Effects of Porosity on the Flow of Third Grade Nanofluid with Space Dependent Viscosity

By

SALMAN AZIZ

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan

2011

--Mathematics

2-

Effects of Porosity on the Flow of Third Grade Nanofluid with Space Dependent Viscosity

By

SALMAN AZIZ

Supervised by

Dr. RAHMAT ELLAHI

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2011

Effects of Porosity on the Flow of Third Grade Nanofluid with Space Dependent Viscosity

By

SALMAN AZIZ

A Thesis
Submitted in the Partial Fulfillment of the Requirements for the Degree of
MASTER OF SCIENCE
In
MATHEMATICS

Supervised by

Dr. RAHMAT ELLAHI

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2011

Certificate

Effects of Porosity on the Flow of Third Grade Nanofluid with Space Dependent Viscosity

By

SALMAN AZIZ

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE **MASTRER OF SCIENCE** IN **MATHEMATICS**

We accept this thesis as conforming to the required standard.

1. Dr. Irshad Ahmad Arshad
(Chairman)

2. Dr. Rahmat Ellahi
(Supervisor)

3. Prof. Dr. Muhammad Ayub
(External Examiner)

4. Dr. Ambreen Afsar
(Internal Examiner)

Department of Mathematics and Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan
2011

Dedicated to

Dr. Abdul Qadir Khan who devoted his life for the glory of Muslim world.

Acknowledgements

Praise is to Allah Almighty who created us and blessed us with intellect so that we can discover and serve his creations.

Foremost, I am always grateful to **Almighty ALLAH**, who made human being, the best creation of all the living species and made them understand to write with pen. He provided me the boldness and capability to achieve this task.

I am deeply obliged to **Dr. Rahmat Ellahi** for giving me an opportunity to work under his supervision and guidance. I will always be proud of working under the supervision of **Dr. Rahmat Ellahi** who indeed is a source of motivation and inspiration for every student. His ideology and concepts have a remarkable impact on my research contrivances. He also arranged some suitable facilities, without which my objective might not be attained.

I offer special thanks to all my friends and my class fellows, who really helped me to their best throughout my research period. I also express special thanks to my close friend **Mr. Mohsin Raza**, who helped me throughout my work, whenever I faced any difficulty relating my problem and consigned a lot of his precious time with me. I don't think that I would have been able to do it up to the caliber without his help.

Special thanks to Mr. Mohsan Hassan, Mr. Raheel Ahmed and Miss. Abeer. I express my sincere gratitude to all of them for their support and help. I thank my parents and family for their prayers and support. I am grateful to all my teachers who have trained me to face the challenges yet to come ahead.

Finally; I am honored to write about a great source of inspiration **Prof. Dr. Rasheed Ahmed (Late)**, who had lived his life as a model of generosity and humbleness, who had thrown light on various dimensions of mathematics. Whose knowledge, skills and kindness was recognized, and appreciated throughout the world, and he ruled the world of mathematics in Pakistan

Salman Aziz

Preface

The flow of non-Newtonian fluids have a variety of practical application in industry and engineering [1-4]. Nanofluid which is one of the most important branch of non-Newtonian fluid has been widely used in industry due to unique physical and chemical properties of nanometers/nanoparticles. The nano-particles are ultra fine particles in the size of nanometer. The term "nanofluid" refers to a liquid containing a suspension of metallic or non-metallic nanometer-sized solid particles and fibers (nanoparticles). Historically, Choi [5] was first to study the behavior of nanofluids. The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Massoudi et al. [6]. This phenomenon suggests the possibility of using nanofluids in a variety of engineering applications, including advanced nuclear systems (see Buongiorno et al. [7]). The general topics on heat transfer analysis in nanofluids have been surveyed by many researchers, for instance see [8-12].

Despite the above studies, nanotechnology is considered by many to be one of the significant forces that derived the next major industrial revolution of this century. It represents the most relevant technological cutting edge currently being explored. It aims the manipulating the structure of the meter at the molecular level with the goal for innovation in virtually every industry and public endeavor including biological sciences, physical sciences, electronic cooling, transportation, the environment and national security etc. Some numerical and experimental studies on nanofluid can be found in references [13, 14].

Moreover, recent several decades, flow of fluid in porous media has intensively, been studied and it has become a very productive field of research this topic has many widespread particle applications in modern industries and in many environmental issues such as, nuclear waste management, building thermal insulation spread of pollutants, geothermal power plants, grain storage, packed- bed chemical reactors, oil recovery and ceramic processing etc. In the fluid mechanics there is sufficient literature available in witch the porosity parameter is used as a major part [15-18].

Keeping in view the above importance of nanofluid and porosity, the present thesis is arranged as follows:

In chapter one, the basic definitions of fluids, and equation of motion for non-Newtonian nanofluid of third grade are given. Basic idea of HAM is also presented here.

Chapter two is devoted to study the influence of variable viscosity and viscous dissipation on non-Newtonian flow which is review of paper by Ellahi et al. [19]. This chapter concerns with the effect of constant and variable viscosity on velocity and temperature distributions for a third grade non-Newtonian fluid is studied.

Chapter three is a new contribution in the literature in which we consider flow of third grade non-Newtonian nanofluid between coaxial cylinders with constant and variable viscosity when the outer cylinder is porous. The nonlinear coupled governing equations for nanoparticles concentration are solved with the help of Homotopy Analysis Method (HAM) [20-22]. The graphical results are displayed and analyzed for different emerging parameters.

Declaration

The work described in this thesis was carried out under the supervision and direction of Dr. Rahmat Ellahi, Department of Mathematics and Statistics, International Islamic University. No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning. The thesis is my original work except where due reference and credit is given.

Signature: ____________

Salman aziz

MS (Mathematics)

Contents

Chapter 1

Basic of Fluid Mechanics

In this chapter present some basic definitions and important concepts of fluid mechanics which we will be useful in succeeding chapters.

1.1 Flow

A material that deforms continuously when different forces act upon it. If the deformation continuously increases without limit,this phenomenon is know as flow.

1.2 Fluid

Any liquid or gas that cannot sustain a shearing force when at rest and that undergoes a continuous change in shape when subjected to such as stress.

1.3 Nanoparticles

The nanoparticles are ultrafine particles in the size of nanometer order. "Nano" is a prefix denoting the minus 9th power of ten, namely one billionth. Here it means nanometer (nm) applied for the length. One nm is extremely small length corresponding to one billionth of one m, one millionth of one mm, or one thousandth of one μm.

1.4 Nanofluid

Nanofluid is a fluid containing nanoparticles. These fluids are engineered colloidal suspensions of nanoparticles in a base fluid.

1.5 Porous Medium

A porous medium or a porous material is a solid permeated by an interconnected network of pores (voids) filled with a fluid (liquid or gas). Natural porous media include soil, sand, mineral salts, sponge, wood and others. Synthetic porous media include paper, cloth filters, chemical reaction catalyst and membranes.

1.6 Porosity

Porosity or void fraction is a measure of the void spaces in a material and is a fraction of the volume of voids over the total volume, i.e.,

$$\varphi = \frac{V_V}{V_T}, \tag{1.1}$$

where V_V is the volume of void-space (such as fluids) and V_T is the total or bulk volume of material, including the solid and void components.

1.7 Pressure

Pressure is an effect which occurs when a force is applied on a surface per unit area. Mathematically,

$$P = \frac{F}{A}, \tag{1.2}$$

where P is pressure, F is the normal force and A is the area. The SI unit for pressure is Pascal (Pa), equal to one Newton per square meter (Nm^{-2} or $kgm^{-1}s^{-2}$).

1.8 Velocity Field

In dealing with fluids in motion, we shall necessarily be concerned with the description of a velocity field. At a given instant the velocity field, **V**, is a function of the space coordinates (x, y, z) and time t. The velocity at any point in the flow field might vary from one instant to another. Thus the complete representation of velocity is given by

$$\mathbf{V} = \mathbf{V}(x, y, z, t) \tag{1.3}$$

1.9 Viscosity

The internal friction of a fluid, produced by the movement of its molecules against each other. Viscosity causes the fluid to resist flowing.

$$\text{Viscosity} = \frac{\text{Shear Stress}}{\text{Rate of Shear Strain}} \tag{1.4}$$

1.10 Density

Density of a fluid is defined as the mass per unit volume. Mathematically, it is denoted by ρ and defined as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}, \tag{1.5}$$

where δv is the total volume element around the point C and δm is the mass of the fluid within δv.

1.11 Flow Visualization

There are four types of flow lines that may help to describe a flow field.

1.11.1 Streamline

A streamline is a line that is everywhere tangent to the velocity vector at a given instant of time. A streamline is hence an instantaneous pattern. Let u, v and w be the components of velocity in the x, y and z direction respectively. The equation of stream line for three dimensional flow is

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \tag{1.6}$$

and for two-dimensional flow the stream line equations are

$$\frac{dx}{u} = \frac{dy}{v}. \tag{1.7}$$

1.11.2 Streakline

A streakline is an instantaneous line whose points are occupied by particles which have earlier passed through a prescribed point in space. A streak line is hence an integrated pattern.

1.11.3 Pathlines

Pathlines are the trajectories that individual fluid particles follow. The direction the path takes will be determined by the streamlines of the fluid at each moment in time.

1.11.4 Timeline

A timeline is a set of fluid particles that form a line segment at a given instant of time.

1.12 Prandtl Number

It is the ratio of the product of dynamic viscosity and specific heat to the thermal conductivity and denoted by the symbol P_r and is given by

$$P_r = \frac{\mu c_p}{k}, \tag{1.8}$$

where μ is dynamic viscosity c_p is specific heat capacity and k is permeability of free space.

1.13 Reynolds Number

It is the ratio of inertia force to the viscous force. It is denoted by the symbol Re and is given by

$$\mathrm{Re} = \frac{VL}{\nu}, \tag{1.9}$$

where L denotes the characteristics length and ν is kinematic viscosity.

1.14 Fundamental Equations of Fluids

1.14.1 Equation of Continuity

In any closed system, the mass is always invariant regardless of its changes in shape when external forces are absent or the principle that matter cannot be created or destroyed. In fluid mechanics, this law is named as equation of continuity. In other words the mass of the system remains conserved. Mathematically, it is described as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{1.10}$$

In cylindrical coordinates, this equation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0. \tag{1.11}$$

1.14.2 Equation of Momentum

When some bodies constituting an isolated system act upon one another, the total momentum of the system remains same. In an inertial frame of reference, the general form of equations of fluid motion or the law of conservation of momentum is

$$\rho \frac{D\mathbf{V}}{Dt} = \nabla \cdot \mathbf{T} + \rho \mathbf{f}, \tag{1.12}$$

where $\mathbf{T}$ is the Cauchy stress tensor, D/Dt is the total material derivative and $\mathbf{f}$ is the body force. In cylindrical coordinates the momentum equation in components forms is given by.

r- component:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2}{r}+w\frac{\partial w}{\partial z}\right)=\begin{array}{c}-\frac{\partial p}{\partial r}+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}(ru)\right)\right]\\+\eta\left[\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}\right]+\rho f_r,\end{array} \tag{1.13}$$

θ- component:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}+w\frac{\partial v}{\partial z}\right)=\begin{array}{c}-\frac{1}{r}\frac{\partial p}{\partial r}+\eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}(rv)\right)\right]\\+\eta\left[\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}+\frac{\partial^2 u}{\partial z^2}\right]+\rho f_\theta\end{array} \tag{1.14}$$

z- component:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+w\frac{\partial w}{\partial z}\right)=\begin{array}{c}-\frac{\partial p}{\partial r}+\eta\left[\left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r})\right)\right]\\+\eta\left[\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right]+\rho fz\end{array}. \tag{1.15}$$

1.14.3 Equation of Energy

The law of conservation of energy states that energy may neither be created nor destroyed. Therefore, the sum of all the energies in the system is a constant. The energy equation is described as

$$\rho c_p\frac{D\theta}{Dt}=\mathbf{T}.\mathbf{L}-\boldsymbol{\nabla}.\mathbf{q}, \tag{1.16}$$

in which

$$\mathbf{L}=\boldsymbol{\nabla}\mathbf{V}. \tag{1.17}$$

In cylindrical coordinates, it is given as

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+\frac{v}{r}\frac{\partial T}{\partial \theta}+w\frac{\partial T}{\partial z}\right)=k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}\right]+\mu\phi, \tag{1.18}$$

where ϕ is the viscous dissipation function.

1.15 Solution Methodology

To describe the solution methodology we use the basic ideas of the HAM, we consider the following differential equation:

$$\mathcal{N}[u(r)] = 0, \tag{1.19}$$

where N is a nonlinear operator, t denotes the independent variable, $u(r)$ is an unknown function. By means of generalizing the traditional homotopy method, Liao [23] constructs the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[u^*(r;p) - u_0(r)] = p\hbar\{N[u^*(r;p)]\}, \tag{1.20}$$

where $p \in [0,1]$ is an embedding parameter, $\hbar$ is a nonzero auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(r)$ is an initial guess of $u(r)$ and $u^*(r;p)$ is an unknown function. It is important to note that one has great freedom to choose auxiliary objects such as $\hbar$ and $\mathcal{L}$ in HAM. Obviously, when $p = 0$ and $p = 1$, we get

$$u^*(r;0) = u_0(r), \qquad u^*(r;1) = u(r) \tag{1.21}$$

hold. Thus as p increases from 0 to 1, the solution $u^*(r;p)$ varies from the initial guess $u_0(r)$ to the solution $u(r)$. Expanding $u^*(r;p)$ in Taylor series with respect to p, one has

$$u^*(r;p) = u_0(r) + \sum_{m=1}^{\infty} u_m(r)p^m, \tag{1.22}$$

where

$$u_m(r) = \frac{1}{m!}\frac{\partial u^*(r;p)}{\partial pm}\bigg|_{p=0}. \tag{1.23}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen, then the series (1.16) converges at $p = 1$ and one can get

$$u^*(r;1) = u_0(r) + \sum_{m=1}^{\infty} u_m(r), \tag{1.24}$$

which must be one of the solutions of the original nonlinear equation, if $\hbar = -1$, Eq. (1.14) becomes

$$(1-p)\mathcal{L}[u^*(r;p) - u_0(r)] + p\{N[u^*(r;p)]\} = 0, \tag{1.25}$$

which is used mostly in the HPM. In view of (1.17), the governing equations can be deduced from the zero-order deformation Eq. (1.14). We define the vectors

$$u_i = \{u_0(r), u_0(r), \ldots, u_0(r)\}. \tag{1.26}$$

Differentiating Eq. (1.14) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m - \chi_m u_{m-1}] = \hbar Rm(u_{m-1}), \tag{1.27}$$

where

$$\begin{aligned} Rm(u_{m-1}) &= \frac{1}{(m-1)!} \frac{\partial^{m-1}\{N[u^*(r;p)]\}}{\partial p^{m-1}}\bigg|_{p=0}, \\ \chi_m &= \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}. \end{aligned} \tag{1.28}$$

It should be emphasized that $u_m (m \geq 1)$ are governed by the linear Eq. (1.21) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as MAPLE and MATHEMATICA.

1.16 Cylindrical Coordinates

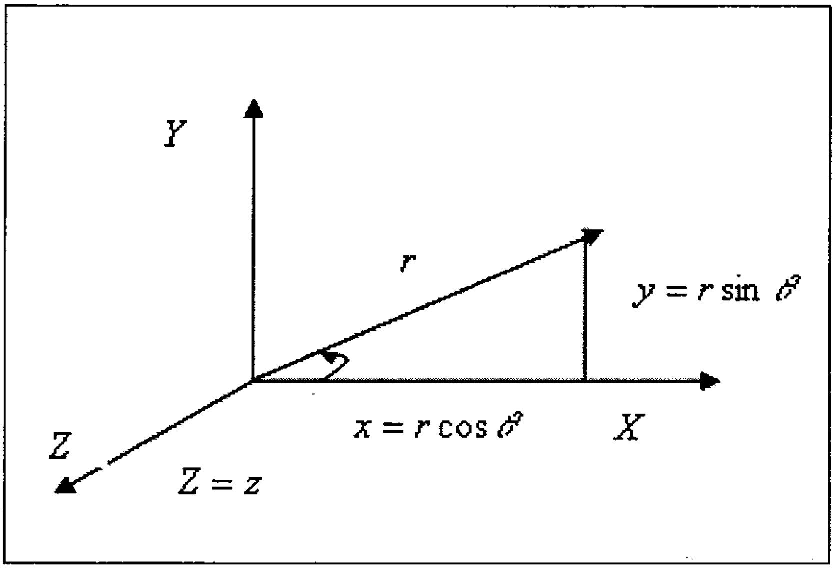

Fig. 1.1 : Cylindrical frame of reference.

In cylindrical coordinate system, points are located by giving them the values to $\{r, \theta, z\}$ see Fig. 1.1, which are related to $\{x = x_1, y = x_2, z = x_3\}$ by

$$\left.\begin{array}{c} x = \cos\theta,\ \ y = r\sin\theta,\ \ z = z \text{ and} \\ r = \left(x^2 + y^2\right)^{\frac{1}{2}},\ \ \theta = \tan^{-1}\left(\frac{y}{x}\right), z = z \end{array}\right\}. \tag{1.29}$$

The basis vectors in this frame are related to the Cartesian ones by

$$\left.\begin{array}{c} \mathbf{e}_r = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y, \mathbf{e}_x = \sin\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta, \\ \mathbf{e}_\theta = -\ \sin\theta\mathbf{e}_x + \cos\theta\mathbf{e}_y, \mathbf{e}_y = \sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta \end{array}\right\}. \tag{1.30}$$

The velocity $\mathbf{V}$, a tensor $\mathbf{S}$, gradient operator, $grad\mathbf{V}$ and $div\mathbf{V}$ in terms of these coordinates are respectively given by

$$\mathbf{V} = V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\mathbf{e}_z = (V_r, V_\theta, V_z), \tag{1.31}$$

$$\mathbf{S} = \left.\begin{array}{c} S_{rr}\mathbf{e}_r\mathbf{e}_r + S_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + S_{rz}\mathbf{e}_r\mathbf{e}_z \\ +S_{\theta r}\mathbf{e}_\theta\mathbf{e}_r + S_{\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + S_{\theta z}\mathbf{e}_\theta\mathbf{e}_z \\ +S_{zr}\mathbf{e}_z\mathbf{e}_r + S_{z\theta}\mathbf{e}_z\mathbf{e}_\theta + S_{zz}\mathbf{e}_z\mathbf{e}_z \end{array}\right\}, \tag{1.32}$$

where a matrix representation of $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix}, \tag{1.33}$$

$$\boldsymbol{\nabla} = \left.\begin{array}{c} (\cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) + \\ (\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) + \mathbf{e}_z\frac{\partial}{\partial z} \end{array}\right\}, \tag{1.34}$$

$$\boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right), \tag{1.35}$$

$$\boldsymbol{\nabla}\mathbf{V} = \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\mathbf{e}_z), \tag{1.36}$$

$$\boldsymbol{\nabla}\mathbf{V} = \left.\begin{array}{c} \mathbf{e}_r\mathbf{e}_r\frac{\partial V_r}{\partial r} + \mathbf{e}_r\mathbf{e}_\theta\frac{\partial V_\theta}{\partial r} + \mathbf{e}_r\mathbf{e}_z\frac{\partial V_z}{\partial r} \\ +\mathbf{e}_\theta\mathbf{e}_r\left(\frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r}\right) + \mathbf{e}_\theta\mathbf{e}_\theta\left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r}\right) \\ +\mathbf{e}_\theta\mathbf{e}_z\frac{1}{r}\frac{\partial V_z}{\partial\theta} + \mathbf{e}_z\mathbf{e}_r\frac{\partial V_r}{\partial z} + \mathbf{e}_z\mathbf{e}_\theta\frac{\partial V_\theta}{\partial z} + \mathbf{e}_z\mathbf{e}_z\frac{\partial V_z}{\partial z} \end{array}\right\}, \tag{1.37}$$

where a matrix representation of $\boldsymbol{\nabla}\mathbf{V}$ in cylindrical coordinates is given by

$$\boldsymbol{\nabla}\mathbf{V} = \begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_z}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} & \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} & \frac{1}{r}\frac{\partial V_z}{\partial\theta} \\ \frac{\partial V_r}{\partial z} & \frac{\partial V_\theta}{\partial z} & \frac{\partial V_z}{\partial z} \end{bmatrix}, \tag{1.38}$$

$$\boldsymbol{\nabla}.\mathbf{V} = \frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_r}{\partial\theta} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z}, \tag{1.39}$$

where we have used the following relations

$$\left.\begin{array}{c}\frac{\partial}{\partial r}\mathbf{e}_r = 0, \frac{\partial}{\partial r}\mathbf{e}_\theta = 0, \frac{\partial}{\partial r}\mathbf{e}_z = 0, \\ \frac{\partial}{\partial \theta}\mathbf{e}_r = \mathbf{e}_\theta, \frac{\partial}{\partial \theta}\mathbf{e}_\theta = -\mathbf{e}_r, \frac{\partial}{\partial \theta}\mathbf{e}_z = 0, \\ \frac{\partial}{\partial z}\mathbf{e}_r = 0, \frac{\partial}{\partial z}\mathbf{e}_\theta = 0, \frac{\partial}{\partial z}\mathbf{e}_z = 0\end{array}\right\}. \tag{1.40}$$

Chapter 2

The Influence of Variable Viscosity and Viscous Dissipation on the Non-Newtonian Flow: An Analytical Solution

2.1 Introduction

In this chapter, we review the work of Ellahi et al. [19]. We consider a flow of third grade fluid in a pipe with the viscous dissipation. The temperature of the pipe is higher than the temperature of the fluid. The governing equations are formulated mathematically. The non-linear governing equations are solved analytically by Homotopy Analysis Method (HAM). Convergence of the obtained solutions are properly discussed by help of $\hbar-$curves, explained the velocity and temperature profiles with the help of graphs taking different values of different pertinent parameters.

2.2 Formulation of the Problem

We consider the steady flow of an incompressible, third grade fluid in a pipe. The $z-$axis is taken along the axis of the flow. The velocity field in cylindrical coordinates is given by

$$\mathbf{V} = [0, 0, u(r)]\,. \tag{2.1}$$

By definition of incompressible fluids, the continuity Eq. (1.18) becomes

$$\nabla \cdot \mathbf{V} = 0. \tag{2.2}$$

Using Eq. (1.16) in Eq. (1.20), we obtain

$$\rho c_p \frac{D\theta}{Dt} = \mathbf{T} \cdot \mathbf{L} + k\nabla^2\theta. \tag{2.3}$$

For third grade fluid stress tenser is defined by

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1, \tag{2.4}$$

where p_1 is hydrostatic pressure, $\mathbf{I}$ is the identity tensor and $\alpha_i (i = 1, 2)$ and $\beta_j (j = 1, 2)$ are material constants. The *Rivlin-Ericksen* tensors are defined by the following general relations

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^t, \tag{2.5}$$

$$\mathbf{A}_\mathbf{n} = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^t\mathbf{A}_{n-1}, \qquad n > 1. \tag{2.6}$$

Thermodynamical limitations [24] comprise

$$\mu \geq 0,\ \alpha_1 \geq 0,\ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},\ \beta_1 = \beta_2 = 0,\ \beta_3 \geq 0. \tag{2.7}$$

In view of Eq. (2.7), Eq. (2.4), takes the following form

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{2.8}$$

Using the velocity field given in Eq. (2.1), we obtain

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{du}{dr} & 0 & 0 \end{bmatrix}, \qquad \mathbf{L}^t = \begin{bmatrix} 0 & 0 & \frac{du}{dr} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.9}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^t = \begin{bmatrix} 0 & 0 & \frac{du}{dr} \\ 0 & 0 & 0 \\ \frac{du}{dr} & 0 & 0 \end{bmatrix}, \tag{2.10}$$

$$\mathbf{A}_1\mathbf{L} + \mathbf{L}^t\mathbf{A}_1 = \begin{bmatrix} 2\left(\frac{du}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.11}$$

For steady flow

$$\frac{\partial \mathbf{A}_1}{\partial t} = 0 \tag{2.12}$$

and

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^t\mathbf{A}_1 = \begin{bmatrix} 2\left(\frac{du}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.13}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{du}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \left(\frac{du}{dr}\right)^2 \end{bmatrix}, \tag{2.14}$$

$$tr(\mathbf{A}_1^2)\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 2\left(\frac{du}{dr}\right)^3 \\ 0 & 0 & 0 \\ 2\left(\frac{du}{dr}\right)^3 & 0 & 0 \end{bmatrix}, \tag{2.15}$$

$$\left.\begin{aligned} \tau_{rr} &= -p_1 + (2\alpha_1 + \alpha_2)\left(\frac{du}{dr}\right)^2, \ \tau_{r\theta} = 0 = \tau_{\theta r}, \\ \tau_{rz} &= \mu\frac{du}{dr} + 2\beta_3\left(\frac{du}{dr}\right)^3 = \tau_{zr}, \tau_{\theta\theta} = -p_1, \\ \tau_{\theta z} &= 0 = \tau_{z\theta}, \ \tau_{zz} = -p_1 + \alpha_2\left(\frac{du}{dr}\right)^2. \end{aligned}\right\}. \tag{2.16}$$

In the absence of body forces and using cylindrical coordinates for the flow in a pipe, the momentum Eq. (1.19) will be simplified in the following form

$$\frac{1}{r}\frac{d}{dr}\left[r\mu\left(\frac{du}{dr}\right)\right]+\frac{2\beta_3}{r}\frac{d}{dr}\left[r\left(\frac{du}{dr}\right)^3\right]=\frac{\partial\hat{p}}{\partial z}, \tag{2.17}$$

subject to the boundary conditions

$$u(R)=0, \qquad \frac{du}{dr}(0)=0, \tag{2.18}$$

where

$$\hat{p}=p_1-\alpha_2\left(\frac{du}{dr}\right)^2 \tag{2.19}$$

is the modified pressure. Now using the definition of product of two tensors, we have

$$\mathbf{T.L}=\tau_{zr}\frac{du}{dr}, \tag{2.20}$$

$$\mathbf{T.L}=\mu\left(\frac{du}{dr}\right)^2+2\beta_3\left(\frac{du}{dr}\right)^4, \tag{2.21}$$

$$\nabla^2\theta=\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right). \tag{2.22}$$

The energy Eq. (2.3) becomes

$$\mu\left(\frac{du}{dr}\right)^2+2\beta_3\left(\frac{du}{dr}\right)^4+k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right]=0. \tag{2.23}$$

The relating boundary conditions are

$$\theta(R)=0, \qquad \frac{d\theta}{dr}(0)=0. \tag{2.24}$$

Using non- dimensionalization criteria, we set

$$u=\frac{\overline{u}}{u_0},\ r=\frac{\overline{r}}{R},\ \mu=\frac{\overline{\mu}}{\mu_0},\ \theta=\frac{\overline{\theta}-\theta_0}{\theta_1-\theta_0} \tag{2.25}$$

$$c_1 = \frac{\partial \hat{p}}{\partial z}, \; c = \frac{c_1 R^2}{u_0 \mu_0}, \; \Lambda = \frac{2\beta_3 u_0^2}{\mu_0 R^2}, \; \Gamma = \frac{\mu_0 u_0^2}{k(\theta_1 - \theta_0)} \tag{2.26}$$

boundary value problems consisting of Eqs. (2.17), (2.18), (2.23) and (2.24) become

$$\frac{1}{r}\frac{d}{dr}\left[r\mu\left(\frac{du}{dr}\right)\right] + \frac{\Lambda}{r}\frac{d}{dr}\left[r\left(\frac{du}{dr}\right)^3\right] = c, \tag{2.27}$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{du}{dr}\right)^2\left[\mu + \Lambda\left(\frac{du}{dr}\right)^2\right] = 0, \tag{2.28}$$

$$u(1) = \theta(1) = 0, \qquad \frac{du}{dr}(0) = \frac{d\theta}{dr}(0) = 0, \tag{2.29}$$

in which R, u_0, μ_0, θ_0, $\overline{\theta}$ and θ_1 are the radius, reference velocity, reference viscosity, reference temperature, pipe and fluid temperatures, respectively. Also, c_1 is the axial pressure drop, Λ is third grade parameter and Γ is related to the Prandtl and Eckert numbers. For simplicity we have omitted the bar symbols.

2.3 Solution of the Problem

We use homotopy analysis method (HAM) to solve the problem under consideration.

Case I: For the Constant Viscosity

When we take $\mu = 1$, the governing Eqs. (2.27) and (2.28) in simplified form reduce to

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + 3\Lambda\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 = c, \tag{2.30}$$

and

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{du}{dr}\right)^2 + \Lambda\Gamma\left(\frac{du}{dr}\right)^4 = 0, \tag{2.31}$$

respectively. We use the method of higher order differential mapping [25], to choose the linear operator $\mathcal{L}$, i.e.,

$$\mathcal{L}_1 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}. \tag{2.32}$$

The above operator satisfies the following relation

$$\mathcal{L}_1[C_1 + C_2 \ln r] = 0, \tag{2.33}$$

where C_1 and C_2 are the arbitrary constants. Integrating the linear part of Eq. (2.30), we get

$$u_0(r) = \frac{1}{4}c(r^2 - 1) \tag{2.34}$$

as the initial approximation of velocity u, which satisfies the linear operator $\mathcal{L}_1$ and boundary conditions too.

Zeroth order deformation equation

For non-zero auxiliary parameter $\hbar$ and an embedding parameter $p \in [0, 1]$, the zeroth order deformation equation in HAM is given by the following relation

$$(1-p)\mathcal{L}_1[u^*(r,p) - u_0(r)] = p\hbar \begin{bmatrix} \frac{d^2u^*}{dr^2} + \frac{1}{r}\frac{du^*}{dr} + 3\Lambda\left(\frac{du^*}{dr}\right)^2 \frac{d^2u^*}{dr^2} \\ +\frac{\Lambda}{r}\left(\frac{du^*}{dr}\right)^3 - c \end{bmatrix}, \tag{2.35}$$

subject to the following boundary conditions

$$u^*(1,p) = 0, \qquad \frac{du^*}{dr}(0,p) = 0. \tag{2.36}$$

$m-$th order deformation equation

If we differentiate $m-$times the zeroth order deformation Eqs. (2.35) and (2.36) with respect to p, dividing by $m!$ and finally taking $p = 0$, we have the mth order deformation equation, of the following form

$$\mathcal{L}_1\left[u_m - \chi_m u_{m-1}\right] = \hbar R_m(r), \tag{2.37}$$

where

$$R_m(r) = u''_{m-1} + \frac{1}{r}u'_{m-1} + \Lambda \sum_{k=0}^{m-1} u'_{m-1-k} \sum_{j=0}^{k} u'_{k-j}(\frac{1}{r}u'_j + 3u''_j) - c(1 - \chi_m). \tag{2.38}$$

Corresponding boundary conditions take the following form

$$u'_m(0) = u_m(1) = 0, \tag{2.39}$$

where prime denotes the differentiation with respect to r.

From Eq. (2.35) by setting $p = 0$, it can be shown that

$$u^*(r,p) = u_0(r). \tag{2.40}$$

By the definition of homotopy, as p varies from 0 to 1, $u^*(r,p)$ varies from initial guess $u_0(r)$ to the exact solution $u(r)$, that is for properly chosen $\hbar$, we get

$$u^*(r,p) = u(r) \qquad \text{for } p = 1. \tag{2.41}$$

Then employing the Taylor's theorem, we can write

$$u^*(r,p) = u_0(r) + \sum_{m=1}^{\infty} u_m(r)p^m, \tag{2.42}$$

where

$$u_m(r) = \frac{1}{m!}\frac{\partial^m u^*(r,p)}{\partial p^m}\bigg|_{p=0}. \tag{2.43}$$

Now using Eq. (2.41) in Eq. (2.42), we get

$$u(r) = u_0(r) + \sum_{m=1}^{\infty} u_m(r). \tag{2.44}$$

Differentiate Eq. (2.35) with respect to p and set $p = 0$, then after solving the resulting equation we obtain the following

$$u_1(r) = \frac{1}{32}\hbar\Lambda c^3(r^4 - 1). \tag{2.45}$$

Again differentiating Eq. (2.35) with respect to p, putting $p = 0$ and using the similar procedure, we get

$$u_2(r) = \frac{1}{32}\hbar\Lambda c^3(r^4 - 1)(\hbar + 1) + \frac{1}{64}\hbar^2\Lambda^2 c^5\left(r^6 - 1\right). \tag{2.46}$$

Now from Taylor series, we have the three terms solution as

$$u(r) = u_0(r) + u_1(r) + u_2(r). \tag{2.47}$$

Finally, inserting Eqs. (2.34), (2.46) and (2.47) in Eq. (2.48), we get the expression for velocity as follows

$$u(r) = \frac{1}{4}c(r^2-1) + \frac{1}{32}\hbar\Lambda c^3(r^4-1)(\hbar+2) + \frac{1}{64}\hbar^2\Lambda^2c^5(r^6-1). \tag{2.48}$$

Now using Eqs. (2.31) and (2.49), with boundary conditions Eq. (2.29), we can find θ by using Cauchy-Euler equation and computer software, 'MATHEMATICA'. The result is given below

$$\left.\begin{array}{c} \theta(r) = A_1(r^4-1) + A_2(r^6-1) + A_3(r^8-1) + A_4(r^{10}-1) \\ +A_5(r^{12}-1) + A_6(r^{14}-1) + A_7(r^{16}-1) + A_8(r^{18}-1) \\ +A_9(r^{20}-1) + A_{10}(r^{22}-1) \end{array}\right\}. \tag{2.49}$$

The calculated values of coefficients $A_i (i = 1, 2, ...10)$ are given in Appendix A.

Case II: For the Variable Viscosity

Let us now assume that the viscosity is space dependent and choose $\mu = r$. From Eq. (2.26), we have

$$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \frac{\Lambda}{r}\frac{d}{dr}\left[r\left(\frac{du}{dr}\right)^3\right] = c, \tag{2.50}$$

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} + 3\Lambda\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 = c, \tag{2.51}$$

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \frac{3\Lambda}{r}\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2} + \frac{\Lambda}{r^2}\left(\frac{du}{dr}\right)^3 = \frac{c}{r}, \tag{2.52}$$

along with the same boundary conditions given in Eq. (2.29). Similarly Eq. (2.28) simplifies to

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\left(\frac{d\theta}{dr}\right) + \Gamma r\left(\frac{du}{dr}\right)^2 + \Gamma\Lambda\left(\frac{du}{dr}\right)^4 = 0, \tag{2.53}$$

which corresponds to Eq. (2.26). The linear operator in this case will be

$$\mathcal{L}_2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}, \tag{2.54}$$

provided that

$$\mathcal{L}_2\left[C_3 + \frac{C_4}{r}\right] = 0, \tag{2.55}$$

where C_3 and C_4 are constants of integration. Thus the initial approximation for the velocity u is

$$u_0(r) = \frac{1}{6}c(r^2 - 1). \tag{2.56}$$

Zeroth order deformation equation

With the use of Eq. (2.53), one can define the zeroth order deformation equation for u as

$$(1-p)\mathcal{L}_2[u^*(r,p) - u_0(r)] = p\hbar\begin{bmatrix} \frac{d^2u^*}{dr^2} + \frac{2}{r}\frac{du^*}{dr} + \frac{3\Lambda}{r}\left(\frac{du^*}{dr}\right)^2\frac{d^2u^*}{dr^2} \\ +\frac{\Lambda}{r^2}\left(\frac{du^*}{dr}\right)^3 - \frac{c}{r}\end{bmatrix}. \tag{2.57}$$

$m-$th order deformation equation

The mth order deformation problem can be written as

$$\mathcal{L}_2\left[u_m - \chi_m u_{m-1}\right] = \hbar R_m(r), \tag{2.58}$$

where

$$R_m(r) = u''_{m-1} + \frac{2}{r}u'_{m-1} + \Lambda\sum_{k=0}^{m-1}u'_{m-1-k}\sum_{j=0}^{k}u'_{k-j}\left(\frac{1}{r^2}u'_j + 3u''_j\right) - c(1-\chi_m). \tag{2.59}$$

The expression for θ can also be defined in the same manner. The mth order deformation equation can be obtained by using similar procedure like that of given in case I. Following the same procedure, we find three terms series solution of u as follows

$$\left.\begin{array}{c} u(r) = \frac{1}{6}c(r^2-1) + \frac{1}{6}\hbar c(r^2-1)(\hbar+2) + \frac{2}{81}\hbar\Lambda c^3(2\hbar+1)(r^3-1) \\ -\frac{1}{2}\hbar c(\hbar+2)(r-1) + \frac{1}{324}\hbar^2\Lambda^2c^5(r^4-1) - \frac{1}{12}\hbar^2\Lambda c^3(r^2-1)\end{array}\right\}. \tag{2.60}$$

For finding the solution of temperature θ, we use 'MATHEMATICA' to solve the Cauchy-Euler equation. Then we obtain

$$\left.\begin{array}{c}\theta(r) = A_{11}(r^2-1) + A_{12}(r^3-1) + A_{13}(r^4-1) + A_{14}(r^5-1) + A_{15}(r^6-1) + \\ A_{16}(r^7-1) + A_{17}(r^8-1) + A_{18}(r^9-1) + A_{19}(r^{10}-1) + A_{20}(r^{11}-1) + \\ A_{21}(r^{12}-1) + A_{22}(r^{13}-1) + A_{23}(r^{14}-1)\end{array}\right\}, \quad (2.61)$$

where the coefficients $A_j (j = 11, 12, ...23)$ are given in Appendix A.

2.4 Convergence of the Solution

It is noticed that the HAM method strictly depend upon the auxiliary parameter $\hbar$. As specified by Liao [26], the convergence region and rate of approximations given by the HAM are strongly dependent upon $\hbar$. Figs. 2.1 and 2.2 potray the $\hbar-$curves of velocity and temperature profiles, respectively just to find the range of $\hbar$ in case of variable viscosity. The range for admissible values of $\hbar$ for velocity in this case of constant viscosity is $-0.8 \leq \hbar \leq 0.1$ and for temperature is $-0.8 \leq \hbar \leq 0$. Figs. 2.3 and 2.4 represent the $\hbar-$curves for variable viscosity when $\mu = r$. The admissible ranges for both velocity and temperature profiles are $-0.3 \leq \hbar \leq 0.1$ and $-0.3 \leq \hbar \leq -0.1$, respectively.

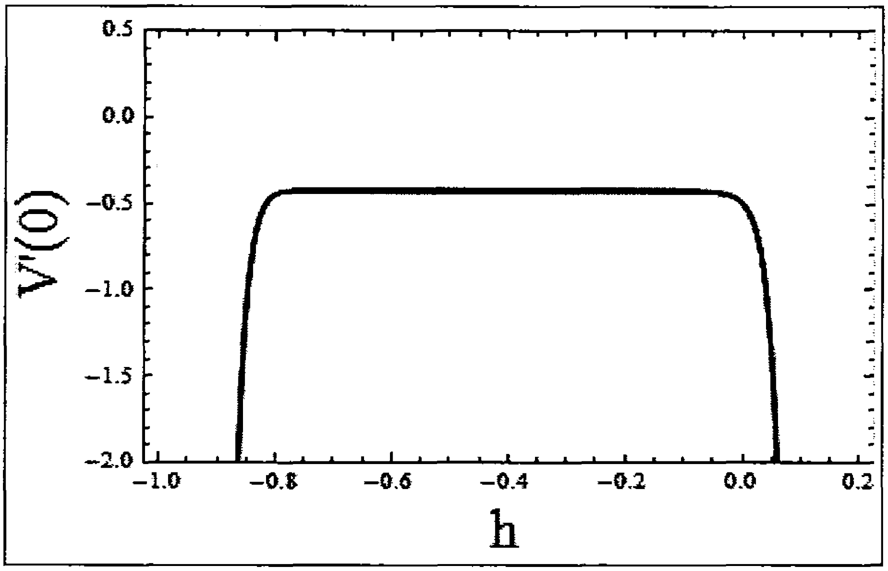

Fig. 2.1: $\hbar$−curve for velocity profile in case of constant viscosity at $30th$ order approximation.

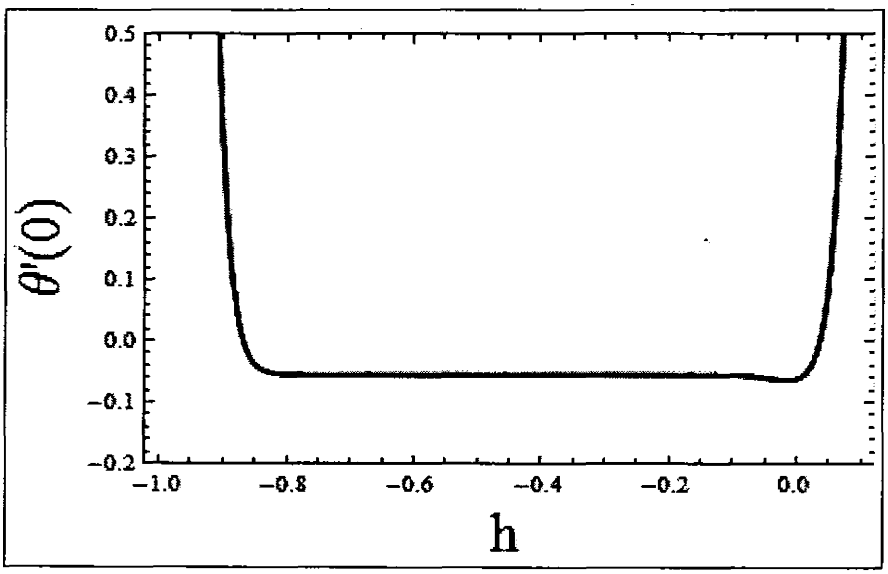

Fig. 2.2: $\hbar$−curve for temperature profile in case of constant viscosity at $30th$ order approximation.

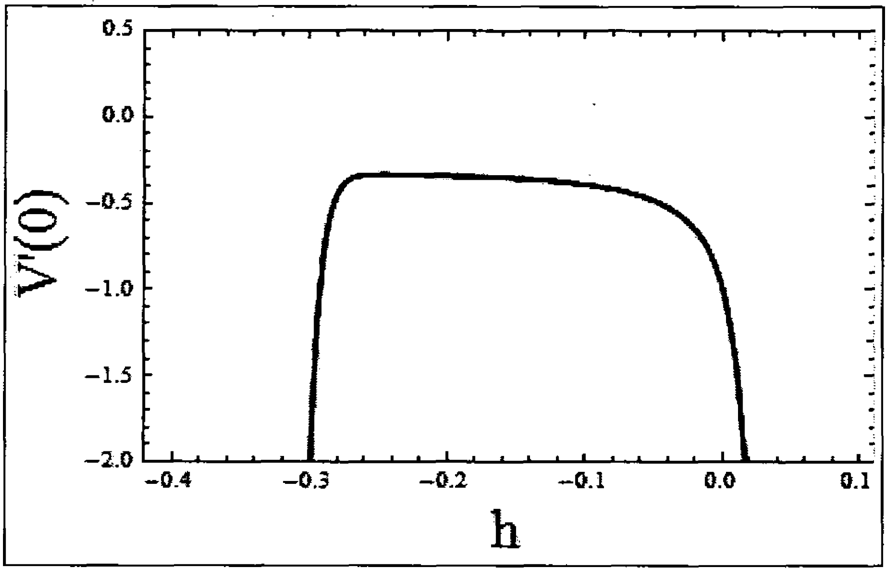

Fig. 2.3: $\hbar$−curve for velocity profile in case of variable viscosity at $30th$ order approximation.

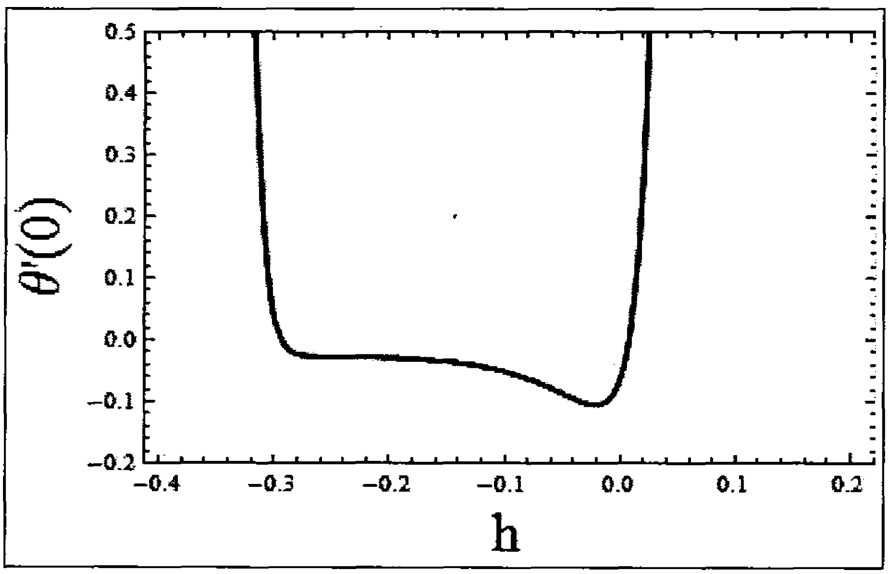

Fig. 2.4: $\hbar$−curve for temperature profile in case of variable viscosity at $30th$ order approximation.

2.5 Graphs

In this section, we will discuss the results of velocity and temperature profiles for both constant and variable viscosity with the help of graphs.

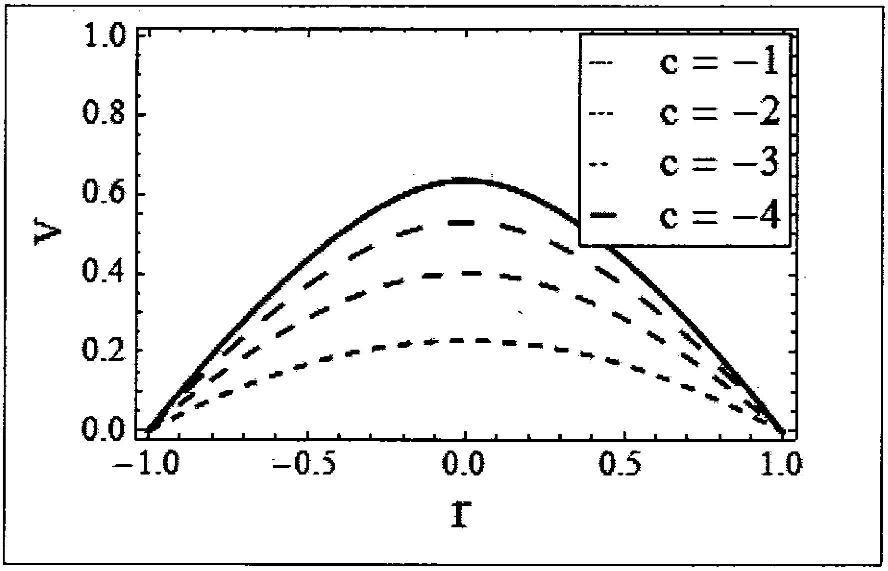

Fig. 2.5: Influence of c on velocity when $\Lambda = 1$ and $\Gamma = 1$.

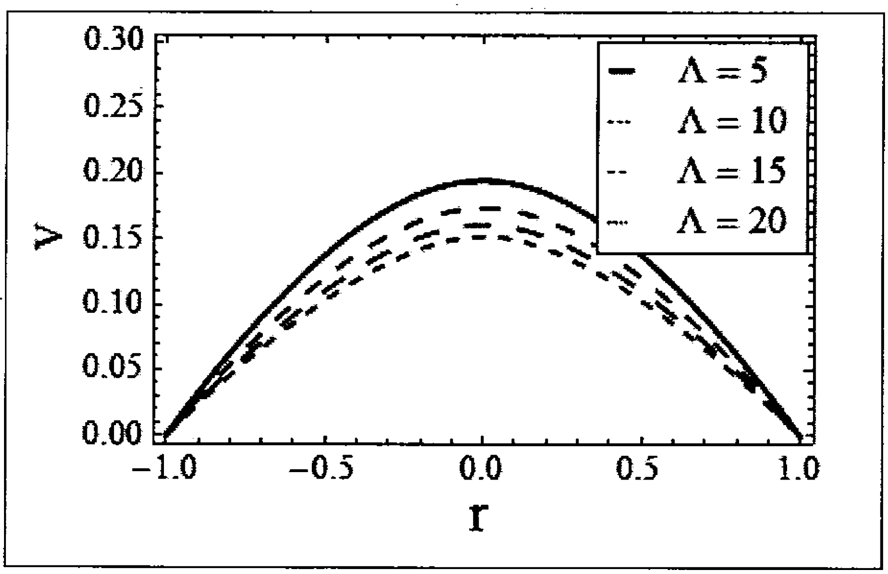

Fig. 2.6: Influence of Λ on velocity when $c = -1$ and $\Gamma = 1$.

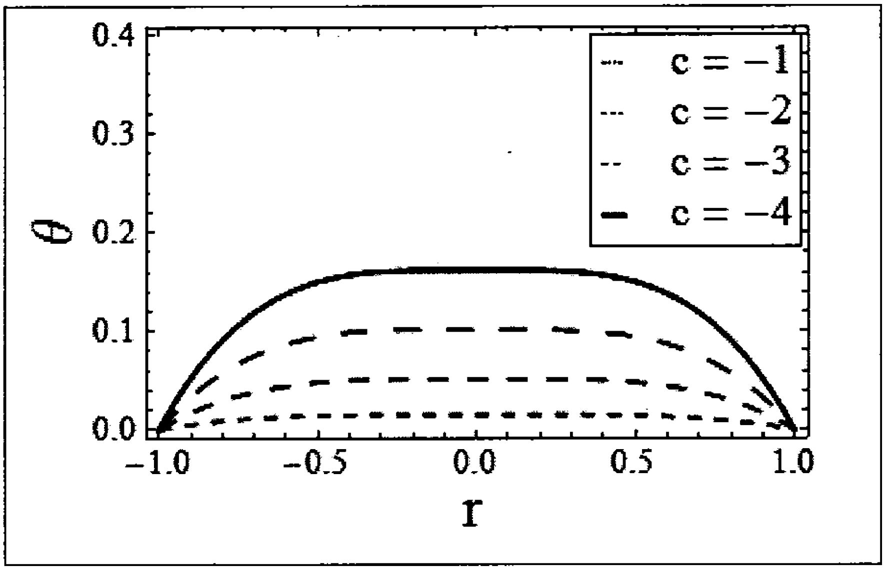

Fig. 2.7: Influence of c on temperature when $\Gamma = 1$ and $\Lambda = 1$.

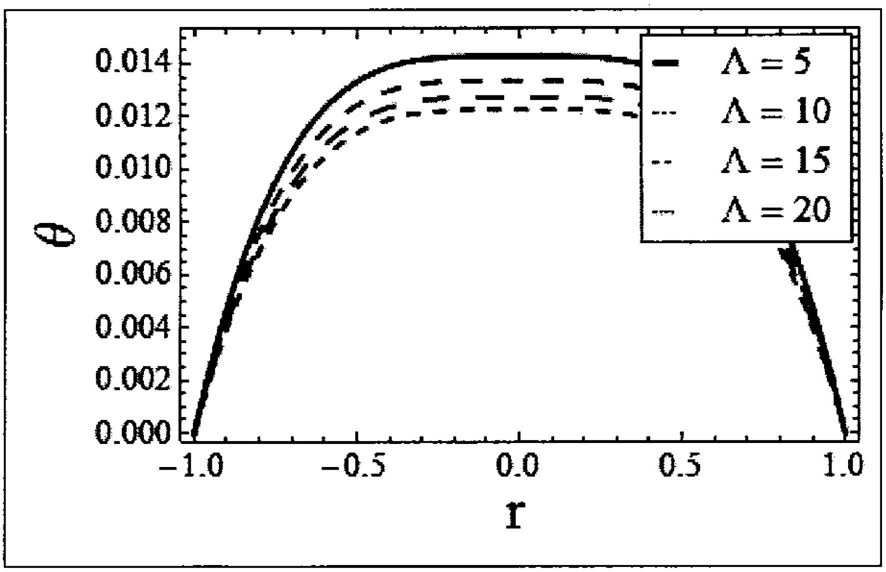

Fig. 2.8: Influence of Λ on temperature when $c = -1$and $\Gamma = 1$.

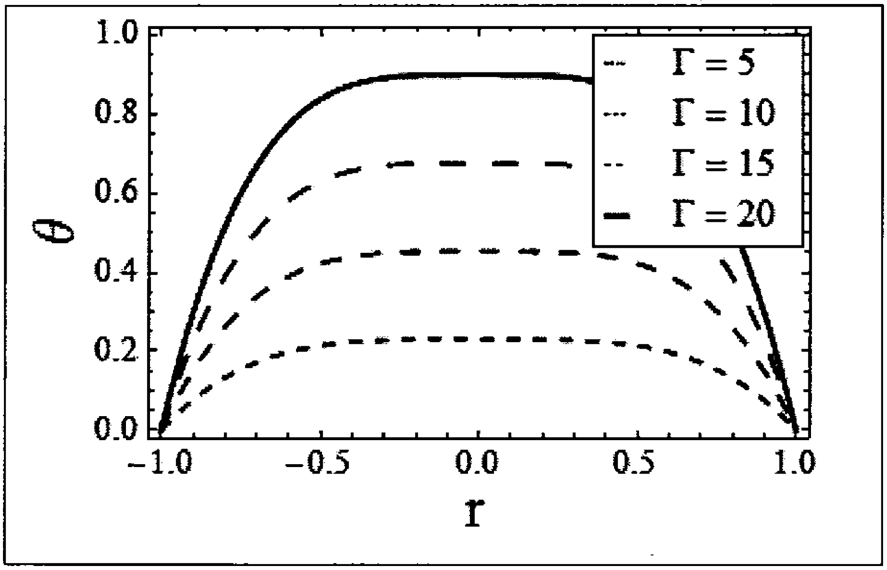

Fig. 2.9: Influence of Γ on temperature when $c = -1$ and $\Lambda = 1$.

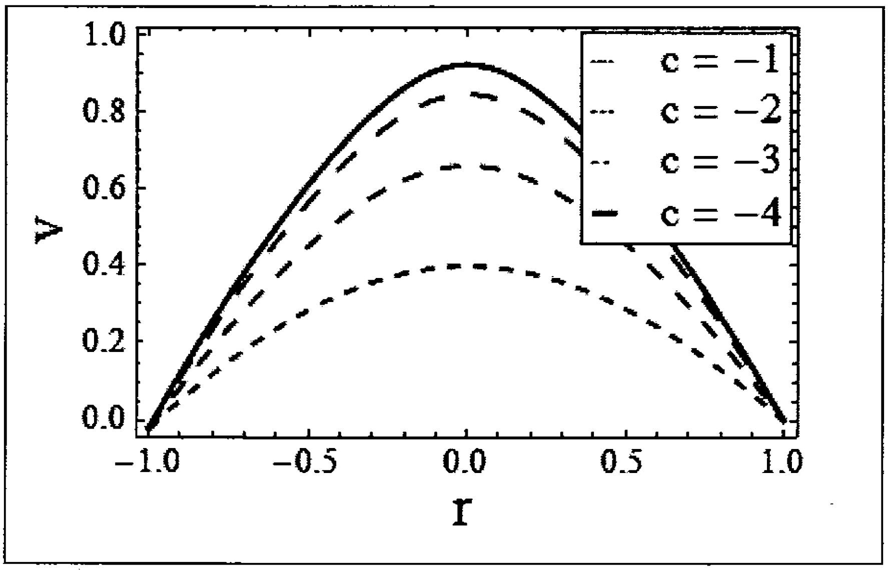

Fig. 2.10: Influence of c on velocity when $\Lambda = 1$ and $\Gamma = 1$.

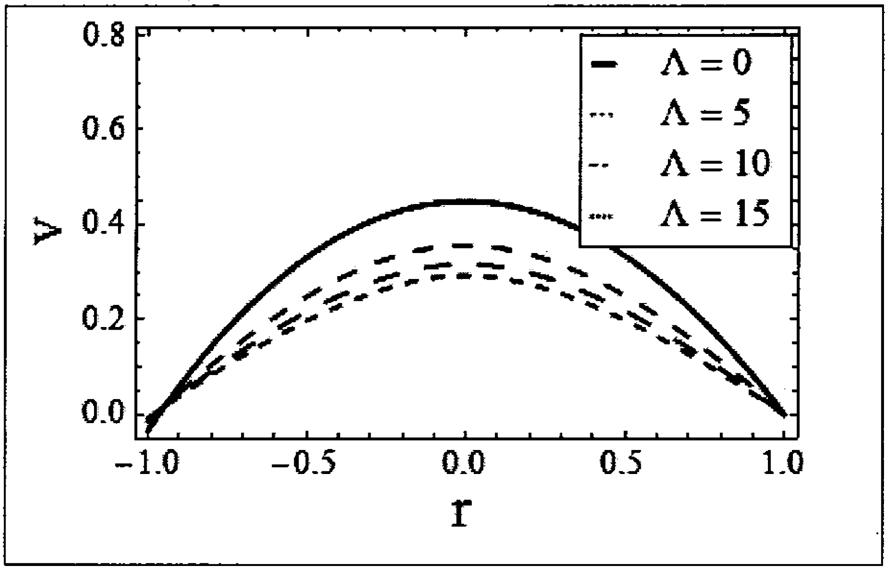

Fig. 2.11: Influence of Λ on velocity when $c = -1$ and $\Lambda = 1$.

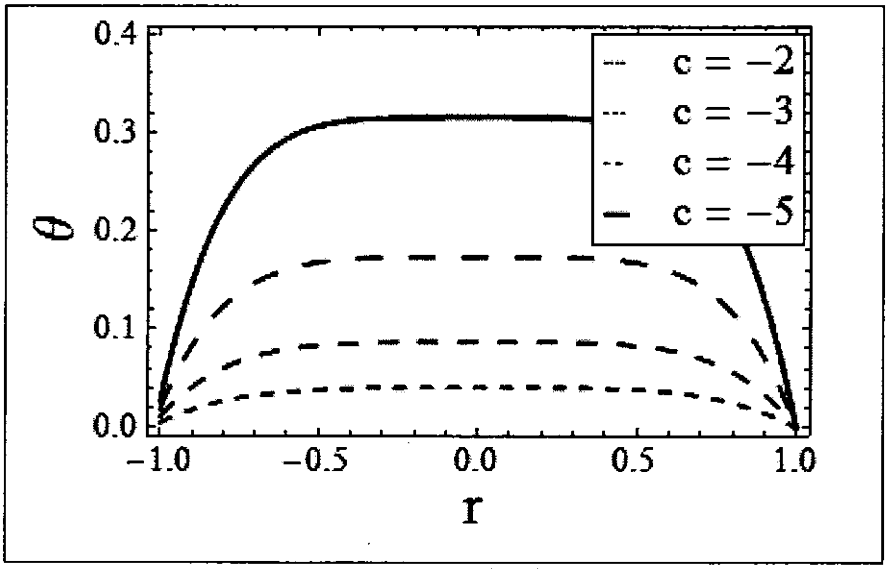

Fig. 2.12: Influence of c on temperature when $\Gamma = 1$ and $\Lambda = 1$.

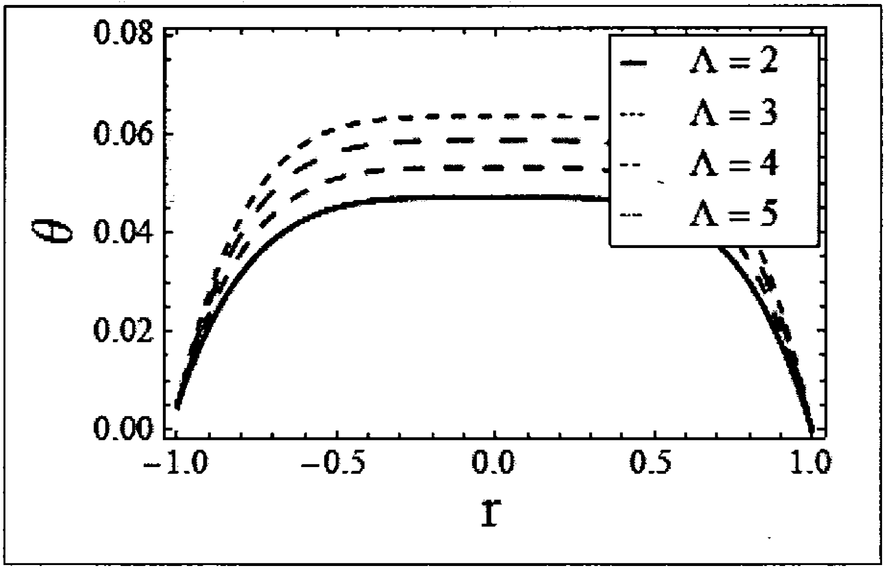

Fig. 2.13: Influence of Λ on temperature when $\Gamma = 1$ and $c = -2$.

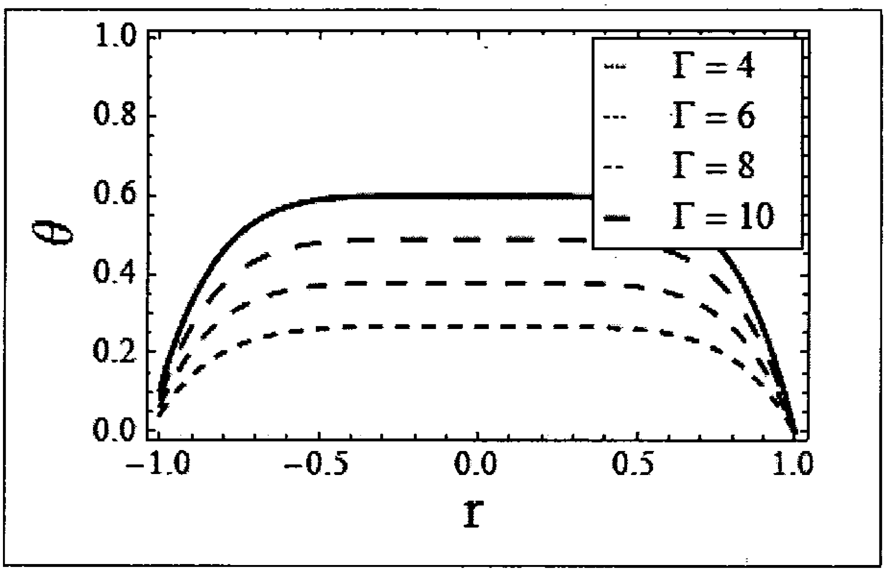

Fig. 2.14: Influence of Γ on temperature when $c = -3$ and $\Lambda = 1.5$.

2.6 Discussion

In this chapter, the solution for the velocity and temperature distributions are plotted against the pipe radius. Figs. 2.5 to 2.9 show the variation of velocity and temperature profiles for constant viscosity case and for space dependent viscosity, Figs. 2.10 to 2.14 are presented. In these figures, the variation of the velocity u and temperature θ with the emerging parameters Λ, c and Γ is revealed.

In Fig. 2.5, the effect of pressure gradient c is depicted (when $\hbar$ is approximately equal to -0.05). It is clear that the velocity approaches its maximums at the center of the pipe and varies inversely with c. Also, the effect of c on θ (in Fig. 2.7) is similar to that of velocity. The effect of third grade parameter Λ on the velocity and temperature distributions are shown in Figs. 2.6 and 2.8 respectively. As expected, an increase in Λ results in a decrease in both velocity and temperature. However, the temperature profile is more flatter than the velocity profile for same values of Λ. Fig. 2.9 illustrates the effect of the parameter Γ on temperature distribution θ. It is concluded that θ increases with the increase of Γ and hence the thermal boundary layer thickness decreases.

So far, we disclosed the results of the velocity and temperature for constant viscosity model. Now we turn our consideration to the discussion of above mentioned parameters for space dependent viscosity. Figs. 2.10 to 2.14 represent the influence of all dealing parameters (c, Λ and Γ) on both, velocity and temperature solutions when viscosity is depending upon space. From these figures, it is observed that the impact of c, Λ and Γ on u and θ (when $\hbar$ is nearly equal to -0.01) is similar to that of constant viscosity case.

Chapter 3

Effects of Porosity on the Flow of Third Grade Nanofluid with Space Dependent Viscosity

3.1 Introduction

Chapter three is a new contribution in literature in which the flow of third grade nanofluid between coaxial cylinders with constant and variable viscosity is considered when the outer cylinder is porous. To drive the solution of governing nonlinear boundary value problem, we have used one of the most modern perturbation methods, Homotopy Analysis Method (HAM). The physical features of the pertinent parameters are presented in graphical forms. A brief conclusion is also given at the end of the chapter.

3.2 Formulation of the Problem

Consider a unidirectional, an incompressible third grade nanofluid between two infinite coaxial cylinders. The outer cylinder is porous the flow is induced by a constant pressure gradient and motion of an inner cylinder with no-slip conditions. The heat transfer analysis and nanoparticale concentration equations are also taken into account. The $z-$axis is taken along the axis of the flow. The velocity field in cylindrical coordinates is given by

$$\mathbf{V} = [0, 0, u(r)]. \tag{3.1}$$

By definition of incompressible fluids, the continuity Eq. (1.18) becomes

$$\nabla \cdot \mathbf{V} = 0. \tag{3.2}$$

The non-dimensional quantities are defined by the following relations

$$\begin{aligned} \phi &= \frac{\overline{\phi} - \phi_w}{\phi_m - \phi_w}, \ B_r = \frac{(\rho_p - \rho_w) R^2 (\phi_m - \phi_w) G_r}{\mu_0 u_0}, \\ G_r &= \frac{(\theta_m - \theta_w) \rho_{fw} R^2 (1 - \phi_w) G}{\mu_0 u_0}, \ P = \frac{R^2 \varphi}{k_1}. \end{aligned} \tag{3.3}$$

In view of Eq. (3.1-3.3), we get the non dimensional problem of following form

$$\begin{aligned} \frac{d\mu}{dr}\frac{du}{dr} + \frac{\mu}{r}\frac{du}{dr} + \mu\frac{d^2u}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 + 3\Lambda\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2} \\ = c + P\left(\mu + \Lambda\left(\frac{du}{dr}\right)^2\right)u - G_r\theta - B_r\phi \end{aligned}, \tag{3.4}$$

$$(\alpha + \alpha_1 N_t)\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + N_b\frac{d\theta}{dr}\frac{d\phi}{dr} = 0, \tag{3.5}$$

$$N_b\left(\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr}\right) + N_t\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) = 0, \tag{3.6}$$

subject to boundary conditions

$$\begin{aligned} u(1) = 1, \ u(2) = 0, \\ \theta(1) = 1, \ \theta(2) = 0, \\ \phi(1) = 1, \ \phi(2) = 0. \end{aligned} \tag{3.7}$$

where G_r is thermophoresis diffusion constant and B_r is Brownian diffusion constant P is porosity parameter, N_t and N_b are thermophoresis parameter and Brownian diffusion coefficient respectively.

3.3 Solution of the Problem.

In this section, we discuss two models of viscosity namely; constant and variable viscosity. By using Homotopy Analysis Method (HAM), we find the series solutions of the nonlinear governing equations

Case I: Constant Viscosity

For constant viscosity model we choose

$$\mu = 1, \tag{3.8}$$

Making use of Eq. (3.8) in Eq. (3.4) we get

$$\left.\begin{array}{c} \frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 + 3\Lambda\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2} \\ = c + P\left[1 + \Lambda\left(\frac{du}{dr}\right)^2\right]u - G_r\theta - B_r\phi \end{array}\right\}, \tag{3.9}$$

whereas Eq. (3.5) and Eq. (3.6) remains same because viscosity is involve there. The solution of above equation has been obtained analytically by homotopy analysis method.

For HAM solution we select

$$u_0(r) = \frac{(128 - r^7)}{127},\ \theta_0(r) = \frac{(128 - r^7)}{127},\ \phi_0(0) = \frac{(128 - r^7)}{127}, \tag{3.10}$$

as the initial approximation of u , θ and ϕ. Further we choose the following auxiliary linear operator $\mathcal{L}_3$

$$\mathcal{L}_3 = \frac{d^2}{dr^2}, \tag{3.11}$$

which satisfies the following relation

$$\mathcal{L}_3[C_5 + C_6 \ln r] = 0, \tag{3.12}$$

where C_5 and C_6 are arbitrary constants.

Zeroth order deformation equation:

For embedding parameter $p \in [0, 1]$, auxiliary parameters is $\hbar$, the zeroth order deformation equation in HAM is given by the following relations

$$(1-p)\mathcal{L}_3[u^*(r,p)-u_0(r)] = p\hbar\mathcal{N}_1\ [u^*(r,p),\theta^*(r,p),\phi^*(r,p)], \tag{3.13}$$

$$(1-p)\mathcal{L}_3[\theta^*(r,p)-\theta_0(r)] = p\hbar\mathcal{N}_2\ [u^*(r,p),\theta^*(r,p),\phi^*(r,p)], \tag{3.14}$$

$$(1-p)\mathcal{L}_3[\phi^*(r,p)-\phi_0(r)] = p\hbar\mathcal{N}_3\ [\phi^*(r,p),\theta^*(r,p),\phi^*(r,p)], \tag{3.15}$$

$$\begin{aligned} u^*(1,p) &= 1,\ u^*(2,p)=0,\\ \theta^*(1,p) &= 1,\ \theta^*(2,p)=0,\\ \phi^*(1,p) &= 1,\ \phi^*(2,p)=0. \end{aligned} \tag{3.16}$$

In above equations the nonlinear operators for velocity, temperature and nanoparticles are defined as

$$\mathcal{N}_1\ [u^*(r,p),\theta^*(r,p),\phi^*(r,p)] = \left.\begin{array}{c} \frac{1}{r}\frac{du^*}{dr}+\frac{d^2u^*}{dr^2}+\frac{\Lambda}{r}\left(\frac{du^*}{dr}\right)^3+3\Lambda\left(\frac{du^*}{dr}\right)^2\frac{d^2u^*}{dr^2} \\ -c-P\left[1+\Lambda\left(\frac{du^*}{dr}\right)^2\right]u^*+G_r\theta^*+B_r\phi^* \end{array}\right\}, \tag{3.17}$$

$$\mathcal{N}_2\ [u^*(r,p),\theta^*(r,p),\phi^*(r,p)] = (\alpha+\alpha_1N_t)\frac{d^2\theta^*}{dr^2}+\frac{1}{r}\frac{d\theta^*}{dr}+N_b\frac{d\theta^*}{dr}\frac{d\phi^*}{dr}, \tag{3.18}$$

$$\mathcal{N}_3\ [u^*(r,p),\theta^*(r,p),\phi^*(r,p)] = N_b\left(\frac{d^2\theta^*}{dr^2}+\frac{1}{r}\frac{d\theta^*}{dr}\right)+N_t\left(\frac{d^2\phi^*}{dr^2}+\frac{1}{r}\frac{d\phi^*}{dr}\right). \tag{3.19}$$

$m-$th order deformation equation:

mth order deformation problem of above equations are

$$\mathcal{L}_2\left[u_m-\chi_m u_{m-1}\right] = \hbar\mathcal{R}1_m(r), \tag{3.20}$$

$$\mathcal{L}_2\left[\theta_m-\chi_m \theta_{m-1}\right] = \hbar\mathcal{R}2_m(r)$$

$$\mathcal{L}_2\left[\phi_m-\chi_m \phi_{m-1}\right] = \hbar\mathcal{R}3_m(r), \tag{3.21}$$

$$
\begin{aligned}
u_m(1) &= 0,\ u(2) = 0,\\
\theta_m(1) &= 0,\ \theta_m(2) = 0,\\
\phi_m(1) &= 0,\ \phi_m(2) = 0.
\end{aligned} \tag{3.22}
$$

where

$$
\mathcal{R}1_m(r) = \left.\begin{matrix}
\frac{d^2u_{m-1}}{dr^2} + \frac{1}{r}\frac{du_{m-1}}{dr}\\
+\frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}\frac{du_l}{dr}\\
+3\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}\frac{d^2u_l}{dr^2}\\
-Pu_{m-1} + BP\sum_{k=0}^{m-1}u_{m-1-k}\theta_k\\
-P\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}u_l\\
-c(1-\chi_m) + G_r\theta_{m-1} + B_r\phi_{m-1}
\end{matrix}\right\}, \tag{3.23}
$$

$$
\mathcal{R}2_m(r) = (\alpha + \alpha_1 N_t)\frac{d^2\theta_{m-1}}{dr^2} + \frac{1}{r}\frac{d\theta_{m-1}}{dr} + N_b\sum_{k=0}^{m-1}\frac{d\theta_{m-1-i}}{dr}\frac{d\phi_i}{dr}, \tag{3.24}
$$

$$
\mathcal{R}3_m(r) = N_b\left(\frac{d^2\theta_{m-1}}{dr^2} + \frac{1}{r}\frac{d\theta_{m-1}}{dr}\right) + N_t\left(\frac{d^2\phi_{m-1}}{dr^2} + \frac{1}{r}\frac{d\phi_{m-1}}{dr}\right). \tag{3.25}
$$

Case-II: Variable Viscosity

For variable viscosity is defined as

$$
\mu = r, \tag{3.26}
$$

In view of Eqs. (3.26) and (3.4) we get the nonlinear terms as following

$$
\mathcal{N}_4\left[u^*(r,p),\theta^*(r,p),\phi^*(r,p)\right] = \begin{matrix}
2\frac{du}{dr} + r\frac{d^2u}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 + 3\Lambda\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2}\\
-c - P\left(r + \Lambda\left(\frac{du}{dr}\right)^2\right)u + G_r\theta + B_r\phi
\end{matrix}, \tag{3.27}
$$

$$
\mathcal{N}_5\left[u^*(r,p),\theta^*(r,p),\phi^*(r,p)\right] = (\alpha + \alpha_1 N_t)\frac{d^2\theta^*}{dr^2} + \frac{1}{r}\frac{d\theta^*}{dr} + N_b\frac{d\theta^*}{dr}\frac{d\phi^*}{dr}, \tag{3.28}
$$

$$\mathcal{N}_6\left[u^*(r,p),\theta^*(r,p),\phi^*(r,p)\right]=N_b\left(\frac{d^2\theta^*}{dr^2}+\frac{1}{r}\frac{d\theta^*}{dr}\right)+N_t\left(\frac{d^2\phi^*}{dr^2}+\frac{1}{r}\frac{d\phi^*}{dr}\right). \tag{3.29}$$

mth order deformation equation:

mth order deformation problem of above equations are

$$\mathcal{L}_2\left[u_m-\chi_m u_{m-1}\right]=\hbar\mathcal{R}4_m(r), \tag{3.30}$$

$$\mathcal{L}_2\left[\theta_m-\chi_m\theta_{m-1}\right]=\hbar\mathcal{R}5_m(r),$$

$$\mathcal{L}_2\left[\phi_m-\chi_m\phi_{m-1}\right]=\hbar\mathcal{R}6_m(r), \tag{3.31}$$

$$\begin{aligned} u_m(1)&=0,\ u(2)=0,\\ \theta_m(1)&=0,\ \theta_m(2)=0,\\ \phi_m(1)&=0,\ \phi_m(2)=0. \end{aligned} \tag{3.32}$$

where

$$\mathcal{R}4_m(r)=\left.\begin{array}{c} r\frac{d^2u_{m-1}}{dr^2}+2\frac{du_{m-1}}{dr}-Pru_{m-1}\\ +\frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}\frac{du_l}{dr}\\ +3\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}\frac{d^2u_l}{dr^2}\\ +G_r\theta_{m-1}+B_r\phi_{m-1}-c(1-\chi_m)\\ -P\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}u_l \end{array}\right\}, \tag{3.33}$$

$$\mathcal{R}5_m(r)=(\alpha+\alpha_1N_t)\frac{d^2\theta_{m-1}}{dr^2}+\frac{1}{r}\frac{d\theta_{m-1}}{dr}+N_b\sum_{k=0}^{m-1}\frac{d\theta_{m-1-i}}{dr}\frac{d\phi_i}{dr}, \tag{3.34}$$

$$\mathcal{R}6_m(r)=N_b\left(\frac{d^2\theta_{m-1}}{dr^2}+\frac{1}{r}\frac{d\theta_{m-1}}{dr}\right)+N_t\left(\frac{d^2\phi_{m-1}}{dr^2}+\frac{1}{r}\frac{d\phi_{m-1}}{dr}\right). \tag{3.35}$$

Case-III: Variable Exponential Viscosity

For variable parabolic viscosity is defined as

$$\mu=e^r\approx 1+r+\frac{r^2}{2}+O(r^3), \tag{3.36}$$

In view of Eqs. (3.36) and (3.4) we get the nonlinear terms as following

$$\begin{aligned}\mathcal{N}_7\left[u^*(r,p),\theta^*(r,p),\phi^*(r,p)\right] = {} & (1+r)\frac{du}{dr} + \frac{(1+r+\frac{r^2}{2})}{r}\frac{du}{dr} + (1+r+\tfrac{r^2}{2})\frac{d^2u}{dr^2} + \frac{\Lambda}{r}\left(\frac{du}{dr}\right)^3 + 3\Lambda\left(\frac{du}{dr}\right)^2\frac{d^2u}{dr^2} \\ & -c - P\left((1+r+\tfrac{r^2}{2}) + \Lambda\left(\tfrac{du}{dr}\right)^2\right)u + G_r\theta + B_r\phi\end{aligned} \tag{3.37}$$

$$\mathcal{N}_8\left[u^*(r,p),\theta^*(r,p),\phi^*(r,p)\right] = (\alpha+\alpha_1 N_t)\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + N_b\frac{d\theta}{dr}\frac{d\phi}{dr}, \tag{3.38}$$

$$\mathcal{N}_9\left[u^*(r,p),\theta^*(r,p),\phi^*(r,p)\right] = N_b\left(\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr}\right) + N_t\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right). \tag{3.39}$$

mth order deformation equation:

mth order deformation problem of above equations are

$$\mathcal{L}_2\left[u_m - \chi_m u_{m-1}\right] = \hbar\mathcal{R}7_m(r), \tag{3.40}$$

$$\mathcal{L}_2\left[\theta_m - \chi_m \theta_{m-1}\right] = \hbar\mathcal{R}8_m(r), \tag{3.41}$$

$$\mathcal{L}_2\left[\phi_m - \chi_m \phi_{m-1}\right] = \hbar\mathcal{R}9_m(r), \tag{3.42}$$

$$\begin{aligned} u_m(1) = 0,&\ u(2) = 0, \\ \theta_m(1) = 0,&\ \theta_m(2) = 0, \\ \phi_m(1) = 0,&\ \phi_m(2) = 0. \end{aligned} \tag{3.43}$$

where

$$\mathcal{R}7_m(r) = \left.\begin{array}{c} (1+r+\frac{r^2}{2})\frac{d^2u_{m-1}}{dr^2} + (1+r)\frac{du_{m-1}}{dr} \\ +\frac{(1+r+\frac{r^2}{2})}{r}\frac{du_{m-1}}{dr} - P(1+r+\frac{r^2}{2})u_{m-1} \\ +\frac{\Lambda}{r}\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}\frac{du_l}{dr} \\ +3\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}\frac{d^2u_l}{dr^2} \\ +G_r\theta_{m-1} + B_r\phi_{m-1} - c(1-\chi_m) \\ -P\Lambda\sum_{k=0}^{m-1}\sum_{l=0}^{k}\left(\frac{du_{m-1-k}}{dr}\right)\frac{du_{m-k-l}}{dr}u_l \end{array}\right\}, \tag{3.44}$$

$$\mathcal{R}8_m(r) = (\alpha+\alpha_1 N_t)\frac{d^2\theta_{m-1}}{dr^2} + \frac{1}{r}\frac{d\theta_{m-1}}{dr} + N_b\sum_{k=0}^{m-1}\frac{d\theta_{m-1-i}}{dr}\frac{d\phi_i}{dr}, \tag{3.45}$$

$$\mathcal{R}9_m(r) = N_b \left(\frac{d^2\theta_{m-1}}{dr^2} + \frac{1}{r}\frac{d\theta_{m-1}}{dr} \right) + N_t \left(\frac{d^2\phi_{m-1}}{dr^2} + \frac{1}{r}\frac{d\phi_{m-1}}{dr} \right). \tag{3.46}$$

3.4 Convergence of the Solution

It is noticed that the HAM method strictly depend upon the auxiliary parameter $\hbar$. As specified by Liao [26], the convergence region and rate of approximations given by the HAM are strongly dependent upon $\hbar$. Figs. 3.1 to 3.3 potray the $\hbar-$curves of velocity profile for all three cases above and for the temperature and nanoparticles concentration profiles for all above cases are shown in Figs. 3.4 to 3.5 , just to find the range of $\hbar$ in case of constant and variable viscosity. The range for admissible values of $\hbar$ for velocity in this case of constant viscosity is $-0.3 \leq \hbar \leq 0.1$, for variable viscosity when $\mu = r$ is $-0.9 \leq \hbar \leq 0.9$ and for $\mu = 1 + r + r^2/2$ is $-0.8 \leq \hbar \leq 0.8$ similarly for temperature and concentration profile admissible values of for $\hbar$ are $-1.3 \leq \hbar \leq -0.5$ and $-2.5 \leq \hbar \leq 1$, respectively.

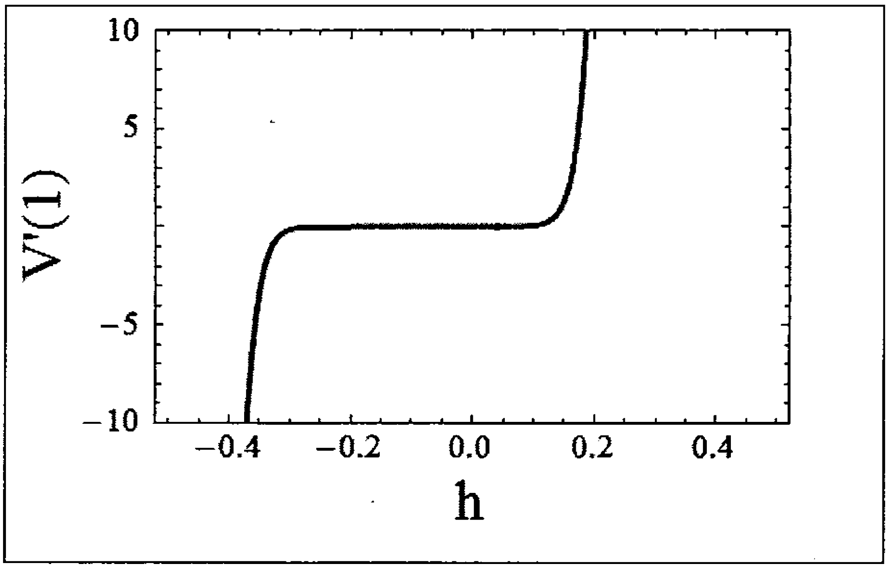

Fig. 3.1: $\hbar-$curve for velocity profile for the case 1 at $15th$ order approximation.

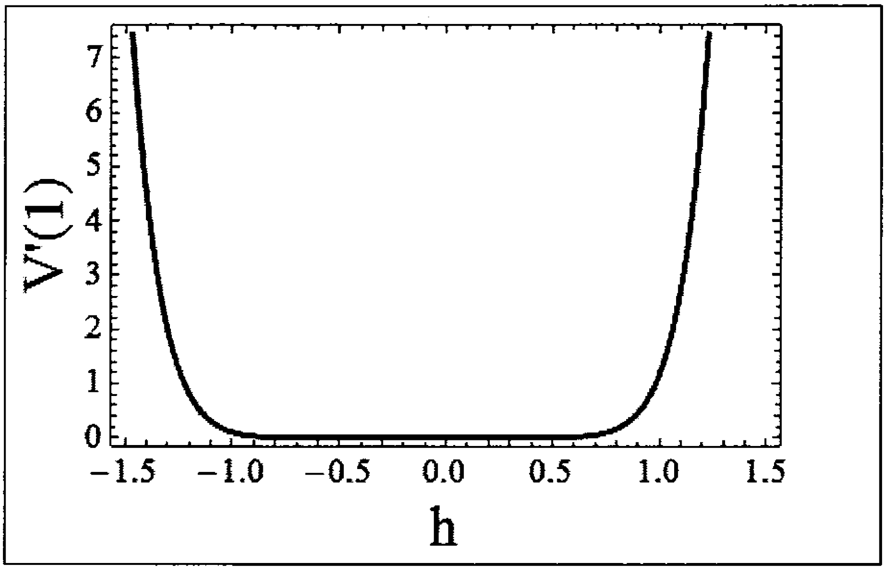

Fig. 3.2: $\hbar-$curve for velocity profile for the case 2 at $15th$ order approximation.

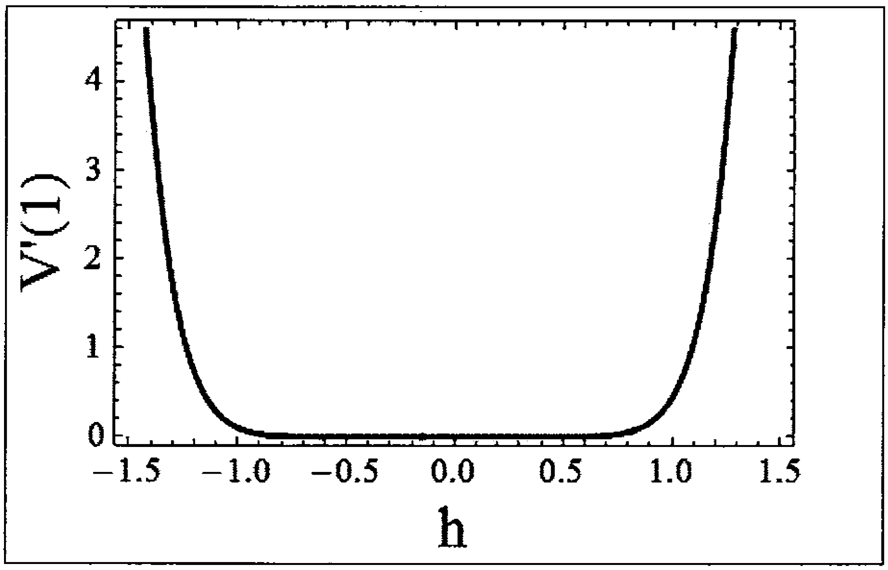

Fig. 3.3: $\hbar-$curve for velocity profile for the case 3 at $15th$ order approximation.

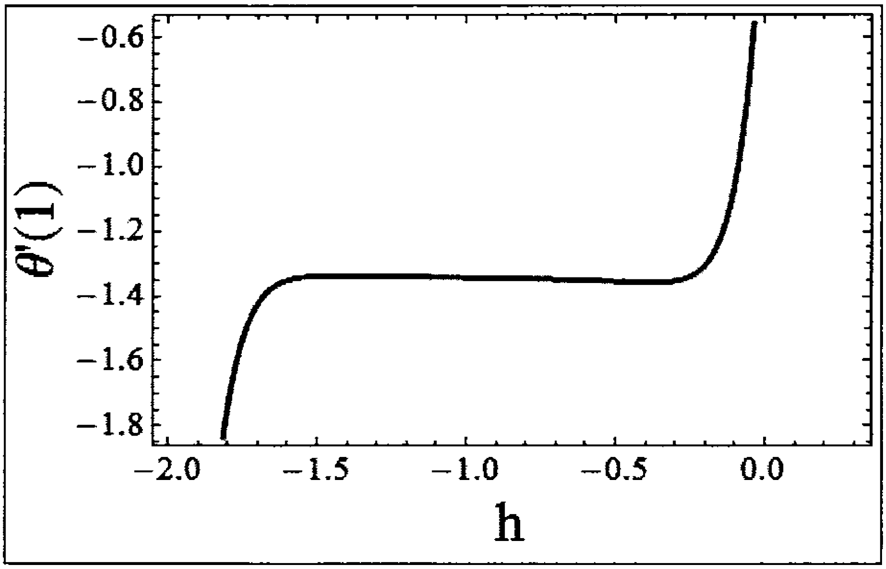

Fig. 3.4: $\hbar$−curve for temperature profile for the case 1 to case 3 at $15th$ order approximation.

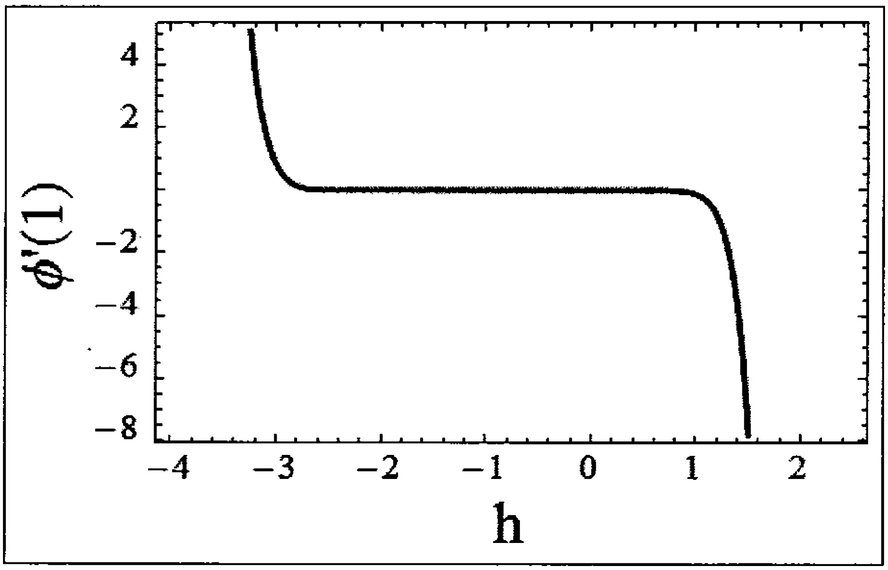

Fig. 3.5: $\hbar$−curve for nanoparticale concentration profile for the case 1 to case 3 at $15th$ order approximation.

3.5 Graphs

In this section, we will discuss the results of velocity and temperature profiles for both constant and variable viscosity with the help of graphs. It is noted that we have fixed the some parameters for instance B_r =10, $c = -1$, $G_r = 10$, $\alpha = 1$ and $\alpha_1 = 1$.

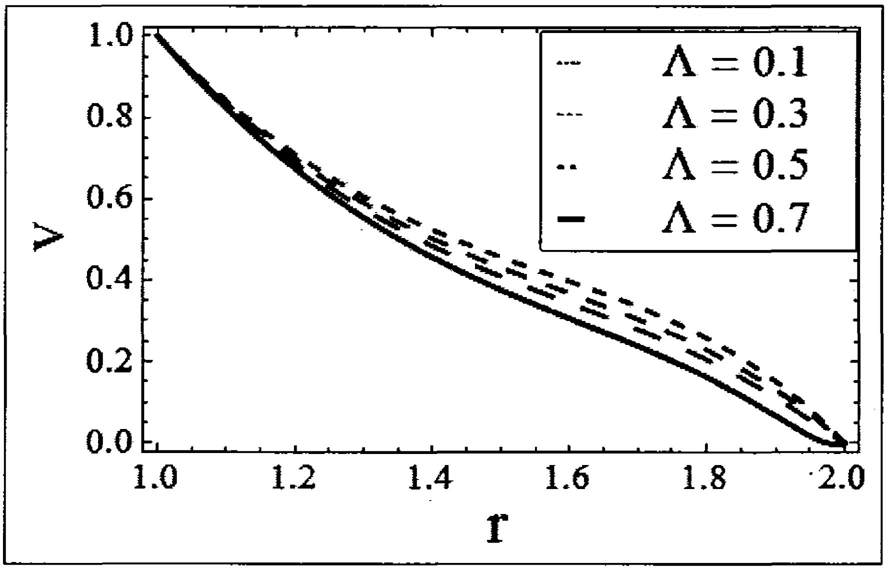

Fig. 3.6: Effect of Λ on velocity for case 1 when $P = 1$, $N_t = 1$, $N_b = 1$ and $\hbar = -0.06$.

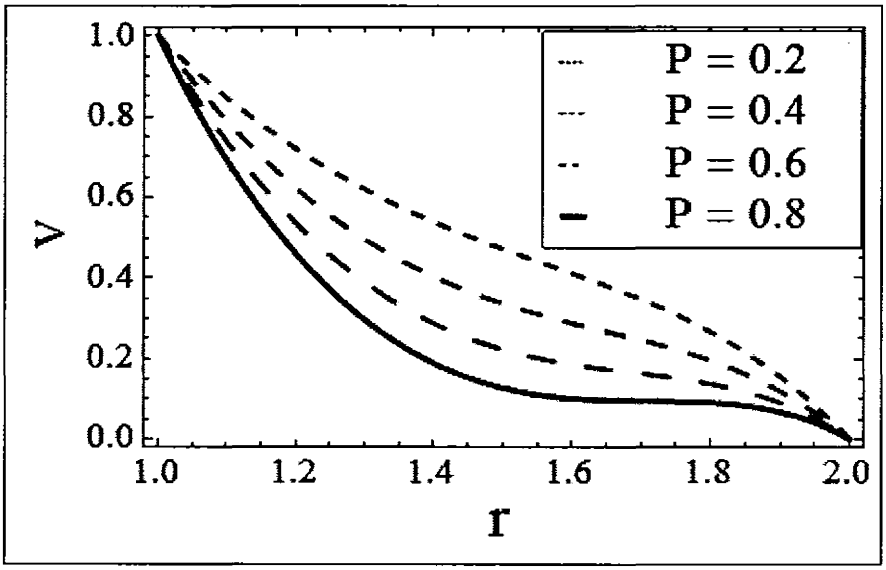

Fig. 3.7: Effect of P on velocity for case 1 when $\Lambda = 0.1$, $N_t = 1$ $N_b = 1$ and $\hbar = -0.06$.

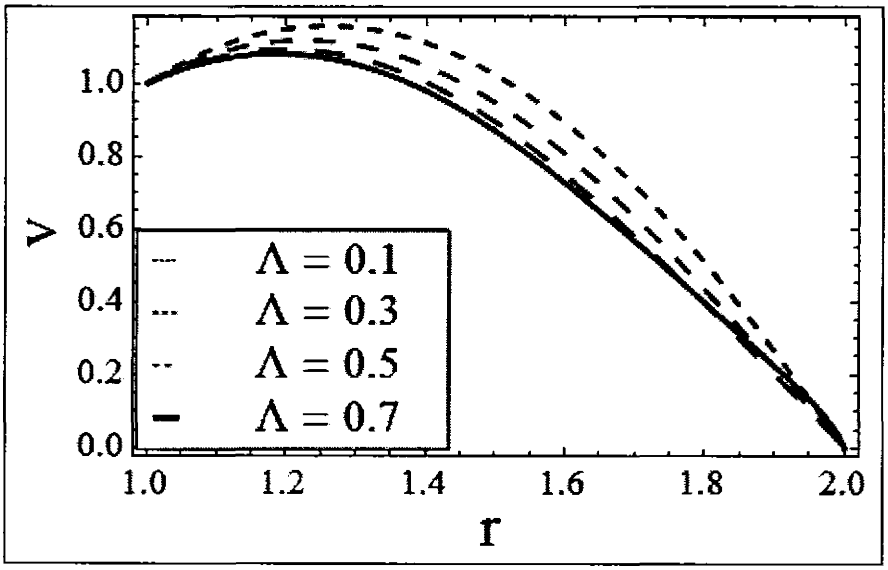

Fig. 3.8: Effect of Λ on velocity for case 2 when $P = 1$, $N_t = 1$, $N_b = 1$ and $\hbar = -0.06$.

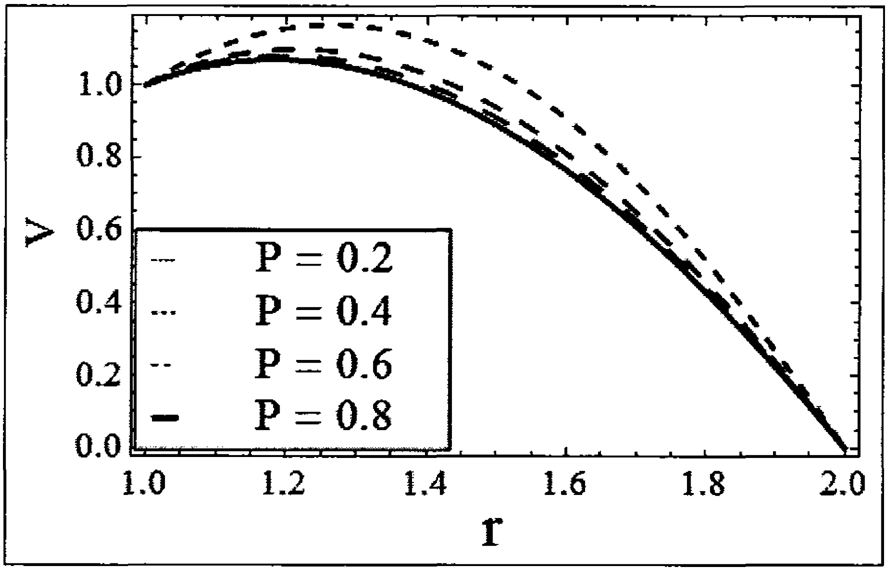

Fig. 3.9: Effect of P on velocity for case 2 when $\Lambda = 0.1$, $N_t = 1$, $N_b = 1$ and $\hbar = -0.06$.

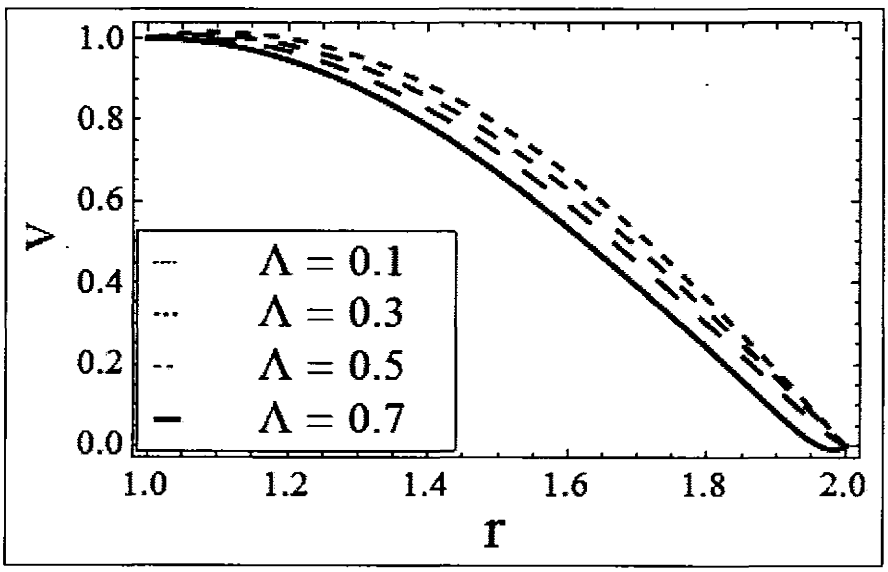

Fig. 3.10: Effect of Λ on velocity for case 3 when $P = 1$, $N_t = 1$ $N_b = 1$ and $\hbar = -0.06$.

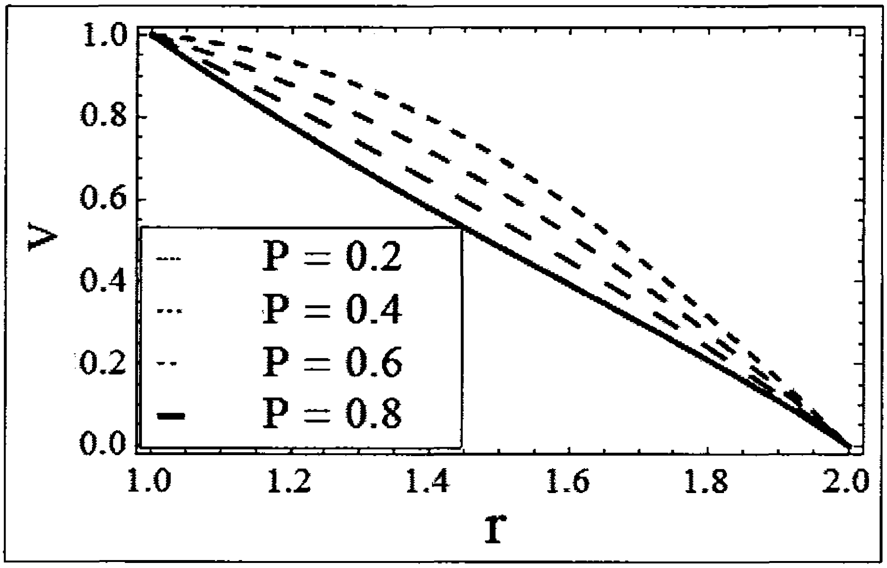

Fig. 3.11: Effect of P on velocity for case 3 when

$\Lambda = 0.1,\ N_t = 1,\ N_b = 1$ and $\hbar = -0.06$.

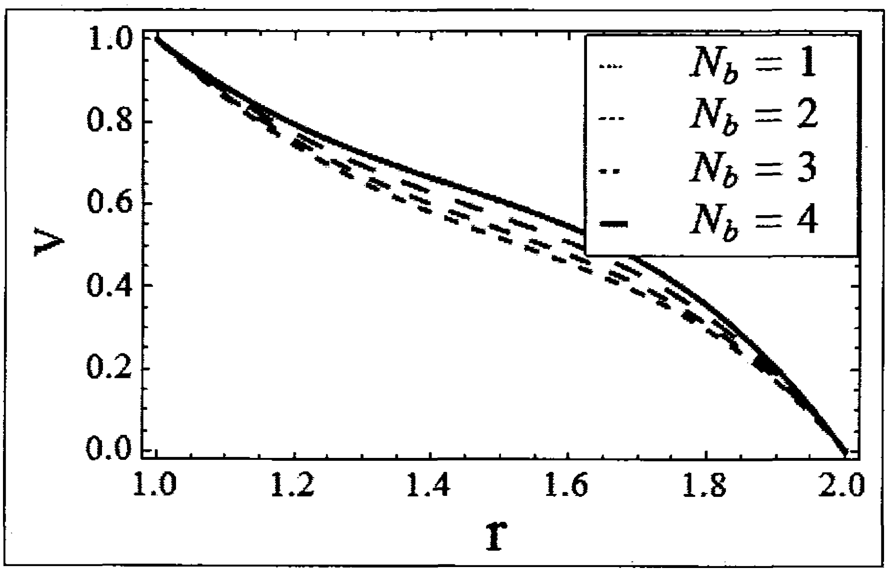

Fig. 3.12: Effect of N_b on velocity for case 1 when

$\Lambda = 0.1,\ N_t = 1,\ P = 1$ and $\hbar = -0.06$.

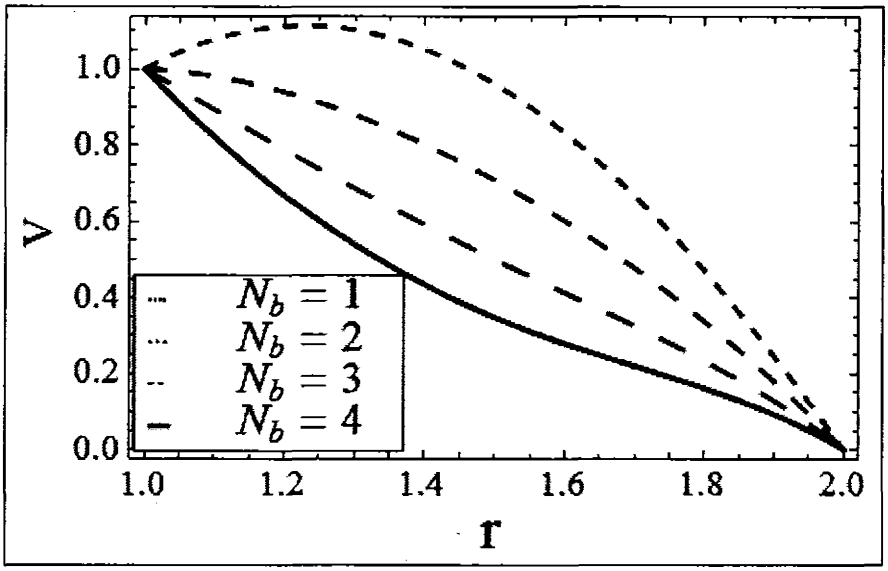

Fig. 3.13: Effect of N_b on velocity for case 2 when

$\Lambda = 0.1,\ N_t = 1,\ P = 1$ and $\hbar = -0.06.$

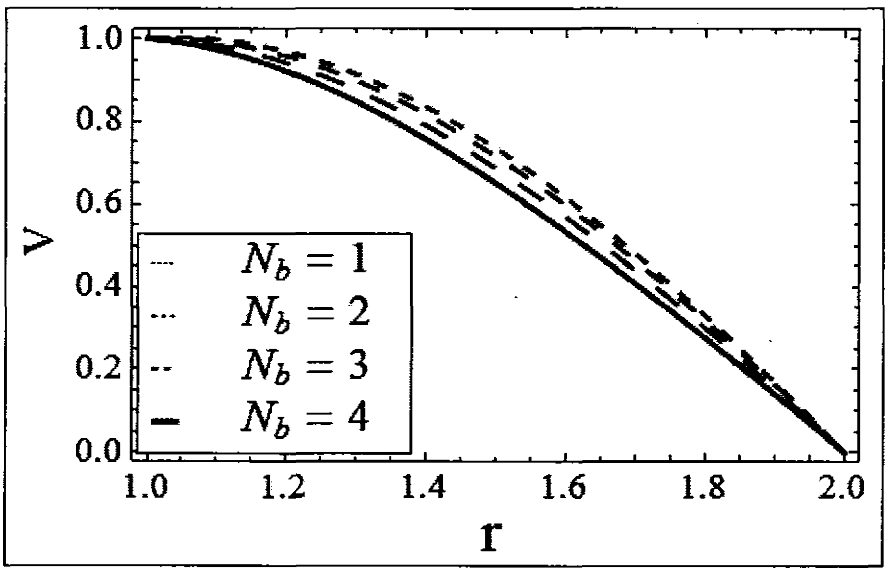

Fig. 3.14: Effect of N_b on velocity for case 3 when

$\Lambda = 0.1,\ N_t = 1,\ P = 1$ and $\hbar = -0.06.$

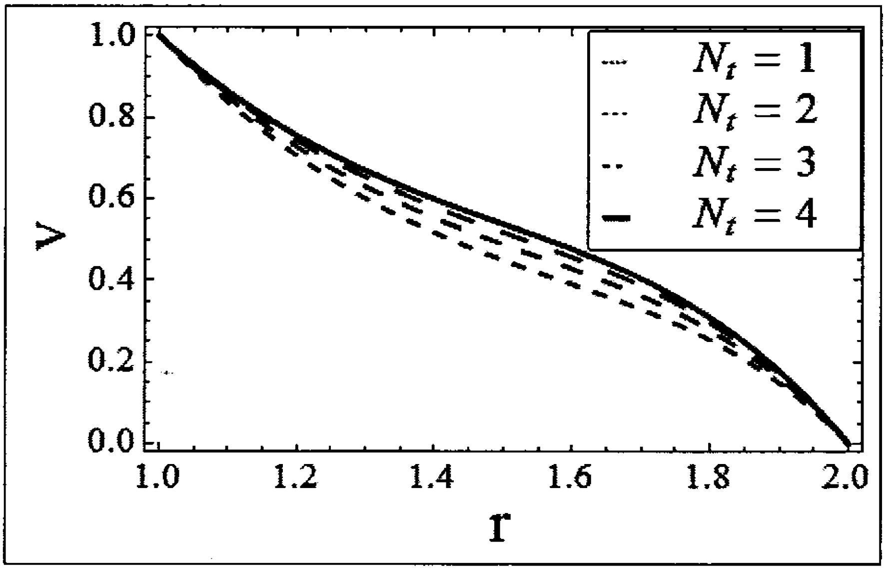

Fig. 3.15: Effect of N_t on velocity for case 1 when

$\Lambda = 0.1,\ N_b = 1,\ P = 1$ and $\hbar = -0.06$.

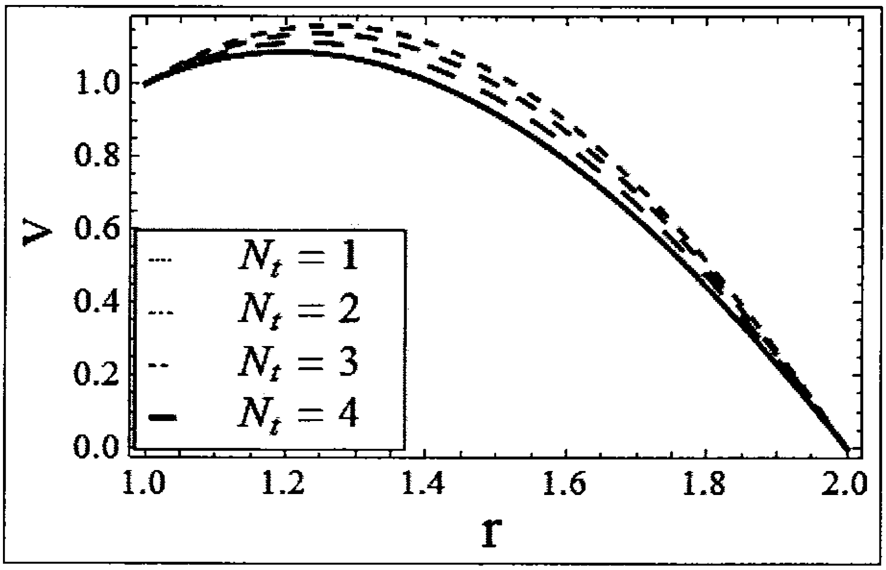

Fig. 3.16: Effect of N_t on velocity for case 2 when

$\Lambda = 0.1,\ N_b = 1,\ P = 1$ and $\hbar = -0.06$.

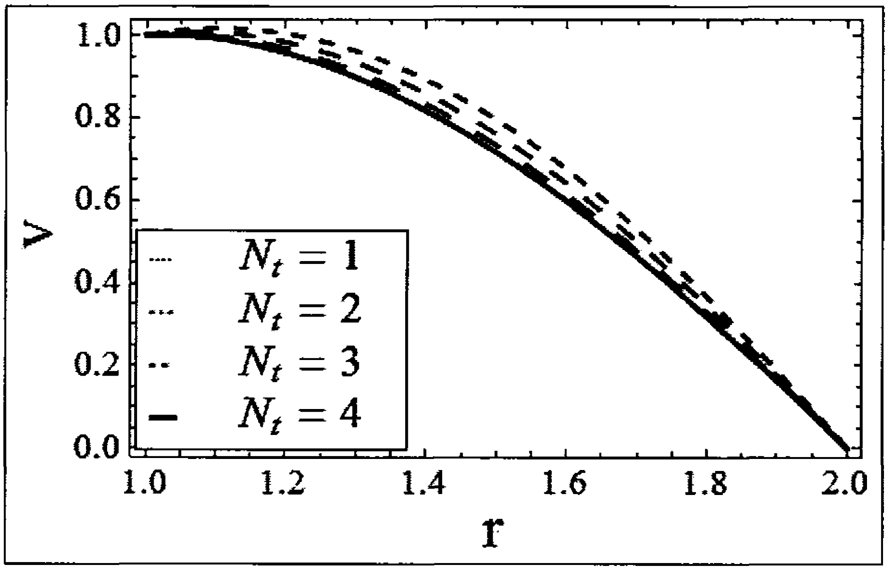

Fig. 3.17: Effect of N_t on velocity for case 3 when

$\Lambda = 0.1,\ N_b = 1,\ P = 1$ and $\hbar = -0.06$.

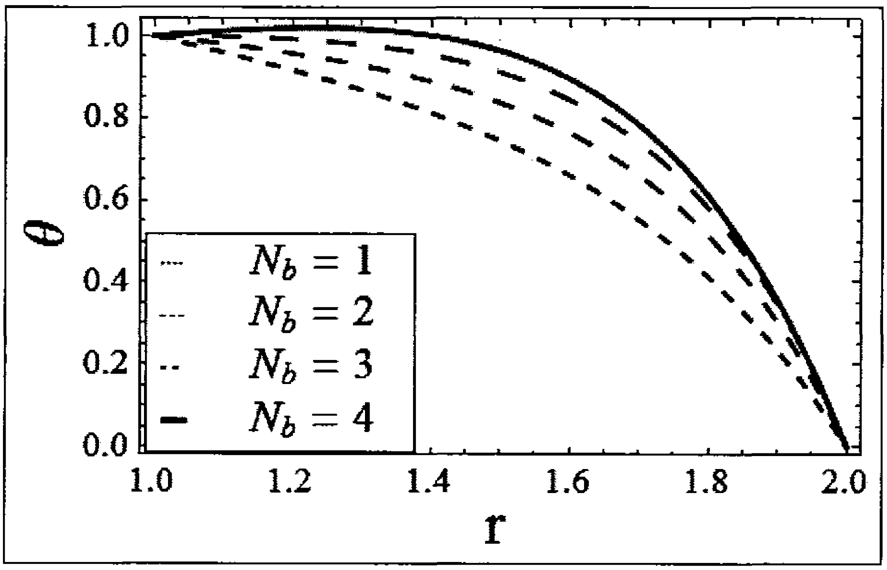

Fig. 3.18: Effect of N_b on temperature for case 1 to case 3 when

$N_t = 1$ and $\hbar = -1$.

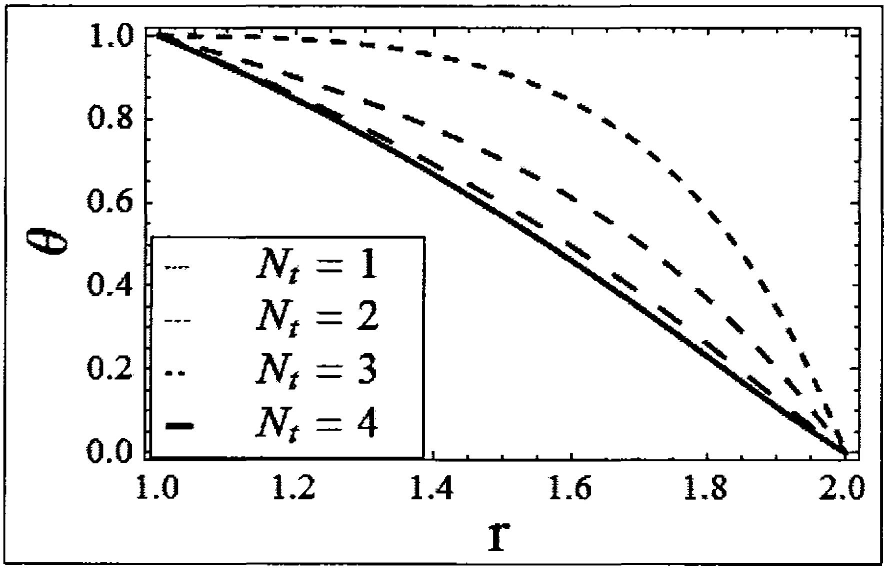

Fig. 3.19: Effect of N_t on temperature for case 1 to case 3 when $N_b = 1$ and $\hbar = -1$.

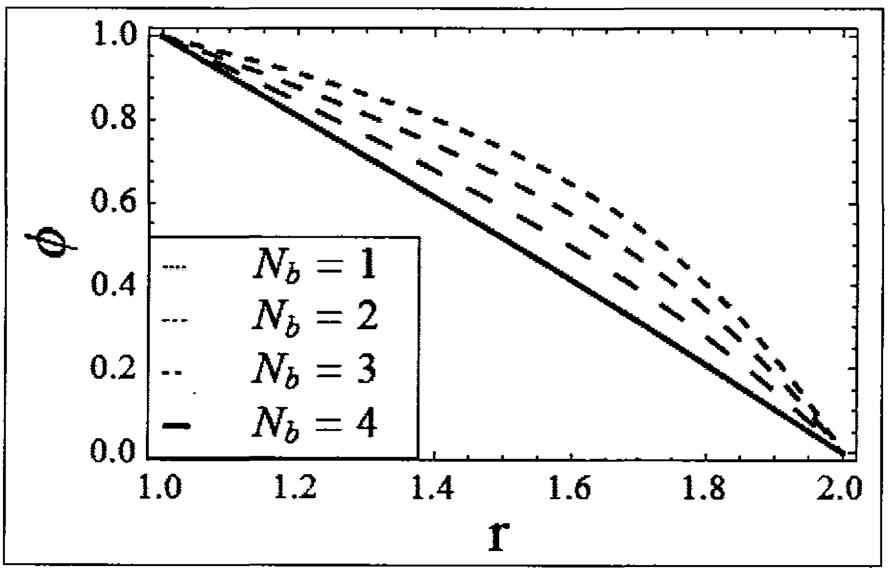

Fig. 3.20: Effect of N_b on nanoparticale concentration for case 1 to case 3 when $N_t = 1$ and $\hbar = -1$.

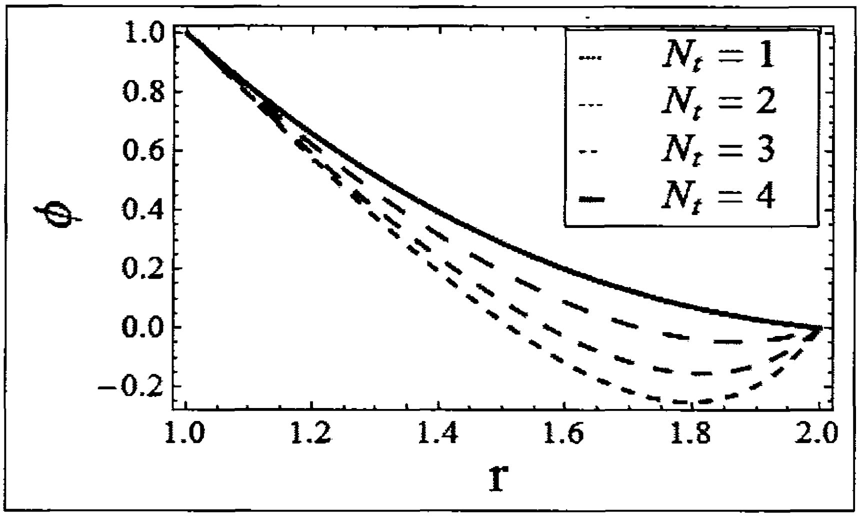

Fig. 3.21: Effect of N_t on nanoparticale concentration for case 1 to case 3 when $N_b = 1$ and $\hbar = -1$.

3.6 Discussion

In this chapter, flow of third grade nanofluid in a coaxial cylinders is examined. An analytical series solutions for velocity and temperature are obtained by Homotopy Analysis Method. Special emphasis has been focussed here to see the behavior for third grade non-Newtonion nanofluid. The behavior of the velocity, temperature and nanoparticale concentration against r are showing in Figs. 3.6 to 3.21 in order to see the variation of each of the sundry parameters. The convergence of the solution is also discussed explicitly in Figs. 3.1 to 3.5.

The following conclusions may be extracted from the graphical results.

1. In Figs. 3.6 to 3.11, it is found that the velocity decreases by increases the values of third grade and porosity parameters.

2. From Figs. 3.12 to 3.14, we seen that the velocity is decreases by increasing N_b.

3. Figs. 3.15 to 3.17, we perceived that the velocity increases by increases of N_t.

4. Figs. 3.18 to 3.21, show the influence of temperature and nanoparticale concentration for different values of N_b and N_t.

Appendix A

The related constants of calculation are given by

$$A_1 = \frac{1}{64}c^2\Gamma,$$

$$A_2 = \frac{1}{576}c^4\Gamma\Lambda + \frac{1}{96}c^4\hbar\Gamma\Lambda + \frac{1}{144}c^4\hbar^2\Gamma\Lambda,$$

$$A_3 = \frac{3}{1024}c^6\hbar\Gamma\Lambda^2 + \frac{29}{4096}c^6\hbar^2\Gamma\Lambda^2 + \frac{3}{1024}c^6\hbar^3\Gamma\Lambda^2 + \frac{1}{1024}c^6\hbar^4\Gamma\Lambda^2,$$

$$B_4 = \frac{39}{12800}c^8\hbar^2\Gamma\Lambda^3 + \frac{27}{6400}c^8\hbar^3\Gamma\Lambda^3 + \frac{3}{1600}c^8\hbar^4\Gamma\Lambda^3,$$

$$A_5 = \frac{9}{4096}c^{10}\hbar^3\Gamma\Lambda^4 + \frac{11}{4096}c^{10}\hbar^4\Gamma\Lambda^4 + \frac{1}{1024}c^{10}\hbar^5\Gamma\Lambda^4 + \frac{1}{4608}c^{10}\hbar^6\Gamma\Lambda^4,$$

$$A_6 = \frac{135}{114688}c^{12}\hbar^4\Gamma\Lambda^5 + \frac{135}{100352}c^{12}\hbar^5\Gamma\Lambda^5 + \frac{9}{14336}c^{12}\hbar^6\Gamma\Lambda^5 + \frac{3}{25088}c^{12}\hbar^7\Gamma\Lambda^5 + \frac{1}{50176}c^{12}\hbar^8\Gamma\Lambda^5,$$

$$A_7 = \frac{243}{524288}c^{14}\hbar^5\Gamma\Lambda^6 + \frac{135}{262144}c^{14}\hbar^6\Gamma\Lambda^6 + \frac{27}{131072}c^{14}\hbar^7\Gamma\Lambda^6 + \frac{3}{65536}c^{14}\hbar^8\Gamma\Lambda^6,$$

$$A_8 = \frac{13}{98304}c^{16}\hbar^6\Gamma\Lambda^7 + \frac{1}{8192}c^{16}\hbar^7\Gamma\Lambda^7 + \frac{1}{24576}c^{16}\hbar^8\Gamma\Lambda^7,$$

$$A_9 = \frac{81}{3276800}c^{18}\hbar^7\Gamma\Lambda^8 + \frac{27}{1638400}c^{18}\hbar^8\Gamma\Lambda^8,$$

$$A_{10} = \frac{81}{31719424}c^{20}\hbar^8\Gamma\Lambda^9,$$

$$A_{11} = \frac{1}{4}c^4\hbar^4\Lambda\Gamma + \frac{1}{2}c^4\hbar^5\Lambda\Gamma + \frac{3}{8}c^4\hbar^6\Lambda\Gamma + \frac{1}{8}c^4\hbar^7\Lambda\Gamma + \frac{1}{64}c^4\hbar^8\Lambda\Gamma,$$

$$\begin{aligned} A_{12} \;=\;& \frac{1}{9}c^2\hbar^2\Gamma + \frac{1}{9}c^2\hbar^3\Gamma + \frac{1}{36}c^2\hbar^4\Gamma - \frac{4}{27}c^4\hbar^3\Lambda\Gamma - \frac{14}{27}c^4\hbar^4\Lambda\Gamma - \frac{19}{27}c^4\hbar^5\Lambda\Gamma - \frac{25}{54}c^4\hbar^6\Lambda\Gamma - \\ & \frac{4}{27}c^4\hbar^7\Lambda\Gamma - \frac{1}{54}c^4\hbar^8\Lambda\Gamma + \frac{2}{27}c^6\hbar^5\Lambda^2\Gamma + \frac{1}{9}c^6\hbar^6\Lambda^2\Gamma + \frac{1}{18}c^6\hbar^7\Lambda^2\Gamma + \frac{1}{108}c^6\hbar^8\Lambda^2\Gamma, \end{aligned}$$

$$\begin{aligned} A_{13} \;=\;& -\frac{1}{24}c^2\hbar\Gamma - \frac{5}{48}c^2\hbar^2\Gamma - \frac{1}{12}c^2\hbar^3\Gamma - \frac{1}{48}c^2\hbar^4\Gamma + \frac{1}{24}c^4\hbar^2\Lambda\Gamma + \frac{11}{48}c^4\hbar^3\Lambda\Gamma + \frac{7}{16}c^4\hbar^4\Lambda\Gamma + \\ & \frac{11}{24}c^4\hbar^5\Lambda\Gamma + \frac{13}{48}c^4\hbar^6\Lambda\Gamma + \frac{1}{12}c^4\hbar^7\Lambda\Gamma + \frac{1}{96}c^4\hbar^8\Lambda\Gamma - \frac{13}{216}c^6\hbar^4\Lambda^2\Gamma - \frac{41}{216}c^6\hbar^5\Lambda^2\Gamma - \\ & \frac{59}{288}c^6\hbar^6\Lambda^2\Gamma - \frac{5}{54}c^6\hbar^7\Lambda^2\Gamma - \frac{13}{864}c^6\hbar^8\Lambda^2\Gamma + \frac{1}{96}c^8\hbar^6\Lambda^3\Gamma + \frac{1}{96}c^8\hbar^7\Lambda^3\Gamma + \frac{1}{384}c^8\hbar^8\Lambda^3\Gamma, \end{aligned}$$

$$
\begin{aligned}
A_{14} = {} & \frac{1}{225}c^2\Gamma + \frac{4}{225}c^2\hbar\Gamma + \frac{2}{75}c^2\hbar^2\Gamma + \frac{4}{225}c^2\hbar^3\Gamma + \frac{1}{225}c^2\hbar^4\Gamma - \frac{4}{675}c^4\hbar\Lambda\Gamma - \frac{11}{225}c^4\hbar^2\Lambda\Gamma - \\
& \frac{88}{675}c^4\hbar^3\Lambda\Gamma - \frac{13}{75}c^4\hbar^4\Lambda\Gamma - \frac{4}{27}c^4\hbar^5\Lambda\Gamma - \frac{2}{25}c^4\hbar^6\Lambda\Gamma - \frac{16}{675}c^4\hbar^7\Lambda\Gamma - \frac{2}{675}c^4\hbar^8\Lambda\Gamma + \\
& \frac{14}{675}c^6\hbar^3\Lambda^2\Gamma + \frac{271}{2700}c^6\hbar^4\Lambda^2\Gamma + \frac{122}{675}c^6\hbar^5\Lambda^2\Gamma + \frac{106}{675}c^6\hbar^6\Lambda^2\Gamma + \frac{44}{675}c^6\hbar^7\Lambda^2\Gamma + \\
& \frac{7}{675}c^6\hbar^8\Lambda^2\Gamma - \frac{1}{81}c^8\hbar^5\Lambda^3\Gamma - \frac{43}{1350}c^8\hbar^6\Lambda^3\Gamma - \frac{16}{675}c^8\hbar^7\Lambda^3\Gamma - \frac{11}{2025}c^8\hbar^8\Lambda^3\Gamma + \\
& \frac{1}{1350}c^{10}\hbar^7\Lambda^4\Gamma + \frac{1}{2700}c^{10}\hbar^8\Lambda^4\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
A_{15} = {} & \frac{1}{2916}c^4\Lambda\Gamma + \frac{1}{243}c^4\hbar\Lambda\Gamma + \frac{11}{729}c^4\hbar^2\Lambda\Gamma + \frac{19}{729}c^4\hbar^3\Lambda\Gamma + \frac{13}{486}c^4\hbar^4\Lambda\Gamma + \frac{14}{729}c^4\hbar^5\Lambda\Gamma + \\
& \frac{7}{729}c^4\hbar^6\Lambda\Gamma + \frac{2}{729}c^4\hbar^7\Lambda\Gamma + \frac{1}{2916}c^4\hbar^8\Lambda\Gamma - \frac{5}{1458}c^6\hbar^2\Lambda^2\Gamma - \frac{17}{729}c^6\hbar^3\Lambda^2\Gamma - \\
& \frac{19}{324}c^6\hbar^4\Lambda^2\Gamma - \frac{56}{729}c^6\hbar^5\Lambda^2\Gamma - \frac{83}{1458}c^6\hbar^6\Lambda^2\Gamma - \frac{16}{729}c^6\hbar^7\Lambda^2\Gamma - \frac{5}{1458}c^6\hbar^8\Lambda^2\Gamma + \\
& \frac{97}{17496}c^8\hbar^4\Lambda^3\Gamma + \frac{101}{4374}c^8\hbar^5\Lambda^3\Gamma + \frac{25}{729}c^8\hbar^6\Lambda^3\Gamma + \frac{46}{2187}c^8\hbar^7\Lambda^3\Gamma + \frac{79}{17496}c^8\hbar^8\Lambda^3\Gamma - \\
& \frac{1}{648}c^{10}\hbar^6\Lambda^4\Gamma - \frac{2}{729}c^{10}\hbar^7\Lambda^4\Gamma - \frac{1}{972}c^{10}\hbar^8\Lambda^4\Gamma + \frac{1}{46656}c^{12}\hbar^8\Lambda^5\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
A_{16} = {} & \frac{8}{35721}c^6\hbar\Lambda^2\Gamma + \frac{74}{35721}c^6\hbar^2\Lambda^2\Gamma + \frac{244}{35721}c^6\hbar^3\Lambda^2\Gamma + \frac{422}{35721}c^6\hbar^4\Lambda^2\Gamma + \frac{440}{35721}c^6\hbar^5\Lambda^2\Gamma + \\
& \frac{32}{3969}c^6\hbar^6\Lambda^2\Gamma + \frac{104}{35721}c^6\hbar^7\Lambda^2\Gamma + \frac{16}{35721}c^6\hbar^8\Lambda^2\Gamma - \frac{40}{35721}c^8\hbar^3\Lambda^3\Gamma - \frac{233}{35721}c^8\hbar^4\Lambda^3\Gamma - \\
& \frac{520}{35721}c^8\hbar^5\Lambda^3\Gamma - \frac{572}{35721}c^8\hbar^6\Lambda^3\Gamma - \frac{304}{35721}c^8\hbar^7\Lambda^3\Gamma - \frac{62}{35721}c^8\hbar^8\Lambda^3\Gamma + \frac{34}{35721}c^{10}\hbar^5\Lambda^4\Gamma + \\
& \frac{38}{11907}c^{10}\hbar^6\Lambda^4\Gamma + \frac{40}{11907}c^{10}\hbar^7\Lambda^4\Gamma + \frac{38}{35721}c^{10}\hbar^8\Lambda^4\Gamma - \frac{4}{35721}c^{12}\hbar^7\Lambda^5\Gamma - \frac{1}{10206}c^{12}\hbar^8\Lambda^5\Gamma,
\end{aligned}
$$

$$
\begin{aligned}
A_{17} = {} & \frac{1}{11664}c^8\hbar^2\Lambda^3\Gamma + \frac{23}{34992}c^8\hbar^3\Lambda^3\Gamma + \frac{23}{11664}c^8\hbar^4\Lambda^3\Gamma + \frac{1}{324}c^8\hbar^5\Lambda^3\Gamma + \frac{97}{34992}c^8\hbar^6\Lambda^3\Gamma + \\
& \frac{23}{17496}c^8\hbar^7\Lambda^3\Gamma + \frac{1}{3888}c^8\hbar^8\Lambda^3\Gamma - \frac{151}{629856}c^{10}\hbar^4\Lambda^4\Gamma - \frac{47}{39366}c^{10}\hbar^5\Lambda^4\Gamma - \frac{229}{104976}c^{10}\hbar^6\Lambda^4\Gamma - \\
& \frac{34}{19683}c^{10}\hbar^7\Lambda^4\Gamma - \frac{307}{629856}c^{10}\hbar^8\Lambda^4\Gamma + \frac{5}{46656}c^{12}\hbar^6\Lambda^5\Gamma + \frac{1}{3888}c^{12}\hbar^7\Lambda^5\Gamma + \frac{13}{93312}c^{12}\hbar^8\Lambda^5\Gamma - \\
& \frac{1}{279936}c^{14}\hbar^8\Lambda^6\Gamma,
\end{aligned}
$$

$$A_{18} = \frac{104}{4782969}c^{10}\hbar^3\Lambda^4\Gamma + \frac{697}{4782969}c^{10}\hbar^4\Lambda^4\Gamma + \frac{1808}{4782969}c^{10}\hbar^5\Lambda^4\Gamma + \frac{2368}{4782969}c^{10}\hbar^6\Lambda^4\Gamma + \frac{1544}{4782969}c^{10}\hbar^7\Lambda^4\Gamma + \frac{400}{4782969}c^{10}\hbar^8\Lambda^4\Gamma - \frac{172}{4782969}c^{12}\hbar^5\Lambda^5\Gamma - \frac{230}{1594323}c^{12}\hbar^6\Lambda^5\Gamma - \frac{304}{1594323}c^{12}\hbar^7\Lambda^5\Gamma - \frac{386}{4782969}c^{12}\hbar^8\Lambda^5\Gamma + \frac{4}{531441}c^{14}\hbar^7\Lambda^6\Gamma + \frac{5}{531441}c^{14}\hbar^8\Lambda^6\Gamma,$$

$$A_{19} = \frac{107}{26572050}c^{12}\hbar^4\Lambda^5\Gamma + \frac{302}{13286025}c^{12}\hbar^5\Lambda^5\Gamma + \frac{43}{885735}c^{12}\hbar^6\Lambda^5\Gamma + \frac{614}{13286025}c^{12}\hbar^7\Lambda^5\Gamma + \frac{443}{26572050}c^{12}\hbar^8\Lambda^5\Gamma - \frac{11}{2952450}c^{14}\hbar^6\Lambda^6\Gamma - \frac{16}{1476225}c^{14}\hbar^7\Lambda^6\Gamma - \frac{23}{2952450}c^{14}\hbar^8\Lambda^6\Gamma + \frac{1}{3936600}c^{16}\hbar^8\Lambda^7\Gamma,$$

$$A_{20} = \frac{104}{192913083}c^{14}\hbar^5\Lambda^6\Gamma + \frac{160}{64304361}c^{14}\hbar^6\Lambda^6\Gamma + \frac{248}{64304361}c^{14}\hbar^7\Lambda^6\Gamma + \frac{400}{192913083}c^{14}\hbar^8\Lambda^6\Gamma - \frac{16}{64304361}c^{16}\hbar^7\Lambda^7\Gamma - \frac{26}{64304361}c^{16}\hbar^8\Lambda^7\Gamma,$$

$$A_{21} = \frac{1}{19131876}c^{16}\hbar^6\Lambda^7\Gamma + \frac{5}{28697814}c^{16}\hbar^7\Lambda^7\Gamma + \frac{1}{6377292}c^{16}\hbar^8\Lambda^7\Gamma - \frac{1}{114791256}c^{18}\hbar^8\Lambda^8\Gamma,$$

$$A_{22} = \frac{8}{2424965283}c^{18}\hbar^7\Lambda^8\Gamma + \frac{16}{2424965283}c^{18}\hbar^8\Lambda^8\Gamma,$$

$$A_{23} = \frac{1}{8437157316}c^{20}\hbar^8\Lambda^9\Gamma.$$

3.7 References

1. Ellahi, R., "Steady and unsteady flow for Newtonian and non-Newtonian fluid", Basics concepts and methods VDM Germany, 2009.

2. Tan, W. C. and Masuoka, T., "Stokes first problem for second grade fluid in a porous half space with in heated boundary" , Int. J. Non-Linear Mech. 40; 512-522, 2005.

3. Fetecau, C. and Fetecau, C., "Starting solutions for the motion of second grade fluid due to longitudinal and torsional oscillations of a circular cylinder", Int. J. Eng. Sci. 44; 788-796, 2006

4. Hameed, M. and Nadeem, S., "Unsteady MHD flow of non-Newtonian fluid on porous plate", Journ. of Math. Anal. and Appli. 325; 724-733, 2007

5. Choi, S. U. S., "Enhancing thermal conductivity of fluids with nanoparticles", the proceedings of the a995 ASME International Mechanical Engineering Congress and Exposition, San Francisco, 66; 99-105, 1995.

6. Massoudi, M. and Christie, I., "Effects of variable viscosity and viscous dissipation on the flow of a third grade fluid in a pipe", Int. J. Non-Linear Mech., 30; 687-699, 1995.

7. Buongiorno, J., Hu, L. W., Rubner, M., Cohen, R., Forrest, E. and Williamson, E., "Augmentation of Nucleate Boiling Heat Transfer and Critical Heat Flux Using Nanoparticle Thin-Film Coatings", Int. J. Heat Mass Transfer, 53, 58–67, 2010.

8. Das, S. K. and Choi, S. U. S., "Heat Transfer in Nanofluids", Advances in Heat Transfer, 41; 81-191, 2009.

9. Strandberg, R. T. and Das, D. K., "Heat Transfer Performance of Nanofluids", Applications in Facility Heating, LAP Lambert Academic Publishing, ISBN:978-3-8383-9842-6; 136, 2010.

10. Feng,Y., Yu, B., Peng, X. and Zou, M., "The effective thermal conductivity of nanofluids based on the nanolayer and the aggregation of nanoparticles", J. Phys. D 40; 3164-3171, 2007.

11. Xu, J., Yu, B., Zou, M. and Peng, X., "A new model for heat conduction of nanofluids based on fractal distributions of nanoparticles", J. Phys. D 39; 4486-4490, 2006.

12. Peng, X., Wu, Z., Mujumdar, A. S. and Yu, B., "Innovative hydrocyclone inlet designs to reduce erosion-induced wear in mineral dewater processes", Drying Technology 27; 201-211, 2009.

13. Lotfi, R., Saboohi, Y. and Rashidi, A. M. "Numerical study of forced convective heat transfer of Nanofluids: Comparison of different approach", Int. Comm. in Heat and Mass Transfer, 37; 74-78, 2010.

14. Matthews, M. T. and Hill, J. M. "Nanofluidics and the Navier boundary condition", Int. J. Nanotechnol., 5; 218–242, 2008.

15. Bujurke, N. M., Patail, H. P. and Bhavi, S. G. "Porous slider bearing with couple stress fluid", Acta Mechanica, 85; 99-113, 2005.

16. Ahmad, S. and Pop, I., "Mixed convection boundary layer flow from a vertical flat plate embedded in a porous medium filled with nanofluids", Int. Comm. in Heat and Mass Transfer, 37; 987-991, 2010.

17. Andrew, D., Rees, S., and Pop, I. "The effect of large-amplitude g-jitter vertical free convection boundary-layer flow in porous media", International Journal of Heat and Mass Transfer, 46; 1097-1102, 2003.

18. Tan, W. C. and Masuoka, T., "Stability analysis of a Maxwell fluid in a porous medium heated from below", Physics Letters A 360; 454-460, 2007.

19. Hayat, T., Ellahi, R. and Asghar, S. "The influence of variable viscosity and viscous dissipation on the non-Newtonian flow: An analytic solution", Communication in Nonlinear science and Numerical Simulation, 12; 300-313, 2007.

20. Hilton, P. J., "An introduction to homotopy theory", (Cambridge University Press), 1953.

21. Hayat, T., Ellahi, R., Ariel, P. D. and Asghar, S., "Homotopy solution for the channel flow of a third grade fluid", Nonlinear Dynamics, 45; 55-64, 2006.

22. Abbasbandy, S., "The application of homotopy analysis method to non linear equations arising in heat transfer", Phys. Lett., A, 360; 109-113, 2010.

23. Liao, S. J., "The proposed homotopy analysis technique for the solution of nonlinear problems", PhD thesis, Shanghai Jiao Tong University, 1992.

24. Fosdick, R. L. and Rajagopal, K. R., "Thermodynamics and stability of fluids of third grade", Proc Roy Soc London, 77; 339-351, 1980.

25. Van Gorder, R. A. and Vajravelu, K., "On the selection of auxiliary functions, operator and convergence control parameter in the application of Homotopy Analysis Method to nonlinear differential equations", A general approach. Commun. Nonlinear Sci. Numer. Simul. 14; 4078-4089, 2009.

26. Liao, S. J., "An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude", Int. J. Non-Linear Mech. 38; 1173-1183, 2003.