Best Proximity Points for Some Classes of Nonlinear Operators

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***DOCTOR OF PHILOSOPHY IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD.

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Khalil Javed

List of Publications

The list of research articles, deduced from the work presented in this thesis, published/accepted in the international journals (ESCI & SCI) is given below.

1. **K. Javed**, M. Nazam, O. Acar, M. Arshad, On the best proximity point theorems for interpolative proximal contractions with applications, **Filomat** (2025), 39:8, 2817-2830. https://doi.org/10.2298/FIL2508817J.
2. **K. Javed**, M. Nazam, M. Arshad, M. De La Sen, Applications and Theoretical Foundations of Best Proximity Points in Generalized Interpolative Proximal Contractions, **European Journal of Pure and Applied Mathematics** (2025), Vol.18, issue 14, https://doi.org/10.29020/nybg.ejpam.v18i4.6841.
3. **K. Javed**, M. Nazam, F. Jahangeer, M. Arshad, M. De La Sen, A new approach to generalized interpolative proximal contraction-archimedean fuzzy metric spaces. **AIMS Mathematics** 8, no. 2 (2023): 2891-2909. https://doi.org/10.3934/math.2023151.
4. **K. Javed**, M. Lashin, M. Nazam, H. Sulami, A. Hussain and M. Arshad, Best Proximity Point Theorems for the Generalized Fuzzy Interpolative Proximal Contractions, **Fractal and Fractional**, (2022), 8(7), 455. https://doi.org/10.3390/fractalfract6080455.

Contents

Introduction

Best proximity points have widespread applications in optimization, economics, and various engineering disciplines, where exact fixed points are elusive, and optimal approximations are sought. Future research may extend these concepts to more complex structures, such as partial metric spaces or ordered metric spaces, broadening the scope and applicability of these results. In optimization and fixed point theory, best proximity points are crucial when dealing with non-self mappings where fixed points do not exist. The classical Banach contraction principle has seen various extensions to accommodate different contractions and more general settings. This note focuses on a specific generalization: generalized interpolative proximal contractions.

BPP theory is an area of mathematical analysis and optimization that focuses on finding points in one set that are closest to points in another set when a contractive mapping is involved. In metric fixed point theory, the concept of best proximity points plays a crucial role, particularly when dealing with mappings that do not necessarily have fixed points. This note delves into the best proximity points for generalized interpolative proximal contractions, an important class of mappings in metric spaces.

A metric space approach is a crucial technique in many mathematics branches and especial in fixed point theory. Recently, different development of a metric spaces have been developed. Fixed point theory is well-known and established concept in mathematical field and also a large firm of utilization. Banach fixed point theorem is the most crucial development in the study of presence and identification of solution of non-linear problems arising in mathematics and its applications to enginering and life sciences. Fixed point results is offensive on the presence and identification of the explanation of a easy equation $p(\varsigma) = \varsigma$ and mentioned mapping p is a self-mapping. Consequently, "fixed point theory" is taken into account in the concrete solution of such equations. The best proximity point becomes a fixed point when the mapping in question is a self-mapping. Analyzing various proximal contractions [1, 2, 3, 4, 5] reveals the bpp. The product of distances with exponents that satisfy a few conditions constitutes the basis of the interpolative contraction principles. The well-known mathematician Erdal Karapinar coined the term "interpolative contraction" in his 2018 paper [6]. The interpolative contraction is defined as follows:

A map S defined on a m.s $(\Omega, \eth^d)$ is refered to as interpolative contr., if $\nu \in (0,1]$, $K \in [0,1)$ s.t

$$\eth^d(Se, Sr) \leq K\left(\eth^d(e,r)\right)^{\nu}, \forall e, r \in \Omega.$$

Note $\nu = 1$, S is a BC. If S defined on a MS $(\Omega, \eth^d)$ satisfies:

$$\begin{aligned}
\eth^d(S\varsigma^{kj}, Sr) &\leq K\left(\eth^d(\varsigma^{kj}, S\varsigma^{kj})\right)^{\nu}\left(\eth^d(r, Sr)\right)^{1-\nu}, \\
\eth^d(S\varsigma^{kj}, Sr) &\leq K\left(\eth^d(r, S\varsigma^{kj})\right)^{\nu}\left(\eth^d(\varsigma^{kj}, Sr)\right)^{1-\nu}, \\
\eth^d(S\varsigma^{kj}, Sr) &\leq K\left(\eth^d(\varsigma^{kj}, r)\right)^{\eta}\left(\eth^d(\varsigma^{kj}, S\varsigma^{kj})\right)^{\nu}\left(\eth^d(r, Sr)\right)^{1-\nu-\eta}, \ \nu+\eta<1 \\
\eth^d(S\varsigma^{kj}, Sr) &\leq K\left(\eth^d(\varsigma^{kj}, r)\right)^{\nu}\left(\eth^d(\varsigma^{kj}, S\varsigma^{kj})\right)^{\eta}\left(\eth^d(r, Sr)\right)^{\gamma}\left(\frac{1}{2}(\eth^d(\varsigma^{kj}, Sr)+\eth^d(r, S\varsigma^{kj}))\right)^{1-\eta-\nu-\gamma},
\end{aligned}$$

for all $\varsigma^{kj}, r \in \otimes$, then S is called an IKTC, an ICTC, an IĆRRTC and an IHRTC, respectively. Interpolation has been used to revisit several classical and advanced contractions (see [7, 8, 9, 10]).

Altun *et al.* [11], recently defined interpolative proximal contractions and reviewed all interpolative contractions. On such contractions, they presented the best proximity theorems. This aims to establish the best proximity point theorems for interpolative proximal contractions in the case of non-self mappings.

Gabeleh [13] demonstrated the existence and uniqueness of a best proximity point for weak proximal contractions and introduced a new class of non-self mappings. Some utilization of best proximity points has been discussed in (see [14, 15, 16]).

Proinov [10](2020) provided a number of fixed-point theorems that expanded on earlier work in [5]. First, Karapinar introduced the idea of interpolation contraction in his work [9] published in 2018, then Proinov gave the second idea in his paper [10] published in 2020. Recently, Altun and Taşdemir [23] have utilized the interpolative proximal contraction to produce some best proximity point theorems.

Finding an element ς in R that is as close to $S(\varsigma)$ in G as possible, is of great interest, since a non-self mapping need not have a fixed point. In other words, it is considered to find an approximation solution ς in R such that the error $\eth^d(\varsigma, S(\varsigma))$ is smallest, where φ is the distance function, if the fixed point equation $S(\varsigma) = \varsigma$ has no exact solution. In fact, best proximity point theorems look into the possibility of such best proximity point for approximate solutions to the fixed point equation $S(\varsigma) = \varsigma$ in the absence of a precise solution.

Chapter 1 contains the basic concepts and introduction of best proximity point theory. It defines important and fundamental notions of bpp and fbpp.

Chapter 2 seeks to provide bppt for contractive non-self mappings using interpolation, leading to global optimal approximate solutions to specific fixed point equations. Iterative strategies are also provided to find such ideal approximative proving the presence of bpp. Also, we introducing $(\hat{H}_m, \Phi)$-interpolative proximal contraction, which generalize and and establishing the optimal proximity point theorems for them. We look for various conditions on the functions to introduce presence of bppt of improved pc, improved Ćirić-Reich-Rus interpolative proximal contraction, improved Hardy Rogers interpolative proximal contraction. These results have published in **Filomat (2025), 39:8,** 2817-2830.

In **Chapter 3,** we investigate optimal solutions for best proximity points through the framework of generalized interpolative proximal contractions. We introduce a new method that uses interpolation techniques to handle a wider class of mappings by expanding the concepts of classical proximal contraction. In the absence of a precise solution, bppt investigate the existence of such best proximity points for approximate solutions to the fixed point problem. This chapter aims to develop the bppt for contractive non-self mappings via interpolation. We illustrate the utility of our findings with a few instances. The value of our research is illustrated with a few examples and applications. These results have published in **European Journal of Pure and Applied Mathematics (2025), 18:4,** 1-23.

In **Chapter 4,** we establish certain bppt for such pc. These results have published in **Fractal and fractional (2022), 1(2):** 1-19.

In **Chapter 5,** these results have published in **Aims Mathematics (2025), 8(2):** 2891-2909.

Chapter 1

Preliminaries

A few basic definitions, results and examples related to metric spaces and its generalized form were discussed in the current chapter which will support us in next chapters.

1.1 Some Basic Concepts

1.1.1 Definition [1]

Let (J, θ) be a complete metric space (in short CMS). Let $U \neq \phi$ and $V \neq \phi$ are closed subset of J. Let $\Gamma : U \to V$ be a mapping. A point $s \in U$ is the BPP of Γ if it satisfies,

$$\theta\left(s, \Gamma s\right) = \theta\left(U, V\right)$$

We proceed with the following notations that are used in the sequel.

$$L_0 = \{s \in L : \theta(s, t) = \theta\left(U, V\right), \text{ for some } t \in V\}$$

$$V_0 = \{t \in V : \theta(s,t) = \theta(U,V), \text{ for some } s \in U\}.$$

1.1.2 Definition

Let $R, G \subseteq (\Omega, \eth^d)$. A mapping $S : R \to G$ satisfying

$$\left.\begin{array}{l} \eth^d(\varsigma^{kj}, S(\mathfrak{q}_1)) = \eth^d(R,G) \\ \eth^d(\zeta^{jk}, S(\mathfrak{q}_2)) = \eth^d(R,G) \end{array}\right\} \Rightarrow \eth^d(\varsigma^{kj}, \zeta^{jk}) \leq k\eth^d(\mathfrak{q}_1, \mathfrak{q}_2) \tag{1.1}$$

for all $\varsigma^{kj}, \zeta^{jk}, q_1, q_2 \in R$ such that $\varsigma^{kj} \neq \zeta^{jk}$ and $k \in [0,1)$ is called PC-I.

Every PC-I can be modified to a Banach contraction.

1.1.3 Definition

Let $R, G \subseteq (\Omega, \eth^d)$. A mapping $S : R \to G$ satisfying

$$\left.\begin{array}{l} \eth^d(\varsigma^{kj}, S(\mathfrak{q}_1)) = \eth^d(R,G) \\ \eth^d(\zeta^{jk}, S(\mathfrak{q}_2)) = \eth^d(R,G) \end{array}\right\} \Rightarrow \eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right) \leq k\eth^d(S\mathfrak{q}_1, S\mathfrak{q}_2),$$

for all $\varsigma^{kj}, \zeta^{jk}, q_1, q_2 \in R$ such that $S\varsigma^{kj} \neq S\zeta^{jk}$, and $k \in [0,1)$ is said to be a PC-II.

For a self-mapping $S : R \to R$ to be a PC-II, it needs to satisfy the following inequality:

$$\eth^d\left(S^2\mathfrak{q}_1, S^2\mathfrak{q}_2\right) \leq k\eth^d(S\mathfrak{q}_1, S\mathfrak{q}_2), \text{ for all } \mathfrak{q}_1, \mathfrak{q}_2 \in R.$$

1.1.4 Remark

Every contraction is a PC-II but the converse is not true. Indeed, the mapping $S:[0,1]\to[0,1]$ defined by

$$S(\varsigma)=\begin{cases}0 & \text{if } \varsigma \text{ is rational}\\ \\ 1 & \text{otherwise}\end{cases}$$

is a PC-II but not a contraction in $(R,\eth^{d})$.

1.1.5 Definition [23]

If each sequence $\{s_n\}$ in V holding the condition that $\theta(t,s_n)\to\theta(t,V)$ for some $t\in U$, then there is a subsequence of $\{s_{n_k}\}$ s.t $s_{n_k}\to s\in V$, so V is A-Compact with respect to (in short w.r.t) U. Each compact subset of U is AC w.r.t any subsets as well as A-Compact w.r.t itself.

1.1.6 Lemma [18]

Let (J,θ) be a CMS. Let U and $V\in P_{CB}(J)$ and $s\in U$, then for each $h>1$, there $t\in V$ with

$$\eth^{d}(s,t)\leq hH(s,t). \tag{1.2}$$

1.1.7 Lemma [18]

Let (J,θ) be a MS. Let U and $V\in P_{CB}(J)$ and $s\in U$,

(i) For each $\varepsilon > 0$, there exist $t \in V$ s.t;

$$\theta(s,t) \leq H(U,V) + \varepsilon.$$

(ii) For each $h > 1$, there exist $t \in V$ s.t;

$$\theta(s,t) \leq hH(U,V) + \varepsilon.$$

1.1.8 Definition [11]

Let (J,θ) CMS. Let $U \neq \phi$ and $V \neq \phi$ are closed subset of J. Then a mapping $\Gamma : U \to V$ be a IKTPC. If $\exists$ a real number $\kappa \in [0,1)$ and $l \in (0,1)$ s.t

$$\theta(s_1,s_2) \leq \kappa \left(\theta(s_1,t_1)\right)^l \left(\theta(s_2,t_2)\right)^{1-l},$$

for all $s_1, s_2, t_1$and $t_2 \in U$ with $s_i \neq t_i$ for $i \in \{1,2\}$.

1.1.9 Definition [11]

Let (J,θ) be a CMS. Let $U \neq \phi$ and $V \neq \phi$ are subset of J. Then a mapping $\Gamma : U \to V$ be a IRRCTPC of first kind. If $\exists$ a real number $\kappa \in [0,1)$ and $l_1, l_2 \in (0,1)$ s.t,

$$\theta(s_1,s_2) \leq k \left(\theta(t_1,t_2)\right)^{l_1} \left(\theta(s_1,t_1)\right)^{l_2} \left(\theta(s_2,t_2)\right)^{1-l_1-l_2},$$

for all s_1, s_2, t_1 and $t_2 \in U$ with $s_i \neq t_i$ for $i \in \{1,2\}$.

1.1.10 Definition [11]

Let (J,θ) be a CMS. Let $U \neq \phi$ and $V \neq \phi$ are closed subset of J. Then a mapping $\Gamma : U \to V$ be a IHRTPC of first kind. If $\exists$ a real number $\lambda \in [0,1)$ and $l_1, l_2, l_2 \in (0,1)$

s.t,

$$\theta\left(s_1,s_2\right)\leq\lambda\left(\theta\left(t_1,t_2\right)\right)^{l_1}\left(\theta\left(s_1,t_1\right)\right)^{l_2}\left(\theta\left(s_2,t_2\right)\right)^{l_3}\left(\frac{1}{2}\left(\theta\left(s_1,t_1\right)+\theta\left(s_2,t_2\right)\right)\right)^{1-l_1-l_2-l_3}$$

for all $s_1, s_2, t_1$and $t_2 \in U$ with $s_i \neq t_i$ for $i \in \{1,2\}$ w.r.t $\theta\left(s_1, \Gamma t_1\right) = \theta\left(U,V\right)$ and $\theta\left(s_2, \Gamma t_2\right) = \theta\left(U,V\right)$.

1.1.11 Lemma [10]

Let $\{l_n\}$ be a sequence in $(\Omega, \eth^d)$ verifying $\lim_{n\to\infty} \eth^d(l_n, l_{n+1}) = 0$. If the seqence $\{l_n\}$ is not cauchy, then there are sub-sequences $\{l_{n_k}\}$, $\{l_{m_k}\}$ and $\varsigma^{kj} > 0$ such that

$$\lim_{k\to\infty} \eth^d(l_{n_k+1}, l_{m_k+1}) = \epsilon + \text{some term}(s). \tag{1.3}$$

$$\lim_{k\to\infty} \eth^d(l_{n_k}, l_{m_k}) = \lim_{k\to\infty} \eth^d(l_{n_k+1}, l_{m_k}) = \lim_{k\to\infty} \eth^d(l_{n_k}, l_{m_k+1}) = \epsilon. \tag{1.4}$$

1.1.12 Lemma [10]

Let $\hat{H}_m : (0,\infty) \to R$. Then the axioms (i)-(iii) are equivalent.

(i) $\inf_{\mathfrak{z}>\varepsilon} \hat{H}_m\left(\mathfrak{z}\right) > -\infty$ for every $\varepsilon > 0$.

(ii) $\lim_{\mathfrak{z}\to\varepsilon+} \inf \hat{H}_m\left(\mathfrak{z}\right) > -\infty$ for every $\varepsilon > 0$.

(iii) $\lim_{n\to\infty} \hat{H}_m\left(\mathfrak{z}_n\right) = -\infty$ implies that $\lim_{n\to\infty} z_n = 0$.

1.1.13 Lemma [5]

Let $\hat{H}_m : (0,1] \to R$. Then following are equivalent:

(i) $\inf_{t>\varepsilon} \hat{H}_m\left(t\right) > -\infty$ for every $\varepsilon \in (0,1)$.

(ii) $\lim_{t\to\varepsilon-} \inf \hat{H}_m\left(t\right) > -\infty$ for any $\varepsilon \in (0,1)$.

(*iii*) $\lim_{n\to\infty} \hat{H}_m(t_n) = -\infty$ implies that $\lim_{n\to\infty} t_n = 1$.

Chapter 2

Existence of best proximity point with applications

2.1 Introduction

We introduce some new generalized proximal interpolative contraction principles that produce corresponding proximal interpolative contraction principles and proximal contraction principles as special cases. We prove various best proximity point theorems using introduced generalized proximal interpolative contraction principles. Some examples and applications are given to demonstrate the usefulness of our results.

We proceed with the following notations that are used in the sequel.

$$\begin{aligned} \eth^d(R,G) &= \inf\{\eth^d(\varsigma,\mathfrak{q}) : \varsigma \in R \wedge \mathfrak{q} \in G\}, \\ R_0 &= \{\varsigma \in R : \eth^d(\varsigma,\mathfrak{q}) = \eth^d(R,G) \text{ for some } \mathfrak{q} \in G\}, \\ G_0 &= \{\mathfrak{q} \in G : \eth^d(\varsigma,\mathfrak{q}) = \eth^d(R,G) \text{ for some } \varsigma \in R\}, \end{aligned}$$

where $\left(\Omega, \eth^d\right)$ is a metric space and $R, G \subseteq \left(\Omega, \eth^d\right)$.

2.2 Improved proximal contractions.

In this section, we define improved PC and show that it generalizes PC (1.1). We state and prove some existence of bpp theorems for improved PC and improved interpolative PC in a CMS.

2.2.1 Definition

Let R, G be subsets of $\left(\Omega, \eth^d\right)$. A mapping $S : R \to G$ satisfying

$$\left.\begin{aligned} \eth^d\left(\varsigma^{kj}, S\mathfrak{q}_1\right) &= \eth^d\left(R, G\right) \\ \eth^d\left(\zeta^{jk}, S\mathfrak{q}_2\right) &= \eth^d\left(R, G\right) \end{aligned}\right\} \Rightarrow \hat{H}_m\hat{H}_m\left(\eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right)\right) \le \Phi\left(\eth^d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)\right), \tag{2.1}$$

for all $\varsigma^{kj}, \zeta^{jk}, q_1, q_2 \in R$ such that $\varsigma^{kj} \neq \zeta^{jk}$, is called an improved PC-II, where the maps $\hat{H}_m, \Phi : (0, \infty) \to (-\infty, \infty)$ such that $\hat{H}_m$ is non-decreasing function and $\Phi\left(t\right) < \hat{H}_m(t)$ for all $t > 0$.

The following example shows the significance of improved PC-II.

2.2.2 Example

Let $\eth^d : R^2 \times R^2 \to [0, \infty)$ be defined by

$$\eth^d((\varsigma, \mathfrak{q}), (u, v)) = |\varsigma - u| + |\mathfrak{q} - \hat{\jmath}| \text{ for all } (\varsigma, y), (u, \hat{\jmath}) \in \Omega.$$

Let R, G be the subsets of Ω defined by

$$R = \{(0, \mathfrak{q}); \mathfrak{q} \in \mathbb{R}\}, \;\; G = \{(1, \mathfrak{q}); \mathfrak{q} \in \mathbb{R}\}, \text{ then } \eth^d(R, G) = 1.$$

Define the functions $\hat{H}_m, \Phi : R^+ \to R$ by

$$\hat{H}_m(t) = 2t \text{ and } \Phi(t) = t, t \in \mathbb{R}^+.$$

Define the mapping $S : R \to G$ by $S((0, r)) = (1, \frac{r}{4})$ for all $(0, r) \in R$. We show that S is an improved PC-II. For $\varsigma^{kj} = (0, \varsigma)$, $\zeta^{jk} = (0, u)$ and $q_1 = (0, 4\varsigma)$, $q_2 = (0, 4u) \in R$ we have,

$$\begin{aligned} \eth^d(\varsigma^{kj}, S\mathfrak{q}_1) &= \eth^d((0,\varsigma), S(0,4\varsigma)) = 1 = \eth^d(R,G), \\ \eth^d(\zeta^{jk}, S\mathfrak{q}_2) &= \eth^d((0,u), S(0,4u)) = 1 = \eth^d(R,G). \end{aligned}$$

This implies that

$$\hat{H}_m\left(\eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right)\right) \leq \Phi\left(\eth^d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)\right),$$

This shows that S is an improved PC-II. However, the following calculations show that it is not a PC-II. We know that

$$\begin{aligned} \eth^d(\varsigma^{kj}, S\mathfrak{q}_1) &= 1 = \eth^d(R,G) \\ \eth^d(\zeta^{jk}, S\mathfrak{q}_2) &= 1 = \eth^d(R,G). \end{aligned}$$

If there exists $k \in (0, 1)$ such that

$$\eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right) \leq k\eth^d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right),$$

then, $k = \frac{1}{6}$, a contradiction. Hence, S is not a PC-II.

The following lemmas are integral part of this paper and have an impact on further investigations.

2.2.3 Lemma

Let $\{\varsigma_n^{kj}\}$ be a sequence in $(\Omega, \eth^d)$ obeying the equation $\lim_{n\to\infty} \eth^d(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}) = 0$. Suppose that the mapping $S : R \to G$ satisfies (2.1) and the maps $\hat{H}_m, \Phi : (0, \infty) \to R$ such that

$$\limsup_{t\to\epsilon+} \Phi(\mathfrak{z}) < \hat{H}_m(\epsilon+) \tag{2.2}$$

for any $\epsilon > 0$. Then $\{\varsigma_n^{kj}\}$ is a cauchy sequence.

Proof. First, we consider $\{\varsigma_n^{kj}\}$ is not cauchy, then by Lemma 1.1.11, $\exists$ two subsequence $\{\varsigma_{n_k}\}$, $\{\varsigma_{m_k}\}$ of $\{\varsigma_n^{kj}\}$ and $\epsilon > 0$ so that (1.3) and (1.4) hold. By (1.3), we get that $\eth^d(\varsigma_{n_k+1}, \varsigma_{m_k+1}) > \epsilon$ and

$$\begin{aligned} \eth^d(\varsigma_{n_k+1}, S(\varsigma_{m_k})) &= \eth^d(R, G), \\ \eth^d(\varsigma_{m_k+1}, S(\varsigma_{n_k})) &= \eth^d(R, G), \text{ for all } k \geq 1. \end{aligned}$$

Thus, by (2.1), we have

$$\hat{H}_m(\eth^d(\varsigma_{n_k+1}, \varsigma_{m_k+1})) \leq \Phi(\eth^d(\varsigma_{n_k}, \varsigma_{m_k})), \text{ for any } k \geq 1. \tag{2.3}$$

Putting $c_k = \eth^d(\varsigma_{n_k+1}, \varsigma_{m_k+1})$ and $\varsigma_k^{kj} = \eth^d(\varsigma_{n_k}, \varsigma_{m_k})$ in (2.3), we have

$$\hat{H}_m(c_k) \leq \Phi(\varsigma_k^{kj}), \text{ for any } k \geq 1. \tag{2.4}$$

By (1.3) and (1.4), $\lim_{k\to\infty} c_k = \epsilon+$ some term(s) and $\lim_{k\to\infty} \varsigma_k^{kj} = \epsilon$. By (2.4), we get

$$\hat{H}_m(\epsilon+) = \lim_{k\to\infty} \hat{H}_m(c_k) \leq \limsup_{k\to\infty} \Phi(\varsigma_k^{kj}) \leq \limsup_{p\to\epsilon} \Phi(p). \tag{2.5}$$

This is a contradiction to the assumption (2.2). Consequently, $\{\varsigma_n^{kj}\}$ is a cauchy sequence in G. ■

2.2.4 Theorem

Let $R, G \subseteq (\Omega, \eth^d)$ with the property that "R is a-compact w.r.t G" and $(\Omega, \eth^d)$ be a cms and $S\colon R \to G$ be a continuous improved PC-II verifying conditions

(i) $\hat{H}_m$ is non-decreasing (nd) function and $\limsup_{t\to\epsilon+} \Phi(t) < \hat{H}_m(\epsilon+)$ for any $\epsilon > 0$,

(ii) R_0 is non-empty subset of R obeying $S(R_0) \subseteq G_0$.

Then S has a bpp.

Proof. Consider $\varsigma_0^{kj} \in R_0$. Since $S(\varsigma_0^{kj}) \in S(R_0) \subseteq G_0$, there exists $\varsigma^{kj} \in R_0$ satisfying $\eth^d(\varsigma^{kj}, S(\varsigma_0^{kj})) = \eth^d(R, G)$. Also, we have $S(\varsigma^{kj}) \in S(R_0) \subseteq G_0$, there exists $\zeta^{jk} \in R_0$ so that $\eth^d(\zeta^{jk}, S(\varsigma^{kj})) = \eth^d(R, G)$. We build a series by continuing this approach such that $\{\varsigma_n^{kj}\}$ in R_0 satisfies the following equation:

$$\eth^d(\varsigma_n^{kj}, S(\varsigma_{n-1}^{kj})) = \eth^d(R, G), \text{for all } n \in \mathbb{N}. \tag{2.6}$$

If $n \in \mathbb{N}$ s.t $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$, then ς_n^{kj} is a bpp of the mapping S. If $\varsigma_{n-1}^{kj} \neq \varsigma_n^{kj}$ for all $n \in \mathbb{N}$, then we have

$$\begin{aligned}
\eth^d(\varsigma_n^{kj}, S(\varsigma_{n-1}^{kj})) &= \eth^d(R, G),\\
\eth^d(\varsigma_{n+1}^{kj}, S(\varsigma_n^{kj})) &= \eth^d(R, G), \text{ for all } n \geq 1.
\end{aligned}$$

Thus, by (2.1), we have

$$\hat{H}_m(\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj})) \leq \Phi(\eth^d(S\varsigma_{n-1}^{kj}, S\varsigma_n^{kj})).$$

Let $\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}) = \mathfrak{d}_n$. We know that, $\Phi(t) < \hat{H}_m(t)$ for all $t > 0$, we have

$$\hat{H}_m(\mathfrak{d}_n) \leq \Phi(\mathfrak{d}_{n-1}) < \hat{H}_m(\mathfrak{d}_{n-1}). \tag{2.7}$$

Given that $\hat{H}_m$ is nd, by (2.7), $d_n < d_{n-1} \; \forall \; n \in N$. Thus, it converges to some element $d \geq 0$. We claim that $d = 0$. If $d > 0$, by (2.7), we obtain the following:

$$\hat{H}_m(\mathfrak{d}+) = \lim_{n\to\infty} \hat{H}_m(\mathfrak{d}_n) \leq \lim_{n\to\infty} \Phi(\mathfrak{d}_{n-1}) \leq \lim_{t\to\mathfrak{d}+} \sup \Phi(t).$$

This contradicts (i), hence, $d = 0$ and $\lim_{n\to\infty} \eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}) = 0$. By using (i) and Lemma 2.2.3, we conclude that $\{S(\varsigma_n^{kj})\}$ is a cauchy sequence. Since G is a closed subset of complete metric space $(\Omega, \eth^d)$, there exists $q^* \in G$ such that $\lim_{n\to\infty} \eth^d(S\varsigma_n^{kj}, q^*) = 0$. Moreover,

$$\begin{aligned}
\eth^d(\mathfrak{q}^*, R) &\leq \eth^d(\mathfrak{q}^*, \varsigma_n^{kj}) \\
&\leq \eth^d(\mathfrak{q}^*, S(\varsigma_n^{kj})) + \eth^d(S(\varsigma_{n-1}^{kj}), \varsigma_n^{kj}) \\
&\leq \eth^d(\mathfrak{q}^*, S(\varsigma_{n-1}^{kj})) + \eth^d(R, G) \\
&\leq \eth^d(\mathfrak{q}^*, S(\varsigma_{n-1}^{kj})) + \eth^d(\mathfrak{q}^*, R).
\end{aligned}$$

Thus, $\eth^d(q^*, \varsigma_n^{kj}) \to \eth^d(q^*, R)$ as $n \to \infty$. Since R is a-compact w.r.t G, $\exists \; \{(\varsigma_{n_k})\}$ of $\{(\varsigma_n^{kj})\}$ converging to $\varsigma^* \in R$(say). We infer the following equation:

$$\eth^d(\varsigma^*, \mathfrak{q}^*) = \eth^d(\varsigma_{n_k}, S(\varsigma_{n_k-1})) = \eth^d(R, G). \tag{2.8}$$

Due to the continuity of S, we have $S(\varsigma_{n_k-1}) \to S(\varsigma^*)$. Thus,

$$\eth^d(\varsigma^*, S(\varsigma^*)) = \eth^d(R, G).$$

Let $R, G \subseteq (\Omega, \eth^d)$ with the property that "R is approximately compact with respect to G" and $(\Omega, \eth^d)$ be a complete metric space. ■

2.2.5 Theorem

Let $R, G \subseteq (\Omega, \eth^d)$ with the property that "R is a-compact w.r.t G" and $(\Omega, \eth^d)$ be a cms and $S\colon R \to G$ be a improved PC-I verifying conditions

(i) $\hat{H}_m$ is nd function and $\limsup_{t\to\epsilon+} \Phi\,(t) < \hat{H}_m(\epsilon+)$ for any $\epsilon > 0$,

(ii) R_0 is non-empty subset of R obeying $S(R_0) \subseteq G_0$. Then S has a bpp.

We omit the proof of Theorem 2.2.4, as it follows from the previous one.

2.2.6 Theorem

Let $R, G \subseteq (\Omega, \eth^d)$ with the property that "R is a-compact w.r.t G" and $(\Omega, \eth^d)$ be a cms and $S\colon R \to G$ be a improved PC-II verifying conditions

(i) $\hat{H}_m$ is non-decreasing and if $\{\hat{H}_m(t_n)\}$ and $\{\Phi(t_n)\}$ are convergent sequence satisfying $\lim_{n\to\infty} \hat{H}_m(t_n) = \lim_{n\to\infty} \Phi(t_n)$, then $\lim_{n\to\infty} t_n = 0$,

(ii) R_0 is non-empty subset of R obeying $S(R_0) \subseteq G_0$. Then S has a bpp.

Proof. Following the procedure used in the proof of Theorem 2.2.4, we have

$$\hat{H}_m(\mathfrak{d}_n) \leq \Phi(\mathfrak{d}_{n-1}) < \hat{H}_m(\mathfrak{d}_{n-1}). \tag{2.9}$$

By (2.9), we have $\{\hat{H}_m\,(\mathfrak{d}_n)\}$ is strictly decreasing seq. If $\{\hat{H}_m\,(w_n)\}$ is not bounded below, then

$$\inf_{w_n > \varepsilon} \hat{H}_m\,(\mathfrak{d}_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

From, Lemma 1.1.12, then $d_n \to 0$ as $n \to \infty$. Secondly, if seq $\{\hat{H}_m(d_n)\}$ is bounded below, then, it is a cgt seq. By (2.9), the seq $\{\Phi(d_n)\}$ also cgs. By (i), we have $\lim_{n\to\infty} d_n = 0$, or $\lim_{n\to\infty} \eth^d \left(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}\right) = 0$, for any seq $\left\{\varsigma_n^{kj}\right\}$ in R. Theorem

2.2.4, we have

$$\eth^{d}(\varsigma^{*}, S(\varsigma^{*})) = \eth^{d}(R, G).$$

Hence, ς^{*} is a bpp of the mapping S. ■

2.3 Improved Ćirić-Reich-Rus interpolative proximal contraction

2.3.1 Definition

Let $\left(\Omega, \eth^{d}\right)$ be a cms, and R, G be a pair of non-empty subsets of Ω. Let $\hat{H}_{m}, \Phi : (0, \infty) \rightarrow R$ be two functions. A mapping $S : R \rightarrow G$ is said to be an improved Ćirić-Reich-Rus interpolative PC-II if there exist $\alpha, \beta \in (0, 1)$; $\alpha + \beta < 1$ satisfying

$$\hat{H}_{m}\left(\eth^{d}\left(S\varsigma^{kj}, S\zeta^{jk}\right)\right) \leq \Phi \left(\begin{array}{c} \left(\eth^{d}\left(S\mathfrak{q}_{1}, S\mathfrak{q}_{2}\right)\right)^{\alpha}\left(\eth^{d}\left(S\mathfrak{q}_{1}, S\varsigma^{kj}\right)\right)^{\beta} \\ \left(\eth^{d}\left(S\mathfrak{q}_{2}, S\zeta^{jk}\right)\right)^{1-\alpha-\beta} \end{array} \right), \tag{2.10}$$

whenever $\eth^{d}\left(\varsigma^{kj}, S\mathfrak{q}_{1}\right) = \eth^{d}(R, G)$ and $\eth^{d}\left(\zeta^{jk}, S\mathfrak{q}_{2}\right) = \eth^{d}(R, G)$ for all distinct $\varsigma^{kj}, \zeta^{jk} q_{1}, q_{2} \in R$.

2.3.2 Example

Let $\eth^{d} : R^{2} \times R^{2} \rightarrow R$ be the Euclidean metric on R^{2} and R, G be the subsets of R^{2} defined by

$$R = \left\{(\varsigma, \mathfrak{q}) : \mathfrak{q} = \sqrt{9 - \varsigma^{2}}\right\}; G = \left\{(\varsigma, \jmath) : \mathfrak{q} = \sqrt{16 - \varsigma^{2}}\right\} \text{ then } \eth^{d}(R, G) = 1.$$

Define the functions $\hat{H}_m, \Phi : R^+ \to R$ and $S : R \to G$ by

$$\Phi(\mathfrak{z}) = \sqrt{\mathfrak{z}} \text{ and } \hat{H}_m(\mathfrak{z}) = \mathfrak{z}, \text{ for all } \mathfrak{z} \in \mathbb{R}^+.$$

$$S(\zeta) = S(\varsigma, \hat{\jmath}) = \begin{cases} \left(\frac{\varsigma}{2}, \frac{\mathfrak{q}}{2}\right) & \text{for } \varsigma \geq 0, \\ (-1, 0) & \text{for } \varsigma < 0, \end{cases} \quad \text{for all } (\varsigma, y) \in R.$$

The following information shows that S generalizes the interpolative Ćirić-Reich-Rus type proximal contraction [23]. Indeed, for $\varsigma^{kj} = (1,0), \zeta^{jk} = (1,2), q_1 = (2,2), q_2 = (0,4)$, we have $\eth^d(\varsigma^{kj}, Sq_1) = 1 = \eth^d(R,G)$, $\eth^d(\zeta^{jk}, Sq_2) = 1 = \eth^d(R,G)$, and for $\alpha = \frac{1}{2}, \beta = \frac{1}{3}$,

$$\begin{aligned} \hat{H}_m\left(\eth^d\left(S(1,0), S(1,2)\right)\right) &\leq \Phi\left(\begin{array}{c} \left(\eth^d\left(S(0,4), S(2,2)\right)\right)^{\frac{1}{2}} \left(\eth^d\left(S(2,2), S(1,0)\right)\right)^{\frac{1}{3}} \\ \left(\eth^d\left(S(0,4), S(1,2)\right)\right)^{1-\frac{1}{2}-\frac{1}{3}} \end{array}\right), \\ \hat{H}_m(1) &\leq \Phi(1.2573) \Rightarrow 1 \leq 1.1213. \end{aligned}$$

Thus,

$$\hat{H}_m\left(\eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right)\right) \leq \Phi\left(\begin{array}{c} \left(\eth^d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)\right)^{\alpha} \left(\eth^d\left(S\mathfrak{q}_1, S\varsigma^{kj}\right)\right)^{\beta} \\ \left(\eth^d\left(S\mathfrak{q}_2, S\zeta^{jk}\right)\right)^{1-\alpha-\beta} \end{array}\right).$$

This shows that S is an improved interpolative Ćirić-Reich-Rus PC-II. However, for $\varsigma^{kj} = (1,0), \zeta^{jk} = (1,2), q_1 = (2,2), q_2 = (0,4)$, if there exists some k satisfying the following inequality:

$$\begin{aligned} \eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right) &\leq k\left(\eth^d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)\right)^{\alpha} \left(\eth^d\left(S\mathfrak{q}_1, S\varsigma^{kj}\right)\right)^{\beta} \left(\eth^d\left(S\mathfrak{q}_2, S\zeta^{jk}\right)\right)^{1-\alpha-\beta} \\ \eth^d\left(S(1,0), S(1,2)\right) &\leq k\left(\eth^d\left(S(0,4), S(2,2)\right)\right)^{\frac{1}{2}} \left(\eth^d\left(S(2,2), S(1,0)\right)\right)^{\frac{1}{3}} \\ &\quad \left(\eth^d\left(S(0,4), S(1,2)\right)\right)^{1-\frac{1}{2}-\frac{1}{3}}. \end{aligned}$$

Then, $k \in \left[\frac{1}{1.2573}, \infty\right)$, a contradiction. Hence, S is not interpolative Ćirić-Reich-Rus PC-II. We note that for $\varsigma \geq 0$, there is $\zeta = (\varsigma, \hat{\jmath}) \in R$ such that $\eth^d(\zeta, S(\zeta)) = \eth^d(R, G) = 1$.

The criteria for the existence of bpp of the improved Ćirić-Reich-Rus interpolative PC-II are stated in the following two theorems.

2.3.3 Theorem

Let $R, G \subseteq (\Omega, \eth^d)$ with the property that "R is approximately compact with respect to G" and $(\Omega, \eth^d)$ be a complete metric space. If $S\colon R \to G$ is a continuous improved Ćirić-Reich-Rus type interpolative PC-II satisfying the following assumptions: $\hat{H}_m$ is non-decreasing function and $\limsup_{t\to\epsilon+} \Phi(t) < \hat{H}_m(\epsilon+)$ for any $\epsilon > 0$. R_0 is non-empty subset of R such that $S(R_0) \subseteq G_0$. Then S has a bpp.

Proof. Consider an arbitrary initial guess $\varsigma_o^{kj} \in R_0$. Since $S(\varsigma_o^{kj}) \in S(R_0) \subseteq G_0$, there exists $\varsigma^{kj} \in R_0$ such that

$$\eth^d(\varsigma^{kj}, S(\varsigma_o^{kj})) = \eth^d(R, G).$$

Also, $S(\varsigma^{kj}) \in S(R_0) \subseteq G_0$, there exists $\zeta^{jk} \in R_0$ such that

$$\eth^d(\zeta^{jk}, S(\varsigma^{kj})) = \eth^d(R, G).$$

We build a series by continuing this approach such that $\{\varsigma_n^{kj}\}$ in R_0 satisfies the following equation:

$$\eth^d(\varsigma_{n+1}^{kj}, S(\varsigma_n^{kj})) = \eth^d(R, G), \text{ for all } n \in \mathbb{N}. \tag{2.11}$$

Now, if $\exists\ n \in \mathbb{N}$ s.t $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$, the point ς_n^{kj} is a bpp of the mapping S. Assume that $\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}$ $\forall\ n \in \mathbb{N}$ and using (2.11), we have

$$\eth^d(\varsigma_n^{kj}, S(\varsigma_{n-1}^{kj})) = \eth^d(R, G),$$

and

$$\eth^d(\varsigma_{n+1}^{kj}, S(\varsigma_n^{kj})) = \eth^d(R, G), \text{ for all } n \geq 1.$$

By (2.10), we have

$$\hat{H}_m(\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj})) \leq \Phi\left(\left(\eth^d\left(S\varsigma_{n-1}^{kj}, S\varsigma_n^{kj}\right)\right)^{\alpha}\left(\eth^d\left(S\varsigma_{n-1}^{kj}, S\varsigma_n\right)\right)^{\beta}\left(\eth^d\left(S\varsigma_n, S\varsigma_{n+1}^{kj}\right)\right)^{1-\alpha-\beta}\right), \tag{2.12}$$

for all distinct $\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \varsigma_{n+1}^{kj} \in R$. Given that $\Phi(t) < \hat{H}_m(t)$ for all $t > 0$, by (2.12), we have

$$\hat{H}_m(\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj})) < \hat{H}_m\left(\left(\eth^d\left(S\varsigma_{n-1}^{kj}, S\varsigma_n^{kj}\right)\right)^{\alpha}\left(\eth^d\left(S\varsigma_{n-1}^{kj}, S\varsigma_n^{kj}\right)\right)^{\beta}\left(\eth^d\left(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}\right)\right)^{1-\alpha-\beta}\right).$$

Since $\hat{H}_m$ is a nd,

$$\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}) < \left(\eth^d\left(S\varsigma_{n-1}^{kj} S\varsigma_n^{kj}\right)\right)^{\alpha+\beta}(\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}))^{1-\alpha-\beta}.$$

This implies that

$$(\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}))^{\alpha+\beta} < \left(\eth^d\left(S\varsigma_{n-1}^{kj}, S\varsigma_n^{kj}\right)\right)^{\alpha+\beta}.$$

This shows that the sequence $\{\eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}) = d_n\}$ converges to some element $d \geq 0$. We claim that $d = 0$. If $d > 0$, by (2.12), we obtain the following:

$$\hat{H}_m(\mathfrak{d}+) = \lim_{n\to\infty} \hat{H}_m(\mathfrak{d}_n) \leq \lim_{n\to\infty} \Phi\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{1-\alpha-\beta}\right) \leq \lim_{\mathfrak{z}\to\mathfrak{d}+} \sup \Phi(\mathfrak{z}).$$

This contradicts (i), hence, $d = 0$ and $\lim_{n\to\infty} \eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}) = 0$. By using (i) and Lemma 2.2.3, we conclude that $\{S\varsigma_n^{kj}\}$ is a cauchy sequence. Since G is a closed subset of cms $(\Omega, \eth^d)$, there exists $q^* \in G$, such that $\lim_{n\to\infty} \eth^d(S\varsigma_n^{kj}, q^*) = 0$. Now, we can obtain the desired result by following the reasoning used in the proof of Theorem 2.2.4. ■

2.3.4 Theorem

Let $R, G \subseteq (\Omega, \eth^d)$ with the property that "R is a-compact with repect to G" and $(\Omega, \eth^d)$ be a cms. If $S\colon R \to G$ is a continuous improved Ćirić-Reich-Rus type interpolative PC-II verifying (i)-(ii) $\hat{H}_m$ is non-decreasing and if $\{\hat{H}_m(z_n)\}$ and $\{\Phi(z_n)\}$ are convergent sequences satisfying $\lim_{n\to\infty} \hat{H}_m(z_n) = \lim_{n\to\infty} \Phi(z_n)$, then $\lim_{n\to\infty} z_n = 0$, R_0 is non-void subset of R obeying $S(R_0) \subseteq G_0$. Then the mapping S has a bpp.

Proof. Following the procedure used in the proof of Theorem 2.3.3, we have

$$\hat{H}_m(\mathfrak{d}_n) \leq \Phi\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{1-\alpha-\beta}\right) < \hat{H}_m\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{1-\alpha-\beta}\right). \tag{2.13}$$

By (2.13), we infer that $\left\{\hat{H}_m(\mathfrak{d}_n)\right\}$ is strictly decreasing sequence. If $\left\{\hat{H}_m(\mathfrak{d}_n)\right\}$ is not bounded below, then

$$\inf_{\mathfrak{d}_n > \varepsilon} \hat{H}_m(\mathfrak{d}_n) > -\infty \text{ for every } \varepsilon > 0,\ n \in \mathbb{N}.$$

It follows by Lemma 1.1.12, that $d_n \to 0$ as $n \to \infty$. Secondly, if the seq $\left\{\hat{H}_m(\mathfrak{d}_n)\right\}$ is bounded below, then, it is cgt seq. By (2.13) the seq $\{\Phi(\mathfrak{d}_n)\}$ also cgs. By (i), we have $\lim_{n\to\infty} d_n = 0$ for any seq $\{\varsigma_n^{kj}\}$ in R. The proof of Theorem 2.3.3 leads to the rest of the proof. ■

Note that, if S is a self-mapping defined on R, then best proximity point is a fixed point of S.

2.4 Improved Hardy Rogers interpolative proximal contraction

2.4.1 Definition

Let $R, G \subseteq (\Omega, \eth^d)$. A mapping $S : R \to G$ satisfying

$$\hat{H}_m\left(\eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right)\right) \leq \Phi\left(\begin{array}{c} \eth^d\left(S\mathfrak{q}_1, S\mathfrak{q}_2\right)^{\alpha}\eth^d\left(S\mathfrak{q}_1, S\varsigma^{kj}\right)^{\beta}\eth^d\left(S\hat{\jmath}_2, S\zeta^{jk}\right)^{\gamma} \\ \left(\frac{1}{2}\left(\eth^d\left(S\mathfrak{q}_1, S\zeta^{jk}\right)+\eth^d\left(S\hat{\jmath}_2, S\varsigma^{kj}\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array}\right), \tag{2.14}$$

whenever, $\eth^d\left(\varsigma^{kj}, S\mathfrak{q}_1\right) = \eth^d(R, G)$; $\eth^d\left(\zeta^{jk}, Sq_2\right) = \eth^d(R, G)$, is called an improved Hardy Rogers interpolative PC-II, where $\alpha, \beta, \gamma \in (0, 1)$ such that $\alpha + \beta + \gamma < 1$, $\varsigma^{kj}, \zeta^{jk}, q_1, q_2 \in R$ and $\hat{H}_m, \Phi : R^+ \to R$.

The following example shows that improved Hardy Rogers type interpolative PC-II generalizes the Hardy Rogers type interpolative PC-II appeared in [23].

2.4.2 Example

Let $\eth^d : R^2 \to R$ be a usual metric and R, G be subsets of Ω defined as

$$R = \{1, 2, 3, 4, 5, 6, 7\}, G = \{0, 1, 2, 3, 4, 5\} \text{ then } \eth^d(R, G) = 0.$$

Define the functions $\hat{H}_m, \Phi : R^+ \to R$ and $S : R \to G$ by

$$\hat{H}_m(\varsigma) = \begin{cases} \varsigma + 1 \text{ for } \varsigma = 2, \\ \varsigma + 10 \text{ for } \varsigma \neq 2, \end{cases} \qquad \Phi(\varsigma) = \begin{cases} \frac{\varsigma}{2} \text{ for } \varsigma = 2, \\ \varsigma + 5 \text{ otherwise,} \end{cases}$$

and $S(\varsigma) = \varsigma - 1$ for all $\varsigma \in R$. We show that S is an improved interpolative Hardy Rogers PC-II. Indeed, for $\varsigma^{kj} = 2, \zeta^{jk} = 4, y_1 = 3, y_2 = 5$, and $\alpha = 0.2, \beta = 0.3, \gamma = 0.4$ we have $\eth^d(\varsigma^{kj}, S\mathfrak{q}_1) = 0 = \eth^d(R, G)$, $\eth^d(\zeta^{jk}, S\mathfrak{q}_2) = 0 = \eth^d(R, G)$ and

$$\begin{aligned} \hat{H}_m(2) &\leq \Phi\left((2)^{\alpha} (1)^{\beta} (1)^{\gamma} \left(\tfrac{1}{2}(3+1)\right)^{1-\alpha-\beta-\gamma} \right) \\ \hat{H}_m(2) &\leq \Phi\left((2)^{0.2} (1)^{0.3} (1)^{0.4} (2)^{0.1}\right) \\ &= \Phi(0.7764) \Rightarrow 3 < 5.7764. \end{aligned}$$

Hence,

$$\hat{H}_m\left(\eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right)\right) \leq \Phi\left(\begin{array}{c} \eth^d(S\mathfrak{q}_1, S\mathfrak{q}_2)^{\alpha} \eth^d\left(S\mathfrak{q}_1, S\varsigma^{kj}\right)^{\beta} \eth^d\left(S\mathfrak{q}_2, S\zeta^{jk}\right)^{\gamma} \\ \left(\frac{1}{2}\left(\eth^d\left(S\mathfrak{q}_1, S\zeta^{jk}\right) + \eth^d\left(S\mathfrak{q}_2, S\varsigma^{kj}\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array} \right).$$

Suppose there is some k satisfying the following inequality:

$$\eth^d\left(S\varsigma^{kj}, S\zeta^{jk}\right) \leq k \left(\begin{array}{c} \eth^d(S\mathfrak{q}_1, S\mathfrak{q}_2)^{\alpha} \eth^d\left(S\mathfrak{q}_1, S\varsigma^{kj}\right)^{\beta} \eth^d\left(S\mathfrak{q}_2, S\zeta^{jk}\right)^{\gamma} \\ \left(\frac{1}{2}\left(\eth^d\left(S\mathfrak{q}_1, S\zeta^{jk}\right) + \eth^d\left(S\mathfrak{q}_2, S\varsigma^{kj}\right)\right)\right)^{1-\alpha-\beta-\gamma} \end{array} \right).$$

Then, $k \in \left[\frac{2}{0.7764}, \infty\right)$, which is a contradiction to the assumption that $k \in (0, 1)$. Hence, S is not an interpolative Hardy Rogers PC-II.

The criteria for the existence of the bpp of improved interpolative Hardy Rogers PC S are stated theorems. The proofs are very identical to the proofs of Theorems 2.2.5 and 2.3.3. We only write the distinct parts of the proof.

2.4.3 Theorem

Let $R,G\subseteq(\Omega,\eth^d)$ with the property that "R is a-compact with repect to G" and $(\Omega,\eth^d)$ be a cms. If $S\colon R\to G$ is a continuous improved interpolative Hardy Rogers PC-II verifying conditions

(i) $\hat{H}_m$ is non-decreasing function and $\limsup_{t\to\epsilon+}\Phi(t)<\hat{H}_m(\epsilon+)$ for any $\epsilon>0$,

(ii) R_0 is non-empty subset of R obeying $S(R_0)\subseteq G_0$.

Then S has a bpp.

Proof. Starting with the initial input the Theorem 2.2.5, we have

$$\eth^d(\varsigma_n^{kj},S(\varsigma_{n-1}^{kj}))=\eth^d(R,G),$$

$$\eth^d(\varsigma_{n+1}^{kj},S(\varsigma_n^{kj}))=\eth^d(R,G),\ \text{for all } n\geq 1.$$

Thus by (2.14) we can write

$$\hat{H}_m(\eth^d(S\varsigma_n^{kj},S\varsigma_{n+1}^{kj}))\ \leq\ \Phi\left(\begin{array}{c}\left(\eth^d\left(S\varsigma_{n-1}^{kj}S\varsigma_n^{kj}\right)\right)^{\alpha}\left(\eth^d(S\varsigma_{n-1}^{kj},S\varsigma_n^{kj})\right)^{\beta}\left(\eth^d\left(S\varsigma_n^{kj},S\varsigma_{n+1}^{kj}\right)\right)^{\gamma}\\ \left(\frac{1}{2}\left(\eth^d\left(S\varsigma_{n-1}^{kj},S\varsigma_{n+1}^{kj}\right)+\eth^d\left(S\varsigma_n,S\varsigma_n^{kj}\right)\right)\right)^{1-\alpha-\beta-\gamma}\end{array}\right)$$

$$\hat{H}_m\left(\eth^d\left(S\varsigma_n^{kj},S\varsigma_{n+1}^{kj}\right)\right)\ \leq\ \Phi\left(\begin{array}{c}\left(\eth^d\left(S\varsigma_{n-1}^{kj},S\varsigma_n^{kj}\right)\right)^{\alpha}\left(\eth^d\left(S\varsigma_{n-1}^{kj},S\varsigma_n^{kj}\right)\right)^{\beta}\left(\eth^d\left(S\varsigma_n,S\varsigma_{n+1}^{kj}\right)\right)^{\gamma}\\ \left(\frac{1}{2}\eth^d\left(S\varsigma_{n-1}^{kj},S\varsigma_{n+1}^{kj}\right)\right)^{1-\alpha-\beta-\gamma}\end{array}\right)$$

$$\hat{H}_m\left(\eth^d\left(S\varsigma_n^{kj},S\varsigma_{n+1}^{kj}\right)\right)\ \leq\ \Phi\left(\begin{array}{c}\left(\eth^d\left(S\varsigma_{n-1}^{kj}S\varsigma_n^{kj}\right)\right)^{\alpha}\left(\eth^d(S\varsigma_{n-1}^{kj},S\varsigma_n)\right)^{\beta}\left(\eth^d\left(S\varsigma_n,S\varsigma_{n+1}^{kj}\right)\right)^{\gamma}\\ \left(\frac{1}{2}\left(\eth^d\left(S\varsigma_{n-1}^{kj},S\varsigma_n^{kj}\right)+\eth^d\left(S\varsigma_n,S\varsigma_{n+1}^{kj}\right)\right)\right)^{1-\alpha-\beta-\gamma}\end{array}\right)$$

$$\hat{H}_m\left(\eth^d\left(S\varsigma_n,S\varsigma_{n+1}^{kj}\right)\right)\ \leq\ \Phi\left(\begin{array}{c}\left(\eth^d\left(S\varsigma_{n-1}^{kj},S\varsigma_n\right)\right)^{\alpha+\beta}\left(\eth^d\left(S\varsigma_n,S\varsigma_{n+1}^{kj}\right)\right)^{\gamma}\\ \left(\frac{1}{2}\left(\eth^d\left(S\varsigma_{n-1}^{kj},S\varsigma_n\right)+\eth^d\left(S\varsigma_n,S\varsigma_{n+1}^{kj}\right)\right)\right)^{1-\alpha-\beta-\gamma}\end{array}\right),$$

for all distinct$\varsigma_{n-1}^{kj},\varsigma_n^{kj},\varsigma_{n+1}^{kj}\in R$. Let $\eth^d(S\varsigma_n^{kj},S\varsigma_{n+1}^{kj})=\Bbbk_n^{sw}$. Since $\Phi(t)<\hat{H}_m(t)$ for all $t > 0$, we get

$$\hat{H}_m(\mathfrak{d}_n) < \hat{H}_m\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{\gamma}\left(\frac{1}{2}(\mathfrak{d}_{n-1}+\mathfrak{d}_n)\right)^{1-\alpha-\beta-\gamma}\right). \qquad (2.15)$$

If $d_{n-1} < d_n$ for some $n \geq 1$ and by monotonicity of $\hat{H}_m$, we have

$$(\mathfrak{d}_n)^{\alpha+\beta} < (\mathfrak{d}_n)^{\alpha+\beta},$$

which is a false statement, $d_n < d_{n-1}$ for all $n \in N$. This implies $d_n < d_{n-1}$ for all $n \in N$. Thus, it converges to some element $d \geq 0$. Suppose $d > 0$, then

$$\hat{H}_m(\mathfrak{d}+) = \lim_{n\to\infty} \hat{H}_m(\mathfrak{d}_n) \leq \lim_{n\to\infty} \Phi\left((\mathfrak{d}_{n-1})^{\alpha+\beta}(\mathfrak{d}_n)^{\gamma}\left(\frac{1}{2}(\mathfrak{d}_n+\mathfrak{d}_{n-1})\right)^{1-\alpha-\beta-\gamma}\right) \leq \lim_{t\to\Bbbk^{sw}+} \Phi(t).$$

This contradicts (i), hence, $d = 0$ and $\lim_{n\to\infty} \eth^d(S\varsigma_n^{kj}, S\varsigma_{n+1}^{kj}) = 0$. We omit the remaining details as they are similar to proof of Theorem 2.2.5. ■

2.4.4 Theorem

Let $R, G \subseteq (\Omega, \eth^d)$ with the property that "R is a-compact with repect to G" and $(\Omega, \eth^d)$ be a cms. If $S\colon R \to G$ is a continuous improved interpolative Hardy Rogers PC-II verifying

(i) $\hat{H}_m$ is non-decreasing and if $\{\hat{H}_m(z_n)\}$ and $\{\Phi(z_n)\}$ are convergent sequences satisfying $\lim_{n\to\infty} \hat{H}_m(z_n) = \lim_{n\to\infty} \Phi(z_n)$, then $\lim_{n\to\infty} z_n = 0$,

(ii) R_0 is non-void subset of R obeying $S(R_0) \subseteq G_0$. Then the mapping S has a bpp.

Proof. This proof follows from the proof of Theorem 2.3.3 and Theorem 2.3.4. ■

2.4.5 Remark

If $S\colon R \to R$ $(G = R)$, then the best proximity point is a fixed point and Theorem 2.2.4, Theorem 2.2.5, Theorem 2.3.3, Theorem 2.3.4, Theorem 2.4.3 and Theorem 2.4.4 are fixed point theorems.

2.5 Application to integral equations

We intend to apply Theorem 2.2.5 (for $R \subseteq G$) to show the existence of the solution to the following nonlinear Volterra type integral equations:

$$f(k) = \int_0^k H_\varsigma(k, h, f)\, dh, \tag{2.16}$$

for all $k \in [0, 1]$, $\varsigma \in \Theta$, and H_ς is a function defined on $[0, 1]^2 \times C([0, 1], R_+)$ to R. We show the existence to the solution of (2.16). For $f \in C([0, 1], R_+)$, the norm as: $\|f\|_\tau = \sup_{k\in[0,1]} |f(k)|\, \varsigma^{kj-\tau k}$, $\tau > 0$. Define

$$\eta_\tau(f, \varkappa) = \left[\sup_{k\in[0,1]} |f(k) - \varkappa(k)|\, \varsigma^{kj-\tau k}\right] = \|f - \varkappa\|_\tau$$

for all $f, \varkappa \in C([0, 1], R_+)$, with these settings, $(C([0, 1], R_+), \eta_\tau)$ represents a cms. Now, we show the following theorem to clarify that the solution of integral equation exists.

2.5.1 Theorem

Suppose that the mapping $H_\varsigma : [0, 1] \times [0, 1] \times C([0, 1], R_+) \to R$ is a continuous mapping:

$$|H_\varsigma(k, h, f) - H_\varsigma(k, h, c)| \le \frac{\tau\eta_\tau(\tilde{A}_M, c)}{\tau\eta_\tau(f, c) + 1}\varsigma^{kj\tau h} \tag{2.17}$$

for every $h, k \in [0,1]$ and $f, c \in C([0,1], R)$. Then, integral equation (2.16) has at most one solution in $C([0,1], R_{+})$ or equivalently the associated operator $L_{\varsigma} : R \to R$ defined by

$$(L_{\varsigma} f)(k) = \int_{0}^{k} H_{\varsigma}(k, h, f)\, dh, \tag{2.18}$$

admits a best proximity point.

Proof. By (2.17) and (2.18), we has the following information.

$$\begin{aligned}
|L_{\varsigma} f - L_{\varsigma} \varkappa| &= \int_{0}^{k} |H_{\varsigma}(k, h, f) - H_{\varsigma}(k, h, \varkappa)|\, dh, \\
&\leq \int_{0}^{k} \frac{\tau \eta_{\tau}(f, \varkappa)}{\tau \eta_{\tau}(f, \varkappa) + 1} \varsigma^{kj\tau h} dh \\
&\leq \frac{\tau \eta_{\tau}(f, \varkappa)}{\tau \eta_{\tau}(f, \varkappa) + 1} \int_{0}^{k} \varsigma^{kj\tau h} dh \\
&\leq \frac{\eta_{\tau}(f, \varkappa)}{\tau \eta_{\tau}(f, \varkappa) + 1} \varsigma^{kj\tau k}.
\end{aligned}$$

This implies

$$\begin{aligned}
|L_{\varsigma} f - L_{\varsigma} \varkappa|\, \varsigma^{kj - \tau k} &\leq \frac{\eta_{\tau}(f, \varkappa)}{\tau \eta_{\tau}(f, \varkappa) + 1} \\
\|L_{\varsigma} f - L_{\varsigma} \varkappa\|_{\tau} &\leq \frac{\eta_{\tau}(f, \varkappa)}{\tau \eta_{\tau}(f, \varkappa) + 1} \\
\frac{\tau \eta_{\tau}(f, \varkappa) + 1}{\eta_{\tau}(f, \varkappa)} &\leq \frac{1}{\|L_{\varsigma} f - L_{\varsigma} \varkappa\|_{\tau}} \\
\tau + \frac{1}{\eta_{\tau}(f, \varkappa)} &\leq \frac{1}{\|L_{\varsigma} f - L_{\varsigma} \varkappa\|_{\tau}}
\end{aligned}$$

which further implies

$$\tau - \frac{1}{\|L_{\varsigma} f - L_{\varsigma} \varkappa\|_{\tau}} \leq \frac{-1}{\eta_{\tau}(f, \varkappa)}.$$

So all the conditions of Theorem 2.2.5 are satisfied for $\hat{H}_{m}(\varkappa) = \frac{-1}{\varkappa}$; $\varkappa > 0$ and $\Phi(\varkappa) = \hat{H}_{m}(\varkappa) - \tau$. Hence, the integral equation (2.16) admits a solution. ∎

2.6 Conclusion

The theorems provided here establish a broad criterion a bpp of improved IPC-II. The results will extend earlier results of Basha [1], Altun and Taşdemir [23], Beg et al. [9], Espinola et al. [5], Suzuki [4] and others.

Chapter 3

Best proximity point results for a class of nonlinear contractions in metric spaces

3.1 Introduction

Best proximity point theorems look into the possibility of such *best proximity point* for approximate solutions to the fixed point equation $F(\hbar) = \hbar$ in the absence of a precise solution. In order to produce global optimal approximate solutions to some fixed point equations, aims to establish best proximity point theorems for contractive non-self mappings via interpolation. Iterative strategies are also provided to find such ideal approximative solutions in addition to proving the presence of best proximity points.

3.2 Modified proximal contractions

In this section, we explain modified proximal contractions and show that it generalizes proximal contractions. We prove the existence of the best proximity points of modified proximal contractions in a complete metric space.

3.2.1 Definition

Let $(\mathfrak{W}, \vartheta)$ be a complete metric space, and $\mathfrak{C}, \mathfrak{D}$ are subsets of $\mathfrak{W}$. A mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ is said to be a $(\mathfrak{J}, \pounds)$-prox contrs if

$$\left.\begin{array}{l} \vartheta\left(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1\right) = \vartheta\left(\mathfrak{C}, \mathfrak{D}\right) \\ \vartheta\left(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2\right) = \vartheta\left(\mathfrak{C}, \mathfrak{D}\right) \end{array}\right\} \Rightarrow \mathfrak{J}\left(\vartheta\left(\mathfrak{b}_1, \mathfrak{b}_2\right)\right) \leq \pounds\left(\vartheta\left(\mathfrak{m}_1, \mathfrak{m}_2\right)\right) \tag{3.1}$$

for all $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{m}_1, \mathfrak{m}_2 \in \mathfrak{C}$ with $\mathfrak{b}_1 \neq \mathfrak{b}_2$, where $\mathfrak{J}, \pounds : \mathbb{R}^+ \to \mathbb{R}$ are two mappings.

3.2.2 Example

Let $\mathfrak{W} = \mathbb{R}^2$ and define the function $\vartheta : \mathfrak{W} \times \mathfrak{W} \to [0, \infty)$ by

$$\vartheta((\mathfrak{b}, \mathfrak{m}), (\mathfrak{u}, \mathfrak{v})) = |\mathfrak{b} - \mathfrak{u}| + |\mathfrak{m} - \mathfrak{v}| \text{ for all } (\mathfrak{b}, \mathfrak{m}), (\mathfrak{u}, \mathfrak{v}) \in \mathfrak{W}.$$

Then $(\mathfrak{W}, \vartheta)$ is a $m.s.$ Let $\mathfrak{C}, \mathfrak{D}$ be the subsets of $\mathfrak{W}$ defined by

$$\mathfrak{C} = \{(0, \mathfrak{m}); 0 \leq \mathfrak{m} \leq 1\}, \ \ \mathfrak{D} = \{(1, \mathfrak{m}); 0 \leq \mathfrak{m} \leq 1\}, \text{ then } \vartheta(\mathfrak{C}, \mathfrak{D}) = 1.$$

Define the functions $\mathfrak{J}, \mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathfrak{J}(\mathfrak{z}) = \mathfrak{z} \text{ and } \pounds(\mathfrak{z}) = \mathfrak{z} - \frac{\mathfrak{z}^2}{2}, \mathfrak{z} \in \mathbb{R}^+.$$

Define the mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ by $\mathfrak{P}((0,r)) = (1, r - \frac{r^2}{2})$ for all $(0,r) \in \mathfrak{C}$. We show that $\mathfrak{P}$ is a $(\mathfrak{J}, \pounds)$-prox contrs. For $\mathfrak{b} = (0, \mathfrak{b}_1)$, $\mathfrak{u} = (0, \mathfrak{b}_2)$ and $\mathfrak{m}_1 = (0, \mathfrak{a}_1)$, $\mathfrak{m}_2 = (0, \mathfrak{a}_2)$ (let $\mathfrak{a}_1 > \mathfrak{a}_2$), we have

$$\vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{m}_1) = \vartheta(\mathfrak{C}, \mathfrak{D}) \tag{3.2}$$

$$\vartheta(\mathfrak{u}, \mathfrak{P}\mathfrak{m}_2) = \vartheta(\mathfrak{C}, \mathfrak{D}). \tag{3.3}$$

We note that the equations (3.2) and (3.3) can further be simplified to have the following information:

$$\begin{aligned} \mathfrak{b}_1 &= \mathfrak{a}_1 - \frac{\mathfrak{a}_1^2}{2}, \\ \mathfrak{b}_2 &= \mathfrak{a}_2 - \frac{\mathfrak{a}_2^2}{2}. \end{aligned}$$

This implies that

$$\begin{aligned} \mathfrak{J}(\vartheta(\mathfrak{b}, \mathfrak{u})) &= \mathfrak{J}(\vartheta((0, \mathfrak{b}_1), (0, \mathfrak{b}_2))) = (\mid 0 - 0 \mid + \mid \mathfrak{b}_1 - \mathfrak{b}_2 \mid) \\ &\leq (\mathfrak{a}_1 - \mathfrak{a}_2) - \frac{1}{2}(\mathfrak{a}_1 - \mathfrak{a}_2)^2 \\ &= \vartheta(\mathfrak{m}_1, \mathfrak{m}_2) - \frac{1}{2}\left(\vartheta(\mathfrak{m}_1, \mathfrak{m}_2)\right)^2 = \pounds(\vartheta(\mathfrak{m}_1, \mathfrak{m}_2)) \end{aligned}$$

This shows that $\mathfrak{P}$ is a $(\mathfrak{J}, \pounds)$-prox contrs. Next, we show that it is not a prox contraction. Since

$$\begin{aligned} \vartheta(\mathfrak{b}, \mathfrak{P}\mathfrak{m}_1) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \\ \vartheta(\mathfrak{u}, \mathfrak{P}\mathfrak{m}_2) &= \vartheta(\mathfrak{C}, \mathfrak{D}). \end{aligned}$$

If there exists $\mathfrak{k} \in (0,1)$ such that

$$\vartheta\left(\mathfrak{b}, \mathfrak{u}\right) \leq \mathfrak{k}\vartheta\left(\mathfrak{m}_1, \mathfrak{m}_2\right).$$

Then,

$$\begin{aligned}
\vartheta((0,\mathfrak{b}_1),(0,\mathfrak{b}_2)) &\leq \mathfrak{k}\vartheta((0,\mathfrak{a}_1),(0,\mathfrak{a}_2)) \\
(|\, 0-0 \,| + |\, \mathfrak{b}_1 - \mathfrak{b}_2 \,|) &\leq \mathfrak{k}(|\, 0-0 \,| + |\, \mathfrak{a}_1 - \mathfrak{a}_2 \,|) \\
\mathfrak{a}_1 - \frac{\mathfrak{a}_1^2}{2} - \mathfrak{a}_2 + \frac{\mathfrak{a}_2^2}{2} &\leq \mathfrak{k}(\mathfrak{a}_1 - \mathfrak{a}_2) \\
1 + \frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2} &\leq \mathfrak{k}.
\end{aligned}$$

This is a contradiction. Hence, $\mathfrak{P}$ is not a prox contraction.

3.2.3 Theorem

Let $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ be a $(\mathfrak{J}, \pounds)$-prox contrs defined on a complete *m.s* $(\mathfrak{W}, \vartheta)$ and $\mathfrak{C}, \mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is a-compact *w.r.t* $\mathfrak{C}$. If $\mathfrak{J}$ is nd and $\limsup_{t\to\epsilon+} \pounds(t) < \mathfrak{J}(\epsilon+)$ for any $\epsilon > 0$. $\mathfrak{C}_0$ is non-void subset of $\mathfrak{C}$ such that $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.Then $\mathfrak{P}$ has a *bpp*.

Proof. Let $b_0 \in C_0$. Since $P(b_0) \in P(C_0) \subseteq D_0$, there exists $b_1 \in C_0$ such that, $\vartheta(b_1, P(b_0)) = \vartheta(C, D)$. Also we have $P(b_1) \in P(C_0) \subseteq D_0$, so, there exist $b_2 \in C_0$ such that $\vartheta(b_2, P(b_1)) = \vartheta(C, D)$. Then C_0 implies to have a seq $\{b_n\} \subseteq C_0$ such that

$$\vartheta(\mathfrak{b}_n, \mathfrak{P}(\mathfrak{b}_{n-1})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

If $\exists\, n \in N$ such that $b_n = b_{n+1}$, then by (3.4), then b_n is a bpp of the mapping P. If $b_{n-1} \neq b_n \;\forall\, n \in N$, then by (3.4), we have

$$\begin{aligned}
\vartheta(\mathfrak{b}_n, \mathfrak{P}(\mathfrak{b}_{n-1})) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \\
\vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) &= \vartheta(\mathfrak{C}, \mathfrak{D}), \text{ for all } n \geq 1.
\end{aligned}$$

Thus, by (3.1), we have

$$\mathfrak{J}(\vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1})) \leq \pounds(\vartheta(\mathfrak{b}_{n-1}, \mathfrak{b}_n)), \text{ for all } \mathfrak{b}_{n-1}, \mathfrak{b}_n, \mathfrak{b}_{n+1} \in \mathfrak{C}.$$

Let $\vartheta(b_n, b_{n+1}) = \theta_n$,

$$\mathfrak{J}(\theta_n) \leq \mathcal{F}(\theta_{n-1}) < \mathfrak{J}(\theta_{n-1}). \tag{3.5}$$

If $\theta > 0$, so that, by (3.5), we obtain the following:

$$\mathfrak{J}(\theta+) = \lim_{n\to\infty} \mathfrak{J}(\theta_n) \leq \lim_{n\to\infty} \mathcal{F}(\theta_{n-1}) \leq \lim_{t\to\theta+} \sup \mathcal{F}(t).$$

This defies presumption (i), hence, $\theta = 0$ and $\lim_{n\to\infty} \vartheta(b_n, b_{n+1}) = 0$. Now (i) and Lemma 1.1.12, we conclude that $\{b_n\}$ is a Cauchy seq. Since (W, ϑ) is a complete ms. Then $\exists\, b^* \in C$, st $\lim_{n\to\infty} \vartheta(b_n, b^*) = 0$. Moreover,

$$\begin{aligned} \vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) \\ &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{C}, \mathfrak{D}) \\ &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}^*, \mathfrak{D}). \end{aligned}$$

Therefore, $\vartheta(b^*, P(b_n)) \to \vartheta(b^*, D)$ as $n \to \infty$. Since D is a-compact w.r.t C, there exists a subseq $\{P(b_{n_\mathfrak{k}})\}$ of $\{P(b_n)\}$. Such that $P(b_{n_\mathfrak{k}}) \to m^* \in D$ as $k \to \infty$. Thus, by solving the following eqation with $k \to \infty$,

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \tag{3.6}$$

we have,

$$\vartheta(\mathfrak{b}^*, \theta^*) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Since, $l^* \in C_0$, so, $P(b^*) \in P(C_0) \subseteq D_0$ and $p \in C_0$

$$\vartheta(\mathfrak{p}, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}). \tag{3.7}$$

Now, (3.6) and (3.7), by (3.1) we have

$$\mathfrak{J}(\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p})) \leq \mathfrak{b}(\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*)) < \mathfrak{J}(\vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*)), \text{ for all } \mathfrak{k} \in \mathbb{N}.$$

Since, J is nd,

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p}) < \vartheta(\mathfrak{b}_{n_{\mathfrak{k}}}, \mathfrak{b}^*)$$

Thus, as $k \to \infty$, we have $\vartheta(b^*, p) = 0$ or $b^* = p$. Finally, by (3.7) we have

$$\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Hence, b^* is a bpp of the mapping P. ∎

3.2.4 Theorem

Let $P\colon C \to D$ be a $(\mathfrak{J}, \pounds)$-prox contrs defined on a complete m.s (W, ϑ) and C, D be nonvoid, closed subsets of W such that D is a-compact w.r.t C. If J is nd and $\{J(t_n)\}$ and $\{\pounds(t_n)\}$ are cgt seqs st $\lim_{n\to\infty} J(t_n) = \lim_{n\to\infty} \pounds(t_n)$, then $\lim_{n\to\infty} t_n = 0$. C_0 is non-empty subset of C st $P(C_0) \subseteq D_0$.Then P admits a bpp.

Proof. As Theorem 3.2.3, we have

$$\mathfrak{J}(\theta_n) \leq \pounds(\theta_{n-1}) < \mathfrak{J}(\theta_{n-1}). \tag{3.8}$$

By (3.8), then $\{J(\theta_n)\}$ is a strictly decreasing seq.

$$\inf_{\theta_n > \varepsilon} \mathfrak{J}(\theta_n) > -\infty \text{ for every } \varepsilon > 0, n \in \mathbb{N}.$$

From lemma 1.1.20, indicated that $\theta_n \to 0$ as $n \to \infty$. Second, the seq $\{J(\theta_n)\}$ is cgt if it is bounded below. The seq $\{\pounds(\theta_n)\}$ likewise cgs by (3.8). Using (i), we have

$\lim_{n\to\infty}\theta_n = 0$, for any seq $\{\mathfrak{b}_n\}$ in C. Now, the rest of the proof aligns with the methodology outlined in Theorem 3.2.3, we have

$$\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Hence, b^* is a bpp of the mapping P. ■

3.2.5 Example

Let $\mathfrak{W} = \mathbb{R}^2$ and define the function $\vartheta : \mathfrak{W} \times \mathfrak{W} \to [0, \infty)$ by

$$\vartheta((\mathfrak{b}, \mathfrak{m}), (\mathfrak{u}, \mathfrak{v})) = |\mathfrak{b} - \mathfrak{u}| + |\mathfrak{m} - \mathfrak{v}| \text{ for all } (\mathfrak{b}, \mathfrak{m}), (\mathfrak{u}, \mathfrak{v}) \in \mathfrak{W}.$$

Then $(\mathfrak{W}, \vartheta)$ is a complete *m.s.* Let $\mathfrak{C}, \mathfrak{D}$ be the subsets of $\mathfrak{W}$ defined by

$$\mathfrak{C} = \{(0, \mathfrak{m}); 0 \leq \mathfrak{m} \leq 1\},\ \ \mathfrak{D} = \{(1, \mathfrak{m}); 0 \leq \mathfrak{m} \leq 1\},\ \text{then } \vartheta(\mathfrak{C}, \mathfrak{D}) = 1.$$

Here $\mathfrak{C}_0 = \mathfrak{C}$ and $\mathfrak{D}_0 = \mathfrak{D}$. Define the mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ by $\mathfrak{P}((0, r)) = (1, \frac{r}{2})$ for all $(0, r) \in \mathfrak{C}$. Thus $\mathfrak{P}(\mathfrak{C}_0) = \mathfrak{D}_0$. Define the functions $\mathfrak{J}, \pounds : \mathbb{R}^+ \to \mathbb{R}$ by

$$\mathfrak{J}(\mathfrak{b}) = 2\mathfrak{b} \text{ and } \pounds(\mathfrak{b}) = \mathfrak{b};\ \mathfrak{b} \in \mathbb{R}^+.$$

As $\mathfrak{J}(\mathfrak{b}) > \pounds(\mathfrak{b})$ for every $\mathfrak{b} \geq t > 0$. Also $\lim_{s\to\varepsilon^+} \mathfrak{J}(\mathfrak{b}) > \lim_{\mathfrak{b}\to\varepsilon^+} \sup \pounds(\mathfrak{b})$. We need to check whether $\mathfrak{P}$ is a $(\mathfrak{J}, \pounds)$-prox contrs or not.

For $\mathfrak{u}_1 = (0, \mathfrak{b})$, $\mathfrak{u}_2 = (0, \mathfrak{m})$ and $\mathfrak{v}_1 = (0, 2\mathfrak{b})$,$\mathfrak{v}_2 = (0, 2\mathfrak{m})$

$$\begin{aligned} \vartheta(\mathfrak{u}_1, \mathfrak{P}\mathfrak{v}_1) &= \vartheta((0, \mathfrak{b}), \mathfrak{P}(0, 2\mathfrak{b})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \\ \vartheta(\mathfrak{u}_2, \mathfrak{P}\mathfrak{v}_2) &= \vartheta((0, \mathfrak{m}), \mathfrak{P}(0, 2\mathfrak{m})) = \vartheta(\mathfrak{C}, \mathfrak{D}). \end{aligned}$$

This implies that,

$$\mathfrak{J}(\vartheta(\mathfrak{u}_1, \mathfrak{u}_2)) \leq \pounds(\vartheta(\mathfrak{v}_1, \mathfrak{v}_2))$$

Therefore, the $(\mathfrak{J},\pounds)$-prox contrs is fulfilled. Also, $(0,0)$ is the *bpp* of the mapping $\mathfrak{P}$. Hence, all the conditions of the Theorem 3.2.3 are hold.

3.3 Modified Hardy Rogers type proximal contraction

3.3.1 Definition

Let $(\mathfrak{W},\vartheta)$ be a complete *m.s*, and $\mathfrak{C},\mathfrak{D}$ be a pair of nonvoid subsets of $\mathfrak{W}$. A mapping $\mathfrak{P}:\mathfrak{C}\to\mathfrak{D}$ is said to be a $(\mathfrak{J},\pounds)$-intplv H-R type prox contrs if there exist $\alpha,\beta,\gamma,\delta\in(0,1)$ satisfying $\alpha+\beta+\gamma+\delta<1$ such that

$$\left.\begin{array}{l}\vartheta\left(\mathfrak{b}_1,\mathfrak{P}\mathfrak{m}_1\right)=\vartheta\left(\mathfrak{C},\mathfrak{D}\right)\\ \vartheta\left(\mathfrak{b}_2,\mathfrak{P}\mathfrak{m}_2\right)=\vartheta\left(\mathfrak{C},\mathfrak{D}\right)\end{array}\right\}\Rightarrow\mathfrak{J}\left(\vartheta\left(\mathfrak{b}_1,\mathfrak{b}_2\right)\right)\leq\pounds\left(\begin{array}{l}\vartheta\left(\mathfrak{m}_1,\mathfrak{m}_2\right)^{\alpha}\vartheta\left(\mathfrak{m}_1,\mathfrak{b}_1\right)^{\beta}\vartheta\left(\mathfrak{m}_2,\mathfrak{b}_2\right)^{\gamma}\\ \left(\frac{1}{2}\left(\vartheta\left(\mathfrak{m}_1,\mathfrak{b}_2\right)+\vartheta\left(\mathfrak{m}_2,\mathfrak{b}_1\right)\right)\right)^{1-\alpha-\beta-\gamma}\end{array}\right), \tag{3.8}$$

for all distinct $\mathfrak{b}_1,\mathfrak{b}_2,\mathfrak{m}_1,\mathfrak{m}_2\in\mathfrak{C}$ and $\mathfrak{b}_i\neq\mathfrak{m}_i,\ i\in\{1,2\}$ with $\vartheta\left(\mathfrak{P}\mathfrak{b},\mathfrak{P}\mathfrak{m}\right)>0$; $\mathfrak{J},\pounds:\mathbb{R}^+\to\mathbb{R}$ are two functions.

The following example shows that $(\mathfrak{J},\pounds)$-H-R type intplv prox contrs generalizes the H-R type intplv prox contrs [23].

3.3.2 Example

Let $\mathfrak{W}=\mathbb{R}$ and define the function $\vartheta:\mathfrak{W}\times\mathfrak{W}\to\mathbb{R}$ by

$$\vartheta\left(\mathfrak{b},\mathfrak{m}\right)=\mid\mathfrak{b}-\mathfrak{m}\mid$$

Then $(\mathfrak{W}, \vartheta)$ is a $m.s.$ Let $\mathfrak{C}, \mathfrak{D}$ be the subsets of $\mathfrak{W}$ defined as

$$\mathfrak{C} = \{1,2,3,4,5\}, \mathfrak{D} = \{1,2,3,4,5,6,7\} \text{ then } \vartheta(\mathfrak{C}, \mathfrak{D}) = 0$$

Define the functions $\mathfrak{J}, \pounds : \mathbb{R}^+ \to \mathbb{R}$

$$\mathfrak{J}(\mathfrak{b}) = \begin{cases} \mathfrak{b}+1 & \text{for } \mathfrak{b}=2 \\ \mathfrak{b}+10 & \text{for } \mathfrak{b} \neq 2 \end{cases} \text{ and } \pounds(\mathfrak{b}) = \left\{ \begin{array}{ll} \frac{\mathfrak{b}}{2} & \text{for } \mathfrak{b}=2 \\ \mathfrak{b}+5 & \text{otherwise} \end{array} \right\}$$

Define the mapping $\mathfrak{P} : \mathfrak{C} \to \mathfrak{D}$ by $\mathfrak{P}(\mathfrak{b}) = \mathfrak{b}+1$ for all $\mathfrak{b} \in \mathfrak{C}$. We show that $\mathfrak{P}$ is a $(\mathfrak{J}, \pounds)$-intplv H-R type prox contrs. For $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{m}_1$, $\mathfrak{m}_2 \in \mathfrak{C}$,and $\alpha = \frac{1}{8}, \beta = \frac{1}{7}, \gamma = \frac{1}{6}$

$$\begin{aligned} \vartheta(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \\ \vartheta(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \end{aligned}$$

implies

$$\mathfrak{J}(\vartheta(\mathfrak{b}_1, \mathfrak{b}_2)) \leq \pounds\left(\vartheta(\mathfrak{m}_1, \mathfrak{m}_2)^{\alpha} \vartheta(\mathfrak{m}_1, \mathfrak{b}_1)^{\beta} \vartheta(\mathfrak{m}_2, \mathfrak{b}_2)^{\gamma} \left(\frac{1}{2}(\vartheta(\mathfrak{m}_1, \mathfrak{b}_2) + \vartheta(\mathfrak{m}_2, \mathfrak{b}_1))\right)^{1-\alpha-\beta-\gamma}\right).$$

This shows that $\mathfrak{P}$ is a $(\mathfrak{J}, \pounds)$-H-R intplv type prox contraction. However, the following calculation shows that it is not an intplv H-R type prox contrs. We know that

$$\begin{aligned} \vartheta(\mathfrak{b}_1, \mathfrak{P}\mathfrak{m}_1) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \\ \vartheta(\mathfrak{b}_2, \mathfrak{P}\mathfrak{m}_2) &= \vartheta(\mathfrak{C}, \mathfrak{D}) \end{aligned}$$

If there exists $\mathfrak{k} \in (0,1)$ such that

$$\begin{aligned} \vartheta(\mathfrak{b}_1, \mathfrak{b}_2) &\leq \mathfrak{k}\left(\vartheta(\mathfrak{m}_1, \mathfrak{m}_2)^{\alpha} \vartheta(\mathfrak{m}_1, \mathfrak{b}_1)^{\beta} \vartheta(\mathfrak{m}_2, \mathfrak{b}_2)^{\gamma} \left(\frac{1}{2}(\vartheta(\mathfrak{m}_1, \mathfrak{b}_2) + \vartheta(\mathfrak{m}_2, \mathfrak{b}_1))\right)^{1-\alpha-\beta-\gamma}\right) \\ 2 &\leq \mathfrak{k}\left((2)^{\frac{1}{8}} (1)^{\frac{1}{7}} (1)^{\frac{1}{6}} \left(\frac{1}{2}(3+1)\right)^{1-\frac{1}{8}-\frac{1}{7}-\frac{1}{6}}\right) \\ 2 &\leq \mathfrak{k}(1.6138), \end{aligned}$$

a contradiction. Hence, $\mathfrak{P}$ is not an intplv H-R type prox contraction.

3.3.3 Theorem

Let $(\mathfrak{W},\vartheta)$ be a complete *m.s* and $\mathfrak{C},\mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is a-compact *w.r.t* $\mathfrak{C}$. Let $\mathfrak{P}\colon\mathfrak{C}\to\mathfrak{D}$ be an $(\mathfrak{J},\pounds)-$ intplv H-R type prox contrs. If $\mathfrak{J}$ is nd and for any $\varepsilon>0$,

$$\lim_{t\to\varepsilon+}\sup\pounds\left(t\right)<\mathfrak{J}\left(\varepsilon+\right).$$

$\mathfrak{C}_0$ is nonvoid subset of $\mathfrak{C}$ such that $\mathfrak{P}\left(\mathfrak{C}_0\right)\subseteq\mathfrak{D}_0$.Then $\mathfrak{P}$ has a *bpp*.

Proof. Let $\mathfrak{b}_0\in\mathfrak{C}_0$. Since $\mathfrak{P}(\mathfrak{b}_0)\in\mathfrak{P}(\mathfrak{C}_0)\subseteq\mathfrak{D}_0$, there exist $\mathfrak{b}_1\in\mathfrak{C}_0$ such that, $\vartheta(\mathfrak{b}_1,\mathfrak{P}(\mathfrak{b}_0))=\vartheta(\mathfrak{C},\mathfrak{D})$.Similarly, for $\mathfrak{P}(\mathfrak{b}_1)\in\mathfrak{P}(\mathfrak{C}_0)\subseteq\mathfrak{D}_0$, there exists $\mathfrak{b}_2\in\mathfrak{C}_0$ such that $\vartheta(\mathfrak{b}_2,\mathfrak{P}(\mathfrak{b}_1))=\vartheta(\mathfrak{C},\mathfrak{D})$. Then $\mathfrak{C}_0$ implies to have a *seq* $\{\mathfrak{b}_n\}\subseteq\mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{b}_{n+1},\mathfrak{P}(\mathfrak{b}_n))=\vartheta(\mathfrak{C},\mathfrak{D})\tag{3.9}$$

so, $\mathfrak{b}_n=\mathfrak{b}_{n+1}$, then $\mathfrak{b}_n$ is a *bpp* of the mapping $\mathfrak{P}$ (see (3.9)). Assume that $\mathfrak{b}_{n+1}\neq\mathfrak{b}_n$ for all $n\in\mathbb{N}$, then by (3.9) we have

$$\begin{aligned}\vartheta(\mathfrak{b}_n,\mathfrak{P}(\mathfrak{b}_{n-1}))&=\vartheta(\mathfrak{C},\mathfrak{D}),\\ \vartheta(\mathfrak{b}_{n+1},\mathfrak{P}(\mathfrak{b}_n))&=\vartheta(\mathfrak{C},\mathfrak{D}),\ \text{for all } n\geq 1.\end{aligned}$$

Thus by (3.8), we have

$$\begin{aligned}
\mathfrak{J}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})) &\leq \pounds\begin{pmatrix} (\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n))^{\alpha}(\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n))^{\beta}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}\left(\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_{n+1})+\vartheta(\mathfrak{b}_n,\mathfrak{b}_n)\right)\right)^{1-\alpha-\beta-\gamma} \end{pmatrix} \\
\mathfrak{J}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})) &= \pounds\begin{pmatrix} (\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n))^{\alpha}(\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n))^{\beta}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}\left(\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_{n+1})\right)\right)^{1-\alpha-\beta-\gamma} \end{pmatrix} \\
\mathfrak{J}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})) &\leq \pounds\begin{pmatrix} (\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n))^{\alpha}(\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n))^{\beta}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}\left(\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n)+\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})\right)\right)^{1-\alpha-\beta-\gamma} \end{pmatrix} \\
\mathfrak{J}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})) &\leq \pounds\begin{pmatrix} (\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n))^{\alpha+\beta}(\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1}))^{\gamma} \\ \left(\frac{1}{2}\left(\vartheta(\mathfrak{b}_{n-1},\mathfrak{b}_n)+\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})\right)\right)^{1-\alpha-\beta-\gamma} \end{pmatrix},
\end{aligned}$$

for all distinct $\mathfrak{b}_{n-1},\mathfrak{b}_n,\mathfrak{b}_{n+1}\in\mathfrak{C}$. Let $\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})=\theta_n$. Since, $\pounds(t)<\mathfrak{J}(t)$ for all $t>0$, so we get

$$\mathfrak{J}(\theta_n)<\mathfrak{J}\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{\gamma}\left(\frac{1}{2}(\theta_n+\theta_{n-1})\right)^{1-\alpha-\beta-\gamma}\right). \tag{3.10}$$

Assume that for some $n\geq 1$, $\theta_{n-1}<\theta_n$. According to (3.10), we have $(\theta_n)^{\alpha+\beta}<(\theta_n)^{\alpha+\beta}$ since $\mathfrak{J}$ is non- decreasing. As a result, for every $n\in\mathbb{N}$, we obtain $\theta_n<\theta_{n-1}$. This indicates a strictly decreasing *seq* $\{\theta_n\}$. As a result, it approaches an element $\theta\geq 0$. Consequently, $\theta=0$, in case $\theta>0$, we can derive the following via (3.10):

$$\mathfrak{J}(\theta+)=\lim_{n\to\infty}\mathfrak{J}(\theta_n)\leq\lim_{n\to\infty}\pounds\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{\gamma}\left(\frac{1}{2}(\theta_n+\theta_{n-1})\right)^{1-\alpha-\beta-\gamma}\right)\leq\lim_{t\to\theta+}\mathfrak{b}(t)$$

This contradicts (i), hence, $\theta=0$ and $\lim_{n\to\infty}\vartheta(\mathfrak{b}_n,\mathfrak{b}_{n+1})=0$. we conclude that $\{\mathfrak{b}_n\}$ is a Cauchy *seq*. Since $(\mathfrak{W},\vartheta)$ is a complete *m.s* and $\mathfrak{C}$ is a closed subset of $\mathfrak{W}$, so, there exists $\mathfrak{b}^* \in \mathfrak{C}$, such that $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}^*) = 0$. Moreover,

$$\begin{aligned}\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}_{n+1}, \mathfrak{P}(\mathfrak{b}_n)) \\ &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{C}, \mathfrak{D}) \\ &\leq \vartheta(\mathfrak{b}^*, \mathfrak{b}_{n+1}) + \vartheta(\mathfrak{b}^*, \mathfrak{D}).\end{aligned}$$

Thus, $\vartheta(\mathfrak{b}^*, \mathfrak{P}(\mathfrak{b}_n)) \to \vartheta(\mathfrak{b}^*, \mathfrak{D})$ as $n \to \infty$. Since $\mathfrak{D}$ is a-compact *w.r.t* $\mathfrak{C}$, there exists a sub*seq* $\{\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})\}$ of $\{\mathfrak{P}(\mathfrak{b}_n)\}$ such that $\mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}}) \to \mathfrak{m}^* \in \mathfrak{D}$ as $\mathfrak{k} \to \infty$. Letting $\mathfrak{k} \to \infty$ in the following equation:

$$\vartheta(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{P}(\mathfrak{b}_{n_\mathfrak{k}})) = \vartheta(\mathfrak{C}, \mathfrak{D}), \tag{3.11}$$

we have,

$$\vartheta(\mathfrak{b}^*, \mathfrak{m}^*) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Since, $\mathfrak{b}^* \in \mathfrak{C}_0$, so $\mathfrak{P}(\mathfrak{b}^*) \in \mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$, there exists $\mathfrak{p} \in \mathfrak{C}_0$ such that

$$\vartheta(\mathfrak{p}, \mathfrak{P}(\mathfrak{b}^*)) = \vartheta(\mathfrak{C}, \mathfrak{D}). \tag{3.12}$$

Now, using (3.8) in association with (3.9) and (3.10), for all $\mathfrak{k} \in \mathbb{N}$, we have

$$\begin{aligned}\mathfrak{J}\left(\vartheta\left(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p}\right)\right) &\leq \pounds\begin{pmatrix} \left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}^*\right)\right)^\alpha \left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)^\beta \left(\vartheta\left(\mathfrak{b}^*, \mathfrak{p}\right)\right)^\gamma \\ \left(\frac{1}{2}\left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{p}\right) + \vartheta\left(\mathfrak{b}^*, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)\right)^{1-\alpha-\beta-\gamma}\end{pmatrix} \\ &< \mathfrak{J}\begin{pmatrix} \left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}^*\right)\right)^\alpha \left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)^\beta \left(\vartheta\left(\mathfrak{b}^*, \mathfrak{p}\right)\right)^\gamma \\ \left(\frac{1}{2}\left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{p}\right) + \vartheta\left(\mathfrak{b}^*, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)\right)^{1-\alpha-\beta-\gamma}\end{pmatrix}.\end{aligned}$$

By using the monotonicity of $\mathfrak{J}$, for all $\mathfrak{k} \in \mathbb{N}$, we have

$$\vartheta\left(\mathfrak{b}_{n_{\mathfrak{k}+1}}, \mathfrak{p}\right) \leq \left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}^*\right)\right)^\alpha \left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)^\beta \left(\vartheta\left(\mathfrak{b}^*, \mathfrak{p}\right)\right)^\gamma \left(\frac{1}{2}\left(\vartheta\left(\mathfrak{b}_{n_\mathfrak{k}}, \mathfrak{p}\right) + \vartheta\left(\mathfrak{b}^*, \mathfrak{b}_{n_{\mathfrak{k}+1}}\right)\right)\right)^{1-\alpha-\beta-\gamma}.$$

Thus, as $\mathfrak{k} \to \infty$, $\mathfrak{b}^* = \mathfrak{p}$. Finally, by (3.12) we have

$$\vartheta\left(\mathfrak{b}^*, \mathfrak{P}\left(\mathfrak{b}^*\right)\right) = \vartheta\left(\mathfrak{C}, \mathfrak{D}\right).$$

Hence, $\mathfrak{b}^*$ is a *bpp* of the mapping $\mathfrak{P}$. ■

3.3.4 Theorem

Let $(\mathfrak{W}, \vartheta)$ be a complete *m.s* and $\mathfrak{C}, \mathfrak{D}$ be nonvoid, closed subsets of $\mathfrak{W}$ such that $\mathfrak{D}$ is a-compact *w.r.t* $\mathfrak{C}$. Let $\mathfrak{P}\colon \mathfrak{C} \to \mathfrak{D}$ be an $(\mathfrak{J}, \pounds)$-intplv H-R type prox contrs. If $\mathfrak{J}$ is non-decreasing and $\{\mathfrak{J}(t_n)\}$ and $\{\mathfrak{b}(t_n)\}$ are convergent *seq*s such that

$$\lim_{n\to\infty} \mathfrak{J}(t_n) = \lim_{n\to\infty} \mathfrak{b}(t_n),$$

then $\lim_{n\to\infty} t_n = 0.\mathfrak{C}_0$ is nonvoid subset of $\mathfrak{C}$ such that $\mathfrak{P}(\mathfrak{C}_0) \subseteq \mathfrak{D}_0$.Then $\mathfrak{P}$ has a *bpp*.

Proof. The proof aligns with the methodology outlined in Theorem 3.3.3, we have

$$\begin{aligned} \mathfrak{J}(\theta_n) &\le \pounds\left((\theta_{n-1})^{\alpha+\beta} (\theta_n)^{\gamma} \left(\frac{1}{2}(\theta_n + \theta_{n-1})\right)^{1-\alpha-\beta-\gamma} \right) \\ &< \mathfrak{J}\left((\theta_{n-1})^{\alpha+\beta} (\theta_n)^{\gamma} \left(\frac{1}{2}(\theta_n + \theta_{n-1})\right)^{1-\alpha-\beta-\gamma} \right). \end{aligned} \tag{3.13}$$

We establish that $\{\mathfrak{J}(\theta_n)\}$ is a strictly decreasing *seq* by (3.13). Lemma 1.1.12, indicates that $\theta_n \to 0$ as n approaches to ∞. Second, the *seq* $\{\mathfrak{J}(\theta_n)\}$ is cgt if it is bounded below. The *seq* $\{\mathfrak{b}(\theta_n)\}$ likewise cgs by (3.13), and, both have the same limit. For each *seq* $\{\mathfrak{b}_n\}$ in $\mathfrak{C}$ we have $\lim_{n\to\infty} \vartheta(\mathfrak{b}_n, \mathfrak{b}_{n+1}) = 0$ according to (i). Now, according to Theorem 3.3.3 proof, we have

$$\vartheta(\mathfrak{b}^*, \mathfrak{P}\mathfrak{b}^*) = \vartheta(\mathfrak{C}, \mathfrak{D}).$$

Hence, $\mathfrak{b}^*$ is a *bpp* of the mapping $\mathfrak{P}$. ■

3.4 Conclusion

Generalized interpolative proximal contractions provide a robust framework for solving proximity problems in various mathematical and applied contexts. The established existence and uniqueness results facilitate their practical use, offering significant insights and solutions in various applied mathematics and engineering fields.

Chapter 4

New results on best proximity points via generalized fuzzy interpolative proximal contractions

4.1 Introduction

In this chapter, we define $\left(\hat{H}_m, \Phi\right)$-PC and show that it generalizes PC. We ensure the bpp of $\left(\hat{H}_m, \Phi\right)$-proximal contraction in a complete nafms followed by supporting examples. Moreover, we bpp of complete non-Archimedean fuzzy metric space.

4.2 Proinov type non-Archimedean fuzzy proximal contraction

In this section, we prove the existence of bpp of $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy pc and $\left(\hat{H}_m, \Phi\right)$-interpolative nafms in a complete nafms.

4.2.1 Definition

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms and R, S be subsets of Ω. A mapping $\Upsilon : R \to S$ is called $\left(\hat{H}_m, \Phi\right)$-nafpc of the first kind

$$\left.\begin{array}{l} \widehat{\Sigma}\left(\varsigma^{kj}, \Upsilon v_1, \Bbbk^{sw}\Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right) \\ \widehat{\Sigma}\left(\zeta^{jk}, \Upsilon v_2, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right) \end{array}\right\} \Rightarrow \hat{H}_m\left(\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}\right)\right) \geq \Phi\left(\widehat{\Sigma}\left(v_1, v_2, \Bbbk^{sw}\right)\right), \tag{4.1}$$

for all distinct $\varsigma^{kj}, \zeta^{jk}, \jmath_1, v_2 \in R$ with $\varsigma^{kj} \neq \zeta^{jk}$, where $\hat{H}_m, \Phi : (0, 1] \to \mathbb{R}$ are two functions s.t $\Phi(t) > \hat{H}_m(t)\ \forall\ t \in (0, 1)$.

The following example shows that $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy proximal contraction generalizes non-Archimedean fuzzy proximal contraction.

4.2.2 Example

Let $\Omega = \mathbb{R}^2$, $\widehat{\Sigma} : \Omega \times \Omega \times (0, +\infty) \to [0, 1]$ by

$$\widehat{\Sigma}(\varsigma, v, \Bbbk^{sw}) = \varsigma^{kj - \frac{\eth^d\left(\left(\varsigma^{kj}, v_1\right), \left(\zeta^{jk}, v_2\right)\right)}{\Bbbk^{sw}}}$$

$$\eth^d((\varsigma^{kj}, v_1), (\zeta^{jk}, v_2)) = \sqrt[2]{\left(\varsigma^{kj} - \zeta^{jk}\right)^2 + (v_1 - v_2)^2} \text{ for all } \left(\varsigma^{kj}, v_1\right), \left(\zeta^{jk}, v_2\right) \in \Omega.$$

Then $\left(\Omega, \widehat{\Sigma}, *\right)$ is a nafms.

Let R, S be the subsets of Ω defined as

$$R = \{(0, \varsigma); \varsigma \in \mathbb{R}\},\ S = \{(1, \varsigma); \varsigma \in \mathbb{R}\},\ \text{then } \widehat{\Sigma}(R, S, \Bbbk^{sw}) = \varsigma^{kj - \frac{1}{\Bbbk^{sw}}}.$$

Define the functions $\hat{H}_m, \Phi : (0, 1] \to \mathbb{R}$ by

$$\hat{H}_m(s) = \sqrt[2]{s} \text{ and } \Phi(s) = s^2 \text{ for } s \in (0, 1)\,.$$

Define the mapping $\Upsilon : R \to S$ by $\Upsilon\left((0, \gamma)\right) = (1, 2\gamma)$ for all $(0, \gamma) \in R$. Let us consider $\varsigma^{kj} = (0, 2)$, $\upsilon_1 = (0, 1)$ and $\zeta^{jk} = (0, 4)$, $\upsilon_2 = (0, 2), \Bbbk^{sw} = 1$

$$\widehat{\Sigma}(\varsigma^{kj}, \Upsilon\upsilon_1, \Bbbk^{sw}) = \widehat{\Sigma}((0, 2), \Upsilon(0, 1), \Bbbk^{sw}) = \varsigma^{kj - \frac{1}{\Bbbk^{sw}}} = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \tag{4.2}$$

$$\widehat{\Sigma}(\zeta^{jk}, \Upsilon\upsilon_2, \Bbbk^{sw}) = \widehat{\Sigma}((0, 4), \Upsilon(0, 2), \Bbbk^{sw}) = \varsigma^{kj - \frac{1}{\Bbbk^{sw}}} = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \tag{4.3}$$

This implies that

$$\hat{H}_m\left(\widehat{\Sigma}(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw})\right) \geq \Phi\left(\widehat{\Sigma}(\upsilon_1, \upsilon_2, \Bbbk^{sw})\right)$$

$$\begin{aligned} \widehat{\Sigma}(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}) &= \widehat{\Sigma}((0, 2), (0, 4), \Bbbk^{sw}) = 0.1353 \\ \widehat{\Sigma}(\upsilon_1, \upsilon_2, \Bbbk^{sw}) &= \widehat{\Sigma}((0, 1)\,, (0, 2), \Bbbk^{sw}) = 0.3679. \end{aligned}$$

$$\begin{aligned} \hat{H}_m\,(0.1353) &\geq \Phi\,(0.3679) \\ 0.3673 &> 0.1354 \end{aligned}$$

This shows that Υ is a $\left(\hat{H}_m, \Phi\right)$-nafmpc. We know that

$$\begin{aligned} \widehat{\Sigma}(\varsigma^{kj}, \Upsilon\upsilon_1, \Bbbk^{sw}) &= \widehat{\Sigma}((0, 2), \Upsilon(0, 1), \Bbbk^{sw}) = \varsigma^{kj - \frac{1}{\Bbbk^{sw}}} = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \\ \widehat{\Sigma}(\zeta^{jk}, \Upsilon\upsilon_2, \Bbbk^{sw}) &= \widehat{\Sigma}((0, 4), \Upsilon(0, 2), \Bbbk^{sw}) = \varsigma^{kj - \frac{1}{\Bbbk^{sw}}} = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \end{aligned}$$

This implies that

$$\widehat{\Sigma}(\varsigma^{kj},\zeta^{jk},\Bbbk^{sw}) \geq \widehat{\Sigma}(\upsilon_1,\upsilon_2,\Bbbk^{sw})$$
$$0.1353 \geq 0.3679$$

This shows that, Υ is not a nafmpc.

4.2.3 Lemma

Let $\{\varsigma_n^{kj}\}$ be a seq in $\left(\Omega,\widehat{\Sigma},*\right)$ s.t $\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\Bbbk^{sw}\right) > 1-\varepsilon\ \forall\ \Bbbk^{sw}>0$ and $\varepsilon\in(0,1)$ and $\Upsilon:R\to S$ be a map satisfying (4.1). If the functions $\hat{H}_m,\Phi:(0,1]\to R$ are s.t

(1) $\liminf_{t\to\varepsilon-}\Phi(t) > \hat{H}_m(\varepsilon-)$ for any $\varepsilon\in(0,1)$.

Then $\{\varsigma_n^{kj}\}$ is cauchy.

Proof. If $\{\varsigma_{n_k}\},\{\varsigma_{m_k}\}$ and $\varepsilon\in(0,1)$ such that the equations (1.5) and (1.6) hold. By (1.5), we get that $\widehat{\Sigma}\left(\varsigma_{n_{k+1}},\varsigma_{m_{k+1}},\Bbbk^{sw}\right) < 1-\varepsilon$. Since, for $\varsigma_{n_k},\varsigma_{m_k},\varsigma_{m_{k+1}},\varsigma_{n_{k+1}}\in R$, we have

$$\widehat{\Sigma}\left(\varsigma_{n_{k+1}},\Upsilon\varsigma_{n_k},\Bbbk^{sw}\right) = \widehat{\Sigma}\left(R,S,\Bbbk^{sw}\right)$$
$$\widehat{\Sigma}\left(\varsigma_{m_{k+1}},\Upsilon\varsigma_{n_k},\Bbbk^{sw}\right) = \widehat{\Sigma}\left(R,S,\Bbbk^{sw}\right) \text{ for all } k\geq 1$$

Thus, by (4.1) we have

$$\widehat{\Sigma}\left(\widehat{\Sigma}\left(\varsigma_{n_{k+1}},\varsigma_{m_{k+1}},\Bbbk^{sw}\right)\right) \geq \Phi\left(\widehat{\Sigma}\left(\varsigma_{n_k},\varsigma_{m_k},\Bbbk^{sw}\right)\right),\ \text{for any } k\geq 1$$

For if $a_k=\widehat{\Sigma}\left(\varsigma_{n_{k+1}},\varsigma_{m_{k+1}},\Bbbk^{sw}\right)$ and $b_k=\widehat{\Sigma}\left(\varsigma_{n_k},\varsigma_{m_k},\Bbbk^{sw}\right)$, we have

$$\hat{H}_m\left(a_k\right) \geq \Phi\left(b_k\right),\ \text{for any } k\geq 1. \tag{4.4}$$

By (1.5) and (1.6), we have $\lim_{k\to\infty} a_k = \varepsilon-$ and $\lim_{k\to\infty} b_k = \varepsilon$. By (4.4), we get that

$$\hat{H}_m(\varepsilon-) = \lim_{k\to\infty} \hat{H}_m(a_k) \geq \lim \inf_{k\to\infty} \Phi(b_k) \geq \lim \inf_{c\to\varepsilon} \Phi(c) \tag{4.5}$$

This is contradicts to the assumption (1). Consequently, $\left\{\varsigma_n^{kj}\right\}$ is a cau seq in R. ■

4.2.4 Theorem

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafmc and S is a-compact w.r.t R. Let $\Upsilon\colon R \to S$ be an $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fpc of the first kind. If

(i) $\hat{H}_m$ is non-decreasing function and $\lim\inf_{t\to\varepsilon-} \Phi(t) > \hat{H}_m(\varepsilon-)$ for any $\varepsilon \in (0,1)$.

(ii) $\Upsilon(R_0) \subseteq S_0$.

Then Υ admits a bpp.

Proof. Since $\Upsilon(\varsigma_o^{kj}) \in \Upsilon(R_0) \subseteq S_0$, there exists $\varsigma^{kj} \in R_0$ such that,

$$\widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_o^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Also we have $\Upsilon(\varsigma^{kj}) \in \Upsilon(R_0) \subseteq S_0$. So, there exist $\zeta^{jk} \in R_0$ such that

$$\widehat{\Sigma}(\zeta^{jk}, \Upsilon(\varsigma^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}),$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \tag{4.6}$$

In light, $\exists\ n \in \mathbb{N}$ s.t $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$ then from (4.6) the point ς_n^{kj} is a bpp of Υ. If

$\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}$ for all $n \in \mathbb{N}$. Then by (4.6), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \Upsilon(\varsigma_{n-1}^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}),$$

and

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

for all $n \geq 1$. Thus, by (4.1)

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})) \geq \Phi(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})).$$

for all distinct $\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \varsigma_{n+1}^{kj} \in R$. Let $\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) = \theta_n$. We have

$$\hat{H}_m(\theta_n) \geq \Phi(\theta_{n-1}) > \hat{H}_m(\theta_{n-1}). \tag{4.7}$$

Since $\hat{H}_m$ is nd, so, by (4.7), so $\theta_n > \theta_{n-1}$ $\forall$ $n \in \mathbb{N}$. Assume on contrary that $\theta < 1$, so that (4.7), therefore the following holds:

$$\hat{H}_m(\varepsilon-) = \lim_{n\to\infty} \hat{H}_m(\theta_n) \geq \lim_{n\to\infty} \Phi(\theta_{n-1}) \geq \lim_{t\to\varsigma^{kj}-} \inf \Phi(t).$$

Which is not true to condition (i), hence, $\theta = 1$ and $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right) = 1$. The condition (i) and lemma 4.2.3, we conclude that $\left\{\varsigma_n^{kj}\right\}$ is a cau seq. Since

$\left(\Omega, \widehat{\Sigma}, *\right)$ is a cnafms. Then $\exists\ \varsigma \in R$, s.t $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma, \Bbbk^{sw}\right) = 1$. Moreover,

$$\begin{aligned}
\widehat{\Sigma}(R, S, \Bbbk^{sw}) &= \widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) * \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \Upsilon\varsigma_n^{kj}, \Bbbk^{sw}\right) \\
&= \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) * \widehat{\Sigma}(R, S, \Bbbk^{sw}).
\end{aligned}$$

This implies

$$\begin{aligned}
\widehat{\Sigma}(R, S, \Bbbk^{sw}) &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}), \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) * \widehat{\Sigma}(R, S, \Bbbk^{sw}).
\end{aligned}$$

Applying to limit as $n \to \infty$ for above inequality,

$$\begin{aligned}
\widehat{\Sigma}(R, S, \Bbbk^{sw}) &\geq 1 * \lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}), \\
&\geq 1 * 1 * \widehat{\Sigma}(R, S, \Bbbk^{sw}).
\end{aligned}$$

That is,

$$\lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Therefore, $\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma_n^{kj}\right), \Bbbk^{sw}) \to \widehat{\Sigma}(\varsigma, S, \Bbbk^{sw})$. Since S is a-compact w.r.t R, $\exists$ a subseq $\{\Upsilon(\varsigma_{n_k})\}$ of $\{\Upsilon(\varsigma_n^{kj})\}$ s.t $(\Upsilon\varsigma_{n_k}) \to \eta \in S$ as $k \to \infty$. Then, $k \to \infty$

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \Upsilon(\varsigma_{n_k}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \tag{4.8}$$

we have,

$$\widehat{\Sigma}(\varsigma, \eta, \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Since, $\varsigma \in R_0$, so, $\Upsilon(\varsigma) \in \Upsilon(R_0) \subseteq S_0$ there exists $\xi \in R_0$ such that

$$\widehat{\Sigma}(\xi, \Upsilon\varsigma, \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \tag{4.9}$$

Now, having in mind the equations (4.8) and (4.9), by (4.1) we have

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \mathbb{k}^{sw})) \geq \Phi(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma, \mathbb{k}^{sw})) > \hat{H}_m\left(\widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma, \mathbb{k}^{sw}\right)\right)$$

Since, $\hat{H}_m$ is non-decreasing function, so, we have

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \mathbb{k}^{sw}) > \widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma, \mathbb{k}^{sw}\right)$$

Thus, as $k \to \infty$, we have $\widehat{\Sigma}\left(\varsigma, \xi, \mathbb{k}^{sw}\right) = 1$ or $\varsigma = \xi$. Finally, by (4.9) we have

$$\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma\right), \mathbb{k}^{sw}) = \widehat{\Sigma}\left(R, S, \mathbb{k}^{sw}\right).$$

■

4.2.5 Theorem

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms and S is a-compact w.r.t R. Let $\Upsilon\colon R \to S$ be an $\left(\hat{H}_m, \Phi\right)$-fpc of the first kind. If

(*i*) $\hat{H}_m$ is non-decreasing and $\left\{\hat{H}_m\left(t_n\right)\right\}$ and $\left\{\Phi\left(t_n\right)\right\}$ are cgt seq s.t $\lim_{n\to\infty} \hat{H}_m\left(t_n\right) = \lim_{n\to\infty} \Phi\left(t_n\right)$, then $\lim_{n\to\infty}\left(t_n\right) = 1$

(*ii*) R_0 is non-empty subset of R such that $\Upsilon\left(R_0\right) \subseteq S_0$.

Then Υ admits a best proximity point.

4.3 Best proximity point and Proinov type proximal contraction in NAFMS

The aim of this section we introduce new findings on bppt, incorporating the functions $(\hat{H}_m, \Phi) : (0,1] \to \mathbb{R}$. The example below demonstrates that $\left(\hat{H}_m, \Phi\right)$-nafirrc is not equivalent to nafirrctpc.

4.3.1 Definition

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms, and R, S of Ω. A mapping $\Upsilon : R \to S$ is said to be a $\left(\hat{H}_m, \Phi\right)$-nafirrc proximal contraction of the first kind if there exist $(\alpha, \beta) \in (0,1)$ with $\alpha \dotplus \beta < 1$.

$$\left.\begin{array}{l} \widehat{\Sigma}\left(\varsigma^{kj}, \Upsilon v_1, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right) \\ \widehat{\Sigma}\left(\zeta^{jk}, \Upsilon v_2, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right) \end{array}\right\}$$

$$\Rightarrow \hat{H}_m\left(\widehat{\Sigma}\left(\zeta^{jk}, \varsigma^{kj}, \Bbbk^{sw}\right)\right) \geq \Phi\left(\left(\widehat{\Sigma}\left(v_1, v_2, \Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(v_1, \varsigma^{kj}, \Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(v_2, \zeta^{jk}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta}\right). \tag{4.10}$$

for all $\varsigma^{kj}, \zeta^{jk} v_1, v_2 \in A$ and $\varsigma_i \neq v_{i,}$ $i \in \{1,2\}$ with $\widehat{\Sigma}\left(\varsigma, v, \Bbbk^{sw}\right) > 0$ where $\hat{H}_m, \Phi :$ $(0,1] \to \mathbb{R}$ are s.t $\Phi\left(t\right) > \hat{H}_m\left(t\right)$ for $t \in (0,1)$.

4.3.2 Example

Let $\Omega = \mathbb{R}$ and define a function $\widehat{\Sigma} : \Omega \times \Omega \times (0, \infty) \to [0,1]$ by

$$\widehat{\Sigma}\left(\varsigma, v, \Bbbk^{sw}\right) = \frac{\Bbbk^{sw}}{\Bbbk^{sw} + \eth^{d}(\varsigma, v)}.$$

Then $\left(\Omega,\widehat{\Sigma},*\right)$ fms. Let R,S are subsets of Ω and defined as

$$R=\{1,2,3,4,5\},\ S=\{1,2,3,4,5,6,7\},\ \text{then } \widehat{\Sigma}\left(A,B,\Bbbk^{sw}\right)=1.$$

Define the function $\hat{H}_m,\Phi:(0,1]\to\mathbb{R}$ by

$$\hat{H}_m\left(t\right)=\sqrt[2]{t}\text{ and }\Phi\left(t\right)=t\ \forall\ t\in\left(0,1\right).$$

Define the mapping $\Upsilon:R\to S$ by $\Upsilon\left(\varsigma\right)=\varsigma+1$. We show that Υ is a $\left(\hat{H}_m,\Phi\right)$-non-Archimedean fuzzy irrctpc of the first kind. For this consider $\varsigma^{kj}=4,\zeta^{jk}=2$, $\upsilon_1=3,\upsilon_2=1$,and $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$ but $\alpha+\beta<1$. For $\Bbbk^{sw}=1$, we have,

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj},\Upsilon\upsilon_1,\Bbbk^{sw}) &= \widehat{\Sigma}(4,\Upsilon3,\Bbbk^{sw})=\ \widehat{\Sigma}(R,S,\Bbbk^{sw}),\\ \widehat{\Sigma}(\varsigma^{kj},\Upsilon\upsilon_1,\Bbbk^{sw}) &= \widehat{\Sigma}(2,\Upsilon1,\Bbbk^{sw})=\widehat{\Sigma}(R,S,\Bbbk^{sw}).\end{aligned}$$

Hence, we have to prove that

$$\begin{aligned}\hat{H}_m\left(\widehat{\Sigma}\left(\zeta^{jk},\varsigma^{kj},\Bbbk^{sw}\right)\right) &\geq \Phi\left(\left(\widehat{\Sigma}\left(\upsilon_1,\upsilon_2,\Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\upsilon_1,\varsigma^{kj},\Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\upsilon_2,\zeta^{jk},\Bbbk^{sw}\right)\right)^{1-\alpha-\beta}\right),\\ \hat{H}_m\left(0.3333\right) &\geq \Phi\left(\left(0.3333\right)^{\frac{1}{2}}\left(0.5\right)^{\frac{1}{3}}\left(0.3333\right)^{1-\frac{1}{2}-\frac{1}{3}}\right),\\ \hat{H}_m\left(0.3333\right) &\geq \Phi\left(0.4079\right),\\ 0.5773 &\geq 0.4079.\end{aligned}$$

This shows that Υ is a $(\hat{H}_m,\Phi)$-non-Archimedean fuzzy interpolative Rich-Rus Ciric type contraction of the first kind. However, the following calculation shows that it is not a non-Archimedean fuzzy irrctpc of the first kind. We know that

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj},\Upsilon\upsilon_1,\Bbbk^{sw}) &= \widehat{\Sigma}(4,\Upsilon3,\Bbbk^{sw})=\ \widehat{\Sigma}(R,S,\Bbbk^{sw}),\\ \widehat{\Sigma}(\varsigma^{kj},\Upsilon\upsilon_1,\Bbbk^{sw}) &= \widehat{\Sigma}(2,\Upsilon1,\Bbbk^{sw})=\widehat{\Sigma}(R,S,\Bbbk^{sw}).\end{aligned}$$

This implies that,

$$\widehat{\Sigma}\left(\zeta^{jk},\varsigma^{kj},\mathbb{k}^{sw}\right) \geq \left(\left(\widehat{\Sigma}\left(v_1,v_2,\mathbb{k}^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(v_1,\varsigma^{kj},\mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(v_2,\zeta^{jk},\mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta}\right),$$

$$0.3333 > 0.4079.$$

This is contradiction.

4.3.3 Theorem

Let $\left(\Omega,\widehat{\Sigma},*\right)$ be a cnafmc and R,S be non-empty, s.t S is a-compact w.r.t R. Let $\Upsilon\colon R\to S$ be an $\left(\hat{H}_m,\Phi\right)$-nafirrctpc of the first kind. If

(i) $\hat{H}_m$ is non-decreasing function and $\liminf_{t\to\varepsilon-}\Phi\left(t\right)>\hat{H}_m\left(\varepsilon-\right)$ for any $\varepsilon\in(0,1)$;

(ii) R_0 is non-empty subset of R such that $\Upsilon\left(R_0\right)\subseteq S_0$;

Then Υ admits a bpp.

Proof. Let ς_0^{kj} in R_0. Since $\Upsilon(\varsigma_0^{kj})\in\Upsilon(R_0)\subseteq S_0$, there exists $\varsigma^{kj}\in R_0$ such that,

$$\widehat{\Sigma}(\varsigma^{kj},\Upsilon(\varsigma_0^{kj}),\mathbb{k}^{sw})=\widehat{\Sigma}(R,S,\mathbb{k}^{sw}).$$

Also we have $\Upsilon(\varsigma^{kj})\in\Upsilon(R_0)\subseteq S_0$. So, there exist $\zeta^{jk}\in R_0$ such that

$$\widehat{\Sigma}(\zeta^{jk},\Upsilon(\varsigma^{kj}),\mathbb{k}^{sw})=\widehat{\Sigma}(R,S,\mathbb{k}^{sw}).$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj},\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw})=\widehat{\Sigma}(R,S,\mathbb{k}^{sw}). \tag{4.11}$$

for all $n\in\mathbb{N}$. Observe that, if there exist $n\in\mathbb{N}$ s.t $\varsigma_n^{kj}=\varsigma_{n+1}^{kj}$ then from (4.11) the point ς_n^{kj} is a bpp of the mapping Υ. If $\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}$ $\forall$ $n \in \mathbb{N}$. Then by (4.11), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \Upsilon(\varsigma_{n-1}^{kj}), \mathbb{k}^{sw}) = \widehat{\Sigma}(R, S, \mathbb{k}^{sw}), \text{and}$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}) = \widehat{\Sigma}(R, S, \mathbb{k}^{sw}),$$

for all $n \geq 1$. Thus, by (4.13)

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw})) \geq \Phi\left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta}\right). \tag{4.12}$$

for all distinct $\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \varsigma_{n+1}^{kj} \in R$ by (4.12), we have

$$\hat{H}_m\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw})\right) > \hat{H}_m\left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_{n+1}, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta}\right).$$

Since, $\hat{H}_m$ is nd function, we have

$$\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw})\right) > \left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta}\right).$$

This implies that

$$\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw})\right)^{\alpha+\beta} > \left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \mathbb{k}^{sw})\right)^{\alpha+\beta}.$$

Let $\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}\right) = \theta_n$. This implies that

$$\hat{H}_m\left((\theta_n)\right) \geq \Phi\left((\theta_n)^{\alpha+\beta}(\theta_n)^{1-\alpha-\beta}\right) > \hat{H}_m\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{1-\alpha-\beta}\right).$$

Since $\hat{H}_m$ is nd, so, by (4.12), we have $\theta_n > \theta_{n-1}$ for all $n \in \mathbb{N}$. Assume that $\theta < 1$, so that (4.12), we obtain the following:

$$\hat{H}_m(\varepsilon-) = \lim_{n\to\infty} \hat{H}_m(\theta_n) \geq \lim_{n\to\infty} \Phi\left((\theta_{n-1})^{\alpha+\beta}(\theta_n)^{1-\alpha-\beta}\right) \geq \lim_{t\to\varsigma^{kj}-} \inf \Phi(t).$$

This is contradicts assumption (i), hence, $E = 1$ and $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right) = 1$. Now keeping in mind the assumption (i) and Lemma 4.2.3, we conclude that $\left\{\varsigma_n^{kj}\right\}$ is a cau seq. Since $\left(\Omega, \widehat{\Sigma}, *\right)$ is a cnafms. Then $\exists\ \varsigma \in R$, s.t $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma, \Bbbk^{sw}\right) = 1$. Moreover,

$$\begin{aligned}
\widehat{\Sigma}(R, S, \Bbbk^{sw}) &= \widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}), \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}), \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) * \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \Upsilon\varsigma_n, \Bbbk^{sw}\right), \\
&= \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) * \widehat{\Sigma}(R, S, \Bbbk^{sw}).
\end{aligned}$$

This implies

$$\begin{aligned}
\widehat{\Sigma}(R, S, \Bbbk^{sw}) &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) * \widehat{\Sigma}(R, S, \Bbbk^{sw}).
\end{aligned}$$

Applying to limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
\widehat{\Sigma}(R, S, \Bbbk^{sw}) &\geq 1 * \lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) \\
&\geq 1 * 1 * \widehat{\Sigma}(R, S, \Bbbk^{sw}).
\end{aligned}$$

That is,

$$\lim_{n\to\infty} \widehat{\Sigma}(\varsigma^*, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Therefore, $\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma_n^{kj}\right), \Bbbk^{sw}) \to \widehat{\Sigma}(\varsigma, S, \Bbbk^{sw})$ as $n \to \infty$. Since S is a-compact w.r.t R, $\exists$ a subseq $\{\Upsilon(\varsigma_{n_k})\}$ of $\{\Upsilon(\varsigma_n^{kj})\}$ s.t $(\Upsilon\varsigma_{n_k}) \to \eta \in S$ as $k \to \infty$. Therefore, by taking $k \to \infty$ in the following equation,

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \Upsilon(\varsigma_{n_k}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \tag{4.13}$$

we have,

$$\widehat{\Sigma}(\varsigma, \eta, \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Since, $\varsigma \in R_0$, so, $\Upsilon(\varsigma) \in \Upsilon(R_0) \subseteq S_0$ there exists $\xi \in R_0$ such that

$$\widehat{\Sigma}\left(\xi, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \tag{4.14}$$

Now, having in mind the equations (4.13) and (4.14), by (4.13) we have

$$\begin{aligned} \hat{H}_m(\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \Bbbk^{sw})) &\geq \Phi\left(\left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta}\right), \\ &> \hat{H}_m\left(\left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta}\right). \end{aligned}$$

Since, $\hat{H}_m$ is non-decreasing function, so, we have

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \Bbbk^{sw}) > \left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta}.$$

Thus, as $k \to \infty$, we have $\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right) = 1$ or $\varsigma = \xi$. Finally, by (4.14) we have

$$\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma\right), \Bbbk^{sw}) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right).$$

■

4.3.4 Theorem

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms and R, S be non-empty, s.t S is a-compact w.r.t R. Let $\Upsilon\colon R \to S$ be an $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy irrctpc of the first kind. If

(i) $\hat{H}_m$ is non-decreasing and $\left\{\hat{H}_m(t_n)\right\}$ and $\{\Phi(t_n)\}$ s.t that $\lim_{n\to\infty} \hat{H}_m(t_n) = \lim_{n\to\infty} \Phi(t_n)$, then $\lim_{n\to\infty}(t_n) = 1$

(ii) $\Upsilon(R_0) \subseteq S_0$.

Then Υ admits a bpp.

4.3.5 Definition

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms, and R, S of Ω. A mapping $\Upsilon : R \to S$ is said to be $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy interpolative ktpc of the first kind if $\alpha \in (0,1)$ such that

$$\left.\begin{array}{l}\widehat{\Sigma}\left(\varsigma^{kj}, \Upsilon v_1, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right) \\ \widehat{\Sigma}\left(\zeta^{jk}, \Upsilon v_2, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right)\end{array}\right\},$$

$$\Rightarrow \hat{H}_m\left(\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}\right)\right) \geq \Phi\left(\left(\widehat{\Sigma}\left(v_1, \varsigma^{kj}, \Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(v_2, \zeta^{jk}, \Bbbk^{sw}\right)\right)^{1-\alpha}\right). \quad (4.15)$$

for all $\varsigma^{kj}, \zeta^{jk} v_1, v_2 \in R$ and $\varsigma_i \neq v_{i,}\ i \in \{1,2\}$ with $\widehat{\Sigma}(\varsigma, v, \Bbbk^{sw}) > 0$. Where $\hat{H}_m, \Phi : (0,1] \to \mathbb{R}$ s.t $\Phi(t) > \hat{H}_m(t)$ for $t \in (0,1)$.

4.3.6 Example

Let $\Omega = \mathbb{R}$ and define the function $\widehat{\Sigma} : \Omega \times \Omega \times (0,\infty) \to [0,1]$ by

$$\widehat{\Sigma}\left(\varsigma, v, \Bbbk^{sw}\right) = \varsigma^{kj - \frac{\eth^{d}(\varsigma, v)}{\Bbbk^{sw}}}.$$

Then $\left(\Omega,\widehat{\Sigma},*\right)$ is a nafms. Let R,S be the subsets of Ω defined by

$$R=\{1,2,3,4,5\},\ S=\{1,2,3,4,5,6,7\},\ \text{then } \widehat{\Sigma}\left(R,S,\Bbbk^{sw}\right)=1.$$

Define the functions $\hat{H}_m,\Phi:(0,1]\to\mathbb{R}$ by

$$\hat{H}_m\left(s\right)=\sqrt[2]{s}\text{ and }\Phi\left(s\right)=s\text{ for all }s\in\left(0,1\right).$$

Define the mapping $\Upsilon:R\to S$ by $\Upsilon\left(\varsigma\right)=\varsigma+1$ for all $\varsigma\in R$. We show that Υ is a $\left(\hat{H}_m,\Phi\right)$-non-Archimedean fuzzy iktfpc of the first kind. For $\varsigma^{kj}=3$, $\zeta^{jk}=5$, $\upsilon_1=2$, $\upsilon_2=4$, and $\alpha=\frac{1}{2}$, for $\Bbbk^{sw}=1$;

$$\begin{aligned}
\widehat{\Sigma}(\varsigma^{kj},\Upsilon\upsilon_1,\Bbbk^{sw}) &= \widehat{\Sigma}\left(3,\Upsilon 2,\Bbbk^{sw}\right)=1=\widehat{\Sigma}\left(R,S,\Bbbk^{sw}\right),\\
\widehat{\Sigma}(\zeta^{jk},\Upsilon\upsilon_2,\Bbbk^{sw}) &= \widehat{\Sigma}\left(5,\Upsilon 4,\Bbbk^{sw}\right)=1=\widehat{\Sigma}\left(R,S,\Bbbk^{sw}\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
\hat{H}_m\left(\widehat{\Sigma}\left(\varsigma^{kj},\zeta^{jk},\Bbbk^{sw}\right)\right) &\geq \Phi\left(\left(\widehat{\Sigma}\left(\upsilon_1,\varsigma^{kj},\Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\upsilon_2,\zeta^{jk},\Bbbk^{sw}\right)\right)^{1-\alpha}\right),\\
\hat{H}_m\left(\widehat{\Sigma}\left(3,5,1\right)\right) &\geq \Phi\left(\left(\widehat{\Sigma}\left(2,3,1\right)\right)^{\frac{1}{2}}\left(\widehat{\Sigma}\left(4,5,1\right)\right)^{1-\frac{1}{2}}\right),\\
\hat{H}_m\left(0.1353\right) &\geq \Phi\left(\left(0.3678\right)^{\frac{1}{2}}\left(0.3678\right)^{\frac{1}{2}}\right),\\
\hat{H}_m\left(0.1353\right) &\geq \Phi\left(0.3678\right),\\
0.3678 &\geq 0.3678.
\end{aligned}$$

This shows that Υ is a $\left(\hat{H}_m,\Phi\right)$-non-Archimedean fuzzy iktfpc of the first kind. However, the following calculations shows that it is not a non Archimedean fuzzy iktpc of the first kind. We know that

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj}, \Upsilon \upsilon_1, \Bbbk^{sw}) &= \widehat{\Sigma}\left(3, \Upsilon 2, \Bbbk^{sw}\right) = 1 = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right), \\ \widehat{\Sigma}(\zeta^{jk}, \Upsilon \upsilon_2, \Bbbk^{sw}) &= \widehat{\Sigma}\left(5, \Upsilon 4, \Bbbk^{sw}\right) = 1 = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right).\end{aligned}$$

Then,

$$\begin{aligned}\left(\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}\right)\right) &\geq \left(\left(\widehat{\Sigma}\left(\upsilon_1, \varsigma^{kj}, \Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\upsilon_2, \zeta^{jk}, \Bbbk^{sw}\right)\right)^{1-\alpha}\right), \\ 0.1353 &\geq 0.3678.\end{aligned}$$

This is a contradiction.

4.3.7 Theorem

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms and R, S be non-empty, s.t S is a-compact with respect to R. Let $\Upsilon \colon R \to S$ be an $\left(\hat{H}_m, \Phi\right)$-iktpc of the first kind. If

(i) $\hat{H}_m$ is non-decreasing function and $\liminf_{t \to \varepsilon-} \Phi\left(t\right) > \hat{H}_m\left(\varepsilon-\right)$ for any $\varepsilon \in (0, 1)$.

(ii) R_0 is non-empty subset of R such that $\Upsilon\left(R_0\right) \subseteq S_0$.

Then Υ admits a bpp.

Proof. Let ς_o^{kj} in R_0. Since $\Upsilon(\varsigma_o^{kj}) \in \Upsilon(R_0) \subseteq S_0$, there exists $\varsigma^{kj} \in R_0$ such that,

$$\widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_o^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Also we have $\Upsilon(\varsigma^{kj}) \in \Upsilon(R_0) \subseteq S_0$. So, there exist $\zeta^{jk} \in R_0$ such that

$$\widehat{\Sigma}(\zeta^{jk}, \Upsilon(\varsigma^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \tag{4.16}$$

for all $n \in \mathbb{N}$. Observe that, if $\exists\ n \in \mathbb{N}$ s.t $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$ then (4.16) the point ς_n^{kj} is a bpp of the mapping Υ. If $\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}$ for all $n \in \mathbb{N}$. Then by (4.16), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \Upsilon(\varsigma_{n-1}^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}),$$

and

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

for all $n \geq 1$. Thus, by (4.15)

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})) \geq \Phi\left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha}\right). \tag{4.17}$$

for all distinct $\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \varsigma_{n+1}^{kj} \in R$ by (4.17), we have

$$\hat{H}_m\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})\right) > \hat{H}_m\left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha}\right).$$

Since, $\hat{H}_m$ is nd function, we have

$$\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})\right) > \left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha}\right).$$

This implies that

$$\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})\right)^{\alpha} > \left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}.$$

Let $\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)=\theta_n$. This implies that

Since $\hat{H}_m$ is nd, so, by (4.17), $\theta_n>\theta_{n-1}\ \forall\ n\in\mathbb{N}$. Assume that $\theta<1$, so that (4.17), we obtain the following:

$$\hat{H}_m(\varepsilon-)=\lim_{n\to\infty}\hat{H}_m(\theta_n)\geq\lim_{n\to\infty}\Phi\left((\theta_{n-1})^{\alpha}(\theta_n)^{1-\alpha}\right)\geq\lim_{t\to E-}\inf\Phi(t).$$

This is contradicts assumption (i), hence, $\theta=1$ and $\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)=1$. Now keeping in mind the assumption (i) and Lemma 4.2.3, we conclude that $\left\{\varsigma_n^{kj}\right\}$ is a cau seq. Since $\left(\Omega,\widehat{\Sigma},*\right)$ is a cnafms. Then $\exists\ \varsigma\in R$, s.t $\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma,\mathbb{k}^{sw}\right)=1$. Moreover,

$$\begin{aligned}\widehat{\Sigma}(R,S,\mathbb{k}^{sw}) &= \widehat{\Sigma}(\varsigma_{n+1}^{kj},\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw}),\\ &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw}),\\ &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\varsigma_{n+1}^{kj},\mathbb{k}^{sw})*\widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\Upsilon\varsigma_n,\mathbb{k}^{sw}\right),\\ &= \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\varsigma_{n+1}^{kj},\mathbb{k}^{sw})*\widehat{\Sigma}(R,S,\mathbb{k}^{sw}).\end{aligned}$$

Above information implies that

$$\begin{aligned}\widehat{\Sigma}(R,S,\mathbb{k}^{sw}) &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw})\\ &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\varsigma_{n+1}^{kj},\mathbb{k}^{sw})*\widehat{\Sigma}(R,S,\mathbb{k}^{sw}).\end{aligned}$$

Applying to limit as $n\to\infty$ for above inequality,

$$\begin{aligned}\widehat{\Sigma}(R,S,\mathbb{k}^{sw}) &\geq 1*\lim_{n\to\infty}\widehat{\Sigma}(\varsigma,\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw})\\ &\geq 1*1*\widehat{\Sigma}(R,S,\mathbb{k}^{sw}).\end{aligned}$$

That is,

$$\lim_{n\to\infty}\widehat{\Sigma}(\varsigma,\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw})=\widehat{\Sigma}(R,S,\mathbb{k}^{sw}).$$

Therefore, $\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma_n^{kj}\right), \Bbbk^{sw}) \to \widehat{\Sigma}(\varsigma, S, \Bbbk^{sw})$ as $n \to \infty$. Since S is a-compact w.r.t R, there exists a subseq $\{\Upsilon(\varsigma_{n_k}^{kj})\}$ of $\{\Upsilon(\varsigma_n^{kj})\}$ s.t $(\Upsilon\varsigma_{n_k}) \to \eta \in S$ as $k \to \infty$. Therefore, by taking $k \to \infty$ in the following equation,

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \Upsilon(\varsigma_{n_k}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \tag{4.18}$$

We have,

$$\widehat{\Sigma}(\varsigma, \eta, \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Since, $\varsigma^* \in R_0$, so, $\Upsilon\left(\varsigma^*\right) \in \Upsilon\left(R_0\right) \subseteq S_0$ there exists $\xi \in R_0$ such that

$$\widehat{\Sigma}\left(\xi, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \tag{4.19}$$

Now, having in mind the equations (4.18) and (4.18), by (4.15) we have

$$\begin{aligned} \hat{H}_m(\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \Bbbk^{sw})) &\geq \Phi\left(\left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha}\right), \\ &> \hat{H}_m\left(\left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha}\right). \end{aligned}$$

Since, $\hat{H}_m$ is non-decreasing function, so, we have

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \Bbbk^{sw}) > \left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha}.$$

Thus, as $k \to \infty$, we have $\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right) = 1$ or $\varsigma = \xi$. Finally, by (4.19) we have

$$\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma\right), \Bbbk^{sw}) = \widehat{\Sigma}\left(R, S, \Bbbk^{sw}\right).$$

■

4.3.8 Theorem

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms and R, S be non-empty, s.t S is a-compact w.r.t R. Let $\Upsilon\colon R \to S$ be an $\left(\hat{H}_m, \Phi\right)$-nafikpc of the first kind. If

(i) $\hat{H}_m$ is non-decreasing and $\left\{\hat{H}_m\left(t_n\right)\right\}$ and $\left\{\Phi\left(t_n\right)\right\}$ s.t $\lim_{n\to\infty} \hat{H}_m\left(t_n\right) = \lim_{n\to\infty} \Phi\left(t_n\right),$ then $\lim_{n\to\infty}\left(t_n\right) = 1.$

(ii) $\Upsilon\left(R_0\right) \subseteq S_0.$

Then Υ admits a bpp.

4.3.9 Definition

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms, and R, S be a pair of non-empty subsets of Ω. A mapping $\Upsilon : R \to S$ is said to be $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy interpolative Hardy Rogers type pc of the first kind if $\alpha, \beta, \gamma, \delta \in (0,1)$ s.t $\alpha+\beta+\gamma+\delta < 1.$

$$\left.\begin{array}{l} \widehat{\Sigma}\left(\varsigma^{kj}, \Upsilon\upsilon_1, \mathbb{k}^{sw}\right) = \widehat{\Sigma}\left(R, S, \mathbb{k}^{sw}\right) \\ \widehat{\Sigma}\left(\zeta^{jk}, \Upsilon\upsilon_2, \mathbb{k}^{sw}\right) = \widehat{\Sigma}\left(R, S, \mathbb{k}^{sw}\right) \end{array}\right\}$$

$$\Rightarrow \quad \hat{H}_m\left(\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \mathbb{k}^{sw}\right)\right) \geq \Phi\left(\begin{array}{c} \left(\widehat{\Sigma}\left(\upsilon_1, \upsilon_2, \mathbb{k}^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\upsilon_1, \varsigma^{kj}, \mathbb{k}^{sw}\right)\right)^{\beta} \\ \left(\widehat{\Sigma}\left(\upsilon_2, \zeta^{jk}, \mathbb{k}^{sw}\right)\right)^{\gamma}\left(\widehat{\Sigma}\left(\upsilon_1, \zeta^{jk}, \mathbb{k}^{sw}\right)\right)^{\delta} \\ \left(\widehat{\Sigma}\left(\upsilon_2, \varsigma^{kj}, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta} \end{array}\right) \tag{4.20}$$

for all $\varsigma^{kj}, \zeta^{jk}\upsilon_1, \upsilon_2 \in R$ and $\varsigma_i \neq \upsilon_{i,}$ $i \in \{1,2\}$ with $\widehat{\Sigma}\left(\Upsilon\varsigma, \Upsilon\upsilon, \mathbb{k}^{sw}\right) > 0$ where $\hat{H}_m, \Phi : (0,1] \to \mathbb{R}$ s.t $\Phi(t) > \hat{H}_m(t)$ for $t \in (0,1)$.

4.3.10 Example

Let $\Omega = \mathbb{R}^2$ and $\widehat{\Sigma} : \Omega \times \Omega \times (0, \infty) \rightarrow [0, 1]$ by

$$\widehat{\Sigma}(\varsigma, \upsilon, \Bbbk^{sw}) = \frac{\Bbbk^{sw}}{\Bbbk^{sw} + \eth^{d}(\varsigma, \upsilon)} \text{ where } \eth^{d}(\varsigma, \upsilon) = \mid \varsigma^{kj} - \upsilon_1 \mid + \mid \zeta^{jk} - \upsilon_2 \mid,$$

for all $\varsigma^{kj}, \upsilon_1, \zeta^{jk}, \upsilon_2 \in \Omega$. Then $\left(\Omega, \widehat{\Sigma}, *\right)$ is a fms. Let R, S be the subset of Ω defined by

$$R = \{(0, \varsigma), \varsigma \in \mathbb{R}\}, \; S = \{(1, 0), \varsigma \in \mathbb{R}\}, \text{ then } \widehat{\Sigma}(R, S, \Bbbk^{sw}) = \frac{\Bbbk^{sw}}{\Bbbk^{sw} + 1}.$$

Define the functions $\hat{H}_m, \Phi : (0, 1] \rightarrow \mathbb{R}$ by

$$\hat{H}_m(s) = \sqrt[2]{s} \text{ and } \Phi(s) = s^2 \text{ for all } s \in (0, 1).$$

Define the mapping $\Upsilon : R \rightarrow S$ by

$$\Upsilon(s) = \left\{ \begin{array}{lcc} (1, s) & \text{if} & s \in [-1, 1] \\ (1, s^2) & & \text{otherwise} \end{array} \right\} \text{for all } s \in R.$$

We show that Υ is $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy interpolative Hardy Rogers type pc of the first kind. Let $\varsigma = (0, 4), \upsilon = (0, 2), x = (0, 9), y = (0, 3)$ let $\alpha = 0.01, \beta = 0.02, \gamma = 0.03, \delta = 0.04$ and also $\Bbbk^{sw} = 1$ then we have

$$\begin{aligned} \widehat{\Sigma}(\varsigma, \Upsilon\upsilon, \Bbbk^{sw}) &= \widehat{\Sigma}((0, 4), \Upsilon(0, 2), \Bbbk^{sw}) = \frac{\Bbbk^{sw}}{\Bbbk^{sw} + 1} = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \\ \widehat{\Sigma}(x, \Upsilon y, \Bbbk^{sw}) &= \widehat{\Sigma}((0, 9), \Upsilon(0, 3), \Bbbk^{sw}) = \frac{\Bbbk^{sw}}{\Bbbk^{sw} + 1} = \widehat{\Sigma}(R, S, \Bbbk^{sw}). \end{aligned}$$

This implies that

$$
\begin{aligned}
\hat{H}_m\left(\widehat{\Sigma}\left(\varsigma, x, \mathbb{k}^{sw}\right)\right) &\geq \Phi\left(\begin{array}{c}\left(\widehat{\Sigma}\left(v, y, \mathbb{k}^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(v, \varsigma, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(y, x, \mathbb{k}^{sw}\right)\right)^{\gamma} \\ \left(\widehat{\Sigma}\left(v, x, \mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(y, \varsigma, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}\end{array}\right), \\
\hat{H}_m(0.1667) &\geq \Phi\left(\begin{array}{c}(0.5)^{0.01}(0.3333)^{0.02}(0.1429)^{0.03} \\ (0.125)^{0.04}(0.5)^{0.9}\end{array}\right), \\
\hat{H}_m(0.1667) &\geq \Phi(0.4519), \\
0.4082 &\geq 0.2042.
\end{aligned}
$$

This shows that Υ is a $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy interpolative Hardy Rogers type proximal contraction of the first kind. However, the following calculations show that it is not-non-Archimedean fuzzy interpolative Hardy Rogers type pc of the first kind. let $\alpha = 0.01, \beta = 0.02, \gamma = 0.03, \delta = 0.04$ for $\mathbb{k}^{sw} = 1$, We know that

$$
\begin{aligned}
\widehat{\Sigma}\left(\varsigma, \Upsilon v, \mathbb{k}^{sw}\right) &= \widehat{\Sigma}\left((0,4), \Upsilon(0,2), \mathbb{k}^{sw}\right) = \frac{\mathbb{k}^{sw}}{\mathbb{k}^{sw}+1} = \widehat{\Sigma}\left(R, S, \mathbb{k}^{sw}\right), \\
\widehat{\Sigma}\left(x, \Upsilon y, \mathbb{k}^{sw}\right) &= \widehat{\Sigma}\left((0,9), \Upsilon(0,3), \mathbb{k}^{sw}\right) = \frac{\mathbb{k}^{sw}}{\mathbb{k}^{sw}+1} = \widehat{\Sigma}\left(R, S, \mathbb{k}^{sw}\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\left(\widehat{\Sigma}\left(\varsigma, x, \mathbb{k}^{sw}\right)\right) &\geq \begin{array}{c}\left(\widehat{\Sigma}\left(v, y, \mathbb{k}^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(v, \varsigma, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(y, x, \mathbb{k}^{sw}\right)\right)^{\gamma} \\ \left(\widehat{\Sigma}\left(v, x, \mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(y, \varsigma, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}\end{array}, \\
0.1667 &\geq 0.4519.
\end{aligned}
$$

This is a contradiction.

4.3.11 Theorem

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms and R, S be non-empty, closed subsets of Ω such that S is a-compact w.r.t R. Let $\Upsilon\colon R \to S$ be an $\left(\hat{H}_m, \Phi\right)$-interpolative Hardy Rogers type pc of the first kind. If

(i) $\hat{H}_m$ is non-decreasing function and $\liminf_{t\to\varepsilon-}\Phi(t) > \hat{H}_m(\varepsilon-)$ for any $\varepsilon \in (0,1)$.

(ii) $\Upsilon(R_0) \subseteq S_0$.

Then Υ admits a bpp.

Proof. Let ς_o^{kj} in R_0. Since $\Upsilon(\varsigma_o^{kj}) \in \Upsilon(R_0) \subseteq S_0$, there exists $\varsigma^{kj} \in R_0$ such that,

$$\widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_o^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

Also we have $\Upsilon(\varsigma^{kj}) \in \Upsilon(R_0) \subseteq S_0$. So, there exist $\zeta^{jk} \in R_0$, such that

$$\widehat{\Sigma}(\zeta^{jk}, \Upsilon(\varsigma^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}).$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \tag{4.21}$$

for all $n \in \mathbb{N}$. Observe that, if $\exists\ n \in \mathbb{N}$ s.t $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$ then from (4.21) the point ς_n^{kj} is a bpp of the mapping Υ. If $\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}\ \forall\ n \in \mathbb{N}$. Then by (4.21), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \Upsilon(\varsigma_{n-1}^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}), \text{ and}$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, S, \Bbbk^{sw}),$$

$\forall\ n \geq 1$. Thus, by (4.20)

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})) \geq \Phi\left(\begin{array}{c} \left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n, \Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{\gamma} \\ \left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_n, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta} \end{array}\right),$$

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_n,\varsigma_{n+1}^{kj},\mathbb{k}^{sw}))\geq\Phi\left(\begin{array}{c}\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n,\mathbb{k}^{sw}\right)\right)^{\beta}\\ \left(\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\gamma}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}(1)^{1-\alpha-\beta-\gamma-\delta}\end{array}\right),$$

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}))\geq\Phi\left(\begin{array}{c}\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n,\mathbb{k}^{sw}\right)\right)^{\beta}\\ \left(\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\gamma}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\end{array}\right),$$

$$\hat{H}_m(\widehat{\Sigma}(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}))\geq\Phi\left(\begin{array}{c}\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n,\mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n,\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\gamma}\\ \left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma_n,\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\end{array}\right).\tag{4.22}$$

$\forall$ distinct $\varsigma_{n-1}^{kj},\varsigma_n^{kj},\varsigma_{n+1}^{kj}\in R$ by (4.22), we have

$$\hat{H}_m\left(\widehat{\Sigma}(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw})\right)>\hat{H}_m\left(\begin{array}{c}\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n,\mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n,\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\gamma}\\ \left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma_n,\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\end{array}\right).$$

Since, $\hat{H}_m$ is non decreasing function, we have

$$\left(\widehat{\Sigma}(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw})\right)>\begin{array}{c}\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n,\mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n,\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\gamma}\\ \left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma_n,\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\end{array}.$$

This implies that

$$\left(\widehat{\Sigma}(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw})\right)>\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\mathbb{k}^{sw})\right)^{\alpha+\beta+\delta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\gamma+\delta}.$$

Let $\widehat{\Sigma}\left(\varsigma_n,\varsigma_n^{kj},\mathbb{k}^{sw}\right)=\theta_n$. This implies that

$$\hat{H}_m\left((\theta_n)\right)\geq\Phi\left((\theta_{n-1})^{\alpha+\beta+\delta}(\theta_n)^{\gamma+\delta}\right)>\hat{H}_m\left((\theta_n)^{\alpha+\beta+\delta}(\theta_n)^{\gamma+\delta}\right).$$

Suppose that $\theta_{n-1}>\theta_n$ for some $n\geq 1$. Since $\hat{H}_m$ is non- decreasing, we have $(\theta_n)^{\alpha+\beta+\delta}<(\theta_n)^{\alpha+\beta+\delta}$. This is not possible. Consequently, we have $\theta_n>\theta_{n-1}$ for all

$n \in \mathbb{N}$. This implies $\theta_n > \theta_{n-1}$ for all $n \in \mathbb{N}$. Assume contrary $\theta < 1$, we obtain the following:

$$\hat{H}_m(\varepsilon-) = \lim_{n\to\infty} \hat{H}_m(\theta_n) \geq \lim_{n\to\infty} \Phi\left((\theta_{n-1})^{\alpha+\beta+\delta}(\theta_n)^{\gamma+\delta}\right) \geq \lim_{t\to\varsigma^{kj}-} \inf \Phi(t).$$

This is contradicts assumption (i), hence, $\theta = 1$ and $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}\right) = 1$.Now keeping in mind the assumption (i) and Lemma 4.2.3, we conclude that $\left\{\varsigma_n^{kj}\right\}$ is a cau seq. Since $\left(\Omega, \widehat{\Sigma}, *\right)$ is a cfms. Then $\exists\ \varsigma \in R$, s.t $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma, \mathbb{k}^{sw}\right) = 1$. Moreover,

$$\begin{aligned}
\widehat{\Sigma}(R, S, \mathbb{k}^{sw}) &= \widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}), \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}), \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}) * \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \Upsilon\varsigma_n, \mathbb{k}^{sw}\right), \\
&= \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}) * \widehat{\Sigma}(R, S, \mathbb{k}^{sw}).
\end{aligned}$$

This implies

$$\begin{aligned}
\widehat{\Sigma}(R, S, \mathbb{k}^{sw}) &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}), \\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}) * \widehat{\Sigma}(R, S, \mathbb{k}^{sw}).
\end{aligned}$$

Applying to limit as $n \to \infty$ for above inequality

$$\begin{aligned}
\widehat{\Sigma}(R, S, \mathbb{k}^{sw}) &\geq 1 * \lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}) \\
&\geq 1 * 1 * \widehat{\Sigma}(R, S, \mathbb{k}^{sw}).
\end{aligned}$$

That is,

$$\lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}) = \widehat{\Sigma}(R, S, \mathbb{k}^{sw}).$$

Therefore, $\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma_n^{kj}\right), \mathbb{k}^{sw}) \to \widehat{\Sigma}(\varsigma, S, \mathbb{k}^{sw})$ as $n \to \infty$. Since S is a-compact w.r.t R, $\exists$ a subseq $\{\Upsilon(\varsigma_{n_k})\}$ of $\{\Upsilon(\varsigma_n^{kj})\}$ s.t $(\Upsilon\varsigma_{n_k}) \to \eta \in S$ as $k \to \infty$. Therefore, by taking $k \to \infty$ in the following equation,

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \Upsilon(\varsigma_{n_k}), \mathbb{k}^{sw}) = \widehat{\Sigma}(R, S, \mathbb{k}^{sw}). \tag{4.23}$$

we have,

$$\widehat{\Sigma}(\varsigma, \eta, \mathbb{k}^{sw}) = \widehat{\Sigma}(R, S, \mathbb{k}^{sw}).$$

Since, $\varsigma \in R_0$, so, $\Upsilon(\varsigma) \in \Upsilon(R_0) \subseteq S_0$ there exists $\xi \in R_0$ such that

$$\widehat{\Sigma}\left(\xi, \Upsilon\varsigma, \mathbb{k}^{sw}\right) = \widehat{\Sigma}(R, S, \mathbb{k}^{sw}). \tag{4.24}$$

Now, having in mind the equations (4.23) and (4.24), by (4.20) we have

$$\begin{aligned} \hat{H}_m(\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \mathbb{k}^{sw})) &\geq \Phi\left(\begin{array}{c} \left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma, \mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma, \xi, \mathbb{k}^{sw}\right)\right)^{\gamma} \\ \left(\widehat{\Sigma}\left(\varsigma_{n_k}, \xi, \mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma, \varsigma_{n_{k+1}}, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta} \end{array}\right) \\ &> \hat{H}_m\left(\begin{array}{c} \left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma, \mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma, \xi, \mathbb{k}^{sw}\right)\right)^{\gamma} \\ \left(\widehat{\Sigma}\left(\varsigma_{n_k}, \xi, \mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma, \varsigma_{n_{k+1}}, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta} \end{array}\right). \end{aligned}$$

Since, $\hat{H}_m$ is non-decreasing function, so, we have

$$\widehat{\Sigma}(\varsigma_{n_{k+1}}, \xi, \mathbb{k}^{sw}) > \begin{array}{c} \left(\widehat{\Sigma}(\varsigma_{n_k}, \varsigma, \mathbb{k}^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n_k}, \varsigma_{n_{k+1}}, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma, \xi, \mathbb{k}^{sw}\right)\right)^{\gamma} \\ \left(\widehat{\Sigma}\left(\varsigma_{n_k}, \xi, \mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma, \varsigma_{n_{k+1}}, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta} \end{array}.$$

Thus, as $k \to \infty$, we have $\widehat{\Sigma}\left(\varsigma, \xi, \mathbb{k}^{sw}\right) = 1$ or $\varsigma = \xi$. Finally, by (4.24) we have

$$\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma\right), \mathbb{k}^{sw}) = \widehat{\Sigma}\left(R, S, \mathbb{k}^{sw}\right).$$

■

4.3.12 Theorem

Let $\left(\Omega, \widehat{\Sigma}, *\right)$ be a cnafms and R, S be non-empty, s.t S is a-compact w.r.t R. Let $\Upsilon \colon R \to S$ be an $\left(\hat{H}_m, \Phi\right)$-non-Archimedean fuzzy interpolative Hardy Rorgers type pc of the first kind. If

(i) $\hat{H}_m$ is non-decreasing and $\left\{\hat{H}_m\left(t_n\right)\right\}$ and $\left\{\Phi\left(t_n\right)\right\}$ s.t $\lim_{n\to\infty} \hat{H}_m\left(t_n\right) = \lim_{n\to\infty} \Phi\left(t_n\right)$, then $\lim_{n\to\infty}\left(t_n\right) = 1$.

(ii) $\Upsilon\left(R_0\right) \subseteq S_0$.

Then Υ admits a bpp.

4.4 Conclusion

The main aim of our chapter is to present new concepts of bppt for $(\hat{H}_m, \Phi)$-fuzzy ipc, thereby extending Proinov type fpr in a fms principle [25] to the case of non-self mappings.

Chapter 5

Best proximity point results for proximal contractions in fuzzy metric spaces

5.1 Introduction

In this chapter, we introduced a new type of interpolative proximal contractive condition that bpp of fuzzy mappings.We establish certain bppt for such pc. We improve and generalize the fuzzy proximal contractions by introducing fuzzy proximal interpolative contractions. We explain some bpp results in fms by introducing new fuzzy interpolative contraction mappings.

5.2 Modified Kannan type proximal contraction in fuzzy metric space

5.2.1 Definition

Let $(\varsigma, F, *)$ be a cnafms and $R, G \subseteq U$. A mapping $\Upsilon : R \to G$ is said to be iktpc, if there exist $\lambda \in [0,1)$ and $\alpha \in (0,1)$ such that

$$\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \mathbb{k}^{sw}\right) \geq \lambda\left(\left(\widehat{\Sigma}\left(\upsilon_1, \varsigma^{kj}, \mathbb{k}^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\upsilon_2, \zeta^{jk}, \mathbb{k}^{sw}\right)\right)^{1-\alpha}\right), \tag{5.1}$$

for all $\varsigma^{kj}, \zeta^{jk}\upsilon_1, \upsilon_2 \in R$, $\mathbb{k}^{sw} > 0$ and $\varsigma_i \neq \upsilon_{i,}$ $i \in \{1,2\}$ whenever $\widehat{\Sigma}\left(\varsigma^{kj}, \Upsilon\upsilon_1, \mathbb{k}^{sw}\right) = \widehat{\Sigma}\left(R, G, \mathbb{k}^{sw}\right)$, $\widehat{\Sigma}\left(\zeta^{jk}, \Upsilon\upsilon_2, \mathbb{k}^{sw}\right) = \widehat{\Sigma}\left(R, G, \mathbb{k}^{sw}\right)$ and $\widehat{\Sigma}\left(\varsigma, \upsilon, \mathbb{k}^{sw}\right) > 0$.

5.2.2 Example

Let $U = \mathbb{R} \times \mathbb{R}$, $\widehat{\Sigma} : U \times U \times (0, \infty) \to [0,1]$ by

$$\widehat{\Sigma}\left(\varsigma, \jmath, \mathbb{k}^{sw}\right) = \frac{\mathbb{k}^{sw}}{\mathbb{k}^{sw} + \eth^d\left(\left(\varsigma^{kj}, \hat{\jmath}_1\right), \left(\zeta^{jk}, \hat{\jmath}_2\right)\right)},$$

for all $\left(\varsigma^{kj}, \hat{\jmath}_1\right), \left(\zeta^{jk}, \hat{\jmath}_2\right) \in U$ where $\eth^d\left(\left(\varsigma^{kj}, \hat{\jmath}_1\right), \left(\zeta^{jk}, \hat{\jmath}_2\right)\right) = \mid \varsigma^{kj} - \hat{\jmath}_1 \mid + \mid \zeta^{jk} - \hat{\jmath}_2 |$. Then $\left(U, \widehat{\Sigma}, *\right)$ is a nafms. Let $R, G \subseteq U$ defined by

$$\begin{aligned} R &= \left\{\left(0, \frac{1}{n}\right); n \in \mathbb{N}\right\} \cup \{(0,0)\}, \\ G &= \{\left(1, \frac{1}{n}\right); n \in \mathbb{N}\} \cup \{(1,0)\}. \end{aligned}$$

Define $\widehat{\Sigma}(R, G, \mathbb{k}^{sw}) = \sup\{\widehat{\Sigma}(\varsigma, \hat{\jmath}, \mathbb{k}^{sw}) : \varsigma \in R, \hat{\jmath} \in G$ and $\mathbb{k}^{sw} > 0\}$. So, we have $\widehat{\Sigma}(R, G, \mathbb{k}^{sw}) = \frac{\mathbb{k}^{sw}}{\mathbb{k}^{sw}+1}$, $R_0\left(\mathbb{k}^{sw}\right) = R$ and $G_0\left(\mathbb{k}^{sw}\right) = G$. Define the mapping $\Upsilon : R \to$

G by

$$\Upsilon\left(\varsigma^{kj},\zeta^{jk}\right)=\begin{cases}\left(1,\frac{1}{2n}\right), \text{ if } \left(\varsigma^{kj},\zeta^{jk}\right)=\left(0,\frac{1}{n}\right) \text{ for all } n\in\mathbb{N}\\ (1,0), \text{ if } \left(\varsigma^{kj},\zeta^{jk}\right)=(0,0)\end{cases}$$

for all $\left(\varsigma^{kj},\zeta^{jk}\right)\in R$. Then, clearly $\Upsilon(R_0)\subseteq G_0$. Now, we show that Υ is a iktpc. For $\varsigma^{kj}=\left(0,\frac{1}{2}\right)$, $\zeta^{jk}=\left(0,\frac{1}{4}\right)$, $\hat{\jmath}_1=(0,1)$, $\hat{\jmath}_2=\left(0,\frac{1}{2}\right)$, $\alpha=\frac{1}{2}$, $\lambda=\frac{1}{3}$ and $\Bbbk^{sw}=1$.

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj},\Upsilon\hat{\jmath}_1,\Bbbk^{sw}) &= \widehat{\Sigma}\left(\left(0,\frac{1}{2}\right),\Upsilon(0,1),1\right),\\ &= \widehat{\Sigma}(R,G,\Bbbk^{sw}), \text{ and}\end{aligned}$$

$$\begin{aligned}\widehat{\Sigma}(\zeta^{jk},\Upsilon\hat{\jmath}_2,\Bbbk^{sw}) &= \widehat{\Sigma}\left(\left(0,\frac{1}{4}\right),\Upsilon\left(0,\frac{1}{2}\right),1\right),\\ &= \widehat{\Sigma}(R,G,\Bbbk^{sw}).\end{aligned}$$

Above information implies that,

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma^{kj},\zeta^{jk},\Bbbk^{sw}\right) &= \widehat{\Sigma}\left(\left(0,\frac{1}{2}\right),\left(0,\frac{1}{4}\right),1\right),\\ &\geq \lambda\left(\widehat{\Sigma}\left(\hat{\jmath}_1,\varsigma^{kj},\Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\hat{\jmath}_2,\zeta^{jk},\Bbbk^{sw}\right)\right)^{1-\alpha},\\ &\geq \lambda\left(\widehat{\Sigma}\left((0,1),\left(0,\frac{1}{2}\right),1\right)\right)^{\frac{1}{2}}\left(\widehat{\Sigma}\left(\left(0,\frac{1}{2}\right),\left(0,\frac{1}{4}\right),1\right)\right)^{1-\frac{1}{2}},\end{aligned}$$

which yield,

$$0.5714\geq 0.1826.$$

This shows that Υ is a interpolative Kannan type contraction. However, for $\varsigma^{kj}=\left(0,\frac{1}{2}\right)$, $\zeta^{jk}=\left(0,\frac{1}{4}\right)$, $\hat{\jmath}_1=(0,1)$, $\hat{\jmath}_2=\left(0,\frac{1}{2}\right)$, $\lambda=0.499$ and $\Bbbk^{sw}=1$. Now, we have

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj},\Upsilon\hat{\jmath}_1,\Bbbk^{sw}) &= \widehat{\Sigma}\left(\left(0,\frac{1}{2}\right),\Upsilon(0,1),1\right),\\ &= \widehat{\Sigma}(R,G,\Bbbk^{sw}), \text{ and}\end{aligned}$$

$$\begin{aligned}\widehat{\Sigma}(\zeta^{jk}, \Upsilon\hat{\jmath}_2, \Bbbk^{sw}) &= \widehat{\Sigma}\left(\left(0, \frac{1}{4}\right), \Upsilon\left(0, \frac{1}{2}\right), 1\right), \\ &= \widehat{\Sigma}\left(R, G, \Bbbk^{sw}\right).\end{aligned}$$

Above information implies that

$$\begin{aligned}\left(\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}\right)\right) &= \widehat{\Sigma}\left(\left(0, \frac{1}{2}\right), \left(0, \frac{1}{4}\right), 1\right), \\ &\geq \lambda\left(\left(\widehat{\Sigma}\left(\hat{\jmath}_1, \varsigma^{kj}, \Bbbk^{sw}\right)\right) + \left(\widehat{\Sigma}\left(\hat{\jmath}_2, \zeta^{jk}, \Bbbk^{sw}\right)\right)\right), \\ &= \lambda\left(\begin{array}{c}\left(\widehat{\Sigma}\left((0,1), \left(0, \frac{1}{2}\right), 1\right)\right) \\ +\widehat{\Sigma}\left(\left(0, \frac{1}{2}\right), \left(0, \frac{1}{4}\right), 1\right)\end{array}\right),\end{aligned}$$

which yield

$$\begin{aligned}0.5714 &\geq \lambda\left(0.4 + 0.75\right), \\ 0.5714 &\ngeq 0.5739.\end{aligned}$$

This is a contradiction. Hence, Υ is not a Kannan type contraction.

Next, we start our main results:

5.2.3 Theorem

Let $\left(U, \widehat{\Sigma}, *\right)$ be a cnafms and $R, G \subseteq U$ such that G is a-compact with respect to R. Let $\Upsilon \colon R \rightarrow G$ iktpc. If $R_0 \subseteq R$ such that $\Upsilon\left(R_0\right) \subseteq G_0$. Then Υ admits a bpp.

Proof. Let $\varsigma_o^{kj} \in R_0$. Since $\Upsilon(\varsigma_o^{kj}) \in \Upsilon(R_0) \subseteq G_0$ there exist $\varsigma^{kj} \in R_0$ such that,

$$\widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_o^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

Also we have $\Upsilon(\varsigma^{kj}) \in \Upsilon(R_0) \subseteq G_0$. So, there exist $\zeta^{jk} \in R_0$ such that,

$$\widehat{\Sigma}(\zeta^{jk}, \Upsilon(\varsigma^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}), \tag{5.2}$$

$\forall\, n \in \mathbb{N}$. Observe that, if $\exists\, n \in \mathbb{N}$ such that $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$ then from (5.2), the point ς_n^{kj} is a bpp of the mapping Υ. If $\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}$ $\forall\, n \in \mathbb{N}$. Then by (5.2), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \Upsilon(\varsigma_{n-1}^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}),$$

and

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

for all $n \geq 1$. Thus, by (5.1),

$$(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})) \geq \lambda \left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha} \left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha}, \tag{5.3}$$

for all distinct $\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \varsigma_{n+1}^{kj} \in R$. Since, by (5.3), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) \geq \lambda \left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha} \left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha},$$

$$\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})\right)^{\alpha} \geq \lambda \left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}. \tag{5.4}$$

So, by (5.4), let $\theta_n = \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)$. We have $\theta_{n-1} < \theta_n$ $\forall\, n \in \mathbb{N}$. Now from (5.4), we have

$$\begin{aligned} \widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) &\geq \lambda^{\frac{1}{\alpha}} \widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}), \\ &\geq \lambda^{\frac{2}{\alpha}} \widehat{\Sigma}(\varsigma_{n-2}, \varsigma_{n-1}^{kj}, \Bbbk^{sw}), \\ &\vdots \\ &\geq \lambda^{\frac{n}{\alpha}} \widehat{\Sigma}(\varsigma^{kj}, \varsigma_o^{kj}, \Bbbk^{sw}). \end{aligned}$$

Then $\theta_n(\Bbbk^{sw}) > \theta_{n-1}(\Bbbk^{sw})$, that is the sequence $\{\theta_n\}$ is non-decreasing sequence for all $\Bbbk^{sw} > 0$. Consequently, there exist $\theta(\Bbbk^{sw}) \leq 1$ such that $\lim_{n\to\infty}\theta_n(\Bbbk^{sw}) = \theta(\Bbbk^{sw})$. Now, $\theta(\Bbbk^{sw}) = 1$. Suppose, $0 < \theta(\Bbbk_0^{sw}) < 1$ for some $\Bbbk_0^{sw} > 0$. Since $\theta_n(\Bbbk_0^{sw}) \geq \theta(\Bbbk_0^{sw})$, by taking the limit with $\Bbbk^{sw} = \Bbbk_0^{sw}$. We obtain

$$\theta(\Bbbk_0^{sw}) \geq \lambda^{\frac{1}{\alpha}}\theta(\Bbbk_0^{sw}) > \theta(\Bbbk_0^{sw}).$$

Which is contradiction and hence, $\theta(\Bbbk^{sw}) = 1$ for all $\Bbbk^{sw} > 0$. Now, we show $\left\{\varsigma_n^{kj}\right\}$ is a cau seq. Then $\exists\ \epsilon \in (0,1)$ and $\Bbbk_0^{sw} > 0$ s.t $\forall\ k \in \mathbb{N}$, $\exists\ n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj}, \varsigma_{n(k)}^{kj}, \Bbbk_0^{sw}\right) \leq 1-\epsilon.$$

$$\widehat{\Sigma}\left(\varsigma_{m(k)-1}, \varsigma_{n(k)}^{kj}, \Bbbk_0^{sw}\right) > 1-\epsilon.$$

and so $\forall\ k$ we get

$$\begin{aligned} 1-\epsilon &\geq \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj}, \varsigma_{n(k)}^{kj}, \Bbbk^{sw}\right), \\ &\geq \widehat{\Sigma}\left(\varsigma_{m(k)-1}, \varsigma_{m(k)}^{kj}, \Bbbk^{sw}\right) * \widehat{\Sigma}\left(\varsigma_{m(k)-1}, \varsigma_{n(k)}^{kj}, \Bbbk^{sw}\right), \\ &> H_{m(k)}(\Bbbk_0^{sw}) * (1-\epsilon). \end{aligned} \tag{5.5}$$

Putting limit $n \to \infty$ in (5.5), that

$$\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj}, \varsigma_{n(k)}^{kj}, \Bbbk_0^{sw}\right) = 1-\epsilon,$$

from,

$$\begin{aligned} \widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj}, \varsigma_{n(k)+1}^{kj}, \Bbbk_0^{sw}\right) &\geq \widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj}, \varsigma_{m(k)}^{kj}, \Bbbk_0^{sw}\right) * \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj}, \varsigma_{n(k)}^{kj}, \Bbbk_0^{sw}\right) \\ &\quad *\widehat{\Sigma}\left(\varsigma_{n(k)}^{kj}, \varsigma_{n(k)+1}^{kj}, \Bbbk_0^{sw}\right), \end{aligned}$$

and

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right) &\geq \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{m(k)+1}^{kj},\mathbb{k}_0^{sw}\right)*\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\\ &\quad *\widehat{\Sigma}\left(\varsigma_{n(k)+1}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right),\end{aligned}$$

we get,

$$\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)=1-\epsilon.$$

From equation (5.2), we know that

$$\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\Upsilon\varsigma_{m(k)}^{kj},\mathbb{k}_0^{sw}\right)=\widehat{\Sigma}\left(R,G,\mathbb{k}_0^{sw}\right)\text{ and }\widehat{\Sigma}\left(\varsigma_{n(k)+1}^{kj},\Upsilon\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right)=\widehat{\Sigma}\left(R,G,\mathbb{k}_0^{sw}\right).$$

So by (5.1),

$$\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\geq\lambda\left(\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{m(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\right)^{1-\alpha},$$

taking lim $k\to\infty$, we get

$$1-\epsilon\geq\lambda\left(1-\epsilon\right)>1-\epsilon.$$

this is wrong. Then $\{\varsigma_n\}$ is cau seq. Since $\left(\varsigma,\widehat{\Sigma},*\right)$ is a cnafms. Then $\exists\ \varsigma\in R$, s.t $\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_n^{kj},\varsigma,\mathbb{k}^{sw}\right)=1$. Moreover,

$$\begin{aligned}\widehat{\Sigma}(R,G,\mathbb{k}^{sw}) &= \widehat{\Sigma}(\varsigma_{n+1}^{kj},\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw}),\\ &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw}),\\ &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\varsigma_{n+1}^{kj},\mathbb{k}^{sw})*\widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\Upsilon\varsigma_n,\mathbb{k}^{sw}\right),\\ &= \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\varsigma_{n+1}^{kj},\mathbb{k}^{sw})*\widehat{\Sigma}(R,G,\mathbb{k}^{sw}).\end{aligned}$$

This implies,

$$\begin{aligned}\widehat{\Sigma}(R,G,\mathbb{k}^{sw}) &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\Upsilon(\varsigma_n^{kj}),\mathbb{k}^{sw}),\\ &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj},\varsigma,\mathbb{k}^{sw}\right)*\widehat{\Sigma}(\varsigma,\varsigma_{n+1}^{kj},\mathbb{k}^{sw})*\widehat{\Sigma}(R,G,\mathbb{k}^{sw}).\end{aligned}$$

Applying to limit as $n \to \infty$ for above inequality,

$$\begin{aligned}\widehat{\Sigma}(R, G, \Bbbk^{sw}) &\geq 1 * \lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}), \\ &\geq 1 * 1 * \widehat{\Sigma}(R, G, \Bbbk^{sw}).\end{aligned}$$

That is,

$$\lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

Therefore, $\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma_n^{kj}\right), \Bbbk^{sw}) \to \widehat{\Sigma}(\varsigma, G, \Bbbk^{sw})$ as $n \to \infty$. Since G is a-compact w.r.t R, then $\exists\ \xi \in R_0\left(\Bbbk^{sw}\right)$ s.t,

$$\widehat{\Sigma}\left(\xi, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, G, \Bbbk^{sw}) = \widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}). \tag{5.6}$$

We now show that $\varsigma = \xi$. If not, then

$$\widehat{\Sigma}\left(\xi, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right) \geq \lambda \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{\alpha} \left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha},$$

on taking limit as $n \to \infty$ gives

$$\widehat{\Sigma}\left(\xi, \varsigma, \Bbbk^{sw}\right) \geq \lambda \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{\alpha} > \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{\alpha}.$$

Which is contradiction. Hence $\widehat{\Sigma}\left(\varsigma, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, G, \Bbbk^{sw}) = \widehat{\Sigma}\left(\xi, \Upsilon\xi, \Bbbk^{sw}\right)$, that is , ς is the best proximity point. We show that ς is the unique bpp of Υ. Assume, that $0 < \widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) < 1\ \forall\ \Bbbk^{sw} > 0$ and $\hat{\jmath} \neq \varsigma$ is another bpp of Υ, i.e., $\widehat{\Sigma}\left(\varsigma, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, G, \Bbbk^{sw}) = \widehat{\Sigma}\left(\hat{\jmath}, \Upsilon\hat{\jmath}, \Bbbk^{sw}\right)$ then from (5.1),

$$\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) \geq \lambda \left(\widehat{\Sigma}\left(\varsigma, \varsigma, \Bbbk^{sw}\right)\right)^{\alpha} \left(\widehat{\Sigma}\left(\hat{\jmath}, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{1-\alpha} > 1.$$

Which is contradiction and hence $\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) = 1$ for all $\Bbbk^{sw} > 0$, that is $\varsigma = \hat{\jmath}$. ■

5.3 Modified Reich-Rus-Ciric type proximal contraction in fuzzy metric space

5.3.1 Definition

Let $\left(U, \widehat{\Sigma}, *\right)$ be a cnafms, and $R, G \subseteq U$. A mapping $\Upsilon : R \to G$ is called irrpc, if $\exists$ $\alpha, \beta \in (0,1)$ and $\lambda \in [0,1)$ with $\alpha + \beta < 1$.

$$\widehat{\Sigma}\left(u_2, \varsigma^{kj}, \Bbbk^{sw}\right) \geq \lambda \left(\widehat{\Sigma}\left(\hat{\jmath}_1, \hat{\jmath}_2, \Bbbk^{sw}\right)\right)^{\alpha} \left(\widehat{\Sigma}\left(\hat{\jmath}_1, \varsigma^{kj}, \Bbbk^{sw}\right)\right)^{\beta} \left(\widehat{\Sigma}\left(\hat{\jmath}_2, \zeta^{jk}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta}, \tag{5.7}$$

for all $\varsigma^{kj}, \zeta^{jk}\hat{\jmath}_1, \hat{\jmath}_2 \in R$, $\Bbbk^{sw} > 0$ and $\varsigma_i \neq \hat{\jmath}_{i,}$ $i \in \{1,2\}$ whenever $\widehat{\Sigma}\left(\varsigma^{kj}, \Upsilon\hat{\jmath}_1, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, G, \Bbbk^{sw}\right)$, $\widehat{\Sigma}\left(\zeta^{jk}, \Upsilon\hat{\jmath}_2, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, G, \Bbbk^{sw}\right)$ and $\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) > 0$.

5.3.2 Example

Let $U = \mathbb{R}^2$, $\widehat{\Sigma} : U \times U \times (0, +\infty) \to [0,1]$ by

$$\widehat{\Sigma}(\varsigma, \hat{\jmath}, \Bbbk^{sw}) = \frac{\Bbbk^{sw}}{\Bbbk^{sw} + \eth^{d}\left(\varsigma, \hat{\jmath}\right)},$$

where $\eth^{d}((\varsigma^{kj}, \hat{\jmath}_1), (\zeta^{jk}, \hat{\jmath}_2)) = \sqrt[2]{\left(\zeta^{jk} - \varsigma^{kj}\right)^2 + \left(\hat{\jmath}_2 - \hat{\jmath}_1\right)^2}$ for all $\left(\varsigma^{kj}, \hat{\jmath}_1\right), \left(\zeta^{jk}, \hat{\jmath}_2\right) \in U$. Then $\left(U, \widehat{\Sigma}, *\right)$ is a nafms. Let $R, G \subseteq U$ defined as

$$\begin{aligned} R &= \{(0, \varsigma); \varsigma \in \mathbb{R}\}, \\ G &= \{(1, \varsigma); \varsigma \in \mathbb{R}\}. \end{aligned}$$

Define $\widehat{\Sigma}(R, G, \Bbbk^{sw}) = \sup\{\widehat{\Sigma}(\varsigma, \hat{\jmath}, \Bbbk^{sw}) : \varsigma \in R, \hat{\jmath} \in G \text{ and } \Bbbk^{sw} > 0\}$. So we have $\widehat{\Sigma}(R, G, \Bbbk^{sw}) = \frac{\Bbbk^{sw}}{\Bbbk^{sw}+1}$, $R_0\left(\Bbbk^{sw}\right) = R$, $G_0\left(\Bbbk^{sw}\right) = G$. Define the mapping $\Upsilon : R \to G$

by

$$\Upsilon\left((0,\gamma)\right)=(1,2\gamma),$$

for all $(0,\gamma)\in R$. Then clearly $\Upsilon\left(R_0\right)\subseteq G_0$. Now, we show that Υ is a irrpc. For $\varsigma^{kj}=(0,2)$, $\hat{\jmath}_1=(0,1)$, $\zeta^{jk}=(0,4)$, $\hat{\jmath}_2=(0,2)$, $\Bbbk^{sw}=1,\alpha=\frac{1}{2},\beta=\frac{1}{3}$ and $\lambda=0.27$.

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj},\Upsilon\hat{\jmath}_1,\Bbbk^{sw}) &= \widehat{\Sigma}((0,2),\Upsilon(0,1),1),\\ &= \widehat{\Sigma}(R,G,\Bbbk^{sw}),\ \text{and}\end{aligned}$$

$$\begin{aligned}\widehat{\Sigma}(\zeta^{jk},\Upsilon\hat{\jmath}_2,\Bbbk^{sw}) &= \widehat{\Sigma}((0,4),\Upsilon(0,2),1),\\ &= \widehat{\Sigma}(R,G,\Bbbk^{sw}).\end{aligned}$$

Above information implies that,

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj},\zeta^{jk},\Bbbk^{sw}) &= \widehat{\Sigma}((0,2),(0,4),1),\\ &\geq \lambda\left(\left(\widehat{\Sigma}(\hat{\jmath}_1,\hat{\jmath}_2,\Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\hat{\jmath}_1,\varsigma^{kj},\Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\hat{\jmath}_2,\zeta^{jk},\Bbbk^{sw}\right)\right)^{1-\alpha-\beta}\right),\\ &= \lambda\left(\begin{array}{c}\left(\widehat{\Sigma}\left(\left(0,1\right),\left(0,2\right),\Bbbk^{sw}\right)\right)^{\frac{1}{2}}\left(\widehat{\Sigma}\left(\left(0,1\right),\left(0,2\right),1\right)\right)^{\frac{1}{3}}\\ \left(\widehat{\Sigma}\left(\left(0,2\right),\left(0,4\right),1\right)\right)^{1-\frac{1}{2}-\frac{1}{3}}\end{array}\right),\end{aligned}$$

which yield

$$0.3333\geq 0.1557.$$

Then Υ is a irrpc. However, for $u_1=(0,2)$, $\hat{\jmath}_1=(0,1)$ and $\zeta^{jk}=(0,4)$, $\hat{\jmath}_2=(0,2)$, $\lambda=0.27$. Now, we have

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj},\Upsilon\hat{\jmath}_1,\Bbbk^{sw}) &= \widehat{\Sigma}((0,2),\Upsilon(0,1),1),\\ &= \widehat{\Sigma}(R,G,\Bbbk^{sw}),\ \text{and}\end{aligned}$$

$$\begin{aligned}\widehat{\Sigma}(\zeta^{jk}, \Upsilon\hat{\jmath}_2, \Bbbk^{sw}) &= \widehat{\Sigma}((0,4), \Upsilon(0,2), 1),\\ &= \widehat{\Sigma}(R, G, \Bbbk^{sw}).\end{aligned}$$

Above information implies that,

$$\begin{aligned}\widehat{\Sigma}(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}) &= \widehat{\Sigma}((0,2),(0,4),1),\\ &\geq \lambda\left(\widehat{\Sigma}(\hat{\jmath}_1, \hat{\jmath}_2, \Bbbk^{sw}) + \widehat{\Sigma}\left(\hat{\jmath}_1, \varsigma^{kj}, \Bbbk^{sw}\right) + \widehat{\Sigma}\left(\hat{\jmath}_2, \zeta^{jk}, \Bbbk^{sw}\right)\right),\\ &= \lambda\left(\begin{array}{c}\widehat{\Sigma}((0,1),(0,2),\Bbbk^{sw}) + \widehat{\Sigma}((0,1),(0,2),1) + \\ \widehat{\Sigma}((0,2),(0,4),1)\end{array}\right),\end{aligned}$$

which yield,

$$0.3333 \ngtr 0.3599.$$

This is a contradiction.

5.3.3 Theorem

Let $\left(U, \widehat{\Sigma}, *\right)$ be a cnafms and $R, G \subseteq U$ s.t G a-compact w.r.t R. Let $\Upsilon\colon R \to G$ be a irrpc. If $R_0 \subseteq R$ such that $\Upsilon(R_0) \subseteq G_0$. Then Υ admits a bpp.

Proof. Let $\varsigma_o^{kj} \in R_0$. Since $\Upsilon(\varsigma_o^{kj}) \in \Upsilon(R_0) \subseteq G_0$, so $\exists\, \varsigma^{kj} \in R_0$ s.t,

$$\widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_o^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

Also we have $\Upsilon(\varsigma^{kj}) \in \Upsilon(R_0) \subseteq G_0$. So, there exist $\zeta^{jk} \in R_0$ such that,

$$\widehat{\Sigma}(\zeta^{jk}, \Upsilon(\varsigma^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}). \tag{5.8}$$

$\forall\ n \in \mathbb{N}$. Observe that, if $\exists\ n \in \mathbb{N}$ s.t $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$ then from (5.8), the point ς_n^{kj} is a bpp of the mapping Υ. If $\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}$ $\forall\ n \in \mathbb{N}$. Then by (5.8), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \Upsilon(\varsigma_{n-1}^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}), \text{ and}$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}),$$

for all $n \geq 1$. Thus, by (5.7),

$$(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})) \geq \lambda\left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n, \Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta}\right), \tag{5.9}$$

for all distinct $\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \varsigma_{n+1}^{kj} \in R$. Since, by (5.9), we have

$$(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})) \geq \lambda\left(\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n, \Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta}\right),$$

$$\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) \geq \lambda\left(\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw})\right)^{\alpha+\beta}\left(\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw})\right)^{1-\alpha-\beta}. \tag{5.10}$$

So, by (5.10), let $\theta_n = \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)$. We have $\theta_{n-1} < \theta_n$ for all $n \in \mathbb{N}$. Now from (5.10), we have

$$\begin{aligned}
\widehat{\Sigma}(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}) &\geq \lambda^{\frac{1}{\alpha+\beta}}\widehat{\Sigma}(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}),\\
&\geq \lambda^{\frac{2}{\alpha+\beta}}\widehat{\Sigma}(\varsigma_{n-2}, \varsigma_{n-1}^{kj}, \Bbbk^{sw}),\\
&\vdots\\
&\geq \lambda^{\frac{n}{\alpha+\beta}}\widehat{\Sigma}(\varsigma^{kj}, \varsigma_o^{kj}, \Bbbk^{sw}).
\end{aligned}$$

Then $\theta_{n-1}(\Bbbk^{sw}) < \theta_n(\Bbbk^{sw})$, that is the sequence $\{\theta_n\}$ is non-decreasing sequence for all $\Bbbk^{sw} > 0$. Consequently, there exist $\theta(\Bbbk^{sw}) \leq 1$ such that $\lim_{n\to\infty}\theta_n(\Bbbk^{sw}) = \theta(\Bbbk^{sw})$. Now $\theta(\Bbbk^{sw}) = 1$. Suppose, $0 < \theta(\Bbbk_0^{sw}) < 1$ for some $\Bbbk_0^{sw} > 0$. Since $\theta_n(\Bbbk_0^{sw}) \geq$

$\theta\left(\mathbb{k}_0^{sw}\right)$, by taking the limit with $\mathbb{k}^{sw}=\mathbb{k}_0^{sw}$. We obtain

$$\theta\left(\mathbb{k}_0^{sw}\right)\geq\lambda^{\frac{1}{\alpha+\beta}}\theta\left(\mathbb{k}_0^{sw}\right)>\theta\left(\mathbb{k}_0^{sw}\right).$$

Which is contradiction and hence, $\theta\left(\mathbb{k}^{sw}\right)=1$ for all $\mathbb{k}^{sw}>0$. Now, $\left\{\varsigma_n^{kj}\right\}$ is a cau seq. Then $\exists\ \epsilon\in(0,1)$ and $\mathbb{k}_0^{sw}>0$ s.t $\forall\ k\in\mathbb{N}$, there are $n(k),m(k)\in\mathbb{N}$ with $m(k)>n(k)\geq k$ and

$$\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right)\leq 1-\epsilon.$$

$$\widehat{\Sigma}\left(\varsigma_{m(k)-1},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right)>1-\epsilon,$$

$$\begin{aligned}1-\epsilon &\geq \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}^{sw}\right),\\ &\geq \widehat{\Sigma}\left(\varsigma_{m(k)-1},\varsigma_{m(k)}^{kj},\mathbb{k}^{sw}\right)*\widehat{\Sigma}\left(\varsigma_{m(k)-1},\varsigma_{n(k)}^{kj},\mathbb{k}^{sw}\right),\\ &> H_{m(k)}\left(\mathbb{k}_0^{sw}\right)*(1-\epsilon).\end{aligned}\tag{5.11}$$

Putting limit $n\to\infty$ in (5.11),

$$\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right)=1-\epsilon,$$

from

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right) &\geq \widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{m(k)}^{kj},\mathbb{k}_0^{sw}\right)*\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right)\\ &\quad *\widehat{\Sigma}\left(\varsigma_{n(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right),\end{aligned}$$

and

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right) &\geq \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{m(k)+1}^{kj},\mathbb{k}_0^{sw}\right)*\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\\ &\quad *\widehat{\Sigma}\left(\varsigma_{n(k)+1}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right),\end{aligned}$$

we get

$$\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma^{kj}_{m(k)+1}, \varsigma^{kj}_{n(k)+1}, \Bbbk^{sw}_0\right) = 1-\epsilon.$$

From equation (5.8), we know that

$$\widehat{\Sigma}\left(\varsigma^{kj}_{m(k)+1}, \Upsilon\varsigma^{kj}_{m(k)}, \Bbbk^{sw}_0\right) = \widehat{\Sigma}\left(R, G, \Bbbk^{sw}_0\right) \text{ and } \widehat{\Sigma}\left(\varsigma^{kj}_{n(k)+1}, \Upsilon\varsigma^{kj}_{n(k)}, \Bbbk^{sw}_0\right) = \widehat{\Sigma}\left(R, G, \Bbbk^{sw}_0\right).$$

So by (5.7),

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma^{kj}_{m(k)+1}, \varsigma^{kj}_{n(k)+1}, \Bbbk^{sw}_0\right) &\geq \lambda\left(\widehat{\Sigma}\left(\varsigma^{kj}_{m(k)}, \varsigma^{kj}_{n(k)}, \Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma^{kj}_{m(k)}, \varsigma^{kj}_{m(k)+1}, \Bbbk^{sw}_0\right)\right)^{\beta} \\ &\quad \left(\widehat{\Sigma}\left(\varsigma^{kj}_{n(k)}, \varsigma^{kj}_{n(k)+1}, \Bbbk^{sw}_0\right)\right)^{1-\alpha-\beta},\end{aligned}$$

taking lim $k \to \infty$ we get

$$1-\epsilon \geq \lambda(1-\epsilon) > 1-\epsilon.$$

Then $\{\varsigma^{kj}_n\}$ is cau seq. Since $\left(U, \widehat{\Sigma}, *\right)$ is a cnafms and R is closed subset of U. Then $\exists\ \varsigma \in R$, s.t $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma^{kj}_n, \varsigma, \Bbbk^{sw}\right) = 1$. Moreover,

$$\begin{aligned}\widehat{\Sigma}(R, G, \Bbbk^{sw}) &= \widehat{\Sigma}(\varsigma^{kj}_{n+1}, \Upsilon(\varsigma^{kj}_n), \Bbbk^{sw}) \geq \widehat{\Sigma}\left(\varsigma^{kj}_{n+1}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n), \Bbbk^{sw}), \\ &\geq \widehat{\Sigma}\left(\varsigma^{kj}_{n+1}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma^{kj}_{n+1}, \Bbbk^{sw}) * \widehat{\Sigma}\left(\varsigma^{kj}_{n+1}, \Upsilon\varsigma_n, \Bbbk^{sw}\right), \\ &= \widehat{\Sigma}\left(\varsigma^{kj}_{n+1}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma^{kj}_{n+1}, \Bbbk^{sw}) * \widehat{\Sigma}(R, G, \Bbbk^{sw}).\end{aligned}$$

Above information implies that,

$$\begin{aligned}\widehat{\Sigma}(R, G, \Bbbk^{sw}) &\geq \widehat{\Sigma}\left(\varsigma^{kj}_{n+1}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma^{kj}_n), \Bbbk^{sw}), \\ &\geq \widehat{\Sigma}\left(\varsigma^{kj}_{n+1}, \varsigma, \Bbbk^{sw}\right) * \widehat{\Sigma}(\varsigma, \varsigma^{kj}_{n+1}, \Bbbk^{sw}) * \widehat{\Sigma}(R, G, \Bbbk^{sw}).\end{aligned}$$

Applying to limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned}\widehat{\Sigma}(R, G, \Bbbk^{sw}) &\geq 1 * \lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma^{kj}_n), \Bbbk^{sw}), \\ &\geq 1 * 1 * \widehat{\Sigma}(R, G, \Bbbk^{sw}).\end{aligned}$$

That is,

$$\lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

Therefore, $\widehat{\Sigma}(\varsigma, \Upsilon\left(\varsigma_n^{kj}\right), \Bbbk^{sw}) \to \widehat{\Sigma}(\varsigma, G, \Bbbk^{sw})$ as $n \to \infty$. Since G is approximately comact with respec to R, there exist $\xi \in R_0\left(\Bbbk^{sw}\right)$ such that,

$$\widehat{\Sigma}\left(\xi, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, G, \Bbbk^{sw}) = \widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}). \tag{5.12}$$

We show that $\varsigma = \xi$. If not, then

$$\widehat{\Sigma}\left(\xi, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right) \geq \lambda \left(\begin{array}{c} \left(\widehat{\Sigma}\left(\varsigma, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{\alpha} \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{\beta} \\ \left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta} \end{array} \right),$$

taking limit as $n \to \infty$ gives

$$\widehat{\Sigma}\left(\xi, \varsigma, \Bbbk^{sw}\right) \geq \lambda \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{\beta} > \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{\beta}.$$

Which is contradiction. Hence $\widehat{\Sigma}\left(\varsigma, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, G, \Bbbk^{sw}) = \widehat{\Sigma}\left(\xi, \Upsilon\xi, \Bbbk^{sw}\right)$ that is ς is the best proximity point. Next, ς is the ubpp of Υ. Assume, on the contrary, that $0 < \widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) < 1$ for all $\Bbbk^{sw} > 0$ and $\hat{\jmath} \neq \varsigma$ is another bpp of Υ then from (5.7) we have

$$\begin{aligned} \widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) &\geq \lambda \left(\left(\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{\alpha} \left(\widehat{\Sigma}\left(\varsigma, \varsigma, \Bbbk^{sw}\right)\right)^{\beta} \left(\widehat{\Sigma}\left(\hat{\jmath}, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta} \right), \\ &> \left(\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{\alpha}. \end{aligned}$$

Which is contradiction and hence $\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) = 1$ for all $\Bbbk^{sw} > 0$, that is $\varsigma = \hat{\jmath}$. ∎

5.4 Modified Hardy Rogers type contraction in fuzzy metric space

5.4.1 Definition

Let $\left(U, \widehat{\Sigma}, *\right)$ be a cnafms, and $R, G \subseteq U$. A mapping $\Upsilon : R \to G$ is called ihrpc, if $\exists$ $\alpha, \beta, \gamma, \delta \in (0,1)$ s.t $\alpha+\beta+\gamma+\delta<1$, and $\lambda \in [0,1)$.

$$\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}\right) \geq \lambda \left(\begin{array}{c} \left(\widehat{\Sigma}\left(\hat{\jmath}_1, \hat{\jmath}_2, \Bbbk^{sw}\right)\right)^{\alpha} \left(\widehat{\Sigma}\left(\hat{\jmath}_1, \varsigma^{kj}, \Bbbk^{sw}\right)\right)^{\beta} \left(\widehat{\Sigma}\left(\hat{\jmath}_2, \zeta^{jk}, \Bbbk^{sw}\right)\right)^{\gamma} \\ \left(\widehat{\Sigma}\left(\hat{\jmath}_1, \zeta^{jk}, \Bbbk^{sw}\right)\right)^{\delta} \left(\widehat{\Sigma}\left(\hat{\jmath}_2, \varsigma^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta} \end{array} \right), \tag{5.13}$$

for all $\varsigma^{kj}, \zeta^{jk}\hat{\jmath}_1, \hat{\jmath}_2 \in R$, $\Bbbk^{sw} > 0$ and $\varsigma_i \neq \hat{\jmath}_{i,}$ $i \in \{1,2\}$ whenever $\widehat{\Sigma}\left(\varsigma^{kj}, \Upsilon\hat{\jmath}_1, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, G, \Bbbk^{sw}\right)$, $\widehat{\Sigma}\left(\zeta^{jk}, \Upsilon\hat{\jmath}_2, \Bbbk^{sw}\right) = \widehat{\Sigma}\left(R, G, \Bbbk^{sw}\right)$ and $\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) > 0$.

5.4.2 Example

Let $U = \mathbb{R}^2$, $\widehat{\Sigma} : U \times U \times (0,\infty) \to [0,1]$ by

$$\widehat{\Sigma}\left(\varsigma^{kj}, \hat{\jmath}, \Bbbk^{sw}\right) = \frac{\Bbbk^{sw}}{\Bbbk^{sw} + \eth^{d}\left(\left(\varsigma^{kj}, \hat{\jmath}_1\right), \left(\zeta^{jk}, \hat{\jmath}_2\right)\right)},$$

where $\eth^{d}\left(\left(\varsigma^{kj}, \hat{\jmath}_1\right), \left(\zeta^{jk}, \hat{\jmath}_2\right)\right) = \sqrt[2]{\left(\zeta^{jk} - \varsigma^{kj}\right)^2 + \left(\hat{\jmath}_2 - \hat{\jmath}_1\right)^2}$ for all $\left(\varsigma^{kj}, \hat{\jmath}_1\right), \left(\zeta^{jk}, \hat{\jmath}_2\right) \in U$. Then $\left(U, \widehat{\Sigma}, *\right)$ is a nafms. Let R,$G \subseteq U$ defined by

$$\begin{aligned} R &= \left\{\left(0, \varsigma^{kj}\right), \varsigma^{kj} \in \mathbb{R}\right\}, \\ G &= \left\{\left(1, \varsigma^{kj}\right), \varsigma^{kj} \in \mathbb{R}\right\}. \end{aligned}$$

Define $\widehat{\Sigma}(R, G, \Bbbk^{sw}) = \sup\{\widehat{\Sigma}(\varsigma^{kj}, \hat{\jmath}, \Bbbk^{sw}) : \varsigma^{kj} \in R, \hat{\jmath} \in G \text{ and } \Bbbk^{sw} > 0\}$. Then $\widehat{\Sigma}(R, G, \Bbbk^{sw}) = \frac{\Bbbk^{sw}}{\Bbbk^{sw}+1}$, $R_0\left(\Bbbk^{sw}\right) = R$, $G_0\left(\Bbbk^{sw}\right) = G$. Define the mapping $\Upsilon : R \to G$

by

$$\Upsilon\left(0,\varsigma^{kj}\right)=\begin{cases}\left(1,\varsigma^{kj}\right) \text{ if } s\in[-1,1]\\ \left(1,\varsigma^{kj2}\right) \text{ otherwise}\end{cases},$$

for all $\left(0,\varsigma^{kj}\right)\in R$. Then clearly $\Upsilon(R_0)\subseteq G_0$. We show that Υ is interpolative Hardy Rogers type contraction. For $\varsigma^{kj}=(0,4)$, $\hat{\jmath}_1=(0,2)$, $\zeta^{jk}=(0,9)$, $\hat{\jmath}_2=(0,3)$, $\alpha=0.01$, $\beta=0.02$, $\gamma=0.03$, $\delta=0.04$, $\lambda=\frac{1}{4}$ then we have

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma^{kj},\Upsilon\hat{\jmath}_1,\Bbbk^{sw}\right) &= \widehat{\Sigma}\left((0,4),\Upsilon(0,2),1\right),\\ &= \widehat{\Sigma}\left(R,G,\Bbbk^{sw}\right),\end{aligned}$$

and

$$\begin{aligned}\widehat{\Sigma}\left(\zeta^{jk},\Upsilon\hat{\jmath}_2,\Bbbk^{sw}\right) &= \widehat{\Sigma}\left((0,9),\Upsilon(0,3),1\right),\\ &= \widehat{\Sigma}\left(R,G,\Bbbk^{sw}\right).\end{aligned}$$

This implies that,

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma^{kj},\zeta^{jk},\Bbbk^{sw}\right) &= \widehat{\Sigma}\left((0,4),(0,9),1\right),\\ &\geq \lambda\left(\begin{array}{c}\left(\widehat{\Sigma}\left(\hat{\jmath}_1,\hat{\jmath}_2,\Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\hat{\jmath}_1,\varsigma^{kj},\Bbbk^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\hat{\jmath}_2,\zeta^{jk},\Bbbk^{sw}\right)\right)^{\gamma}\\ \left(\widehat{\Sigma}\left(\hat{\jmath}_1,\zeta^{jk},\Bbbk^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\hat{\jmath}_2,\varsigma^{kj},\Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}\end{array}\right),\end{aligned}$$

which yield,

$$0.4082\geq 0.1129.$$

This shows that Υ is a interpolative Hardy Rogers type contraction. However, for $\varsigma^{kj}=(0,4)$, $\hat{\jmath}_1=(0,2)$, $\zeta^{jk}=(0,9)$, $\hat{\jmath}_2=(0,3)$, $\lambda=0.2$ and $\Bbbk^{sw}=1$. We know that

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma^{kj},\Upsilon\hat{\jmath}_1,\Bbbk^{sw}\right) &= \widehat{\Sigma}\left((0,4),\Upsilon(0,2),1\right),\\ &= \widehat{\Sigma}\left(R,G,\Bbbk^{sw}\right),\end{aligned}$$

and,

$$\begin{aligned}\widehat{\Sigma}\left(\zeta^{jk}, \Upsilon \hat{\jmath}_2, \Bbbk^{sw}\right) &= \widehat{\Sigma}\left((0,9), \Upsilon(0,3), 1\right), \\ &= \widehat{\Sigma}\left(R, G, \Bbbk^{sw}\right).\end{aligned}$$

Implies,

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma^{kj}, \zeta^{jk}, \Bbbk^{sw}\right) &= \widehat{\Sigma}\left((0,4), (0,9), 1\right), \\ &\geq \lambda \left(\begin{array}{c}\left(\widehat{\Sigma}\left(\hat{\jmath}_1, \hat{\jmath}_2, \Bbbk^{sw}\right)\right) + \left(\widehat{\Sigma}\left(\hat{\jmath}_1, \varsigma^{kj}, \Bbbk^{sw}\right)\right) + \left(\widehat{\Sigma}\left(\hat{\jmath}_2, \zeta^{jk}, \Bbbk^{sw}\right)\right) \\ + \left(\widehat{\Sigma}\left(\hat{\jmath}_1, \zeta^{jk}, \Bbbk^{sw}\right)\right) + \left(\widehat{\Sigma}\left(\hat{\jmath}_2, \zeta^{jk}, \Bbbk^{sw}\right)\right)\end{array}\right),\end{aligned}$$

which yield,

$$0.1667 \ngeq 0.3201.$$

This is a contradiction.

5.4.3 Theorem

Let $\left(U, \widehat{\Sigma}, *\right)$ be a cnafms, $R, G \subseteq U$ s.t G is a-compact w.r.t R. Let $\Upsilon\colon R \to G$ be a interpolative Hardy Rogers type proximal contraction. If $R_0 \subseteq R$ such that $\Upsilon(R_0) \subseteq G_0$. Then Υ admits a bpp.

Proof. Let $\varsigma_o^{kj} \in R_0$. Since $\Upsilon(\varsigma_o^{kj}) \in \Upsilon(R_0) \subseteq G_0$, there exist $\varsigma^{kj} \in R_0$ such that,

$$\widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_o^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

Also we have $\Upsilon(\varsigma^{kj}) \in \Upsilon(R_0) \subseteq G_0$, so there exist $\zeta^{jk} \in R_0$ such that,

$$\widehat{\Sigma}(\zeta^{jk}, \Upsilon(\varsigma^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}).$$

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}), \tag{5.14}$$

$\forall\ n \in \mathbb{N}$. Observe that, if $\exists\ n \in \mathbb{N}$ s.t $\varsigma_n^{kj} = \varsigma_{n+1}^{kj}$ then from (5.14), the point ς_n^{kj} is a bpp of the mapping Υ. If $\varsigma_n^{kj} \neq \varsigma_{n+1}^{kj}\ \forall\ n \in \mathbb{N}$. Then by (5.14), we have

$$\widehat{\Sigma}(\varsigma_n^{kj}, \Upsilon(\varsigma_{n-1}^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}),$$

and

$$\widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \Bbbk^{sw}) = \widehat{\Sigma}(R, G, \Bbbk^{sw}),$$

for all $n \geq 1$, thus, by (5.13),

$$\begin{aligned} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right) \geq\ & \lambda\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{\beta} \\ & \left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{\gamma}, \left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{\delta} \\ & \left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}, \end{aligned} \tag{5.15}$$

for all distinct $\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \varsigma_{n+1}^{kj} \in R$. Since, by (5.15), we have

$$\begin{aligned} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right) &\geq \lambda\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{\alpha+\beta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{\gamma}\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-} \\ &\geq \lambda\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{\alpha+\beta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{\gamma}, \\ &\quad \left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}, \\ &\geq \lambda\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{1-\gamma-\delta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\delta}, \end{aligned}$$

$$\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)\right)^{\alpha+\beta+\delta} \geq \lambda\left(\widehat{\Sigma}\left(\varsigma_{n-1}^{kj}, \varsigma_n^{kj}, \Bbbk^{sw}\right)\right)^{1-\gamma-\delta}. \tag{5.16}$$

Letting $\theta_n = \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \Bbbk^{sw}\right)$, (5.16), we have $\theta_{n-1} < \theta_n$ for all $n \in \mathbb{N}$. Now from

(5.16), we have

$$\begin{aligned}
\widehat{\Sigma}(\varsigma_n^{kj},\varsigma_{n+1}^{kj},\Bbbk^{sw}) &\geq \lambda^{\frac{1}{\alpha+\beta+\delta}}\widehat{\Sigma}(\varsigma_{n-1}^{kj},\varsigma_n^{kj},\Bbbk^{sw})^{\frac{1-\gamma-\delta}{\alpha+\beta+\delta}}, \\
&\geq \lambda^{\frac{2}{\alpha+\beta+\delta}}\widehat{\Sigma}(\varsigma_{n-2},\varsigma_{n-1}^{kj},\Bbbk^{sw})^{\frac{1-\gamma-\delta}{\alpha+\beta+\delta}}, \\
&\vdots \\
&\geq \lambda^{\frac{n}{\alpha+\beta+\delta}}\widehat{\Sigma}(\varsigma^{kj},\varsigma_o^{kj},\Bbbk^{sw})^{\frac{1-\gamma-\delta}{\alpha+\beta+\delta}}.
\end{aligned}$$

Then $\theta_{n-1}(\Bbbk^{sw}) < \theta_n(\Bbbk^{sw})$, that is the seq $\{\theta_n\}$ is non-decreasing seq for all $\Bbbk^{sw} > 0$. Consequently, there exist $\theta(\Bbbk^{sw}) \leq 1$ such that $\lim_{n\to\infty}\theta_n(\Bbbk^{sw}) = \theta(\Bbbk^{sw})$. Suppose, $0 < \theta(\Bbbk_0^{sw}) < 1$ for some $\Bbbk_0^{sw} > 0$. Since $\theta_n(\Bbbk_0^{sw}) \geq \theta(\Bbbk_0^{sw})$, by taking the limit with $\Bbbk^{sw} = \Bbbk_0^{sw}$. We obtain

$$\theta(\Bbbk_0^{sw}) \geq \lambda^{\frac{1}{\alpha+\beta+\delta}}\theta(\Bbbk_0^{sw}) > \theta(\Bbbk_0^{sw}).$$

Satisfying the above inequality, that is equivalently,

$$\widehat{\Sigma}\left(\varsigma_{m(k)-1}^{kj},\varsigma_{n(k)}^{kj},\Bbbk_0^{sw}\right) > 1-\epsilon,$$

and so for all k we get

$$\begin{aligned}
1-\epsilon &\geq \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\Bbbk^{sw}\right), \\
&\geq \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\Bbbk^{sw}\right) * \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\Bbbk^{sw}\right), \qquad (5.17) \\
&\geq \theta_{m(k)}(\Bbbk_0^{sw}) * (1-\epsilon),
\end{aligned}$$

putting limit $n \to \infty$ in (5.17), we get that

$$\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\Bbbk_0^{sw}\right) = 1-\epsilon,$$

from

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right) &\geq \widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{m(k)}^{kj},\mathbb{k}_0^{sw}\right)*\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right)\\ &\quad *\widehat{\Sigma}\left(\varsigma_{n(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right),\end{aligned}$$

and

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right) &\geq \widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{m(k)+1}^{kj},\mathbb{k}_0^{sw}\right)*\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\\ &\quad *\widehat{\Sigma}\left(\varsigma_{n(k)+1}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right),\end{aligned}$$

we get,

$$\lim_{n\to\infty}\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)=1-\epsilon.$$

From equation (5.14), we know that

$$\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\Upsilon\varsigma_{m(k)}^{kj},\mathbb{k}_0^{sw}\right)=\widehat{\Sigma}\left(R,G,\mathbb{k}_0^{sw}\right) \text{ and } \widehat{\Sigma}\left(\varsigma_{n(k)+1}^{kj},\Upsilon\varsigma_{n(k)}^{kj},\mathbb{k}_0^{sw}\right)=\widehat{\Sigma}\left(R,G,\mathbb{k}_0^{sw}\right),$$

so by (5.13),

$$\begin{aligned}\widehat{\Sigma}\left(\varsigma_{m(k)+1}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right) &\geq \lambda\left(\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)}^{kj},\mathbb{k}^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{m(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\right)^{\beta}\\ &\quad \left(\widehat{\Sigma}\left(\varsigma_{n(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}_0^{sw}\right)\right)^{\gamma},\left(\widehat{\Sigma}\left(\varsigma_{m(k)}^{kj},\varsigma_{n(k)+1}^{kj},\mathbb{k}^{sw}\right)\right)^{\delta}\\ &\quad \left(\widehat{\Sigma}\left(\varsigma_{n(k)}^{kj},\varsigma_{m(k)+1}^{kj},\mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}.\end{aligned}$$

Taking lim $k\to\infty$ we get

$$1-\epsilon\geq\lambda\left(1-\epsilon\right)>1-\epsilon.$$

Which is contradiction. Then $\left\{\varsigma_n^{kj}\right\}$ is cau seq. Since $\left(U,\widehat{\Sigma},*\right)$ is a cnafms and R is closed subset of U. Then $\exists\ \varsigma \in R$, s.t $\lim_{n\to\infty} \widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma, \mathbb{k}^{sw}\right) = 1$. Moreover,

$$\begin{aligned}
\widehat{\Sigma}(R, G, \mathbb{k}^{sw}) &= \widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}),\\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma^{kj}, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}),\\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma^{kj}, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}) * \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \Upsilon\varsigma_n^{kj}, \mathbb{k}^{sw}\right),\\
&= \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma^{kj}, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}) * \widehat{\Sigma}(R, G, \mathbb{k}^{sw}).
\end{aligned}$$

This implies,

$$\begin{aligned}
\widehat{\Sigma}(R, G, \mathbb{k}^{sw}) &\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma^{kj}, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}),\\
&\geq \widehat{\Sigma}\left(\varsigma_{n+1}^{kj}, \varsigma^{kj}, \mathbb{k}^{sw}\right) * \widehat{\Sigma}(\varsigma^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}) * \widehat{\Sigma}(R, G, \mathbb{k}^{sw}).
\end{aligned}$$

Applying to limit as $n \to \infty$ in the above inequality, we get

$$\begin{aligned}
\widehat{\Sigma}(R, G, \mathbb{k}^{sw}) &\geq 1 * \lim_{n\to\infty} \widehat{\Sigma}(\varsigma, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}),\\
&\geq 1 * 1 * \widehat{\Sigma}(R, G, \mathbb{k}^{sw}).
\end{aligned}$$

That is,

$$\lim_{n\to\infty} \widehat{\Sigma}(\varsigma^{kj}, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}) = \widehat{\Sigma}(R, G, \mathbb{k}^{sw}).$$

Therefore, $\widehat{\Sigma}(\varsigma^{kj}, \Upsilon\left(\varsigma_n^{kj}\right), \mathbb{k}^{sw}) \to \widehat{\Sigma}(\varsigma^{kj}, G, \mathbb{k}^{sw})$ as $n \to \infty$. Since G is a-compact w.r.t R, there exist $\xi \in R_0\left(\mathbb{k}^{sw}\right)$ s.t,

$$\widehat{\Sigma}\left(\xi, \Upsilon\varsigma^{kj}, \mathbb{k}^{sw}\right) = \widehat{\Sigma}(R, G, \mathbb{k}^{sw}) = \widehat{\Sigma}(\varsigma_{n+1}^{kj}, \Upsilon(\varsigma_n^{kj}), \mathbb{k}^{sw}). \tag{5.18}$$

We now show that $\varsigma^{kj} = \xi$. If not, then

$$\begin{aligned}
\widehat{\Sigma}\left(\xi, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}\right) \geq\ & \lambda\left(\widehat{\Sigma}\left(\varsigma, \varsigma_n^{kj}, \mathbb{k}^{sw}\right)\right)^{\alpha}\left(\widehat{\Sigma}\left(\varsigma, \xi, \mathbb{k}^{sw}\right)\right)^{\beta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}\right)\right)^{\gamma}\\
&\left(\widehat{\Sigma}\left(\varsigma, \varsigma_{n+1}^{kj}, \mathbb{k}^{sw}\right)\right)^{\delta}\left(\widehat{\Sigma}\left(\varsigma_n^{kj}, \xi, \mathbb{k}^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}.
\end{aligned}$$

on taking limit as $n \to \infty$ gives

$$\widehat{\Sigma}\left(\xi, \varsigma, \Bbbk^{sw}\right) \geq \lambda \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha-\gamma-\delta} > \left(\widehat{\Sigma}\left(\varsigma, \xi, \Bbbk^{sw}\right)\right)^{1-\alpha-\gamma-\delta}.$$

Which is contradiction. Hence $\widehat{\Sigma}\left(\varsigma, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, G, \Bbbk^{sw}) = \widehat{\Sigma}\left(\xi, \Upsilon\xi, \Bbbk^{sw}\right)$ that is, ς is the bpp. We show that ς is the ubpp of Υ. Assume, on the contrary, that $0 < \widehat{\Sigma}\left(\varsigma, \jmath, \Bbbk^{sw}\right) < 1$ for all $\Bbbk^{sw} > 0$ and $\hat{\jmath} \neq \varsigma$ is another bpp of Υ, i.e., $\widehat{\Sigma}\left(\varsigma, \Upsilon\varsigma, \Bbbk^{sw}\right) = \widehat{\Sigma}(R, G, \Bbbk^{sw}) = \widehat{\Sigma}\left(\hat{\jmath}, \Upsilon\hat{\jmath}, \Bbbk^{sw}\right)$ then from (5.13) we have

$$\begin{aligned}
\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) &\geq \lambda \left(\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{\alpha} \left(\widehat{\Sigma}\left(\varsigma, \varsigma, \Bbbk^{sw}\right)\right)^{\beta} \left(\widehat{\Sigma}\left(\hat{\jmath}, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{\gamma} \\
&\quad \left(\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{\delta} \left(\widehat{\Sigma}\left(\hat{\jmath}, \varsigma, \Bbbk^{sw}\right)\right)^{1-\alpha-\beta-\gamma-\delta}, \\
&> \left(\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right)\right)^{1-\beta-\gamma}.
\end{aligned}$$

Which is contradiction and hence $\widehat{\Sigma}\left(\varsigma, \hat{\jmath}, \Bbbk^{sw}\right) = 1 \ \forall \ \Bbbk^{sw} > 0$, that is $\varsigma = \hat{\jmath}$. ■

5.5 Conclusion

We have produced several new type of contractive condition that ensures the existence of bpp in cnafms. According to the nature (linear and nonlinear) of contractions (5.1), (5.7) and (5.13). The study carried out in this paper generalizes the valuable research work presented in [25, 38].

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