

**Distance-Like Functions in Topological Spaces
with Applications in Fixed Point Theory**



Submitted By:

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**Department of Mathematics and Statistics
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International Islamic University, Islamabad
Pakistan
2025**

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A Dissertation

Submitted in the Partial Fulfillment of the

Requirements for the degree of

MASTER of SCIENCE

in

MATHEMATICS

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2025

Certificate

By:

Muhammad Kamran

A dissertation submitted in the partial fulfillment of the requirements for the degree of the **MASTER of SCIENCE in MATHEMATICS**.

We accept this dissertation as conforming to the required standard.

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2025

Dedicated
To
My
Respected Teachers
Grand father (late)
Loving Parents
My Family
And Friends

DECLARATION

I **Muhammad Kamran** son of Allah Ditta Registration No. **969-FOS/MSMA/F23** a student of MS Mathematics Program at International Islamic University, Islamabad do hereby declare that this dissertation entitled “**Distance-like Functions in Topological Spaces with Applications in Fixed Point Theory**” submitted by me in partial fulfillment of MS Mathematics program is my original work and has not been submitted or published earlier. I also solemnly declare that it shall not, in future, be submitted for obtaining any other degree from this or any other university or institution.

I also understand the zero-tolerance policy against plagiarism of the university and the HEC and if my work is found to be plagiarized or copied from someone other's work at any stage, even after the award of the degree, the work may be cancelled and the degree confiscated.

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Abbreviations

Metric space	MS
Topological space	TS
Cauchy sequence	CS
Open ball	OB
Fixed point	FP
Unique fixed point	UFP
Banach contraction	BC
Differential equations	DEs

Preface

Fixed Point Theory has emerged as one of the most vibrant and influential areas of modern mathematics. The fundamental idea underlying this theory is deceptively simple, for a function $\bar{T}: K \rightarrow K$, a fixed point(FP) is a point $l^* \in K$ such that $\bar{T}(l^*) = l^*$. Despite this apparent simplicity, existence and properties of FPs have profound consequences in mathematics.

Historically, the study of FPs is the work of Karl Weierstrass and Henri Poincaré in the nineteenth century. Their investigations into the behavior of continuous function, stability, and qualitative analysis of dynamical systems laid the groundwork for fixed point theory. However, the subject achieved a new level of rigor and generality in the early twentieth century with Banach Contraction Principle (1922). Stefan Banach established that in a complete metric space, any contraction mapping possesses a unique FP, and moreover, successive iterations of the mapping converge to that FP. This theorem, simple yet powerful, became the cornerstone of metric fixed point theorem and inspired extensive research in generalizations, extensions, and applications.

Since Banach's seminal work, fixed point theory has evolved into a highly dynamic field with countless applications. In economics, they underlie equilibrium concepts in markets and most notably in the Nash equilibrium theorem, which itself can be viewed as a FP result. In computer science, fixed points appear in semantics of programming languages and algorithms for recursive definitions. In biology and population dynamics, FPs model equilibrium states of species interactions and ecosystem balance. The wide applicability of fixed point theory demonstrates its universality. often, the existence of a solution to a real-world problem can be reduced to showing the existence of a FP of a suitably defined operator.

Over time, researchers realized that the classical assumptions of metric spaces particularly the requirements of symmetry and the triangular inequality-were sometimes too restrictive. Many natural and practical problems arise in settings where these conditions are not satisfied, yet one still seeks to establish fixed point results. This observation motivated the introduction of generalized metric space structures, including Qusai metric spaces, Partial

metric spaces, and distance-like functions. Each of these generalizations retains some aspects of the metric while relaxing others, thereby enlarging the scope of fixed point theory. Among these, Distance-like functions on topological space have received growing attention. A Distance-like functions $\varphi: K \times K \rightarrow [0, \infty)$ generalizes the concept of distance without requiring strict symmetry or the classical triangular inequality. This flexibility allows the study of fixed point results in environments that go beyond the limitations of metric space. To ensure convergence and stability, the framework instead employs the Asymptotic triangular inequality, a weaker but sufficiently property that replaces the standard triangular inequality. This innovation enables the extension of fixed point results to settings where traditional assumptions no longer hold, thereby opening new directions of research.

The importance of this shift is two fold. First, it allows for the generalization of classical contraction principles such as those of Banach, Kannan, Chatterjea, Hardy Roger contractions, Reich, and Ćirić, all of which can be reformulated in terms of Distance-like functions. Second, it enhances the applicability of fixed point theory to more complex problems in nonlinear analysis, applied mathematics, and beyond. For example, problems in functional equations and integral inclusions often involve operators that do not naturally fit within the framework of metric space but can be accommodated within the more general setting of Distance-like functions.

This thesis is situated within this broad context. Its main objective is to develop FP theorems for mappings defined on topological spaces endowed with Distance-like functions, while employing the Asymptotic Triangular inequality in place of classical restrictions. The results obtained here not only extend and unify known contraction principles but also introduce new avenues for applications. Specifically, we establish generalizations of Banach, Kannan, Chatterjea, Hardy-Rogers, and Reich-type contractions in the framework of Distance-like functions. We also explore the existence of common fixed point(CFP) for pairs of mappings under generalized conditions. These contributions aim to enrich the theoretical foundations of fixed point theory while ensuring greater applicability to nonlinear analysis, optimization, and integral equations. In Chapter 1 we give some primary concepts and preliminaries, including definitions related to metric and

topological spaces. Chapter 2 presents literature review, focusing on extensions of classical contraction principles and related fixed point results. Chapter 3 develops new FP theorems for single and CFP under Distance-like functions with the asymptotic triangular inequality.

Chapter 1

Preliminaries

Some fundamental definitions and ideas are included in this chapter.

1.1 Definition [1]

Let K be a non-empty set. A function $\check{d}: K \times K \rightarrow \mathbb{R}$ is called a metric on K if the following conditions hold

1. $\check{d}(\aleph, \beth) \geq 0$
2. $\check{d}(\aleph, \beth) = 0 \Leftrightarrow \aleph = \beth$
3. $\check{d}(\aleph, \beth) = \check{d}(\beth, \aleph)$
4. $\check{d}(\aleph, \beth) \leq \check{d}(\aleph, \beth) + \check{d}(\beth, \beth)$.

1.2 Open Ball in Metric Space (MS) [1]

Let $(K, \check{d})$ be a MS. A subset of K consisting of all points whose distance from some fixed point $\aleph_0 \in K$ is less than a fixed positive real number r is called an open ball. The fixed point $\aleph_0$ is called the centre, while the fixed number r is the radius of the open ball. An open ball with centre $\aleph_0$ and radius r is denoted by $S_r(\aleph_0)$ or $B(\aleph_0; r)$

$$S_r(\aleph_0) = B(\aleph_0; r) = \{\aleph \in K: \check{d}(\aleph_0, \aleph) < r\}.$$

1.3 Sequence [1]

Let $(K, \check{d})$ be a metric space. A sequence in K is a function $\bar{T}: \mathbb{N} \rightarrow K$, say $\bar{T}(n) = \aleph_n$. $\aleph_n$ is the n -th term and the image set $\{\aleph_1, \aleph_2, \dots\}$ is written as $\{\aleph_n\}$.

1.4 Convergence in Metric Space [1]

A sequence $\{\aleph_n\}$ in $(K, \check{d})$ converges to $\aleph \in K$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ s.t for all $n \geq n_0$ $\check{d}(\aleph_n, \aleph) < \varepsilon$. If $\{\aleph_n\}$ converges to $\aleph$, we write $\aleph_n \rightarrow \aleph$.

1.5 Cauchy Sequence(CS) [1]

A sequence $\{\mathfrak{x}_n\}$ in $(K, \check{d})$ is a CS if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ s.t for all $m, n \geq n_0$

$$\check{d}(\mathfrak{x}_m, \mathfrak{x}_n) < \varepsilon.$$

1.6 Complete Metric Space [1]

A MS $(K, \check{d})$ is complete if every CS in K converges to a point of K .

1.7 Definition [2]

A topological space(TS) is a pair (K, τ) , where K is a set and τ is a collection of subsets of K , called a topology if

1. $\emptyset, K \in \tau$
2. Arbitrary unions of members of τ belong to τ
3. Finite intersections of members of τ belong to τ

1.8 Definition [2]

A TS (K, τ) is a T_0 -space if for each $\mathfrak{x} \neq \mathfrak{y} \in K$, $\exists$ an open set $U \in \tau$ s.t either $\mathfrak{x} \in U$ and $\mathfrak{y} \notin U$ or $\mathfrak{y} \in U$ and $\mathfrak{x} \notin U$.

1.9 Definition [2]

A TS (K, τ) is a T_1 -space if for each $\mathfrak{x} \neq \mathfrak{y} \in K$, $\exists$ an open set containing $\mathfrak{x}$ but not $\mathfrak{y}$, and another open set containing $\mathfrak{y}$ but not $\mathfrak{x}$. Equivalently, every singleton $\{\mathfrak{x}\}$ is closed in K .

Conclusion

This chapter has introduced the fundamental notions required for the development of FP theory in generalized spaces. The discussion began with MSs, which provide a standard framework for studying distance, convergence, and continuity. We then considered TSs, which extend these ideas to a more abstract setting, allowing greater flexibility in analysis. Finally, the basic principles of FP theory were presented, emphasizing their role in mathematical research and practical applications.

The material outlined here serves as the groundwork for the forthcoming chapters. These definitions and concepts will be essential as we move towards generalized approaches, particularly those involving distance-like functions, where the usual conditions of symmetry and the triangle inequality may not hold.

Chapter 2

Fixed Point Results in Topological Spaces and Extension of Banach Contraction Principle

2.1 Definition [4]

Let K be a TS, and let $g: K \times K \rightarrow \mathbb{R}$ be CF. Let $\{\mathfrak{K}_n\}$ be a sequence in K and $\mathfrak{K} \in K$. Then $\{\mathfrak{K}_n\}$ is g -convergent to l if

$$|g(\mathfrak{K}_n, \mathfrak{K})| \rightarrow 0 \text{ when } n \rightarrow \infty,$$

i.e., for a given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ s.t

$$|g(\mathfrak{K}_n, \mathfrak{K})| < \varepsilon, \forall n \geq k.$$

2.2 Definition [4]

Let K be a TS, and let $g: K \times K \rightarrow \mathbb{R}$ be CF. A sequence $\{\mathfrak{K}_n\}$ in K is g -Cauchy if

$$|g(\mathfrak{K}_n, \mathfrak{K}_m)| \rightarrow 0 \text{ as } n, m \rightarrow \infty,$$

i.e., for a given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ s.t

$$|g(\mathfrak{K}_n, \mathfrak{K}_m)| < \varepsilon, \forall n, m \geq k.$$

2.3 Definition [4]

A space K is g -complete if every g -CS $\{\mathfrak{K}_n\}$ is g -convergent to some $\mathfrak{K} \in K$.

2.4 Lemma [4]

Let K be a TS and $g: K \times K \rightarrow \mathbb{R}$ be continuous s.t

$$g(\mathfrak{K}, \mathfrak{J}) = 0 \Rightarrow \mathfrak{K} = \mathfrak{J}, \check{d}(\mathfrak{K}, \mathfrak{E}) \leq \check{d}(\mathfrak{K}, \mathfrak{J}) + \check{d}(\mathfrak{J}, \mathfrak{E}) \quad \forall \mathfrak{K}, \mathfrak{J}, \mathfrak{E} \in K.$$

Then the g -limit of a g -convergent sequence is unique.

2.5 Definition [4]

Let $\bar{T}: A \rightarrow B$ be a mapping, where $A, B \subset K$ and are non-empty. $\bar{T}$ is a topological Banach contraction w.r.t g if there exists $\alpha \in (0, 1)$ s.t

$$|g(\bar{T}(\aleph), \bar{T}(\beth))| \leq \alpha |g(\aleph, \beth)|, \forall \aleph, \beth \in A.$$

2.6 Theorem [4]

Let K be a g -complete TS where $g: K \times K \rightarrow \mathbb{R}$ is CF such that $g(\aleph, \beth) = 0 \Rightarrow \aleph = \beth$, $|g(\aleph, \beth)| = |g(\beth, \aleph)|$, and $\check{d}(\aleph, \beth) \leq \check{d}(\aleph, \beth) + \check{d}(\beth, \beth)$ for all $\aleph, \beth, \beth \in K$. Let $\bar{T}: K \rightarrow K$ be a topological BC mapping w.r.t g . Then $\bar{T}$ has a unique FP and for any $\aleph_0 \in K$, the sequence $\aleph_{n+1} = \bar{T}(\aleph_n), n \geq 0$ converges to the unique FP of $\bar{T}$.

2.12 Example [4]

Let $\mathbb{R}^2$ with the usual topology, and define $g((\aleph, \beth), (u, \beth)) = \beth - \beth$. Let $\bar{T}((\aleph, \beth)) = (\aleph, \beth/2)$. Then $\bar{T}$ is a topological BC w.r.t g . For any g -CS $\{(\aleph_n, \beth_n)\}$, we have $\aleph_n \rightarrow \aleph^*$ and $(\aleph_n, \beth_n) \rightarrow (\aleph_{n_0}, \beth^*)$. The set of FPs is $\{(\aleph, 0): \aleph \in \mathbb{R}\}$, so infinitely many FPs exist.

2.13 Corollary [4]

Let K be g -complete, and $\bar{T}: K \rightarrow K$ such that some iterate $\bar{T}^{n_0}$ is a topological BC. Then $\bar{T}$ has a unique FP.

2.14 Quasi Contraction [3]

Let $(K, \check{d})$ be a MS. A mapping $\bar{T}: K \rightarrow K$ is a quasi-contraction if there exists $\kappa \in [0, 1)$ s.t for all $\aleph, \beth \in K$

$$\check{d}(\bar{T}\aleph, \bar{T}\beth) \leq \kappa \max\{\check{d}(\aleph, \beth), \check{d}(\aleph, \bar{T}\aleph), \check{d}(\beth, \bar{T}\beth), \check{d}(\aleph, \bar{T}\beth), \check{d}(\beth, \bar{T}\aleph)\}.$$

2.14.1 Theorem [3]

Let $(K, \check{d})$ be complete MS, and $\bar{T}$ satisfy the quasi-contraction condition. Then $\bar{T}$ has a unique FP $\aleph^* \in K$.

Proof

Let $\aleph_0 \in K$ and define $\aleph_{n+1} = \bar{T}\aleph_n, n = 0, 1, 2, \dots$. Let $\check{d}_n = \check{d}(\aleph_{n+1}, \aleph_n)$. From the quasi-contraction condition,

$$\check{d}_n \leq \kappa \max\{\check{d}_{n-1}, \check{d}_n, \check{d}_{n-1} + \check{d}_n\} \Rightarrow (1 - \kappa)\check{d}_n \leq \kappa\check{d}_{n-1} \Rightarrow \check{d}_n \leq \frac{\kappa}{1 - \kappa} \check{d}_{n-1}.$$

$$\hat{d}_n \leq \frac{\kappa}{1 - \kappa} \hat{d}_{n-1} \leq \left(\frac{\kappa}{1 - \kappa}\right)^{n-1} \hat{d}_1.$$

Since $0 \leq \kappa < 1$, it follows that $\hat{d}_n \rightarrow 0$. Hence,

$$\hat{d}(\mathfrak{X}_{n+1}, \mathfrak{X}_n) \rightarrow 0.$$

For $m > n$,

$$\hat{d}(\mathfrak{X}_m, \mathfrak{X}_n) \leq \hat{d}(\mathfrak{X}_n, \mathfrak{X}_{n+1}) + \hat{d}(\mathfrak{X}_{n+1}, \mathfrak{X}_{n+2}) + \cdots + \hat{d}(\mathfrak{X}_{m-1}, \mathfrak{X}_m) = \sum_{i=n}^{m-1} \hat{d}_i.$$

Since $\hat{d}_i \rightarrow 0$ geometrically, the series converges. Therefore, $\{\mathfrak{X}_n\}$ is CS. Since $\mathbb{K}$ is complete, there exists $\mathfrak{X}^* \in \mathbb{K}$ s.t

$$\mathfrak{X}_n \rightarrow \mathfrak{X}^*$$

Taking limits in $\mathfrak{X}_{n+1} = \bar{T}\mathfrak{X}_n$, we get

$$\mathfrak{X}^* = \lim \mathfrak{X}_{n+1} = \lim \bar{T}\mathfrak{X}_n = \bar{T}(\lim \mathfrak{X}_n) = \bar{T}\mathfrak{X}^*.$$

Hence $\mathfrak{X}^*$ is a FP of $\bar{T}$.

If $\mathfrak{J}^*$ is another FP, $\mathfrak{J}^* \neq \mathfrak{X}^*$, then using the quasi-contraction condition:

$$\hat{d}(\mathfrak{X}^*, \mathfrak{J}^*) = \hat{d}(\bar{T}\mathfrak{X}^*, \bar{T}\mathfrak{J}^*) \leq \kappa \max\{\hat{d}(\mathfrak{X}^*, \mathfrak{J}^*), 0, 0, \hat{d}(\mathfrak{X}^*, \mathfrak{J}^*), \hat{d}(\mathfrak{J}^*, \mathfrak{X}^*)\} = \kappa \hat{d}(\mathfrak{X}^*, \mathfrak{J}^*),$$

which implies

$$(1 - \kappa)\hat{d}(\mathfrak{X}^*, \mathfrak{J}^*) \leq 0 \Rightarrow \hat{d}(\mathfrak{X}^*, \mathfrak{J}^*) = 0 \Rightarrow \mathfrak{X}^* = \mathfrak{J}^*.$$

Hence, the FP is unique.

2.15 Definition (Bianchini Contraction) [3]

Let $(\mathbb{K}, \hat{d})$ be a MS. A mapping $\bar{T}: \mathbb{K} \rightarrow \mathbb{K}$ is called a Bianchini contraction if there exists $\beta \in [0, 1)$ s.t for all $\mathfrak{X}, \mathfrak{J} \in \mathbb{K}$,

$$\hat{d}(\bar{T}\aleph, \bar{T}\beth) \leq \beta \max\{\hat{d}(\aleph, \bar{T}\aleph), \hat{d}(\beth, \bar{T}\beth)\}.$$

2.15.1 Theorem [3]

Let $(K, \hat{d})$ be complete MS, and $\bar{T}$ a Bianchini contraction. Then $\bar{T}$ has a unique FP $\aleph^* \in K$.

Proof

Let $\aleph_0 \in K$ and define the sequence $\{\aleph_n\}$ by

$$\aleph_{n+1} = \bar{T}\aleph_n, n \geq 0$$

Distance between consecutive terms

$$\hat{d}(\aleph_{n+1}, \aleph_n) = \hat{d}(\bar{T}\aleph_n, \bar{T}\aleph_{n-1}) \leq \beta \max\{\hat{d}(\aleph_n, \aleph_{n+1}), \hat{d}(\aleph_{n-1}, \aleph_n)\}.$$

Two cases

$$1: \hat{d}(\aleph_n, \aleph_{n+1}) \leq \beta \hat{d}(\aleph_{n-1}, \aleph_n)$$

$$2: \hat{d}(\aleph_n, \aleph_{n+1}) \leq \beta \hat{d}(\aleph_n, \aleph_{n+1}) \text{ a contradiction. Hence Case 1 holds for all } n,$$

$$\hat{d}(\aleph_{n+1}, \aleph_n) \leq \beta \hat{d}(\aleph_n, \aleph_{n-1}) \leq \dots \leq \beta^n \hat{d}(\aleph_1, \aleph_0).$$

For $m > n$, using the triangular inequality

$$\hat{d}(\aleph_n, \aleph_m) \leq \sum_{i=n}^{m-1} \hat{d}(\aleph_{i+1}, \aleph_i) \leq (\beta^n + \dots + \beta^{m-1}) \hat{d}(\aleph_1, \aleph_0) \leq \frac{\beta^n}{1-\beta} \hat{d}(\aleph_1, \aleph_0) \rightarrow 0 \text{ as}$$

$n \rightarrow \infty$

So $\{\aleph_n\}$ is Cauchy $\rightarrow$ convergent to $\aleph^*$. Taking limits in $\aleph_{n+1} = \bar{T}\aleph_n$ gives $\aleph^* = \bar{T}\aleph^*$.

If $\beth^* \neq \aleph^*$ is another FP,

$$\hat{d}(\aleph^*, \beth^*) = \hat{d}(\bar{T}\aleph^*, \bar{T}\beth^*) \leq \beta \max\{0, 0\} = 0 \Rightarrow \aleph^* = \beth^*.$$

2.16 Definition (Reich Contraction) [3]

A mapping $\bar{T}: K \rightarrow K$ is a Reich contraction if there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$ s.t for all $\aleph, \beth \in K$

$$\hat{d}(\bar{T}\aleph, \bar{T}\beth) \leq \alpha \hat{d}(\aleph, \bar{T}\aleph) + \beta \hat{d}(\beth, \bar{T}\beth) + \gamma \hat{d}(\aleph, \beth).$$

2.16.1 Theorem [3]

Let $(K, \hat{d})$ be complete, and $\bar{T}$ a Reich contraction. Then $\bar{T}$ has a UFP.

Proof

Define $\aleph_{n+1} = \bar{T}\aleph_n$. Applying the Reich condition,

$$\hat{d}(\aleph_{n+1}, \aleph_n) \leq \alpha \hat{d}(\aleph_n, \aleph_{n+1}) + (\beta + \gamma) \hat{d}(\aleph_n, \aleph_{n-1}) \Rightarrow \hat{d}(\aleph_{n+1}, \aleph_n) \leq \eta \hat{d}(\aleph_n, \aleph_{n-1}),$$

where $\eta = (\beta + \gamma)/(1 - \alpha) < 1$.

Inductively

$$\hat{d}(\aleph_{n+1}, \aleph_n) \leq \eta^n \hat{d}(\aleph_1, \aleph_0).$$

For $m > n$

$$\hat{d}(\aleph_m, \aleph_n) \leq \sum_{k=n}^{m-1} \hat{d}(\aleph_{k+1}, \aleph_k) \leq \frac{\eta^n}{1 - \eta} \hat{d}(\aleph_1, \aleph_0) \rightarrow 0.$$

So $\{\aleph_n\}$ is Cauchy $\rightarrow$ convergent to $\aleph^*$. Taking limits gives $\bar{T}\aleph^* = \aleph^*$. Uniqueness follows similarly.

2.17 Definition (Sehgal Contraction) [3]

$\bar{T}: K \rightarrow K$ is a Sehgal contraction if for all $\aleph, \beth \in K, \aleph \neq \beth$

$$\hat{d}(\bar{T}\aleph, \bar{T}\beth) < \max\{\hat{d}(\aleph, \bar{T}\aleph), \hat{d}(\beth, \bar{T}\beth), \hat{d}(\aleph, \beth)\}.$$

2.17.1 Theorem [3]

Let $(K, \hat{d})$ be complete, and $\bar{T}$ a Sehgal contraction. Then $\bar{T}$ has a UFP.

Proof.

Define $\aleph_{n+1} = \bar{T}\aleph_n$. If $\aleph_n = \aleph_{n+1}$ for some n , we are done. Otherwise, the sequence

$\hat{d}(\aleph_n, \aleph_{n+1})$ is strictly decreasing $\rightarrow$ converges to 0. Hence $\{\aleph_n\}$ is Cauchy $\rightarrow$ convergent to $\aleph^*$. Taking limits in $\aleph_{n+1} = \bar{T}\aleph_n$ gives $\aleph^* = \bar{T}\aleph^*$.

Taking $n \rightarrow \infty$, we get

$$\check{d}(\aleph^*, \bar{T}(\aleph^*)) \leq \max\{0, \check{d}(\aleph^*, \bar{T}(\aleph^*)), 0\}.$$

This forces

$$\check{d}(\aleph^*, \bar{T}(\aleph^*)) = 0 \implies \bar{T}(\aleph^*) = \aleph^*.$$

So $\aleph^*$ is a FP of $\bar{T}$. Assume $\beth^*$ is another FP, $\beth^* \neq \aleph^*$. Applying the Sehgal condition to $\aleph = \aleph^*, \beth = \beth^*$, we have

$$\check{d}(\bar{T}(\aleph^*), \bar{T}(\beth^*)) < \max\{\check{d}(\aleph^*, \bar{T}(\aleph^*)), \check{d}(\beth^*, \bar{T}(\beth^*)), \check{d}(\aleph^*, \beth^*)\}.$$

But $\bar{T}(\aleph) = \aleph$ and $\bar{T}(\beth) = \beth$, so

$$\check{d}(\aleph^*, \beth^*) < \max\{0, 0, \check{d}(\aleph^*, \beth^*)\} = \check{d}(\aleph^*, \beth^*),$$

which is a contradiction. Hence, the FP $\aleph^*$ is unique.

2.18 Definition (Rakotch Contraction) [3]

Let $(K, \check{d})$ be a MS. A mapping $\bar{T}: K \rightarrow K$ is called a Rakotch contraction if there exists a monotone non increasing function

$$\alpha: (0, \infty) \rightarrow [0, 1), (\alpha(t) < 1 \text{ for all } t > 0)$$

such that for every pair of distinct points $\aleph, \beth \in K$,

$$\check{d}(\bar{T}\aleph, \bar{T}\beth) < \alpha(\check{d}(\aleph, \beth)).$$

2.18.1 Theorem [3]

Let $(K, \check{d})$ be complete and $\bar{T}$ a Rakotch contraction. Then $\bar{T}$ has a unique FP $\aleph^* \in K$.

Proof

Pick arbitrary $\aleph_0 \in K$ and define

$$\aleph_{n+1} = \bar{T}(\aleph_n), n \geq 0.$$

If for some n , $\aleph_n = \aleph_{n+1}$, we are done. Otherwise, set

$$\delta_n = \check{d}(\mathfrak{K}_n, \mathfrak{K}_{n+1}), n > 0.$$

Applying the Rakotch condition to $\mathfrak{K}_n, \mathfrak{K}_{n+1}$ gives

$$\delta_{n+1} = \check{d}(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2}) = \check{d}(\bar{T}(\mathfrak{K}_n), \bar{T}(\mathfrak{K}_{n+1})) < \alpha(\delta_n).$$

Since $0 \leq \alpha(\delta_n) < 1$, the sequence (δ_n) is monotone nonincreasing and has a limit

$$\mathfrak{K} = \lim_{n \rightarrow \infty} \delta_n \geq 0.$$

Assume $\mathfrak{K} > 0$. Taking limits,

$$\mathfrak{K} \leq \alpha(\mathfrak{K})\mathfrak{K} \Rightarrow 1 \leq \alpha(l),$$

contradicting $\alpha(\mathfrak{K}) < 1$. Hence $\mathfrak{K} = 0$, i.e., $\delta_n \rightarrow 0$.

Choose $\varepsilon > 0$. Since $\delta_n \rightarrow 0$, there exists N such that $\delta_N < \varepsilon$. Monotonicity of α implies for $k \geq N$,

$$\alpha(\delta_k) \leq \alpha(\delta_N) \leq \alpha(\varepsilon) = q < 1.$$

By induction

$$\delta_{N+m} \leq q^m \delta_N$$

For $m > n \geq N$,

$$\check{d}(\mathfrak{K}_n, \mathfrak{K}_m) \leq \sum_{k=n}^{m-1} \delta_k \leq \delta_n \sum_{j=0}^{\infty} q^j = \frac{\delta_n}{1-q} \rightarrow 0.$$

Hence $(\mathfrak{K}_n)$ is Cauchy $\rightarrow$ convergent to $\mathfrak{K}^*$. Taking limits in $\mathfrak{K}_{n+1} = \bar{T}(\mathfrak{K}_n)$ gives $\bar{T}(\mathfrak{K}^*) = \mathfrak{K}^*$.

Suppose $\mathfrak{K}^* \neq \mathfrak{K}$ is another FP. Apply Rakotch,

$$\check{d}(\bar{T}(\mathfrak{K}^*), \bar{T}(\mathfrak{K})) < \alpha(\check{d}(\mathfrak{K}^*, \mathfrak{K})).$$

Left side = $\check{d}(\aleph^*, \beth^*)$. Dividing by $\check{d}(\aleph^*, \beth^*) > 0$ gives $1 < \alpha \left(\check{d}(\aleph^*, \beth^*) \right)$, contradiction.

Hence the FP is unique.

2.19 Definition (Kannan Contraction) [3]

Let $(K, \check{d})$ be a MS. A mapping $\bar{T}: K \rightarrow K$ is a Kannan contraction if there exists a constant $a \in (0, 1/2)$ s.t for all $\aleph, \beth \in K$,

$$\check{d}(\bar{T}\aleph, \bar{T}\beth) \leq a[\check{d}(\aleph, \bar{T}\aleph) + \check{d}(\beth, \bar{T}\beth)].$$

2.19.1 Theorem [3]

Let (K, d) be complete and $\bar{T}$ a Kannan contraction with $a \in (0, 1/2)$. Then $\bar{T}$ has a FP $\aleph^* \in K$.

Proof

Choose $\aleph_0 \in K$ and define $\aleph_{n+1} = \bar{T}\aleph_n$. Let $s_n = \check{d}(\aleph_n, \aleph_{n+1})$. Applying the Kannan inequality

$$s_{n+1} = \check{d}(\aleph_{n+1}, \aleph_{n+2}) = \check{d}(\bar{T}\aleph_n, \bar{T}\aleph_{n+1}) \leq a(s_n + s_{n+1}) \Rightarrow s_{n+1} \leq \frac{a}{1-a} s_n.$$

Set $r = \frac{a}{1-a} \in (0, 1)$. Then $s_n \leq r^n s_0$. For $m \geq 1$,

$$\check{d}(\aleph_n, \aleph_{n+m}) \leq \sum_{k=0}^{m-1} s_{n+k} \leq s_0 \frac{r^n}{1-r} \rightarrow 0.$$

Thus $(\aleph_n)$ is Cauchy $\rightarrow$ convergent to $\aleph^*$. Applying Kannan condition to $\aleph^*, \aleph_n$ and taking limits,

$$\check{d}(\bar{T}\aleph^*, \aleph^*) \leq a\check{d}(\bar{T}\aleph^*, \aleph^*) \Rightarrow \check{d}(\bar{T}\aleph^*, \aleph^*) = 0 \Rightarrow \bar{T}\aleph^* = \aleph^*.$$

If $\aleph_1^*, \aleph_2^*$ are fixed points,

$$\check{d}(\aleph_1^*, \aleph_2^*) \leq a(0 + 0) = 0 \Rightarrow \aleph_1^* = \aleph_2^*.$$

Thus, the FP is unique

Conclusion

In this chapter, we have reviewed significant contributions concerning best FP results in TSs and the extensions of the BC principle. The Banach contraction principle, being fundamental in FP theory, has inspired numerous generalizations by relaxing classical requirements such as symmetry, convexity, and reliance on metric structures. Best proximity point theory has emerged as an important parallel to FP theory, addressing cases where FPs may not exist and offering approximate solutions, particularly in the study of non-self mappings.

The survey of existing research shows a clear evolution from MS frameworks toward broader topological and distance-like settings. These developments highlight the relevance and applicability of FP results in mathematical analysis, optimization, and other applied areas. This literature review sets the stage for the forthcoming chapters, where new generalizations of fixed point theorems will be established within topological structures using distance-like functions.

Chapter 3

Distance-Like Functions in Topological spaces with Applications in Fixed Point Theory

About our Distance-like functions. Why we use it?

Because, previously it was not fully utilized or explored.

3.1 Example

Let K be a TS and $g: K \times K \rightarrow \mathbb{R}$ be a CF. Define

$$|g(\aleph, \beth)| = \aleph^2 + 3\beth^3, \forall \aleph, \beth \in K.$$

Then

$$|g(1,2)| = 1 + 24 = 25, |g(2,1)| = 4 + 3 = 7, |g(1,2)| \neq |g(2,1)|.$$

Example 2

Let K be a TS and $g: K \times K \rightarrow \mathbb{R}$ be a CF. Define

$$|g(\aleph, \beth)| = 2\aleph + \beth, \forall \aleph, \beth \in K.$$

Then:

$$|g(1,2)| = 2 + 2 = 4, |g(2,1)| = 4 + 1 = 5, |g(1,2)| \neq |g(2,1)|.$$

These examples show that distance-like functions may not be symmetric.

3.2 Left Open Ball Using Distance-Like Function

Let K be a non-empty set with a distance-like function $g: K \times K \rightarrow [0, \infty)$. For $\aleph \in K$ and $r > 0$, the left open ball(OB) centered at $\aleph$ with radius r is defined as

$$B_L(\aleph, r) = \{\beth \in K: |g(\aleph, \beth)| < r\}.$$

3.3 Right Open Ball Using Distance-Like Function

Similarly, the right open ball centered at $\aleph$ with radius r is

$$B_R(r, \aleph) = \{\beth \in \aleph: |g(\beth, \aleph)| < r\}.$$

3.4 Left Open Set Using Distance-Like Function

A subset $U_L \subseteq \aleph$ is left open if, for every point $\aleph \in U_L$, there exists $r > 0$ s.t

$$B_L(\aleph, r) \subseteq U_L.$$

3.5 Right Open Set Using Distance-Like Function

A subset $U_R \subseteq \aleph$ is right open if, for every $\aleph \in U_R$, there exists $r > 0$ s.t

$$B_R(r, \aleph) \subseteq U_R.$$

3.6 Left g -Convergence

A sequence $\{\aleph_n\} \subset \aleph$ is left g -convergent to $\aleph \in \aleph$ if

$$|g(\aleph_n, \aleph)| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

i.e., for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ s.t

$$|g(\aleph_n, \aleph)| < \epsilon, \forall n \geq k.$$

3.7 Right g -Convergence

A sequence $\{\aleph_n\} \subset \aleph$ is right g -convergent to $\aleph \in \aleph$ if

$$|g(\aleph, \aleph_n)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

i.e., for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$|g(\aleph, \aleph_n)| < \epsilon, \forall n \geq k.$$

3.8 Left g -Cauchy Sequence

A sequence $\{\aleph_n\}$ is left g -Cauchy if

$$|g(\mathfrak{X}_n, \mathfrak{X}_m)| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

i.e., for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ s.t

$$|g(\mathfrak{X}_n, \mathfrak{X}_m)| < \epsilon, \forall n \geq k.$$

3.9 Right g -Cauchy Sequence

A sequence $\{\mathfrak{X}_n\}$ is right g -Cauchy if

$$|g(\mathfrak{X}_m, \mathfrak{X}_n)| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

i.e., for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ s.t

$$|g(\mathfrak{X}_m, \mathfrak{X}_n)| < \epsilon, \forall n \geq k.$$

3.10 Asymptotic Triangular Inequality

For sequences $\{\mathfrak{X}_n\}, \{\mathfrak{Z}_n\}, \{\mathfrak{E}_n\} \subset \mathbb{K}$,

$$|g(\mathfrak{X}_n, \mathfrak{Z}_n)| \rightarrow 0 \text{ and } |g(\mathfrak{Z}_n, \mathfrak{E}_n)| \rightarrow 0 \Rightarrow |g(\mathfrak{X}_n, \mathfrak{E}_n)| \rightarrow 0.$$

3.11 Theorem (Distance-Like Functions Form a Left T_1 Space)

Let $\mathbb{K}$ be non-empty with a distance-like function g (without assuming symmetry or triangle inequality). Let

$$B_L(\mathfrak{X}; r) = \{\mathfrak{Z} \in X : |g(\mathfrak{X}, \mathfrak{Z})| < r\}.$$

Define τ as the collection of all such open sets. Then $(\mathbb{K}, \tau)$ is a left T_1 space.

Proof:

U is called open set if for each $\mathfrak{X} \in U$ their exist $B(\mathfrak{X}; r) \subseteq U$. Also we consider τ is collection of open set. $\emptyset \in \tau$ trivially. For some $\mathfrak{X} \in X$, then for $r > 0$, $B(\mathfrak{X}; r) \subseteq X$

So $X \in \tau$.

Let U_α be collection of open sets. To prove UU_α is open set.

Let $\mathfrak{X} \in UU_\alpha \Rightarrow \mathfrak{X} \in U'_\alpha$ for some α'

$$\Rightarrow \text{there exist } B(\aleph, r) \subseteq U'_\alpha \subseteq UU'_\alpha$$

$$\Rightarrow UU_\alpha \in \tau$$

Let $U_1, U_2 \in \tau$, we have to prove that $U_1 \cap U_2 \in \tau$. Let $l \in U_1 \cap U_2$ such that $\aleph \in U_1$, and $\aleph \in U_2$. As U_1 is open, so $\aleph \in B(\aleph^*, r_1) \subseteq U_1$. Also U_2 is open, so $\aleph \in B(\aleph, r_2) \subseteq U_2$.

$$r_3 = \min\{r_1, r_2\}$$

$$\aleph \in B(\aleph, r_3) \subseteq U_1 \cap U_2$$

So finite intersection belong to τ . Thus collection of open ball with no symmetric property and no triangular inequality makes left topology. Now we want to prove this a left T_1 space. As we know, a topological space is T_1 space iff each singleton subset is closed. Let $F \subseteq X$, $F = \{l\}$ and $F^c = X - F$ or $X - \{l\}$.

If $\beth \in F^c$, $\beth \notin F$ ($\beth \neq \aleph$) implies that

$$\Rightarrow B(\beth, r) \subseteq F^c$$

$$\Rightarrow B(\beth, r) \cap F = \emptyset$$

$$\Rightarrow B(\beth, r) \subseteq F^c$$

So, $B(\beth, r)$ is open.

Therefore, $F^c = X - F$ or $X - \{\aleph\}$ is open. Hence X is a left T_1 space.

Similarly, we can make right T_1 space.

3.12 Distance-Like Quasi Contraction

A mapping $\bar{T}: \mathcal{K} \rightarrow \mathcal{K}$ is a Distance-like Quasi contraction if there exists a constant $k \in (0,1)$ such that, for all $\aleph, \beth \in \mathcal{K}$,

$$|g(\bar{T}\aleph, \bar{T}\beth)| \leq k \max\{|g(\aleph, \beth)|, |g(\aleph, \bar{T}\aleph)|, |g(\beth, \bar{T}(\beth))|, |g(\aleph, \bar{T}(\beth))|, |g(\beth, \bar{T}(\aleph))|\}.$$

3.12.1 Theorem

Let $\mathcal{K}$ be a non-empty set with a Distance-like function $g: \mathcal{K} \times \mathcal{K} \rightarrow [0, \infty)$ satisfying all the conditions of Distance-like Quasi contraction, and suppose that g satisfies the asymptotic triangular inequality. Then $\bar{T}$ has a unique FP in $\mathcal{K}$.

Proof

Choose any $\mathfrak{K}_0 \in \mathbb{K}$, and define the sequence $\{\mathfrak{K}_n\}$ by

$$\mathfrak{K}_{n+1} = \bar{T}(\mathfrak{K}_n), \forall n \in \mathbb{N}.$$

We want to show that

$$|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| \rightarrow 0.$$

Applying the Distance-like Quasi contraction to $\mathfrak{K}_{n-1}$ and $\mathfrak{K}_n$, we obtain

$$\begin{aligned} |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| &= |g(\bar{T}(\mathfrak{K}_{n-1}), \bar{T}(\mathfrak{K}_n))| \\ &\leq k \max\{|g(\mathfrak{K}_{n-1}, \mathfrak{K}_n)|, |g(\mathfrak{K}_{n-1}, \bar{T}(\mathfrak{K}_{n-1}))|, |g(\mathfrak{K}_n, \bar{T}(\mathfrak{K}_n))|, \\ &\quad |g(\mathfrak{K}_{n-1}, \bar{T}(\mathfrak{K}_n))|, |g(\mathfrak{K}_n, \bar{T}(\mathfrak{K}_{n-1}))|\}. \end{aligned}$$

Note that

$$|g(\mathfrak{K}_n, \bar{T}(\mathfrak{K}_n))| = |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|, |g(\mathfrak{K}_{n-1}, \bar{T}(\mathfrak{K}_{n-1}))| = |g(\mathfrak{K}_{n-1}, \mathfrak{K}_n)|.$$

Let

$$d_n = |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|.$$

Then

$$d_n \leq k \max\{d_{n-1}, d_n, |g(\mathfrak{K}_{n-1}, \mathfrak{K}_{n+1})|\}$$

Let

$$M_n = \max\{d_{n-1}, d_n, |g(\mathfrak{K}_{n-1}, \mathfrak{K}_{n+1})|\}$$

so

$$d_n \leq kM_n$$

Assume

$$\limsup d_n = \delta > 0$$

Then infinitely often $d_n \geq \delta - \varepsilon$ for some $\varepsilon > 0$, which implies

$$\delta \leq k\delta \Rightarrow (1 - k)\delta \leq 0,$$

a contradiction. Hence

$$d_n = |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| \rightarrow 0.$$

We now show that $\{\mathfrak{K}_n\}$ is a g -left CS, i.e.

$$|g(\mathfrak{K}_n, \mathfrak{K}_m)| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Fix $\varepsilon > 0$. Since

$$|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| \rightarrow 0,$$

there exists $N \in \mathbb{N}$ s.t

$$|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| < \frac{\varepsilon}{3} \quad \forall n \geq N.$$

Let $m > n \geq N$. Using intermediate points $\mathfrak{K}_n = \mathfrak{K}_{n+1}$, we have

$$|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| < \varepsilon/3, |g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})| < \varepsilon/3, \text{ etc.}$$

Applying the asymptotic triangular inequality repeatedly, we obtain

$$|g(\mathfrak{K}_n, \mathfrak{K}_m)| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Thus $\{\mathfrak{K}_n\}$ is a g -left CS.

Assume (K, g) is sequentially complete. Then there exists $\mathfrak{K}^* \in K$ s.t

$$\mathfrak{K}_n \rightarrow \mathfrak{K}^*.$$

Since

$$\mathfrak{K}_{n+1} = \bar{T}(\mathfrak{K}_n) \rightarrow \bar{T}(\mathfrak{K}^*)$$

by sequential continuity of $\bar{T}$, we get

$\mathfrak{K}^* = \bar{T}(\mathfrak{K}^*)$. so $\mathfrak{K}^*$ is a FP. Let $\mathfrak{K}^*, \mathfrak{J}^* \in K$ be two FPs. Then

$$\begin{aligned}
|g(\aleph^*, \beth^*)| &= |g(\bar{T}(\aleph^*), \bar{T}(\beth^*))| \\
&\leq k \max\{|g(\aleph^*, \beth^*)|, 0, 0, |g(\aleph^*, \beth^*)|, |g(\aleph^*, \beth^*)|\} \\
&= k |g(\aleph^*, \beth^*)|.
\end{aligned}$$

Hence

$$(1 - k)|g(\aleph^*, \beth^*)| \leq 0 \Rightarrow |g(\aleph^*, \beth^*)| = 0 \Rightarrow \aleph^* = \beth^*.$$

Therefore, $\bar{T}$ has a unique FP in K .

3.13 Definition

Let K be a non empty set and let $g: K \times K \rightarrow [0, \infty)$ be a Distance-like function satisfying,

$$\begin{aligned}
|g(\aleph, \aleph)| &= 0 \quad \forall \aleph \in K, \\
|g(\aleph, \beth)| &= 0 \Rightarrow \aleph = \beth \quad \text{for all } \aleph, \beth \in K.
\end{aligned}$$

A mapping $\bar{T}: K \rightarrow K$ is called a **Distance-like Bianchini contraction** if there exist a constant $\lambda \in [0, 1)$ s.t, for all $\aleph, \beth \in K$,

$$|g(\bar{T}\aleph, \bar{T}\beth)| \leq \lambda \cdot \max\{|g(\aleph, \bar{T}\aleph)|, |g(\beth, \bar{T}\beth)|\}.$$

3.13.1 Theorem

Let K be a non empty set with a Distance-like function $g: K \times K \rightarrow [0, \infty)$ satisfying all the conditions of Distance-like Bianchini contraction and suppose g satisfies the asymptotic triangular inequality. Then $\bar{T}$ has a unique FP in K .

Proof

Let $\aleph_0 \in K$ be arbitrary. Define a sequence $\{\aleph_n\}$ by $\aleph_{n+1} = \bar{T}(\aleph_n)$, for all $n \in \mathbb{N}$. Apply the contraction to consecutive terms. But

$$\begin{aligned}
|g(\aleph_n, \bar{T}(\aleph_n))| &= |g(\aleph_n, \aleph_{n+1})| \\
|g(\aleph_{n+1}, \bar{T}(\aleph_{n+1}))| &= |g(\aleph_{n+1}, \aleph_{n+2})|
\end{aligned}$$

Thus

$$|g(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2})| = |g(T(\mathfrak{x}_n), T(\mathfrak{x}_{n+1}))| \leq \lambda \cdot \max\{|g(\mathfrak{x}_n, \mathfrak{x}_{n+1})|, |g(\mathfrak{x}_{n+1}, \mathfrak{x}_{n+2})|\}.$$

Let

$$a_n = |g(\mathfrak{x}_n, \mathfrak{x}_{n+1})|.$$

Then

$$a_{n+1} \leq \lambda \max\{a_n, a_{n+1}\}.$$

We analyze this inequality. Suppose $a_{n+1} \geq a_n$, then

$$a_{n+1} \leq \lambda a_{n+1} \Rightarrow (1 - \lambda)a_{n+1} \leq 0 \Rightarrow a_{n+1} = 0.$$

Otherwise, if $a_{n+1} < a_n$, then

$$a_{n+1} \leq \lambda a_n.$$

Thus, in both cases

$$a_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $\varepsilon > 0$. Since $a_n \rightarrow 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$|g(\mathfrak{x}_n, \mathfrak{x}_{n+1})| < \varepsilon/2$. Now define sequences

$$\mathfrak{x}_n, \mathfrak{x}_{n+1}, \mathfrak{x}_{n+2}, \dots, \mathfrak{x}_{n+k}.$$

We know $|g(\mathfrak{x}_n, \mathfrak{x}_{n+1})| \rightarrow 0$ for all n , so applying the asymptotic triangular inequality repeatedly,

$$|g(\mathfrak{x}_n, \mathfrak{x}_{n+k})| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Similarly,

$$|g(\mathfrak{x}_n, \mathfrak{x}_m)| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Thus, $\{\mathfrak{x}_n\}$ is a g -left CS. We will show $\mathfrak{x}^* = \bar{T}(\mathfrak{x}^*)$. Let $\mathfrak{x}_n \rightarrow \mathfrak{x}^*$, and

$$\mathfrak{x}_{n+1} = \bar{T}(\mathfrak{x}_n) \rightarrow \bar{T}(\mathfrak{x}^*)$$

Then by asymptotic triangular inequality,

$$|g(\bar{T}(\aleph^*), \aleph^*)| = \lim_{n \rightarrow \infty} |g(\bar{T}(\aleph_n), \aleph_n)| = \lim_{n \rightarrow \infty} |g(\aleph_{n+1}, \aleph_n)| = 0.$$

Hence, $|g(\bar{T}(\aleph^*), \aleph^*)| = 0 \Rightarrow \bar{T}(\aleph^*) = \aleph^*$. Let there be another FP $\beth^* \neq \aleph^*$. Then

$$|g(\bar{T}(\aleph^*), \bar{T}(\beth^*))| \leq \lambda \max\{|g(\aleph^*, \bar{T}(\aleph^*))|, |g(\beth^*, \bar{T}(\beth^*))|\} = \lambda \max\{0, 0\} = 0.$$

So

$$|g(\aleph^*, \beth^*)| = 0 \Rightarrow \aleph^* = \beth^*.$$

Thus mapping T has a unique FP.

3.14 Distance-like Reich contraction

Let K be a non empty set and let $g: K \times K \rightarrow [0, \infty)$ be a Distance-like function satisfying

$$\begin{aligned} |g(\aleph, \aleph)| &= 0, \forall \aleph \in K \\ |g(\aleph, \beth)| &= 0 \Rightarrow \aleph = \beth \end{aligned}$$

A mapping $\bar{T}: K \rightarrow K$ is called a Distance-like Reich contraction if there exist constants $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$, s.t for each $\aleph, \beth \in K$,

$$|g(\bar{T}\aleph, \bar{T}\beth)| \leq \alpha |g(\aleph, \bar{T}\aleph)| + \beta |g(\beth, \bar{T}(\beth))| + \gamma |g(\aleph, \beth)|.$$

3.14.1 Theorem

Let K be a non empty set and $g: K \times K \rightarrow [0, \infty)$ be a Distance-like function satisfying all the conditions of Distance-like Reich contraction, also g satisfies the asymptotic triangular inequality. Then $\bar{T}$ has a unique FP in K .

3.14.2 Example

Let $K = [1, \infty)$, and define a Distance-like function $g: K \times K \rightarrow [0, \infty)$ by

$$g(\aleph, f) = \begin{cases} \ln(\beth/\aleph), & \text{if } \beth \geq \aleph, \\ 2\ln(\aleph/\beth), & \text{if } \beth < \aleph. \end{cases}$$

Define a self-mapping $\bar{T}: K \rightarrow K$ by

$$\bar{T}(\aleph) = \sqrt{\aleph}.$$

Let $\aleph_0 \in \aleph$ be arbitrary and construct the sequence

$$\aleph_{n+1} = \bar{T}(\aleph_n) = \sqrt{\aleph_n}.$$

Then

$$\aleph_n = \aleph_0^{1/2^n} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

So $\{\aleph_n\}$ is a g -left CS and

$$|g(\aleph_n, \aleph_{n+1})| = \ln(\aleph_n) \rightarrow 0.$$

Thus $\aleph_n \rightarrow 1$, and since

$$\bar{T}(1) = \sqrt{1} = 1,$$

the point 1 is the unique FP of $\bar{T}$.

3.15 Definition

Let $\aleph$ be a non empty set with a Distance-like function $g: \aleph \times \aleph \rightarrow [0, \infty)$ satisfying

$$|g(\aleph, \aleph)| = 0 \quad \forall \aleph \in \aleph$$

$$|g(\aleph, \beth)| = 0 \Rightarrow \aleph = \beth.$$

But g has no symmetric and no triangular inequality property.

A mapping $\bar{T}: \aleph \rightarrow \aleph$ satisfies **Distance-like Sehgal contraction** if there exist a constant $k \in [0, 1)$ s.t, for all $\aleph, \beth \in \aleph$,

$$|g(\bar{T}\aleph, \bar{T}\beth)| < \max\{|g(\aleph, \bar{T}\aleph)|, |g(\beth, \bar{T}(\beth))|, |g(\aleph, \beth)|\}.$$

3.15.1 Theorem

Let $\mathbb{K}$ be a non empty set with a Distance-like function $g: \mathbb{K} \times \mathbb{K} \rightarrow [0, \infty)$ satisfying the conditions of Distance-like Sehgal contraction. Also, g satisfies the Asymptotic triangular inequality. Then $\bar{T}$ has a UFP in $\mathbb{K}$.

Proof

Assume $\mathfrak{K}_0 \in \mathbb{K}$ be arbitrary & define $\{\mathfrak{K}_n\}$ as $\mathfrak{K}_{n+1} = T(\mathfrak{K}_n), \forall n \in \mathbb{N}$. We want to show that $|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|$ is decreasing and converges to 0 . Using Distance-like Sehgal contraction,

$$\begin{aligned} |g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})| &= |g(\bar{T}(\mathfrak{K}_n), \mathfrak{K}(\mathfrak{K}_{n+1}))| \\ &< \max\{|g(\mathfrak{K}_n, \bar{T}(\mathfrak{K}_n))|, |g(\mathfrak{K}_{n+1}, \bar{T}(\mathfrak{K}_{n+1}))|, |g(\mathfrak{K}_n, \mathfrak{K}_n)|\} \end{aligned}$$

But $\mathfrak{K}_{n+1} = \bar{T}(\mathfrak{K}_n)$, so

$$|g(\mathfrak{K}_n, \bar{T}(\mathfrak{K}_n))| = |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|,$$

and similarly for the next terms. Hence,

$$|g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})| < \max\{|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|, |g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})|, |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|\}.$$

This implies that

$$|g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})| < |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|, \forall n$$

Thus, $\{|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|\}$ is a monotonically decreasing sequence bounded below by 0 , so it converges,

$$\lim_{n \rightarrow \infty} |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| = r \geq 0.$$

Suppose $r > 0$. Then from the contraction condition,

$$|g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})| < \max\{|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|, |g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})|, |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})|\} = r,$$

Contradicting $\lim |g(\mathfrak{K}_{n+1}, \mathfrak{K}_{n+2})| = r$. Therefore, $r = 0$, i.e.

$$\lim_{n \rightarrow \infty} |g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| = 0.$$

Let $\varepsilon > 0$. Since

$|g(\mathfrak{X}_n, \mathfrak{X}_{n+1})| \rightarrow 0$, there exists $N \in \mathbb{N}$ s.t for all $n \geq N$,

$$|g(\mathfrak{X}_n, \mathfrak{X}_{n+1})| < \varepsilon/3.$$

Now, by induction and repeated application of the Asymptotic triangular inequality,

$$|g(\mathfrak{X}_n, \mathfrak{X}_{n+1})| \rightarrow 0, |g(\mathfrak{X}_{n+1}, \mathfrak{X}_{n+2})| \rightarrow 0,$$

hence

$$|g(\mathfrak{X}_n, \mathfrak{X}_{n+2})| \rightarrow 0.$$

Similarly,

$$|g(\mathfrak{X}_n, \mathfrak{X}_m)| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Thus, $\{\mathfrak{X}_n\}$ is a g -left CS. Using only the Asymptotic triangular inequality, it converges under g , there exists $\mathfrak{X}^* \in \mathbb{K}$ s.t

$$\lim_{n \rightarrow \infty} |g(\mathfrak{X}_n, \mathfrak{X}^*)| = 0.$$

Now we want to prove $\mathfrak{X}^*$ is a FP of $\bar{T}$. From the contraction condition and limit behavior,

$$|g(\bar{T}(\mathfrak{X}_n), \bar{T}(\mathfrak{X}^*))| < \max\{|g(\mathfrak{X}_n, \bar{T}(\mathfrak{X}_n))|, |g(\mathfrak{X}^*, \bar{T}(\mathfrak{X}^*))|, |g(\mathfrak{X}_n, \mathfrak{X}^*)|\}.$$

Now,

$$\begin{aligned} |g(\mathfrak{X}_n, \bar{T}(\mathfrak{X}_n))| &= |g(\mathfrak{X}_n, \mathfrak{X}_{n+1})| \rightarrow 0, \\ |g(\mathfrak{X}_n, \mathfrak{X}^*)| &\rightarrow 0. \end{aligned}$$

So all terms in the maximum go to 0 . Therefore,

$$|g(\mathfrak{X}_{n+1}, \bar{T}(\mathfrak{X}^*))| = |g(\bar{T}(\mathfrak{X}_n), \bar{T}(\mathfrak{X}^*))| \rightarrow 0.$$

But $\mathfrak{X}_{n+1} \rightarrow \mathfrak{X}^*$, so by the Asymptotic triangular inequality,

$$|g(\aleph^*, \bar{T}(\aleph^*))| = 0 \Rightarrow \bar{T}(\aleph^*) = \aleph^*.$$

Assume there are two FPs $\aleph^* \neq \beth^*$ such that $\bar{T}(\aleph^*) = \aleph^*, \bar{T}(\beth^*) = \beth^*$. Then

$$|g(\aleph^*, \beth^*)| = |g(T(\aleph^*), T(\beth^*))| < \max\{|g(\aleph^*, T(\aleph^*))|, |g(\beth^*, T(\beth^*))|, |g(\aleph^*, \beth^*)|\}.$$

But

$$|g(\aleph^*, \bar{T}(\aleph^*))| = |g(\aleph^*, \aleph^*)| = 0,$$

and similarly for $\beth^*$. Hence,

$$|g(\aleph^*, \beth^*)| < |g(\aleph^*, \beth^*)|,$$

a contradiction unless

$$|g(\aleph^*, \beth^*)| = 0 \Rightarrow \aleph^* = \beth^*$$

Thus, mapping $\bar{T}$ has a UFP.

3.15.2 Example

Let $K = [0, \infty)$, and define a function $g: K \times K \rightarrow [0, \infty)$ by

$$g(\aleph, \beth) = \begin{cases} \beth - \aleph + 1, & \text{if } \beth > \aleph, \\ 0, & \text{if } \beth \leq \aleph. \end{cases}$$

Define a self-map $\bar{T}: K \rightarrow K$ as $\bar{T}(\aleph) = \aleph$. Also $\bar{T}$ is a Distance-like Sehgal contraction on (K, g) . Then $\bar{T}$ has a UFP in K .

Solution

$$\begin{aligned} |g(\aleph, \beth)| &\geq 0, \forall \aleph, \beth \in K, \\ |g(\aleph, \aleph)| &= 0. \end{aligned}$$

$$\begin{aligned} \text{If } |g(\aleph, \beth)| &= 0, \text{ then } \aleph = \beth. \\ |g(2, 4)| &= 3 \text{ but } |g(4, 2)| = 0. \\ |g(l, \dot{w}_n)| &\not\leq |g(l, \beth)| + |g(\beth, \dot{w}_n)|. \end{aligned}$$

Thus, g is a Distance-like function. So, Distance-like Sehgal contraction is,

$$|g(\bar{T}\aleph, \bar{T}\beth)| < \max\{|g(\aleph, \bar{T}\aleph)|, |g(\beth, \bar{T}\beth)|, |g(\aleph, \beth)|\}, \forall \aleph, \beth \in \mathbb{K}, \aleph \neq \beth.$$

Let $\aleph = 4, \beth = 9$. Then

$$\begin{aligned} \bar{T}(\aleph) &= 4, \bar{T}(\beth) = 9, \\ |g(\bar{T}\aleph, \bar{T}\beth)| &= |g(4, 9)| = 6, \\ |g(\aleph, \bar{T}\aleph)| &= |g(4, 4)| = 0, |g(\beth, \bar{T}\beth)| = |g(9, 9)| = 0, \\ |g(\aleph, \beth)| &= 6. \end{aligned}$$

So,

$$6 < \max\{0, 0, 6\},$$

which fails. To fix this, redefine $\bar{T}$ as

$$\bar{T}(\aleph) = \frac{\aleph}{2} + 1, \forall \aleph \in \mathbb{K}.$$

Now take $\aleph = 4, \beth = 9$. Then

$$\begin{aligned} \bar{T}(\aleph) &= 3, \bar{T}(\beth) = 5.5 \\ |g(\bar{T}\aleph, \bar{T}\beth)| &= 3.5 \\ |g(\aleph, \bar{T}\aleph)| &= 0, |g(\beth, \bar{T}\beth)| = 0 \\ |g(\aleph, \beth)| &= 6 \end{aligned}$$

Thus,

$$3.5 < 6$$

Hence $T(\aleph) = \aleph/2 + 1$ is a Distance-like Sehgal contraction. Let $\aleph_0 \in \mathbb{K}$ and define

$$\aleph_{n+1} = \bar{T}(\aleph_n) = \frac{\aleph_n}{2} + 1$$

Take $\aleph_0 = 16$. Then

$$\aleph_1 = 9, \aleph_2 = 5.5, \aleph_3 = 3.75, \aleph_4 = 2.875, \aleph_5 = 2.4375, \dots$$

Clearly $\{\aleph_n\}$ is decreasing and bounded below hence converges. Let $\lim \aleph_n = \aleph$. Then

$$\aleph = \frac{\aleph}{2} + 1 \Rightarrow \aleph = 2.$$

Let $\aleph^* = 2$. Then

$$\bar{T}(\aleph^*) = \frac{2}{2} + 1 = 2.$$

Hence $\aleph^* = 2$ is a FP. Suppose $\beth^* \neq 2$ is another FP. Then

$$\frac{\beth^*}{2} + 1 = \beth^* \Rightarrow \beth^* = 2$$

a contradiction. Thus there is a UFP.

3.16 Definition

Let $\mathcal{K}$ be a non-empty set equipped with a Distance-like function $g: \mathcal{K} \times \mathcal{K} \rightarrow [0, \infty)$,

and let $\bar{T}, H: \mathcal{K} \rightarrow \mathcal{K}$. A point $\aleph^* \in \mathcal{K}$ is called a common FP of the pair $(\bar{T}, H)$ if

$$\bar{T}(\aleph^*) = \aleph^* = H(\aleph^*)$$

The pair $(\bar{T}, H)$ is said to satisfy a **Distance-like Sehgal contraction** condition if there exists a constant $0 \leq k < 1$ such that, for all $\aleph, \beth \in \mathcal{K}$,

$$|g(\bar{T}\aleph, H\beth)| < \max\{|g(\aleph, \bar{T}\aleph)|, |g(\beth, H\beth)|, |g(\aleph, \beth)|\}.$$

3.16.1 Theorem

Let $\mathcal{K}$ be a non-empty set equipped with a Distance-like function $g: \mathcal{K} \times \mathcal{K} \rightarrow [0, \infty)$

satisfying all the conditions of a Distance-like Sehgal contraction. Assume further that g satisfies the Asymptotic triangular inequality. Suppose that $\bar{T}(\mathcal{K}) \subseteq H(\mathcal{K})$, and either $\bar{T}$ or H is continuous. Then $\bar{T}$ and H have a unique common FP in $\mathcal{K}$ that is, there exists $\aleph^* \in \mathcal{K}$ s.t

$$\bar{T}(\aleph^*) = H(\aleph^*) = \aleph^*.$$

Proof

Choose an arbitrary $\aleph_0 \in \aleph$. Since $\bar{T}(\aleph) \subseteq H(\aleph)$, there exists $\aleph_1 \in \aleph$ s.t

$$\bar{T}(\aleph_0) = H(\aleph_1).$$

Inductively define a sequence $\{\aleph_n\} \subset \aleph$ such that $\bar{T}(\aleph_n) = H(\aleph_{n+1}), \forall n \in \mathbb{N}$.

Denote $\beth_n = \bar{T}(\aleph_n) = H(\aleph_{n+1}), \forall n \in \mathbb{N}$. Using the Distance-like Sehgal contraction condition, we obtain

$$|g(\beth_n, \beth_{n+1})| < \max\{|g(\aleph_n, \beth_n)|, |g(\aleph_{n+1}, \beth_{n+1})|, |g(\aleph_n, \aleph_{n+1})|\}.$$

Let

$$\delta_n = |g(\beth_n, \beth_{n+1})|.$$

Then $\{\delta_n\}$ is a non-negative sequence and $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. By repeated application of the Asymptotic triangular inequality, $|g(\beth_n, \beth_m)| \rightarrow 0$ as $n, m \rightarrow \infty$. Hence $\{\beth_n\}$ is a g -left Cauchy sequence. Since either $\bar{T}$ or H is continuous and the image is closed under g -limits, there exists $\aleph^* \in \aleph$ s.t

$$\beth_n = \bar{T}(\aleph_n) \rightarrow \aleph^*.$$

$\aleph^*$ is a FP of $\bar{T}$. Since $\bar{T}$ is continuous, $\bar{T}(\beth_n) = \bar{T}(\bar{T}(\aleph_n)) \rightarrow \bar{T}(\aleph^*)$. But $\beth_{n+1} \rightarrow \aleph^*$, hence $|g(\bar{T}(\aleph^*), \aleph^*)| = 0 \Rightarrow \bar{T}(\aleph^*) = \aleph^*$. $\aleph^*$ is a FP of H . Since $\beth_n = H(\aleph_{n+1}) \rightarrow \aleph^*$ and H is continuous,

$$H(\aleph^*) = \aleph^*.$$

Thus $\bar{T}(\aleph^*) = H(\aleph^*) = \aleph^*$. Assume there exists another common FP $\beth^* \neq \aleph^*$.

Then

$$|g(\aleph^*, \beth^*)| = |g(\bar{T}(\aleph^*), \bar{T}(\beth^*))| < \max\{0, 0, |g(\aleph^*, \beth^*)|\},$$

which is a contradiction unless

$$|g(\aleph^*, \beth^*)| = 0 \Rightarrow \aleph^* = \beth^*.$$

Hence the common FP is unique.

3.16.2 Example

Let $K = [0, \infty)$ and define $g(\aleph, \beth) = \aleph + 3\beth, \forall \aleph, \beth \in K$. Clearly, $|g(\aleph, \aleph)| = 4\aleph \geq 0, |g(1,2)| = 7, |g(2,1)| = 5$, so g is a Distance-like function.

Define mappings $\bar{T}, H: K \rightarrow K$ by

$$\bar{T}(\aleph) = \frac{\aleph + 2}{3}, \hat{H}(\aleph) = \frac{2\aleph + 1}{4}.$$

For $\aleph = 1, \beth = 2$,

$$|g(\bar{T}1, \bar{T}2)| = 5 < \max\{4, 6, 7\}.$$

Hence $\bar{T}$ satisfies the Distance-like Sehgal contraction; similarly H .

$$\bar{T}(\aleph^*) = H(\aleph^*)$$

gives $\aleph^* = \frac{5}{2}$. Then $\bar{T}\left(\frac{5}{2}\right) = H\left(\frac{5}{2}\right) = \frac{3}{2}$. Let $G(\ell) = \bar{T}(\aleph) = H(\aleph)$. Solving

$$G(\aleph^*) = \aleph^*$$

Yields $\aleph^* = 1$. Hence the unique common FP is

$$\aleph^* = 1.$$

3.17 Definition

Let K be a non empty set with a Distance-like function g and let $\bar{T}, H: K \rightarrow K$.

The pair $(\bar{T}, H)$ has common fixed point for **Distance-like Rakotch contraction** if there exist a function $\alpha: [0, \infty) \rightarrow [0, 1)$ s.t α is non-decreasing, $\alpha(t) < 1$ for all $t > 0$, and the following contraction holds $|g(\bar{T}\aleph, H\beth)| \leq \alpha|g(\aleph, \beth)|, \forall \aleph, \beth \in K$ with $|g(\aleph, \beth)| > 0$.

3.17.1 Theorem

Let (K, g) be a set equipped with a Distance-like function $g: K \times K \rightarrow [0, \infty)$

satisfying the conditions of Distance-like Rakotch contraction. Also, g satisfies the Asymptotic triangular inequality. Then $\bar{T}$ has a UFP in K .

Proof

Suppose $\aleph_0 \in K$ is arbitrary. Define $\aleph_{n+1} = \bar{T}(\aleph_n)$, for $n \in \mathbb{N}$. Define the successive distances $s_n = |g(\aleph_n, \aleph_{n+1})| = |g(\aleph_n, \bar{T}(\aleph_n))|$. To show $\{s_n\}$ is decreasing and converges to 0. Using the Distance-like Rakotch contraction,

$$\begin{aligned} s_{n+1} &= |g(\aleph_{n+1}, \aleph_{n+2})| \\ &= |g(\bar{T}(\aleph_n), \bar{T}(\aleph_{n+1}))| \\ &\leq \phi(|g(\aleph_n, \aleph_{n+1})|) \\ &= \phi(s_n) \end{aligned}$$

Since $\phi(s_n) < 1$ for $s_n > 0$, and ϕ is non-decreasing, $s_{n+1} \leq \phi(s_n) < 1$. So the sequence $\{s_n\}$ is decreasing and bounded below by 0. Let $s_n \rightarrow s \geq 0$. Then taking limits, $\lim s_{n+1} = \lim \phi(s_n) = \phi(s)$. Thus $s = \phi(s)$. But $\phi(t) < t$ for all $t > 0$, so this forces $s = 0$. Therefore $|g(\aleph_n, \aleph_{n+1})| \rightarrow 0$. We have $|g(\aleph_n, \aleph_{n+2})| \rightarrow 0$.

Repeated use of the Asymptotic triangular inequality gives

$$|g(\aleph_n, \aleph_m)| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Hence, $\{\aleph_n\}$ is a g -left CS with respect to g . By assumption, (K, g) is complete under g , so there exists $\aleph^* \in K$ s.t $\aleph_n \rightarrow \aleph^*$ with respect to g . To prove $\bar{T}(\aleph^*) = \aleph^*$, note that $|g(\aleph_n, \aleph_{n+1})| = |g(\aleph_n, \bar{T}(\aleph_n))| \rightarrow 0$, $\aleph_n \rightarrow \aleph^*$. We aim to show $g(\bar{T}(\aleph^*), \aleph^*) = 0$. By the Asymptotic triangular inequality,

$$|g(\bar{T}(\aleph_n), \bar{T}(\aleph^*))| \leq \phi(|g(\aleph_n, \aleph^*)|) \rightarrow \phi(0^+).$$

Also, $|g(\aleph_{n+1}, \bar{T}(\aleph^*))| = |g(\bar{T}(\aleph_n), \bar{T}(\aleph^*))| \rightarrow 0$. By the Asymptotic triangular inequality again,

$$|g(\aleph_n, \aleph^*)| \rightarrow 0.$$

Thus, $|g(\aleph^*, \bar{T}(\aleph^*))| = 0 \Rightarrow \bar{T}(\aleph^*) = \aleph^*$. So $\aleph^*$ is a FP. Suppose there is another fixed point $\beth^* \in K$, $\beth^* \neq \aleph^*$ s.t $\bar{T}(\beth^*) = \beth^*$. Then

$$|g(\aleph^*, \beth^*)| = |g(\bar{T}(\aleph^*), \bar{T}(\beth^*))| \leq \phi(|g(\aleph^*, \beth^*)|).$$

If $|g(\aleph^*, \beth^*)| > 0$ then $\phi(|g(\aleph^*, \beth^*)|) < |g(\aleph^*, \beth^*)|$, a contradiction. Therefore,

$|g(\aleph^*, \beth^*)| = 0 \Rightarrow \aleph^* = \beth^*$. Thus, the mapping $\bar{T}$ has a UFP.

3.17.2 Example

Let $K = [0, \infty)$ and define a Distance-like function $g: K \times K \rightarrow [0, \infty)$ by

$$g(\aleph, \beth) = \begin{cases} \aleph - \beth, & \text{if } \beth \geq \aleph, \\ 2(\aleph - \beth), & \text{if } \aleph > \beth. \end{cases}$$

Then the mapping has a UFP.

Solution

Check Distance-like function properties $|g(\aleph, \aleph)| = 0, \forall \aleph \in K$.

$$|g(2, 1)| = 2(2 - 1) = 2, |g(1, 2)| = 2 - 1 = 1.$$

Consider $\aleph = 4, f = 1, z = 0$. Then

$$|g(4, 0)| = 2(4 - 0) = 8, |g(4, 1)| + |g(1, 0)| = 2(4 - 1) + 2(1 - 0) = 8.$$

Define self-maps $\hat{T}, H: K \rightarrow K$ by $\bar{T}(\aleph) = \aleph/2, H(\beth) = \beth/4$. Check the Distance-like Rakotch contraction. Let $\aleph, \beth \in K$. Then $|g(\bar{T}\aleph, \hat{H}\beth)| = g(\aleph/2, \beth/4)$.

Case 1: $\beth/4 \geq \aleph/2 \Rightarrow \beth \geq 2\aleph$

$$|g(\bar{T}\aleph, H\beth)| = \frac{\beth - 2\aleph}{4}.$$

If $\beth \geq \aleph$, then $|g(\aleph, \beth)| = \beth - \aleph$, so

$$\frac{|g(\bar{T}\aleph, H\beth)|}{|g(\aleph, \beth)|} = \frac{\beth - 2\aleph}{4(\beth - \aleph)} = \alpha |g(\aleph, \beth)| < 1 \text{ as } \beth \rightarrow \infty.$$

Case 2: $\beth/4 < \aleph/2 \Rightarrow \beth < 2\aleph$ $|g(\bar{T}\aleph, H\beth)| = 2\left(\frac{\aleph}{2} - \frac{\beth}{4}\right) = \aleph - \frac{\beth}{2}$. If $\aleph > \beth$, then

$|g(\aleph, \beth)| = 2(\aleph - \beth)$, so

$$\frac{|g(\bar{T}\aleph, H\beth)|}{|g(\aleph, \beth)|} = \frac{2\aleph - \beth}{4(\aleph - \beth)} = \alpha |g(\aleph, \beth)| < 1 \text{ as } \aleph \rightarrow \beth^+.$$

Define the contraction control function $\alpha(t) = \frac{1}{2} < 1$. Let $\aleph_n = \frac{1}{2^n}$. Then

$$|g(\aleph_{n+1}, \aleph_n)| = \frac{1}{2^{n+1}} \rightarrow 0. \text{ Hence } \{\aleph_n\} \text{ is a } g\text{-left CS. We compute } \bar{T}(0) = 0, \hat{H}(0) = 0.$$

So 0 is a common fixed point of $\bar{T}$ and H . Suppose $\beth^* \neq 0$ is another common FP. Then $\beth^*/2 = \beth^*/4 \Rightarrow \beth^* = 0$, a contradiction. Thus, 0 is the unique common FP.

3.18 Definition

Let K be a non empty set and $g: K \times K \rightarrow [0, \infty)$ be a Distance-like function satisfying

$$\begin{aligned} |g(\aleph, \aleph)| &= 0 \quad \forall \aleph \in K, \\ |g(\aleph, \beth)| &= 0 \Rightarrow \aleph = \beth. \end{aligned}$$

A map $\bar{T}: K \rightarrow K$ is a **Distance-like Kannan contraction** if there exists $k \in [0, 1/2)$ such that for all $\aleph, \beth \in K$ $|g(\bar{T}\aleph, \bar{T}\beth)| \leq k(|g(\aleph, \bar{T}\aleph)| + |g(\beth, \bar{T}\beth)|)$.

3.18.1 Theorem

If $\bar{T}: K \rightarrow K$ is a Distance-like Kannan contraction with constant $k \in [0, 1/2)$, then T has a UFP $\aleph^* \in K$.

Proof

For each $\aleph_0 \in K$, the Picard iterates $\aleph_{n+1} = \bar{T}\aleph_n$ converge to $\aleph^*$. Fix $\aleph_0 \in K$ and define

$$\aleph_{n+1} = \bar{T}\aleph_n, n \geq 0.$$

Apply the Distance-like Kannan inequality to the pair $(\aleph_{n-1}, \aleph_n)$ for $n \geq 1$:

$$|g(\aleph_n, \aleph_{n+1})| = |g(\bar{T}\aleph_{n-1}, \bar{T}\aleph_n)| \leq k(|g(\aleph_{n-1}, \aleph_n)| + |g(\aleph_n, \aleph_{n+1})|).$$

Thus,

$$(1 - k)|g(\aleph_n, \aleph_{n+1})| \leq k|g(\aleph_{n-1}, \aleph_n)|.$$

Let $\lambda = \frac{k}{1-k}$. Since $0 \leq k < 1/2$, we have $0 \leq \lambda < 1$. Hence,

$$|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| \leq \lambda |g(\mathfrak{K}_{n-1}, \mathfrak{K}_n)| \leq \lambda^n |g(\mathfrak{K}_0, \mathfrak{K}_1)|.$$

Therefore, $|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| \rightarrow 0$ as $n \rightarrow \infty$. Let $\varepsilon > 0$. Choose N s.t for all $t \geq N$,

$$|g(\mathfrak{K}_t, \mathfrak{K}_{t+1})| < \varepsilon/(2k).$$

For $m > n \geq N + 1$, applying the Kannan inequality to $(\mathfrak{K}_{n-1}, \mathfrak{K}_{m-1})$.

$$|g(\mathfrak{K}_n, \mathfrak{K}_m)| \leq k(|g(\mathfrak{K}_{n-1}, \mathfrak{K}_n)| + |g(\mathfrak{K}_{m-1}, \mathfrak{K}_m)|) < \varepsilon.$$

Hence $\{\mathfrak{K}_n\}$ is a g -left CS. By completeness, there exists $\mathfrak{K}^* \in \mathfrak{K}$ s.t

$$|g(\mathfrak{K}_n, \mathfrak{K}^*)| \rightarrow 0.$$

Now apply the Kannan inequality to $(\mathfrak{K}_n, \mathfrak{K}^*)$. $|g(\mathfrak{K}_{n+1}, \bar{T}\mathfrak{K}^*)| \leq k(|g(\mathfrak{K}_n, \mathfrak{K}_{n+1})| + |g(\mathfrak{K}^*, \bar{T}\mathfrak{K}^*)|)$. Taking lim sup as $n \rightarrow \infty$ $|g(\mathfrak{K}^*, \bar{T}\mathfrak{K}^*)| \leq k|g(\mathfrak{K}^*, \bar{T}\mathfrak{K}^*)|$. Since $0 \leq k < 1$, we obtain $|g(\mathfrak{K}^*, \bar{T}\mathfrak{K}^*)| = 0$, which implies $\bar{T}\mathfrak{K}^* = \mathfrak{K}^*$. if $\mathfrak{J}^*$ is another FP, then

$$|g(\mathfrak{K}^*, \mathfrak{J}^*)| = |g(\bar{T}\mathfrak{K}^*, \bar{T}\mathfrak{J}^*)| \leq k(0 + 0) = 0,$$

$$\text{so } \mathfrak{K}^* = \mathfrak{J}^*.$$

3.18.2 Example

Let $\mathcal{K} = [0,1]$. Define

$$g(\ell, \mathfrak{J}) = \frac{|f^2 - \mathfrak{K}^2|}{1 + \mathfrak{K}}, \mathfrak{K}, \mathfrak{J} \in [0,1],$$

And $\bar{T}(\mathfrak{K}) = \frac{\mathfrak{K}}{4}$. Clearly,

$$|g(\mathfrak{K}, \mathfrak{J})| = 0 \Rightarrow \mathfrak{K} = \mathfrak{J},$$

and g is non-symmetric. Let sequences $\{\mathfrak{K}_n\}, \{\mathfrak{J}_n\}, \{\mathfrak{E}_n\} \subset [0,1]$ satisfy

$$|g(\mathfrak{K}_n, \mathfrak{J}_n)| \rightarrow 0, |g(\mathfrak{J}_n, \mathfrak{E}_n)| \rightarrow 0.$$

Then

$$\mathfrak{E}_n^2 - \mathfrak{K}_n^2 = (\mathfrak{E}_n^2 - \mathfrak{J}_n^2) + (\mathfrak{J}_n^2 - \mathfrak{K}_n^2) \rightarrow 0,$$

which implies

$$|g(\mathfrak{K}_n, \mathfrak{E}_n)| \rightarrow 0.$$

Moreover, for $\mathfrak{K}, \mathfrak{J} \in [0,1]$.

$$|g(\bar{T}\mathfrak{K}, \bar{T}\mathfrak{J})| = \frac{|(\mathfrak{J}/4)^2 - (\mathfrak{K}/4)^2|}{1 + \mathfrak{K}/4} \leq \frac{1}{16}(\mathfrak{K}^2 + \mathfrak{J}^2).$$

Also,

$$|g(\mathfrak{K}, \bar{T}\mathfrak{K})| + |g(\mathfrak{J}, \bar{T}\mathfrak{J})| \geq \frac{15}{32}(\mathfrak{K}^2 + \mathfrak{J}^2).$$

Thus, choosing $\alpha = 0.18 < 1/2$,

$$|g(T\mathfrak{K}, T\mathfrak{J})| \leq \alpha(|g(\mathfrak{K}, T\mathfrak{K})| + |g(\mathfrak{J}, T\mathfrak{J})|).$$

Hence $\bar{T}$ is a Distance-like Kannan contraction. Solving $\bar{T}\mathfrak{K} = \mathfrak{K}$ gives $\mathfrak{K} = 0$. $\mathfrak{K}^* = 0$ is the UFP.

Conclusion

This chapter has been devoted to the study of FP theorems in TS defined by distance-like functions. Unlike the classical setting of metric spaces, where symmetry and the triangle inequality are indispensable, distance-like functions provide a more general environment that retains the essential properties required for FP analysis while relaxing unnecessary restrictions. Within this framework, we successfully extended several well-known contraction principles—such as those of Bianchini, Kannan, Shegal, Rakotch and Reich—to spaces where neither symmetry nor the usual triangle inequality are assumed. Our arguments were constructed through iterative techniques supported by the asymptotic triangular inequality in its sequential form, which allowed us to establish the convergence of sequences toward unique fixed points. By adopting weaker assumptions, we demonstrated that many classical fixed point results remain valid in broader topological structures. Moreover, common FP theorems were obtained for mappings satisfying generalized conditions, further enlarging the scope and applicability of the theory. The findings of this chapter reveal that distance-like functions offer a unifying and flexible framework for the development of fixed point results. They not only generalize existing theorems but also provide new insights that can be applied in functional analysis, nonlinear analysis, optimization, DEs, and in dynamical systems, particularly in contexts where the underlying structure is not strictly metric. In summary, Chapter 3 establishes that distance-like functions play a central role in extending fixed point theory beyond metric spaces, thereby contributing to both theoretical advancements and potential applications in various branches of mathematics.

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