak{u} \right)(\ell) = G(\ell, \mathfrak{u}(\ell)) + \tau(\ell), \text{ for all } \ell \in [0, \mathsf{T}].$$

From Remark 2.2, we have

$$\begin{aligned} \mathfrak{u}(\ell) &= \delta(\ell) + \int_{0}^{\mathsf{T}} \Psi(\ell, \varsigma) G(\varsigma, \mathfrak{u}(\varsigma)) \, d\varsigma \\ &\quad + \int_{0}^{\mathsf{T}} \Psi(\ell, \varsigma) \tau(\varsigma) \, d\varsigma + \mathfrak{Z}_1 \mathfrak{p}_1 \int_{0}^{\mathsf{T}} E(\varsigma, \mathfrak{u}(\varsigma)) d\varsigma \\ &\quad + \mathfrak{Z}_2 \mathfrak{p}_2 \left(\xi_1 \mathsf{T} + (1 - \xi_1)\ell \right) \int_{0}^{\mathsf{T}} \eta(\varsigma, \mathfrak{u}(\varsigma)) d\varsigma, \end{aligned}$$

which implies

$$\left|\begin{array}{c} \mathfrak{u}(\ell)-\delta(\ell)-\int\limits_0^{\mathsf{T}}\Psi(\ell,\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma-\mathfrak{z}_1\mathfrak{p}_1\int\limits_0^{\mathsf{T}}E(\varsigma,\mathfrak{u}(\varsigma))d\varsigma \\ -\mathfrak{z}_2\mathfrak{p}_2\left(\xi_1\mathsf{T}+(1-\xi_1)\ell\right)\int\limits_0^{\mathsf{T}}\eta(\varsigma,\mathfrak{u}(\varsigma))d\varsigma \end{array}\right|\leq\varrho\mathsf{T}\epsilon, \qquad (2.14)$$

where ϱ is defined in Remark 2.2.

Now assume that $\vartheta(\ell)\in X$ is a unique solution of AB-Caputo fractional BVP (2.1)-(2.2). We consider

$$\begin{aligned} \left|\mathfrak{u}(\ell)-\vartheta(\ell)\right| &= \left|\mathfrak{u}(\ell)-\delta(\ell)-\int\limits_0^{\mathsf{T}}\Psi(\ell,\varsigma)G(\varsigma,\vartheta(\varsigma))\,d\varsigma-\mathfrak{z}_1\mathfrak{p}_1\int\limits_0^{\mathsf{T}}E(\varsigma,\vartheta(\varsigma))d\varsigma\right. \\ &\quad\left.-\mathfrak{z}_2\mathfrak{p}_2\left(\xi_1\mathsf{T}+(1-\xi_1)\ell\right)\int\limits_0^{\mathsf{T}}\eta(\varsigma,\vartheta(\varsigma))d\varsigma\right| \\ &\leq \left|\mathfrak{u}(\ell)-\delta(\ell)-\int\limits_0^{\mathsf{T}}\Psi(\ell,\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma-\mathfrak{z}_1\mathfrak{p}_1\int\limits_0^{\mathsf{T}}E(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right. \\ &\quad\left.-\mathfrak{z}_2\mathfrak{p}_2\left(\xi_1\mathsf{T}+(1-\xi_1)\ell\right)\int\limits_0^{\mathsf{T}}\eta(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right| \\ &\quad+\left|\int\limits_0^{\mathsf{T}}\Psi(\ell,\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma-\int\limits_0^{\mathsf{T}}\Psi(\ell,\varsigma)G(\varsigma,\vartheta(\varsigma))\,d\varsigma\right| \\ &\quad+\left|\mathfrak{z}_1\mathfrak{p}_1\left(\int\limits_0^{\mathsf{T}}E(\varsigma,\mathfrak{u}(\varsigma))d\varsigma-\int\limits_0^{\mathsf{T}}E(\varsigma,\vartheta(\varsigma))d\varsigma\right)\right| \\ &\quad+\left|\mathfrak{z}_2\mathfrak{p}_2\left(\xi_1\mathsf{T}+(1-\xi_1)\ell\right)\left(\int\limits_0^{\mathsf{T}}\eta(\varsigma,\mathfrak{u}(\varsigma))d\varsigma-\int\limits_0^{\mathsf{T}}\eta(\varsigma,\vartheta(\varsigma))d\varsigma\right)\right|, \end{aligned}$$

from (2.14), we write

$$\begin{aligned} \left|\mathfrak{u}(\ell)-\vartheta(\ell)\right| &\leq \varrho\mathsf{T}\epsilon+\varrho\mathsf{T}\mathfrak{L}_1\left|\mathfrak{u}(\ell)-\vartheta(\ell)\right|+\mathsf{T}\left|\mathfrak{z}_1\mathfrak{p}_1\right|\mathfrak{L}_{21}\left|\mathfrak{u}(\ell)-\vartheta(\ell)\right| \\ &\quad+\mathsf{T}\left|\mathfrak{z}_2\mathfrak{p}_2\left(\xi_1\mathsf{T}+(1-\xi_1)\ell\right)\right|\mathfrak{L}_{31}\left|\mathfrak{u}(\ell)-\vartheta(\ell)\right|. \end{aligned}$$

taking $\sup\limits_{\ell\in[0,\mathsf{T}]}$ on both sides, we get

$$\begin{aligned}\|\mathfrak{u}-\vartheta\| &\leq \varrho\mathsf{T}\epsilon+\varrho\mathsf{T}\mathfrak{L}_1\|\mathfrak{u}-\vartheta\|+\mathsf{T}\mathfrak{z}_1\mathfrak{p}_1\mathfrak{L}_2\|\mathfrak{u}-\vartheta\| \\ &\quad +\mathfrak{z}_2\mathfrak{p}_2(1+\xi_1)\mathsf{T}^2\mathfrak{L}_3\|\mathfrak{u}-\vartheta\|\end{aligned}$$

$$\|\mathfrak{u}-\vartheta\|\left\{1-(\varrho\mathsf{T}\mathfrak{L}_1-\Theta_2\mathfrak{L}_2-\Theta_3\mathfrak{L}_3)\right\}\leq\varrho\mathsf{T}\epsilon.$$

which implies

$$\|\mathfrak{u}-\vartheta\|\leq\frac{\varrho\mathsf{T}\epsilon}{\left\{1-(\varrho\mathsf{T}\mathfrak{L}_1-\Theta_2\mathfrak{L}_2-\Theta_3\mathfrak{L}_3)\right\}}=\lambda\epsilon.$$

where

$$\lambda=\frac{\varrho\mathsf{T}}{\left\{1-(\varrho\mathsf{T}\mathfrak{L}_1-\Theta_2\mathfrak{L}_2-\Theta_3\mathfrak{L}_3)\right\}}.$$

provided

$$\left\{1-(\varrho\mathsf{T}\mathfrak{L}_1-\Theta_2\mathfrak{L}_2-\Theta_3\mathfrak{L}_3)\right\}\neq 0.$$

Therefore AB-Caputo fractional BVP (2.1)-(2.2) is HU stable. ■

Next, we provide example to validate the conditions of Theorem 2.2 and Theorem 2.3, to get unique and stable solution.

Example 2.1. Consider, the following AB-Caputo fractional BVP

$$\left({}^{ABC}{}_{0}D^{1.5}\mathfrak{u}\right)(\ell)=\frac{1}{8+\ell}\left(1-\frac{\mathfrak{u}(\ell)}{e^{2\ell}}\right),\quad 1<\sigma\leq 2,\ \ell\in[0,T], \tag{2.15}$$

with

$$\begin{cases}\mathfrak{u}(0)=\frac{1}{4}\mathfrak{u}(1)+\frac{2}{7}\int\limits_0^1\frac{\sin(\varsigma)}{7}\left(1-\frac{\mathfrak{u}(\varsigma)}{e^{3\varsigma}}\right)d\varsigma. \\ \\ \mathfrak{u}'(0)=\frac{1}{5}\mathfrak{u}'(1)+\frac{3}{8}\int\limits_0^1\frac{\varsigma}{6}\left(1-\frac{\mathfrak{u}(\varsigma)}{e^{4\varsigma}}\right)d\varsigma.\end{cases} \tag{2.16}$$

From (2.16), we have

$$\begin{aligned}\xi_1 &= \frac{1}{4},\ \xi_2=\frac{1}{5},\ \mathfrak{p}_1=\frac{2}{7},\ \mathfrak{p}_2=\frac{3}{8}.\ \mathfrak{z}_1=\frac{4}{3}. \\ \mathfrak{z}_2 &= \frac{5}{3},\ T=1 \text{ and } AB(\sigma-1)=1,\ \text{for all } 1<\sigma\leq 2.\end{aligned}$$

As

$$\begin{aligned}
|G(\ell,\mathrm{u}(\ell))-G(\ell,\vartheta(\ell))| &\leq \frac{1}{8}\,|\mathrm{u}(\ell)-\vartheta(\ell)|,\\
|E(\ell,\mathrm{u}(\ell))-E(\ell,\vartheta(\ell))| &\leq \frac{1}{7}\,|\mathrm{u}(\ell)-\vartheta(\ell)|,\\
|\eta(\ell,\mathrm{u}(\ell))-\eta(\ell,\vartheta(\ell))| &\leq \frac{1}{6}\,|\mathrm{u}(\ell)-\vartheta(\ell)|.
\end{aligned}$$

From above inequalities, we have $\mathfrak{L}_1=\frac{1}{8},\ \mathfrak{L}_2=\frac{1}{7},\ \mathfrak{L}_3=\frac{1}{6}$. Hence

$$\mathfrak{S}_2=\begin{cases}
\frac{\mathfrak{L}_1}{AB(\sigma-1)}[\{\mathsf{T}(|\xi_2\mathfrak{z}_2(1+\xi_1)|+|\mathfrak{z}_1\xi_1|+1)\}\\
+\frac{1}{\Gamma(\sigma+1)}\{(\mathsf{T})^{\sigma}(|\xi_2\mathfrak{z}_2(1+\xi_1)|\sigma+|\mathfrak{z}_1\xi_1|+1)\}] = 0.587\\
+\mathfrak{L}_2\,|\mathfrak{z}_1\mathfrak{p}_1|\,\mathsf{T}+\mathfrak{L}_3\,|\mathfrak{p}_2\mathfrak{z}_2(1+\xi_1)|\,\mathsf{T}^2
\end{cases}$$

and

$$\{1-(\varrho\mathsf{T}\mathfrak{L}_1-\Theta_2\mathfrak{L}_2-\Theta_3\mathfrak{L}_3)\}\neq 0.$$

Hence all conditions of Theorems 2.2 and Theorem 2.3 are satisfied. So, we get unique and stable solution of given AB-Caputo fractional BVP (2.15) – (2.16).

2.5 Conclusion

In this chapter, the AB-Caputo fractional BVP involving non-separated boundary conditions of fractional order $\sigma\in(1,2]$ are discussed. In Section 2.2, first we find solution of linear AB-Caputo fractional BVP in Lemma 2.1. After that in Theorem 2.1, existence result is presented in which we obtain atleast one solution of (2.1) – (2.2) using Krasnoselskii's fixed point theorem. In Section 2.3 (Theorem 2.2), unique solution of AB-Caputo fractional BVP (2.1) – (2.2) is obtained via Banach contraction principle. In Section 2.4 (Theorem 2.3), the criteria for HU stability of the solution of AB-Caputo fractional BVP (2.1) – (2.2) is discussed. In the end of Section 2.4, an example is provided in which a unique and stable solution is obtained by satisfying the conditions of Theorem 2.2 and Theorem 2.3.

Chapter 3

Existence and Stability Results of FDEs Involving Integral Type Boundary Conditions

3.1 Introduction

The classical boundary conditions are used to identify the behavior of solution at specific or end-points however integral boundary conditions which are used to find the solution in the specified region, so the problems involving integral boundary conditions are more general than ordinary boundary conditions. Integral boundary conditions are also used where the classical conditions failed to develop the mathematical models. Integral boundary conditions has attained much importance because of numerous applications in many fields. As Nicoud and Schonfeld [70], used integral boundary conditions in unsteady biomedical computational fluid dynamics (BMCFD). The decomposition of solution into waves is done by using this strategy. It can be applied to any numerical scheme to solve hyperbolic equations, involving Navier Stokes compressible equations or incompressible counterpart, when solved with an artificial compressibility technique. This technique is also validated to compute both pulsated channel and steady flows. Some other applications of integral type boundary conditions can be seen in the areas of thermo-elasticity, chemical engineering, blood flow problems and population dynamics etc.

Recently integral boundary conditions are widely used in FDEs. Some researchers also develop stability, uniqueness and existence results of FDEs having various fractional operators with the help of integral type boundary conditions. For instance Ahmad et al. [71], discussed the approximate solutions of converging quadratically and monotonically to unique solution of forced Duffing equation having integral type boundary conditions. They presented an algorithm that allow to examine the numerous practical phenomena like predicting the possible occurrence of vascular diseases and identifying the emergence of chaos in speech etc. After that Wang et al. [72], studied integro-differential equations including deviating arguments with first order. They investigated existence results of given problem including integral type boundary conditions by developing comparison result and using monotone iterative method. To find comparison result, they discussed differential inequalities of first order including deviating argument. Later Cabada and Wang [73], presented existence of the positive solution for nonlinear BVP having integral type boundary conditions including Caputo fractional derivative. Guo Krasnoselskii's fixed point theorem are applied to obtain existence of the positive solution. Recently Luo et al. [74], presented analysis of implicit FDEs including integral type boundary conditions having Stieltjes integral. They found existence and uniqueness results of given implicit FDEs. They used Schaefer fixed point theorem and Leray Schauder theorem of cone type result for existence results. They obtained uniqueness via Banach contraction principle. They discussed HU stability, HU Rassias stability, generalized HU stability and generalized HU Rassias stability to get stable solution. Some other results can also be seen in [75, 76, 77, 78, 79, 80, 81].

Motivated from the above work, we investigate non-linear AB-Caputo fractional BVP in this chapter,

$$ {}^{ABC}_{\ \ 0}D^{\sigma}(\mathrm{u}(\ell)) = G(\ell, \mathrm{u}(\ell)) \tag{3.1} $$

with integral type boundary conditions

$$ \mathrm{u}(0) = 0 = \mathrm{u}''(0), \ \hbar \mathrm{u}(1) = \mathfrak{f}\int_0^1 \delta_1(s)\mathrm{u}(s)\,ds. \tag{3.2} $$

where $2 < \sigma \leq 3$, $\delta_1 : [0,1] \to \mathbb{R}^+$ and $\mathfrak{f}, \hbar > 0$.

For our next discussion in this chapter, the following assumptions are useful.

$$|G(\ell,\mathrm{u}(\ell)) - G(\ell,\vartheta(\ell))| \leq \mathfrak{L}\,|\mathrm{u}(\ell) - \vartheta(\ell)|,\ \mathfrak{L} > 0,\ \ell \in [0,1]. \tag{H5}$$

$$|G(\ell,\mathrm{u})| \leq \xi(\ell),\ \xi \in C([0,1],\mathbb{R}^+). \tag{H6}$$

$$\begin{cases} (i)\ \int_0^1 |\Psi_1(\ell,\varsigma)|\,d\varsigma \leq \varrho_1. \\ \\ (ii)\ \int_0^1 |\Psi_2(\ell,\varsigma)|\,d\varsigma \leq \varrho_2. \end{cases} \tag{H7}$$

$$\begin{cases} (i)\ \Phi = \int_0^1 \varsigma\delta_1(\varsigma)d\varsigma. \\ (ii)\ \mathfrak{S} = \frac{\mathfrak{j}}{[h-\Phi\mathfrak{j}]} \text{ with } [h-\Phi\mathfrak{j}] \neq 0,\ \mathfrak{j}, h > 0. \end{cases} \tag{H8}$$

3.2 Existence Results

This section presents existence results of the AB-Caputo fractional BVP (3.1) - (3.2) using Theorem 1.2 and Theorem 1.3. First, we derive our main lemma for the solution of linear AB-Caputo BVP

$$^{ABC}{}_{0}D^{\sigma}(\mathrm{u}(\ell)) = Q(\ell) \tag{3.3}$$

with integral boundary conditions (3.2).

Lemma 3.1. Suppose $Q \in (C[0,1],\mathbb{R})$ then the solution of AB-Caputo fractional BVP (3.3) – (3.2) is given as

$$\mathrm{u}(\ell) = \int_0^1 \Psi_1(\ell,\varsigma)Q(\varsigma)d\varsigma + \mathfrak{S}\left\{\ell\int_0^1 \Psi_2(\ell,\varsigma)Q(\varsigma)d\varsigma\right\}. \tag{3.4}$$

where

$$\Psi_1(\ell,\varsigma) = \begin{cases} +\frac{(3-\sigma)}{AB(\sigma-2)}(\ell-\varsigma) - \frac{(3-\sigma)\ell}{AB(\sigma-2)}(1-\varsigma) + \frac{(\sigma-2)}{AB(\sigma-2)\Gamma(\sigma)}(\ell-\varsigma)^{\sigma-1} \\ \qquad - \frac{(\sigma-2)\ell}{AB(\sigma-2)\Gamma(\sigma)}(1-\varsigma)^{\sigma-1}, \quad 0 \leq \varsigma \leq \ell, \\ \\ -\frac{(3-\sigma)\ell}{AB(\sigma-2)}(1-\varsigma) - \frac{(\sigma-2)\ell}{AB(\sigma-2)\Gamma(\sigma)}(1-\varsigma)^{\sigma-1}, \quad \ell \leq \varsigma \leq 1. \end{cases} \tag{3.5}$$

and

$$\Psi_2(\ell,\varsigma) = \int_0^1 \delta_1(\ell)\Psi_1(\ell,\varsigma)d\ell. \tag{3.6}$$

Proof. Applying ${}^{AB}I^{\sigma}$ on both sides and using Proposition 1.1, we get

$$\begin{aligned}\mathfrak{u}(\ell) &= c_1 + c_2\ell + c_3\frac{\ell^2}{2} + \frac{(3-\sigma)}{AB(\sigma-2)}\int_0^\ell (\ell-\varsigma)Q(\varsigma)d\varsigma \\ &+\frac{(\sigma-2)}{AB(\sigma-2)\Gamma(\sigma)}\int_0^\ell (\ell-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma.\end{aligned} \tag{3.7}$$

Putting $\mathfrak{u}(0) = 0$ in (3.7) which implies that $c_1 = 0$, so

$$\begin{aligned}\mathfrak{u}(\ell) &= c_2\ell + c_3\frac{\ell^2}{2} + \frac{(3-\sigma)}{AB(\sigma-2)}\int_0^\ell (\ell-\varsigma)Q(\varsigma)d\varsigma \\ &+\frac{(\sigma-2)}{AB(\sigma-2)\Gamma(\sigma)}\int_0^\ell (\ell-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma.\end{aligned} \tag{3.8}$$

Using $\mathfrak{u}''(0) = 0$ and $\hbar\mathfrak{u}(1) = \mathfrak{f}\int_0^1\delta_1(\varsigma)\mathfrak{u}(\varsigma)d\varsigma$, we obtain $c_3 = 0$ and

$$\begin{aligned}c_2 &= -\frac{(3-\sigma)}{AB(\sigma-2)}\int_0^1 (1-\varsigma)Q(\varsigma)d\varsigma - \frac{(\sigma-2)}{AB(\sigma-2)\Gamma(\sigma)}\int_0^1 (1-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma \\ &+\frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varsigma)\mathfrak{u}(\varsigma)d\varsigma.\end{aligned}$$

Now put values of (c_2) and (c_3) in (3.8), we get

$$\begin{aligned}\mathfrak{u}(\ell) &= -\frac{(3-\sigma)\ell}{AB(\sigma-2)}\int_0^1 (1-\varsigma)Q(\varsigma)d\varsigma - \frac{(\sigma-2)\ell}{AB(\sigma-2)\Gamma(\sigma)}\int_0^1 (1-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma \\ &+\ell\frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varsigma)\mathfrak{u}(\varsigma)d\varsigma + \frac{(3-\sigma)}{AB(\sigma-2)}\int_0^\ell (\ell-\varsigma)Q(\varsigma)d\varsigma\end{aligned}$$

$$+\frac{(\sigma-2)}{AB\,(\sigma-2)\,\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma. \tag{3.9}$$

which implies

$$\begin{aligned}
\mathrm{u}\,(\ell) &= \int_0^1 \Psi_1(\ell,\varsigma)Q(\varsigma)d\varsigma+\ell\frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varsigma)\mathrm{u}(\varsigma)d\varsigma \\
&= \int_0^1 \Psi_1(\ell,\varsigma)Q(\varsigma)d\varsigma+\ell\Omega,
\end{aligned} \tag{3.10}$$

here, we have

$$\begin{aligned}
\Omega &= \frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varsigma)\mathrm{u}(\varsigma)d\varsigma=\frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varkappa)\mathrm{u}(\varkappa)d\varkappa \\
&= \frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varkappa)\left\{\int_0^1 \Psi_1(\varkappa,\varsigma)Q(\varsigma)d\varsigma+\varkappa\frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varsigma)\mathrm{u}(\varsigma)d\varsigma\right\}d\varkappa \\
&= \frac{\mathfrak{f}}{\hbar}\int_0^1\int_0^1 \delta_1(\varkappa)\Psi_1(\varkappa,\varsigma)Q(\varsigma)d\varkappa d\varsigma+\frac{\mathfrak{f}}{\hbar}\Phi\frac{\mathfrak{f}}{\hbar}\int_0^1 \delta_1(\varsigma)\mathrm{u}(\varsigma)d\varsigma,
\end{aligned}$$

which implies

$$\begin{aligned}
\Omega &= \frac{\hbar}{[\hbar-\Phi\mathfrak{f}]}\frac{\mathfrak{f}}{\hbar}\int_0^1\int_0^1 \delta_1(\varkappa)\Psi_1(\varkappa,\varsigma)Q(\varsigma)d\varkappa d\varsigma. \\
&= \mathfrak{S}\int_0^1\int_0^1 \delta_1(\ell)\Psi_1(\ell,\varsigma)Q(\varsigma)d\ell d\varsigma.
\end{aligned} \tag{3.11}$$

Now replace value of (3.11) in (3.10), we get the required solution. ■

Next, we consider a Banach space $X=C([0,1],\mathbb{R})$ of all continuous functions having norm defined as

$$\|\mathrm{u}\|=\sup_{\ell\in[0,1]}|\mathrm{u}\,(\ell)|\,.$$

We transform our given AB-Caputo fractional BVP (3.1) – (3.2) into fixed point problem as

$$\mathrm{u} = \mathcal{F}\mathrm{u}, \tag{3.12}$$

where the operator $\mathcal{F} : X \to X$ is defined as

$$\begin{aligned}(\mathcal{F}\mathrm{u})(\ell) &= \int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma \\ &\quad +\mathfrak{S}\left\{\ell\int_0^1 \Psi_2(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma\right\}.\end{aligned} \tag{3.13}$$

Theorem 3.1 Suppose $G : [0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous function and the conditions $(H6)$ - $(H7)$ hold. Then the AB-Caputo fractional BVP (3.1) – (3.2). has atleast one solution

Proof. Consider

$$\mathfrak{B}_{r_1} = \{\mathrm{u}\in X : \|\mathrm{u}\| \le r_1 \ge \|\xi\|\{\varrho_1 + \mathfrak{S}\varrho_2\}\}. \tag{3.14}$$

is closed, bounded, convex and nonempty subset of X. Where $\mathfrak{S}$ is given in $(H8)$.

Step 1. In this step, we show $\mathcal{F}(\mathfrak{B}_{r_1}) \subseteq \mathfrak{B}_{r_1}$. For this we consider

$$\begin{aligned}|(\mathcal{F}\mathrm{u})(\ell))| &= \left|\int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma \right. \\ &\quad \left. +\mathfrak{S}\left\{\ell\int_0^1 \Psi_2(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma\right\}\right| \\ &\le \int_0^1 |\Psi_1(\ell,\varsigma)|\,|G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma \\ &\quad +\mathfrak{S}\ell\int_0^1 |\Psi_2(\ell,\varsigma)|\,|G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma \\ &\le |\xi(\ell)|\left\{\int_0^1 |\Psi_1(\ell,\varsigma)|\,d\varsigma + \mathfrak{S}\ell\int_0^1 |\Psi_2(\ell,\varsigma)|\,d\varsigma\right\}.\end{aligned}$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides, we get

$$\|(\mathcal{F}\mathrm{u})\| \leq \|\xi\| \left\{\varrho_1 + \mathfrak{S}\varrho_2\right\} \leq r_1.$$

Therefore $\mathcal{F}(\mathfrak{B}_{r_1}) \subseteq \mathfrak{B}_{r_1}$.

The continuity of G implies the continuity of $\mathcal{F}$. As

$$\|(\mathcal{F}\mathrm{u})\| \leq r_1.$$

Hence $\mathcal{F}(\mathfrak{B}_{r_1})$ is uniformly bounded.

Step 2. Next, we prove $\mathcal{F}(\mathfrak{B}_{r_1})$ is equicontinuous. For this, we consider $0 \leq \ell_1 \leq \ell_2 \leq 1$,

$$\begin{aligned}
|(\mathcal{F}\mathrm{u})(\ell_2) - (\mathcal{F}\mathrm{u})(\ell_1)| &= \left\{\left|\int\limits_0^1 \{\Psi_1(\ell_2,\varsigma) - \Psi_1(\ell_1,\varsigma)\} G(\varsigma,\mathrm{u}(\varsigma))\, d\varsigma\right.\right. \\
&\quad + \mathfrak{S}\left(\ell_2\int\limits_0^1 \Psi_2(\ell_2,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\, d\varsigma\right. \\
&\quad \left.\left.\left. -\ell_1\int\limits_0^1 \Psi_2(\ell_1,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\, d\varsigma\right)\right\}\right| \\
&\leq \left\{\int\limits_0^1 |\Psi_1(\ell_2,\varsigma) - \Psi_1(\ell_1,\varsigma)|\, |G(\varsigma,\mathrm{u}(\varsigma))|\, d\varsigma\right. \\
&\quad + \mathfrak{S}\left(\left|\ell_2\int\limits_0^1 \Psi_2(\ell_2,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\, d\varsigma\right.\right. \\
&\quad \left.\left.\left. -\ell_1\int\limits_0^1 \Psi_2(\ell_1,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\, d\varsigma\right|\right)\right\},
\end{aligned}$$

as $\ell_2 \to \ell_1$ then right side of inequalities approaches to zero. Therefore $\mathcal{F}(\mathfrak{B}_{r_1})$ is equicontinuous. Hence from Arzela-Ascoli theorem (Lemma 1.3), $\mathcal{F}(\mathfrak{B}_{r_1})$ is relatively compact. So from Theorem 1.2, the AB-Caputo fractional BVP (3.1) – (3.2) has atleast one solution. ■

Theorem 3.2. Suppose $G : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous function and satisfying $(H5)$ – $(H7)$. Then there exists atleast one solution of AB-Caputo fractional BVP (3.1) – (3.2) if

$$\mathfrak{L}\mathfrak{S}\varrho_2 < 1,$$

where $\mathfrak{S}$ is given in $(H8)$.

Proof. Consider $\mathfrak{B}_{r_1}$ (defined in (3.14)) is closed, convex, bounded and nonempty subset of X. We decompose $\mathcal{F}$ into sum of two different operators $\mathcal{F}_1$ and $\mathcal{F}_2$ as,

$$(\mathcal{F}_1 \mathrm{u})(\ell) = \int_0^1 \Psi_1(\ell,\varsigma) G(\varsigma, \mathrm{u}(\varsigma))\, d\varsigma$$

and

$$(\mathcal{F}_2 \mathrm{u})(\ell) = \mathfrak{S}\left\{\ell \int_0^1 \Psi_2(\ell,\varsigma) G(\varsigma, \mathrm{u}(\varsigma))\right\} d\varsigma.$$

Step 1. In this step, we show $\mathcal{F}_1\mathrm{u} + \mathcal{F}_2\vartheta \in \mathfrak{B}_{r_1}$. Consider $\mathrm{u}, \vartheta \in \mathfrak{B}_{r_1}$

$$\begin{aligned} |(\mathcal{F}_1\mathrm{u})(\ell) + (\mathcal{F}_2\vartheta)(\ell)| &\leq \left\{\int_0^1 |\Psi_1(\ell,\varsigma)|\,|G(\varsigma, \mathrm{u}(\varsigma))|\, d\varsigma \right.\\ &\quad \left. + \mathfrak{S}\ell \int_0^1 |\Psi_2(\ell,\varsigma)|\,|G(\varsigma, \vartheta(\varsigma))|\, d\varsigma\right\} \\ &\leq |\xi(\ell)| \left\{\int_0^1 |\Psi_1(\ell,\varsigma)|\, d\varsigma + \mathfrak{S}\ell \int_0^1 |\Psi_2(\ell,\varsigma)|\, d\varsigma\right\}. \end{aligned}$$

taking $\sup\limits_{\ell\in[0,1]}$ both sides

$$\|(\mathcal{F}_1\mathrm{u}) + (\mathcal{F}_2\vartheta)\| \leq \|\xi\| \left\{\varrho_1 + \mathfrak{S}\varrho_2\right\} \leq r_1.$$

Thus $\mathcal{F}_1\mathrm{u} + \mathcal{F}_2\vartheta \in \mathfrak{B}_{r_1}$.

Step 2. Now we prove $\mathcal{F}_2$ is a contraction in this step. For this, consider

$$|(\mathcal{F}_2\mathrm{u})(\ell) - (\mathcal{F}_2\vartheta)(\ell)| \leq \left\{\mathfrak{S}\ell \int_0^1 |\Psi_2(\ell,\varsigma)|\,|G(\varsigma, \mathrm{u}(\varsigma)) - G(\varsigma, \vartheta(\varsigma))|\, d\varsigma\right\}$$

$$\leq \left\{ \mathfrak{S}\ell\mathfrak{L}\left|\mathfrak{u}(\ell)-\vartheta(\ell)\right| \int_0^1 \left|\Psi_2(\ell,\varsigma)\right| d\varsigma \right\}$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides

$$\left\|(\mathcal{F}_2\mathfrak{u})-(\mathcal{F}_2\vartheta)\right\| \leq \mathfrak{L}\mathfrak{S}\varrho_2\left\|\mathfrak{u}-\vartheta\right\|.$$

as $\mathfrak{L}\mathfrak{S}\varrho_2<1$, hence $\mathcal{F}_2$ is a contraction.

Step 3. In this step, we prove the continuity and compactness of $\mathcal{F}_1$. The continuity of G implies the continuity of $\mathcal{F}_1$. Since

$$\begin{aligned}\left\|(\mathcal{F}_1\mathfrak{u})\right\| &\leq \sup_{\ell\in[0,1]}\int_0^1 \left|\Psi_1(\ell,\varsigma)\right|\left|G\left(\varsigma,\mathfrak{u}\left(\varsigma\right)\right)\right| d\varsigma\\ &\leq \varrho_1\left\|\xi\right\| \leq r_1.\end{aligned}$$

Therefore $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is uniformly bounded.

Next, we prove $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is equicontinuous. For this, we consider $0<\ell_1<\ell_2<1$

$$\begin{aligned}\left|(\mathcal{F}_1\mathfrak{u})(\ell_2)-(\mathcal{F}_1\mathfrak{u})(\ell_1)\right| &= \left|\int_0^1 \Psi_1(\ell_2,\varsigma)G\left(\varsigma,\mathfrak{u}\left(\varsigma\right)\right) d\varsigma - \int_0^1 \Psi_1(\ell_1,\varsigma)G\left(\varsigma,\mathfrak{u}\left(\varsigma\right)\right) d\varsigma\right|\\ &\leq \int_0^1 \left\{\left|\Psi_1(\ell_2,\varsigma)-\Psi_1(\ell_1,\varsigma)\right|\right\}\left|G\left(\varsigma,\mathfrak{u}\left(\varsigma\right)\right)\right| d\varsigma \longrightarrow 0,\end{aligned}$$

when $\ell_1\rightarrow\ell_2$. Therefore $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is equicontinuous. Thus $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is uniformly bounded and equicontinuous. So by using Arzela-Ascoli theorem (Lemma 1.3), $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is relatively compact. Since $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is relatively compact so $\overline{\mathcal{F}_1(\mathfrak{B}_{r_1})}$ is compact. As $\mathcal{F}_1(\mathfrak{B}_{r_1})\subset\overline{\mathcal{F}_1(\mathfrak{B}_{r_1})}\subset X$, which implies $\mathcal{F}_1$ is is compact. Hence all conditions of Theorem 1.3, are satisfied. Therefore the AB-Caputo fractional BVP (3.1) – (3.2) has atleast one solution. ■

3.3 Uniqueness

This section provides unique solution of AB-Caputo fractional BVP (3.1) – (3.2). To find unique solution of AB-Caputo fractional BVP (3.1) – (3.2), we apply Banach contraction principle

ciple (Theorem 1.1).

Theorem 3.3. Suppose $G : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $(H5)$, $(H7)$ hold. Then the AB-Caputo fractional BVP $(3.1) - (3.2)$ has a unique solution if

$$\mathfrak{L}\{\varrho_1 + \mathfrak{S}\varrho_2\} < 1,$$

where $\mathfrak{S}$ is given in $(H8)$.

Proof. Consider a closed ball

$$\mathfrak{B}_{r2} = \left\{ \mathrm{u} \in X : \|\mathrm{u}\| \leq r_2 \geq \frac{\Im(\varrho_1 + \mathfrak{S}\varrho_2)}{1 - \mathfrak{L}(\varrho_1 + \mathfrak{S}\varrho_2)} \right\}.$$

Where $1 - \mathfrak{L}(\varrho_1 + \mathfrak{S}\varrho_2) \neq 0$, set $\Im = \sup\limits_{\ell \in [0,1]} \{|G(\ell, 0)|\}$ and note that

$$\begin{aligned} |G(\ell, \mathrm{u}(\ell))| &= |G(\ell, \mathrm{u}(\ell)) - G(\ell, 0) + G(\ell, 0)| \\ &\leq |G(\ell, \mathrm{u}(\ell)) - G(\ell, 0)| + |G(\ell, 0)| \\ &\leq \mathfrak{L}\,\|\mathrm{u}\| + \Im \leq \mathfrak{L}r_2 + \Im. \end{aligned}$$

Step 1. In this step, we show that $\mathcal{F}(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$. Consider

$$\begin{aligned} |(\mathcal{F}\mathrm{u})(\ell)| &= \left| \int_0^1 \Psi_1(\ell, \varsigma) \{G(\varsigma, \mathrm{u}(\varsigma))\} d\varsigma + \mathfrak{S}\ell \int_0^1 \Psi_2(\ell, \varsigma) \{G(\varsigma, \mathrm{u}(\varsigma))\} d\varsigma \right| \\ &\leq \left\{ \int_0^1 |\Psi_1(\ell, \varsigma)| \{|G(\varsigma, \mathrm{u}(\varsigma)) - G(\varsigma, 0)| + |G(\varsigma, 0)|\} d\varsigma \right. \\ &\quad \left. + \mathfrak{S}\ell \int_0^1 |\Psi_2(\ell, \varsigma)| \{|G(\varsigma, \mathrm{u}(\varsigma)) - G(\varsigma, 0)| + |G(\varsigma, 0)|\} d\varsigma \right\}, \end{aligned}$$

taking $\sup\limits_{\ell \in [0,1]}$ on both sides

$$\|(\mathcal{F}\mathrm{u})\| \leq (\mathfrak{L}r_1 + \Im)(\varrho_1 + \mathfrak{S}\varrho_2) \leq r_2.$$

Therefore $\mathcal{F}(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$.

Step 2. In this step, we prove $\mathcal{F}$ is contraction mapping. For this, we consider

$$\begin{aligned}
\left|(\mathcal{F}\mathrm{u})(\ell)-(\mathcal{F}\vartheta)(\ell)\right| &= \Bigg|\int_0^1 \Psi_1(\ell,\varsigma)\left\{G(\varsigma,\mathrm{u}(\varsigma))\right\}d\varsigma+\mathfrak{S}\ell\int_0^1 \Psi_2(\ell,\varsigma)\left\{G(\varsigma,\mathrm{u}(\varsigma))\right\}d\varsigma \\
&\quad -\int_0^1 \Psi_1(\ell,\varsigma)\left\{G(\varsigma,\vartheta(\varsigma))\right\}d\varsigma-\mathfrak{S}\ell\int_0^1 \Psi_2(\ell,\varsigma)\left\{G(\varsigma,\vartheta(\varsigma))\right\}d\varsigma\Bigg| \\
&\leq \left\{\int_0^1 \left|\Psi_1(\ell,\varsigma)\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,\vartheta(\varsigma))\right|d\varsigma\right. \\
&\quad \left.+\mathfrak{S}\ell\int_0^1 \left|\Psi_2(\ell,\varsigma)\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,\vartheta(\varsigma))\right|d\varsigma\right\},
\end{aligned}$$

taking $\sup\limits_{\ell\in[0,1]}$ both sides

$$\|(\mathcal{F}\mathrm{u})-(\mathcal{F}\vartheta)\|\leq\mathfrak{L}\,\|\mathrm{u}-\vartheta\|\left\{\varrho_1+\mathfrak{S}\varrho_2\right\}.$$

as $\mathfrak{L}\{\varrho_1+\mathfrak{S}\varrho_2\}<1$. So $\mathcal{F}$ is contraction mapping. Hence by using Theorem 1.1, we get a unique solution of AB-Caputo fractional BVP (3.1) – (3.2). ■

3.4 Hyers-Ulam Stability

To investigate the stability of AB-Caputo fractional BVP (3.1)-(3.2), we use HU stability in this section.

Remark 3.1. The solution $\mathrm{u}(\ell)$ of AB-Caputo fractional BVP (3.1) – (3.2) is given by.

$$\mathrm{u}(\ell)=\int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\,d\varsigma+\mathfrak{S}\left\{\ell\int_0^1 \Psi_2(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma))\right\}d\varsigma.$$

Definition 3.1. The AB-Caputo fractional BVP (3.1)-(3.2) is HU stable if there exists a constant $\lambda>0$, such that for every $\epsilon>0$ and each solution $\mathrm{u}(\ell)\in X$, satisfying

$$\left|({}^{ABC}{}_0D^{\sigma}\mathrm{u})(\ell)-G(\ell,\mathrm{u}(\ell))\right|\leq\epsilon, \tag{3.15}$$

for all $\ell \in [0,1]$. Then there exists a solution $\vartheta(\ell) \in X$ of given AB-Caputo fractional BVP (3.1) – (3.2), such that

$$|\mathrm{u}(\ell) - \vartheta(\ell)| \leq \lambda\epsilon, \text{ for all } \ell \in [0,1]. \tag{3.16}$$

Remark 3.2. $\mathrm{u}(\ell) \in X$ is the solution of (3.15) if and only if there exists $\tau(\ell) \in X$ such that
(i) $|\tau(\ell)| \leq \epsilon$, for all $\ell \in [0,1]$.
(ii) $({}^{ABC}{}_{0}D^{\sigma}\mathrm{u})(\ell) = G(\ell, \mathrm{u}(\ell)) + \tau(\ell)$, for all $\ell \in [0,1]$.

In next theorem, we find conditions under which the solution of given AB-Caputo fractional BVP (3.1) – (3.2) is *HU* stable.

Theorem 3.4. Suppose $G : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfying $(H5), (H7)$. If

$$(1 - \varrho_1 \mathfrak{L} - \mathfrak{S}\varrho_2 \mathfrak{L}) \neq 0,$$

then the AB-Caputo fractional BVP (3.1) – (3.2) is HU stable.

Proof. Suppose $\vartheta(\ell) \in X$ is solution of (3.15) then by Remark 3.2,

$$({}^{ABC}{}_{0}D^{\sigma}\mathrm{u})(\ell) = G(\ell, \mathrm{u}(\ell)) + \tau(\ell), \text{ for all } \ell \in [0,1].$$

Using Remark 3.1, we write

$$\begin{aligned} \mathrm{u}(\ell) &= \int_0^1 \Psi_1(\ell,\varsigma) G(\varsigma, \mathrm{u}(\varsigma))\, d\varsigma + \mathfrak{S}\left\{\ell \int_0^1 \Psi_2(\ell,\varsigma) G(\varsigma, \mathrm{u}(\varsigma))\, d\varsigma\right\} \\ &\quad + \int_0^1 \{\Psi_1(\ell,\varsigma) + \ell\mathfrak{S}\Psi_2(\ell,\varsigma)\}\, \tau(\varsigma)\, d\varsigma, \end{aligned}$$

which implies

$$\left| \mathrm{u}(\ell) - \int_0^1 \Psi_1(\ell,\varsigma) G(\varsigma, \mathrm{u}(\varsigma))\, d\varsigma - \mathfrak{S}\left\{\ell \int_0^1 \Psi_2(\ell,\varsigma) G(\varsigma, \mathrm{u}(\varsigma))\, d\varsigma\right\} \right| \leq \varrho_1\epsilon + \mathfrak{S}\ell\varrho_2\epsilon.$$

We consider

$$\begin{aligned} \left|\mathfrak{u}(\ell)-\vartheta(\ell)\right| &= \left|\mathfrak{u}(\ell)-\int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\vartheta(\varsigma))\,d\varsigma-\mathfrak{S}\left\{\ell\int_0^1 \Psi_2(\ell,\varsigma)G(\varsigma,\vartheta(\varsigma))\,d\varsigma\right\}\right| \\ &\le \left|\mathfrak{u}(\ell)-\int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma-\mathfrak{S}\left\{\ell\int_0^1 \Psi_2(\ell,\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma\right\}\right| \\ &\quad+\left|\int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma-\int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\vartheta(\varsigma))\,d\varsigma\right| \\ &\quad+\left|\mathfrak{S}\left\{\ell\left(\int_0^1 \Psi_2(\ell,\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))\,d\varsigma-\int_0^1 \Psi_2(\ell,\varsigma)G(\varsigma,\vartheta(\varsigma))\,d\varsigma\right)\right\}\right|, \end{aligned}$$

taking $\sup\limits_{\ell\in[0,1]}$ both sides

$$\|\mathfrak{u}-\vartheta\| \le \varrho_1\epsilon+\mathfrak{S}\varrho_2\epsilon+\varrho_1\mathfrak{L}\|\mathfrak{u}-\vartheta\|+\mathfrak{S}\varrho_2\mathfrak{L}\|\mathfrak{u}-\vartheta\|$$

$$\|\mathfrak{u}-\vartheta\|\{1-\varrho_1\mathfrak{L}-\mathfrak{S}\varrho_2\mathfrak{L}\} \le \varrho_1\epsilon+\mathfrak{S}\varrho_2\epsilon,$$

which implies

$$\|\mathfrak{u}-\vartheta\| \le \frac{(\varrho_1+\mathfrak{S}\varrho_2)\epsilon}{(1-\varrho_1\mathfrak{L}-\mathfrak{S}\varrho_2\mathfrak{L})}=\lambda\epsilon.$$

where $\lambda=\frac{\varrho_1+\mathfrak{S}\varrho_2}{(1-\varrho_1\mathfrak{L}-\mathfrak{S}\varrho_2\mathfrak{L})}$. Since $(1-\varrho_1\mathfrak{L}-\mathfrak{S}\varrho_2\mathfrak{L})\neq 0$ therefore the AB-Caputo fractional BVP (3.1) – (3.2) is HU stable. ■

Next, we provide example to validate the conditions of Theorem 3.3 and Theorem 3.4, to get a unique and stable solution.

Example 3.1. Consider the following AB-Caputo fractional BVP

$$^{ABC}{}_0D^{2.5}\mathfrak{u}(\ell)=\left\{\frac{8\log(\ell+2)-\cos\mathfrak{u}(\ell)}{2e^{\ell}}\right\},\quad 2<\sigma\le 3,\ \ell\in[0,1], \tag{3.17}$$

with boundary conditions

$$\mathfrak{u}(0)=0=\mathfrak{u}''(0),\ 4\mathfrak{u}(1)=3\int_0^1 \delta_1(\varsigma)\mathfrak{u}(\varsigma)d\varsigma. \tag{3.18}$$

Here

$$G(\ell,\mathrm{u}(\ell))=\left\{\frac{8\log(\ell+2)-\cos\mathrm{u}(\ell)}{2e^{\ell}}\right\},\ \mathfrak{f}=3,\hbar=4,\ \Phi=\int\limits_{0}^{1}\varsigma\sec^{2}x\,d\varsigma$$

and

$$\begin{aligned}|G(\ell,\mathrm{u}(\ell))-G(\ell,\vartheta(\ell))| &= \left|\frac{8\log(\ell+2)-\cos\mathrm{u}(\ell)}{7e^{\ell}}-\frac{8\log(\ell+2)-\cos\vartheta(\ell)}{7e^{\ell}}\right|\\ &\leq \frac{1}{7e^{\ell}}|\cos\mathrm{u}(\ell)-\cos\vartheta(\ell)|\leq\frac{1}{7}\|\mathrm{u}-\vartheta\|.\end{aligned}$$

From above we have $\mathfrak{L}=\frac{1}{7}>0$. After calculation, we get

$$\begin{aligned}\mathfrak{L}(\varrho_1+\mathfrak{S}\varrho_2) &= \frac{\{0.80091+0.0139797615(2.5539304990)\}}{7}\\ &= 0.1195161917<1,\end{aligned}$$

and

$$\begin{aligned}(1-\varrho_1\mathfrak{L}-\mathfrak{S}\varrho_2\mathfrak{L}) &= 1-\frac{0.80091}{7}-\frac{0.0139797615(2.5539304990)}{7}\\ &= 0.8805130174\neq 0.\end{aligned}$$

Hence, we get unique and stable solution of AB-Caputo fractional BVP (3.17) – (3.18) by satisfying all conditions of Theorem 3.3 and Theorem 3.4.

3.5 Conclusion

We investigate the AB-Caputo fractional BVP with integral type boundary conditions of order $2<\sigma\leq 3$. In Section 3.2, first we provide our main result in Lemma 3.1 in which we find solution of given AB-Caputo fractional BVP. After that we get atleast one solution of AB-Caputo fractional BVP (3.1) – (3.2) in Theorem 3.1 and Theorem 3.2 by using Schauder and Krasnoselskii's fixed point theorems. In Section 3.3 (Theorem 3.3), we obtain unique solution via Banach contraction principle. In Section 3.4 (Theorem 3.4), we use HU stability to investigate the stability of given AB-Caputo fractional BVP (3.1)–(3.2). In the end of Section 3.4, Example 3.1 is provided for the validity of Theorem 3.3 and Theorem 3.4.

Chapter 4

Existence Results of Multi-term AB-Caputo Fractional BVP

4.1 Introduction

The equations consisting of one differential term are called single-term equations,

$$D^{\sigma}(\mathrm{u}(\ell)) + G(\ell, \mathrm{u}(\ell)) = 0,$$

and these equations are used frequently in various physical and numerical problems of mathematics. But these equations are not able to solve all complex physical and numerical problems Therefore we use sometimes multi-term equations, involving more than one differential term. A famous example of multi-term equations is the Bagley Torvik equation,

$$\lambda D^{2}(\mathrm{u}(\ell)) + \beta D^{\frac{3}{2}}(\mathrm{u}(\ell)) + \eta(\mathrm{u}(\ell)) = G(\ell),$$

where G is function and λ, β,η are constants. This equation use in the modelling of motion of rigid plate immersed in Newtonian fluid and presented in [82]. The Basset equation is another example of multi-term equation,

$$\lambda D^{1}(\mathrm{u}(\ell)) + \beta D^{n}(\mathrm{u}(\ell)) + \eta(\mathrm{u}(\ell)) = G(\ell), \quad \mathrm{u}(0) = \mathrm{u}_0,$$

where $0 < n < 1$. It explains the forces that arise, when the spherical object sinks in the incompressible viscous fluid and given in [83]. Multi-term equations are also used in magnetic resonance imaging, unsteady flows of viscoelastic fluid, atmospheric remote sensing, diffusion wave equations, low-temperature collisional plasmas, prevention from cardiovascular diseases etc. Some results which are related to multi-term can be seen in [84, 85, 86, 87, 88, 89].

Multi-term FDEs are generalization of ordinary differential equations (ODEs) in which fractional derivatives of different orders are involved. Multi-term FDEs are included in study of various mathematical analysis and solution techniques of FDEs. Few years earlier, many researchers find existence and uniqueness of solution of multi-term FDEs involving various types of fractional operators using different fixed point theorems. For instance, Baleanu et al. [90] derived existence and uniqueness results of nonlinear multi-term FDEs involving Caputo fractional derivative. They used Schauder fixed point theorem for existence results and Banach contraction principle for uniqueness. After that Agarwal et al. [91], presented results regarding existence and uniqueness of multi-term FDEs with non-local boundary conditions involving Caputo fractional derivative. They derived existence results with help of nonlinear alternative Leray Schauder type fixed point theorem and obtained uniqueness via Banach contraction principle. Later Ahmad et al. [92], explored nonlinear multi-term FDEs with Riemann-Stieltjes (R-S) integro-multipoint boundary conditions involving Caputo fractional derivative. They derived existence results via Sadovskii theorem and obtained unique solution of given BVP using Banach contraction principle. Recently Chen and Dong [93], investigated nonlinear multi-term FDEs including infinite delay with Caputo fractional derivatives. They found existence results via Leray Schauder alternative fixed point theorem and used Banach contraction principle for uniqueness. Some other related results are given in [94, 95, 96, 97, 98, 99].

For the motivation of above work, we present the following multi-term AB-Caputo fractional BVP

$$\left(\delta_2 \, {}^{ABC}_{\;\;0}D^{\sigma+2} + \delta_1 \, {}^{ABC}_{\;\;0}D^{\sigma+1} + \delta_0 \, {}^{ABC}_{\;\;0}D^{\sigma}\right) \mathfrak{u}(t) = G(t, \mathfrak{u}(t)), \tag{4.1}$$

with non-local boundary conditions

$$\mathfrak{u}(0) = 0, \;\; \mathfrak{u}(\mathfrak{p}) = \sum_{i=1}^{n} \jmath_i \mathfrak{u}(\eta_i), \;\; \mathfrak{u}(1) = \lambda \int_0^3 \mathfrak{u}(\varkappa) d\varkappa, \tag{4.2}$$

where $G : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function and $0 < \sigma < 1$, $\ell \in [0,1]$, $0 < \mathfrak{p} < \eta_i < \mathfrak{z} < 1$, $j_i, \lambda > 0$, where $i = 1, 2, 3, ...n$ and $\delta_k \in \mathbb{R}$, where $k = 0, 1, 2$ with $\delta_2 \neq 0$.

For our next discussion, we use $(H5) - (H6)$ assumptions, which are presented in Chapter 3.

4.2 Existence Results

This section presents existence results by using Theorem 1.3, for multi-term AB-Caputo fractional BVP. First we discuss the following linear multi-term AB-Caputo fractional BVP

$$\left(\delta_2 \ {}^{ABC}_{\ \ 0}D^{\sigma+2} + \delta_1 \ {}^{ABC}_{\ \ 0}D^{\sigma+1} + \delta_0 \ {}^{ABC}_{\ \ 0}D^{\sigma}\right) \mathfrak{u}(\ell) = Q(\ell), \tag{4.3}$$

for three different cases $\delta_1^2 - 4\delta_0\delta_2 = 0$, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$ with boundary conditions (4.2).

Here $Q : [0,1] \longrightarrow \mathbb{R}$. After taking ${}^{AB}I^{\sigma}$ on both sides of (4.3), we get

$$\left(\delta_2 \ D^2 + \delta_1 \ D + \delta_0 \ \right) \mathfrak{u}\,(\ell) = c_1 + \frac{(1-\sigma)}{AB\,(\sigma)} Q(\ell) + \frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)} \int\limits_0^{\ell} (\ell - \varkappa)^{\sigma-1} Q(\varkappa) d\varkappa. \tag{4.4}$$

where D represents ordinary derivative. As general solution of the above equation consists of two parts that are complementary solution and particular solution. To get particular solution of (4.4), we use the variation of parameters method.

The following equations of different cases are useful in our next discussion.

Case 1. In first case, we take $\delta_1^2 - 4\delta_0\delta_2 = 0$.

$$\begin{aligned}
\mathfrak{u}\,(\ell) \ &= \ \kappa_1(\ell) \left[\frac{(1-\sigma)}{AB\,(\sigma)} \left\{ \sum_{i=1}^{n} j_i \int\limits_0^{\eta_i} (\eta_i - \varsigma) e^{\mathfrak{m}(\eta_i - \varsigma)} Q(\varsigma) d\varsigma - \int\limits_0^{\mathfrak{p}} (\mathfrak{p} - \varsigma) e^{\mathfrak{m}(\mathfrak{p} - \varsigma)} Q(\varsigma) d\varsigma \right\} \right. \\
&+ \frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)} \left\{ \sum_{i=1}^{n} j_i \int\limits_0^{\eta_i} \int\limits_0^{\varkappa} (\eta_i - \varkappa) e^{\mathfrak{m}(\eta_i - \varkappa)} (\varkappa - \varsigma)^{\sigma-1} Q(\varsigma) d\varsigma d\varkappa \right. \\
&\left. \left. - \int\limits_0^{\mathfrak{p}} \int\limits_0^{\varkappa} (\mathfrak{p} - \varkappa) e^{\mathfrak{m}(\mathfrak{p} - \varkappa)} (\varkappa - \varsigma)^{\sigma-1} Q(\varsigma) d\varsigma d\varkappa \right\} \right]
\end{aligned}$$

$$+\kappa_2(\ell)\left[\frac{(1-\sigma)}{AB(\sigma)}\left\{\lambda\int_0^3\int_0^{\varkappa}(\varkappa-\varsigma)e^{\mathrm{m}(\varkappa-\varsigma)}Q(\varsigma)d\varsigma d\varkappa-\int_0^1(1-\varsigma)e^{\mathrm{m}(1-\varsigma)}Q(\varsigma)d\varsigma\right\}\right.$$

$$+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\left\{\lambda\int_0^3\int_0^{\varkappa}\left(\frac{\mathrm{m}e^{\mathrm{m}(3-\varkappa)}(3-\varkappa)-e^{\mathrm{m}(3-\varkappa)}+1}{\mathrm{m}^2}\right)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right.$$

$$\left.\left.-\int_0^1\int_0^{\varkappa}(1-\varkappa)e^{\mathrm{m}(1-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right\}\right]$$

$$+\frac{(1-\sigma)}{AB(\sigma)}\int_0^{\ell}(\ell-\varsigma)e^{\mathrm{m}(\ell-\varsigma)}Q(\varsigma)d\varsigma$$

$$+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_0^{\ell}\int_0^{\varkappa}(\ell-\varkappa)e^{\mathrm{m}(\ell-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa. \tag{1.5}$$

where $\mathrm{m}=\frac{-\delta_1}{2\delta_2}$, $\mu_1=\varpi_1\varpi_4-\varpi_2\varpi_3\neq 0$, $Z_1(\ell)=\ell e^{\mathrm{m}\ell}$, $Z_2(\ell)=$ [illegible].

$$\kappa_1(\ell)=\frac{\varpi_4Z_1(\ell)-\varpi_3Z_2(\ell)}{\mu_1},\quad \kappa_2(\ell)=\frac{\varpi_1Z_2(\ell)-\varpi_2Z_1(\ell)}{\mu_1}.$$

$$\varpi_1=\mathrm{p}e^{\mathrm{mp}}-\sum_{i=1}^{n}j_i\eta_ie^{\mathrm{m}\eta_i},\quad \varpi_2=\frac{1}{\mathrm{m}^2}\left(\mathrm{mp}e^{\mathrm{mp}}+1-e^{\mathrm{mp}}-\sum_{i=1}^{n}j_i(\mathrm{m}\eta_ie^{\mathrm{m}\eta_i}+1-e^{\mathrm{m}\eta_i})\right),$$

$$\varpi_3=\left(\frac{\mathrm{m}(\mathrm{m}e^{\mathrm{m}}+1-e^{\mathrm{m}})-\lambda\left\{\mathrm{m}3e^{\mathrm{m}3}-2e^{\mathrm{m}3}+\mathrm{m}3+2\right\}}{\mathrm{m}^3}\right),$$

$$\varpi_4=\left(\frac{\mathrm{m}^2e^{\mathrm{m}}-\lambda(\mathrm{m}3e^{\mathrm{m}3}-e^{\mathrm{m}3}+1)}{\mathrm{m}^2}\right).$$

Case 2. In this case we take $\delta_1^2-4\delta_0\delta_2>0$.

$$\mathrm{u}(\ell)=\frac{1}{\delta}\left[\hbar_1(\ell)\left\{\frac{(1-\sigma)}{AB(\sigma)}\left(\sum_{i=1}^{n}j_i\int_0^{\eta_i}\left(e^{\mathrm{m}_2(\eta_i-\varsigma)}-e^{\mathrm{m}_1(\eta_i-\varsigma)}\right)Q(\varsigma)d\varsigma\right.\right.\right.$$

$$\left.-\int_0^{\mathrm{p}}\left(e^{\mathrm{m}_2(\mathrm{p}-\varsigma)}-e^{\mathrm{m}_1(\mathrm{p}-\varsigma)}\right)Q(\varsigma)d\varsigma\right)$$

$$+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\left(\sum_{i=1}^{n}j_i\int_0^{\eta_i}\int_0^{\varkappa}\left(e^{\mathrm{m}_2(\eta_i-\varkappa)}-e^{\mathrm{m}_1(\eta_i-\varkappa)}\right)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right.$$

$$
\begin{aligned}
&-\int_0^{\mathfrak{p}}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(\mathfrak{p}-\varkappa)}-e^{\mathfrak{m}_1(\mathfrak{p}-\varkappa)}\right)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\Bigg)\Bigg\}\\
&+\hbar_2(\ell)\left\{\frac{(1-\sigma)}{AB\,(\sigma)}\left(\lambda\int_0^{3}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(\varkappa-\varsigma)}-e^{\mathfrak{m}_1(\varkappa-\varsigma)}\right)Q(\varsigma)d\varsigma d\varkappa\right.\right.\\
&\left.-\int_0^{1}\left(e^{\mathfrak{m}_2(1-\varsigma)}-e^{\mathfrak{m}_1(1-\varsigma)}\right)Q(\varsigma)d\varsigma\right)\\
&+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\left\{\left(\lambda\int_0^{3}\int_0^{\varkappa}\left(\frac{(e^{\mathfrak{m}_2(3-\varkappa)}-1)}{\mathfrak{m}_2}-\frac{(e^{\mathfrak{m}_1(3-\varkappa)}-1)}{\mathfrak{m}_1}\right)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right.\right.\\
&\left.\left.\left.-\int_0^{1}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(1-\varkappa)}-e^{\mathfrak{m}_1(1-\varkappa)}\right)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right)\right\}\right\}\\
&+\frac{(1-\sigma)}{AB\,(\sigma)}\left(\int_0^{\ell}\left(e^{\mathfrak{m}_2(\ell-\varsigma)}-e^{\mathfrak{m}_1(\ell-\varsigma)}\right)Q(\varsigma)d\varsigma\right)\\
&+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\left(\int_0^{\ell}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(\ell-\varkappa)}-e^{\mathfrak{m}_1(\ell-\varkappa)}\right)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right)\Bigg]
\end{aligned}
$$

where $\mathfrak{m}_1=\frac{-\delta_1-\sqrt{\delta_1^2-4\delta_0\delta_2}}{2\delta_2}$, $\mathfrak{m}_2=\frac{-\delta_1+\sqrt{\delta_1^2-4\delta_0\delta_2}}{2\delta_2}$, $\delta=\mathfrak{m}_2-\mathfrak{m}_1$, $\frac{\delta_0}{\delta_2}=\mathfrak{m}_1\mathfrak{m}_2$.

$$
\mathfrak{S}_1(\ell)=\left\{\frac{\delta_2}{\delta_0}(\mathfrak{m}_2(1-e^{\mathfrak{m}_1\ell})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\ell}))\right\},\ \mathfrak{S}_2(\ell)=\delta(e^{\mathfrak{m}_1\ell}-e^{\mathfrak{m}_2\ell}).
$$

$$
\hbar_1(\ell)=\frac{\{w_4\mathfrak{S}_1(\ell)-w_3\mathfrak{S}_2(\ell)\}}{\mu_2},\ \hbar_2(\ell)=\frac{\{w_1\mathfrak{S}_2(\ell)-w_2\mathfrak{S}_1(\ell)\}}{\mu_2},\ \mu_2=w_1w_4-w_2w_3\neq 0,
$$

$$
w_1=\frac{\delta_2}{\delta_0}\left[(\mathfrak{m}_2(1-e^{\mathfrak{m}_1\mathfrak{p}})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\mathfrak{p}}))-\sum_{i=1}^{n}j_i(\mathfrak{m}_2(1-e^{\mathfrak{m}_1\eta_i})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\eta_i}))\right],
$$

$$
w_2=\delta\left[(e^{\mathfrak{m}_1\mathfrak{p}}-e^{\mathfrak{m}_2\mathfrak{p}})-\sum_{i=1}^{n}j_i(e^{\mathfrak{m}_1\eta_i}-e^{\mathfrak{m}_2\eta_i})\right],
$$

$$
w_3=\left[\begin{array}{c}(\mathfrak{m}_2(1-e^{\mathfrak{m}_1})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2}))\\ -\lambda\frac{\delta_2}{\delta_0}\left\{\mathfrak{m}_2^2\mathfrak{m}_1 3-\mathfrak{m}_2^2e^{\mathfrak{m}_1 3}+\mathfrak{m}_2^2-\mathfrak{m}_1^2\mathfrak{m}_2 3-\mathfrak{m}_1^2e^{\mathfrak{m}_2 3}\quad \mathfrak{m}_1^2\right\}\end{array}\right].
$$

$$
w_4=\left[(e^{\mathfrak{m}_1}-e^{\mathfrak{m}_2})-\lambda\frac{\delta_2}{\delta_0}\left\{\mathfrak{m}_1(1-e^{\mathfrak{m}_2 3})-\mathfrak{m}_2(1-e^{\mathfrak{m}_1 3})\right\}\right].
$$

Case 3. In this case we take $\delta_1^2 - 4\delta_0\delta_2 < 0$.

$$\begin{aligned}
\mathfrak{u}(\ell) = \frac{1}{b}\Bigg[& \tau_1(\ell)\Bigg\{\frac{(1-\sigma)}{AB(\sigma)}\Bigg(\sum_{i=1}^{n} j_i \int_0^{\eta_i} e^{-a(\eta_i-\varsigma)}\sin b(\eta_i-\varsigma)Q(\varsigma)d\varsigma \\
& -\int_0^{\mathfrak{p}} e^{-a(\mathfrak{p}-\varsigma)}\sin b(\mathfrak{p}-\varsigma)Q(\varsigma)d\varsigma\Bigg) \\
& +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\Bigg(\sum_{i=1}^{n} j_i \int_0^{\eta_i}\int_0^{\varkappa} e^{-a(\eta_i-\varkappa)}\sin b(\eta_i-\varkappa)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa \\
& -\int_0^{\mathfrak{p}}\int_0^{\varkappa} e^{-a(\mathfrak{p}-\varkappa)}\sin b(\mathfrak{p}-\varkappa)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\Bigg)\Bigg\} \\
& +\tau_2(\ell)\Bigg\{\frac{(1-\sigma)}{AB(\sigma)}\Bigg(\lambda\int_0^{\mathfrak{z}}\int_0^{\varkappa} e^{-a(\varkappa-\varsigma)}\sin b(\varkappa-\varsigma)Q(\varsigma)d\varsigma d\varkappa \\
& -\int_0^{1} e^{-a(1-\varsigma)}\sin b(1-\varsigma)Q(\varsigma)d\varsigma\Bigg) \\
& +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\Bigg(\lambda\int_0^{\mathfrak{z}}\int_0^{\varkappa}\Big\{\{-ae^{-a(\mathfrak{z}-\varkappa)}\sin b(\mathfrak{z}-\varkappa) \\
& -be^{-a(\mathfrak{z}-\varkappa)}\cos b(\mathfrak{z}-\varkappa)+b(\varkappa-\varsigma)^{\sigma-1}\}Q(\varsigma)d\varsigma d\varkappa\Big\} \\
& -\int_0^{1}\int_0^{\varkappa} e^{-a(1-\varkappa)}\sin b(1-\varkappa)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\Bigg)\Bigg\} \\
& +\frac{(1-\sigma)}{AB(\sigma)}\Bigg(\int_0^{\ell}(\ell-\varsigma)e^{\mathfrak{m}(\ell-\varsigma)}Q(\varsigma)d\varsigma\Bigg) \\
& +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\Bigg(\int_0^{\ell}\int_0^{\varkappa}(\ell-\varkappa)e^{\mathfrak{m}(\ell-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\Bigg)\Bigg], \qquad (4.7)
\end{aligned}$$

where $\mathfrak{m} = -a + ib,\ a = \frac{\delta_1}{2\delta_2},\ b = \frac{\sqrt{4\delta_0\delta_2-\delta_1^2}}{2\delta_2}$,

$$\begin{aligned}
\tau_1(\ell) &= \frac{\{q_4\mathfrak{J}_1(\ell) - q_3\mathfrak{J}_2(\ell)\}}{\mu_3}, \\
\tau_2(\ell) &= \frac{\{q_1\mathfrak{J}_2(\ell) - q_2\mathfrak{J}_1(\ell)\}}{\mu_3}.
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{I}_1(\ell) &= \frac{-ae^{-a\ell}\sin b\ell + b - be^{-a\ell}\cos b\ell}{a^2+b^2},\\
\mathfrak{I}_2(\ell) &= be^{-a\ell}\sin b\ell. \quad \mu_3 = q_1q_4 - q_2q_3 \neq 0,\\
q_1 &= \frac{1}{a^2+b^2}\Bigg[-ae^{-a\mathfrak{p}}\sin b\mathfrak{p} + b - be^{-a\mathfrak{p}}\cos b\mathfrak{p}\\
&\quad - \sum_{i=1}^{n} j_i\left(-ae^{-a\eta_i}\sin b\eta_i + b - be^{-a\eta_i}\cos b\eta_i\right)\Bigg].\\
q_2 &= \left[be^{-a\mathfrak{p}}\sin b\mathfrak{p} - \sum_{i=1}^{n} j_ie^{-a\eta_i}\sin b\eta_i\right],\\
q_3 &= \frac{1}{(a^2+b^2)}\Bigg[\left(-ae^{-a}\sin b + b - be^{-a}\cos b\right) - \lambda\Bigg[\left\{\left(\frac{a^2-b^2}{a^2+b^2}\right)\left(e^{-a\mathfrak{z}}\sin b\mathfrak{z}\right)\right\}\\
&\quad + \left(\frac{2ab}{a^2+b^2}\right)\left(e^{-a\mathfrak{z}}\cos b\mathfrak{z} - 1\right) + b\mathfrak{z}\Bigg]\Bigg],\\
q_4 &= b\left[e^{-a}\sin b + \frac{\lambda}{(a^2+b^2)}\left\{ae^{-a\mathfrak{z}}\sin b\mathfrak{z} + be^{-a\mathfrak{z}}\cos b\mathfrak{z} - b\right\}\right].
\end{aligned}
$$

Lemma 4.1. For $Q \in C([0,1],\mathbb{R})$, the solution of linear multi-term AB-Caputo fractional BVP (4.3) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 = 0$ is given in (4.5).

Proof. We have

$$
\left(\delta_2\ {}^{ABC}_{\ \ \ 0}D^{\sigma+2} + \delta_1\ {}^{ABC}_{\ \ \ 0}D^{\sigma+1} + \delta_0\ {}^{ABC}_{\ \ \ 0}D^{\sigma}\right)\mathrm{u}(\ell) = Q(\ell).
$$

Taking ${}^{AB}I^{\sigma}$ and from Proposition 1.1. we get (4.4). Now use the method of variation of parameters, we get

$$
\begin{aligned}
\mathrm{u}(\ell) &= c_3e^{\mathrm{m}\ell} + c_2\ell e^{\mathrm{m}\ell} + \int_0^{\ell}(\ell-\varkappa)e^{\mathrm{m}(\ell-\varkappa)}\Bigg(c_1 + \frac{(1-\sigma)}{AB(\sigma)}Q(\varkappa)\\
&\quad + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_0^{\varkappa}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\Bigg). \qquad (4.8)
\end{aligned}
$$

Using $\mathrm{u}(0)=0$ in (4.8), we get $c_3 = 0$. Putting value of c_3 in (4.8),we obtain

$$
\mathrm{u}(\ell) = c_2\ell e^{\mathrm{m}\ell} + c_1\left(\frac{\mathrm{m}\ell e^{\mathrm{m}\ell} + 1 - e^{\mathrm{m}\ell}}{\mathrm{m}^2}\right) + \frac{(1-\sigma)}{AB(\sigma)}\int_0^{\ell}(\ell-\varkappa)e^{\mathrm{m}(\ell-\varkappa)}Q(\varkappa)d\varkappa
$$

$$+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_0^{\ell}\int_0^{\varkappa}(\ell-\varkappa)e^{\mathfrak{m}(\ell-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa. \qquad (4.9)$$

Using $\mathfrak{u}(\mathfrak{p})=\sum_{i=1}^{n} j_i\mathfrak{u}(\eta_i)$ and $\mathfrak{u}(1)=\lambda\int_0^3\mathfrak{u}(\varkappa)d\varkappa$ in (4.9) we get

$$c_1\varpi_1+c_2\varpi_2=V_1+V_2 \qquad (4.10)$$

and

$$c_1\varpi_3+c_2\varpi_4=V_3+V_4, \qquad (4.11)$$

where

$$\begin{aligned}
V_1 &= \frac{(1-\sigma)}{AB(\sigma)}\left\{\sum_{i=1}^{n} j_i\int_0^{\eta_i}(\eta_i-\varkappa)e^{\mathfrak{m}(\eta_i-\varkappa)}Q(\varkappa)d\varkappa-\int_0^{\mathfrak{p}}(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varkappa)}Q(\varkappa)d\varkappa\right\},\\
V_2 &= \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\left\{\sum_{i=1}^{n} j_i\int_0^{\eta_i}\int_0^{\varkappa}(\eta_i-\varkappa)e^{\mathfrak{m}(\eta_i-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right.\\
&\quad\left.-\int_0^{\mathfrak{p}}\int_0^{\varkappa}(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varkappa)}\left((\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma\right)d\varkappa\right\}.\\
V_3 &= \frac{(1-\sigma)}{AB(\sigma)}\left\{\lambda\int_0^{3}\int_0^{\varkappa}(\varkappa-\varsigma)e^{\mathfrak{m}(\varkappa-\varsigma)}Q(\varsigma)d\varsigma d\varkappa-\int_0^{1}(1-\varkappa)e^{\mathfrak{m}(1-\varkappa)}Q(\varkappa)d\varkappa\right\}
\end{aligned}$$

and

$$\begin{aligned}
V_4 &= \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\left\{\lambda\int_0^{3}\int_0^{\varkappa}\left(\frac{\mathfrak{m}e^{\mathfrak{m}(3-\varkappa)}(3-\varkappa)-e^{\mathfrak{m}(3-\varkappa)}+1}{\mathfrak{m}^2}\right)(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right.\\
&\quad\left.-\int_0^{1}\int_0^{\varkappa}(1-\varkappa)e^{\mathfrak{m}(1-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa\right\}.
\end{aligned}$$

From (4.10) and (4.11), we obtain

$$c_1=\frac{(V_1+V_2)\varpi_4-(V_3+V_4)\varpi_2}{\mu_1}$$

and

$$c_2 = \frac{(V_3+V_4)\varpi_1 - (V_1+V_2)\varpi_3}{\mu_1}.$$

Substituting the values of c_1 and c_2 in (4.9),

$$\begin{aligned}
\mathrm{u}(\ell) &= \{(V_1+V_2)\kappa_1(\ell) + (V_3+V_4)\kappa_2(\ell)\} \\
&+ \frac{(1-\sigma)}{AB(\sigma)}\int\limits_0^{\ell} (\ell-\varkappa)e^{\mathrm{m}(\ell-\varkappa)}Q(\varkappa)d\varkappa \\
&+ \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_0^{\ell}\int\limits_0^{\varkappa} (\ell-\varkappa)e^{\mathrm{m}(\ell-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}Q(\varsigma)d\varsigma d\varkappa.
\end{aligned}$$

replacing values of V_1, V_2, V_3 and V_4, required result is obtained. ■

Lemma 4.2. For $Q \in C([0,1],\mathbb{R})$, the solution of linear multi-term AB-Caputo fractional BVP (4.3) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 > 0$ is given in (4.6).

Proof. This lemma can be proved in a same way as the proof of Lemma 4.1. ■

Lemma 4.3. For $Q \in C([0,1],\mathbb{R})$, the solution of linear multi-term AB-Caputo fractional BVP (4.3) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 < 0$ is given in (4.7).

Proof. This lemma can be proved in a same way as the proof of Lemma 4.1. ■

Next, consider a Banach space $X = C([0,1],\mathbb{R})$ of all continuous functions having norm defined as

$$\|\mathrm{u}\| = \sup\{|\mathrm{u}(\ell)| : \ell \in [0,1]\}.$$

We transform our given multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 =$ 0 into fixed point problem as

$$\mathrm{u} = \mathcal{F}\mathrm{u}. \tag{4.12}$$

where $\mathcal{F} : X \longrightarrow X$ is defined as

$$\begin{aligned}
(\mathcal{F}\mathrm{u})(\ell) &= \kappa_1(\ell)\left[\frac{(1-\sigma)}{AB(\sigma)}\left\{\sum_{i=1}^{n} j_i \int\limits_0^{\eta_i} (\eta_i-\varsigma)e^{\mathrm{m}(\eta_i-\varsigma)}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma \right.\right. \\
&\left. - \int\limits_0^{\mathfrak{p}} (\mathfrak{p}-\varkappa)e^{\mathrm{m}(\mathfrak{p}-\varsigma)}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma \right\}
\end{aligned}$$

$$+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\left\{\sum_{i=1}^{n} j_i\int_0^{\eta_i}\int_0^{\varkappa}(\eta_i-\varkappa)e^{\mathfrak{m}(\eta_i-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right.$$

$$\left.-\int_0^{\mathfrak{p}}\int_0^{\varkappa}(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right\}\Bigg]$$

$$+\kappa_2(\ell)\left[\frac{(1-\sigma)}{AB\,(\sigma)}\left\{\lambda\int_0^{3}\int_0^{\varkappa}(\varkappa-\varsigma)e^{\mathfrak{m}(\varkappa-\varsigma)}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right.\right.$$

$$\left.-\int_0^{1}(1-\varsigma)e^{\mathfrak{m}(1-\varsigma)}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right\}+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}$$

$$\left\{\lambda\int_0^{3}\int_0^{\varkappa}\left(\frac{\mathfrak{m}e^{\mathfrak{m}(3-\varkappa)}(\mathfrak{z}-\varkappa)-e^{\mathfrak{m}(3-\varkappa)}+1}{\mathfrak{m}^2}\right)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right.$$

$$\left.-\int_0^{1}\int_0^{\varkappa}(1-\varkappa)e^{\mathfrak{m}(1-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right\}\Bigg]$$

$$+\frac{(1-\sigma)}{AB\,(\sigma)}\int_0^{\ell}(\ell-\varsigma)e^{\mathfrak{m}(\ell-\varsigma)}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma$$

$$+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\int_0^{\ell}\int_0^{\varkappa}(\ell-\varkappa)e^{\mathfrak{m}(\ell-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa, \tag{4.13}$$

where we take $\hat{\kappa}_1=\max\limits_{\ell\in[0,1]}\kappa_1(\ell)$, $\hat{\kappa}_2=\max\limits_{\ell\in[0,1]}\kappa_2(\ell)$, $\varepsilon=\max\limits_{\ell\in[0,1]}\{1-e^{\mathfrak{m}\ell}(1-\mathfrak{m}\ell)\}$,

$$\Sigma_1=\frac{(1-\sigma)}{AB\,(\sigma)\,|\mathfrak{m}^2|}\left\{\varepsilon+\hat{\kappa}_1\left(\left|\sum_{i=1}^{n}j_i\right||1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)|+|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})|\right)\right.$$

$$\left.+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\left|\mathfrak{m}\mathfrak{z}-2(e^{\mathfrak{m}3}-1)+\mathfrak{m}\mathfrak{z}e^{\mathfrak{m}3}\right|+|1-e^{\mathfrak{m}}(1-\mathfrak{m})|\right)\right\}. \tag{4.14}$$

$$\Omega_1=\frac{\sigma}{|\mathfrak{m}^2|\,AB\,(\sigma)\,\Gamma(\sigma+1)}\left\{\varepsilon+\hat{\kappa}_1\left(\begin{array}{c}\left|\sum\limits_{i=1}^{n}j_i\right||\eta_i^{\sigma}|\,|1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)|\\+\mathfrak{p}^{\sigma}\,|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})|\end{array}\right)\right.$$

$$\left.+\hat{\kappa}_2\left(\begin{array}{c}\frac{1}{|\mathfrak{m}|}|\lambda|\,|\mathfrak{z}^{\sigma}|\,|2(1-e^{\mathfrak{m}\mathfrak{z}})+\mathfrak{m}\mathfrak{z}(1+e^{\mathfrak{m}\mathfrak{z}})|\\+|1-e^{\mathfrak{m}}(1-\mathfrak{m})|\end{array}\right)\right\} \tag{4.15}$$

and

$$\Lambda_1 = \Sigma_1 + \Omega_1 - \frac{\varepsilon}{AB(\sigma)\,|\mathfrak{m}^2|}\left\{(1-\sigma) + \frac{\sigma}{\Gamma(\sigma+1)}\right\}. \tag{4.16}$$

In the next theorem, we prove existence results of given multi-term AB-Caputo fractional BVP (1.1) – (1.2) using Theorem 1.3

Theorem 4.1. Suppose $G : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function and satisfying $(H5)$-$(H6)$. Then the multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 = 0$ has atleast one solution if

$$\mathfrak{L}\Lambda_1 < 1.$$

where Λ_1 is defined in (4.16).

Proof. Consider

$$\mathfrak{B}_{r_1} = \{\mathfrak{u} \in X : \|\mathfrak{u}\| \leq r_1 \geq \|\varsigma\|(\Sigma_1 + \Omega_1)\}.$$

is a closed, bounded, convex and nonempty subset of X. We decompose $\mathcal{F}$ into sum of two different operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathfrak{B}_{r_1}$ as follows,

$$\begin{aligned}(\mathcal{F}_1\mathfrak{u})(\ell) &= \frac{1}{AB(\sigma)}\left\{(1-\sigma)\int_0^{\ell}(\ell-\varsigma)e^{\mathfrak{m}(\ell-\varsigma)}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right.\\ &\left.+\frac{\sigma}{\Gamma(\sigma)}\int_0^{\ell}\int_0^{\varkappa}(\ell-\varkappa)e^{\mathfrak{m}(\ell-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right\}\end{aligned}$$

and

$$\begin{aligned}(\mathcal{F}_2\mathfrak{u})(\ell) &= \kappa_1(\ell)\left[\frac{(1-\sigma)}{AB(\sigma)}\left\{\sum_{i=1}^{n} j_i\int_0^{\eta_i}(\eta_i-\varsigma)e^{\mathfrak{m}(\eta_i-\varsigma)}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right.\right.\\ &\left.-\int_0^{\mathfrak{p}}(\mathfrak{p}-\varsigma)e^{\mathfrak{m}(\mathfrak{p}-\varsigma)}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right\} + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\\ &\left\{\sum_{i=1}^{n} j_i\int_0^{\eta_i}\int_0^{\varkappa}(\eta_i-\varkappa)e^{\mathfrak{m}(\eta_i-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right.\\ &\left.\left.-\int_0^{\mathfrak{p}}\int_0^{\varkappa}(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right\}\right]\end{aligned}$$

$$
\begin{aligned}
&+\kappa_2(\ell)\left[\frac{(1-\sigma)}{AB\,(\sigma)}\left\{\lambda\int_0^3\int_0^{\varkappa}(\varkappa-\varsigma)e^{\mathrm{m}(\varkappa-\varsigma)}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma d\varkappa\right.\right.\\
&\left.-\int_0^1(1-\varsigma)e^{\mathrm{m}(1-\varsigma)}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma\right\}+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\\
&\left\{\lambda\int_0^3\int_0^{\varkappa}\left(\frac{\mathrm{m}e^{\mathrm{m}(3-\varkappa)}(3-\varkappa)-e^{\mathrm{m}(3-\varkappa)}+1}{\mathrm{m}^2}\right)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma d\varkappa\right.\\
&\left.\left.-\int_0^1\int_0^{\varkappa}(1-\varkappa)e^{\mathrm{m}(1-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathrm{u}(\varsigma))d\varsigma d\varkappa\right\}\right].
\end{aligned}
$$

Step 1. In this step, we show that $\mathcal{F}_1\mathrm{u}+\mathcal{F}_2v\in\mathfrak{B}_{r_1}$. For this, consider

$$
\begin{aligned}
|(\mathcal{F}_1\mathrm{u})(\ell)+(\mathcal{F}_2v)(\ell)| \leq\ & \left[\left\{\frac{(1-\sigma)}{AB(a)}\int_0^{\ell}\left|(\ell-\varsigma)e^{\mathrm{m}(\ell-\varsigma)}G(\varsigma,\mathrm{u}(\varsigma))\right|d\varsigma\right.\right.\\
&\left.+\frac{\sigma}{AB(a)\Gamma(\sigma)}\int_0^{\ell}\int_0^{\varkappa}\left|(\ell-\varkappa)e^{\mathrm{m}(\ell-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathrm{u}(\varsigma))\right|d\varsigma d\varkappa\right\}\\
&+|\kappa_1(\ell)|\left\{\frac{(1-\sigma)}{AB\,(\sigma)}\left(\left|\sum_{i=1}^{n}j_i\right|\int_0^{\eta_i}\left|(\eta_i-\varsigma)e^{\mathrm{m}(\eta_i-\varsigma)}G(\varsigma,v(\varsigma))\right|d\varsigma\right.\right.\\
&\left.+\int_0^{\mathfrak{p}}\left|(\mathfrak{p}-\varkappa)e^{\mathrm{m}(\mathfrak{p}-\varsigma)}G(\varsigma,v(\varsigma))\right|d\varsigma\right)+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\\
&\left(\left|\sum_{i=1}^{n}j_i\right|\int_0^{\eta_i}\int_0^{\varkappa}\left|(\eta_i-\varkappa)e^{\mathrm{m}(\eta_i-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right.\\
&\left.\left.+\int_0^{\mathfrak{p}}\int_0^{\varkappa}\left|(\mathfrak{p}-\varkappa)e^{\mathrm{m}(\mathfrak{p}-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right)\right\}\\
&+|\kappa_2(\ell)|\left\{\frac{(1-\sigma)}{AB\,(\sigma)}\left(|\lambda|\int_0^3\int_0^{\varkappa}\left|(\varkappa-\varsigma)e^{\mathrm{m}(\varkappa-\varsigma)}G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right.\right.\\
&\left.+\int_0^1\left|(1-\varsigma)e^{\mathrm{m}(1-\varsigma)}G(\varsigma,v(\varsigma))\right|d\varsigma\right)+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}
\end{aligned}
$$

$$\left(|\lambda|\int_0^3\int_0^{\varkappa}\left|\left(\frac{\mathfrak{m}e^{\mathfrak{m}(3-\varkappa)}(3-\varkappa)-e^{\mathfrak{m}(3-\varkappa)}+1}{\mathfrak{m}^2}\right)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\upsilon(\varsigma))\right|d\varsigma d\varkappa\right.$$
$$\left.\left.\left.+\int_0^1\int_0^{\varkappa}\left|(1-\varkappa)e^{\mathfrak{m}(1-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\upsilon(\varsigma))\right|d\varsigma d\varkappa\right)\right\}\right].$$

taking $\sup\limits_{\ell\in[0,1]}$ both sides, we get

$$\begin{aligned}
\|(\mathcal{F}_1\mathfrak{u})+(\mathcal{F}_2\upsilon)\| \le \frac{\|\xi\|}{AB(\sigma)}&\left[\frac{(1-\sigma)}{|\mathfrak{m}^2|}\left\{\varepsilon\right.\right.\\
&+\hat{\kappa}_1\left(\left|\sum_{i=1}^{n}j_i\right|\left|1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)\right|\left|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})\right|\right)\\
&\left.+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\left|\mathfrak{m}3-2(e^{\mathfrak{m}3}-1)+\mathfrak{m}3e^{\mathfrak{m}3}\right|+\left|1-e^{\mathfrak{m}}(1-\mathfrak{m})\right|\right)\right\}\\
&+\frac{\sigma}{|\mathfrak{m}^2|\Gamma(\sigma+1)}\left\{\varepsilon\right.\\
&+\hat{\kappa}_1\left(\left|\sum_{i=1}^{n}j_i\right|\left|\eta_i^{\sigma}\right|\left|1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)\right|+\left|\mathfrak{p}^{\sigma}\right|\left|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})\right|\right)\\
&\left.\left.+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\left|3^{\sigma}\right|\left|2(1-e^{\mathfrak{m}3})+\mathfrak{m}3(1+e^{\mathfrak{m}3})\right|+\left|1-e^{\mathfrak{m}}(1-\mathfrak{m})\right|\right)\right\}\right].
\end{aligned}$$

from (4.14) and (4.15), we have

$$\|\mathcal{F}_1\mathfrak{u}+\mathcal{F}_2\upsilon\| \le \|\xi\|\left(\Sigma_1+\Omega_1\right) \le r_1.$$

Thus $\mathcal{F}_1\mathfrak{u}+\mathcal{F}_2\upsilon \in \mathfrak{B}_{r_1}$.

Step 2. In this step, we show $\mathcal{F}_2$ is a contraction. For this, we consider

$$\begin{aligned}
&|(\mathcal{F}_2\mathfrak{u})(\ell)-\mathcal{F}_2\upsilon)(\ell)|\\
&\le \left[|\kappa_1(\ell)|\left\{\frac{(1-\sigma)}{AB(\sigma)}\left(\left|\sum_{i=1}^{n}j_i\right|\int_0^{\eta_i}\left|(\eta_i-\varsigma)e^{\mathfrak{m}(\eta_i-\varsigma)}\right|\left|G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,\upsilon(\varsigma))\right|d\varsigma\right.\right.\right.\\
&\left.+\int_0^{\mathfrak{p}}\left|(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varsigma)}\right|\left|G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,\upsilon(\varsigma))\right|d\varsigma\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\Bigg\{\left|\sum_{i=1}^{n} j_i\right|\int_0^{\eta_i}\int_0^{\varkappa}\left|(\eta_i-\varkappa)e^{\mathfrak{m}(\eta_i-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}\right||G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,v(\varsigma))|\,d\varsigma d\varkappa \\
&+\int_0^{\mathfrak{p}}\int_0^{\varkappa}\left|(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}\right||G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,v(\varsigma))|\,d\varsigma d\varkappa\Bigg\}\Bigg] \\
&+|\kappa_2(\ell)|\Bigg[\frac{(1-\sigma)}{AB(\sigma)}\Bigg(|\lambda|\int_0^{3}\int_0^{\varkappa}\left|(\varkappa-\varsigma)e^{\mathfrak{m}(\varkappa-\varsigma)}\right||G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,v(\varsigma))|\,d\varsigma d\varkappa \\
&+\int_0^{1}\left|(1-\varsigma)e^{\mathfrak{m}(1-\varsigma)}\right||G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,v(\varsigma))|\,d\varsigma\Bigg)+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \\
&\Bigg\{|\lambda|\int_0^{3}\int_0^{\varkappa}\left|\left(\frac{\mathfrak{m}e^{\mathfrak{m}(3-\varkappa)}(3-\varkappa)-e^{\mathfrak{m}(3-\varkappa)}+1}{\mathfrak{m}^2}\right)(\varkappa-\varsigma)^{\sigma-1}\right||G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,v(\varsigma))|\,d\varsigma d\varkappa \\
&+\int_0^{1}\int_0^{\varkappa}\left|(1-\varkappa)e^{\mathfrak{m}(1-\varkappa)}(\varkappa-\varsigma)^{\sigma-1}\right||G(\varsigma,\mathfrak{u}(\varsigma))-G(\varsigma,v(\varsigma))|\,d\varsigma d\varkappa\Bigg\}\Bigg].
\end{aligned}
$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides,

$$
\begin{aligned}
\|(\mathcal{F}_2\mathfrak{u})-\mathcal{F}_2v)\| \leq \mathfrak{L}&\Bigg[\frac{(1-\sigma)}{AB(\sigma)|\mathfrak{m}^2|}\Bigg\{\hat{\kappa}_1\left(\left|\sum_{i=1}^{n} j_i\right||1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)|+|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})|\right) \\
&+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\left|\mathfrak{m}3-2(e^{\mathfrak{m}3}-1)+\mathfrak{m}3e^{\mathfrak{m}3}\right|+|1-e^{\mathfrak{m}}(1-\mathfrak{m})|\right)\Bigg\} \\
&+\frac{\sigma}{AB(\sigma)|\mathfrak{m}^2|\Gamma(\sigma+1)}\Bigg\{\hat{\kappa}_1\left(\begin{array}{c}\left|\sum\limits_{i=1}^{n} j_i\right||\eta_i^{\sigma}||1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)| \\ +|\mathfrak{p}^{\sigma}||1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})|\end{array}\right) \\
&+\hat{\kappa}_2\left(\begin{array}{c}\frac{1}{|\mathfrak{m}|}|\lambda||3^{\sigma}||2(1-e^{\mathfrak{m}3})+\mathfrak{m}3(1+e^{\mathfrak{m}3})| \\ +|1-e^{\mathfrak{m}}(1-\mathfrak{m})|\end{array}\right)\Bigg\}\Bigg]\|\mathfrak{u}-v\|.
\end{aligned}
$$

from (4.16), we have

$$\|\mathcal{F}_2\mathfrak{u}-\mathcal{F}_2v)\| \leq \mathfrak{L}\Lambda_1\|\mathfrak{u}-v\|.$$

as $\mathfrak{L}\Lambda_1<1$, hence $\mathcal{F}_2$ is contraction.

Step 3. In this step, we prove the compactness and continuity of $\mathcal{F}_1$. As the continuity of

G implies continuity of $\mathcal{F}_1$. Since

$$\|(\mathcal{F}_1\mathrm{u})\| \leq \frac{\varepsilon\,\|\xi\|}{AB\,(\sigma)\,|\mathrm{m}^2|}\left\{(1-\sigma)+\frac{\sigma}{\Gamma(\sigma+1)}\right\} \leq r_1.$$

therefore $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is uniformly bounded.

Now we prove $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is equicontinuous. For this, we consider $0 \leq \ell_1 < \ell_2 \leq 1$

$$\begin{aligned}
& |(\mathcal{F}_1\mathrm{u})(\ell_2)-(\mathcal{F}_1\mathrm{u})(\ell_1)| \\
\leq\ & \left[\frac{1}{AB\,(\sigma)}\left\{(1-\sigma)\int\limits_0^{\ell_1}\left|(\ell_2-\varsigma)e^{\mathrm{m}(\ell_2-\varsigma)}-(\ell_1-\varsigma)e^{\mathrm{m}(\ell_1-\varsigma)}\right||G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma\right.\right. \\
& \left.+\frac{\sigma}{\Gamma(\sigma)}\int\limits_0^{\ell_1}\int\limits_0^{\varkappa}\left|(\ell_2-\varkappa)e^{\mathrm{m}(\ell_2-\varkappa)}-(\ell_1-\varkappa)e^{\mathrm{m}(\ell_1-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right||G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma d\varkappa\right\} \\
& +\frac{1}{AB\,(\sigma)}\left\{(1-\sigma)\int\limits_{\ell_1}^{\ell_2}\left|(\ell_2-\varsigma)e^{\mathrm{m}(\ell_2-\varsigma)}\right||G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma\right. \\
& \left.\left.+\frac{\sigma}{\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell_2}\int\limits_0^{\varkappa}\left|(\ell_2-\varkappa)e^{\mathrm{m}(\ell_2-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right||G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma d\varkappa\right\}\right] \\
\leq\ & \frac{\xi(\ell)}{AB\,(\sigma)}\left[(1-\sigma)\left|\left(\frac{-(\ell_2-\ell_1)e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}}+\frac{e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}^2}+0-\frac{1}{\mathrm{m}^2}\right)\right.\right. \\
& \left.-\left(-\frac{\ell_2 e^{\mathrm{m}\ell_2}}{\mathrm{m}}+\frac{e^{\mathrm{m}\ell_2}}{\mathrm{m}^2}+\frac{\ell_1 e^{\mathrm{m}\ell_1}}{\mathrm{m}}-\frac{e^{\mathrm{m}\ell_1}}{\mathrm{m}^2}\right)\right| \\
& +\frac{\sigma}{\Gamma(\sigma+1)}|\ell_1^{\sigma}|\left|\left(\frac{-(\ell_2-\ell_1)e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}}+\frac{e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}^2}+0-\frac{1}{\mathrm{m}^2}\right)\right. \\
& \left.-\left(-\frac{\ell_2 e^{\mathrm{m}\ell_2}}{\mathrm{m}}+\frac{e^{\mathrm{m}\ell_2}}{\mathrm{m}^2}+\frac{\ell_1 e^{\mathrm{m}\ell_1}}{\mathrm{m}}-\frac{e^{\mathrm{m}\ell_1}}{\mathrm{m}^2}\right)\right| \\
& +(1-\sigma)\left|\left(0+\frac{1}{\mathrm{m}^2}\right)-\left(\frac{-(\ell_2-\ell_1)e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}}+\frac{e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}^2}\right)\right| \\
& \left.+\frac{\sigma}{\Gamma(\sigma+1)}|\ell_2^{\sigma}|\left|\left(0+\frac{1}{\mathrm{m}^2}\right)-\left(\frac{-(\ell_2-\ell_1)e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}}+\frac{e^{\mathrm{m}(\ell_2-\ell_1)}}{\mathrm{m}^2}\right)\right|\right].
\end{aligned}$$

as $\ell_2 \to \ell_1$, therefore

$$|(\mathcal{F}_1\mathrm{u})(\ell_2)-(\mathcal{F}_1\mathrm{u})(\ell_1)| \to 0.$$

Therefore $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is equicontinuous. From the statement of Arzela-Ascoli theorem (Lemma 1.3), $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is relatively compact. Since $\mathcal{F}_1(\mathfrak{B}_{r_1})$ is relatively compact. Hence $\overline{\mathcal{F}_1(\mathfrak{B}_{r_1})}$ is compact.

As $\mathcal{F}_1(\mathfrak{B}_{r_1}) \subset \overline{\mathcal{F}_1(\mathfrak{B}_{r_1})} \subset X$, which implies $\mathcal{F}_1$ is compact. So all conditions of Theorem 1.3 are verified. Hence the given multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 = 0$ has atleast one solution. ■

In view of Lemma 4.2, the solution of multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 > 0$ can be transformed into a fixed point problem as

$$\mathfrak{u} = \digamma\mathfrak{u}, \tag{4.17}$$

where $\digamma : X \longrightarrow X$ is defined as

$$\begin{aligned}
(\digamma\mathfrak{u})(\ell) = \frac{1}{\delta}\Bigg[&\hbar_1(\ell)\left\{\frac{(1-\sigma)}{AB(\sigma)}\left(\sum_{i=1}^{n} j_i \int_0^{\eta_i}\left(e^{\mathfrak{m}_2(\eta_i-\varsigma)} - e^{\mathfrak{m}_1(\eta_i-\varsigma)}\right)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right.\right.\\
&\left.-\int_0^{\mathfrak{p}}\left(e^{\mathfrak{m}_2(\mathfrak{p}-\varsigma)} - e^{\mathfrak{m}_1(\mathfrak{p}-\varsigma)}\right)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right) + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\\
&\left(\sum_{i=1}^{n} j_i \int_0^{\eta_i}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(\eta_i-\varkappa)} - e^{\mathfrak{m}_1(\eta_i-\varkappa)}\right)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right.\\
&\left.\left.-\int_0^{\mathfrak{p}}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(\mathfrak{p}-\varkappa)} - e^{\mathfrak{m}_1(\mathfrak{p}-\varkappa)}\right)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right)\right\}\\
&+\hbar_2(\ell)\left\{\frac{(1-\sigma)}{AB(\sigma)}\left(\lambda\int_0^{3}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(\varkappa-\varsigma)} - e^{\mathfrak{m}_1(\varkappa-\varsigma)}\right)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right.\right.\\
&\left.-\int_0^{1}\left(e^{\mathfrak{m}_2(1-\varsigma)} - e^{\mathfrak{m}_1(1-\varsigma)}\right)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right) + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\\
&\left(\lambda\int_0^{3}\int_0^{\varkappa}\left(\frac{(e^{\mathfrak{m}_2(3-\varkappa)}-1)}{\mathfrak{m}_2} - \frac{(e^{\mathfrak{m}_1(3-\varkappa)}-1)}{\mathfrak{m}_1}\right)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right.\\
&\left.\left.-\int_0^{1}\left(e^{\mathfrak{m}_2(1-\varkappa)} - e^{\mathfrak{m}_1(1-\varkappa)}\right)\int_0^{\varkappa}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right)\right\}
\end{aligned}$$

$$+\frac{(1-\sigma)}{AB(\sigma)}\left(\int_0^{\ell}\left(e^{\mathfrak{m}_2(\ell-\varsigma)}-e^{\mathfrak{m}_1(\ell-\varsigma)}\right)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\right)$$

$$\left.+\frac{\sigma}{AB(\sigma)\,\Gamma(\sigma)}\left(\int_0^{\ell}\int_0^{\varkappa}\left(e^{\mathfrak{m}_2(\ell-\varkappa)}-e^{\mathfrak{m}_1(\ell-\varkappa)}\right)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\right)\right]. \tag{4.18}$$

Where we set $\varepsilon_1=\max\limits_{\ell\in[0,1]}\left|\mathfrak{m}_2(1-e^{\mathfrak{m}_1\ell})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\ell})\right|$, $\hat{\hbar}_1=\max\limits_{\ell\in[0,1]}|\hbar_1(\ell)|$, $\hat{\hbar}_2=\max\limits_{\ell\in[0,1]}|\hbar_1(\ell)|$.

$$\Phi=\frac{|\delta_2|}{|\delta_0\delta|}\frac{(1-\sigma)}{AB(\sigma)}\Bigg\{\varepsilon_1$$

$$+\hat{\hbar}_1\left(\sum_{i=1}^{n}|j_i|\,|\mathfrak{m}_2(1-e^{\mathfrak{m}_1\eta_i})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\eta_i})|+|\mathfrak{m}_2(1-e^{\mathfrak{m}_1\mathfrak{p}})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\mathfrak{p}})|\right)$$

$$+\hat{\hbar}_2\left(\frac{|\delta_2|}{|\delta_0|}|\lambda|\left|\mathfrak{m}_1\mathfrak{m}_2(\mathfrak{m}_2-\mathfrak{m}_1)\mathfrak{Z}-\{\mathfrak{m}_2^2(e^{\mathfrak{m}_1\mathfrak{Z}}-1)-\mathfrak{m}_1^2(e^{\mathfrak{m}_2\mathfrak{Z}}-1)\}\right|\right.$$

$$\left.+|\mathfrak{m}_2(1-e^{\mathfrak{m}_1})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2})|\right)\Bigg\}. \tag{4.19}$$

$$\Psi=\frac{|\delta_2|}{|\delta_0\delta|}\frac{\sigma}{AB(\sigma)\,\Gamma(\sigma+1)}\Bigg\{\varepsilon_1$$

$$+\hat{\hbar}_1\left(\sum_{i=1}^{n}|j_i|\,|\eta_i^{\sigma}|\,|\mathfrak{m}_2(1-e^{\mathfrak{m}_1\eta_i})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\eta_i})|+|\mathfrak{p}^{\sigma}|\,|\mathfrak{m}_2(1-e^{\mathfrak{m}_1\mathfrak{p}})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2\mathfrak{p}})|\right)$$

$$+\hat{\hbar}_2\left(\frac{|\delta_2|}{|\delta_0|}|\lambda|\,|\mathfrak{Z}^{\sigma}|\left|\mathfrak{m}_2^2(1-e^{\mathfrak{m}_1\mathfrak{Z}}+\mathfrak{m}_1\mathfrak{Z})-\mathfrak{m}_1^2(1-e^{\mathfrak{m}_2\mathfrak{Z}}+\mathfrak{m}_2\mathfrak{Z})\right|\right.$$

$$\left.+|\mathfrak{m}_2(1-e^{\mathfrak{m}_1})-\mathfrak{m}_1(1-e^{\mathfrak{m}_2})|\right)\Bigg\}, \tag{4.20}$$

and

$$\Phi_1=\Phi+\Psi-\frac{|\delta_2|}{|\delta_0\delta|\,AB(\sigma)}\varepsilon_1\left((1-\sigma)+\frac{\sigma}{\Gamma(\sigma+1)}\right). \tag{4.21}$$

Theorem 4.2. Suppose $G:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous function and satisfying $(H5)-(H6)$. Then the multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2-4\delta_0\delta_2>0$, has atleast one solution if

$$\mathfrak{L}\Phi_1<1,$$

where Φ_1 is defined in (4.21).

Proof. This theorem can be proved in a same way as the proof of Theorem 4.1. ■

In view of Lemma 4.3, the solution of AB-Caputo fractional BVP (4.1) – (4.2) with δ

$4\delta_0\delta_2 < 0$ can be transformed into a fixed point problem

$$\mathfrak{u} = P\mathfrak{u}, \tag{4.22}$$

where the operator $P : X \longrightarrow X$ is defined by

$$
\begin{aligned}
(P\mathfrak{u})(\ell) = \frac{1}{b}\Bigg[&\tau_1(\ell)\Bigg\{\frac{(1-\sigma)}{AB(\sigma)}\Bigg(\sum_{i=1}^{n} j_i\int_0^{\eta_i} e^{-a(\eta_i-\varsigma)}\sin b(\eta_i-\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma \\
&-\int_0^{\mathfrak{p}} e^{-a(\mathfrak{p}-\varkappa)}\sin b(\mathfrak{p}-\varkappa)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\Bigg) + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \\
&\Bigg(\sum_{i=1}^{n} j_i\int_0^{\eta_i}\int_0^{\varkappa} e^{-a(\eta_i-\varkappa)}\sin b(\eta_i-\varkappa)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa \\
&-\int_0^{\mathfrak{p}}\int_0^{\varkappa} e^{-a(\mathfrak{p}-\varkappa)}\sin b(\mathfrak{p}-\varkappa)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\Bigg)\Bigg\} \\
&+\tau_2(\ell)\Bigg\{\frac{(1-\sigma)}{AB(\sigma)}\Bigg(\lambda\int_0^{3}\int_0^{\varkappa} e^{-a(\varkappa-\varsigma)}\sin b(\varkappa-\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa \\
&-\int_0^{1} e^{-a(1-\varsigma)}\sin b(1-\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\Bigg) \\
&+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\Bigg(\lambda\int_0^{3}\int_0^{\varkappa}\Big\{-ae^{-a(3-\varkappa)}\sin b(3-\varkappa) \\
&-be^{-a(3-\varkappa)}\cos b(3-\varkappa)+b\Big\}(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa \\
&-\int_0^{1}\int_0^{\varkappa} e^{-a(1-\varkappa)}\sin b(1-\varkappa)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\Bigg)\Bigg\} \\
&+\frac{(1-\sigma)}{AB(\sigma)}\Bigg(\int_0^{\ell} e^{-a(\ell-\varsigma)}\sin b(\ell-\varsigma)G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma\Bigg) \\
&+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_0^{\ell}\int_0^{\varkappa} e^{-a(\ell-\varkappa)}\sin b(\ell-\varkappa)(\varkappa-\varsigma)^{\sigma-1}G(\varsigma,\mathfrak{u}(\varsigma))d\varsigma d\varkappa\Bigg]
\end{aligned} \tag{4.23}
$$

Where we set $\hat{\tau}_1 = \max\limits_{\ell\in[0,1]} \tau_1(\ell)$, $\hat{\tau}_2 = \max\limits_{\ell\in[0,1]} \tau_2(\ell)$, $\varepsilon_2 = \max\limits_{\ell\in[0,1]} \{b - ae^{-a\ell}\sin b\ell - be^{-a\ell}\cos b\ell\}$

$$\begin{aligned}
\mathfrak{R}_1 &= \frac{(1-\sigma)}{b(a^2+b^2)AB\,(\sigma)}\Big\{\,\varepsilon_2 \\
&\quad +\hat{\tau}_1\left(\left|\sum_{i=1}^{n} j_i\right|\,\left|\left(b - ae^{-a\eta_i}\sin b\eta_i - be^{-a\eta_i}\cos b\eta_i\right)\right| + \left|\left(b - ae^{-a\mathfrak{p}}\sin b\mathfrak{p} - be^{-a\mathfrak{p}}\cos b\mathfrak{p}\right)\right|\right) \\
&\quad +\hat{\tau}_2\left(\ |\lambda|\,\frac{1}{a^2+b^2}\left|(a^2-b^2)e^{-a\mathfrak{z}}\sin b\mathfrak{z} + 2ab\left(e^{-a\mathfrak{z}}\cos b\mathfrak{z} - 1\right) + b(a^2+b^2)\mathfrak{z}\right|\right. \\
&\quad \left. + \left|\left(b - ae^{-a}\sin b - be^{-a}\cos b\right)\right|\ \right)\Big\}, && (4.24) \\
\Upsilon_1 &= \frac{\sigma}{b(a^2+b^2)AB\,(\sigma)\,\Gamma(\sigma+1)}\Big\{\,\varepsilon_2 \\
&\quad +\hat{\tau}_1\left(\left|\sum_{i=1}^{n} j_i\right|\,|\eta_i^{\sigma}|\,\left|\left(b - ae^{-a\eta_i}\sin b\eta_i - be^{-a\eta_i}\cos b\eta_i\right)\right|\right. \\
&\quad \left. + \mathfrak{p}^{\sigma}\left|\left(b - ae^{-a\mathfrak{p}}\sin b\mathfrak{p} - be^{-a\mathfrak{p}}\cos b\mathfrak{p}\right)\right|\ \right) \\
&\quad +\hat{\tau}_2\left(\ |\lambda|\,|\mathfrak{z}^{\sigma}|\,\left|\left((a^2-b^2)e^{-a\mathfrak{z}}\sin b\mathfrak{z} - 2ab\left(1 - e^{-a\mathfrak{z}}\cos b\mathfrak{z}\right) + b(a^2+b^2)\mathfrak{z}\right)\right.\right. \\
&\quad \left. + \left|\left(b - ae^{-a}\sin b - be^{-a}\cos b\right)\right|\ \right)\Big\} && (4.25)
\end{aligned}$$

and

$$\lambda_1 = \mathfrak{R}_1 + \Upsilon_1 - \frac{1}{b(a^2+b^2)AB\,(\sigma)}\varepsilon_2\left\{(1-\sigma) + \frac{\sigma}{\Gamma(\sigma+1)}\right\}. \tag{4.26}$$

Theorem 4.3. Suppose $G : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function and satisfying $(H5)$ – $(H6)$. Then the multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 < 0$ has atleast one solution if

$$\mathfrak{L}\lambda_1 < 1,$$

where λ_1 is defined in (4.26).

Proof. This theorem can be proved in a same way as the proof of Theorem 4.1. ∎

4.3 Uniqueness

This section provides unique solution of multi-term AB-Caputo fractional BVP (4.1) – (4.2) with three different cases $\delta_1^2 - 4\delta_0\delta_2 = 0$, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$. To obtain unique solution, Theorem 1.1 is applied.

Theorem 4.4. Suppose $G : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function and satisfying (H5). Then the AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2 - 4\delta_0\delta_2 = 0$ has a unique solution with

$$\mathfrak{L}\left(\Sigma_1 + \Omega_1\right) < 1,$$

where Σ_1 and Ω_1 are given in (4.14) and (4.15).

Proof. Consider

$$\mathfrak{B}_{r_2} = \{\mathrm{u} \in X : \|\mathrm{u}\| \leq \ r_2 \geq (\mathfrak{L}r_2 + M)\left(\Sigma_1 + \Omega_1\right)\},$$

and we set $\sup\limits_{\ell\in[0,1]} |G(\ell,0)| = M$, then

$$\begin{aligned} |G(\ell,\mathrm{u}(\ell))| &= |G(\ell,\mathrm{u}(\ell)) - G(\ell,0) + G(\ell,0)| \\ &\leq |G(\ell,\mathrm{u}(\ell)) - G(\ell,0)| + |G(\ell,0)| \\ &\leq \mathfrak{L}\,\|\mathrm{u}\| + M \leq \mathfrak{L}r_2 + M. \end{aligned}$$

Step 1. In this step, we show that $\mathcal{F}\mathfrak{B}_{r_2} \subset \mathfrak{B}_{r_2}$. For this, we consider

$$\begin{aligned} |(\mathcal{F}\mathrm{u})(\ell)| \leq & \left[\left\{\frac{(1-\sigma)}{AB\,(\sigma)}\int\limits_0^\ell \left|(\ell-\varsigma)e^{\mathfrak{m}(\ell-\varsigma)}\right| |G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma\right.\right. \\ & \left.+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\int\limits_0^\ell\int\limits_0^\varkappa \left|(\ell-\varkappa)e^{\mathfrak{m}(\ell-\varkappa)}\right| \left|(\varkappa-\varsigma)^{\sigma-1}\right| |G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma d\varkappa\right\} \\ & +|\kappa_1(\ell)|\left\{\frac{(1-\sigma)}{AB\,(\sigma)}\left(\left|\sum_{i=1}^n j_i\right|\int\limits_0^{\eta_i}\left|(\eta_i-\varsigma)e^{\mathfrak{m}(\eta_i-\varsigma)}\right| |G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma\right.\right. \\ & \left.+\int\limits_0^{\mathfrak{p}}\left|(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varsigma)}\right| |G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma\right) + \frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)} \\ & \left(\left|\sum_{i=1}^n j_i\right|\int\limits_0^{\eta_i}\int\limits_0^{\varkappa}\left|(\eta_i-\varkappa)e^{\mathfrak{m}(\eta_i-\varkappa)}\right| \left|(\varkappa-\varsigma)^{\sigma-1}\right| |G(\varsigma,\mathrm{u}(\varsigma))|\,d\varsigma d\varkappa\right. \end{aligned}$$

$$+\int_0^{\mathfrak{p}}\int_0^{\varkappa}\left|(\mathfrak{p}-\varkappa)e^{\mathfrak{m}(\mathfrak{p}-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathfrak{u}(\varsigma)))\right|d\varsigma d\varkappa\Bigg)\Bigg\}$$

$$+\left|\kappa_2(\ell)\right|\Bigg\{\frac{(1-\sigma)}{AB\,(\sigma)}\Bigg(|\lambda|\int_0^{3}\int_0^{\varkappa}\left|(\varkappa-\varsigma)e^{\mathfrak{m}(\varkappa-\varsigma)}\right|\left|G(\varsigma,\mathfrak{u}(\varsigma))\right|d\varsigma d\varkappa$$

$$+\int_0^{1}\left|(1-\varsigma)e^{\mathfrak{m}(1-\varsigma)}\right|\left|G(\varsigma,\mathfrak{u}(\varsigma))\right|d\varsigma\Bigg)+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}$$

$$\Bigg(|\lambda|\int_0^{3}\int_0^{\varkappa}\left|\left(\frac{\mathfrak{m}e^{\mathfrak{m}(3-\varkappa)}(3-\varkappa)-e^{\mathfrak{m}(3-\varkappa)}+1}{\mathfrak{m}^2}\right)\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathfrak{u}(\varsigma))\right|d\varsigma d\varkappa$$

$$+\int_0^{1}\int_0^{\varkappa}\left|(1-\varkappa)e^{\mathfrak{m}(1-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathfrak{u}(\varsigma))\right|d\varsigma d\varkappa\Bigg)\Bigg\}\Bigg].$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides,

$$\|(\mathcal{F}\mathfrak{u})\|\leq(\mathfrak{L}r_2+M)\Bigg[\frac{(1-\sigma)}{AB\,(\sigma)\,|\mathfrak{m}^2|}\Bigg\{\varepsilon_1$$

$$+\hat{\kappa}_1\left(\left|\sum_{i=1}^{n}j_i\right|\left|1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)\right|+\left|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})\right|\right)$$

$$+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\left|\mathfrak{m}3-2(e^{\mathfrak{m}3}-1)+\mathfrak{m}3e^{\mathfrak{m}3}\right|+\left|1-e^{\mathfrak{m}}(1-\mathfrak{m})\right|\right)\Bigg\}$$

$$+\frac{\sigma}{\mathfrak{m}^2AB\,(\sigma)\,\Gamma(\sigma+1)}\Bigg\{\varepsilon_1$$

$$+\hat{\kappa}_1\left(\left|\sum_{i=1}^{n}j_i\right|\left|\eta_i^{\sigma}\right|\left|1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)\right|+\left|\mathfrak{p}^{\sigma}\right|\left|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})\right|\right)$$

$$+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\left|3^{\sigma}\right|\left|2(1-e^{\mathfrak{m}3})+\mathfrak{m}3(1+e^{\mathfrak{m}3})\right|+\left|1-e^{\mathfrak{m}}(1-\mathfrak{m})\right|\right)\Bigg\}\Bigg]$$

from (4.14) and (4.15)

$$\|(\mathcal{F}\mathfrak{u})\|\leq(\mathfrak{L}r_2+M)\,(\Sigma_1+\Omega_1)\leq\ r_2.$$

Hence $\mathcal{F}\mathfrak{B}_{r_2}\subset\mathfrak{B}_{r_2}$.

Step 2. Now we prove $\mathcal{F}$ is contraction in this step. For this, consider

$$
\begin{aligned}
&|(\mathcal{F}\mathrm{u})(\ell)-(\mathcal{F}v)(\ell)| \\
&\leq \left[\frac{1}{AB\,(\sigma)}\left\{(1-\sigma)\int\limits_0^{\ell}\left|(\ell-\varsigma)e^{\mathrm{m}(\ell-\varsigma)}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma\right.\right. \\
&\left.+\frac{\sigma}{\Gamma(\sigma)}\int\limits_0^{\ell}\int\limits_0^{\varkappa}\left|(\ell-\varkappa)e^{\mathrm{m}(\ell-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right\} \\
&+|\kappa_1(\ell)|\left\{\frac{(1-\sigma)}{AB\,(\sigma)}\left(\left|\sum_{i=1}^{n}j_i\right|\int\limits_0^{\eta_i}\left|(\eta_i-\varsigma)e^{\mathrm{m}(\eta_i-\varsigma)}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma\right.\right. \\
&\left.+\int\limits_0^{\mathfrak{p}}\left|(\mathfrak{p}-\varkappa)e^{\mathrm{m}(\mathfrak{p}-\varsigma)}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma\right) \\
&+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)}\left(\left|\sum_{i=1}^{n}j_i\right|\int\limits_0^{\eta_i}\int\limits_0^{\varkappa}\left|(\eta_i-\varkappa)e^{\mathrm{m}(\eta_i-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right. \\
&\left.\left.+\int\limits_0^{\mathfrak{p}}\int\limits_0^{\varkappa}\left|(\mathfrak{p}-\varkappa)e^{\mathrm{m}(\mathfrak{p}-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right)\right\} \\
&+|\kappa_2(\ell)|\left\{\frac{(1-\sigma)}{AB\,(\sigma)}\left(|\lambda|\int\limits_0^{3}\int\limits_0^{\varkappa}\left|(\varkappa-\varsigma)e^{\mathrm{m}(\varkappa-\varsigma)}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right.\right. \\
&\left.+\int\limits_0^{1}\left|(1-\varsigma)e^{\mathrm{m}(1-\varsigma)}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma\right)+\frac{\sigma}{AB\,(\sigma)\,\Gamma(\sigma)} \\
&\left(|\lambda|\int\limits_0^{3}\int\limits_0^{\varkappa}\left|\left(\frac{\mathrm{m}e^{\mathrm{m}(3-\varkappa)}(3-\varkappa)-e^{\mathrm{m}(3-\varkappa)}+1}{\mathrm{m}^2}\right)\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right. \\
&\left.\left.\left.+\int\limits_0^{1}\int\limits_0^{\varkappa}\left|(1-\varkappa)e^{\mathrm{m}(1-\varkappa)}\right|\left|(\varkappa-\varsigma)^{\sigma-1}\right|\left|G(\varsigma,\mathrm{u}(\varsigma))-G(\varsigma,v(\varsigma))\right|d\varsigma d\varkappa\right)\right\}\right],
\end{aligned}
$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides,

$$
\|(\mathcal{F}\mathrm{u})-(\mathcal{F}v)\|\leq\mathfrak{L}\left[\frac{(1-\sigma)}{AB\,(\sigma)\,\mathrm{m}^2}\left\{\right.\right.
$$

$$+\hat{\kappa}_1\left(\left|\sum_{i=1}^{n} j_i\right| \left|1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)\right| + \left|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})\right|\right)$$

$$+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\left|\mathfrak{m}3-2(e^{\mathfrak{m}3}-1)+\mathfrak{m}3e^{\mathfrak{m}3}\right| + \left|1-e^{\mathfrak{m}}(1-\mathfrak{m})\right|\right)\Bigg\}$$

$$+\frac{\sigma}{|\mathfrak{m}^2|\,AB(\sigma)\,\Gamma(\sigma+1)}\Bigg\{\varepsilon_1$$

$$+\hat{\kappa}_1\left(\left|\sum_{i=1}^{n} j_i\right| |\eta_i^\sigma|\left|1-e^{\mathfrak{m}\eta_i}(1-\mathfrak{m}\eta_i)\right| + |\mathfrak{p}^\sigma|\left|1-e^{\mathfrak{m}\mathfrak{p}}(1-\mathfrak{m}\mathfrak{p})\right|\right)$$

$$+\hat{\kappa}_2\left(\frac{1}{|\mathfrak{m}|}|\lambda|\,|3^\sigma|\left|2(1-e^{\mathfrak{m}3})+\mathfrak{m}3(1+e^{\mathfrak{m}3})\right| + \left|1-e^{\mathfrak{m}}(1-\mathfrak{m})\right|\right)\Bigg\}\Bigg]\,\|\mathfrak{u}-\upsilon\|,$$

which implies

$$\|(\mathcal{F}\mathfrak{u})-(\mathcal{F}\upsilon)\| \leq \mathfrak{L}\left(\Sigma_1+\Omega_1\right)\|\mathfrak{u}-\upsilon\|,$$

as $\mathfrak{L}\left(\Sigma_1+\Omega_1\right)<1$. Thus $\mathcal{F}$ is a contraction. Hence from Theorem 1.1. we get a unique solution of given multi-term AB-Caputo fractional BVP (4.1) – (4.2) for the case $\delta_1^2-4\delta_0\delta_2=0$. ■

Similarly, unique solution of multi-term AB-Caputo fractional BVP (4.1) – (4.2) with the $\delta_1^2-4\delta_0\delta_2>0$ and $\delta_1^2-4\delta_0\delta_2<0$, can be obtained with the help of Theorem 1.1. Here we just state the uniqueness theorems for both cases. Their proofs can be done in the same way as the proof of Theorem 4.4.

Theorem 4.5. Suppose $G:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous function and satisfying $(H5)$. Then the multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2-4\delta_0\delta_2>0$ has a unique solution with

$$\mathfrak{L}(\Phi+\Psi)<1,$$

where Φ and Ψ are given in (4.19) and (4.20).

Theorem 4.5. Suppose $G:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous function and satisfying $(H5)$. Then the multi-term AB-Caputo fractional BVP (4.1) – (4.2) with $\delta_1^2-4\delta_0\delta_2<0$ has a unique solution with

$$\mathfrak{L}(\mathfrak{R}_1+\Upsilon_1)<1,$$

where $\mathfrak{R}_1$ and Υ_1 are given in (4.24) and (4.25).

Next, we discuss examples to validate the all conditions of Theorem 4.3 and Theorem 4.4 to get a unique solution.

Example 4.1. Take the following multi-term AB-Caputo fractional BVP for $\delta_1^2 - 4\delta_0\delta_2 = 0$,

$$\left(2\ {}^{ABC}_{\ \ 0}D^{2.7} + 4\ {}^{ABC}_{\ \ 0}D^{1.7} + 2\ {}^{ABC}_{\ \ 0}D^{0.7}\right)\mathrm{u}(\ell) = G(\ell, \mathrm{u}(\ell)), \quad 0 < \ell < 1, \tag{4.27}$$

having boundary conditions

$$\mathrm{u}(0) = 0,\ \mathrm{u}\left(\frac{1}{7}\right) = 2\mathrm{u}\left(\frac{1}{5}\right) + 3\mathrm{u}\left(\frac{2}{9}\right),\ \mathrm{u}(1) = \frac{3}{8}\int_0^{\frac{4}{5}} \mathrm{u}(\varkappa)d\varkappa. \tag{4.28}$$

Consider

$$G(\ell, \mathrm{u}(\ell)) = \frac{W}{9}\left[\ell^3 e^{-4\ell} + \cos\mathrm{u}\right]. \tag{4.29}$$

From (4,27), we have $\sigma = 0.7,\ \delta_2 = 2,\ \delta_1 = 4,\ \delta_0 = 2$, so $\delta_1^2 - 4\delta_0\delta_2 = 0$ and $\mathrm{m} = \frac{-\delta_1}{2\delta_2} = -1$. From (4.28), we have $\eta_1 = \frac{1}{5},\ \eta_2 = \frac{2}{9},\ j_1 = 2,\ j_2 = 3,\ \mathsf{p} = \frac{1}{7},\ \mathfrak{z} = \frac{4}{5},\ \lambda = \frac{3}{8}$.

Now we calculate remaining values

$\mu_1 = 0.03780962416,\ \varepsilon = 0.2642411177,\ \varpi_1 = -0.7371775365,\ \varpi_2 =$ [illegible], $\varpi_3 = 0.2424457053,\ \varpi_4 = 0.296176492,\ \varkappa_1 = 1.80771573 2,\ \varkappa_2 = 6.828239891,\ \Sigma_1 = 0.9572450864,\ \Omega_1 = 0.5142844518$ and $W < 6.11608518$.

Thus

$$|G(\ell, \mathrm{u}(\ell)) - G(\ell, v(\ell)| = \frac{W}{9}\left|\cos\mathrm{u}(\ell) - \cos v(\ell)\right| \le \frac{W}{9}\left|\mathrm{u}(\ell) - v(\ell)\right|$$

and from (4.29), we have $\mathfrak{L} = \frac{W}{9}$.Taking $W < 6.11608518$, we obtain

$$\mathfrak{L}(\Sigma_1 + \Omega_1) < 1.$$

Hence, we get a unique solution of multi-term AB-Caputo fractional BVP (4.27)−(4.28) for $\delta_1^2 - 4\delta_0\delta_2 = 0$, by satisfying all conditions of Theorem 4.4.

Example 4.2. Take the following multi-term AB–Caputo fractional BVP

$$\left(3\ {}^{ABC}_{\ \ 0}D^{2.7} + 5\ {}^{ABC}_{\ \ 0}D^{1.7} + 2\ {}^{ABC}_{\ \ 0}D^{0.7}\right)\mathrm{u}(\ell) = G(\ell, \mathrm{u}(\ell)), \quad 0 < \ell < 1. \tag{4.30}$$

having boundary conditions (4.28) and $G(\ell, \mathrm{u}(\ell))$ is given in (4.29). From (4.30), we have $\delta_2 = 3,\ \delta_1 = 5,\ \delta_0 = 2$, so $\delta_1^2 - 4\delta_0\delta_2 = 1 > 0$.

Now we calculate remaining values

$\mathfrak{m}_1 = -1$, $\mathfrak{m}_2 = -\frac{2}{3}$, $\delta = \frac{1}{3}$, $\varepsilon_1 = 0.06516917508$, $\hbar_1 = 0.2356511266$, $w_1 =$ 0.03061514[illegible] $w_2 = 0.08510148352$, $w_3 = 0.6465354933$, $w_4 = 0.2934754621$, $\hbar_2 = 0.106730035$, $\Phi =$ 1.14793904, $\Psi = 0.1870472739$, $\mu_2 = -0.0640148074$.

Thus

$$|G(\ell, \mathfrak{u}(\ell)) - G(\ell, v(\ell)| = \frac{W}{9} |\cos \mathfrak{u}(\ell) - \cos v(\ell)| \leq \frac{W}{9} |\mathfrak{u}(\ell) - v(\ell)|$$

where $\mathfrak{L} = \frac{W}{9}$. Taking $W < 6.749163282$, we obtain

$$\mathfrak{L}(\Phi + \Psi) < 1.$$

Hence, we get a unique solution of multi-term AB-Caputo fractional BVP (4.30) – (4.28) for $\delta_1^2 - 4\delta_0\delta_2 > 0$, by satisfying all conditions of Theorem 4.5.

In next example, we find exact solution of case $\delta_1^2 - 4\delta_0\delta_2 = 0$.

Example 4.3. Take the following fractional differential equation

$$\begin{aligned} & {}^{ABC}_{\ \ 0}D^{\sigma} \left(\delta_2 \ D^2 + \delta_1 \ D + \delta_0 \ \right) \mathfrak{u}(\ell) \\ = \ & 2\sum_{k=1}^{\infty} \ell^k E_{\frac{1}{2},k-1}(-\ell)^{\frac{1}{2}} + 2\sum_{k=1}^{\infty} (-1)^{k-1} \ell^{2k-1} E \ \text{[illegible]} \end{aligned}$$

having boundary conditions

$$\mathfrak{u}(0) = 0, \ \mathfrak{u}(1) = 2e^{\ell}, \tag{4.32}$$

with case $\delta_1^2 - 4\delta_0\delta_2 = 0$. Here we take $\sigma = \frac{1}{2}$, $AB\left(\frac{1}{2}\right) = 1$, $\delta_1 = -2$ and $\delta_2 = \delta_0 = 1$, now we apply ${}^{AB}I^{\sigma}$ on both sides of (4.31) and using calculation in [12], we get

$$\left(\ D^2 - 2 \ D + 1 \ \right) \mathfrak{u}(\ell) = e^{\ell} + \sin \ell.$$

Using the method of variation of parameters, we obtain

$$\mathfrak{u}(\ell) = c_1 e^{\ell} + c_2 \ell e^{\ell} + \frac{\ell^2}{2} e^{\ell} + \frac{1}{2} \cos \ell. \tag{4.33}$$

Using (4.32), we find $c_1 = -\frac{1}{2}$ and $c_2 = 2e^{\ell-1}$. Putting these values in (4.33), we get [illegible] solution,

$$u(t) = -\frac{1}{2}e^{t} + 2te^{2t-1} - \frac{t^2}{2}e^{t} + \frac{1}{2}\cos t$$

4.4 Conclusion

We investigate the multi-term AB-Caputo fractional BVP (4.1) – (4.2) of order $0 < \alpha \leq 1$ with different cases ($\delta_1^2 - 4\delta_0\delta_2 = 0$, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$) in this chapter. In Section 4.2, main results are presented for all cases ($\delta_1^2 - 4\delta_0\delta_2 = 0, \delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$) in Lemmas 4.1, 4.2, 4.3. After that, we discuss existence results of given multi-term AB-Caputo fractional BVP (4.1) – (4.2) for all cases ($\delta_1^2 - 4\delta_0\delta_2 = 0$, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$) using Krasnoselskii's fixed point theorem which are given in Theorems 4.1, 4.2 and 4.3 respectively. In Section 4.3, we obtain unique solution via Banach contraction principle for all case ($\delta_1^2 - 4\delta_0\delta_2 = 0$, $\delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$). In the end of Section 4.3, three examples are given. In Example 4.1 and Example 4.2, we obtain a unique solution by validating conditions of Theorem 4.4 and Theorem 4.5. In Example 4.3, we find exact solution of multi-term AB-Caputo fractional BVP (4.31) – (4.32) for the case $\delta_1^2 - 4\delta_0\delta_2 = 0$.

Chapter 5

Existence and Stability Results of Coupled System

5.1 Introduction

A Coupled system consisting of two differential equations or FDEs with two dependent variables and one independent variable. Coupled systems are used in rheology, aerodynamics, viscoelasticity, physics and different engineering problems. [100]. practical and theoretical features of solving coupled models in engineering with numerical and analytical methods are provided. These models are great solver methods to reduce the many difficulties. He provided coupled systems to solve different problems such as (*i*) micro- and macroscale problems (coupling separate scales) (*ii*) multiscale problems (homogenization of scales) (*iii*) multiphysics problems (multiple physical logical scales). To solve multiscale and multiphysics problems, they used embedded and iterative schemes. They also provided applications related to muti-scale for instance Fluid dynamics, Plasma dynamics, Reaction-diffusion, Transport-reaction, theory of rigid body and Sputtering applications etc.

Coupled FDEs are one of the useful tools in mathematics due to involvement in developing different types of models in highly complex-systems. Complex systems models have demonstrated their effectiveness in real-life like prediction and statistics of earthquake, the climate, stock markets dynamics, freeway traffic, human brain, biological cellular networks etc. Although there are few overlap in their methodologies and scope, it is possible to distinguish the [illegible] and concepts in following categories: synergetics, catastrophes, turbulence, nonlinear dynamics, adaptive systems, instabilities, chaos, cellular automata, stochastic processes, networks and graphs, computational intelligence and genetic algorithms. The analysis and the design of fractional control systems are heavily rely on coupled system. Fractional control systems are used to adjust complex processes through time delays and anomalous diffusion.

In recent time, some researchers work on existence and stability theory of coupled system of FDEs involving various fractional operators. For instance, Ahmad and Nieto [101], studied existence results of coupled system with three-points boundary conditions involving Caputo fractional derivative. To obtain existence results, they used Schauder fixed point theorem. After that Ntouyas and Obaid [102], presented uniqueness and existence of the solution of coupled system having integral type boundary conditions including Caputo fractional derivative. They used Leray-Schauder fixed point theorem for existence and obtained unique solution via Banach contraction principle. Later on Abdellaoui et al. [103], discussed existence results of coupled system with arbitrary order involving Caputo fractional derivative. They applied Krasnoselskii's and Schaefer fixed point theorems to find existence results of coupled system. Recently Kouachi and Guezane-Lakoud [104], studied uniqueness and existence of the solution of coupled system for FDEs including p-Laplacian operator involving Caputo fractional derivative. They applied Green functions to transferred coupled system into integral system. They used topological degree theory and Leray Schauder type fixed point theorem to get existence results and obtained unique solution via Banach contraction principle, of given coupled system. They also find stable solution by using HU stability. Some other results related to coupled systems and their applications can be seen in [105, 106, 107, 108, 109, 110, 111, 112, 113].

For motivation of above work, we present the following nonlinear AB-Caputo fractional coupled BVP

$$\begin{cases} {}^{ABC}_{\quad 0}D^{\alpha}\mathrm{u}(\ell) = G(\ell, \mathrm{u}(\ell), \vartheta(\ell)), \quad 1 < \alpha \leq 2, \\ \\ {}^{ABC}_{\quad 0}D^{\sigma}\vartheta(\ell) = E(\ell, \mathrm{u}(\ell), \vartheta(\ell)), \quad 1 < \sigma \leq 2, \end{cases} \tag{5.1}$$

with boundary conditions

$$\begin{cases} \mathrm{u}(0) = 0, \ \theta \mathrm{u}'(\mathfrak{S}) = \phi \mathrm{u}'(1), \\ \vartheta(0) = 0, \ \theta \vartheta'(\mathfrak{S}) = \phi \vartheta'(1), \end{cases} \tag{5.2}$$

where $G, E : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are both continuous functions, for all $\ell \in [0,1]$ and here we take $\theta, \phi > 0$ $(\theta \neq \phi)$, $0 \leq \ell \leq \mathfrak{S} \leq 1$.

For our onward discussion, we use the following assumptions which are crucial for our results

$$\begin{cases} (i).\ |G(\ell, \mathfrak{u}_1(\ell), \vartheta_1(\ell)) - G(\ell, \mathfrak{u}_2(\ell), \vartheta_2(\ell))| \\ \qquad \leq \{\mathfrak{L}_1 |\mathfrak{u}_1(\ell) - \mathfrak{u}_2(\ell)| + \mathfrak{L}_2 |\vartheta_1(\ell) - \vartheta_2(\ell)|\}, \text{ where } \mathfrak{L}_1, \mathfrak{L}_2 > 0. \\ (ii).\ |E(\ell, \mathfrak{u}_1(\ell), \vartheta_1(\ell)) - E(\ell, \mathfrak{u}_2(\ell), \vartheta_2(\ell))| \\ \qquad \leq \{\mathfrak{I}_1 |\mathfrak{u}_1(\ell) - \mathfrak{u}_2(\ell)| + \mathfrak{I}_2 |\vartheta_1(\ell) - \vartheta_2(\ell)|\}, \text{ where } \mathfrak{I}_1, \mathfrak{I}_2 > 0. \end{cases} \tag{H9}$$

$$\begin{cases} (i).\ |G(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \leq \xi_1(\ell), \text{ where } \xi_1(\ell) \in (C[0,1], \mathbb{R}^+). \\ (ii).\ |E(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \leq \xi_2(\ell), \text{ where } \xi_2(\ell) \in (C[0,1], \mathbb{R}^+). \end{cases} \tag{H10}$$

$$\begin{cases} (i).\ \sup\limits_{\ell \in [0,1]} \int\limits_0^1 |\Psi_1(\ell, \varsigma)| \, d\varsigma \leq \varrho_1. \\ (ii).\ \sup\limits_{\ell \in [0,1]} \int\limits_0^1 |\Psi_2(\ell, \varsigma)| \, d\varsigma \leq \varrho_2. \end{cases} \tag{H11}$$

5.2 Existence Results

This section presents existence of the solution of AB-Caputo fractional coupled BVP [illegible] (5.2) with the help of Theorem 1.3. First, we derive our main lemma for the solution of the following linear AB-Caputo fractional coupled BVP

$$\begin{cases} {}^{ABC}{}_0D^{\alpha}\mathfrak{u}(\ell) = Q_1(\ell), \quad 1 < \alpha \leq 2, \\ {}^{ABC}{}_0D^{\sigma}\vartheta(\ell) = Q_2(\ell), \quad 1 < \sigma \leq 2. \end{cases} \tag{5.3}$$

with boundary conditions (5.2), for all $\ell \in [0,1]$.

Lemma 5.1. Suppose $Q_1, Q_2 \in (C[0,1], \mathbb{R})$ then solution of (5.3) with boundary conditions (5.2) is given as

$$\begin{cases} \mathfrak{u}(\ell) = \ell \frac{(2-\alpha)}{(\theta-\phi)AB(\alpha-1)} (\phi Q_1(1) - \theta Q_1(\mathfrak{S})) + \int\limits_0^1 \Psi_1(\ell, \varsigma) Q_1(\varsigma) d\varsigma. \\ \\ \vartheta(\ell) = \ell \frac{(2-\sigma)}{(\theta-\phi)AB(\sigma-1)} (\phi Q_2(1) - \theta Q_2(\mathfrak{S})) + \int\limits_0^1 \Psi_2(\ell, \varsigma) Q_2(\varsigma) d\varsigma. \end{cases} \tag{5.4}$$

where

$$\Psi_1(\ell,\varsigma)=\begin{cases}\frac{\phi(\alpha-1)^2\ell}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}(1-\varsigma)^{\alpha-2}-\frac{\theta(\alpha-1)^2\ell}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}(\mathfrak{S}-\varsigma)^{\alpha-2} & 0\leq\varsigma\leq\ell.\\ \quad+\frac{(2-\alpha)}{AB(\alpha-1)}+\frac{(\alpha-1)}{AB(\alpha-1)\Gamma(\alpha)}(\ell-\varsigma)^{\alpha-1}, & \\ +\frac{\phi(\alpha-1)^2\ell}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}(1-\varsigma)^{\alpha-2}-\frac{\theta(\alpha-1)^2\ell}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}(\mathfrak{S}-\varsigma)^{\alpha-2}, & \ell<\varsigma\leq\mathfrak{S}.\\ \quad+\frac{\phi(\alpha-1)^2\ell}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}(1-\varsigma)^{\alpha-2}, & \mathfrak{S}\leq\varsigma\leq1.\end{cases}\tag{5.5}$$

and

$$\Psi_2(\ell,\varsigma)=\begin{cases}\frac{\phi(\sigma-1)^2\ell}{(\theta-\phi)AB(\sigma-1)\Gamma(\sigma)}(1-\varsigma)^{\sigma-2}-\frac{\theta(\sigma-1)^2\ell}{(\theta-\phi)AB(\sigma-1)\Gamma(\sigma)}(\mathfrak{S}-\varsigma)^{\sigma-2} & 0\leq\varsigma\leq\ell.\\ \quad+\frac{(2-\sigma)}{AB(\sigma-1)}+\frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}(\ell-\varsigma)^{\sigma-1}, & \\ +\frac{\phi(\sigma-1)^2\ell}{(\theta-\phi)AB(\sigma-1)\Gamma(\sigma)}(1-\varsigma)^{\sigma-2}-\frac{\theta(\sigma-1)^2\ell}{(\theta-\phi)AB(\sigma-1)\Gamma(\sigma)}(\mathfrak{S}-\varsigma)^{\sigma-2}, & \ell<\varsigma\leq\mathfrak{S}.\\ \quad+\frac{\phi(\sigma-1)^2\ell}{(\theta-\phi)AB(\sigma-1)\Gamma(\sigma)}(1-\varsigma)^{\sigma-2}, & \mathfrak{S}\leq\varsigma\leq1.\end{cases}\tag{5.6}$$

Proof. Applying ${}^{AB}I^{\alpha}$, ${}^{AB}I^{\sigma}$ on both sides of (5.3) and using Proposition 1.1, we obtain

$$\begin{cases}\mathrm{u}(\ell)=c_1+c_2\ell+\frac{(2-\alpha)}{AB(\alpha-1)}\int_0^{\ell}Q_1(\varsigma)d\varsigma+\frac{(\alpha-1)}{AB(\alpha-1)\Gamma(\alpha)}\int_0^{\ell}(\ell-\varsigma)^{\alpha-1}Q_1(\varsigma)d\varsigma.\\ \vartheta(\ell)=c_3+c_4\ell+\frac{(2-\sigma)}{AB(\sigma-1)}\int_0^{\ell}Q_2(\varsigma)d\varsigma+\frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}Q_2(\varsigma)d\varsigma.\end{cases}\tag{5.7}$$

Now taking derivative on both sides of (5.7), we get

$$\begin{cases}\mathrm{u}'(\ell)=c_2+\frac{(2-\alpha)}{AB(\alpha-1)}Q_1(\ell)+\frac{(\alpha-1)^2}{AB(\alpha-1)\Gamma(\alpha)}\int_0^{\ell}(\ell-\varsigma)^{\alpha-2}Q_1(\varsigma)d\varsigma.\\ \vartheta'(\ell)=c_4+\frac{(2-\sigma)}{AB(\sigma-1)}Q_2(\ell)+\frac{(\sigma-1)^2}{AB(\sigma-1)\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-2}Q_2(\varsigma)d\varsigma.\end{cases}\tag{5.8}$$

Using boundary conditions (5.2), we obtain $c_1=0,\ c_3=0,$

$$\begin{aligned}c_2 &= \frac{(2-\alpha)}{(\theta-\phi)AB(\alpha-1)}(\phi Q_1(1)-\theta Q_1(\mathfrak{S}))\\ &+\frac{\phi(\alpha-1)^2}{(\theta-\phi AB(\alpha-1)\Gamma(\alpha)}\int_0^1(1-\varsigma)^{\alpha-2}Q_1(\varsigma)d\varsigma\\ &-\frac{\theta(\alpha-1)^2}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}\int_0^{\mathfrak{S}}(\mathfrak{S}-\varsigma)^{\alpha-2}Q_1(\varsigma)d\varsigma.\end{aligned}$$

$$\begin{aligned} c_4 &= \frac{(2-\sigma)}{(\theta-\phi)AB\,(\sigma-1)}(\phi Q_2(1)-\theta Q_2(\mathfrak{S})) \\ &+\frac{\phi(\sigma-1)^2}{(\theta-\phi AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^1(1-\varsigma)^{\sigma-2}Q_2(\varsigma)d\varsigma \\ &-\frac{\theta(\sigma-1)^2}{(\theta-\phi)AB\,(\sigma-1)\,\Gamma(\sigma)}\int_0^{\mathfrak{S}}(\mathfrak{S}-\varsigma)^{\sigma-2}Q_2(\varsigma)d\varsigma \end{aligned}$$

Putting values of c_1, c_2, c_3 and c_4 in (5.7), we obtain

$$\begin{cases} \mathrm{u}\,(\ell) = \ell\frac{(2-\alpha)}{(\theta-\phi)AB(\alpha-1)}(\phi Q_1(1)-\theta Q_1(\mathfrak{S})) + \frac{\phi(\alpha-1)^2\ell}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}\int_0^1(1-\varsigma)^{\alpha-2}Q_1(\varsigma)d\varsigma \\ \quad -\frac{\theta(\alpha-1)^2\ell}{(\theta-\phi)AB(\alpha-1)\Gamma(\alpha)}\int_0^{\mathfrak{S}}(\mathfrak{S}-\varsigma)^{\alpha-2}Q_1(\varsigma)d\varsigma + \frac{(2-\alpha)}{AB(\alpha-1)}\int_0^{\ell}Q_1(\varsigma)d\varsigma \\ \quad +\frac{(\alpha-1)}{AB(\alpha-1)\Gamma(\alpha)}\int_0^{\ell}(\ell-\varsigma)^{\alpha-1}Q_1(\varsigma)d\varsigma. \\ \\ \vartheta\,(\ell) = \ell\frac{(2-\sigma)}{(\theta-\phi)AB(\sigma-1)}(\phi Q_2(1)-\theta Q_2(\mathfrak{S})) + \frac{\phi(\sigma-1)^2\ell}{(\theta-\phi)AB(\sigma-1)\Gamma(\sigma)}\int_0^1(1-\varsigma)^{\sigma-2}Q_2(\varsigma)d\varsigma \\ \quad -\frac{\theta(\sigma-1)^2\ell}{(\theta-\phi)AB(\sigma-1)\Gamma(\sigma)}\int_0^{\mathfrak{S}}(\mathfrak{S}-\varsigma)^{\sigma-2}Q_2(\varsigma)d\varsigma + \frac{(2-\sigma)}{AB(\sigma-1)}\int_0^{\ell}Q_2(\varsigma)d\varsigma \\ \quad +\frac{(\sigma-1)}{AB(\sigma-1)\Gamma(\sigma)}\int_0^{\ell}(\ell-\varsigma)^{\sigma-1}Q_2(\varsigma)d\varsigma. \end{cases}$$

After simplification, we get required result (5.4). ■

Next, we consider, $X \times X = (C[0,1],\mathbb{R}) \times (C[0,1],\mathbb{R})$ is Banach space along with norm defined as,

$$\|(\mathrm{u},\vartheta)\|_{X\times X} = \|\mathrm{u}\|_X + \|\vartheta\|_X\,.$$

Define $\mathcal{F} : X \times X \to X \times X$ as,

$$\mathcal{F}(\mathrm{u},\vartheta)(\ell) = \begin{pmatrix} \mathcal{F}_1(\mathrm{u},\vartheta)(\ell) \\ \mathcal{F}_2(\mathrm{u},\vartheta)(\ell) \end{pmatrix}, \tag{5.9}$$

where

$$\begin{cases} \mathcal{F}_1(\mathrm{u},\vartheta)(\ell) = \delta_1^* + \int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))d\varsigma. \\ \mathcal{F}_2(\mathrm{u},\vartheta)(\ell) = \delta_2^* + \int_0^1 \Psi_2(\ell,\varsigma)E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))d\varsigma. \end{cases} \tag{5.10}$$

and

$$\begin{cases} \delta_1^* = \ell\frac{(2-\alpha)}{(\theta-\phi)AB(\alpha-1)}\left\{\phi G(1,\mathrm{u}(1),\vartheta(1)) - \theta G(\mathfrak{S},\mathrm{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\right\}. \\ \delta_2^* = \ell\frac{(2-\sigma)}{(\theta-\phi)AB(\sigma-1)}\left\{\phi E(1,\mathrm{u}(1),\vartheta(1)) - \theta E(\mathfrak{S},\mathrm{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\right\}. \end{cases} \tag{5.11}$$

Theorem 5.1. Suppose $G, E : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, both are continuous and satisfying $(H9)$ - $(H11)$. Then the AB-Caputo fractional coupled BVP (5.1) - (5.2) has atleast one solution, if

$$(\varrho_1 + \varrho_2)(\mathfrak{L}_1 + \mathfrak{L}_2) < 1.$$

Proof. Consider

$$\mathfrak{B}_{r_1} = \left\{(\mathrm{u},\vartheta) \in X \times X : \|(\mathrm{u},\vartheta)\| \le r_1 \ge (\Omega_1 + \varrho_1)\xi_1 + (\Omega_2 + \varrho_2)\xi_2\right\},$$

$\Omega_1 = \frac{2}{(\theta-\phi)AB(\alpha-1)}[\phi+\theta]$ and $\Omega_2 = \frac{2}{(\theta-\phi)AB(\sigma-1)}[\phi+\theta]$, which is closed, bounded, convex and nonempty subset of $X \times X$. Now we decompose $\mathcal{F}$ into $\mathcal{F}_1(\mathrm{u},\vartheta)(\ell)$ and $\mathcal{F}_2(\mathrm{u},\vartheta)(\ell)$ as,

$$\begin{cases} \mathcal{F}_1(\mathrm{u},\vartheta)(\ell) = \mathcal{F}_{1A}(\mathrm{u},\vartheta)(\ell) + \mathcal{F}_{1B}(\mathrm{u},\vartheta)(\ell), \\ \mathcal{F}_2(\mathrm{u},\vartheta)(\ell) = \mathcal{F}_{2A}(\mathrm{u},\vartheta)(\ell) + \mathcal{F}_{2B}(\mathrm{u},\vartheta)(\ell), \end{cases}$$

where

$$\begin{cases} \mathcal{F}_{1A}(\mathrm{u},\vartheta)(\ell) = \delta_1^* = \ell\frac{(2-\alpha)}{(\theta-\phi)AB(\alpha-1)}\left\{(\phi G(1,\mathrm{u}(1),\vartheta(1)) - \theta G(1,\mathrm{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\right\} \\ \mathcal{F}_{2A}(\mathrm{u},\vartheta)(\ell) = \delta_2^* = \ell\frac{(2-\sigma)}{(\theta-\phi)AB(\sigma-1)}\left\{(\phi E(1,\mathrm{u}(1),\vartheta(1)) - \theta E(1,\mathrm{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\right\}. \end{cases}$$

and

$$\begin{cases} \mathcal{F}_{1B}(\mathrm{u},\vartheta)(\ell) = \int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))d\varsigma. \\ \mathcal{F}_{2B}(\mathrm{u},\vartheta)(\ell) = \int_0^1 \Psi_2(\ell,\varsigma)E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))d\varsigma. \end{cases}$$

Step 1. In this step,we prove that $\mathcal{F}_1(\mathrm{u},\vartheta)(\ell) + \mathcal{F}_2(\mathrm{u}_1,\vartheta_1)(\ell)$ is self mapping on $\mathfrak{B}_{r_1}$. For this, consider (u,ϑ), $(\mathrm{u}_1,\vartheta_1) \in \mathfrak{B}_{r_1}$, we have

$$\begin{aligned} |\{\mathcal{F}_{1A}(\mathrm{u},\vartheta)(\ell) + \mathcal{F}_{1B}(\mathrm{u}_1,\vartheta_1)(\ell)\}| &= \left|\ell\frac{(2-\alpha)}{(\theta-\phi)AB(\alpha-1)}\{(\phi G(1,\mathrm{u}(1),\vartheta(1))\right. \\ &\quad \left. -\theta G(\mathfrak{S},\mathrm{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\} + \int_0^1 \Psi_1(\ell,\varsigma)G(\varsigma,\mathrm{u}_1(\varsigma),\vartheta_1(\varsigma))d\varsigma\right| \end{aligned}$$

$$\leq \frac{(2-\alpha)\ell}{(\theta-\phi)AB\,(\alpha-1)}\left|\phi G(1,\mathfrak{u}(1),\vartheta(1))-\theta G(\mathfrak{S},\mathfrak{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\right| + \int_0^1 |\Psi_1(\ell,\varsigma)|\,|G(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma,$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides,

$$\|\{\mathcal{F}_{1A}(\mathfrak{u},\vartheta)+\mathcal{F}_{1B}(\mathfrak{u}_1,\vartheta_1)\}\| \leq \frac{2}{(\theta-\phi)AB\,(\alpha-1)}\,[\phi+\theta]\,\xi_1+\xi_1\varrho_1$$

$$=(\Omega_1+\varrho_1)\xi_1.$$

where $\Omega_1=\frac{2}{(\theta-\phi)AB(\alpha-1)}\,[\phi+\theta]$ with $\theta\neq\phi$.

Similarly

$$\|\mathcal{F}_{2A}(\mathfrak{u},\vartheta)+\mathcal{F}_{2B}(\mathfrak{u}_1,\vartheta_1)\|\leq(\Omega_2+\varrho_2)\xi_2.$$

So

$$\begin{aligned}\|\mathcal{F}_1(\mathfrak{u},\vartheta)+\mathcal{F}_2(\mathfrak{u}_1,\vartheta_1)\| &\leq \|\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{1B}(\mathfrak{u}_1,\vartheta_1)(\ell)\| \\ &\quad+\|\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2B}(\mathfrak{u}_1,\vartheta_1)(\ell)\| \\ &\leq (\Omega_1+\varrho_1)\xi_1+(\Omega_2+\varrho_2)\xi_2<r_1.\end{aligned}$$

Hence $\mathcal{F}_1(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_2(\mathfrak{u}_1,\vartheta_1)(\ell)$ is self mapping on $\mathfrak{B}_{r_1}$.

Step 2. Next we prove $\mathcal{F}_{1B}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2B}(\mathfrak{u},\vartheta)(\ell)$ is contraction. For this, we consider

$$\begin{aligned}&\left|\{\mathcal{F}_{1B}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2B}(\mathfrak{u},\vartheta)(\ell)\}-\{\mathcal{F}_{1B}(\mathfrak{u}_1,\vartheta_1)(\ell)+\mathcal{F}_{2B}(\mathfrak{u}_1,\vartheta_1)(\ell)\}\right| \\ &\leq \int_0^1 |\Psi_1(\ell,\varsigma)|\,|G(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))-G(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma \\ &\quad+\int_0^1 |\Psi_2(\ell,\varsigma)|\,|E(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))-E(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma,\end{aligned}$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides

$$\|\{\mathcal{F}_{1B}(\mathfrak{u},\vartheta)+\mathcal{F}_{2B}(\mathfrak{u},\vartheta)\}-\{\mathcal{F}_{1B}(\mathfrak{u}_1,\vartheta_1)+\mathcal{F}_{2B}(\mathfrak{u}_1,\vartheta_1)\}\|$$

$$
\begin{aligned}
&\leq \varrho_1\mathfrak{L}_1\{\|\mathfrak{u}-\mathfrak{u}_1\|+\|\vartheta-\vartheta_1\|\}+\varrho_2\mathfrak{L}_2\{\|\mathfrak{u}-\mathfrak{u}_1\|+\|\vartheta-\vartheta_1\|\}\\
&= (\varrho_1\mathfrak{L}_1+\varrho_2\mathfrak{L}_2)\{\|\mathfrak{u}-\mathfrak{u}_1\|+\|\vartheta-\vartheta_1\|\}\\
&\leq (\varrho_1+\varrho_2)(\mathfrak{L}_1+\mathfrak{L}_2)\{\|\mathfrak{u}-\mathfrak{u}_1\|+\|\vartheta-\vartheta_1\|\},
\end{aligned}
$$

as $(\varrho_1+\varrho_2)(\mathfrak{L}_1+\mathfrak{L}_2)<1$, hence $\mathcal{F}_{1B}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2B}(\mathfrak{u},\vartheta)(\ell)$ is contraction.

Step 3. Now we show $\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)$ is continuous and compact. As G and E are both continuous, this implies that $\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)$ is continuous. As

$$
\begin{aligned}
\|(\mathcal{F}_{1A}(\mathfrak{u},\vartheta)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)\| &\leq \frac{2}{(\theta-\phi)AB\,(\alpha-1)}\{\phi\|\xi_1\|+\theta\|\xi_1\|\}\\
&\quad+\frac{2}{(\theta-\phi)AB\,(\sigma-1)}\{\phi\|\xi_2\|+\theta\|\xi_2\|\}\\
&\leq (\Omega_1+\Omega_2)\left(\|\xi_1\|+\|\xi_2\|\right)\leq r_1.
\end{aligned}
$$

Therefore $[\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)]\,(\mathfrak{B}_{r_1})$ is uniformly bounded.

Next, we prove $[\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)]\,(\mathfrak{B}_{r_1})$ is equicontinuous.

For this, we take $0\leq\ell_1\leq\ell_2\leq\mathfrak{S}\leq 1$,

$$
\begin{aligned}
&|\{(\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell_2)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell_2)\}-\{(\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell_1)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell_1)\}|\\
=&\ \Bigg|(\ell_2-\ell_1)\frac{(2-\alpha)}{(\theta-\phi)AB\,(\alpha-1)}\{(\phi G(1,\mathfrak{u}(1),\vartheta(1))-\theta G(1,\mathfrak{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\}\\
&+(\ell_2-\ell_1)\frac{(2-\sigma)}{(\theta-\phi)AB\,(\sigma-1)}\{(\phi E(1,\mathfrak{u}(1),\vartheta(1))-\theta E(1,\mathfrak{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\}\Bigg|\\
\leq&\ \frac{(2-\alpha)}{(\theta-\phi)AB\,(\alpha-1)}|(\ell_2-\ell_1)\{(\phi G(1,\mathfrak{u}(1),\vartheta(1))-\theta G(1,\mathfrak{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\}|\\
&+\frac{(2-\sigma)}{(\theta-\phi)AB\,(\sigma-1)}|(\ell_2-\ell_1)\{(\phi E(1,\mathfrak{u}(1),\vartheta(1))-\theta E(1,\mathfrak{u}(\mathfrak{S}),\vartheta(\mathfrak{S}))\}|.
\end{aligned}
$$

when $\ell_1\to\ell_2$, so $|\{(\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell_2)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell_2)\}-\{(\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell_1)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell_1)\}|\to 0$. Therefore $[\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)]\,(\mathfrak{B}_{r_1})$ is equicontinuous. Now by using Lemma 1.3, $[\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)]\,(\mathfrak{B}_{r_1})$ is relatively compact. So

$$
[\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)]\,(\mathfrak{B}_{r_1})\subseteq\overline{[\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)]\,(\mathfrak{B}_{r_1})}\subseteq X,
$$

therefore $[\mathcal{F}_{1A}(\mathfrak{u},\vartheta)(\ell)+\mathcal{F}_{2A}(\mathfrak{u},\vartheta)(\ell)]$ is compact. Thus all the conditions of Theorem 1.3 are satisfied. Hence given AB-Caputo fractional coupled BVP (5.1) – (5.2) has atleast one solution.

■

5.3 Uniqueness

This section provides unique solution of given AB-Caputo fractional coupled BVP (5.1) – (5.2).
To get unique solution, Banach contraction principle (Theorem 1.1) is used.

Theorem 5.2. Suppose $G, E : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ both are continuous and satisfying $(H9)$ – [illegible]
Then the AB-Caputo fractional coupled BVP (5.1) – (5.2) has unique solution if

$$\{(\Omega_1+\varrho_1)(\mathfrak{L}_1+\mathfrak{L}_2)+(\Omega_2+\varrho_2)(\mathfrak{I}_1+\mathfrak{I}_2)\}<1.$$

Proof. Consider the mapping $\Gamma : X\times X \to X\times X$ as given in above (5.9). Consider a closed ball

$$\mathfrak{B}_{r_2}=\{(\mathrm{u},\vartheta)\in X\times X : \|(\mathrm{u},\vartheta)\|\le r_2\},$$

where $r_2 \ge (\Omega_1+\varrho_1)\{(\mathfrak{L}_1+\mathfrak{L}_2)r_2+\mathfrak{R}_1\}+(\Omega_2+\varrho_2)\{(\mathfrak{I}_1+\mathfrak{I}_2)r_2+\mathfrak{R}_2\}$.
Set $\mathfrak{R}_1=\sup\limits_{\ell\in[0,1]}\{|G(\ell,0,0)|\}$ and $\mathfrak{R}_2=\sup\limits_{\ell\in[0,1]}\{|E(\ell,0,0)|\}$. Consider

$$\begin{aligned}|G(\ell,\mathrm{u}(\ell),\vartheta(\ell))| &= |G(\ell,\mathrm{u}(\ell),\vartheta(\ell))-G(\ell,0,0)+G(\ell,0,0)|\\ &\le |G(\ell,\mathrm{u}(\ell),\vartheta(\ell))-G(\ell,0,0)|+|G(\ell,0,0)|\\ &\le (\mathfrak{L}_1\|\mathrm{u}\|+\mathfrak{L}_2\|\vartheta\|)+\mathfrak{R}_1\\ &\le (\mathfrak{L}_1 r_2+\mathfrak{L}_2 r_2)+\mathfrak{R}_1\end{aligned}$$

and

$$\begin{aligned}|E(\ell,\mathrm{u}(\ell),\vartheta(\ell))| &= |E(\ell,\mathrm{u}(\ell),\vartheta(\ell))-E(\ell,0,0)+E(\ell,0,0)|\\ &\le |E(\ell,\mathrm{u}(\ell),\vartheta(\ell))-E(\ell,0,0)|+|E(\ell,0,0)|\\ &\le (\mathfrak{I}_1\|\mathrm{u}\|+\mathfrak{I}_2\|\vartheta\|)+\mathfrak{R}_2\\ &\le (\mathfrak{I}_1 r_2+\mathfrak{I}_2 r_2)+\mathfrak{R}_2.\end{aligned}$$

Step 1. In this step, we show that $\mathcal{F}(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$. For this, we take $(\mathrm{u}, \vartheta) \in \mathfrak{B}_{r_2}$,

$$\begin{aligned}
|\mathcal{F}_1(\mathrm{u}, \vartheta)| &= \Bigg| \frac{(2-\alpha)}{(\theta-\phi)AB(\alpha-1)} \{\ell(\phi G(1, \mathrm{u}(1), \vartheta(1)) - \theta G(\mathfrak{S}, \mathrm{u}(\mathfrak{S}), \vartheta(\mathfrak{S}))\} \\
&\quad + \int_0^1 \Psi_1(\ell, \varsigma) G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)) d\varsigma \Bigg| \\
&\leq \frac{(2-\alpha)\ell}{(\theta-\phi)AB(\alpha-1)} [\phi \{(|(G(1, \mathrm{u}(1), \vartheta(1)) - G(1, 0, 0)|)\} \\
&\quad + |G(1,0,0)|\} - \theta\{|(G(\mathfrak{S}, \mathrm{u}(\mathfrak{S}), \vartheta(\mathfrak{S})) - G(\mathfrak{S}, 0, 0)| + |G(\mathfrak{S}, 0, 0)|\} \\
&\quad + \int_0^1 |\Psi_1(\ell, \varsigma)| \{|G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)) - G(\varsigma, 0, 0)| + |G(\varsigma, 0, 0)|\} d\varsigma.
\end{aligned}$$

taking $\sup\limits_{\ell \in [0,1]}$ both sides

$$\begin{aligned}
\|\mathcal{F}_1(\mathrm{u}, \vartheta)\| &\leq \frac{2}{(\theta-\phi)AB(\alpha-1)} [\phi + \theta] \{(\mathfrak{L}_1 \|\mathrm{u}\| + \mathfrak{L}_2 \|\vartheta\|) + \mathfrak{R}_1\} \\
&\quad + \varrho_1 \{(\mathfrak{L}_1 \|\mathrm{u}\| + \mathfrak{L}_2 \|\vartheta\|) + \mathfrak{R}_1\} \\
&\leq (\Omega_1 + \varrho_1)\{(\mathfrak{L}_1 + \mathfrak{L}_2) r_2 + \mathfrak{R}_1\}.
\end{aligned}$$

Similarly

$$\|\mathcal{F}_2(\mathrm{u}, \vartheta)\| \leq (\Omega_2 + \varrho_2)\{(\mathfrak{I}_1 + \mathfrak{I}_2) r_2 + \mathfrak{R}_2\}.$$

Therefore

$$\begin{aligned}
\|\mathcal{F}(\mathrm{u}, \vartheta)\| &\leq \|\mathcal{F}_1(\mathrm{u}, \vartheta)\| + \|\mathcal{F}_2(\mathrm{u}, \vartheta)\| \\
&\leq (\Omega_1 + \varrho_1)\{(\mathfrak{L}_1 + \mathfrak{L}_2) r_2 + \mathfrak{R}_1\} \\
&\quad + (\Omega_2 + \varrho_2)\{(\mathfrak{I}_1 + \mathfrak{I}_2) r_2 + \mathfrak{R}_2\} \\
&\leq r_2.
\end{aligned}$$

Hence $\mathcal{F}(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$.

Step 2. In this step, we prove a mapping $\mathcal{F}$ is contraction. For this, consider

$$|\mathcal{F}_1(\mathrm{u}_2, \vartheta_2)(\ell) - \mathcal{F}_1(\mathrm{u}_1, \vartheta_1)(\ell)| \leq \frac{2}{(\theta-\phi)AB(\alpha-1)} [\{\phi |(G(1, \mathrm{u}_2(1), \vartheta_2(1))$$

$$-G(1,\mathfrak{u}_1(1),\vartheta_1(1))|\} + \{\theta\,|(G(\mathfrak{S},\mathfrak{u}_2(\mathfrak{S}),\vartheta_2(\mathfrak{S})) - G(\mathfrak{S},\mathfrak{u}_1(\mathfrak{S}),\vartheta_1(\mathfrak{S}))|\}]$$
$$+\int_0^1 |\Psi_1(\ell,\varsigma)|\,|G(\varsigma,\mathfrak{u}_2(\varsigma),\vartheta_2(\varsigma)) - G(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma,$$

taking $\sup\limits_{\ell\in[0,1]}$ on both sides

$$\begin{aligned}
\|\mathcal{F}_1(\mathfrak{u}_2,\vartheta_2) - \mathcal{F}_1(\mathfrak{u}_1,\vartheta_1)\| &\leq \frac{2}{(\theta-\phi)AB(\alpha-1)}[\phi+\theta]\{\mathfrak{L}_1\|\mathfrak{u}_2-\mathfrak{u}_1\| + \mathfrak{L}_2\|\vartheta_2-\vartheta_1\|\} \\
&\quad +\varrho_1\{\mathfrak{L}_1\|\mathfrak{u}_2-\mathfrak{u}_1\| + \mathfrak{L}_2\|\vartheta_2-\vartheta_1\|\} \\
&\leq (\Omega_1+\varrho_1)(\mathfrak{L}_1+\mathfrak{L}_2)\{\|\mathfrak{u}_2-\mathfrak{u}_1\| + \|\vartheta_2-\vartheta_1\|\}.
\end{aligned}$$

Similarly

$$\|\mathcal{F}_2(\mathfrak{u}_2,\vartheta_2)(\ell) - \mathcal{F}_2(\mathfrak{u}_1,\vartheta_1)(\ell)\| \leq (\Omega_2+\varrho_2)(\mathfrak{I}_1+\mathfrak{I}_2)\{\|\mathfrak{u}_2-\mathfrak{u}_1\| + \|\vartheta_2-\vartheta_1\|\}.$$

Thus

$$\begin{aligned}
\|\mathcal{F}(\mathfrak{u}_2,\vartheta_2)(\ell) - \mathcal{F}(\mathfrak{u}_1,\vartheta_1)(\ell)\| &\leq \|\mathcal{F}_1(\mathfrak{u}_2,\vartheta_2)(\ell) - \mathcal{F}_1(\mathfrak{u}_1,\vartheta_1)(\ell)\| \\
&\quad + \|\mathcal{F}_2(\mathfrak{u}_2,\vartheta_2)(\ell) - \mathcal{F}_2(\mathfrak{u}_1,\vartheta_1)(\ell)\| \\
&\leq \left\{\begin{array}{c} (\Omega_1+\varrho_1)(\mathfrak{L}_1+\mathfrak{L}_2) \\ +(\Omega_2+\varrho_2)(\mathfrak{I}_1+\mathfrak{I}_2) \end{array}\right\}\{\|\mathfrak{u}_2-\mathfrak{u}_1\| + \|\vartheta_2-\vartheta_1\|\}.
\end{aligned}$$

Since $\{(\Omega_1+\varrho_1)(\mathfrak{L}_1+\mathfrak{L}_2) + (\Omega_2+\varrho_2)(\mathfrak{I}_1+\mathfrak{I}_2)\} < 1$, thus $\mathcal{F}$ is contraction mapping. Therefore using Theorem 1.1, the AB-Caputo fractional coupled BVP (5.1) – (5.2) has unique solution.

■

5.4 Hyers-Ulam Stability

The HU stability of AB-Caputo fractional coupled BVP (5.1) – (5.2) is presented, in this section. First some remarks and definitions are presented, which are crucial in proof of our main theorem in this section.

Remark. 5.1. The solution $(\mathfrak{u},\vartheta) \in X \times X$ of AB-Caputo fractional coupled BVP (5.1)-(5.2) is given as

$$\begin{cases} \mathrm{u}(\ell) = \delta_1^* + \int\limits_0^1 \Psi_1(\ell,\varsigma) G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)) d\varsigma, \\ \\ \vartheta(\ell) = \delta_2^* + \int\limits_0^1 \Psi_2(\ell,\varsigma) E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)) d\varsigma. \end{cases}$$

Definition 5.1. The given AB-Caputo fractional coupled BVP (5.1) − (5.2) is HU stable if there exists constant $\lambda > 0$, such that for every $\epsilon_1, \epsilon_2 > 0$ and each solution $(\mathrm{u}, \vartheta)(\ell) \in X \times X$, satisfying

$$\left|({}^{ABC}{}_0D^{\alpha}\mathrm{u})(\ell) - G(\ell, \mathrm{u}(\ell), \vartheta(\ell))\right| \leq \epsilon_1 \text{ and } \left|({}^{ABC}{}_0D^{\sigma}\vartheta)(\ell) - E(\ell, \mathrm{u}(\ell), \vartheta(\ell))\right| \leq \epsilon_2, \quad (5.12)$$

for all $\ell \in [0,1]$. Then there exists $(\mathrm{u}_1, \vartheta_1)(\ell) \in X \times X$ a solution of given AB-Caputo fractional coupled BVP (5.1) − (5.2), such that

$$|(\mathrm{u}, \vartheta)(\ell) - (\mathrm{u}_1, \vartheta_1)(\ell)| \leq \lambda\epsilon \text{ for all } \ell \in [0,1]. \quad (5.13)$$

Remark 5.2. $(\mathrm{u}, \vartheta)(\ell) \in X \times X$ is the solution of (5.12) if and only if there exist $\tau_1, \tau_2 \in X$, such that

(*i*) $|\tau_1(\ell)| \leq \epsilon_1$ and $|\tau_2(\ell)| \leq \epsilon_2$, for all $\ell \in [0,1]$.

(*ii*) $({}_0^{ABC}D^{\alpha}\mathrm{u})(\ell) = G(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_1(\ell)$ and

$({}^{ABC}{}_0D^{\sigma}\vartheta)(\ell) = E(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_2(\ell)$, for all $\ell \in [0,1]$.

Theorem 5.3. Suppose $G, E : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ both are continuous functions and satisfying $(H9), (H11)$. Then the solution of AB-Caputo fractional coupled BVP (5.1) − (5.2) is HU stable if

$$1 - \varrho_1(\mathfrak{L}_1 + \mathfrak{L}_2) - \varrho_2(\mathfrak{I}_1 + \mathfrak{I}_2) \neq 0.$$

Proof. Suppose $(\mathrm{u}, \vartheta)(\ell) \in X \times X$ is any solution of (5.12) and from Remark 6.2.we have

$$\begin{cases} ({}^{ABC}{}_0D^{\alpha}\mathrm{u})(\ell) = G(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_1(\ell), \text{ for all } \ell \in [0,1], \\ \\ ({}^{ABC}{}_0D^{\sigma}\vartheta)(\ell) = E(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_2(\ell), \text{ for all } \ell \in [0,1]. \end{cases}$$

Now from Remark 5.1, we can write

$$\begin{cases} \mathrm{u}(\ell) = \delta_1^* + \int\limits_0^1 \Psi_1(\ell,\varsigma)G\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma + \int\limits_0^1 \Psi_1(\ell,\varsigma)\tau_1\left(\varsigma\right)d\varsigma. \\ \\ \vartheta(\ell) = \delta_2^* + \int\limits_0^1 \Psi_2(\ell,\varsigma)E\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma + \int\limits_0^1 \Psi_2(\ell,\varsigma)\tau_2\left(\varsigma\right)d\varsigma. \end{cases}$$

which implies

$$\begin{cases} \left|\mathrm{u}(\ell) - \delta_1^* - \int\limits_0^1 \Psi_1(\ell,\varsigma)G\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma\right| \leq \int\limits_0^1 |\Psi_1(\ell,\varsigma)\tau_1(\varsigma)|\,d\varsigma \leq \omega_1\epsilon_1. \\ \\ \left|\vartheta(\ell) - \delta_2^* - \int\limits_0^1 \Psi_2(\ell,\varsigma)E\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma\right| \leq \int\limits_0^1 |\Psi_2(\ell,\varsigma)\tau_2(\varsigma)|\,d\varsigma \leq \omega_2\epsilon_2. \end{cases}$$

Now assume $(\mathrm{u}_1, \vartheta_1)(\ell) \in X$ is unique solution of AB-Caputo fractional coupled BVP (5.1)-(5.2). Then we have,

$$\begin{cases} \left|\mathrm{u}\left(\ell\right) - \mathrm{u}_1\left(\ell\right)\right| = \left|\mathrm{u}\left(\ell\right) - \delta_1^* - \int\limits_0^1 \Psi_1(\ell,\varsigma)G\left(\varsigma, \mathrm{u}_1(\varsigma), \vartheta_1(\varsigma)\right)d\varsigma\right|. \\ \\ \left|\vartheta\left(\ell\right) - \vartheta_1\left(\ell\right)\right| = \left|\vartheta\left(\ell\right) - \delta_2^* - \int\limits_0^1 \Psi_2(\ell,\varsigma)E\left(\varsigma, \mathrm{u}_1(\varsigma), \vartheta_1(\varsigma)\right)d\varsigma\right|. \end{cases} \tag{5.14}$$

After simplifying,

$$\begin{cases} \left|\mathrm{u}\left(\ell\right) - \mathrm{u}_1\left(\ell\right)\right| \leq \left|\mathrm{u}\left(\ell\right) - \delta_1^* - \int\limits_0^1 \Psi_1(\ell,\varsigma)G\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma\right| \\ + \left|\int\limits_0^1 \Psi_1(\ell,\varsigma)G\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma - \int\limits_0^1 \Psi_1(\ell,\varsigma)G\left(\varsigma, \mathrm{u}_1(\varsigma), \vartheta_1(\varsigma)\right)d\varsigma\right|. \\ \\ \left|\vartheta\left(\ell\right) - \vartheta_1\left(\ell\right)\right| \leq \left|\vartheta\left(\ell\right) - \delta_2^* - \int\limits_0^1 \Psi_2(\ell,\varsigma)E\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma\right| \\ + \left|\int\limits_0^1 \Psi_2(\ell,\varsigma)E\left(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma)\right)d\varsigma - \int\limits_0^1 \Psi_2(\ell,\varsigma)E\left(\varsigma, \mathrm{u}_1(\varsigma), \vartheta_1(\varsigma)\right)d\varsigma\right|. \end{cases}$$

from $(H9)$, we get

$$\begin{cases} |\mathfrak{u}(\ell)-\mathfrak{u}_1(\ell)| \leq \varrho_1\epsilon_1+\varrho_1\left\{\mathfrak{L}_1|\mathfrak{u}(\ell)-\mathfrak{u}_1(\ell)|+\mathfrak{L}_2|\vartheta(\ell)-\vartheta_1(\ell)|\right\}. \\ \\ |\vartheta(\ell)-\vartheta_1(\ell)| \leq \varrho_2\epsilon_2+\varrho_2\left\{\mathfrak{I}_1|\mathfrak{u}(\ell)-\mathfrak{u}_1(\ell)|+\mathfrak{I}_2|\vartheta(\ell)-\vartheta_1(\ell)|\right\}. \end{cases}$$

Which implies

$$\begin{cases} \|\mathfrak{u}-\mathfrak{u}_1\| \leq \varrho_1\epsilon_1+\varrho_1\left\{\mathfrak{L}_1\|\mathfrak{u}-\mathfrak{u}_1\|+\mathfrak{L}_2\|\vartheta-\vartheta_1\|\right\}. \\ \\ \|\vartheta-\vartheta_1\| \leq \varrho_2\epsilon_2+\varrho_2\left\{\mathfrak{I}_1\|\mathfrak{u}-\mathfrak{u}_1\|+\mathfrak{I}_2\|\vartheta-\vartheta_1\|\right\}. \end{cases}$$

Adding both terms

$$\begin{aligned} \|\mathfrak{u}-\mathfrak{u}_1\|+\|\vartheta-\vartheta_1\| &\leq \varrho_1\epsilon_1+\varrho_1\left\{\mathfrak{L}_1\|\mathfrak{u}-\mathfrak{u}_1\|+\mathfrak{L}_2\|\vartheta-\vartheta_1\|\right\} \\ &\quad +\varrho_2\epsilon_2+\varrho_2\left\{\mathfrak{I}_1\|\mathfrak{u}-\mathfrak{u}_1\|+\mathfrak{I}_2\|\vartheta-\vartheta_1\|\right\} \\ \|(\mathfrak{u},\vartheta)-(\mathfrak{u}_1,\vartheta_1)\| &\leq \varrho_1\epsilon_1+\varrho_1(\mathfrak{L}_1+\mathfrak{L}_2)\{\|\mathfrak{u}-\mathfrak{u}_1\|+\|\vartheta-\vartheta_1\|\} \\ &\quad +\varrho_2\epsilon_2+\varrho_2(\mathfrak{I}_1+\mathfrak{I}_2)\left\{\|\mathfrak{u}-\mathfrak{u}_1\|+\|\vartheta-\vartheta_1\|\right\}, \end{aligned}$$

hence

$$\|(\mathfrak{u},\vartheta)-(\mathfrak{u}_1,\vartheta_1)\| \leq \lambda_1\epsilon_1+\lambda_2\epsilon_2 \leq \lambda\epsilon$$

where $\lambda\epsilon=\max\{\lambda_1\epsilon_1,\lambda_2\epsilon_2\}$, $\lambda_1=\frac{\varrho_1}{1-\varrho_1(\mathfrak{L}_1+\mathfrak{L}_2)-\varrho_2(\mathfrak{I}_1+\mathfrak{I}_2)}$ and $\lambda_2=\frac{\varrho_2}{1-\varrho_1(\mathfrak{L}_1+\mathfrak{L}_2)-\varrho_2(\mathfrak{I}_1+\mathfrak{I}_2)}$ with $1-\varrho_1(\mathfrak{L}_1+\mathfrak{L}_2)-\varrho_2(\mathfrak{I}_1+\mathfrak{I}_2)\neq 0$.

Therefore the given AB-Caputo fractional coupled BVP (5.1) – (5.2) is HU stable. ■

Next we provide two examples. In Example 5.1. we find unique and stable solution of AB-Caputo fractional coupled BVP, to validate the conditions of Theorem 5.2 and Theorem 5.3. In Example 5.2, we present nonlinear integral equations in which we find exact and approximation solutions. The graphs of exact and approximate solutions with errors are presented.

Example 5.1. Consider the following AB-Caputo fractional coupled BVP

$$\begin{cases} {}^{ABC}_{\ \ \ 0}D^{1.3}\mathfrak{u}(\ell)=(\ell^2+1)+\frac{1}{16(\ell+1)}\left\{\frac{1}{2}\sin\mathfrak{u}(\ell)+\frac{1}{4}\cos\vartheta(\ell)\right\}. \\ \\ {}^{ABC}_{\ \ \ 0}D^{1.4}\vartheta(\ell)=\ell^3e^{2\ell}+\frac{1}{7e^{\ell}}\left\{\frac{1}{4}\cos\mathfrak{u}(\ell)+\frac{1}{8}\sin\vartheta(\ell)\right\}, \end{cases} \tag{5.15}$$

with boundary conditions

$$\begin{cases} \mathrm{u}(0) = 0, \ \frac{7}{4}\mathrm{u}'\left(\frac{6}{7}\right) = \frac{5}{9}\mathrm{u}'(1) \\ \\ \vartheta(0) = 0, \ \frac{7}{4}\vartheta'\left(\frac{6}{7}\right) = \frac{5}{9}\vartheta'(1), \end{cases} \tag{5.16}$$

for all $\ell \in [0,1]$. From (5.15) and (5.16), we have $\alpha = 1.3$, $\sigma = 1.4$, $\theta = \frac{7}{4}$, $\varphi = \frac{5}{9}$, $\mathfrak{S} = \frac{6}{7}$ and

$$\begin{cases} |G(\ell, \mathrm{u}_1(\ell), \vartheta_1(\ell)) - G(\ell, \mathrm{u}_2(\ell), \vartheta_2(\ell))| \leq \left|\frac{1}{16(\ell+1)}\right| \left\{ \left| \begin{array}{c} \left[\frac{1}{2}\left\{\sin \mathrm{u}_1(\ell) - \sin \mathrm{u}_2(\ell)\right\}\right] \\ + \left[\frac{1}{4}\left\{\cos \vartheta_1(\ell) - \cos \vartheta_2(\ell)\right\}\right] \end{array} \right| \right\}. \\ \\ |E(\ell, \mathrm{u}_1(\ell), \vartheta_1(\ell)) - E(\ell, \mathrm{u}_2(\ell), \vartheta_2(\ell))| \leq \left|\frac{1}{7e^{\ell}}\right| \left\{ \left| \begin{array}{c} \left[\frac{1}{4}\left\{\cos \mathrm{u}_1(\ell) - \cos \mathrm{u}_2(\ell)\right\}\right] \\ + \left[\frac{1}{8}\left\{\sin \vartheta_1(\ell) - \sin \vartheta_2(\ell)\right\}\right] \end{array} \right| \right\}. \end{cases}$$

which implies

$$\begin{aligned} \|G(\ell, \mathrm{u}_1, \vartheta_1) - G(\ell, \mathrm{u}_2, \vartheta_2)\| &\leq \frac{1}{32}\|\mathrm{u}_1 - \mathrm{u}_2\| + \frac{1}{64}\|\vartheta_1 - \vartheta_2\| \\ \|E(\ell, \mathrm{u}_1, \vartheta_1) - E(\ell, \mathrm{u}_2, \vartheta_2)\| &\leq \frac{1}{28}\|\mathrm{u}_1 - \mathrm{u}_2\| + \frac{1}{56}\|\vartheta_1 - \vartheta_2\|. \end{aligned}$$

From calculation we get

$\Omega_1 = 4.823888293764$, $\Omega_2 = 4.947244078833$, $\varrho_1 = 1.97455133228$,

$\varrho_2 = 2.246071439244$, $\mathfrak{L}_1 = \frac{1}{32}$, $\mathfrak{L}_2 = \frac{1}{64}$, $\mathfrak{I}_1 = \frac{1}{28}$, $\mathfrak{I}_2 = \frac{1}{56}$.

Hence

$$\left\{ \begin{array}{c} (\Omega_1 + \varrho_1)(\mathfrak{L}_1 + \mathfrak{L}_2) \\ +(\Omega_2 + \varrho_2)(\mathfrak{I}_1 + \mathfrak{I}_2) \end{array} \right\} = \left[\begin{array}{c} \{4.823888293764 + 1.97455133228\}\left\{\frac{1}{32} + \frac{1}{64}\right\} \\ +\{4.947244078833 + 2.246071439244\}\left\{\frac{1}{28} + \frac{1}{56}\right\} \end{array} \right]$$
$$\doteq 0.70404 < 1.$$

and

$$1 - \varrho_1(\mathfrak{L}_1 + \mathfrak{L}_2) - \varrho_2(\mathfrak{I}_1 + \mathfrak{I}_2) \neq 0.$$

Hence, we get unique and stable solution of given AB-Caputo fractional coupled BVP (5.15) – (5.16) by satisfying all conditions of Theorem 5.2 and Theorem 5.3.

Example 5.2. Consider the following nonlinear integral equations

$$\begin{cases} \mathrm{u}(\ell) = (1+\sin\ell) + \theta e^{-\ell}(\mathrm{u}(\ell) - \vartheta(\ell)) - \theta\left(\int\limits_0^1((e^{-\ell})(\mathrm{u}(\ell) - \vartheta(\ell)))\frac{\sqrt{\mathrm{u}(\varsigma)}}{\sqrt{1+\sin\varsigma}}d\varsigma\right), \\ \\ \vartheta(\ell) = (3+\cos\ell) + \phi\tan\ell(\mathrm{u}(\ell) - \vartheta(\ell)) - \phi\left(\int\limits_0^1((\tan\ell)(\mathrm{u}(\ell) - \vartheta(\ell)))\frac{\sqrt{\vartheta(\varsigma)}}{\sqrt{3+\cos\varsigma}}d\varsigma\right), \end{cases} \tag{5.17}$$

where $\theta = 0.2, \phi = 0.1$. Note that exact solutions are $\mathrm{u}(\ell) = (1+\sin\ell)$ and $\vartheta(\ell) = (3+\cos\ell)$. Which satisfy $(H9)$, as

$$\begin{cases} |G(\ell, \mathrm{u}_1(\ell), \vartheta_1) - G(\ell, \mathrm{u}_2(\ell), \vartheta_2(\ell))| \\ \qquad \leq \left|\frac{e^{-\ell}}{\sqrt{1+\sin\ell}}\right| \left\{\left|\{(\mathrm{u}_1(\ell) - \vartheta_1(\ell))\sqrt{\mathrm{u}_1(\ell)}\} - \{(\mathrm{u}_2(\ell) - \vartheta_2(\ell))\sqrt{\mathrm{u}_2(\ell)}\}\right|\right\}. \\ \\ |E(\ell, \mathrm{u}_1(\ell), \vartheta_1(\ell)) - E(\ell, \mathrm{u}_2(\ell), \vartheta_2(\ell))| \\ \qquad \leq \left|\frac{\tan\ell}{\sqrt{3+\cos\ell}}\right| \left\{\left|\{(\mathrm{u}_1(\ell) - \vartheta_1(\ell))\sqrt{\vartheta_1(\ell)})\} - \{(\mathrm{u}_2(\ell) - \vartheta_2(\ell))\sqrt{\vartheta_2(\ell)})\}\right|\right\}. \end{cases}$$

$$\begin{cases} |G(\ell, \mathrm{u}_1(\ell), \vartheta_1) - G(\ell, \mathrm{u}_2(\ell), \vartheta_2(\ell)))| \\ \qquad \leq \left|\frac{e^{-\ell}}{\sqrt{1+\sin\ell}}\right| \left\{|\{(\mathrm{u}_1(\ell) - \mathrm{u}_2(\ell)) - (\vartheta_1(\ell) - \vartheta_2(\ell))\}| \left|\sqrt{\mathrm{u}_1(\ell)} - \sqrt{\mathrm{u}_2(\ell)}\right|\right\}. \\ \\ |E(\ell, \mathrm{u}_1(\ell), \vartheta_1(\ell)) - E(\ell, \mathrm{u}_2(\ell), \vartheta_2(\ell))| \\ \qquad \leq \left|\frac{\tan\ell}{\sqrt{3+\cos\ell}}\right| \left\{|\{(\mathrm{u}_1(\ell) - \mathrm{u}_2(\ell)) - (\vartheta_1(\ell) - \vartheta_2(\ell))\}| \left|\sqrt{\vartheta_1(\ell)} - \sqrt{\vartheta_2(\ell)}\right|\right\}. \end{cases}$$

Which implies

$$\begin{cases} \|G(\ell, \mathrm{u}_1, \vartheta_1) - G(\ell, \mathrm{u}_2, \vartheta_2)\| \leq \left\|\frac{e^{-\ell}}{\sqrt{1+\sin\ell}}\right\| \|\{(\mathrm{u}_1 - \mathrm{u}_2) - (\vartheta_1 - \vartheta_2)\}\| M. \\ \\ \|E(\ell, \mathrm{u}_1, \vartheta_1) - E(\ell, \mathrm{u}_2, \vartheta_2)\| \leq \left\|\frac{\tan\ell}{\sqrt{3+\cos\ell}}\right\| \|\{(\mathrm{u}_1 - \mathrm{u}_2) - (\vartheta_1 - \vartheta_2)\}\| H. \end{cases}$$

where

$$\begin{aligned} M &= \sup_{\ell\in[0,1]} \left|\sqrt{\mathrm{u}_1(\ell)} - \sqrt{\mathrm{u}_2(\ell)}\right|. \\ H &= \sup_{\ell\in[0,1]} \left|\sqrt{\vartheta_1(\ell)} - \sqrt{\vartheta_2(\ell)}\right|. \end{aligned}$$

finally we get,

$$\begin{cases} \|G(\ell,\mathfrak{u}_1,\vartheta_1)-G(\ell,\mathfrak{u}_2,\vartheta_2)\| \le \mathfrak{L}_1\|(\mathfrak{u}_1-\mathfrak{u}_2)\|+\mathfrak{L}_2\|\vartheta_1-\vartheta_2\|, \\ \\ \|E(\ell,\mathfrak{u}_1,\vartheta_1)-E(\ell,\mathfrak{u}_2,\vartheta_2)\| \le \mathfrak{J}_1\|(\mathfrak{u}_1-\mathfrak{u}_2)\|+\mathfrak{J}_2\|\vartheta_1-\vartheta_2\|, \end{cases}$$

where $\mathfrak{L}_1=\mathfrak{L}_2=M$ and $\mathfrak{J}_1=\mathfrak{J}_2=5H$. Now we take initial guess $\mathfrak{u}_0=1+\ell-\frac{\ell^3}{6}$ and $\vartheta_0=4-\frac{\ell^2}{2}$ and using the iterative process

$$\begin{cases} \mathfrak{u}_{n+1}(\ell)=(1+\sin\ell)+0.2(e^{-\ell})(\mathfrak{u}_n(\ell)-\vartheta_n(\ell))-0.2\int\limits_0^1\left(((e^{-\ell})(\mathfrak{u}_n(\ell)-\vartheta_n(\ell)))\frac{\sqrt{\mathfrak{u}(\varsigma)}}{\sqrt{1+\sin\varsigma}}\right)d\varsigma, \\ \vartheta_{n+1}(\ell)=(3+\cos\ell)+0.1(\tan\ell)(\mathfrak{u}_n(\ell)-\vartheta_n(\ell))-0.1\int\limits_0^1\left(((\tan\ell)(\mathfrak{u}_n(\ell)-\vartheta_n(\ell)))\frac{\sqrt{\vartheta(\varsigma)}}{\sqrt{3+\cos\varsigma}}\right)d\varsigma, \end{cases}$$

we get the approximation solutions after one iteration by

$$\begin{cases} \mathfrak{u}_1(\ell)=(1+\sin\ell)+0.2(e^{-\ell})\left(-\frac{1}{6}\ell^3+\frac{1}{2}\ell^2+\ell-3\right) \\ -0.2\int\limits_0^1\left(\frac{1}{6}(e^{-\ell})\left(-\frac{1}{6}\ell^3+\frac{1}{2}\ell^2+\ell-3\right)\frac{\sqrt{-6\varsigma^3+36\varsigma+36}}{\sqrt{1+\sin\varsigma}}\right)d\varsigma, \\ \vartheta_1(\ell)=(3+\cos\ell)+0.1(\tan\ell)\left(-\frac{1}{6}\ell^3+\frac{1}{2}\ell^2+\ell-3\right) \\ -0.1\int\limits_0^1(\tan\ell)\left(-\frac{1}{6}\ell^3+\frac{1}{2}\ell^2+\ell-3\right)\frac{\sqrt{16-2\varsigma^2}}{\sqrt{3+\cos\varsigma}}d\varsigma. \end{cases}$$

The followings graphs represent approximate and exact solutions of (5.17) with errors.

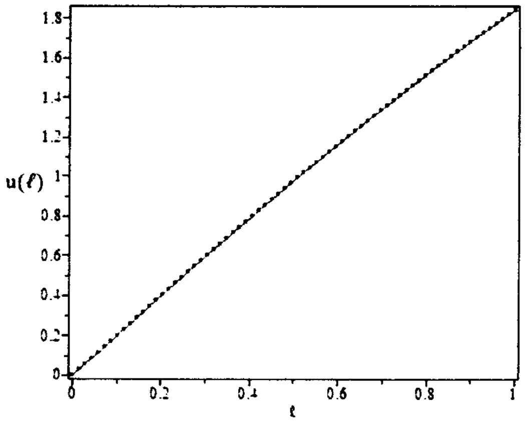

Fig 5.1. Exact (___) and Approximate (.) Solution of u(ℓ)

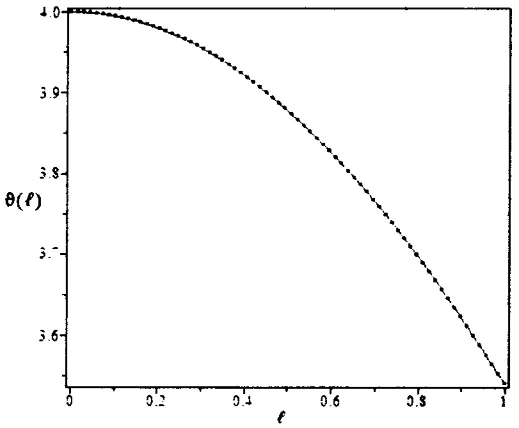

Fig 5.2. Exact ( ) and Approximate Solution ( ) of θ(ℓ)

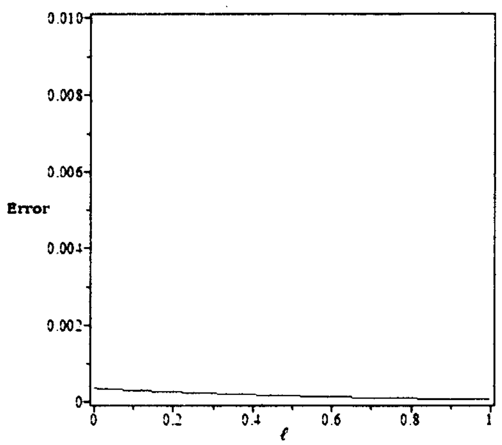

Fig 5.3. Error between Exact and Approximate Solution of $u(\ell)$

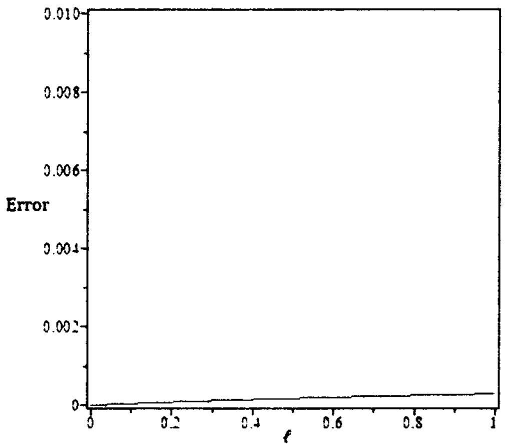

Fig 5.4. Error between Exact and Approximate Solution of $\theta(\ell)$

In Figure 5.1, X-axis present different values of $\ell \in [0,1]$ and Y-axis present the exact and approximate solution of $u(\ell)$. In Figure 5.2, X-axis present different values of $\ell \in [0,1]$ and Y-axis present the exact and approximate solution of $\vartheta(\ell)$. Errors between both (exact and approximate solution) of $u(\ell)$ and $\vartheta(\ell)$ are seen in Figure 5.3 and Figure 5.4 for $\ell \in [0,1]$.

5.5 Conclusion

In this chapter, we discuss the AB-Caputo fractional coupled BVP (5.1) – (5.2). In Section 5.2, we provide our main result in Lemma 5.1 in which we find solution of AB-Caputo fractional coupled BVP (5.3) – (5.2). After that in Theorem 5.1, we obtain atleast one solution of AB-Caputo fractional coupled BVP (5.1)–(5.2) using Krasnoselskii's fixed point theorem. In Section 5.3, Banach contraction principle is used to find unique solution in Theorem 5.2. In Section 5.4 (Theorem 5.3), we use HU stability to investigate the stability of AB-Caputo fractional coupled BVP (5.1) – (5.2). In the end of Section 5.4, two examples are established. In Example 5.1, we find unique and stable solution of AB-Caputo fractional coupled BVP (5.15) – (5.16), by satisfying conditions of Theorem 5.2 and Theorem 5.3. In Example 5.2, we present an equivalent nonlinear integral equations (5.17) of given coupled BVP. In which we calculate exact and approximation solutions (Figure 5.1 and Figure 5.2) with errors (Figure 5.3 and Figure 5.4).

Chapter 6

Mathematical Analysis of Dynamical Systems Involving AB-Piecewise Fractional Derivative

6.1 Introduction

The dynamic term denotes to phenomena that generate time-changing patterns, the qualities of the pattern at particular time are interconnect with those observed at different times. The term can be effectively interchange with pattern of change or time-evolution . It denotes the unfolding of events within ongoing evolutionary process. Almost all phenomena observed in scientific research or in our daily lives have crucial dynamic features. Some specific examples can arise in physical system, social system and life system. In physical system we deal with system of home heating and traveling space vehicle. In social system, economic structure behavior or system of evolution of tribal class and movement in an organizational hierarchy, are discussed. In life system we deal with genetic transference and population growth. These examples indicate potential value of developing facility and illustrate pervasiveness of dynamic situations. It is important to note that concept of dynamics extends beyond the context of process and specific origin. The dynamical system in broader sense is a function from input signals space to output signals space. Here the word signal mean the real vector-valued function of time variable. There are two major types of stability notions for an arbitrary dynamical system, (*i*) internal stability (*ii*) external stability. For internal stability we consider trajectory autonomous system having no inputs and outputs, which is internal dynamics system property. On the other hand, in external stability we explain the degree which the system amplifies signals. In nature everything is evolving and continuously changing. Thus dynamical system is a system in which situations varies with respect to time. The general patterns involved in the systems of nonlinear equations for finding their solution are being explained in theory of dynamical systems. Sometimes inputs and outputs in dynamical systems are involved in control systems, which is considered very essential in the field of engineering. Applications of dynamical systems are used in the different fields of sciences and engineering such as fault-tolerant control systems, computer-based digital control system, automation, aeronautics, mechatronics, ground and air transportation systems, adaptive control system, signal forecast process, fault and supervision detection etc. Some results and applications related to dynamic system can be seen in [114, 115, 116, 117, 118, 119, 120, 121, 122, 123].

Atangana and Araz [13] in 2021, introduced different types of piecewise fractional derivatives and integrals. They combined classical derivatives with fractional derivatives to get piecewise fractional derivatives. These piecewise fractional derivatives are used to deal with crossover behaviors exhibited problems. Numerical scheme for Cauchy type problems and applications presented in which they provided some models such as Zika Virus, Dadras-Momeni and Dequan-Li attractor. Later Heydari and Razzaghi [124], presented piecewise Caputo fractional derivatives to define novel category of fractional control and optimal problems. To constructed numerical method, they used piecewise Chebyshev cardinal functions (PCCFs) to get approximate solution. After that Shah et al. [125], performed analysis of Cauchy type dynamical systems involving the piecewise Caputo fractional derivative. To find existence results, they used Schauder fixed point theorem and obtained unique solution via Banach contraction principle. They presented stability analysis by using conditions of HU stability. They applied numerical scheme in which Newton polynomials interpolation is used to get numerical solution. Recently Arık and Araz [126], combined three different models, which are helpful in the treatment processes of the tumor growth. Some other results which are related to piecewise fractional derivatives are given in [127, 128, 129, 130, 131, 132].

For motivation of above work, we present the following AB-piecewise fractional differential system (FDS),

$$\begin{cases} {}^{PAB}{}_{0}D^{\sigma}\{\mathfrak{u}(\ell)\} = G(\ell, \mathfrak{u}(\ell), \vartheta(\ell)), \\ {}^{PAB}{}_{0}D^{\sigma}\{\vartheta(\ell)\} = E(\ell, \mathfrak{u}(\ell), \vartheta(\ell)), \end{cases} \quad \ell \in [0, \mathsf{T}], \tag{6.1}$$

with initial conditions

$$\begin{cases} \mathfrak{u}(0) = \mathfrak{S}_1 + \int_0^{\mathfrak{z}_1} \eta(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma, \\ \vartheta(0) = \mathfrak{S}_2 + \int_0^{\mathfrak{z}_2} \xi(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma, \end{cases} \tag{6.2}$$

where ${}^{PAB}{}_{0}D^{\sigma}$ represent AB-piecewise derivative of order $0 < \sigma < 1$. $G, E : [0, \mathsf{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\eta : [0, \mathfrak{z}_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\xi : [0, \mathfrak{z}_2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, where $\mathsf{T} > 0$, $0 \le \mathfrak{z}_1, \mathfrak{z}_2 \le \ell_1$. Now we use some notations which are useful for our next discussion.

$$\begin{cases} (i). \ |G(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \hbar_1 |\mathfrak{u}(\ell)| + \hbar_2 |\vartheta(\ell)| + \mathfrak{L}_1, \ \text{where } \hbar_1, \hbar_2, \mathfrak{L}_1 > 0. \\ (ii). \ |E(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \hbar_3 |\mathfrak{u}(\ell)| + \hbar_4 |\vartheta(\ell)| + \mathfrak{L}_2, \ \text{where } \hbar_3, \hbar_4, \mathfrak{L}_2 > 0. \\ (iii). \ |\eta(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \mu_1 |\mathfrak{u}(\ell)| + \mu_2 |\vartheta(\ell)| + \mathfrak{I}_1, \ \text{where } \mu_1, \mu_2, \mathfrak{I}_1 > 0. \\ (iv). \ |\xi(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \mu_3 |\mathfrak{u}(\ell)| + \mu_4 |\vartheta(\ell)| + \mathfrak{I}_2, \ \text{where } \mu_3, \mu_4, \mathfrak{I}_2 > 0. \end{cases} \tag{H12}$$

$$\begin{cases} (i). \ |G(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - G(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \mathfrak{q}_1 [|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|], \ \text{where } \mathfrak{q}_1 > 0. \\ (ii). \ |E(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - E(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \mathfrak{q}_2 [|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|], \ \text{where } \mathfrak{q}_2 > 0. \\ (iii). \ |\eta(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - \eta(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \mathfrak{p}_1 [|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|], \ \text{where } \mathfrak{p}_1 > 0, \\ (iv). \ |\xi(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - \xi(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \le \mathfrak{p}_2 [|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|], \ \text{where } \mathfrak{p}_2 > 0. \end{cases} \tag{H13}$$

6.2 Existence Results

This section provides the existence of the solution of non-linear AB-piecewise FDS (6.1) – (6.2) via Schauder fixed point theorem (Theorem 1.2). First, we provide some basic Lemmas.

Lemma 6.1. Assume that $Q_1 : [0, \mathsf{T}] \to \mathbb{R}$ and $\Psi_1 : [0, \mathfrak{z}_1] \to \mathbb{R}$ are continuous functions. Then the solution of AB-piecewise FDE

$${}^{PAB}{}_{0}D^{\sigma}(\mathfrak{u}(\ell)) = Q_1(\ell), \ \ell \in [0, \mathsf{T}],$$

with

$$\mathrm{u}(0) = \mathfrak{S}_1 + \int_0^{\mathfrak{z}_1} \Psi_1(\varsigma)d\varsigma,$$

is given as

$$\mathrm{u}(\ell) = \begin{cases} \mathfrak{S}_1 + \int_0^{\mathfrak{z}_1} \Psi_1(\varsigma)d\varsigma + \int_0^{\ell} Q_1(\varsigma)d\varsigma \text{ if } 0 \leq \ell \leq \ell_1, \\ \mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)}Q_1(\ell) \\ +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} Q_1(\varsigma)(\ell-\varsigma)^{\sigma-1}d\varsigma, \text{ if } \ell_1 \leq \ell \leq \mathsf{T}. \end{cases}$$

Lemma 6.2. Assume that $G, E : [0,\mathsf{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\eta : [0,\mathfrak{z}_1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\xi : [0,\mathfrak{z}_2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Then the solution of AB-piecewise FDS (6.1)−(6.2) is given as

$$\begin{cases} \mathrm{u}(\ell) = \begin{cases} \mathfrak{S}_1 + \int_0^{\mathfrak{z}_1} \eta(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma \\ +\int_0^{\ell} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma, \text{ if } 0 \leq \ell \leq \ell_1, \\ \mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)}G(\ell, \mathrm{u}(\ell), \vartheta(\ell)) \\ +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma, \text{ if } \ell_1 \leq \ell \leq \mathsf{T} \end{cases} \\ \vartheta(\ell) = \begin{cases} \mathfrak{S}_2 + \int_0^{\mathfrak{z}_2} \xi(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma \\ +\int_0^{\ell} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma, \text{ if } 0 \leq \ell \leq \ell_1, \\ \vartheta(\ell_1) + \frac{1-\sigma}{AB(\sigma)}E(\ell, \mathrm{u}(\ell), \vartheta(\ell)) \\ +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma, \text{ if } \ell_1 \leq \ell \leq \mathsf{T}. \end{cases} \end{cases} \tag{6.3}$$

Next, we take $X \times X = (C[0,\mathsf{T}],\mathbb{R}) \times (C[0,\mathsf{T}],\mathbb{R})$ is Banach space having norm defined as,

$$\|(\mathrm{u},\vartheta)\|_{X\times X} = \|\mathrm{u}\|_X + \|\vartheta\|_X \,.$$

Define $\Gamma : X \times X \to X \times X$ as,

$$\Gamma(\mathrm{u},\vartheta)(\ell) = (\Gamma_1(\mathrm{u},\vartheta)(\ell),\ \Gamma_2(\mathrm{u},\vartheta)(\ell)). \tag{6.4}$$

where

$$\begin{cases} \mathcal{F}_1(\mathrm{u},\vartheta)(\ell) = \begin{cases} \mathfrak{S}_1 + \int\limits_0^{\mathfrak{J}_1} \eta(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int\limits_0^{\ell} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma, \text{ if } 0 \leq \ell \leq \ell_1, \\ \mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)} G(\ell, \mathrm{u}(\ell), \vartheta(\ell)) \\ + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int\limits_{\ell_1}^{\ell} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell - \varsigma)^{\sigma-1} d\varsigma, \text{ if } \ell_1 \leq \ell \leq \mathsf{T}. \end{cases} \\ \mathcal{F}_2(\mathrm{u},\vartheta)(\ell) = \begin{cases} \mathfrak{S}_2 + \int\limits_0^{\mathfrak{J}_2} \xi(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int\limits_0^{\ell} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma \text{ if } 0 \leq \ell \leq \ell_1, \\ \vartheta(\ell_1) + \frac{1-\sigma}{AB(\sigma)} E(\ell, \mathrm{u}(\ell), \vartheta(\ell)) \\ + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int\limits_{\ell_1}^{\ell} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell - \varsigma)^{\sigma-1} d\varsigma, \text{ if } \ell_1 \leq \ell \leq \mathsf{T}. \end{cases} \end{cases} \tag{6.5}$$

The following assumptions are useful in our next discussion.

$$\begin{cases} (i).\ D = |\mathfrak{S}_1| + |\mathfrak{S}_2| \ \ (ii).\ \Omega_1 = \{(\Lambda_1 + \Lambda_2) + (\Lambda_3 + \Lambda_4) \\ (iii).\ \Omega_2 = \{(S_1 + S_2) + (S_3 + S_4)\} \ \ (iv).\ \Omega_3 = (\Omega_1 + \Omega_2) \\ (v).\ W_1 = (\mathfrak{R}_1 + \mathfrak{R}_2) \ \ (vi).\ W_2 = (\mathfrak{R}_3 + \mathfrak{R}_4) \ \ (vii).\ Z = |\mathrm{u}(\ell_1)| + |\vartheta(\ell_1)| \\ (viii).\ W_3 = (W_1 + W_2) \ \ (iX).\ \beta_1 = \{(\mu_1 + \mu_2) + (\mu_3 + \mu_4)\} \\ (X).\ \beta_2 = \{(\hbar_1 + \hbar_2) + (\hbar_3 + \hbar_4)\} \ \ (Xi).\ \beta_3 = \beta_1 + \beta_2 \\ (Xii).\ \gamma_1 = (\mathfrak{L}_1 + \mathfrak{L}_2) \ \ (Xiii).\ \gamma_2 = (\mathfrak{J}_1 + \mathfrak{J}_2) \\ (Xiv).\ \gamma_3 = \gamma_1 + \gamma_2 \ \ (Xv).\ \frac{1}{AB(\sigma)} \left\{ 1 + \frac{\sigma}{\Gamma(\sigma+1)} \mathsf{T}^{\sigma} \right\} = \Theta. \end{cases}$$

Theorem 6.1. Assume that all conditions of $(H12)$ hold. Then AB-piecewise FDS (6.1) – (6.2) has atleast one solution.

Proof. Consider

$$\mathfrak{B}_{r_1} = \{(\mathrm{u},\vartheta) \in X \times X : \|(\mathrm{u},\vartheta)\| \leq r_1\},$$

a closed, bounded, convex and nonempty subset of $X \times X$. Where $r_1 \geq \frac{D + \ell_1 \gamma_3}{1 - \ell_1 \beta_3}$ with $1 - \ell_1 \beta_3 \neq 0$ for $0 \leq \ell \leq \ell_1$ and $r_1 \geq \frac{Z + \gamma_1 \Theta}{1 - \beta_2 \Theta}$ with $1 - \beta_2 \Theta \neq 0$ for $\ell_1 \leq \ell \leq \mathsf{T}$.

Step 1. In this step, we show $\mathcal{F}(\mathfrak{B}_{r_1}) \subseteq (\mathfrak{B}_{r_1})$.

Case 1. For $0 \leq \ell \leq \ell_1$. Consider

$$\begin{aligned}
|\mathcal{F}_1(\mathrm{u},\vartheta)(\ell)| &= \left|\mathfrak{S}_1 + \int_0^{\mathfrak{z}_1} \eta(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma\right| \\
&\leq |\mathfrak{S}_1| + \int_0^{\mathfrak{z}_1} |\eta(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))|\, d\varsigma + \int_0^{\ell} |G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))|\, d\varsigma,
\end{aligned}$$

taking $\sup\limits_{\ell \in [0,\ell_1]}$ on both sides

$$\begin{aligned}
\|\mathcal{F}_1(\mathrm{u},\vartheta)\| &\leq |\mathfrak{S}_1| + \ell_1[\{\mu_1 \|\mathrm{u}\| + \mu_2 \|\vartheta\| + \mathfrak{J}_1\} + \{\hbar_1 \|\mathrm{u}\| + \hbar_2 \|\vartheta\| + \mathfrak{L}_1\}] \\
&= |\mathfrak{S}_1| + \ell_1[\|\mathrm{u}\| \{\mu_1 + \hbar_1\} + \|\vartheta\| \{\mu_2 + \hbar_2\} + (\mathfrak{J}_1 + \mathfrak{L}_1)] \\
&\leq |\mathfrak{S}_1| + \ell_1[r_1\{(\mu_1 + \mu_2) + (\hbar_1 + \hbar_2)\} + (\mathfrak{J}_1 + \mathfrak{L}_1)].
\end{aligned}$$

Similarly

$$\begin{aligned}
|\mathcal{F}_2(\mathrm{u},\vartheta)(\ell)| &= \left|\mathfrak{S}_2 + \int_0^{\mathfrak{z}_2} \xi(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))d\varsigma\right| \\
&\leq |\mathfrak{S}_2| + \int_0^{\mathfrak{z}_2} |\xi(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))|\, d\varsigma + \int_0^{\ell} |E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))|\, d\varsigma.
\end{aligned}$$

taking $\sup\limits_{\ell \in [0,\ell_1]}$ on both sides

$$\begin{aligned}
\|\mathcal{F}_2(\mathrm{u},\vartheta)\| &\leq |\mathfrak{S}_2| + \ell_1[\{\mu_3 \|\mathrm{u}\| + \mu_4 \|\vartheta\| + \mathfrak{J}_2\} + \{\hbar_3 \|\mathrm{u}\| + \hbar_4 \|\vartheta\| + \mathfrak{L}_2\}] \\
&= |\mathfrak{S}_2| + \ell_1[\|\mathrm{u}\| \{\mu_3 + \hbar_3\} + \|\vartheta\| \{\mu_4 + \hbar_4\} + (\mathfrak{J}_2 + \mathfrak{L}_2)] \\
&\leq |\mathfrak{S}_2| + \ell_1[r_1\{(\mu_3 + \mu_4) + (\hbar_3 + \hbar_4)\} + (\mathfrak{J}_2 + \mathfrak{L}_2)].
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathcal{F}(\mathrm{u},\vartheta)\| &\leq \|\mathcal{F}_1(\mathrm{u},\vartheta)\| + \|\mathcal{F}_2(\mathrm{u},\vartheta)\| \\
&\leq |\mathfrak{S}_1| + |\mathfrak{S}_2| + \ell_1[r_1\{(\mu_1 + \mu_2) + (\mu_3 + \mu_4) + (\hbar_1 + \hbar_2) + (\hbar_3 + \hbar_4)\} \\
&\quad + (\mathfrak{J}_1 + \mathfrak{J}_2) + (\mathfrak{L}_1 + \mathfrak{L}_2)]
\end{aligned}$$

$$= D + \ell_1\{r_1(\beta_3) + \gamma_3\} \leq r_1.$$

Where D, β_3 and γ_3 are defined in (6.6). So $\mathcal{F}(\mathfrak{B}_{r_1}) \subseteq (\mathfrak{B}_{r_1})$.

The continuity of G, E implies continuity of $\mathcal{F}$. Since

$$\|\mathcal{F}(\mathrm{u}, \vartheta)\| \leq r_1.$$

Therefore $\mathcal{F}(\mathfrak{B}_{r_1})$ is uniformly bounded for $0 \leq \ell \leq \ell_1$.

Case 2. For $\ell_1 \leq \ell \leq \mathsf{T}$. Consider

$$\begin{aligned}
|\mathcal{F}_1(\mathrm{u}, \vartheta)(\ell)| &= \left| \begin{array}{c} \mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)} G(\ell, \mathrm{u}(\ell), \vartheta(\ell)) \\ + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int\limits_{\ell_1}^{\ell} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell - \varsigma)^{\sigma-1} d\varsigma \end{array} \right| \\
&\leq |\mathrm{u}(\ell_1)| + \frac{1-\sigma}{AB(\sigma)} |G(\ell, \mathrm{u}(\ell), \vartheta(\ell))| \\
&\quad + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int\limits_{\ell_1}^{\ell} |G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))| \, |(\ell - \varsigma)^{\sigma-1}| \, d\varsigma.
\end{aligned}$$

taking $\sup\limits_{\ell \in [\ell_1, \mathsf{T}]}$ on both sides

$$\begin{aligned}
\|\mathcal{F}_1(\mathrm{u}, \vartheta)\| &\leq |\mathrm{u}(\ell_1)| + \frac{1-\sigma}{AB(\sigma)} \{\hbar_1 \|\mathrm{u}\| + \hbar_2 \|\vartheta\| + \mathfrak{L}_1\} \\
&\quad + \frac{\sigma \mathsf{T}^{\sigma}}{AB(\sigma)\Gamma(\sigma+1)} \{\hbar_1 \|\mathrm{u}\| + \hbar_2 \|\vartheta\| + \mathfrak{L}_1\} \\
&\leq |\mathrm{u}(\ell_1)| + \frac{1}{AB(\sigma)} \left\{1 + \frac{\sigma \mathsf{T}^{\sigma}}{\Gamma(\sigma+1)}\right\} \{\hbar_1 r_1 + \hbar_2 r_1 + \mathfrak{L}_1\} \\
&= |\mathrm{u}(\ell_1)| + \Theta\{(\hbar_1 + \hbar_2)\} r_1 + \mathfrak{L}_1 \Theta.
\end{aligned}$$

Similarly

$$\begin{aligned}
|\mathcal{F}_2(\mathrm{u}, \vartheta)(\ell)| &= \left| \begin{array}{c} \vartheta(\ell_1) + \frac{1-\sigma}{AB(\sigma)} E(\ell, \mathrm{u}(\ell), \vartheta(\ell)) \\ + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int\limits_{\ell_1}^{\ell} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell - \varsigma)^{\sigma-1} d\varsigma \end{array} \right| \\
&\leq |\vartheta(\ell_1)| + \frac{1-\sigma}{AB(\sigma)} |E(\ell, \mathrm{u}(\ell), \vartheta(\ell))|
\end{aligned}$$

$$+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} |E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))| |(\ell-\varsigma)^{\sigma-1}| d\varsigma.$$

taking $\sup\limits_{\ell\in[\ell_1,\mathsf{T}]}$ on both sides

$$\begin{aligned}
\|\mathcal{F}_2(\mathrm{u},\vartheta)\| &\leq |\vartheta(\ell_1)| + \frac{1-\sigma}{AB(\sigma)}\{\hbar_3\|\mathrm{u}\| + \hbar_4\|\vartheta\| + \mathfrak{L}_2\} \\
&\quad + \frac{\sigma\mathsf{T}^{\sigma}}{AB(\sigma)\Gamma(\sigma+1)}\{\hbar_3\|\mathrm{u}\| + \hbar_4\|\vartheta\| + \mathfrak{L}_2\} \\
&\leq |\vartheta(\ell_1)| + \frac{1}{AB(\sigma)}\{\hbar_3 r_1 + \hbar_4 r_1 + \mathfrak{L}_2\} \\
&\quad + \frac{\sigma\mathsf{T}^{\sigma}}{AB(\sigma)\Gamma(\sigma+1)}\{\hbar_3 r_1 + \hbar_4 r_1 + \mathfrak{L}_2\} \\
&= |\vartheta(\ell_1)| + \Theta\{(\hbar_3+\hbar_4)\}r_1 + \Theta\mathfrak{L}_2.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathcal{F}(\mathrm{u},\vartheta)\| &\leq \|\mathcal{F}_1(\mathrm{u},\vartheta)\| + \|\mathcal{F}_2(\mathrm{u},\vartheta)\| \\
&\leq |\mathrm{u}(\ell_1)| + \{\Theta(\hbar_1+\hbar_2)\}r_1 + \Theta\mathfrak{L}_1 \\
&\quad + |\vartheta(\ell_1)| + \{\Theta(\hbar_3+\hbar_4)\}r_1 + \Theta\mathfrak{L}_2 \\
&= Z + \Theta[\{(\hbar_1+\hbar_2) + (\hbar_3+\hbar_4)\}]r_1 + \Theta(\mathfrak{L}_1 + \mathfrak{L}_2) \\
&= Z + \Theta\beta_2 r_1 + \gamma_1\Theta \leq r_1.
\end{aligned}$$

So $\mathcal{F}(\mathfrak{B}_{r_1}) \subseteq (\mathfrak{B}_{r_1})$, for $\ell_1 \leq \ell \leq \mathsf{T}$.

The continuity of G, E implies continuity of $\mathcal{F}$. Since

$$\|\mathcal{F}(\mathrm{u},\vartheta)\| \leq r_1.$$

Therefore $\mathcal{F}(\mathfrak{B}_{r_1})$ is uniformly bounded for $\ell_1 \leq \ell \leq \mathsf{T}$.

Hence $\mathcal{F}(\mathfrak{B}_{r_1})$ is uniformly bounded for all $0 \leq \ell \leq \mathsf{T}$.

Step 2. Next, we show that $\mathcal{F}(\mathfrak{B}_{r_1})$ is equicontinuous.

Case 1. For $0 \leq \ell \leq \ell_1$. For this, we consider $0 \leq \ell_n < \ell_m \leq \ell_1$, so clearly

$$\begin{aligned}
|\mathcal{F}_1(u,\vartheta)(\ell_n) - \mathcal{F}_1(u,\vartheta)(\ell_m)| &= \left| \int_0^{\ell_m} G(\varsigma, u(\varsigma), \vartheta(\varsigma))dx + \int_{\ell_m}^{\ell_n} G(\varsigma, u(\varsigma), \vartheta(\varsigma))dx \right. \\
&\quad \left. - \int_0^{\ell_m} G(\varsigma, u(\varsigma), \vartheta(\varsigma))dx \right|, \\
&\leq \int_{\ell_m}^{\ell_n} |G(\varsigma, u(\varsigma), \vartheta(\varsigma))|\, dx \to 0,
\end{aligned}$$

when $\ell_n \to \ell_m$. Also

$$\begin{aligned}
|\mathcal{F}_2(u,\vartheta)(\ell_n) - \mathcal{F}_2(u,\vartheta)(\ell_m)| &= \left| \int_0^{\ell_m} E(\varsigma, u(\varsigma), \vartheta(\varsigma))dx + \int_{\ell_m}^{\ell_n} E(\varsigma, u(\varsigma), \vartheta(\varsigma))dx \right. \\
&\quad \left. - \int_0^{\ell_m} E(\varsigma, u(\varsigma), \vartheta(\varsigma))dx \right|, \\
&\leq \int_{\ell_m}^{\ell_n} |E(\varsigma, u(\varsigma), \vartheta(\varsigma))|\, dx \to 0.
\end{aligned}$$

when $\ell_n \to \ell_m$. Hence

$$|\mathcal{F}(u,\vartheta)(\ell_n) - \mathcal{F}(u,\vartheta)(\ell_m)| \leq \left\{ \begin{array}{c} |\mathcal{F}_1(u,\vartheta)(\ell_n) - \mathcal{F}_1(u,\vartheta)(\ell_m)| \\ + |\mathcal{F}_2(u,\vartheta)(\ell_n) - \mathcal{F}_2(u,\vartheta)(\ell_m)| \end{array} \right\} \to 0,$$

as $\ell_n \to \ell_m$. Therefore $\mathcal{F}(\mathfrak{B}_{r_1})$ is equicontinuous for $0 \leq \ell \leq \ell_1$.

Case 2. For $\ell_1 \leq \ell \leq \mathsf{T}$. For this, we consider $\ell_1 \leq \ell_n < \ell_m \leq \mathsf{T}$,

$$\begin{aligned}
|\mathcal{F}_1(u,\vartheta)(\ell_n) - \mathcal{F}_1(u,\vartheta)(\ell_m)| &= \left| \frac{1-\sigma}{AB(\sigma)} G(\ell_n, u(\ell_n), \vartheta(\ell_n)) \right. \\
&\quad + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)} \int_{\ell_1}^{\ell_n} G(\varsigma, u(\varsigma), \vartheta(\varsigma))(\ell_n - \varsigma)^{\sigma-1} d\varsigma \\
&\quad - \frac{1-\sigma}{AB(\sigma)} G(\ell_m, u(\ell_m), \vartheta(\ell_m))
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell_m} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell_m-\varsigma)^{\sigma-1}d\varsigma\Bigg| \\
= &\Bigg|\frac{1-\sigma}{AB(\sigma)}G(\ell_n, \mathrm{u}(\ell_n), \vartheta(\ell_n)) \\
&+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_m}^{\ell_n} G(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell_n-\varsigma)^{\sigma-1}d\varsigma \\
&-\frac{1-\sigma}{AB(\sigma)}G(\ell_m, \mathrm{u}(\ell_m), \vartheta(\ell_m))\Bigg|,
\end{aligned}
$$

then

$$|\digamma_1(\mathrm{u},\vartheta)(\ell_n)-\digamma_1(\mathrm{u},\vartheta)(\ell_m)| \to 0,$$

when $\ell_n \to \ell_m$.

Similarly

$$
\begin{aligned}
|\digamma_2(\mathrm{u},\vartheta)(\ell_n)-\digamma_2(\mathrm{u},\vartheta)(\ell_m)| = &\Bigg|\frac{1-\sigma}{AB(\sigma)}E(\ell_n, \mathrm{u}(\ell_n), \vartheta(\ell_n)) \\
&+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell_n} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell_n-\varsigma)^{\sigma-1}d\varsigma \\
&-\frac{1-\sigma}{AB(\sigma)}E(\ell_m, \mathrm{u}(\ell_m), \vartheta(\ell_m)) \\
&-\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell_m} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell_m-\varsigma)^{\sigma-1}d\varsigma\Bigg| \\
= &\Bigg|\frac{1-\sigma}{AB(\sigma)}E(\ell_n, \mathrm{u}(\ell_n), \vartheta(\ell_n)) \\
&+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_m}^{\ell_n} E(\varsigma, \mathrm{u}(\varsigma), \vartheta(\varsigma))(\ell_n-\varsigma)^{\sigma-1}d\varsigma \\
&-\frac{1-\sigma}{AB(\sigma)}E(\ell_m, \mathrm{u}(\ell_m), \vartheta(\ell_m))\Bigg|,
\end{aligned}
$$

then

$$|\digamma_2(\mathrm{u},\vartheta)(\ell_n)-\digamma_2(\mathrm{u},\vartheta)(\ell_m)| \to 0.$$

as $\ell_n \to \ell_m$. Hence

$$|\mathcal{F}(u,\vartheta)(\ell_n) - \mathcal{F}(u,\vartheta)(\ell_m)| \le \left\{ \begin{array}{c} |\mathcal{F}_1(u,\vartheta)(\ell_n) - \mathcal{F}_1(u,\vartheta)(\ell_m)| \\ + |\mathcal{F}_2(u,\vartheta)(\ell_n) - \mathcal{F}_2(u,\vartheta)(\ell_m)| \end{array} \right\} \to 0,$$

when $\ell_n \to \ell_m$. Therefore $\mathcal{F}(\mathfrak{B}_{r_1})$ is equicontinuous for $\ell_1 \le \ell \le$T. Hence $\mathcal{F}(\mathfrak{B}_{r_1})$ is equicontinuous for all $0 \le \ell \le$T. So by using Arzela-Ascoli theorem (Lemma 1.3), the set $\mathcal{F}(\mathfrak{B}_{r_1})$ is relative compact. Hence from Theorem 1.2, $\mathcal{F}$ has atleast one fixed point which is solution of (6.1) – (6.2) on [0,T]. ■

6.3 Uniqueness

This section provides unique solution of AB-piecewise FDS (6.1) – (6.2). To find unique solution of AB-piecewise FDS (6.1) – (6.2), we apply Banach contraction principle (Theorem 1.1).

Theorem 6.2. Assume that all conditions of ($H13$) hold. Then AB-piecewise FDS (6.1) – (6.2) has unique solution if

$$\max\{\varrho_1, \varrho_2\} < 1,$$

where $\varrho_1 = \ell_1[\{p_1 + q_1\} + \{p_2 + q_2\}] < 1$ and $\varrho_2 = \{\Theta(q_1 + q_2)\} < 1$.

Proof. Consider a closed ball

$$\mathfrak{B}_{r_2} = \{(u,\vartheta) \in X \times X : \|(u,\vartheta)\| \le r_2\}.$$

where $r_2 \ge \frac{D+\ell_1 W_3}{1-\ell_1\Omega_3}$, where $1 - \ell_1\Omega_3 \ne 0$ for $0 \le \ell \le \ell_1$ and $r_2 \ge \frac{E+W_4\Theta}{(1-\Omega_3\Theta)}$, where $(1 - \Omega_3\Theta) \ne 0$ for $\ell_1 \le \ell \le$T. We set $\mathfrak{R}_1 = \sup_{\ell\in[0,T]} \{|G(\ell,0,0)|\}$, $\mathfrak{R}_2 = \sup_{\ell\in[0,T]} \{|E(\ell,0,0)|\}$, $\mathfrak{R}_3 = \sup_{\ell\in[0,\ell_1]} |\eta(\ell,0,0)|$ and $\mathfrak{R}_4 = \sup_{\ell\in[0,\ell_1]} \{|\xi(\ell,0,0)|\}$. Now consider

$$\begin{aligned} |G(\ell,u(\ell),\vartheta(\ell))| &\le |G(\ell,u(\ell),\vartheta(\ell)) - G(\ell,0,0)| + |G(\ell,0,0)| \\ &\le (S_1\|u\| + S_2\|\vartheta\|) + \mathfrak{R}_1 \\ &\le (S_1 + S_2)r_2 + \mathfrak{R}_1. \end{aligned}$$

Similarly

$$|E(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \leq (S_3 + S_4)r_2 + \mathfrak{R}_2,$$

$$|\eta(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \leq (\Lambda_1 + \Lambda_2)r_2 + \mathfrak{R}_3$$

and

$$|\xi(\ell, \mathfrak{u}(\ell), \vartheta(\ell))| \leq (\Lambda_3 + \Lambda_4)r_2 + \mathfrak{R}_1.$$

Step 1. In this step, we show that $\mathcal{F}(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$.

Case 1. For $0 \leq \ell \leq \ell_1$. Consider,

$$\begin{aligned}
|\mathcal{F}_1(\mathfrak{u}, \vartheta)(\ell)| &= \left|\mathfrak{S}_1 + \int_0^{\mathfrak{z}_1} \eta(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} G(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma\right| \\
&\leq |\mathfrak{S}_1| + \int_0^{\mathfrak{z}_1} \{|\eta(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma)) - \eta(\varsigma, 0, 0)| + |\eta(\varsigma, 0, 0)|\} \, d\varsigma \\
&\quad + \int_0^{\ell} \{|G(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma)) - G(\varsigma, 0, 0)| + |G(\varsigma, 0, 0)|\} \, d\varsigma,
\end{aligned}$$

taking $\sup\limits_{\ell \in [0, \ell_1]}$ on both sides

$$\|\mathcal{F}_1(\mathfrak{u}, \vartheta)\| \leq |\mathfrak{S}_1| + [\{(\Lambda_1 + \Lambda_2)r_2 + \mathfrak{R}_3\} + \{(S_1 + S_2)r_2 + \mathfrak{R}_1\}]\ell_1.$$

Similarly

$$\begin{aligned}
|\mathcal{F}_2(\mathfrak{u}, \vartheta)(\ell)| &= \left|\mathfrak{S}_2 + \int_0^{\mathfrak{z}_2} \xi(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} E(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma\right| \\
&\leq |\mathfrak{S}_2| + \int_0^{\mathfrak{z}_2} \{|\xi(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma)) - \xi(\varsigma, 0, 0)| + |\xi(\varsigma, 0, 0)|\} \, d\varsigma \\
&\quad + \int_0^{\ell} \{|E(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma)) - E(\varsigma, 0, 0)| + |E(\varsigma, 0, 0)|\} \, d\varsigma,
\end{aligned}$$

taking $\sup\limits_{\ell \in [0, \ell_1]}$ on both sides

$$\|\varGamma_2(\mathrm{u},\vartheta)\| \leq |\mathfrak{S}_2| + [\{(\Lambda_3+\Lambda_4)r_2 + \mathfrak{N}_4\} + \{(S_3+S_4)r_2 + \mathfrak{N}_2\}]\ell_1.$$

So

$$\begin{aligned}
\|\varGamma(\mathrm{u},\vartheta)\| &\leq \|\varGamma_1(\mathrm{u},\vartheta)\| + \|\varGamma_2(\mathrm{u},\vartheta)\| \\
&\leq |\mathfrak{S}_1| + |\mathfrak{S}_2| + [\{(\Lambda_1+\Lambda_2)r_2 + \mathfrak{N}_3\} + \{(S_1+S_2)r_2 + \mathfrak{N}_1\}]\ell_1 \\
&\quad +[\{(\Lambda_3+\Lambda_4)r_2 + \mathfrak{N}_4\} + \{(S_3+S_4)r_2 + \mathfrak{N}_2\}]\ell_1 \\
&= D + \ell_1[\{(\Lambda_1+\Lambda_2) + (\Lambda_3+\Lambda_4) + (S_1+S_2) + (S_3+S_4)\}r_2 \\
&\quad +(\mathfrak{N}_1+\mathfrak{N}_2) + (\mathfrak{N}_3+\mathfrak{N}_4)] \\
&= D + \ell_1\{\Omega_3 r_2 + W_3\} \leq r_2.
\end{aligned}$$

Hence $\varGamma(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$, for $0 \leq \ell \leq \ell_1$.

Case 2. For $\ell_1 \leq \ell \leq \mathsf{T}$. Consider

$$\begin{aligned}
|\varGamma_1(\mathrm{u},\vartheta)(\ell)| &= \left|\mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)}G(\ell,\mathrm{u}(\ell),\vartheta(\ell)) + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma\right| \\
&\leq |\mathrm{u}(\ell_1)| + \frac{1}{AB(\sigma)}\ [\{|G(\ell,\mathrm{u}(\ell),\vartheta(\ell)) - G(\ell,0,0)| + |G(\ell,0,0)|\}] \\
&\quad + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} \{|G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma)) - G(\varsigma,0,0)| + |G(\varsigma,0,0)|\}\,|(\ell-\varsigma)^{\sigma-1}|\,d\varsigma
\end{aligned}$$

taking $\sup\limits_{\ell\in[\ell_1,\mathsf{T}]}$ on both sides

$$\begin{aligned}
\|\varGamma_1(\mathrm{u},\vartheta)\| &\leq |\mathrm{u}(\ell_1)| + \frac{1}{AB(\sigma)}\{(S_1+S_2)r_2 + \mathfrak{N}_1\} \\
&\quad + \frac{\sigma\mathsf{T}^{\sigma}}{AB(\sigma)\Gamma(\sigma+1)}\ \{(S_1+S_2)r_2 + \mathfrak{N}_1\} \\
&= |\mathrm{u}(\ell_1)| + \{(S_1+S_2)r_2 + \mathfrak{N}_1\}\frac{1}{AB(\sigma)}\left[1 + \frac{\sigma\mathsf{T}^{\sigma}}{\Gamma(\sigma+1)}\right] \\
&= |\mathrm{u}(\ell_1)| + \{(S_1+S_2)\Theta\}r_2 + \mathfrak{N}_1\Theta.
\end{aligned}$$

Similarly

$$\begin{aligned}
|\mathcal{F}_2(\mathrm{u},\vartheta)(\ell)| &= \left|\vartheta(\ell_1) + \frac{1-\sigma}{AB(\sigma)}E(\ell,\mathrm{u}(\ell),\vartheta(\ell)) + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell} E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma\right| \\
&\leq |\vartheta(\ell_1)| + \frac{1}{AB(\sigma)}\ [\{|E(\ell,\mathrm{u}(\ell),\vartheta(\ell)) - E(\ell,0,0)| + |E(\ell,0,0)|\}] \\
&\quad + \frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell} \{|E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma)) - E(\varsigma,0,0)| + |E(\varsigma,0,0)|\}(\ell-\varsigma)^{\sigma-1}d\varsigma
\end{aligned}$$

taking $\sup\limits_{\ell\in[\ell_1,\mathsf{T}]}$ on both sides

$$\begin{aligned}
\|\mathcal{F}_2(\mathrm{u},\vartheta)\| &\leq |\vartheta(\ell_1)| + \frac{1}{AB(\sigma)}\{(S_3+S_4)r_2 + \mathfrak{R}_2\} \\
&\quad + \frac{\sigma\mathsf{T}^{\sigma}}{AB(\sigma)\Gamma(\sigma+1)}\ \{(S_3+S_4)r_2 + \mathfrak{R}_2\} \\
&= |\vartheta(\ell_1)| + \{(S_3+S_4)\Theta\}r_2 + \mathfrak{R}_2\Theta.
\end{aligned}$$

So

$$\begin{aligned}
\|\mathcal{F}(\mathrm{u},\vartheta)\| &\leq \|\mathcal{F}_1(\mathrm{u},\vartheta)\| + \|\mathcal{F}_2(\mathrm{u},\vartheta)\| \\
&\leq Z + \{(S_1+S_2)\Theta\}r_2 + \mathfrak{R}_1\Theta \\
&\quad \{(S_3+S_4)\Theta\}r_2 + \mathfrak{R}_2\Theta \\
&= Z + (\Omega_2 r_2 + W_1)\Theta \leq r_2.
\end{aligned}$$

Therefore $\mathcal{F}(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$, for $\ell_1 \leq \ell \leq \mathsf{T}$. Hence $\mathcal{F}(\mathfrak{B}_{r_2}) \subseteq \mathfrak{B}_{r_2}$, for both cases.

Step 2. Now we show a mapping $\mathcal{F}$ is contraction .

Case 1 For $0 \leq \ell \leq \ell_1$, consider

$$\begin{aligned}
|\mathcal{F}_1(\mathrm{u},\vartheta)(\ell) - \mathcal{F}_1(\mathrm{u}_1,\vartheta_1)(\ell)| &\leq \left\{\int\limits_0^{\mathfrak{z}_1} |\eta(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma)) - \eta(\varsigma,\mathrm{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma\right. \\
&\quad \left. + \int\limits_0^{\ell} |G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma)) - G(\varsigma,\mathrm{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma\right\},
\end{aligned}$$

taking $\sup\limits_{\ell\in[0,\ell_1]}$ on both sides

$$\|\mathcal{F}_1(\mathrm{u},\vartheta)-\mathcal{F}_1(\mathrm{u}_1,\vartheta_1)\| \leq \ell_1\{\mathfrak{p}_1+\mathfrak{q}_1\}\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}.$$

Similarly

$$\begin{aligned}|\mathcal{F}_2(\mathrm{u},\vartheta)(\ell)-\mathcal{F}_2(\mathrm{u}_1,\vartheta_1)(\ell)| &\leq \Bigg\{\int\limits_0^{3_2}|\xi(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))-\xi(\varsigma,\mathrm{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma\\ &\quad+\int\limits_0^{\ell}|E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))-E(\varsigma,\mathrm{u}_1(\varsigma),\vartheta_1(\varsigma))|\,d\varsigma\Bigg\}.\end{aligned}$$

taking $\sup\limits_{\ell\in[0,\ell_1]}$ on both sides

$$\|\mathcal{F}_2(\mathrm{u},\vartheta)-\mathcal{F}_2(\mathrm{u}_1,\vartheta_1)\| \leq \ell_1\{\mathfrak{p}_2+\mathfrak{q}_2\}\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}.$$

Hence

$$\begin{aligned}\|\mathcal{F}(\mathrm{u},\vartheta)-\mathcal{F}(\mathrm{u}_1,\vartheta_1)\| &\leq \|\mathcal{F}_1(\mathrm{u},\vartheta)-\mathcal{F}_1(\mathrm{u}_1,\vartheta_1)\|+\|\mathcal{F}_2(\mathrm{u},\vartheta)-\mathcal{F}_2(\mathrm{u}_1,\vartheta_1)\|\\ &\leq \ell_1\{(\mathfrak{p}_1+\mathfrak{q}_1)+(\mathfrak{p}_2+\mathfrak{q}_2)\}\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\},\end{aligned}$$

as $\varrho_1=\ell_1[\{\mathfrak{p}_1+\mathfrak{q}_1\}+\{\mathfrak{p}_2+\mathfrak{q}_2\}] < 1$, then

$$\|\mathcal{F}(\mathrm{u},\vartheta)(\ell)-\mathcal{F}(\mathrm{u}_1,\vartheta_1)(\ell)\| \leq \varrho_1\,\|(\mathrm{u},\vartheta)-(\mathrm{u}_1,\vartheta_1)\|.$$

Case 2: For $\ell_1 \leq \ell \leq \mathsf{T}$. Consider

$$\begin{aligned}|\mathcal{F}_1(\mathrm{u},\vartheta)(\ell)-\mathcal{F}_1(\mathrm{u}_1,\vartheta_1)(\ell)| &\leq |\mathrm{u}(\ell)-\mathrm{u}_1(\ell)|+\frac{1-\sigma}{AB(\sigma)}\\ &\quad|G(\ell,\mathrm{u}(\ell),\vartheta(\ell))-G(\ell,\mathrm{u}_1(\ell),\vartheta_1(\ell))|+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\\ &\quad\int\limits_{\ell_1}^{\ell}|G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))-G(\varsigma,\mathrm{u}_1(\varsigma),\vartheta_1(\varsigma))|\left|(\ell-\varsigma)^{\sigma-1}\right|d\varsigma\end{aligned}$$

taking $\sup\limits_{\ell\in[\ell_1,\mathrm{T}]}$ on both sides,

$$\begin{aligned}\|\mathcal{F}_1(\mathrm{u},\vartheta)-\mathcal{F}_1(\mathrm{u}_1,\vartheta_1)\| &\leq \|\mathrm{u}-\mathrm{u}_1\|+\frac{1}{AB(\sigma)}\mathrm{q}_1\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}\\ &\quad+\frac{\sigma}{AB(\sigma)\Gamma(\sigma+1)}\mathrm{q}_1\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}\mathrm{T}^{\sigma}\\ &\leq \Theta\mathrm{q}_1\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\},\end{aligned}$$

where Θ is given in (6.6). So

$$\|\mathcal{F}_1(\mathrm{u},\vartheta)-\mathcal{F}_1(\mathrm{u}_1,\vartheta_1)\|\leq\Theta\mathrm{q}_1\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}.$$

Similarly

$$\begin{aligned}|\mathcal{F}_2(\mathrm{u},\vartheta)(\ell)-\mathcal{F}_2(\mathrm{u}_1,\vartheta_1)(\ell)| &\leq |\vartheta(\ell)-\vartheta_1(\ell)|+\frac{1-\sigma}{AB(\sigma)}\\ &\quad|\{E(\ell,\mathrm{u}(\ell),\vartheta(\ell))\}-E(\ell,\mathrm{u}_1(\ell),\vartheta_1(\ell))|+\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\\ &\quad\int_{\ell_1}^{\ell}|E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))-E(\varsigma,\mathrm{u}_1(\varsigma),\vartheta_1(\varsigma))|\,\left|(\ell-\varsigma)^{\sigma-1}\right|d\varsigma,\end{aligned}$$

taking $\sup\limits_{\ell\in[\ell_1,\mathrm{T}]}$ on both sides,

$$\begin{aligned}\|\mathcal{F}_2(\mathrm{u},\vartheta)-\mathcal{F}_2(\mathrm{u}_1,\vartheta_1)\| &\leq \|\vartheta-\vartheta_1\|+\frac{1}{AB(\sigma)}\mathrm{q}_2\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}\\ &\quad+\frac{\sigma}{AB(\sigma)\Gamma(\sigma+1)}\mathrm{q}_2\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}\mathrm{T}^{\sigma}\\ &\leq \Theta\mathrm{q}_2\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}.\end{aligned}$$

Then

$$\begin{aligned}\|\mathcal{F}(\mathrm{u},\vartheta)-\mathcal{F}(\mathrm{u}_1,\vartheta_1)\| &\leq \|\mathcal{F}_1(\mathrm{u},\vartheta)-\mathcal{F}_1(\mathrm{u}_1,\vartheta_1)\|+\|\mathcal{F}_2(\mathrm{u},\vartheta)-\mathcal{F}_2(\mathrm{u}_1,\vartheta_1)\|\\ &\leq \Theta(\mathrm{q}_1+\mathrm{q}_2)\{\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|\}\end{aligned}$$

where $\varrho_2 = \{\Theta(\mathfrak{q}_1 + \mathfrak{q}_2)\} < 1$, so

$$\|\mathcal{F}(\mathfrak{u},\vartheta) - \mathcal{F}(\mathfrak{u}_1,\vartheta_1)\| \leq \varrho_2 \|(\mathfrak{u},\vartheta) - (\mathfrak{u}_1,\vartheta_1)\|.$$

Hence

$$\|\mathcal{F}(\mathfrak{u},\vartheta) - \mathcal{F}(\mathfrak{u}_1,\vartheta_1)\| \leq \begin{cases} \varrho_1 \|(\mathfrak{u},\vartheta) - (\mathfrak{u}_1,\vartheta_1)\|, \text{ if } \ell \in [0,\ell_1] \\ \varrho_2 \|(\mathfrak{u},\vartheta) - (\mathfrak{u}_1,\vartheta_1)\|, \text{ if } \ell \in [\ell_1,\mathsf{T}]. \end{cases}$$

So

$$\|\mathcal{F}(\mathfrak{u},\vartheta) - \mathcal{F}(\mathfrak{u}_1,\vartheta_1)\| \leq \max\{\varrho_1,\varrho_2\} \|(\mathfrak{u},\vartheta) - (\mathfrak{u}_1,\vartheta_1)\|.$$

As $\max\{\varrho_1,\varrho_2\} < 1$. From Banach contraction principle (Theorem 1.1), we get a unique solution of AB-piecewise FDS (6.1) – (6.2). ∎

6.4 Hyers-Ulam Stability

The HU stability of AB-piecewise FDS (6.1) – (6.2) is presented, in this section. First some useful remarks and definitions are given, which are crucial in proof of theorem in this section.

Remark 6.1. The solution $(\mathfrak{u},\vartheta)(\ell) \in X \times X$ of AB-piecewise FDS (6.1) – (6.2) is given as

$$\begin{cases} \mathfrak{u}(\ell) = \begin{cases} \mathfrak{S}_1 + \int_0^{\mathfrak{z}_1} \eta(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))d\varsigma \\ +\int_0^{\ell} G(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))d\varsigma, \text{ if } 0 \leq \ell \leq \ell_1, \\ \mathfrak{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)}G(\ell,\mathfrak{u}(\ell),\vartheta(\ell)) \\ +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} G(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma, \text{ if } \ell_1 \leq \ell \leq \mathsf{T}. \end{cases} \\ \\ \vartheta(\ell) = \begin{cases} \mathfrak{S}_2 + \int_0^{\mathfrak{z}_2} \xi(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))d\varsigma \\ +\int_0^{\ell} E(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))d\varsigma \text{ if } 0 \leq \ell \leq \ell_1, \\ \vartheta(\ell_1) + \frac{1-\sigma}{AB(\sigma)}E(\ell,\mathfrak{u}(\ell),\vartheta(\ell)) \\ +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} E(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma \text{ if } \ell_1 \leq \ell \leq \mathsf{T}. \end{cases} \end{cases}$$

Definition 6.1. The AB-piecewise FDS (6.1) – (6.2) is HU stable if there exists a constant $\lambda > 0$, such that for every $\epsilon_1, \epsilon_2 > 0$ and each solution $(\mathrm{u}, \vartheta)(\ell) \in X \times X$, satisfying

$$\left|({}^{PAB}{}_{0}D^{\sigma}\mathrm{u})(\ell) - G(\ell, \mathrm{u}(\ell), \vartheta(\ell))\right| \leq \epsilon_1 \text{ and } \left|({}^{PAB}{}_{0}D^{\sigma}\vartheta)(\ell) - E(\ell, \mathrm{u}(\ell), \vartheta(\ell))\right| \leq \epsilon_2, \qquad (6.7)$$

for all $\ell \in$ [0,T]. Then there exists $(\mathrm{u}_1, \vartheta_1)(\ell) \in X \times X$ a solution of given AB-piecewise FDS (6.1) – (6.2), such that

$$|(\mathrm{u}, \vartheta)(\ell) - (\mathrm{u}_1, \vartheta_1)(\ell)| \leq \lambda\epsilon, \text{ for all } \ell \in [0, \mathsf{T}]. \qquad (6.8)$$

Where

$$\lambda = \max\left\{\frac{\ell_1}{1 - \varrho_1}, \frac{\mathsf{T}}{1 - \varrho_2}\right\} \text{ and } \epsilon = \max\{\epsilon_1, \epsilon_2\}.$$

Remark 6.2. $(\mathrm{u}, \vartheta)(\ell) \in X \times X$ is solution of (6.7) if and only if there exists constants $\tau_1(\ell), \tau_2(\ell) \in X$, such that

(i) $|\tau_1(\ell)| \leq \epsilon_1$ and $|\tau_2(\ell)| \leq \epsilon_2$, for all $\ell \in [0, \mathsf{T}]$.

(ii) $({}^{PAB}{}_{0}D^{\sigma}\mathrm{u})(\ell) = G(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_1(\ell)$ and $({}^{PAB}{}_{0}D^{\varrho}\vartheta)(\ell) = E(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_2(\ell)$, for all $\ell \in [0, \mathsf{T}]$.

Theorem 6.3. Assume that G, E : $[0,\mathsf{T}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, both are continuous functions and $(H13)$ hold. Then AB-Piecewise FDS (6.1) – (6.2) is HU stable if

$$1 - \varrho_1 \neq 0 \text{ and } 1 - \varrho_2 \neq 0.$$

Proof. Suppose $(\mathrm{u}, \vartheta)(\ell) \in X \times X$ is solution of (6.7), from Remark 6.2,

$$\begin{cases} ({}^{PAB}{}_{0}D^{\sigma}\mathrm{u})(\ell) = G(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_1(\ell), \\ \\ ({}^{PAB}{}_{0}D^{\sigma}\vartheta)(\ell) = E(\ell, \mathrm{u}(\ell), \vartheta(\ell)) + \tau_2(\ell), \end{cases}$$

for all $\ell \in [0, \mathsf{T}]$.

Case 1 For $0 \leq \ell \leq \ell_1$. Now from Remark 6.1, we can write

$$\begin{cases} \mathfrak{u}(\ell) = \mathfrak{S}_1 + \int_0^{\mathfrak{Z}_1} \eta(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} G(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} \tau_1(\varsigma)\, d\varsigma, \\ \vartheta(\ell) = \mathfrak{S}_2 + \int_0^{\mathfrak{Z}_2} \xi(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} E(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma + \int_0^{\ell} \tau_2(\varsigma)\, d\varsigma. \end{cases}$$

Which implies

$$\begin{cases} \left| \mathfrak{u}(\ell) - \mathfrak{S}_1 - \int_0^{\mathfrak{Z}_1} \eta(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\ell} G(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma \right| \leq \int_0^{\ell} |\tau_1(\varsigma)|\, d\varsigma \\ \qquad \leq \ell\epsilon_1. \\ \left| \vartheta(\ell) - \mathfrak{S}_2 - \int_0^{\mathfrak{Z}_2} \xi(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\ell} E(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma \right| \leq \int_0^{\ell} |\tau_2(\varsigma)|\, d\varsigma \\ \qquad \leq \ell\epsilon_2. \end{cases}$$

Now assume $(\mathfrak{u}_1, \vartheta_1)(\ell) \in X$ is a unique solution $(6.1) - (6.2)$. Then we have.

$$\begin{cases} |\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| = \left| \mathfrak{u}(\ell) - \mathfrak{S}_1 - \int_0^{\mathfrak{Z}_1} \eta(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma - \int_0^{\ell} G(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma \right|. \\ |\vartheta(\ell) - \vartheta_1(\ell)| = \left| \vartheta(\ell) - \mathfrak{S}_2 - \int_0^{\mathfrak{Z}_2} \xi(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma - \int_0^{\ell} E(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma \right|. \end{cases} \tag{6.9}$$

After simplifying, we get

$$\begin{cases} |\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| \leq \left| \mathfrak{u}(\ell) - \mathfrak{S}_1 - \int_0^{\mathfrak{Z}_1} \eta(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\ell} G(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma \right| \\ \qquad + \left| \int_0^{\mathfrak{Z}_1} \eta(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\mathfrak{Z}_1} \eta(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma \right| \\ \qquad + \left| \int_0^{\ell} G(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\ell} G(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma \right|. \\ |\vartheta(\ell) - \vartheta_1(\ell)| \leq \left| \vartheta(\ell) - \mathfrak{S}_2 - \int_0^{\mathfrak{Z}_2} \xi(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\ell} E(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma \right| \\ \qquad + \left| \int_0^{\mathfrak{Z}_2} \xi(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\mathfrak{Z}_2} \xi(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma \right| \\ \qquad + \left| \int_0^{\ell} E(\varsigma, \mathfrak{u}(\varsigma), \vartheta(\varsigma))d\varsigma - \int_0^{\ell} E(\varsigma, \mathfrak{u}_1(\varsigma), \vartheta_1(\varsigma))d\varsigma \right|. \end{cases}$$

From $(H13)$ and taking $\sup\limits_{\ell\in[0,\ell_1]}$ on both sides, we obtain

$$\begin{cases} \|\mathrm{u}-\mathrm{u}_1\| \leq \ell_1\epsilon_1 + \ell_1(\mathrm{p}_1+\mathrm{q}_1)(\|\mathrm{u}-\mathrm{u}_1\| + \|\vartheta-\vartheta_1\|), \\ \|\vartheta-\vartheta_1\| \leq \ell_1\epsilon_2 + \ell_1(\mathrm{p}_2+\mathrm{q}_2)(\|\mathrm{u}-\mathrm{u}_1\| + \|\vartheta-\vartheta_1\|). \end{cases}$$

Adding both above terms

$$\begin{aligned} \|\mathrm{u}-\mathrm{u}_1\| + \|\vartheta-\vartheta_1\| &\leq \ell_1\epsilon_1 + \ell_1(\mathrm{p}_1+\mathrm{q}_1)(\|\mathrm{u}-\mathrm{u}_1\| + \|\vartheta-\vartheta_1\|) \\ &\quad + \ell_1\epsilon_2 + \ell_1(\mathrm{p}_2+\mathrm{q}_2)(\|\mathrm{u}-\mathrm{u}_1\| + \|\vartheta-\vartheta_1\|). \\ &= \ell_1\epsilon + \ell_1\{(\mathrm{p}_1+\mathrm{q}_1) + (\mathrm{p}_2+\mathrm{q}_2)\}(\|\mathrm{u}-\mathrm{u}_1\| + \|\vartheta-\vartheta_1\|). \end{aligned}$$

Then

$$\|\{(\mathrm{u},\vartheta) - (\mathrm{u}_1,\vartheta_1)\}\| \leq \frac{\ell_1\epsilon}{(1-\varrho_1)}.$$

where $\epsilon = \max\{\epsilon_1,\epsilon_2\}$ and $1-\varrho_1 \neq 0$.

Case 2. For $\ell_1 \leq \ell \leq \mathsf{T}$. Now from Remark 6.1, we can write

$$\begin{cases} \mathrm{u}(\ell) = \mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)}G(\ell,\mathrm{u}(\ell),\vartheta(\ell)) \\ +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma + \int_{\ell_1}^{\mathsf{T}} \tau_1(\varsigma)\,d\varsigma, \\ \vartheta(\ell) = \vartheta(\ell_1) + \frac{1-\sigma}{AB(\sigma)}E(\ell,\mathrm{u}(\ell),\vartheta(\ell)) \\ +\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma + \int_{\ell_1}^{\mathsf{T}} \tau_2(\varsigma)\,d\varsigma. \end{cases}$$

Which implies

$$\begin{cases} \left| \begin{array}{l} \mathrm{u}(\ell) - \mathrm{u}(\ell_1) - \frac{1-\sigma}{AB(\sigma)}G(\ell,\mathrm{u}(\ell),\vartheta(\ell)) \\ -\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} G(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma \end{array} \right| \leq \int_{\ell_1}^{\mathsf{T}} |\tau_1(\varsigma)|\,d\varsigma \\ \qquad \leq (\mathsf{T}-\ell_1)\epsilon_1. \\ \left| \begin{array}{l} \vartheta(\ell) - \vartheta(\ell_1) - \frac{1-\sigma}{AB(\sigma)}E(\ell,\mathrm{u}(\ell),\vartheta(\ell)) \\ -\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int_{\ell_1}^{\ell} E(\varsigma,\mathrm{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma \end{array} \right| \leq \int_{\ell_1}^{\mathsf{T}} |\tau_2(\varsigma)|\,d\varsigma \\ \qquad \leq (\mathsf{T}-\ell_1)\epsilon_2. \end{cases}$$

Now assume $(\mathfrak{u}_1,\vartheta_1)(\ell)\in X$ is unique solution (6.1) – (6.2). Then we have.

$$\begin{cases} |\mathfrak{u}(\ell)-\mathfrak{u}_1(\ell)| = \left| \begin{array}{c} \mathfrak{u}(\ell) - \mathfrak{u}_1(\ell_1) - \frac{1-\sigma}{AB(\sigma)}G(\ell,\mathfrak{u}_1(\ell),\vartheta_1(\ell)) \\ -\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell} G(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma \end{array}\right|, \\ |\vartheta(\ell)-\vartheta_1(\ell)| = \left| \begin{array}{c} \vartheta(\ell) - \vartheta_1(\ell_1) - \frac{1-\sigma}{AB(\sigma)}E(\ell,\mathfrak{u}_1(\ell),\vartheta_1(\ell)) \\ -\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell} E(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma \end{array}\right|. \end{cases} \tag{6.10}$$

After simplification,

$$\begin{cases} |\mathfrak{u}(\ell)-\mathfrak{u}_1(\ell)| \leq \Big| \mathfrak{u}(\ell) - \mathfrak{u}(\ell_1) - \frac{1-\sigma}{AB(\sigma)}G(\ell,\mathfrak{u}(\ell),\vartheta(\ell)) \\ \quad -\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell} G(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma \Big| \\ \quad + \left|\frac{1-\sigma}{AB(\sigma)}\left\{G(\ell,\mathfrak{u}(\ell),\vartheta(\ell)) - G(\ell,\mathfrak{u}_1(\ell),\vartheta_1(\ell))\right\}\right| \\ + \left|\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell}\left\{G(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma)) - G(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))\right\}(\ell-\varsigma)^{\sigma-1}d\varsigma\right|, \\ |\vartheta(\ell)-\vartheta_1(\ell)| \leq \Big| \vartheta(\ell) - \vartheta(\ell_1) - \frac{1-\sigma}{AB(\sigma)}E(\ell,\mathfrak{u}(\ell),\vartheta(\ell)) \\ \quad -\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell} E(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma))(\ell-\varsigma)^{\sigma-1}d\varsigma \Big| \\ \quad + \left|\frac{1-\sigma}{AB(\sigma)}\left\{E(\ell,\mathfrak{u}(\ell),\vartheta(\ell)) - E(\ell,\mathfrak{u}_1(\ell),\vartheta_1(\ell))\right\}\right| \\ + \left|\frac{\sigma}{AB(\sigma)\Gamma(\sigma)}\int\limits_{\ell_1}^{\ell}\left\{E(\varsigma,\mathfrak{u}(\varsigma),\vartheta(\varsigma)) - E(\varsigma,\mathfrak{u}_1(\varsigma),\vartheta_1(\varsigma))\right\}(\ell-\varsigma)^{\sigma-1}d\varsigma\right|. \end{cases}$$

From $(H13)$, and taking $\sup\limits_{\ell\in[\ell_1,\mathrm{T}]}$ on both sides

$$\begin{cases} \|\mathfrak{u}-\mathfrak{u}_1\| \leq \mathrm{T}\epsilon_1 + \frac{1-\sigma}{AB(\sigma)}\mathfrak{q}_1(\|\mathfrak{u}-\mathfrak{u}_1\| + \|\vartheta-\vartheta_1\|) \\ \quad +\frac{\sigma}{AB(\sigma)\Gamma(\sigma+1)}\mathrm{T}^{\sigma}\mathfrak{q}_1(\|\mathfrak{u}-\mathfrak{u}_1\| + \|\vartheta-\vartheta_1\|), \\ \|\vartheta-\vartheta_1\| \leq \mathrm{T}\epsilon_2 + \frac{1-\sigma}{AB(\sigma)}\mathfrak{q}_2(\|\mathfrak{u}-\mathfrak{u}_1\| + \|\vartheta-\vartheta_1\|) \\ \quad +\frac{\sigma}{AB(\sigma)\Gamma(\sigma+1)}\mathrm{T}^{\sigma}\mathfrak{q}_2(\|\mathfrak{u}-\mathfrak{u}_1\| + \|\vartheta-\vartheta_1\|). \end{cases}$$

$$\begin{cases} \|\mathfrak{u}-\mathfrak{u}_1\| \leq \mathrm{T}\epsilon_1 + \frac{1}{AB(\sigma)}\left\{1 + \frac{\sigma}{\Gamma(\sigma+1)}\mathrm{T}^{\sigma}\right\} \\ \quad \left\{\mathfrak{q}_1(\|\mathfrak{u}-\mathfrak{u}_1\| + \|\vartheta-\vartheta_1\|)\right\}. \\ \|\vartheta-\vartheta_1\| \leq \mathrm{T}\epsilon_2 + \frac{1}{AB(\sigma)}\left\{1 + \frac{\sigma}{\Gamma(\sigma+1)}\mathrm{T}^{\sigma}\right\} \\ \quad \left\{\mathfrak{q}_2(\|\mathfrak{u}-\mathfrak{u}_1\| + \|\vartheta-\vartheta_1\|)\right\}. \end{cases}$$

Adding both above terms

$$\begin{aligned}\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\| &\leq \mathrm{T}(\epsilon_1+\epsilon_2)\\ &\quad+\Theta(\mathrm{q}_1+\mathrm{q}_2)(\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|)\\ &= \mathrm{T}\epsilon+\{\Theta(\mathrm{q}_1+\mathrm{q}_2)\}(\|\mathrm{u}-\mathrm{u}_1\|+\|\vartheta-\vartheta_1\|).\end{aligned}$$

Then

$$\|\{(\mathrm{u},\vartheta)-(\mathrm{u}_1,\vartheta_1)\}\| \leq \frac{\mathrm{T}\epsilon}{1-\varrho_2},$$

where $\epsilon=\max\{(\epsilon_1,\epsilon_2)\}$ and $1-\varrho_2\neq 0$.

Hence

$$\|\{(\mathrm{u},\vartheta)-(\mathrm{u}_1,\vartheta_1)\}\| \leq \begin{cases}\frac{\ell_1}{(1-\varrho_1)}\epsilon, \text{ if } \ell\in[0,\ell_1],\\ \frac{\mathrm{T}}{1-\varrho_2}\epsilon, \text{ if } \ell\in(\ell_1,\mathrm{T}].\end{cases}$$

So for all $\ell\in[0,\mathrm{T}]$, then

$$\|\{(\mathrm{u},\vartheta)-(\mathrm{u}_1,\vartheta_1)\}\| \leq \max\left\{\frac{\ell_1}{1-\varrho_1},\frac{\mathrm{T}}{1-\varrho_2}\right\}\epsilon = \Lambda\epsilon.$$

Therefore (6.1) – (6.2) is HU stable for $\ell\in[0,\mathrm{T}]$. ■

6.5 Numerical Scheme

This section presents numerical Euler's scheme, which is derived from AB-Taylor series and can be seen in [133]. As

$$\begin{aligned}\mathrm{u}(\ell) &= \sum_{\mathrm{m}=0}^{\mathrm{n}} S_{\alpha,m}(\ell-a)({}^{ABC}{}_0D^{\sigma}_{a^+})^{\mathrm{m}}(\mathrm{u}(a))\\ &\quad+S_{\alpha,\mathrm{n}+1}(\ell-a)({}^{ABC}{}_0D^{\sigma}_{a^+})^{\mathrm{n}+1}(\mathrm{u}(\mathcal{F})).\end{aligned}$$

Then after simplification and neglecting terms involving the powers $2\alpha, 3\alpha$ of h.

$$\mathrm{u}(\ell_{\mathrm{n}+1})=\mathrm{u}(\ell_{\mathrm{n}})+\frac{1-\sigma}{AB(\sigma)}G(\ell_{\mathrm{n}},\mathrm{u}(\ell_{\mathrm{n}}))+\frac{h^{\sigma}}{AB(\sigma)\Gamma(\sigma)}G(\ell_{\mathrm{n}},\mathrm{u}(\ell_{\mathrm{n}}))$$

We use above Euler's scheme for AB-Piecewise FDEs, we get

$$\mathrm{u}(\ell_{\mathrm{n}+1}) = \begin{cases} \mathrm{u}(\ell_{\mathrm{n}}) + hG(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}})), \ 0 \le \ell \le \ell_1 \\ \mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)} G(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}})) + \frac{h^\sigma}{AB(\sigma)\Gamma(\sigma)} G(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}})), \ \ell_1 \le \ell \le \mathsf{T}. \end{cases}$$

Now we write Euler's scheme for AB-Piecewise FDS,

$$\begin{cases} \mathrm{u}(\ell_{\mathrm{n}+1}) = \begin{cases} \mathrm{u}(\ell_{\mathrm{n}}) + hG(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}}), (\vartheta(\ell_{\mathrm{n}})), \ 0 \le \ell \le \ell_1 \\ \mathrm{u}(\ell_1) + \frac{1-\sigma}{AB(\sigma)} G(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}}), \vartheta(\ell_{\mathrm{n}})) \\ + \frac{h^\sigma}{AB(\sigma)\Gamma(\sigma)} G(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}}), \vartheta(\ell_{\mathrm{n}})), \ \ell_1 \le \ell \le \mathsf{T}. \end{cases} \\ \\ \vartheta(\ell_{\mathrm{n}+1}) = \begin{cases} \vartheta(\ell_{\mathrm{n}}) + hE(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}}), \vartheta(\ell_{\mathrm{n}})), \ 0 \le \ell \le \ell_1 \\ \vartheta(\ell_1) + \frac{1-\sigma}{AB(\sigma)} E(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}}), \vartheta(\ell_{\mathrm{n}})) \\ + \frac{h^\sigma}{AB(\sigma)\Gamma(\sigma)} E(\ell_{\mathrm{n}}, \mathrm{u}(\ell_{\mathrm{n}}), \vartheta(\ell_{\mathrm{n}})), \ \ell_1 \le \ell \le \mathsf{T}. \end{cases} \end{cases} \tag{6.11}$$

Next we provide example to validate the conditions of Theorem 6.2 and Theorem 6.3, to get unique and stable solution. Also we get approximate solution with the help of numerical scheme, which is given in 6.11.

Example 6.1. Consider the following non-linear AB-piecewise FDS

$$\begin{cases} {}^{PAB}_{\ \ 0}D^\sigma\{\mathrm{u}(\ell)\} = \frac{\ell+\mathrm{u}(\ell)}{\ell^2+256} + \frac{\vartheta(\ell)+\cos(\ell)}{\ell^2+256}, \\ \\ {}^{PAB}_{\ \ 0}D^\sigma\{\vartheta(\ell)\} = \frac{\sin\ell+\mathrm{u}(\ell)}{\ell^3+128} + \frac{\ell^2+\vartheta(\ell)}{\ell^3+128}, \end{cases} \tag{6.12}$$

and

$$\begin{cases} \mathrm{u}(0) = 0.1 + \int\limits_0^3 \left\{ \frac{\varsigma^6+\mathrm{u}(\varsigma)}{64} + \frac{\varsigma^3+\vartheta(\varsigma)}{64} \right\} d\varsigma. \\ \vartheta(0) = 0.2 + \int\limits_0^4 \left\{ \frac{\varsigma^5+\mathrm{u}(\varsigma)}{56} + \frac{\varsigma^4+\vartheta(\varsigma)}{56} \right\} d\varsigma. \end{cases} \tag{6.13}$$

where

$$\sigma = 0.9, \ \mathfrak{z}_1 = 3, \mathfrak{z}_2 = 4, \ \ell_1 = 10, \ \mathsf{T} = 100 \text{ and } AB(\sigma) = 1, \text{ for all } \sigma \in (0, 1).$$

From (6.12) and (6.13), we have

$$\begin{aligned}
|G(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - G(\ell, \mathfrak{u}_1(\ell), \vartheta_1(\ell))| &\leq \frac{1}{256}\left[|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|\right], \\
|E(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - E(\ell, \mathfrak{u}_1(\ell), \vartheta_1(\ell))| &\leq \frac{1}{128}\left[|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|\right]. \\
|\eta(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - \eta(\ell, \mathfrak{u}_1(\ell), \vartheta_1(\ell))| &\leq \frac{1}{64}\left[|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|\right],
\end{aligned}$$

and

$$|\xi(\ell, \mathfrak{u}(\ell), \vartheta(\ell)) - \xi(\ell, \mathfrak{u}_1(\ell), \vartheta_1(\ell))| \leq \frac{1}{56}\left[|\mathfrak{u}(\ell) - \mathfrak{u}_1(\ell)| + |\vartheta(\ell) - \vartheta_1(\ell)|\right].$$

From above, we get

$$\mathfrak{q}_1 = \frac{1}{256}, \ \mathfrak{p}_1 = \frac{1}{64}, \ \mathfrak{q}_2 = \frac{1}{128}, \ \mathfrak{p}_2 = \frac{1}{56}.$$

Now we calculate,

$$\begin{aligned}
\varrho_1 &= \ell_1[\{\mathfrak{p}_1 + \mathfrak{q}_1\} + \{\mathfrak{p}_2 + \mathfrak{q}_2\}] \\
&= 10\left\{\frac{1}{64} + \frac{1}{256} + \frac{1}{56} + \frac{1}{128}\right\} = 0.4520089286 < 1 \\
\varrho_2 &= \{\Theta(\mathfrak{q}_1 + \mathfrak{q}_2)\} = 0.7036364957 < 1,
\end{aligned}$$

so

$$1 - \varrho_1 \neq 0, \ 1 - \varrho_1 \neq 0.$$

Also

$$\max\{\varrho_1, \varrho_2\} = 0.7036364957 < 0.$$

Hence, we get unique and stable solution of AB-piecewise FDS (6.12) – (6.13) by satisfying all conditions of Theorem 6.2 and Theorem 6.3.

Now we find approximate solution of (6.12) with initial conditions $\mathfrak{u}(0) = 0.1$, $\vartheta(0) = 0.2$, which is taken as special case of (6.13). To find approximate solution of (6.12), numerical scheme is applied which is given in (6.11).

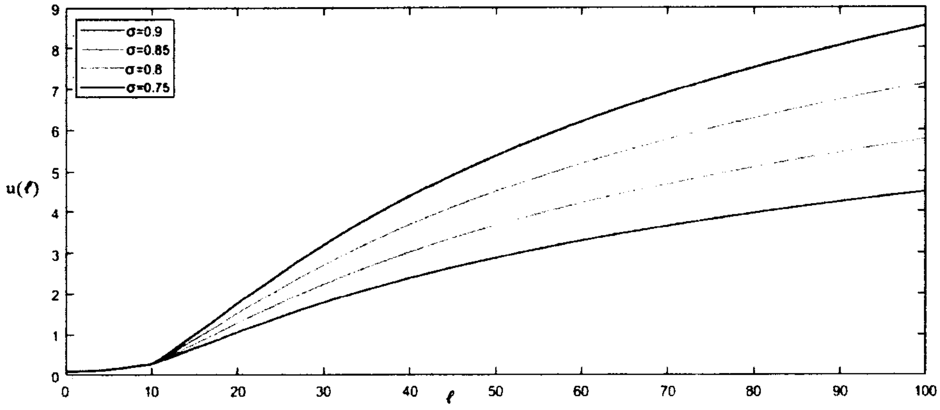

Figure 6.1. Approximate solution u(ℓ) for 0≤ℓ≤10 and 10≤ℓ≤100 with σ=0.9,0.85,0.8,0.75

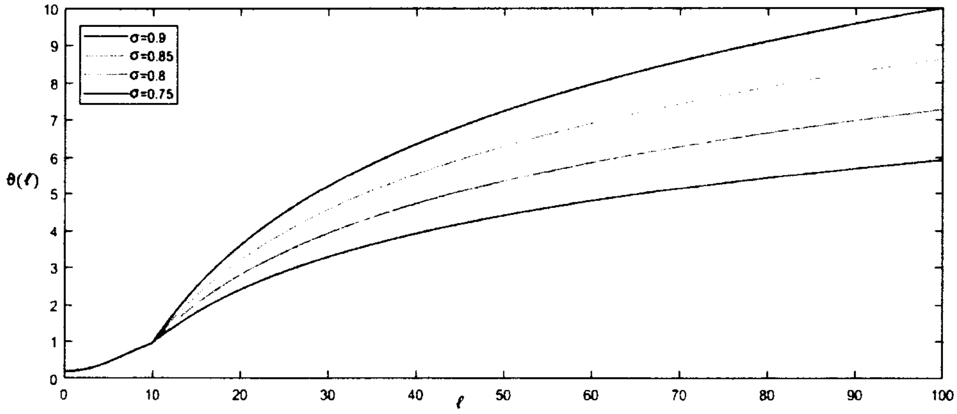

Figure 6.2. Approximate solution θ(ℓ) for 0≤ℓ≤10 and 10≤ℓ≤100 with σ=0.9,0.85,0.8,0.75

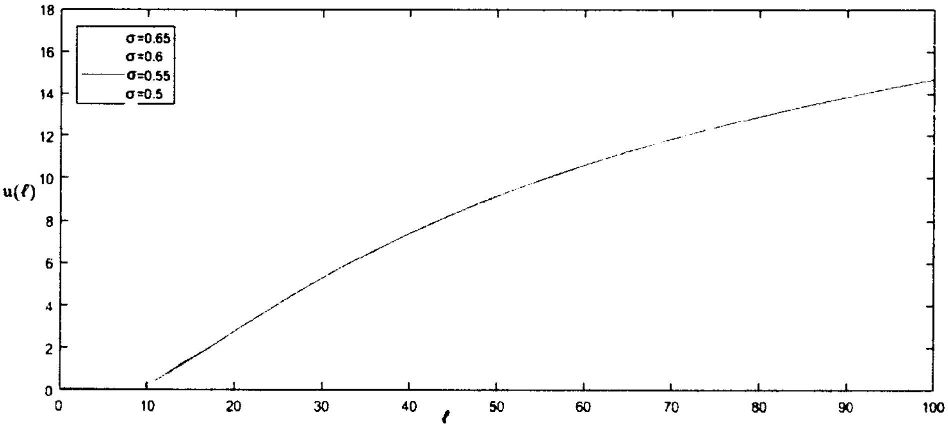

Figure 6.3. Approximate solution u(ℓ) for 0≤ℓ≤10 and 10≤ℓ≤100 with σ=0.65,0.6,0.55,0.5

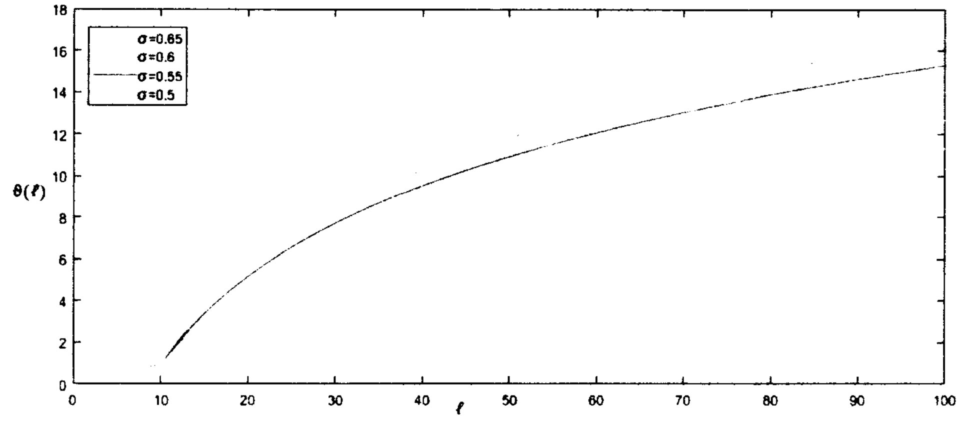

Figure 6.4. Approximate solution $\theta(\ell)$ for $0 \leq \ell \leq 10$ and $10 \leq \ell \leq 100$ with σ=0.65,0.6,0.55,0.5

In Figure 6.1 and Figure 6.2, we present approximate solution $\mathrm{u}(\ell)$ and $\vartheta(\ell)$ respectively, using numerical scheme (6.11) with $h = 0.1$. When take $0 \leq \ell \leq 10$, the values of $\mathrm{u}(\ell)$ and $\vartheta(\ell)$ remain same but the values of $\mathrm{u}(\ell)$ and $\vartheta(\ell)$ changes, when we take $10 \leq \ell \leq 100$ due the involvement of order $\sigma = 0.9, 0.85, 0.8, 0.75$ as show in Figure 6.1 and Figure 6.2. Similarly in Figure 6.3 and Figure 6.4, we present approximate solution $\mathrm{u}(\ell)$ and $\vartheta(\ell)$ using numerical (6.11) with $h = 0.1$ and $\sigma = 0.65, 0.6, 0.55, 0.5$. From above figures it shows that when we reduce the order, the values of $\mathrm{u}(\ell)$ and $\vartheta(\ell)$ increases, and when order become closer to one, the value of $\mathrm{u}(\ell)$ and $\vartheta(\ell)$ decreases. When we take $\sigma = 1$, we get same solution as we get by using classical derivative under classical Euler's formula.

6.6 Conclusion

In this chapter, AB-piecewise FDS (6.1) – (6.2) is presented. In Section 6.2, existence results of AB-piecewise FDS (6.1) – (6.2) is derived in Theorem 6.1 using Schauder fixed point theorem. In Section 6.3 (Theorem 6.2), Banach contraction principle is used to find unique solution. In Section 6.4 (Theorem 6.3), the criteria for HU stability of the solution of AB-piecewise FDS (6.1) – (6.2) is discussed. In Section 6.5, numerical scheme is presented, which is based upon Euler's formula to obtain the approximate solution of given AB-piecewise FDS. In the last, example is provided by satisfying conditions of Theorem 6.2 and Theorem 6.3, to get unique and stable solution. Also we obtain approximate solution using numerical scheme (6.11).

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