Existence and Uniqueness of Solutions for Nonlinear Functional Equations

By

Sami Ullah Khan

43-FBAS/PHDMA/F14

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***DOCTOR OF PHILOSOPHY IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS AND STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY, ISLAMABAD.

Supervised By:

Prof. Dr. Muhammad Arshad

Department of Mathematics & Statistics
Faculty of Basic and Applied Sciences
International Islamic University, Islamabad
Pakistan

2018

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I, **Sami Ullah Khan** Reg. No. **43-FBAS/PHDMA/F14** hereby state that my Ph.D. thesis titled: **Existence and Uniqueness of Solutions for Nonlinear Functional Equations** is my own work and has not been submitted previously by me for taking any degree from this university, **International Islamic University, Sector H-10, Islamabad, Pakistan** or anywhere else in the country/world.

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Student Name: Sami Ullah Khan **Signatures:** ____________

Examination Committee:

a) **External Examiner 1:**
Name/Designation/Office Address **Signatures:** ____________

Prof. Dr. Akbar Azam
Professor of Mathematics,
Department of Mathematics,
COMSATS, IIT, Park Road, Chak Shahzad, Islamabad.

b) **External Examiner 2:**
Name/Designation/Office Address) **Signatures:** ____________

Prof. Dr. Tayyab Kamran
Department of Mathematics,
QAU, Islamabad.

c) **Internal Examiner:**
Name/Designation/Office Address) **Signatures:** ____________

Dr. Maliha Rashid
Assistant Professor

Supervisor Name: **Signatures:** ____________
Prof. Dr. Muhammad Arshad

Name of Dean/HOD: **Signatures:** ____________
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ACKNOWLEDGEMENTS

First and foremost, all praise to 'Allah', lord of the world, the almighty, who gave me the courage and patience to accomplish this task, and peace be upon His beloved apostle Muhammad (PBUH). May Allah guide us and the whole humanity to the right path.

I would like to express my sincere gratitude to my kind nature supervisor Prof. Dr. Muhammad Arshad. Without his sincere piece of advice and valued guidance, this thesis could never have become a reality.

I also would like to express my sincere thanks to Dr. Jamshaid Ahmad, Assistant Prof, Department of Mathematics, University of Jeddha, SA, for his generous assistance and invaluable comments throughout the entire Ph.D. Program. I am indeed grateful to the Chairman Department of Mathematics International Islamic University, Islamabad, Pakistan, for his selfless cooperation throughout my research work. The faculty at International Islamic University, Islamabad, Pakistan, in general and the Department of Mathematics in particular remained encouraging and supportive during my Ph.D. studies for which I am grateful.

In performing this research and writing this thesis, I received grate deal of help from many people in various ways; I am deeply indebted all of them. I greatly acknowledge the cooperation of Dr. Abdullah Shoaib (RIU), Dr. Kifayat Ullah (USTB), Dr. Aftab Hussain (KFUEIT), Dr. Gauhar Rahman (IIUI), Eskandar Ameer (IIUI), Muhammad Nazam (IIUI), Latif ur Rehman (IIUI), Awais Asif (IIUI), Tahir Resham (IIUI) and Qaisar Khan (IIUI).

Finally, my appreciation extends to my father, Mr. Muhammad Alam Khan, Ex. Add. Registrar, Gomal University, my mother, brother, sisters and wife for their patience and inspiration during the course of this work.

List of Publications

The list of the research articles, deduced from the work presented in this thesis, published in the international journals of ISI/ non ISI ranking, is given below.

1. Z. Mustafa, M. Arshad, S. U. Khan, J. Ahmad and M. Jaradat, Common fixed point for multivalued mappings in G-metric spaces with applications. J. Nonlinear Sci. Appl., 10 (2017), 2550-2564.
2. M.M.M. Jaradat, Z. Mustafa, S. U. Khan, M. Arshad and J. Ahmad, Some fixed point results on G-metric and Gb-metric spaces, Demonstr. Math. 50(2017), 190-207.
3. Z. Mustafa, S. U. Khan, M. Jaradat, M. Arshad and H. Jaradat, Fixed point results of F-rational cyclic contractive mappings on 0-complete partial metric spaces, (preprint).
4. S. U. Khan and M. Arshad, Fixed points of multi valued mappings in dualistic partial metric spaces, Bull. Math. Anal. Appl. 8 (2016), 49-58.
5. S. U. Khan, M. Arshad, A. Hussain and M. Nazam, Two new types of fixed point theorems for F-contraction. J. Adv. Stud. Topology 7 (2016), 251-260.
6. S. U. Khan, M. Arshad, T. Rasham and A. Shoaib, Some new common fixed points of generalized rational contractive mappings in dislocated metric spaces with application, Honam Math. J. 39 (2017), 161-174.
7. M. Arshad, S. U. Khan and J. Ahmad, Fixed point results for F-contractions involving some new rational expressions, JP J. Fixed Point Theory Appl. 11 (2016), 79-97.
8. S. U. Khan, J. Ahmad, M. Arshad and M. Zhenhua, Some new fuzzy fixed point results for generalized contractions, (preprint).

DEDICATED TO....

My parents, family, wife, teachers and friends for supporting and encouraging me.

CONTENTS

Preface

In 1922, the Polish mathematician Stefan Banach established a significant fixed point theorem known as the "Banach Contraction Principle" (BCP) which is one of the most prominent results of analysis and considered to be the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics because it requires the structure of complete metric space with a contractive condition on the map which is easy to test in this setting. The Banach Contraction Principle has been expanded in many different directions. In fact, there is a huge amount of literature dealing with extensions/generalizations of this important theorem.

A multivalued function is the one which takes values as a set. In the last forty years, the theory of multivalued functions has progressed in a number of ways. In 1969, the systematic study of Banach type fixed theorems involving multivalued mappings began with the work of Nadler [78], who investigated that a multivalued contractive mapping of a complete metric space X into the family of closed bounded subsets of X has a fixed point.

The study of metric spaces proved a most important tool for many fields both in pure and applied sciences such as biology, medicine, physics and computer science (see [62], [97]). Some generalizations of a metric space have been suggested by some writers, such as rectangular metric spaces, semi metric spaces, pseudo metric spaces, probabilistic metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, D-metric spaces, and cone metric spaces (see [3, 35, 40, 89, 90]) . Branciari [28] brought forward the idea of a generalized metric space replacing the triangle inequality by a rectangular type inequality. Branciari advanced Banach's contraction principle in such spaces.

In 1994, Matthews [65] introduced partial metric spaces and got different fixed point theorems. Actually, he expressed that the BCP can be generalized to the partial metric context for applications in program verification.

Romaguera [91] introduced the idea of 0-Cauchy sequences and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

Mustafa and Sims [69] introduced the $G-$metric spaces as a generalization of the notion of metric spaces. Mustafa and Sims acquired some fixed point theorems for mappings satisfying different contractive conditions for more fixed point results on G-metric space (see [69]-[76]). In 2014, Aghajani et al.[9] presented the notion of G_b-metric spaces and proved that the class of G_b-metric spaces is practically greater than that of G-metric spaces given in [69].

In 2012, Wardowski [103] introduced a new type of contraction called F-contraction and proved a new fixed point theorem concerning F-contraction. Wardowski generalized the BCP. Afterwards Secelean [96] proved fixed point theorems consisting of F-contractions by Iterated function systems. Piri et al.[84] proved a fixed point result for F-Suzuki contractions for some weaker conditions on the self map which generalizes the result of Wardowski. Later on, Acar et al. [8] introduced the idea of generalized multivalued F-contraction mappings. Altun et al. [7] extended multivalued mappings with δ-distance and established fixed point results in complete metric space. Sgroi et al. [98] developed fixed point theorems for multivalued F-contractions and achieved the solution of a few functional and integral equations, which was a suitable generalization of several multivalued fixed point theorems including Nadler's. Lately Ahmad et al. [12, 18, 46] revised the concept of F-contraction to attain some fixed point, and common fixed point results in the discourse of complete metric spaces.

This dissertation consists of five chapters. Each chapter begins with a brief introduction which acts as a summery to the material there in.

Chapter 1 is an overview aimed at explaining the terminology to be used to recall basic definitions and facts.

Chapter 2 is focused to the new concepts called $(g - F)$ contractions and generalized Mizoguchi–Takahashi contractions for complete G-metric spaces and developed some new coincidence points and common fixed point results. Also, we prove some fixed point theorems of $JS - G$-contraction in the setting of generalized metric spaces, and to prove some fixed point results on G_b-complete metric space for a new contraction. Most of these theorems are already known from the literature, we include new alternative proofs and some general investigations regarding the underlying spaces.

Chapter 3 is devoted to single and multivalued F-contraction mappings. We introduce the idea of generalized F-contraction and establish several new fixed point theorems for single and multivalued mappings in the setting of complete metric spaces. We extend the concept of fuzzy fixed points to common α-fuzzy fixed point of generalized F-contraction in the setting of complete metric spaces. Our results unify and generalize different known comparable results from the current literature.

Chapter 4 is devoted to introduce $F-$rational cyclic contraction on partial metric spaces and to present new fixed point results for such cyclic contraction in 0-complete partial metric spaces. We establish a common fixed point theorem for a pair of multivalued $F-\Psi-$proximinal mappings satisfying Ciric-wardowski type contraction in partial metric spaces. We discuss applications of our theorem and obtain the existence and uniqueness of common solution of system of integral equations.

Chapter 5 is focused on the concept of Hausdorff metric on the family of closed bounded subsets of a dualistic partial metric space (DPMS) and establishes a common fixed point theorem of a pair of multivalued mappings satisfying Mizoguchi and Takahashi's contractive conditions. Furthermore, we apply the concept of dislocated metric spaces to obtain theorems asserting the existance of common fixed points for a pair of mappings satisfying new generalized rational contractions in such spaces.

I would like to express my sincere gratitude to my supervisor Prof. Dr. Muhammad Arshad without whose sincere piece of advice and valuable guidance this thesis could never have become a reality. The faculty at International Islamic University, Islamabad, Pakistan, in general and the Department of Mathematics in particular has been of great encouragement and support to me during my Ph.D. studies for which I am thankful. Finally, I thank my family for their affection and support throughout my research.

Sami Ullah Khan.

Chapter 1

Preliminaries

The aim of this chapter is to present some basic concepts and to explain the terminology used throughout this thesis. Some previously known results are given without proof. Section 1.1 is concerned with the introduction of single valued and multivalued contractions. Section 1.2 is devoted to the introductory material on the notions of $G-$metric and G_b-metric spaces. In Section 1.3. we present the concept of cyclic contraction and Mizoguchi-Takahashi function. Section 1.4 deals with the basic concepts of single and multivalued F-contraction mappings.

1.1 Some basic concepts

The contraction mappings are a special type of uniformly continuous functions defined on a metric spaces. Fixed point (FP) results for such mappings play an important role in analysis and applied mathematics.

1.1.1 Definition [4]

Suppose that a set X (nonempty), S and $T : X \longrightarrow X$. Then $x \in X$ is called

(i) FP if image Tx coincides with x ($i.e.$, $Tx = x$);

(ii) common FP of the pair (S, T) if $Sx = Tx = x$;

(iii) coincidence point of the pair (S, T) if $Sx = Tx$;

(iv) point of coincidence of the pair (S, T) for some $y \in X$ s.t, $x = Sy = Ty$.

1.1.2 Definition

Suppose that (X, d) is a metric space. A mapping $T : X \longrightarrow X$ is called
(i) Banach contraction, if there is a positive real number $0 < r < 1$, s.t, $\forall\ x, y \in X$,

$$d(Tx, Ty) \leq rd(x, y);$$

(ii) Edelstein contraction, whenever

$$d(Tx, Ty) < d(x, y) \text{ for } x \neq y,\ \ x, y \in X;$$

(iii) non-expansive whenever

$$d(Tx, Ty) \leq d(x, y),\ \forall\ x, y \in X;$$

(iv) expansive whenever

$$d(Tx, Ty) \geq \eta d(x, y),\ \forall\ x, y \in X \text{ where } \eta > 1;$$

(v) Ciric type whenever

$$d(Tx, Ty) \leq M(x, y),$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}.$$

1.1.3 Definition

Suppose that X is a nonempty set and 2^X be the collection of all nonempty subsets of X. Then $T : X \longrightarrow 2^X$ is called multivalued mapping. A point $x \in X$ is said to be
(i) FP of T if $x \in Tx$;
(ii) coincidence point of a pair of multivalued mappings (T, S) if $Tx \cap Sx \neq \emptyset$;
(iii) common FP of the pair (T, S) if $x \in Tx \cap Sx$.
Suppose that (X, d) be a metric space

$CB(X) =$ the group of nonempty closed and bounded subset of X;

$CL(X) =$ the group of all nonempty closed subsets of X;

$K(X) =$ the family of all nonempty compact subsets of X.

For any A, B in $CB(X)$, define

$$d(A, B) = \inf\{d(x, y) : x \in A,\ y \in B\}.$$

This definition fails to discriminate sufficiently between sets. We would like the distance between two sets to be zero only if the two sets are the same, both in shape and position. For this purpose, the following concept is useful (cf., [57]).

1.1.4 Definition

Suppose that (X, d) is a metric space. For $A, B \in CB(X)$ and $\delta > 0$ the sets $N(\delta, A)$ and $E_{A,B}$ are defined as follows:

$$N(\delta, A) = \{x \in X : d(x, A) < \delta\},$$

$$E_{A,B} = \{\delta : A \subseteq N(\delta, B), B \subseteq N(\delta, A)\},$$

where $d(x, A) = \inf\{d(x, y) : y \in A\}$. The distance function H on $CB(X)$ induced by d is defined as

$$H(A, B) = \inf E_{A,B},$$

which is known as Hausdorff metric on X.

1.1.5 Lemma [78]

Suppose that (X, d) is a metric space. If $A, B \in CB(X)$, then for $r > 0$, $a \in A$ there exists $b \in B$ s.t., $d(a, b) \leq H(A, B) + r$.

1.1.6 Definition [78]

A mapping $T : X \longrightarrow CB(X)$ is said to be a multivalued contraction if there exists a constant r, $0 \leq r < 1$, s.t, $\forall\ x, y \in X$,

$$H(Tx, Ty) \leq rd(x, y).$$

Nadler [78] generalized BCP to multivalued mappings and proved the following important FP result for multivalued contractions.

1.1.7 Theorem [78]

Suppose that (X, d) is a complete metric space and $T : X \longrightarrow CB(X)$ a multi-valued contraction. Then T has a FP.

1.2 Relevant results on $G-$metric and G_b-metric spaces

Mustafa and Sims [69] defined the $G-$metric as fallows:

1.2.1 Definition

Suppose that X is a nonempty set, $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties

$(G1)$ $G(x, y, z) = 0$ if $x = y = z$,

$(G2)$ $0 < G(x, x, y)$ $\forall\ x, y \in X$ with $x \neq y$,

$(G3)$ $G(x, x, y) \leq G(x, y, z)$ $\forall\ x, y, z \in X$ with $y \neq z$,

$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables).

$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ $\forall\ x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, a $G-$metric on X, and the pair (X, G) is called a $G-$metric space.

1.2.2 Definition [69]

Suppose that (X, G) is a G-metric space, and let (x_n) be a sequence of points of X, we say that (x_n) is $G-$convergent to $x \in X$ if $lim_{n,m\rightarrow\infty} G(x, x_n, x_m) = 0$, i.e., for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t, $G(x, x_n, x_m) < \epsilon$, $\forall$, $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $lim_{n\to\infty} x_n = x$.

1.2.3 Proposition [69]

Assume that (X, G) is a G-metric space. The following statements are counterpart:

(1) (x_n) is $G-$convergent to x,

(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$.

(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$,

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

1.2.4 Definition [69]

Suppose that (X, G) is a G-metric space. A sequence (x_n) is called a $G-$Cauchy sequence (C-seq) if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $G(x_n, x_m, x_l) < \epsilon\ \forall\ n, m, l \geq N$, i.e., $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

1.2.5 Definition [69]

A $G-$metric space (X, G) is called $G-$complete if every $G-$C-seq is $G-$convergent in (X, G).

Every G-metric on X defines a metric d_G on X given by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \ \forall\ x, y \in X. \tag{1.1}$$

1.2.6 Example [69]

Suppose that (X, d) is a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\},$$

or

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(z, x),$$

$\forall\ x, y, z \in X$, is a G-metric on X.

1.2.7 Corollary [69])

Assume that (X, d) is a metric space, then (X, d) is complete metric space if and only if (X, G_m) ((X, G_s)) is complete G-metric space.

1.2.8 Corollary [69]

A G-metric space (X, G) is continuous on its three variables.

Recently, Aghajani et al. [9] introduced the concept of G_b-metric spaces as fallows:

1.2.9 Definition [9]

Presume that X is a nonempty set and $s \geq 1$ be a given real number. Suppose that G_b : $X \times X \times X \to \mathbb{R}^+$ is a function satisfying the following properties:

(G_b1) $G_b(x, y, z) = 0$ if $x = y = z$.

(G_b2) $0 < G_b(x, x, y)$ $\forall$ $x, y \in X$ with $x \neq y$,

(G_b3) $G_b(x, x, y) \leq G_b(x, y, z)$ $\forall$ $x, y, z \in X$ with $y \neq z$.

(G_b4) $G_b(x, y, z) = G_b(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry).

(G_b5) $G_b(x, y, z) \leq s(G_b(x, a, a) + G_b(a, y, z))$ $\forall$ $x, y, z, a \in X$ (rectangle inequality). Then the function G_b is called a generalized b-metric, or, a G_b-metric on X, and the pair (X, G) is called a G_b- metric space.

It is cleared that the class of G_b-metric spaces is effectively larger than that of G-metric spaces given in [69]. Indeed, each G-metric space is a G_b-metric space with $s = 1$.

1.2.10 Definition [9]

A G_b-metric space is said to be symmetric if $G_b(x, y, y) = G_b(y, x, x)$ $\forall$ $x, y \in X$.

1.2.11 Proposition [9]

Suppose that X be a G_b-metric space. Then for each $x, y, z, a \in X$ it follows that:

(1) if $G_b(x, y, z) = 0$ then $x = y = z$.

(2) $G_b(x, y, z) \leq s(G_b(x, x, y) + G_b(x, x, z))$.

(3) $G_b(x, y, y) \leq 2sG_b(y, x, x)$.

(4) $G_b(x, y, z) \leq s(G_b(x, a, z) + G_b(a, y, z))$.

1.2.12 Definition [9]

Suppose that (X, G_b) is a G_b-metric space, and (x_n) be a sequence in X. We say that (x_n) is G_b-convergent to $x \in X$ if $lim_{n,m \to \infty} G_b(x, x_n, x_m) = 0$, i.e,, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $G_b(x, x_n, x_m) < \epsilon$, $\forall$, $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $lim_{n \to \infty} x_n = x$.

1.2.13 Proposition [9]

Assume that (X, G_b) is a G_b-metric space. The following statements are counterpart:

(1) (x_n) is G_b-convergent to x,

(2) $G_b(x_n, x_n, x) \to 0$ as $n \to +\infty$,

(3) $G_b(x_n, x, x) \to 0$ as $n \to +\infty$,

(4) $G_b(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

1.2.14 Definition [9]

Assume that X is a G_b-metric space. A sequence (x_n) is called a G_b-C-seq if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ s.t. $G_b(x_n, x_m, x_l) < \epsilon \; \forall \; n, m, l \geq N$, i.e., $G_b(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

1.3 Relevant results in dislocated metric spaces

1.3.1 Definition [42]

Suppose that X is a nonempty set. A mapping $d_l : X \times X \to [0, \infty)$ is called a dislocated metric (or simply d_l-metric) if the following conditions hold:

(1) if $d_l(j, k) = 0$, then $j = k$;

(2) $d_l(j, k) = d_l(k, j)$;

(3) $d_l(j, k) \leq d_l(j, l) + d_l(l, k)$, $j, k, l \in X$.

Then d_l is called a dislocated metric on X, and the pair (X, d_l) is called dislocated metric space or d_l metric space.

1.3.2 Example

If $X = \mathbb{R}^+ \cup \{0\}$, then $d_l(j,k) = j + k$ defines a dislocated metric on X.

1.3.3 Definition [42]

A sequence $\{j_n\}$ in d_l-metric space is called C-seq if for given $\varepsilon > 0$, there corresponds $n_0 \in N$ s.t. $\forall\ n, m \geq n_0$, we have $d_l(j_m, j_n) < \varepsilon$.

1.3.4 Definition [42]

A sequence $\{j_n\}$ in d_l-metric space converges with respect to d_l if there exists $j \in X$ s.t. $d_l(j_n, j) \to 0$ as $n \to \infty$. In this case, j is called limit of $\{j_n\}$ and we write $j_n \to j$.

Every metric space is a dislocated metric space, but the converse may not be true.

1.3.5 Example

Suppose that $X = R$ and $d_l : X \times X \to [0, \infty)$ defined by $d_l(j,k) = |j| + |k|\ \forall\ j, k \in X$.

Note that d_l is a dislocated metric, but not a metric since $d_l(1,1) = 2 > 0$.

1.3.6 Definition [42]

A d_l-metric space (X, d_l) is called complete if every C-seq in X converges to a point in X.

1.3.7 Example

Assume that $X = [0,1]$ and $d_l(j,k) = \max\{j,k\}$, then the pair (X, d_l) is a dislocated metric space, but it is not a metric space.

1.3.8 Definition [42]

Suppose that (X, d_l) is a dislocated metric space. A mapping $T : X \to X$ is called contraction if there exists $0 \leq \lambda < 1$ s.t.

$$d(T(j), T(k)) \leq \lambda d(j,k),\ \forall\ j, k \in X \text{ with } j \neq k.$$

1.3.9 Definition [80]

Suppose that X is a nonempty set. Suppose that the mapping $D : X \times X \to \mathbb{R}$, satisfies:

1. $x = y \Leftrightarrow D(x,x) = D(y,y) = D(x,y)$;

2. $D(x,x) \leq D(x,y)\ \forall\ x, y \in X$;

3. $D(x,y) = D(y,x)\ \forall\ x, y \in X$;

4. $D(x,z) \leq D(x,y) + D(y,z) - D(y,y),\ \forall\ x, y, z \in X$.

Then D is called a dualistic partial metric on X, and (X, D) is called a DPMS.

Note that if $\mathbb{R}$ is replaced by $\mathbb{R}^+$ then D is known as partial metric on X. To make a difference between partial metric and dualistic partial metric, we discuss an example. Suppose that us define $D : X \times X \to \mathbb{R}$ by $D(x,y) = Max\{x,y\}$. Now if $X = \mathbb{R}$, then D is dualistic partial metric but nor partial metric on X, for if $x = -1$ and $y = -3$ then $Max\{-1, -3\} = -1 = D(x,y)$ which is not possible in partial metric. Each dualistic partial metric D on X generates a τ_0 topology $\tau(D)$ on X which has a base topology of open $D-$balls $\{B_D(x,\varepsilon) : x \in X, \varepsilon > 0\}$ and $B_D(x,\varepsilon) = \{y \in X : D(x,y) < \varepsilon + D(x,x)\}$. From this fact it follows that a sequence $(x_n)_n$ in a DPMS converges to a point $x \in X$ if and only if $D(x,x) = \lim_{n\to\infty} D(x, x_n)$.

1.3.10 Definition

Presume that X is a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{R}^+$, satisfies:

1. $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y,\ \forall\ x, y \in X$;

2. $d(x,z) \leq d(x,y) + d(y,z)\ \ \forall\ x, y, z \in X$.

The pair (X, d) is called quasi metric space.

Each quasi metric d on X generates a τ_0 topology $\tau(d)$ on X which has a base topology of open $d-$balls $\{B_d(x,\varepsilon) : x \in X,\ \varepsilon > 0\}$ and $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$.

Moreover if d is quasi metric, then $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ is a metric on X.

Suppose that us define modulus of a dualistic partial metric by

$$|D(x,y)| = \begin{cases} D(x,y) & \text{if } D(x,y) > 0; \\ -D(x,y) & \text{if } D(x,y) < 0. \end{cases}$$

1.3.11 Lemma [82]

If (X, D) is a DPMS, then the function $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x,y) = D(x,y) - D(x,x),$$

$\forall\ x,\ y \in X$, is a quasi metric on X s.t, $\tau(D) = \tau(d_p)$. Now if d_p is quasi metric on X then $d_p^s(x,y) = \max\{d_p(x,y), d_p(y,x)\}$ defines a metric on X.

1.3.12 Lemma [82]

(i) The sequence $\{x_n\}$ in DPMS (X, D) converges to a point x if and only if $D(x,x) = \lim_{n\to\infty} D(x_n, x)$.

(ii) The sequence $\{x_n\}$ in DPMS is called C-seq if $\lim_{n,m\to\infty} D(x_n, x_m)$ exists.

(iii) The DPMS is complete if and only if the metric (X, d_p^s) is complete and further $\lim_{n\to\infty} d_p^s(x_n, x) = 0$ if and only if $D(x,x) = \lim_{n\to\infty} D(x_n, x) = \lim_{n,m\to\infty} D(x_n, x_m)$.

A subset α of X is called closed in (X, D) if it is closed with respect to $\tau(D)$. α is called bounded in (X, D) if there exist $x_0 \in X$ and $M > 0$ s.t, $a \in \beta_D(x_0, M)\ \forall\ a \in \alpha$, i.e.,

$D(x_0, a) < D(x_0, x_0) + M\ \forall\ a \in \alpha$.

Suppose that $CB^D(X)$ is the collection of all nonempty, closed and bounded subsets of X with respect to the dualistic partail metric D. For $\alpha \in CB^D(X)$, we define

$D(x, \alpha) = \inf_{y\in\alpha} D(x,y)$.

For $\alpha, \beta \in CB^D(X)$,

$\delta_D(\alpha, \beta) = \sup_{a\in\alpha} D(a, \beta)$,

$\delta_D(\beta, \alpha) = \sup_{b\in\beta} D(b, \alpha)$,

$H_D(\alpha, \beta) = \max\{\delta_D(\alpha, \beta), \delta_D(\beta, \alpha)\}$.

Note that $D(x, \alpha) = 0 \Longrightarrow d_p^s(x, \alpha) = 0$, where $d_p^s(x, \alpha) = \inf_{y\in\alpha} d_p^s(x, y)$.

1.3.13 Proposition [21]

Suppose that (X, D) is a partial metric space (PMS). For any α, β, $C \in CB^D(X)$, we have

(i) $\delta_D(\alpha, \alpha) = \sup\{D(a, a) : a \in \alpha\}$;

(ii) $\delta_D(\alpha, \alpha) \leq \delta_D(\alpha, \beta)$;

(iii) $\delta_D(\alpha, \beta) = 0 \Longrightarrow \alpha \subseteq \beta$;

(iv) $\delta_D(\alpha, \beta) \leq \delta_D(\alpha, C) + \delta_D(C, \beta) - \inf_{c \in C} D(c, c)$.

1.3.14 Proposition [21]

Suppose that (X, D) is a PMS. For any α, β, $C \in CB^D(X)$, we have

(i) $H_D(\alpha, \alpha) \leq H_D(\alpha, \beta)$;

(ii) $H_D(\alpha, \beta) \leq H_D(\beta, \alpha)$;

(iii) $H_D(\alpha, \beta) \leq H_D(\alpha, C) + H_D(C, \beta) - \inf_{c \in C} D(c, c)$.

1.3.15 Remark [21]

Suppose that (X, D) is a PMS and α be any nonempty set in (X, D), then $a \in \overline{\alpha}$ iff

$D(a, \alpha) = D(a, a)$,

where $\overline{\alpha}$ denotes the clouser of α with respect to partial metric D. Note that α is closed in (X, D) iff $\overline{\alpha} = \alpha$.

1.3.16 Lemma

Suppose that α and β are nonempty, closed and bounded subsets of a DPMS (X, D) and $0 < h \in \mathbb{R}$. Then for every $a \in \alpha$, there exists $b \in \beta$ s.t. $D(a, b) \leq H_D(\alpha, \beta) + h$.

Proof: We argue by contradiction. Suppose there exists $h > 0$, s.t. for any $b \in \beta$ we have

$$D(a, b) > H_D(\alpha, \beta) + h.$$

Then,

$$D(a, \beta) = \inf\{D(a, b),\ b \in \beta\} \geq H_D(\alpha, \beta) + h \geq \delta_D(\alpha, \beta) + h,$$

which is a contradiction. Hence, there exists $b \in \beta$ s.t. $D(a, b) \leq H_D(\alpha, \beta) + h$.

1.3.17 Definition [33]

A function $\varphi : [0, +\infty) \longrightarrow [0, 1)$ is said to be $MT-$function if it satisfies Mizoguchi and Takahashi's conditions (i.e., $\limsup_{r \to \varrho^+} \varphi(r) < 1 \ \forall\ \varrho \in [0, +\infty)$).

1.3.18 Proposition [33]

Suppose that $\varphi : [0, +\infty) \longrightarrow [0, 1)$ is a function. Then the following statements are counterpart.

1. φ is an $MT-$function.

2. For each $\varrho \in [0, \infty)$, there occur $r_\varrho^{(1)} \in [0, 1)$ and $\varepsilon_\varrho^{(1)} > 0$ s.t. $\varphi(s) \leq r_\varrho^{(1)} \ \forall\ s \in \left(\varrho, \varrho + \varepsilon_\varrho^{(1)}\right)$.

3. For each $\varrho \in [0, \infty)$, there occur $r_\varrho^{(2)} \in [0, 1)$ and $\varepsilon_\varrho^{(2)} > 0$ s.t. $\varphi(s) \leq r_\varrho^{(2)} \ \forall\ s \in \left(\varrho, \varrho + \varepsilon_\varrho^{(2)}\right)$.

4. For each $\varrho \in [0, \infty)$, there occur $r_\varrho^{(3)} \in [0, 1)$ and $\varepsilon_\varrho^{(3)} > 0$ s.t. $\varphi(s) \leq r_\varrho^{(3)} \ \forall\ s \in \left(\varrho, \varrho + \varepsilon_\varrho^{(3)}\right)$.

5. For each $\varrho \in [0, \infty)$, there occur $r_\varrho^{(4)} \in [0, 1)$ and $\varepsilon_\varrho^{(4)} > 0$ s.t. $\varphi(s) \leq r_\varrho^{(4)} \ \forall\ s \in \left(\varrho, \varrho + \varepsilon_\varrho^{(4)}\right)$.

6. For any nonincreasing sequence $\{x_n\}_{n \in N}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in N} \varphi(x_n) < 1$.

7. φ is a function of contractive factor [34], i.e., for any strictly decreasing sequence $\{x_n\}_{n \in N}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in N} \varphi(x_n) < 1$.

1.4 Single valued and multivalued F-contraction mappings

Wardowski defined the F-contraction as follows:

1.4.1 Definition [103]

Suppose that (X, d) is a metric space. A mapping $J : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ s.t,

$$\forall x, y \in X,\ d(Jx, Jy) > 0 \Rightarrow \tau + F\left(d(Jx, Jy)\right) \leq F\left(d(x, y)\right), \tag{1.1}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a function satisfying the following conditions:

(F1) F is strictly increasing, i.e. $\forall\ x, y \in \mathbb{R}_+$ s.t, $x < y$, $F(x) < F(y)$;

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers. $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ s.t, $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

Altun et al.[14] modified the above definition by adding a general condition $(F4)$ which is given in this way:

(F4) $F(\inf A) = \inf F(A)\ \forall\ A \subset (0, \infty)$ with $\inf A > 0$.

We represent the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying $(F1)$ - $(F4)$ conditions by F in Section 3.4.

1.4.2 Example [103]

The family of $\mathcal{F}$ is not empty.

1) $F(x) = \ln(x); x > 0$.

2) $F(x) = x + \ln(x); x > 0$.

3) $F(x) = \ln(x^2 + x); x > 0$.

4) $F(x) = \frac{-1}{\sqrt{x}}; x > 0$.

1.4.3 Remark

From (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

Wardowski [103] stated a modified version of the Banach contraction principle as follows.

1.4.4 Theorem

Assume that (X, d) is a complete metric space and let $J : X \to X$ be an F-contraction. Then J has a unique FP $x^* \in X$ and for every $x \in X$ the sequence $\{J^n x\}_{n\in\mathbb{N}}$ converges to x^*.

1.4.5 Theorem [84]

Suppose that J is a self-mapping of a complete metric space X into itself. Suppose F is a continuous mapping satisfying $(F1)$ and $(F2)$. Also there exists $\tau > 0$ s.t,

$$\forall x, y \in X,\ d(Jx, Jy) > 0 \Rightarrow \tau + F\left(d(Jx, Jy)\right) \leq F\left(d(x, y)\right),$$

Then J has a unique FP $x^* \in X$ and for every $x_0 \in X$ the sequence $\{J^n x\}_{n=1}^{\infty}$ converges to x^*.

Acar et al. [8] introduced the concept of generalized multivalued F-contraction mappings and established a FP result, which was a proper generalization of some multivalued FP theorems including Nadler's.

1.4.6 Definition [8]

Presume that (X, d) is a metric space and $J : X \longrightarrow CB(X)$ be a mapping. Then J is said to be a generalized multivalued F-contraction if $F \in \digamma$ and there exists $\tau > 0$ s.t,

$$x, y \in X, \;\; H(Jx, Jy) > 0 \Longrightarrow \tau + F(H(Jx, Jy)) \leq F(M(x, y)).$$

where

$$M(x, y) = \max\{d(x, y), D(x, Jx), D(y, Jy), \frac{1}{2}[D(x, Jy) + D(y, Jx)]\}.$$

1.4.7 Theorem [8]

Suppose that (X, d) be a complete metric space and $J : X \longrightarrow K(X)$ be a generalized multivalued F-contraction. If J or F is continuous, then J has a FP in X.

Chapter 2

Fixed Point Results in Generalized Metric Spaces

In 2006, Mustafa and Sims [69] introduced the concept of G-metric space and prove some reults. Kaewcharoen. [55] proved the common FP results for four mappings in G-metric space. Similarly Nashine [79] established coupled common FP results in ordered G-metric space. Samet et al. [94] gave some remarks on G-metric space. Furthermore G-metric space is improved form of D-metric space and 2-metric space. Because D-metric space and 2-metric space are both discontinuous metric spaces but G-metric space is continuous. For more details in this direction, we refer the reader to [22, 38, 70, 81, 105].

In 2012, Tahat et al. [102] utilized the concept of G-metric spaces and obtained point of coincidence and common FPs of a hybrid pair of single-valued and multi-valued mappings. In 2014, Aghajani et al. [9] introduced the concept of G_b-metric spaces and proved that the class of G_b-metric spaces is effectively larger than that of G-metric spaces given in [69].

In this chapter, it is impossible to cover all the known extensions/generalizations of the Banach Contraction Principle. However, an effort has been made to present some extensions of the Banach Contraction Principle and explore the FP and common FP results in $G-$metric spaces and G_b-metric spaces. We continue these investigations to explore the FP and common FP results in generalized metric spaces. In Section 2.1, we define new notions called $(g-F)$ contractions to prove coincidence and common FP results in $G-$metric spaces with application.

In Section 2.2, we generalized the concept of Mizoguchi–Takahashi Contractions for complete $G-$metric spaces and established some new coincidence points and common FP results. In Section 2.3, we introduce the notion of $JS-G-$contraction and prove some FP theorems in the setting of generalized metric spaces. Section 2.4 is devoted to some FP results on G_b-complete metric space for some new contraction.

2.1 Common FP results in $G-$metric spaces with application

Results given in this section have been published in [73].

The following lemmas of [102] are very crucial to prove our main results.

2.1.1 Lemma

Suppose that (X,G) is a $G-$metric space and $A,B\in CB(X)$. Then for each $a\in A$, we have

$$G(a,B,B)\leq H_G(A,B,B).$$

2.1.2 Lemma

Assume that (X,G) is a $G-$metric space. If $A,B\in CB(X)$ and $a\in A$, then for each $\epsilon>0$, there occurs $b\in B$ s.t.

$$G(a,b,b)\leq H_G(A,B,B)+\epsilon.$$

2.1.3 Proposition([54])

Suppose that X is a given non empty set. Assume that $g:X\longrightarrow X$ and $T:X\longrightarrow 2^X$ are weakly compatible mappings. If g and T have a unique point of coincidence $w=gx\in Tx$, then w is the unique common FP of g and T.

In this way, we define the notion of (g-F) contraction.

2.1.4 Definition

Presume that (X,G) is a G-metric space. Suppose that $T:X\longrightarrow CB(X)$ and $g:X\longrightarrow X$. Then the mapping T is said to be (g-F) contraction if there exist some $F\in\mathcal{F}$ and a constant

$\tau > 0$ s.t,

$$H_G(Tx, Ty, Tz) > 0 \Longrightarrow 2\tau + F(H_G(Tx, Ty, Tz)) \leq F(G(gx, gy, gz)) \tag{2.1}$$

$\forall\, x, y, z \in X$.

2.1.5 Theorem

Suppose that (X, G) is a G-metric space. Suppose that $T : X \longrightarrow CB(X)$ and $g : X \longrightarrow X$ be a $(g\text{-}F)$ contraction. If for any $x \in X$, $Tx \subseteq g(X)$ and $g(X)$ is a G-complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ imply $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$, then
(i) g and T have a unique point of coincidence:
(ii) Furthermore, g and T are weakly compatible, then g and T have a unique common FP.

Proof. Suppose that x_0 be an erratic point of X. Since the range of g contains the range of T, there occurs a point x_1 in X s.t. $gx_1 \in Tx_0$. If $gx_1 = gx_0$, then x_0 is a coincidence point of g and T and the proof is complete, so we assume that $gx_0 \neq gx_1$. Also if $Tx_0 = Tx_1$, then x_1 is a coincidence point of g and T. So we assume that $Tx_0 \neq Tx_1$, which gives that $H_G(Tx_0, Tx_1, Tx_1) > 0$. Now from (2.1) we have

$$2\tau + F(H_G(Tx_0, Tx_1, Tx_1)) \leq F(G(gx_0, gx_1, gx_1)). \tag{2.2}$$

Since F is continuous from the right, there occurs a real number $h > 1$ s.t.

$$F(hH_G(Tx_0, Tx_1, Tx_1)) < F(H_G(Tx_0, Tx_1, Tx_1)) - \tau.$$

As $gx_1 \in Tx_0$ so by Lemma 2.1.1, we have

$$G(gx_1, Tx_1, Tx_1) \leq H_G(Tx_0, Tx_1, Tx_1) < hH_G(Tx_0, Tx_1, Tx_1).$$

where $h > 1$. Now from $G(gx_1, Tx_1, Tx_1) < hH_G(Tx_0, Tx_1, Tx_1)$ and Lemma 2.1.1, we deduce that there exists $x_2 \in X$ with $gx_2 \in Tx_1$ s.t.

$$G(gx_1, gx_2, gx_2) \leq hH_G(Tx_0, Tx_1, Tx_1). \tag{2.3}$$

Consequently, we get

$$F(G(gx_1, gx_2, gx_2)) \leq F(hH_G(Tx_0, Tx_1, Tx_1)) < F(H_G(Tx_0, Tx_1, Tx_1)) + \tau, \tag{2.4}$$

which implies that

$$\begin{aligned} 2\tau + F(G(gx_1, gx_2, gx_2)) &\leq 2\tau + F(H_G(Tx_0, Tx_1, Tx_1)) + \tau \\ &\leq F(G(gx_0, gx_1, gx_1)) + \tau. \end{aligned}$$

Thus

$$\tau + F(G(gx_1, gx_2, gx_2)) \leq F(G(gx_0, gx_1, gx_1)).$$

Continuing in this process, we can define a sequence $\{gx_n\} \subset X$ s.t. $gx_{n+1} \in Tx_n$ with $gx_n \neq gx_{n+1}$, $Tx_n \neq Tx_{n+1}$, and

$$\tau + F(G(gx_n, gx_{n+1}, gx_{n+1})) \leq F(G(gx_{n-1}, gx_n, gx_n))$$

$\forall\ n \in \mathbb{N} \cup \{0\}$. Therefore

$$\begin{aligned} F(G(gx_n, gx_{n+1}, gx_{n+1})) &\leq F(G(gx_{n-1}, gx_n, gx_n)) - \tau \\ &\leq F(G(gx_{n-2}, gx_{n-1}, gx_{n-1})) - 2\tau \\ &\leq \cdots \\ &\leq F(G(gx_0, gx_1, gx_1)) - n\tau \end{aligned} \tag{2.5}$$

$\forall\ n \in \mathbb{N}$. Since $F \in \mathcal{F}$, by taking the limit as $n \longrightarrow \infty$ in (2.5), we have

$$\lim_{n \longrightarrow \infty} F(G(gx_n, gx_{n+1}, gx_{n+1})) = -\infty \text{ Therefore by (F2) } \lim_{n \longrightarrow \infty} G(gx_n, gx_{n+1}, gx_{n+1}) = 0. \tag{2.6}$$

Now from $(F3)$, there exists $0 < k < 1$ s.t.

$$\lim_{n \longrightarrow \infty} [G(gx_n, gx_{n+1}, gx_{n+1})]^k F(G(gx_n, gx_{n+1}, gx_{n+1})) = 0. \tag{2.7}$$

By (2.5), we have

$$[G(gx_n, gx_{n+1}, gx_{n+1})]^k F(G(gx_n, gx_{n+1}, gx_{n+1})) - [G(gx_n, gx_{n+1}, gx_{n+1})]^k F(G(gx_0, gx_1, gx_1))$$

$$\leq [G(gx_n, gx_{n+1}, gx_{n+1})]^k [F(G(gx_0, gx_1, gx_1)) - n\tau - F(G(gx_0, gx_1, gx_1))]$$

$$= -n\tau\, [G(gx_n, gx_{n+1}, gx_{n+1})]^k \leq 0.$$

Therefore,

$$\begin{aligned} & [G(gx_n, gx_{n+1}, gx_{n+1})]^k F(G(gx_n, gx_{n+1}, gx_{n+1})) + n\tau\, [G(gx_n, gx_{n+1}, gx_{n+1})]^k \\ \leq\ & [G(gx_n, gx_{n+1}, gx_{n+1})]^k F(G(gx_0, gx_1, gx_1)). \end{aligned} \tag{2.8}$$

By taking the limit as $n \longrightarrow \infty$ in (2.8) and applying (2.6) and (2.7), we have

$$\lim_{n \longrightarrow \infty} n[G(gx_n, gx_{n+1}, gx_{n+1})]^k = 0. \tag{2.9}$$

It follows from (2.8) that there exists $n_1 \in \mathcal{N}$ s.t,

$$n[G(gx_n, gx_{n+1}, gx_{n+1})]^k \leq 1, \text{ for all } n > n_1. \tag{2.10}$$

So

$$G(gx_n, gx_{n+1}, gx_{n+1})] \leq \frac{1}{n^{\frac{1}{k}}}, \text{ for all } n > n_1. \tag{2.11}$$

Now we prove that $\{gx_n\}$ is a G-C-seq. For $m > n > n_1$ we have

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\leq \sum_{i=n}^{m-1} G(gx_i, gx_{i+1}, gx_{i+1}) \\ &\leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{k}}} \leq \sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{2.12}$$

Since $0 < k < 1$, $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ converges. Therefore, $G(gx_n, gx_m, gx_m) \longrightarrow 0$ as $m, n \longrightarrow \infty$. Thus $\{gx_n\}$ is a G-C-seq in complete subspace $g(X)$, so there exists $q \in g(X)$ s.t.

$$\lim_{n \longrightarrow \infty} G(gx_n, gx_n, q) = \lim_{n \longrightarrow \infty} G(gx_n, q, q) = 0. \tag{2.13}$$

Since $q \in g(X)$, there exists $p \in X$ s.t, $q = gp$. Hence from (2.13), we have

$$\lim_{n \longrightarrow \infty} G(gx_n, gx_n, gp) = \lim_{n \longrightarrow \infty} G(gx_n, gp, gp) = 0.$$

Now we will prove that $gp \in Tp$. Suppose that there exists an increasing sequence $\{n_k\}$ s.t. $gx_{n_k} \in Tp \ \forall\ k \in N$, since Tp is closed and $gx_{n_k} \to gp$, we get $gp \in Tp$ and the proof is complete. So we assume that there exists $n_0 \in N$ s.t. $gx_{n+1} \notin Tp \ \forall\ n \geq n_0$. Since $gx_{n+1} \in Tx_n$, $Tx_n \neq Tp$ $\forall\ n \geq n_0$ and so we have

$$H_G(Tx_n, Tp, Tp) > 0 \text{ for all } n \geq n_0. \tag{2.14}$$

Now as $gx_{n+1} \in Tx_n$, so by Lemma 2.1.1, we have

$$G(gx_{n+1}, Tp, Tp) \leq H_G(Tx_n, Tp, Tp).$$

As F is strictly increasing, so by (2.14), above inequality and (2.1), we get

$$\begin{aligned} F(G(gx_{n+1}, Tp, Tp)) &\leq 2\tau + F(G(gx_{n+1}, Tp, Tp)) \\ &\leq 2\tau + F(H_G(Tx_n, Tp, Tp)) \\ &\leq F(G(gx_n, gp, gp)). \end{aligned}$$

Since F is increasing, we have

$$G(gx_{n+1}, Tp, Tp) \leq G(gx_n, gp, gp). \tag{2.15}$$

putting $n \to \infty$ in previous inequality and using the fact that the function G is continuous on its three variables, we get $G(gp, Tp, Tp) = 0$. Since Tp is closed, we obtain that $gp \in Tp$. That is, p is a coincidence point of T and g. Hence g and T have a point of coincidence w. We will prove the uniqueness of a point of coincidence of g and T. For this we suppose on the contrary that w^* is another point of coincidence of g and T, i.e,, there exists another coincidence point q of g and T s.t, $w^* = gq \in Tq$ with $gp \neq gq$ and $Tp \neq Tq$. Otherwise p and q will not be coincidence points. Then $H_G(Tq, Tp, Tp) > 0$. Thus, we have the following assumption that

$$G(gq, gp, gp) \leq H_G(Tq, Tp, Tp).$$

Since F is increasing, by above inequality and (2.1), we get

$$\begin{aligned} 2\tau + F(G(gq, gp, gp)) &\leq 2\tau + F(H_G(Tq, Tp, Tp)) \\ &\leq F(G(gq, gp, gp). \end{aligned}$$

which further implies that

$$\begin{aligned} F(G(gq, gp, gp)) &\leq F(G(gq, gp, gp) - 2\tau \\ &< F(G(gq, gp, gp). \end{aligned}$$

Since F is strictly increasing, we get

$$G(gq, gp, gp) < G(gq, gp, gp).$$

which is a contradiction. Hence $gp = gq$ and $Tp = Tq$. Hence g and T have a unique point of coincidence. Suppose that g and T are weakly compatible. By applying Proposition 2.1.3, we get that g and T have a unique common FP. ■

2.1.6 Corollary

Assume that (X, G) is a complete $G-$metric space on X, and let $T : X \longrightarrow CB(X)$. If there occur a function $F \in \mathcal{F}$ and a constant $\tau > 0$ s.t. $\forall\ x, y, z \in X$

$$H_G(Tx, Ty, Tz) > 0 \Longrightarrow 2\tau + F(H_G(Tx, Ty, Tz)) \leq F(G(x, y, z))$$

$\forall\ x, y, z \in X$, then T has a FP in X.

Proof. It follows by taking g the identity on X in Theorem 2.1.5.

2.1.7 Example [73]

Suppose that $X = [0, 1]$. Define mapping $T : X \longrightarrow CB(X)$ by $Tx = \left[0, \frac{x}{25}\right]$ and define $g : X \longrightarrow X$ by $g(x) = \frac{3x}{4}$. Define a $G-$metric on X by $G(x, y, z) = |x - y| + |y - z| + |x - z|$. Then

(1) $g(X)$ is a $G-$complete subspace of X;

(2) g and T are weakly compatible;

(3) $Tx \subseteq g(X)$;

(4) T is a $(g - F)$ contraction where $F(\alpha) = \ln(\alpha)$ and $\tau \in \left(0, \ \ln\left(\sqrt{\frac{75}{32}}\right)\right)$.

Proof. The proof of $(1), (2)$ and (3) are clear. We will prove (4).

We have $d_G(x, y) = G(x, y, y) + G(y, x, x) = 4|x - y|\ \forall\ x, y \in X$. To prove (4), let x, y, $z \in X$. If $x = y = z = 0$ then $Tx = Ty = Tz = 0$ and $H_G(Tx, Ty, Tz) = 0$, thus we may presume that x, y and z are not all zero. Without loss of generality we assume that $x < y < z$. Then

$$H_G(Tx, Ty, Tz) = H_G\left(\left[0, \frac{x}{25}\right], \left[0, \frac{y}{25}\right], \left[0, \frac{z}{25}\right]\right)$$
$$= \max\left\{\begin{array}{c} \sup_{0\leq a\leq \frac{x}{25}} G\left(a, \left[0, \frac{y}{25}\right], \left[0, \frac{z}{25}\right]\right), \\ \sup_{0\leq b\leq \frac{y}{25}} G\left(b, \left[0, \frac{x}{25}\right], \left[0, \frac{z}{25}\right]\right), \\ \sup_{0\leq c\leq \frac{z}{25}} G\left(c, \left[0, \frac{x}{25}\right], \left[0, \frac{y}{25}\right]\right) \end{array}\right\}. \qquad (2.16)$$

Since $x < y < z$, $\left[0, \frac{x}{25}\right] \subseteq \left[0, \frac{y}{25}\right] \subseteq \left[0, \frac{z}{25}\right]$ which gives

$$d_G\left(\left[0, \frac{x}{25}\right], \left[0, \frac{y}{25}\right]\right) = d_G\left(\left[0, \frac{y}{25}\right], \left[0, \frac{z}{25}\right]\right) = d_G\left(\left[0, \frac{x}{25}\right], \left[0, \frac{z}{25}\right]\right) = 0.$$

Now for each $0 \leq a \leq \frac{x}{25}$ we have

$$G\left(a, \left[0, \frac{y}{25}\right], \left[0, \frac{z}{25}\right]\right) = d_G\left(a, \left[0, \frac{y}{25}\right]\right) + d_G\left(\left[0, \frac{y}{25}\right], \left[0, \frac{z}{25}\right]\right) + d_G\left(a, \left[0, \frac{z}{25}\right]\right) = 0.$$

Also, for each $0 \leq b \leq \frac{y}{25}$ we have

$$\begin{aligned} G\left(b, \left[0, \frac{x}{25}\right], \left[0, \frac{z}{25}\right]\right) &= d_G\left(b, \left[0, \frac{x}{25}\right]\right) + d_G\left(\left[0, \frac{x}{25}\right], \left[0, \frac{z}{25}\right]\right) + d_G\left(b, \left[0, \frac{z}{25}\right]\right) \\ &= \begin{cases} 0, \text{ if } 0 \leq b \leq \frac{x}{25}; \\ 4b - \frac{4x}{25}, \text{ if } b \geq \frac{x}{25} \end{cases} \end{aligned}$$

which implies that

$$\sup_{0 \leq b \leq \frac{y}{25}} G\left(b, \left[0, \frac{x}{25}\right], \left[0, \frac{z}{25}\right]\right) = \frac{4y - 4x}{25}.$$

Moreover, for each $0 \leq c \leq \frac{z}{25}$ we have

$$\begin{aligned} G\left(c, \left[0, \frac{x}{25}\right], \left[0, \frac{y}{25}\right]\right) &= d_G\left(c, \left[0, \frac{x}{25}\right]\right) + d_G\left(\left[0, \frac{x}{25}\right], \left[0, \frac{y}{25}\right]\right) + d_G\left(c, \left[0, \frac{y}{25}\right]\right) \\ &= \begin{cases} 0, \text{ if } 0 \leq c \leq \frac{x}{25}; \\ 4c - \frac{4x}{25}, \text{ if } \frac{x}{25} \leq c \leq \frac{y}{25}; \\ 8c - \frac{4y}{25} - \frac{4x}{25}, \text{ if } \frac{y}{25} \leq c \leq \frac{z}{25}. \end{cases} \end{aligned}$$

which implies that

$$\sup_{0 \leq c \leq \frac{z}{25}} G\left(c, \left[0, \frac{x}{25}\right], \left[0, \frac{y}{25}\right]\right) = \frac{8z - 4y - 4x}{25}.$$

Thus we deduce that

$$\begin{aligned}
H_G(Tx, Ty, Tz) &= \max\left\{0, \frac{4y-4x}{25}, \frac{8z-4y-4x}{25}\right\} \\
&= \frac{8z-4y-4x}{25} \\
&\leq \frac{8z-8x}{25} \\
&= \frac{8}{25}|z-x| \\
&= \frac{32}{75}\left|\frac{3z}{4} - \frac{3x}{4}\right| \\
&= \frac{32}{75}|gz-gx| \\
&\leq \frac{32}{75}(|gx-gy| + |gy-gz| + |gx-gz|) \\
&= \frac{32}{75}G(gx, gy, gz).
\end{aligned}$$

Therefore,

$$\frac{75}{32}H_G(Tx, Ty, Tz) \leq G(gx, gy, gz).$$

By using $F(\alpha) = \ln(\alpha)$ we get

$$\ln\left(\frac{75}{32}\right) + \ln(H_G(Tx, Ty, Tz)) \leq \ln(G(gx, gy, gz)).$$

Thus, $\forall\, x,\, y,\, z \in X$ with $H_G(Tx, Ty, Tz) > 0$ we have

$$2\tau + F(H_G(Tx, Ty, Tz)) \leq F(G(gx, gy, gz)), \text{ where } 0 < \tau < \ln\left(\sqrt{\frac{75}{32}}\right).$$

Hence, T is a $(g-F)$ contraction. On the other hand it is clear that $x = 0$ is the only coincidence point and all other hypothesis of Theorem 2.1.5 are satisfied. So the mappings T and g have a unique common FP which is $u = 0$. ■

Now, we will use Corollary 2.1.6 to show that there is a solution to the following integral equation:

$$u(\varrho) = \int_a^b H(\varrho, s)K(s, u(s))ds; \quad \varrho \in [a, b]. \tag{2.17}$$

Assume that $X = (C[a,b],\mathbb{R})$ denote the set of all continuous functions from $[a,b]$ to $\mathbb{R}$.

Define a mapping $J : X \to X$ by

$$Ju(\varrho) = \int_a^b H(\varrho, s)K(s, u(s))ds; \quad \varrho \in [a,b]. \tag{2.18}$$

2.1.8 Theorem

Consider (2.17) and suppose:

1. $H : [a,b] \times [a,b] \to [0,\infty)$ is a continuous function,
2. $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ where K is continuous mapping,
3. $\max_{\varrho\in[a,b]} \int_a^b H(\varrho, s)ds < e^{-2\tau}$, for some $\tau \in (0,\infty)$,
4. $\forall\ u(s), v(s) \in X;\ s \in [a,b]$ we have

$$|K(s, u(s)) - K(s, v(s))| \leq |u(s) - v(s)|. \tag{2.19}$$

Then (2.17) has a solution.

Proof. Presume that X and J be as defined above. For all $u, v, w \in X$ define the G-metric on X by

$$G(u, v, w) = d(u, v) + d(v, w) + d(u, w) \tag{2.20}$$

where

$$d(u, v) = \sup_{\varrho\in[a,b]} |u(\varrho) - v(\varrho)|.$$

Clearly that (X, G) is a complete G-metric space, since (X, d) is a complete metric space. ■

Now, Assume that $u(\varrho), v(\varrho) \in X$, then from Definition 2.18. (3) and (4) we have

$$\begin{aligned}
|Ju(\varrho) - Jv(\varrho)| &= \left|\int_a^b H(\varrho,s)\big[K(s,u(s)) - K(s,v(s))\big]ds\right| \\
&\leq \int_a^b H(\varrho,s)|K(s,u(s)) - K(s,v(s)|ds \\
&\leq \int_a^b H(\varrho,s)|u(s) - v(s)|ds \\
&\leq \int_a^b H(\varrho,s) \sup_{s\in[a,b]} |u(s) - v(s)| ds \\
&= \sup_{\varrho\in[a,b]} |u(\varrho) - v(\varrho)| \int_a^b H(\varrho,s)ds \\
&\leq e^{-2\tau} \sup_{\varrho\in[a,b]} |u(\varrho) - v(\varrho)|.
\end{aligned}$$

Hence,

$$\sup_{\varrho\in[a,b]} |Ju(\varrho) - Jv(\varrho)| \leq e^{-2\tau} \sup_{\varrho\in[a,b]} |u(\varrho) - v(\varrho)|. \tag{2.21}$$

Similarly, we have

$$\sup_{\varrho\in[a,b]} |Jv(\varrho) - Jw(\varrho)| \leq e^{-2\tau} \sup_{\varrho\in[a,b]} |v(\varrho) - w(\varrho)| \tag{2.22}$$

and

$$\sup_{\varrho\in[a,b]} |Ju(\varrho) - Jw(\varrho)| \leq e^{-2\tau} \sup_{\varrho\in[a,b]} |u(\varrho) - w(\varrho)|. \tag{2.23}$$

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Therefore, from (2.21), (2.22) and (2.23) we have

Proof.

$$\begin{aligned}
&\sup_{\varrho\in[a,b]} |Ju(\varrho) - Jv(\varrho)| + \sup_{\varrho\in[a,b]} |Jv(\varrho) - Jw(\varrho)| + \sup_{\varrho\in[a,b]} |Ju(\varrho) - Jw(\varrho)| \\
&\leq e^{-2\tau} \Big[\sup_{\varrho\in[a,b]} |u(\varrho) - v(\varrho)| + \sup_{\varrho\in[a,b]} |v(\varrho) - w(\varrho)| + \sup_{\varrho\in[a,b]} |u(\varrho) - w(\varrho)|. \Big]
\end{aligned}$$

which implies

$$G(Ju, Jv, Jw) \leq e^{-2\tau} G(u, v, w). \tag{2.24}$$

Thus,

$$\ln(G(Ju, Jv, Jw)) \leq -2\tau + \ln(G(u, v, w))$$

and so,

$$2\tau + \ln(G(Ju, Jv, Jw)) \leq \ln(G(u, v, w)).$$

Now, we observe that $2\tau + F(G(Ju, Jv, Jw)) \leq F(G(u, v, w))$ is satisfied for $F(\alpha) = \ln(\alpha)$ $\forall$ $\alpha \in X$. Therefore, all conditions of Corollary 2.1.6 are satisfied. As a result of Corollary 2.1.6 the mapping J has a FP in X which is a solution of (2.17). ■

The following example illustrates the validity of Theorem 2.1.8.

2.1.9 Example

The following integral equation has a solution in $X = (C[\ln(2), \ln(3)], \mathbb{R})$.

$$u(\varrho) = \int_{\ln(2)}^{\ln(3)} cosh(s\varrho)\ u(s)ds; \quad \varrho \in [\ln(2), \ln(3)]. \tag{2.25}$$

Proof. Suppose that $J : X \to X$ be defined as $Ju(\varrho) = \int_{\ln(2)}^{\ln(3)} cosh(s\varrho)\ u(s)ds;\ \varrho \in [\ln(2), \ln(3)]$. By specifying $H(\varrho, s) = \cosh(s\varrho)$, $K(s, \varrho) = \varrho$ and $\tau \geq \frac{18}{100}$ in Theorem 2.1.8, we get that: ■

1. the function $H(\varrho, s)$ is continuous on $[\ln(2), \ln(3)] \times [\ln(2), \ln(3)]$.
2. $K(s, \varrho)$ is continuous on $[\ln(2), \ln(3)] \times \mathbf{R}$ $\forall$ $s \in [\ln(2), \ln(3)]$.
3.

$$\begin{aligned}
\max_{\varrho\in[\ln(2),\ln(3)]} \int_{\ln(2)}^{\ln(3)} cosh(s\varrho)ds &= \max_{\varrho\in[\ln(2),\ln(3)]} \frac{sinh(ln(3^{\varrho})) - sinh(ln(2^{\varrho}))}{\varrho} \\
&= \max_{\varrho\in[\ln(2),\ln(3)]} \frac{3^{\varrho} - 3^{-\varrho} - 2^{\varrho} + 2^{-\varrho}}{2\varrho} \\
&< 0.7 \\
&\leq e^{-2\tau},
\end{aligned}$$

4. $\forall$ $u(s), v(s) \in X$ it is clearly that condition (4) in Theorem 2.1.8 is satisfied.

Therefore, all the conditions of Theorem 2.1.8 are satisfied, hence the mapping J has a FP in X, which is a solution to equation (2.25).

2.2 Generalized Mizoguchi-Takahashi's contractions

In 2012, Tahat et al. [102] utilized the concept of G-metric spaces and obtained point of coincidence and common FPs of a hybrid pair of single-valued and multi-valued mappings. They proved the following FP theorem as a main result.

2.2.1 Theorem [102]

Assume that (X, G) is a $G-$metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping and $g : X \rightarrow X$ a self mapping. Assume that there exists a function $\alpha : [0, +\infty) \rightarrow [0, 1)$ satisfying

$$\lim_{r \rightarrow t^+} \sup \alpha(r) < 1$$

for every $t \geq 0$ s.t,

$$H_G(Tx, Ty, Tz) \leq \alpha(G(gx, gy, gz))G(gx, gy, gz) \tag{2.26}$$

$\forall\ x, y, z \in X$. If for any $x \in X$, $Tx \subseteq g(X)$ and $g(X)$ is a $G-$complete subspace of X, then g and T have a point of coincidence in X. Furthermore, if we assume that $gp \in Tp$ and $gq \in Tq$ imply $G(gq, gp, gp) \leq H_G(Tq, Tp, Tp)$,then

(i) g and T have a unique point of coincidence;

(ii) Furthermore, if g and T are weakly compatible, then g and T have a unique common FP.

Recently, Javahernia et al. [51] generalized the above function by introducing the notion of generalized Mizoguchi–Takahashi function in such a way.

2.2.2 Definition [51]

A function $\alpha : R \times R \rightarrow R$ is called a generalized Mizoguchi–Takahashi function (shortly, generalized MT-function) if the following conditions hold:

(a_1) $0 < \alpha(u, v) < 1\ \forall\ u, v > 0$;

(a_2) for any bounded sequence $(u_n) \subset (0, +\infty)$ and any non-increasing sequence $(v_n) \subset$

$(0, +\infty)$, we have

$$\lim_{n\to\infty} \sup \alpha(u_n, v_n) < 1.$$

Consonent with Javahernia et al. [51], we signify by Λ the set of all functions $\alpha : R \times R \to R$ satisfying the conditions (a_1)–(a_2).

The basic aim of this section is to generalize the results of Tahat et al.[102] by utilizing the notion of generalized Mizoguchi–Takahashi function. Now we give the main result of this section.

2.2.3 Theorem [73]

Suppose that (X, G) be a $G-$metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping and $g : X \to X$ a self mapping. If for any $x \in X$, $Tx \subseteq g(X)$ and $g(X)$ is a $G-$complete subspace of X and there exist $\alpha \in \Lambda$ s.t,

$$H_G(Tx, Ty, Tz) \leq \alpha(H_G(Tx, Ty, Tz), G(gx, gy, gz))G(gx, gy, gz) \tag{2.27}$$

$\forall\ x, y, z \in X$. Then g and T have a point of coincidence in X. Furthermore, if we assume that $gx^* \in Tx^*$ and $g\hat{x} \in T\hat{x}$ implies $G(g\hat{x}, gx^*, gx^*) \leq H_G(T\hat{x}, Tx^*, Tx^*)$, then

(i) g and T have a unique point of coincidence.

(ii) Furthermore, if g and T are weakly compatible, then g and T have a unique common FP.

Proof. Suppose that x_0 be an arbitrary point of X. Then by the given assumptions, there occurs a point x_1 in X s.t, $gx_1 \in Tx_0$. If $gx_1 = gx_0$, then we have nothing to prove and x_0 is the required point. So we assume that $gx_0 \neq gx_1$, then $G(gx_0, gx_1, gx_1) > 0$. Now if

$$H_G(Tx_0, Tx_1, Tx_1) = 0,$$

then from (2.27), we get a contradiction to the fact that $gx_0 \neq gx_1$. Thus $H_G(Tx_0, Tx_1, Tx_1) > 0$. From the inequality (2.27), we have

$$H_G(Tx_0, Tx_1, Tx_1) \leq \alpha(H_G(Tx_0, Tx_1, Tx_1), G(gx_0, gx_1, gx_1))\, G(gx_0, gx_1, gx_1).$$

Take

$$\in_1 = \left(\frac{1}{\sqrt{\alpha\left(H_G\left(Tx_0, Tx_1, Tx_1\right), G\left(gx_0, gx_1, gx_1\right)\right)}} - 1 \right) H_G\left(Tx_0, Tx_1, Tx_1\right). \tag{2.28}$$

Then by Lemma 2.1.2 and the inequality (2.28), we have

$$\begin{aligned} G\left(gx_1, gx_2, gx_2\right) &\leq H_G\left(Tx_0, Tx_1, Tx_1\right) + \in_1 \\ &= \frac{H_G\left(Tx_0, Tx_1, Tx_1\right)}{\sqrt{\alpha\left(H_G\left(Tx_0, Tx_1, Tx_1\right), G\left(gx_0, gx_1, gx_1\right)\right)}}. \end{aligned} \tag{2.29}$$

Since $Tx_1 \subseteq g(X)$, there exists a point x_2 in X s.t, $gx_2 \in Tx_1$. If $gx_1 = gx_2$. Then x_1 is the required point. So we assume that $gx_1 \neq gx_2$, then $G\left(gx_1, gx_2, gx_2\right) > 0$. Now if

$$H_G\left(Tx_1, Tx_2, Tx_2\right) = 0.$$

then from the inequality (2.27), we get a contradiction to the fact that $gx_1 \neq gx_2$. Thus,

$$H_G\left(Tx_1, Tx_2, Tx_2\right) > 0.$$

From the inequality (2.27), we have

$$H_G\left(Tx_1, Tx_2, Tx_2\right) \leq \alpha\left(H_G\left(Tx_1, Tx_2, Tx_2\right), G\left(gx_1, gx_2, gx_2\right)\right) G\left(gx_1, gx_2, gx_2\right). \tag{2.30}$$

Take

$$\in_2 = \left(\frac{1}{\sqrt{\alpha(H_G\left(Tx_1, Tx_2, Tx_2\right), G\left(gx_1, gx_2, gx_2\right))}} - 1 \right) H_G\left(Tx_1, Tx_2, Tx_2\right). \tag{2.31}$$

Then by Lemma 2.1.2 and inequality (2.31), we get

$$\begin{aligned} G\left(gx_2, gx_3, gx_3\right) &\leq H_G\left(Tx_1, Tx_2, Tx_2\right) + \in_2 \\ &= \frac{H_G\left(Tx_1, Tx_2, Tx_2\right)}{\sqrt{\alpha\left(H_G\left(Tx_1, Tx_2, Tx_2\right), G\left(gx_1, gx_2, gx_2\right)\right)}}. \end{aligned} \tag{2.32}$$

By repeating the above process, we can construct a sequence $\{gx_k\}$ s.t, $gx_{k+1} \in Tx_k$, where which shows that $\{H_G(Tx_{k-1}, Tx_k, Tx_k)\}$ is a bounded sequence. By (a_2), we have

$$\lim_{n\to\infty} \sup \alpha(H_G(Tx_{k-1}, Tx_k, Tx_k), G(gx_{k-1}, gx_k, gx_k)) < 1. \tag{2.37}$$

Now we claim that $d = 0$. Suppose $d > 0$, then by (2.36), (2.37) and taking the limsup on both sides of (2.35) we get

$$d \le \sqrt{\lim_{k\to\infty} \sup \alpha(H_G(Tx_{k-1}, Tx_k, Tx_k), G(gx_{k-1}, gx_k, gx_k))}\; d < d.$$

So, this contradiction implies that

$$\lim_{k\to\infty} d_k = \inf_{k\in N} d_k = 0. \tag{2.38}$$

Therefore,

$$\lim_{k\to\infty} G(gx_k, gx_{k+1}, gx_{k+1}) = \inf_{k\in N} G(gx_k, gx_{k+1}, gx_{k+1}) = 0. \tag{2.39}$$

Now we prove that $\{gx_k\}$ is a C-seq in X. For each $k \in \mathbb{N}$, let

$$q_k = \sqrt{\alpha(H_G(Tx_{k-1}, Tx_k, Tx_k), G(gx_{k-1}, gx_k, gx_k))}.$$

Then $q_k \in (0, 1)$, $\forall\, k \in \mathbb{N}$. By (2.35), we have

$$G(gx_k, gx_{k+1}, gx_{k+1}) \le q_k G(gx_{k-1}, gx_k, gx_k) \tag{2.40}$$

$\forall\, k \in \mathbb{N}$. From (2.37), we have $\lim_{k\to\infty} \sup q_k < 1$, so there exist $c \in [0, 1)$ and $k_0 \in \mathbb{N}$ s.t. $q_k < c\ \forall\, k \in \mathbb{N}$ with $k \ge k_0$. Since $q_k \in (0, 1)\ \forall\, k \in \mathbb{N}$ and $c \in [0, 1)$, from (2.40) for $k > k_0$, we conclude that

$$\begin{aligned} G(gx_k, gx_{k+1}, gx_{k+1}) &\le q_k G(gx_{k-1}, gx_k, gx_k) \\ &\le q_k q_{k-1} G(gx_{k-2}, gx_{k-1}, gx_{k-1}) \\ &\le \cdots \end{aligned}$$

$$\begin{aligned} &\leq q_k q_{k-1} \cdots q_{k_0} G(gx_0, gx_1, gx_1) \\ &\leq c^{k-k_0+1} G(gx_0, gx_1, gx_1). \end{aligned}$$

Suppose that $\lambda_k = \frac{c^{k-k_0+1}}{1-c} G(gx_0, gx_1, gx_1)$; $k \in \mathbb{N}$. For $k \in \mathbb{N}$ with $k \geq k_0$ and a positive arbitrary number m, then from the last inequality and (G_5), we have

$$G(gx_k, gx_{k+m}, gx_{k+m}) \leq \sum_{i=k}^{k+m-1} G(gx_i, gx_{i+1}, gx_{i+1}) \leq \lambda_k. \tag{2.41}$$

Since $c \in [0,1)$, as a result, $\lim_{k\to\infty} \lambda_k = 0$. Hence $\lim_{k\to\infty} G(gx_k, gx_{k+m}, gx_{k+m}) = 0$. Thus the sequence $\{gx_k\}$ is G-Cauchy in the complete subspace $g(X)$. Thus there exists $x' \in g(X)$ s.t. from Proposition 1.2.3, we have

$$\lim_{k\to\infty} G\left(gx_k, gx_k, x'\right) = \lim_{k\to\infty} G\left(gx_k, x', x'\right) = 0. \tag{2.42}$$

Since $x' \in g(X)$, there exists $x^* \in X$ s.t. $x' = gx^*$. Thus from (2.42), we have

$$\lim_{k\to\infty} G(gx_k, gx_k, gx^*) = \lim_{k\to\infty} G(gx_k, gx^*, gx^*) = 0. \tag{2.43}$$

We claim that $gx^* \in Tx^*$. From (2.27) and (2.43), we have

$$\begin{aligned} \lim_{k\to\infty} G(gx_{k+1}, Tx^*, Tx^*) &\leq \lim_{k\to\infty} H_G(Tx_k, Tx^*, Tx^*) \\ &\leq \lim_{k\to\infty} \alpha(H_G(Tx_{k-1}, Tx^*, Tx^*), G(gx_k, gx^*, gx^*)) G(gx_k, gx^*, gx^*) \\ &= 0. \end{aligned}$$

Hence, $G(gx^*, Tx^*, Tx^*) = 0$, i.e., $gx^* \in Tx^*$. Thus T and g have a point of coincidence x^*. Now we prove that this point of coincidence is unique. We suppose on the contrary that there occurs another $\hat{x}$ s.t, $g\hat{x} \in T\hat{x}$ but $g\hat{x} \neq gx^*$. By (2.27) and this assumption, we have

$$\begin{aligned} G(g\hat{x}, gx^*, gx^*) &\leq H_G(T\hat{x}, Tx^*, Tx^*) \\ &\leq \alpha(H_G(T\hat{x}, Tx^*, Tx^*), G(g\hat{x}, gx^*, gx^*))G(g\hat{x}, gx^*, gx^*). \end{aligned}$$

As $H_G(T\hat{x}, Tx^*, Tx^*) > 0$ and $G(g\hat{x}, gx^*, gx^*) > 0$, so

$$\alpha(H_G(T\hat{x}, Tx^*, Tx^*), G(g\hat{x}, gx^*, gx^*)) < 1.$$

Thus we get

$$G(g\hat{x}, gx^*, gx^*) < G(g\hat{x}, gx^*, gx^*),$$

which is a contradiction to the fact that $g\hat{x} \neq gx^*$. Thus $g\hat{x} = gx^*$. In view of

$$H_G(T\hat{x}, Tx^*, Tx^*) \leq \alpha(H_G(T\hat{x}, Tx^*, Tx^*), G(g\hat{x}, gx^*, gx^*))\, G(g\hat{x}, gx^*, gx^*) = 0$$

we have $T\hat{x} = Tx^*$. Thus, T and g have a unique point of coincidence. Assume that g and T are weakly compatible. By applying Proposition 2.1.3, we obtain that g and T have a unique common FP. ■

2.2.4 Remark

Theorem 2.2.1 follows from Theorem 2.2.3 by taking $\alpha(u, v) = \varphi(v)$.

2.2.5 Remark

Corollary 2.2.4 of [102] can be obtained by taking $a(u, v) = \mu$ in Remark 2.2.4.

2.2.6 Theorem

Assume that (X, G) is a $G-$metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping and $g : X \rightarrow X$ a self mapping. If for any $x \in X$, $Tx \subseteq g(X)$ and $g(X)$ is a $G-$complete subspace of X s.t,

$$H_G(Tx, Ty, Tz) \leq \varphi(G(gx, gy, gz))$$

$\forall\ x, y, z \in X$, where $\varphi : [0, \infty) \rightarrow [0, 1)$ is a function s.t. $\varphi(v) < v$ and $\limsup_{v \rightarrow u^+} \frac{\varphi(v)}{v} < 1$. Then g and T have a point of coincidence in X. Furthermore, if we assume that $gx^* \in Tx^*$ and $gx^* \in T\hat{x}$ imply $G(g\hat{x}, gx^*, gx^*) \leq H_G(T\hat{x}, Tx^*, Tx^*)$, then

(i) g and T have a unique point of coincidence.

(ii) Furthermore, if g and T are weakly compatible, then g and T have a unique common FP.

Proof. Take $\alpha(u, v) = \frac{\varphi(v)}{v}$ in Theorem 2.2.3. ∎

Javahernia et al. [51] also introduced the concept of weak l.s.c. in the following way.

2.2.7 Definition

A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is said to be weak l.s.c. function if for each bounded sequence $\{u_n\} \subset (0, +\infty)$, we have

$$\lim_{n \rightarrow \infty} \inf \phi(u_n) > 0.$$

Consistent with Javahernia et al. [51], we denote by F, the set of all functions $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying the above condition.

2.2.8 Theorem

Assume that (X, G) is a $G-$metric space and let $T : X \longrightarrow CB(X)$ be a multi-valued mapping and $g : X \rightarrow X$ a self mapping. If for any $x \in X$, $Tx \subseteq g(X)$ and $g(X)$ is a $G-$complete subspace of X s.t.

$$H_G(Tx, Ty, Tz) \leq G(gx, gy, gz) - \phi(G(gx, gy, gz))$$

$\forall\ x, y, z \in X$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is s.t. $\phi(0) = 0$, $\phi(v) < v$ and $\phi \in F$. Then g and T have a point of coincidence in X. Furthermore, if we assume that $gx^* \in Tx^*$ and $g\hat{x} \in T\hat{x}$ imply $G(g\hat{x}, gx^*, gx^*) \leq H_G(T\hat{x}, Tx^*, Tx^*)$, then

(i) g and T have a unique point of coincidence,

(ii) Furthermore, if g and T are weakly compatible, then g and T have a unique common FP.

Proof. Define $\alpha(u,v) = 1 - \frac{\phi(u)}{u}$ $\forall$ $u, v > 0$. For each bounded sequence $\{u_n\} \subset (0, +\infty)$, we have $\lim_{n\to\infty} \inf \phi(u_n) > 0$. So $\lim_{n\to\infty} \inf \frac{\phi(u_n)}{u_n} > 0$. Thus

$$\lim_{n\to\infty} \sup(1 - \frac{\phi(u_n)}{u_n}) = 1 - \lim_{n\to\infty} \inf \frac{\phi(u_n)}{u_n} < 0.$$

This shows that $\alpha \in \Lambda$. Also

$$H_G(Tx, Ty, Tz) \leq \alpha(H_G(Tx, Ty, Tz), G(gx, gy, gz))G(gx, gy, gz).$$

Thus by Theorem 2.2.1, we get g and T have a unique common FP. ■

2.3 Fixed point results for new contraction in $G-$metric space

Jleli and Samet [53] introduced a new type of contraction which involves the following set of all functions $\psi : (0, \infty) \to (1, \infty)$ satisfying the conditions:

(ψ_1) ψ is nondecreasing;

(ψ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(ψ_3) there occur $r \in (0, 1)$ and $L \in (0, \infty]$ s.t, $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = L$.

To be consonent with Jleli and Samet [53], we signify by Γ the set of all functions ψ : $(0, \infty) \to (1, \infty)$ satisfying the conditions (ψ_1)-(ψ_3).

Also, they established the following result as a generalization of Banach Contraction Principle.

2.3.1 Theorem [53]

Suppose that (X, d) is a complete metric space and $f : X \to X$ be a mapping. Assume that there occur $\psi \in \Psi$ and $k \in (0, 1)$ s.t,

$$x, y \in X, \quad d(fx, fy) \neq 0 \Longrightarrow \psi(d(fx, fy)) \leq [\psi(d(x, y))]^k.$$

Then f has a unique FP.

In 2015, Hussain et al. [48] modified the above family of functions and proved a FP theorem as a generalization of [53]. They customized the family of functions $\psi : [0, \infty) \to [1, \infty)$ to be as follows:

(ψ_1) ψ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;

(ψ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(ψ_3) there occur $r \in (0, 1)$ and $L \in (0, \infty]$ s.t. $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = L$;

(ψ_4) $\psi(u + v) \le \psi(u) . \psi(v)$ $\forall$ $u, v > 0$.

To be consonent with Hussain et al. [48], we signify by Ψ the set of all functions $\psi : [0, \infty) \to [1, \infty)$ satisfying the conditions (ψ_1)–(ψ_4). For more details in this direction, we directed the reader to [11, 13, 17].

In this section, we introduce a new contraction called JS-G-contraction and prove some FP results of such contraction in the setting of G-metric spaces. The following results have been published in [50].

2.3.2 Definition

Assume that (X, G) is a G-metric space, and let $g : X \to X$ be a self mapping. Then g is said to be a JS-G-contraction whenever there occur a function $\psi \in \Psi$ and positive real numbers r_1, r_2, r_3, r_4 with $0 \le r_1 + 3r_2 + r_3 + 2r_4 < 1$ s.t,

$$\begin{aligned} \psi(G(ga, gb, gc)) &\le [\psi(G(a, b, c))]^{r_1} [\psi(G(a, ga, gc))]^{r_2} [\psi(G(b, gb, gc))]^{r_3} \\ &\quad \times [\psi(G(a, gb, gb) + G(b, ga, ga))]^{r_4}. \end{aligned} \tag{2.44}$$

$\forall$ $a, b, c \in X$.

2.3.3 Theorem

Presume that (X, G) be a complete G-metric space and $g : X \to X$ be a JS-G-contraction. Then g has a unique FP.

Proof. Suppose that $a_0 \in X$ be an erratic point. For $a_0 \in X$, we define the sequence $\{a_n\}$ by $a_n = g^n a_0 = g a_{n-1}$. If there exists $n_0 \in \mathbb{N}$ s.t, $a_{n_0} = a_{n_0+1}$, then a_{n_0} is a FP of g, and we have nothing to prove. Thus, we suppose that $a_n \ne a_{n+1}$, i.e., $G(ga_{n-1}, ga_n, ga_n) > 0$ $\forall$ $n \in \mathbb{N}$

s.t,

$$\left|\frac{\psi\left(G\left(a_n,a_{n+1},a_{n+1}\right)\right)-1}{\left[G\left(a_n,a_{n+1},a_{n+1}\right)\right]^r}-L\right|\leq B_1.$$

$\forall\ n>n_0$. This gives that

$$\frac{\psi\left(G\left(a_{n+1},a_n,a_n\right)\right)-1}{\left[G\left(a_n,a_{n+1},a_{n+1}\right)\right]^r}\geq L-B_1=\frac{L}{2}=B_1,$$

$\forall\ n>n_0$. So

$$n(G\left(a_n,a_{n+1},a_{n+1}\right))^r\leq A_1n[\psi\left(G\left(a_n,a_{n+1},a_{n+1}\right)\right)-1],$$

where $A_1=\frac{1}{B_1}$.

Now for $L=\infty$, let $B_2>0$ be an arbitrary number. From the definition of the limit there occurs $n_1\in\mathbb{N}$ s.t,

$$\frac{\psi\left(G\left(a_n,a_{n+1},a_{n+1}\right)\right)-1}{\left[G\left(a_n,a_{n+1},a_{n+1}\right)\right]^r}\geq B_2.$$

$\forall\ n\geq n_1$. Then

$$n(G\left(a_n,a_{n+1},a_{n+1}\right))^r\leq A_2n[\psi\left(G\left(a_n,a_{n+1},a_{n+1}\right)\right)-1].$$

where $A_2=\frac{1}{B_2}$. Thus, in both cases, there exist $A=\max\{A_1,A_2\}>0$ and $n_*=\max\{n_0,n_1\}\in\mathbb{N}$ s.t,

$$n(G\left(a_n,a_{n+1},a_{n+1}\right))^r\leq An[\psi\left(G\left(a_n,a_{n+1},a_{n+1}\right)\right)-1]\quad\text{for all } n\geq n_*.$$

Hence

$$n(G\left(a_n,a_{n+1},a_{n+1}\right))^r\leq A\,n\left[\left[\psi\left(G\left(a_0,a_1,a_1\right)\right)\right]^{\alpha^n}-1\right],$$

where, $\alpha = \frac{r_1+r_2+r_4}{1-2r_2-r_3-r_4}$. But,

$$\begin{aligned}
&\lim_{n\to\infty} n\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}-1\right]\\
&=\lim_{n\to\infty}\frac{\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}-1\right]}{1/n}\\
&=\lim_{n\to\infty}\frac{\alpha^n\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]}{-1/n^2}\\
&=\lim_{n\to\infty} -n^2\alpha^n\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]\\
&=\lim_{n\to\infty}\frac{-n^2\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]}{\alpha_1^n}\\
&=\lim_{n\to\infty}\frac{-n^2}{\alpha_1^n}\times\lim_{n\to\infty}\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\left[[\psi(G(a_0,a_1,a_1))]^{\alpha^n}\right]\\
&=0\times\ln(\alpha)\ln(\psi(G(a_0,a_1,a_1)))\\
&=0 \quad (\text{where } \alpha_1 = 1/\alpha),
\end{aligned}$$

which implies that $\lim_{n\to\infty} n(G(a_n,a_{n+1},a_{n+1}))^r = 0$. thus there occurs $n_2 \in \mathbb{N}$ s.t.

$$G(a_n,a_{n+1},a_{n+1}) \leq \frac{1}{n^{1/r}}.$$

$\forall\ n > n_2$. Now, for $m > n > n_2$, we have

$$G(a_n,a_m,a_m) \leq \sum_{i=n}^{m-1} G(a_i,a_{i+1},a_{i+1}) \leq \sum_{i=n}^{m-1}\frac{1}{i^{\frac{1}{r}}} \leq \sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}}.$$

Since $0 < r < 1$, $\sum_{i=1}^{\infty}\frac{1}{i^{\frac{1}{r}}}$ is convergent and hence $G(a_n,a_m,a_m) \to 0$ as $m,n \to \infty$. Thus, we proved that $\{a_n\}$ is a G-C-seq. Completeness of (X,G) ensures that there occurs $a^* \in X$ s.t, $a_n \to a^*$ as $n \to \infty$.

Now we shall show that a^* is a FP of g. Using (G5) we get that

$$\begin{aligned}
G(a^*,a^*,ga^*) &\leq G(a^*,a^*,a_{n+1}) + G(a_{n+1},a_{n+1},ga^*)\\
&= G(a^*,a^*,a_{n+1}) + G(ga_n,ga_n,ga^*) \qquad (2.45)
\end{aligned}$$

and

$$G\left(a_n, a_{n+1}, ga^*\right) \leq \left(G\left(a_n, a_{n+1}, a^*\right)\right) + \left(G\left(a^*, a^*, ga^*\right)\right) \tag{2.46}$$

and hence by the properties of ψ we get that

$$\psi(G(a^*, a^*, ga^*)) \leq \psi(G(a^*, a^*, a_{n+1}))\psi(G(ga_n, ga_n, ga^*)). \tag{2.47}$$

$$\psi(G\left(a_n, a_{n+1}, ga^*\right)) \leq \psi(G\left(a_n, a_{n+1}, a^*\right))\psi(G\left(a^*, a^*, ga^*\right)). \tag{2.48}$$

Thus,

$$\left[\psi(G\left(a_n, a_{n+1}, ga^*\right))\right]^{r_2+r_3} \leq \left[\psi(G\left(a_n, a_{n+1}, a^*\right))\right]^{r_2+r_3} \left[\psi(G\left(a^*, a^*, ga^*\right))\right]^{r_2+r_3}. \tag{2.49}$$

But, by using (2.44), (ψ_4) and (2.49), we have

$$\begin{aligned}\psi\left(G\left(a_{n+1}, a_{n+1}, ga^*\right)\right) &= \psi\left(G\left(ga_n, ga_n, ga^*\right)\right) \\ &\leq \left[\psi\left(G\left(a_n, a_n, a^*\right)\right)\right]^{r_1} \left[\psi\left(G\left(a_n, a_{n+1}, ga^*\right)\right)\right]^{r_2} \\ &\times \left[\psi\left(G\left(a_n, a_{n+1}, ga^*\right)\right)\right]^{r_3} \\ &\times \left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right) + G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{r_4} \\ &= \left[\psi\left(G\left(a_n, a_n, a^*\right)\right)\right]^{r_1} \left[\psi\left(G\left(a_n, a_{n+1}, ga^*\right)\right)\right]^{r_2+r_3} \\ &\times \left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{2r_4} \\ &\leq \left[\psi\left(G\left(a_n, a_n, a^*\right)\right)\right]^{r_1} \left[\psi(G\left(a_n, a_{n+1}, a^*\right))\right]^{r_2+r_3} \\ &\left[\psi(G\left(a^*, a^*, ga^*\right))\right]^{r_2+r_3} \left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{2r_4}.\end{aligned} \tag{2.50}$$

Now, substituting (2.50) in (2.47) we get that

$$\begin{aligned}\psi(G(a^*, a^*, ga^*)) \leq \psi(G(a^*, a^*, a_{n+1})) \left[\psi(G\left(a_n, a_n, a^*\right))\right]^{r_1} \left[\psi(G\left(a_n, a_{n+1}, a^*\right))\right]^{r_2+r_3} \\ \left[\psi(G\left(a^*, a^*, ga^*\right))\right]^{r_2+r_3} \left[\psi\left(G\left(a_n, a_{n+1}, a_{n+1}\right)\right)\right]^{2r_4}.\end{aligned} \tag{2.51}$$

Hence.

$$1 \leq [\psi(G(a^*, a^*, ga^*))]^{1-r_2-r_3} \leq \psi(G(a^*, a^*, a_{n+1})) [\psi(G(a_n, a_n, a^*))]^{r_1}$$
$$[\psi(G(a_n, a_{n+1}, a^*))]^{r_2+r_3} [\psi(G(a_n, a_{n+1}, a_{n+1}))]^{2r_4}. \tag{2.52}$$

By taking the limit as $n \to \infty$ and using (ψ_2). Proposition 1.2.3 and the convergence of a_n to a^* in the above equation we get that

$$\psi(G(a^*, a^*, ga^*)) = 1 \tag{2.53}$$

which implies by (ψ_1) that $G(a^*, a^*, ga^*) = 0$ and so $ga^* = a^*$. Thus, a^* is a FP of g.

Finally to show the uniqueness, assume that there occur $a' \neq a^*$ s.t. $a' = ga'$. By (G_2).

$$G(a', a', a^*) = G(ga', ga', ga^*) > 0.$$

Thus. by (2.44) we get

$$\begin{aligned}
\psi(G(a', a', a^*)) = \psi(G(ga', ga', ga^*)) &\leq \left[\psi G(a', a', a^*)\right]^{r_1} \left[\psi(G(a', ga', ga^*))\right]^{r_2} \\
&\times \left[\psi(G(a', ga', ga^*))\right]^{r_3} \left[\psi\left(G(a', ga', ga') + G(a', ga', ga')\right)\right]^{r_4}. \\
&= \left[\psi(G(a', a', a^*))\right]^{r_1} \left[\psi(G(a', a', a^*))\right]^{r_2} \left[\psi(G(a', a', a^*))\right]^{r_3} \\
&\times \left[\psi\left(G(a', a', a') + G(a', a', a')\right)\right]^{r_4}, \\
&= \left[\psi(G(a', a', a^*))\right]^{r_1+r_2+r_3},
\end{aligned}$$

which leads to a contradiction because $r_1 + r_2 + r_3 < 1$. Therefore, g has a unique FP. ■

The following result is a direct consequence of Theorem 2.3.3 by taking $\psi(t) = e^{\sqrt{t}}$ in (2.44).

2.3.4 Corollary

Assume that (X,G) is a complete G-metric space and $g : X \to X$ be a mapping. Suppose that there occur positive real numbers r_1, r_2, r_3, r_4 with $0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1$ s.t.

$$\begin{aligned}\sqrt{G(g\xi, g\gamma, gc)} \leq r_1\sqrt{G(\xi,\gamma,c)} + r_2\sqrt{G(\xi, g\xi, gc)} + r_3\sqrt{G(\gamma, g\gamma, gc)} \\ + r_4\sqrt{G(\xi, g\gamma, g\gamma) + G(\gamma, g\xi, g\xi)}\end{aligned} \tag{2.54}$$

$\forall\ \xi, \gamma, c \in X$. Then g has a unique FP.

2.3.5 Remark

Note that condition (2.54) is equivalent to

$$\begin{aligned}G(g\xi, g\gamma, gc) \leq\ & r_1^2 G(\xi,\gamma,c) + r_2^2 G(\xi, g\xi, gc) + r_3^2 G(\gamma, g\gamma, gc) \\ & + r_4^2\left[G(\xi, g\gamma, g\gamma) + G(\gamma, g\xi, g\xi)\right] \\ & + 2r_1r_2\sqrt{G(\xi,\gamma,c)\,G(\xi, g\xi, gc)} + 2r_1r_3\sqrt{G(\xi,\gamma,c)\,G(\gamma, g\gamma, gc)} \\ & + 2r_1r_4\sqrt{G(\xi,\gamma,c)\left[G(\xi, g\gamma, g\gamma) + G(\gamma, g\xi, g\xi)\right]} \\ & + 2r_2r_3\sqrt{G(\xi, g\xi, gc)\,G(\gamma, g\gamma, gc)} \\ & + 2r_2r_4\sqrt{G(\xi, g\xi, gc)\left[G(\xi, g\gamma, g\gamma) + G(\gamma, g\xi, g\xi)\right]} \\ & + 2r_3r_4\sqrt{G(\gamma, g\gamma, gc)\left[G(\xi, g\gamma, g\gamma) + G(\gamma, g\xi, g\xi)\right]}.\end{aligned}$$

Next, in view of Remark 2.3.5 and by taking $r_2 = r_3 = r_4 = 0$ in Corollary 2.3.4, we obtain the following corollary.

2.3.6 Corollary

Presume that (X,G) is a complete G-metric space and $g : X \to X$ be a mapping. Suppose that there occurs positive real numbers $0 \leq r_1 < 1$, s.t.

$$G(g\xi, g\gamma, gc) \leq r_1^2 G(\xi,\gamma,c) \tag{2.55}$$

$\forall\ \xi, \gamma, c \in X$. Then g has a unique FP.

Finally, by taking $\psi(t) = e^{\sqrt[n]{t}}$ in (2.44), we get the following corollary.

2.3.7 Corollary

Assume that (X, G) is a complete G-metric space and $g : X \to X$ be a mapping. Suppose that there occur positive real numbers r_1, r_2, r_3, r_4 with $0 \leq r_1 + 3r_2 + r_3 + 2r_4 < 1$, s.t.

$$\sqrt[n]{G(g\xi, g\gamma, gc)} \leq r_1 \sqrt[n]{G(\xi, \gamma, c)} + r_2 \sqrt[n]{G(\xi, g\xi, gc)} + r_3 \sqrt[n]{G(\gamma, g\gamma, gc)} \\ + r_4 \sqrt[n]{G(\xi, g\gamma, g\gamma) + G(\gamma, g\xi, g\xi)}$$

$\forall\ \xi, \gamma, c \in X$. Then g has a unique FP.

2.3.8 Remark

By specifying $r_i = 0$ for some $i \in \{1, 2, 3, 4\}$ in Remark 2.3.5 and Corollary 2.3.7, we can get several results.

2.3.9 Example

Suppose that $X = [0, \infty)$ and the G-metric $G_m(\xi, \gamma, c) = \max\{|\xi - \gamma|, |\gamma - c|, |\xi - c|\}$. Define $g : X \to X$ by $g(x) = \frac{x}{8}$ and $\psi(t) = e^{\sqrt{t}}$. Then clearly all the conditions of Theorem 2.3.3 are satisfied with $r_i = \frac{1}{\sqrt{8}}$; $i = 1, 2, 3, 4$, and $x = 0$ is a unique FP of g.

2.4 Fixed point results in G_b-metric spaces

In this section, using the concept of G_b-metric space which was introduced by Aghajani et al. [9] we establish some new FP results in this setting. Ahmad et al. [11] studied JS-contraction and considered a new set of real functions, say, Ω. They replaced condition (ψ_3) by another condition called $(\psi 3)$.

Applying this condition we can have a new range of functions. Thus, consistent with Ahmad et al. [11] we denote by Ω the set of all functions $\theta : [0, \infty) \to [1, \infty)$ satisfying the following conditions:

(ψ_1): θ is nondecreasing and $\Theta(t) = 1$ if and only if $t = 0$;

(ψ_2): for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(ψ_3): θ is continuous.

2.4.1 Example [11]

Suppose that $\theta_1(t) = e^{\sqrt{t}}$, $\theta_2(t) = e^{\sqrt{te^t}}$, $\theta_3(t) = e^t$, $\theta_4(t) = \cosh t$ and $\theta_5(t) = 1 + \ln(1+t)$ $\forall$ $t > 0$. Then $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in \Omega$.

2.4.2 Remark [11]

Note that the conditions (ψ_3) and (Θ_3) are independent of each other. Indeed, for $p \geq 1$, $\theta(t) = e^{t^p}$satisfies the conditions (ψ_1) and (ψ_2) but it does not satisfy (ψ_3), while it satisfies the condition (Θ_3). Therefore $\Omega \not\subseteq \Psi$. Again, for $a > 1$, $m \in (0, \frac{1}{a})$, $\theta(t) = 1 + t^m(1 + [t])$, where $[t]$ denotes the integral part of t, satisfies the conditions (ψ_1) and (ψ_2) but it does not satisfy (Θ_3), while it satisfies the condition (ψ_3) for any $r \in (\frac{1}{a}, 1)$. Therefore $\Psi \not\subseteq \Omega$. Also, if we take $\theta(t) = e^{\sqrt{t}}$, then $\theta \in \Psi$ and $\theta \in \Omega$. Therefore $\Psi \cap \Omega \neq \emptyset$.

2.4.3 Definition [10]

Presume that $g : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$. Then g is called α-admissible if $\forall$ $u, v, w \in X$ with $\alpha(u, v, w) \geq 1$, $\alpha(gu, gv, gw) \geq 1$.

2.4.4 Definition

Presume that $g : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$. Then g is called rectangular-α-admissible if

1. g is α-admissible,
2. $\alpha(u, c, c) \geq 1$ and $\alpha(c, v, w) \geq 1$ imply that $\alpha(u, v, w) \geq 1$

where $u, v, w, c \in X$.

2.4.5 Lemma

Assume that g is a rectangular α-admissible mapping. Suppose that there occurs $a_0 \in X$ s.t. $\alpha(a_0, ga_0, ga_0) \geq 1$. Define the sequence $a_n = g^n a_0$. Then

$$\alpha(a_m, a_n, a_n) \geq 1, \text{ for all } m, n \in N \text{ with } m < n.$$

Proof. Suppose that $a_n = g^n a_0$ and assume that $n = m + k$ for some integer $k \geq 1$. Since $\alpha(a_0, ga_0, ga_0) \geq 1$ and g is α-admissible

$$\alpha(a_1, a_2, a_2) = \alpha(a_1, ga_1, ga_1) = \alpha(ga_0, g^2 a_0, g^2 a_0) \geq 1.$$

Continue in this process we get that $\alpha(a_m, a_{m+1}, a_{m+1}) \geq 1$. Similarly we have

$$\alpha(a_{m+1}, a_{m+2}, a_{m+2}) \geq 1$$

hence by rectangular $\alpha-$admissible we have $\alpha(a_m, a_{m+2}, a_{m+2}) \geq 1$, now repeating the same process we get that $\alpha(a_m, a_n, a_n) = \alpha(a_m, a_{m+k}, a_{m+k}) \geq 1$. ■

Now, we are ready to state our main theorem of this section which has been published in [50].

2.4.6 Theorem

Assume that (X, G_b) be a G_b-complete metric space with $s > 1$. Suppose that $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Assume that there occur $\theta \in \Omega$ and $r \subset (0, 1)$ s.t.

$$\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, v, w) \implies \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right) \leq \left[\theta\left(M(u, v, w)\right)\right]^r \tag{2.56}$$

$\forall\ u, v, w \in X$ with at least two of gu, gv and gw are not equal, where

$$M(u,v,w) = \max \left\{ \begin{array}{c} G_b(u,v,w), \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+s[G_b(u,gu,gw)+G_b(v,gv,gw)]}, \\ \\ \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+G_b(u,gv,gw)+G_b(v,gu,gw)} \end{array} \right\}.$$

Also, suppose that the following assertions hold:

(i) there occurs $a_0 \in X$ s.t, $\alpha(a_0, ga_0, ga_0) \geq 1$;

(ii) for any convergence sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$.

Then g has a FP.

(iii) Moreover, if $\forall\ u, v \in Fix(g)$, $\alpha(u, v, v) \geq 1$, then the FP is unique where $Fix(g) = \{u : gu = u\}$.

Proof. Assume that $a_0 \in X$ be s.t, $\alpha(a_0, ga_0, ga_0) \geq 1$. Define a sequence $\{a_n\}$ by $a_n = g^n a_0\ \forall\ n \in \mathbb{N}$. Since g is an α-admissible mapping and $\alpha(a_0, a_1, a_1) = \alpha(a_0, ga_0, ga_0) \geq 1$, we deduce that $\alpha(a_1, a_2, a_2) = \alpha(ga_0, ga_1, ga_1) \geq 1$. Continuing this process, we get that $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$. Without loss of generality, we assume that $a_n \neq a_{n+1}\ \forall\ n \in \mathbb{N} \cup \{0\}$. We shall proceed in proving the theorem using the following two steps.

Step 1: We shall show that $\lim_{n\to\infty} G_b(a_{n+1}, a_n, a_n) = 0$.

Now,

$$
\begin{aligned}
& M\left(a_{n-1}, a_n, a_n\right) \\
& = \max\left\{\begin{array}{c}
G_b\left(a_{n-1}, a_n, a_n\right), \\
\frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) G_b(a_{n-1}, ga_n, ga_n) + G_b(a_n, ga_n, ga_n) G_b(a_n, ga_{n-1}, ga_{n-1})}{1+s[G_b(a_{n-1}, ga_{n-1}, ga_n) + G_b(a_n, ga_n, ga_n)]}, \\
\frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) G_b(a_{n-1}, ga_n, ga_n) + G_b(a_n, ga_n, ga_n) G_b(a_n, ga_{n-1}, ga_{n-1})}{1+G_b(a_{n-1}, ga_n, ga_n) + G_b(a_n, ga_{n-1}, ga_n)}
\end{array}\right\} \\
& = \max\left\{\begin{array}{c}
G_b\left(a_{n-1}, a_n, a_n\right), \\
\frac{G_b(a_{n-1}, a_n, a_n) G_b(a_{n-1}, a_{n+1}, a_{n+1}) + G_b(a_n, a_{n+1}, a_{n+1}) G_b(a_n, a_n, a_n)}{1+s[G_b(a_{n-1}, a_n, a_{n+1}) + G_b(a_n, a_{n+1}, a_{n+1})]}, \\
\frac{G_b(a_{n-1}, a_n, a_n) G_b(a_{n-1}, a_{n+1}, a_{n+1}) + G_b(a_n, a_{n+1}, a_{n+1}) G_b(a_n, a_n, a_n)}{1+G_b(a_{n-1}, a_{n+1}, a_{n+1}) + G_b(a_n, a_n, a_{n+1})}
\end{array}\right\}
\end{aligned}
$$

$$
= \max\left\{\begin{array}{c}
G_b\left(a_{n-1}, a_n, a_n\right), \\
G_b\left(a_{n-1}, a_n, a_n\right) \frac{G_b(a_{n-1}, a_{n+1}, a_{n+1})}{1+s[G_b(a_{n-1}, a_n, a_{n+1}) + G_b(a_n, a_{n+1}, a_{n+1})]}, \\
G_b\left(a_{n-1}, a_n, a_n\right) \frac{G_b(a_{n-1}, a_{n+1}, a_{n+1})}{1+G_b(a_{n-1}, a_{n+1}, a_{n+1}) + G_b(a_n, a_n, a_{n+1})}
\end{array}\right\} \tag{2.57}
$$

But, from (G_b3), we have $G_b\left(a_{n-1}, a_{n+1}, a_{n+1}\right) \leq G_b\left(a_{n-1}, a_n, a_{n+1}\right)$. and so

$$
\frac{G_b\left(a_{n-1}, a_{n+1}, a_{n+1}\right)}{1+s\left[G_b\left(a_{n-1}, a_n, a_{n+1}\right)+G_b\left(a_n, a_{n+1}, a_{n+1}\right)\right]} \leq 1
$$

and also

$$
\frac{G_b\left(a_{n-1}, a_{n+1}, a_{n+1}\right)}{1+G_b\left(a_{n-1}, a_{n+1}, a_{n+1}\right)+G_b\left(a_n, a_n, a_{n+1}\right)} \leq 1.
$$

Therefore, $M\left(a_{n-1}, a_n, a_n\right) = G_b\left(a_{n-1}, a_n, a_n\right)$.

Since $\alpha\left(a_n, a_{n+1}, a_{n+1}\right) \geq 1$ for each $n \in \mathbb{N}$ and $\frac{1}{3s^2} G_b\left(a_{n-1}, ga_{n-1}, ga_{n-1}\right) \leq G_b\left(a_{n-1}, a_n, a_n\right)$.

as a result by (2.56) we have

$$\begin{aligned}\theta\left(G_b\left(a_n,a_{n+1},a_{n+1}\right)\right) &= \theta\left(G_b\left(ga_{n-1},ga_n,ga_n\right)\right)\\ &\leq \alpha\left(a_{n-1},a_n,a_n\right)\theta\left(s^2G_b\left(ga_{n-1},ga_n,ga_n\right)\right)\\ &\leq \left[\theta\left(M\left(a_{n-1},a_n,a_n\right)\right)\right]^r\\ &= \left[\theta\left(G_b\left(a_{n-1},a_n,a_n\right)\right)\right]^r\\ &< \theta\left(G_b\left(a_{n-1},a_n,a_n\right)\right). \end{aligned} \tag{2.58}$$

Therefore, we have

$$1 < \theta\left(G_b\left(a_n,a_{n+1},a_{n+1}\right)\right) \leq \left[\theta\left(G_b\left(a_{n-1},a_n,a_n\right)\right)\right]^r \leq \cdots \leq \left[\theta\left(G_b\left(a_0,a_1,a_1\right)\right)\right]^{r^n}.$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty}\theta\left(G_b\left(a_n,a_{n+1},a_{n+1}\right)\right) = 1.$$

This gives us, by (θ_2),

$$\lim_{n\to\infty}G_b\left(a_n,a_{n+1},a_{n+1}\right) = 0. \tag{2.59}$$

But $G_b\left(a_{n+1},a_n,a_n\right) \leq 2sG_b\left(a_n,a_{n+1},a_{n+1}\right)$, therefore

$$\lim_{n\to\infty}G_b\left(a_{n+1},a_n,a_n\right) = 0. \tag{2.60}$$

Step 2: We shall prove that the sequence $\{a_n\}$ is a G_b-C-seq. Suppose on the contrary that $\{a_n\}$ is not a G_b-C-seq. Then there occurs $\varepsilon > 0$ for which we can find two subsequences $\{a_{m_i}\}$ and $\{a_{n_i}\}$ of $\{a_n\}$ s.t, n_i is the smallest index for which

$$n_i > m_i > i \text{ and } G_b\left(a_{m_i},a_{n_i},a_{n_i}\right) \geq \varepsilon. \tag{2.61}$$

This means that

$$G_b\left(a_{m_i},a_{n_i-1},a_{n_i-1}\right) < \varepsilon. \tag{2.62}$$

By using (2.61) and (G_b5), we get

$$\varepsilon \leq G_b(a_{m_i}, a_{n_i}, a_{n_i}) \leq sG_b(a_{m_i}, a_{m_i+1}, a_{m_i+1}) + sG_b(a_{m_i+1}, a_{n_i}, a_{n_i}).$$

Taking the upper limit as $i \to \infty$ and using (2.60), we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty} \sup G_b(a_{m_i+1}, a_{n_i}, a_{n_i}). \tag{2.63}$$

Notice that from (2.58) and (θ_1), we get

$$G_b(a_n, a_{n+1}, a_{n+1}) \leq G_b(a_{n-1}, a_n, a_n) \text{ for all } n \in \mathbb{N}. \tag{2.64}$$

Suppose that there occurs $i_0 \in \mathbb{N}$ s.t,

$$\frac{1}{3s^2} G_b\left(a_{m_{i_0}}, ga_{m_{i_0}}, ga_{m_{i_0}}\right) > G_b\left(a_{m_{i_0}}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right)$$

and

$$\frac{1}{3s^2} G_b\left(a_{m_{i_0}+1}, ga_{m_{i_0}+1}, ga_{m_{i_0}+1}\right) > G_b\left(a_{m_{i_0}+1}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right).$$

Then from (G_b5) and (2.64), we have

$$\begin{aligned}
G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) &\leq s\left[G_b\left(a_{m_{i_0}}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right) + G_b\left(a_{n_{i_0}-1}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) \right] \\
&\leq s\left[G_b\left(a_{m_{i_0}}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right) + 2sG_b\left(a_{m_{i_0}+1}, a_{n_{i_0}-1}, a_{n_{i_0}-1}\right) \right] \\
&\leq s\left[\tfrac{1}{3s^2} G_b\left(a_{m_{i_0}}, ga_{m_{i_0}}, ga_{m_{i_0}}\right) + \tfrac{2s}{3s^2} G_b\left(a_{m_{i_0}+1}, ga_{m_{i_0}+1}, ga_{m_{i_0}+1}\right) \right] \\
&= \left[\frac{1}{3s} G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) + \frac{2}{3} G_b\left(a_{m_{i_0}+1}, a_{m_{i_0}+2}, a_{m_{i_0}+2}\right)\right] \\
&\leq \left(\frac{1}{3s} + \frac{2}{3}\right) G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right) \\
&< G_b\left(a_{m_{i_0}}, a_{m_{i_0}+1}, a_{m_{i_0}+1}\right), (\text{ since } s > 1),
\end{aligned} \tag{2.65}$$

which is a contradiction. Hence, either

$$\frac{1}{3s^2}G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right) \leq G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right)$$

or

$$\frac{1}{3s^2}G_b\left(a_{m_i+1}, ga_{m_i+1}, ga_{m_i+1}\right) \leq G_b\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right).$$

holds $\forall\ i \in \mathbb{N}$. First suppose that

$$\frac{1}{3s^2}G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right) \leq G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right). \tag{2.66}$$

From the definition of $M\left(u, v, w\right)$ and using (2.60) and (2.62), we have

$$\begin{aligned}
&\lim_{i\to\infty} \sup M\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right) \\
&= \lim_{i\to\infty} \sup \max \left\{ \begin{array}{c} G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right), \\ \frac{G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right)G_b\left(a_{m_i}, ga_{n_i-1}, ga_{n_i-1}\right)+G_b\left(a_{n_i-1}, ga_{n_i-1}, ga_{n_i-1}\right)G_b\left(a_{n_i-1}, ga_{m_i}, ga_{m_i}\right)}{1+s\left[G_b\left(a_{m_i}, ga_{m_i}, ga_{n_i-1}\right)+G_b\left(a_{n_i-1}, ga_{n_i-1}, ga_{n_i-1}\right)\right]}, \\ \frac{G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right)G_b\left(a_{m_i}, ga_{n_i-1}, ga_{n_i-1}\right)+G_b\left(a_{n_i-1}, ga_{n_i-1}, ga_{n_i-1}\right)G_b\left(a_{n_i-1}, ga_{m_i}, ga_{m_i}\right)}{1+\left[G_b\left(a_{m_i}, ga_{n_i-1}, ga_{n_i-1}\right)+G_b\left(a_{n_i-1}, ga_{m_i}, ga_{n_i-1}\right)\right]}, \end{array} \right\} \\
&= \lim_{i\to\infty} \sup \max \left\{ \begin{array}{c} G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right), \\ \frac{G_b\left(a_{m_i}, a_{m_i+1}, a_{m_i+1}\right)G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right)+G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right)G_b\left(a_{n_i-1}, a_{m_i+1}, a_{m_i+1}\right)}{1+s\left[G_b\left(a_{m_i}, a_{m_i+1}, a_{n_i}\right)+G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right)\right]}, \\ \frac{G_b\left(a_{m_i}, a_{m_i+1}, a_{m_i+1}\right)G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right)+G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right)G_b\left(a_{n_i-1}, a_{m_i+1}, a_{m_i+1}\right)}{1+\left[G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right)+G_b\left(a_{n_i-1}, a_{m_i+1}, a_{n_i}\right)\right]}. \end{array} \right\} \\
&\leq \varepsilon.
\end{aligned}$$

Note that, $m_i \neq n_i - 1$, as otherwise $G_b\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right) = 0$ and so, by (2.66)

$$G_b\left(a_{m_i}, a_{m_i+1}, a_{m_i+1}\right) = G_b\left(a_{m_i}, ga_{m_i}, ga_{m_i}\right) = 0$$

which contradicts our assumption that $a_n \neq a_{n+1}\ \forall\ n \in N$. Hence, $\alpha\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right) \geq 1$. Based on the assumption (2.66), (θ_1), $\alpha\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right) \geq 1$, (2.56), (2.63) and the above inequality, we obtain that

$$\begin{aligned}\theta\left(s^2.\frac{\varepsilon}{s}\right) &\leq \alpha\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right)\theta\left(s^2.\lim_{i\to\infty}\sup G_b\left(a_{m_i+1}, a_{n_i}, a_{n_i}\right)\right)\\ &= \alpha\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right)\theta\left(s^2.\lim_{i\to\infty}\sup G_b\left(ga_{m_i}, ga_{n_i-1}, ga_{n_i-1}\right)\right)\\ &\leq \left[\theta\left(\lim_{i\to\infty}\sup M\left(a_{m_i}, a_{n_i-1}, a_{n_i-1}\right)\right)\right]^r \leq \left[\theta\left(\varepsilon\right)\right]^r.\end{aligned}$$

which implies that $\theta(s\varepsilon) \leq [\theta(\varepsilon)]^r$, a contradiction. Now suppose that

$$\frac{1}{3s^2}G_b\left(a_{m_i+1}, ga_{m_i+1}, ga_{m_i+1}\right) \leq G_b\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) \tag{2.67}$$

holds $\forall\ i \in \mathbb{N}$. Further, from (2.61) and using (G_b5), we get

$$\begin{aligned}\varepsilon \leq G_b\left(a_{m_i}, a_{n_i}, a_{n_i}\right) &\leq sG_b\left(a_{m_i}, a_{m_i+2}, a_{m_i+2}\right) + sG_b\left(a_{m_i+2}, a_{n_i}, a_{n_i}\right).\\ &\leq s^2G_b\left(a_{m_i}, a_{m_i+1}, a_{m_i+1}\right) + s^2G_b\left(a_{m_i+1}, a_{m_i+2}, a_{m_i+2}\right)\\ &+ sG_b\left(a_{m_i+2}, a_{n_i}, a_{n_i}\right).\end{aligned}$$

Taking the upper limit as $i \to \infty$, and using (2.60), we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty}\sup G_b\left(a_{m_i+2}, a_{n_i}, a_{n_i}\right). \tag{2.68}$$

Also, from (G_b5), we get

$$G_b\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) \leq sG_b\left(a_{m_i+1}, a_{n_i}, a_{n_i}\right) + sG_b\left(a_{n_i}, a_{n_i-1}, a_{n_i-1}\right).$$

Taking the upper limit as $i \to \infty$, and using (2.60) and (2.62), we get

$$\lim_{i\to\infty}\sup G_b\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) \leq s\varepsilon. \tag{2.69}$$

From the definition of $M(u, v, w)$ and using (2.60) and (2.69), we have

$\lim_{i\to\infty} \sup M\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) =$

$$\lim_{i\to\infty} \sup \max \left\{ \begin{array}{c} G_b\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right), \\ \frac{G_b\left(a_{m_i+1}, a_{m_i+2}, a_{m_i+2}\right) G_b\left(a_{m_i+1}, a_{n_i}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right) G_b\left(a_{n_i-1}, a_{m_i+2}, a_{m_i+2}\right)}{1+s\left[G_b\left(a_{m_i+1}, a_{m_i+2}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right)\right]}, \\ \frac{G_b\left(a_{m_i+1}, a_{m_i+2}, a_{m_i+2}\right) G_b\left(a_{m_i+1}, a_{n_i}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{n_i}, a_{n_i}\right) G_b\left(a_{n_i-1}, a_{m_i+2}, a_{m_i+2}\right)}{1+\left[G_b\left(a_{m_i+1}, a_{n_i}, a_{n_i}\right) + G_b\left(a_{n_i-1}, a_{m_i+2}, a_{n_i}\right)\right]}. \end{array} \right\}$$

$\leq s\varepsilon.$

Note that, $m_i + 1 \neq n_i - 1$, as otherwise

$$G_b\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) = 0$$

and so, by (2.67), $G_b\left(a_{m_i+1}, a_{m_i+2}, a_{m_i+2}\right) = G_b\left(a_{m_i+1}, ga_{m_i+1}, ga_{m_i+1}\right) = 0$ which contradicts our assumption that $a_n \neq a_{n+1}$ $\forall$ $n \in \mathbb{N}$. Hence, $\alpha\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) \geq 1$.

Based on the assumption (2.67), (θ_1), $\alpha\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) \geq 1$, (2.68), (2.56) and the above inequality we obtain that

$$\begin{aligned} \theta\left(s^2 . \frac{\varepsilon}{s}\right) &\leq \alpha\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) \theta\left(s^2 . \lim_{i\to\infty} \sup G_b\left(a_{m_i+2}, a_{n_i}, a_{n_i}\right)\right) \\ &= \alpha\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right) \theta\left(s^2 . \lim_{i\to\infty} \sup G_b\left(ga_{m_i+1}, ga_{n_i-1}, ga_{n_i-1}\right)\right) \\ &\leq \left[\theta\left(\lim_{i\to\infty} \sup M\left(a_{m_i+1}, a_{n_i-1}, a_{n_i-1}\right)\right)\right]^r \leq \left[\theta\left(s\varepsilon\right)\right]^r . \end{aligned}$$

a contradiction. Therefore, in all cases $\{a_n\}$ is a G_b-C-seq, thus by G_b-completeness of X yields that $\{a_n\}$ is G_b-convergent to a point $a^* \in X$. An argument similar to that in (2.65), we get either

$$\frac{1}{3s^2} G_b\left(a_n, ga_n, ga_n\right) \leq G_b\left(a_n, x^*, a^*\right)$$

or

$$\frac{1}{3s^2} G_b\left(a_{n+1}, ga_{n+1}, ga_{n+1}\right) \leq G_b\left(a_{n+1}, a^*, a^*\right)$$

holds $\forall$ $n \in \mathbb{N}$. First, suppose that

$$\frac{1}{3s^2} G_b\left(a_n, ga_n, ga_n\right) \leq G_b\left(a_n, a^*, a^*\right).$$

Now,

$$M\left(a_n,a^*,a^*\right)=\max\left\{\begin{array}{c} G_b\left(a_n,a^*,a^*\right), \frac{G_b(a_n,ga_n,ga_n)G_b(a_n,ga^*,ga^*)+G_b(a^*,ga^*,ga^*)G_b(a^*,ga_n,ga_n)}{1+s[G_b(a_n,ga_n,ga^*)+G_b(a^*,ga^*,ga^*)]}, \\ \\ \frac{G_b(a_n,ga_n,ga_n)G_b(a_n,ga^*,ga^*)+G_b(a^*,ga^*,ga^*)G_b(a^*,ga_n,ga_n)}{1+[G_b(a_n,ga^*,ga^*)+G_b(a^*,ga_n,ga^*)]} \end{array}\right\}$$

So, $\lim_{n\to\infty} M\left(a_n,a^*,a^*\right)=0$. Hence from (2.56) and assertion (ii) of the theorem, we have

$$\begin{aligned} 1 \leq \theta\left(G_b\left(ga_n,ga^*,ga^*\right)\right) &\leq \theta\left(s^2 G_b\left(ga_n,ga^*,ga^*\right)\right) \\ &\leq \alpha(a_n,a^*,a^*)\theta\left(s^2 G_b\left(ga_n,ga^*,ga^*\right)\right) \\ &\leq \left[\theta\left(M\left(a_n,a^*,a^*\right)\right)\right]^r \end{aligned}$$

$\forall\, n\in\mathbb{N}$. Taking the limit as $n\to\infty$, in the above inequality we get that

$$\lim_{n\to\infty}\theta\left(G_b\left(ga_n,ga^*,ga^*\right)\right)=1.$$

This implies by (Θ_1) that

$$\lim_{n\to\infty} G_b\left(ga_n,ga^*,ga^*\right)=0.$$

Hence, $ga^*=\lim_{n\to\infty} ga_n=\lim_{n\to\infty} a_{n+1}=a^*$. Thus, we deduce that $ga^*=a^*$.

Now if

$$\frac{1}{3s^2}G_b\left(a_{n+1},ga_{n+1},ga_{n+1}\right)\leq G_b\left(a_{n+1},a^*,a^*\right),$$

holds, then by repeating the same process as above we can get $ga^*=a^*$. Therefore, a^* is a FP of g.

Now to prove uniqueness, suppose there occur $u,v\in Fix(g)$ with $u\neq v$, i.e., $u=gu$ and $v=gv$. Therefore by (iii), $\alpha(u,v,v)\geq 1$ and so, by (2.56) and (G_{b2}) we have

$$0=\frac{1}{3s^2}G(u,gu,gu)\leq G(u,v,v)$$

and

$$\begin{aligned}\theta(G_b(u,v,v)) &\le \alpha(u,v,v)\theta(s^2G_b(gu,gv,gv))\\ &\le [\theta(M(u,v,v))]^r\\ &= [\theta(G_b(u,v,v))]^r\\ &< \theta(G_b(u,v,v)).\end{aligned}$$

Thus the contradiction implies that the FP is unique. ■

2.4.7 Theorem

Assume that (X,G_b) is a G_b-complete metric space with $s>1$. Suppose that $\alpha : X\times X\times X \to (0,\infty)$ and g be a rectangular α-admissible mapping. Suppose that there occur $\theta\in\Omega$ and $r\in(0,1)$ s.t,

$$\frac{1}{3s^2}G_b\left(u,gu,gu\right)\le G_b\left(u,v,w\right) \implies \alpha\left(u,v,w\right)\theta\left(s^2G_b\left(gu,gv,gw\right)\right)\le \left[\theta\left(M\left(u,v,w\right)\right)\right]^r \tag{2.70}$$

$\forall\ x,y,z\in X$ with at least two of gx,gy and gz are not equal where

$$M\left(u,v,w\right)=\max\left\{G_b\left(u,v,w\right),\frac{G_b\left(u,gu,gu\right)G_b\left(v,gv,gw\right)}{1+G_b\left(u,v,w\right)},\frac{G_b\left(u,gu,gu\right)G_b\left(v,gv,gw\right)}{1+G_b\left(gu,gv,gw\right)}\right\}.$$

Also, suppose that the following assertions hold:

(i) there occurs $a_0\in X$ s.t, $\alpha\left(a_0,ga_0,ga_0\right)\ge 1$;

(ii) for any convergent sequence $\{a_n\}$ to a with $\alpha\left(a_n,a_{n+1},a_{n+1}\right)\ge 1\ \forall\ n\in\mathbb{N}\cup\{0\}$, we have $\alpha\left(a_n,a,a\right)\ge 1\ \forall\ n\in\mathbb{N}\cup\{0\}$.

Then g has a FP.

(iii) Moreover, if $\forall\ u,v\in Fix(g)$, $\alpha(u,v,v)\ge 1$, then the FP is unique where $Fix(g)=\{u: gu=u\}$.

Proof. Suppose that $a_0\in X$ be s.t. $\alpha\left(a_0,ga_0,ga_0\right)\ge 1$. Define a sequence $\{a_n\}$ by $a_n=g^na_0\ \forall\ n\in\mathbb{N}$. Since g is an α-admissible mapping and $\alpha\left(a_0,a_1,a_1\right)=\alpha\left(a_0,ga_0,ga_0\right)\ge 1$, we deduce that $\alpha\left(a_1,a_2,a_2\right)=\alpha\left(ga_0,ga_1,ga_1\right)\ge 1$. Continuing this process, we get that $\alpha\left(a_n,a_{n+1},a_{n+1}\right)\ge 1\ \forall\ n\in\mathbb{N}\cup\{0\}$. Without loss of generality, assume that $a_n\ne a_{n+1}\ \forall$

$n \in \mathbb{N} \cup \{0\}$. We shall show that $\lim_{n\to\infty} G_b(a_{n+1}, a_n, a_n) = 0$. Now,

$$\begin{aligned}
&M(a_{n-1}, a_n, a_n) \\
&= \max\left\{\begin{array}{c} G_b(a_{n-1}, a_n, a_n), \frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1})G_b(a_n, ga_n, ga_n)}{1+G_b(a_{n-1}, a_n, a_n)}, \\ \frac{G_b(a_{n-1}, ga_{n-1}, ga_{n-1})G_b(a_n, ga_n, ga_n)}{1+G_b(ga_{n-1}, ga_n, ga_n)} \end{array}\right\} \\
&= \max\left\{\begin{array}{c} G_b(a_{n-1}, a_n, a_n), \frac{G_b(a_{n-1}, a_n, a_n)G_b(a_n, a_{n+1}, a_{n-1})}{1+G_b(a_{n-1}, a_n, a_n)}, \\ \frac{G_b(a_{n-1}, a_n, a_n)G_b(a_n, a_{n+1}, a_{n+1})}{1+G_b(a_n, a_{n+1}, a_{n+1})} \end{array}\right\}.
\end{aligned} \tag{2.71}$$

Since, $\frac{G_b(a_{n-1}, a_n, a_n)}{1+G_b(a_{n-1}, a_n, a_n)} < 1$ and $\frac{G_b(a_n, a_{n+1}, a_{n+1})}{1+G_b(a_n, a_{n+1}, a_{n+1})} < 1$.

$$M(a_{n-1}, a_n, a_n) = \max\{G_b(a_{n-1}, a_n, a_n), G_b(a_n, a_{n+1}, a_{n+1})\}.$$

If $\max\{G_b(a_{n-1}, a_n, a_n), G_b(a_n, a_{n+1}, a_{n+1})\} = G_b(a_n, a_{n+1}, a_{n+1})$, then since $\alpha(a_{n-1}, a_n, a_n) \geq 1$ for each $n \in \mathbb{N}$, $\frac{1}{3s^2} G_b(a_{n-1}, ga_{n-1}, ga_{n-1}) \leq G_b(a_{n-1}, a_n, a_n)$ and so by (2.70), we have

$$\begin{aligned}
\theta(G_b(a_n, a_{n+1}, a_{n+1})) &= \theta(G_b(ga_{n-1}, ga_n, ga_n)) \\
&\leq \alpha(a_{n-1}, a_n, a_n)\, \theta\left(s^2 G_b(ga_{n-1}, ga_n, ga_n)\right) \\
&\leq [\theta(M(a_{n-1}, a_n, a_n))]^r \\
&= [\theta(G_b(a_n, a_{n+1}, a_{n+1}))]^r \\
&< \theta(G_b(a_n, a_{n+1}, a_{n+1}))
\end{aligned} \tag{2.72}$$

which is a contradiction since $r \in (0, 1)$. Thus, $M(a_{n-1}, a_n, a_n) = G_b(a_{n-1}, a_n, a_n)$.

The rest of the proof is the same as in the proof of Theorem 2.4.6. ■

Analogously, we can prove the following theorem.

2.4.8 Theorem

Suppose that (X, G_b) is a complete G_b- metric space with $s > 1$. Suppose that $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there occur $\theta \in \Omega$ and

$r \in (0,1)$ s.t,

$$\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, v, w) \implies \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right) \leq [\theta(M(u, v, w))]^r$$

$\forall\ u, v, w \in X$ with at least two of gu, gv and gw are not equal, where

$$M(u, v, w) = \max \left\{ \begin{array}{c} G_b(u, v, w), \frac{G_b(u,gu,gu)G_b(v,gv,gw)}{1+s[G_b(u,v,w)+G_b(v,gu,gu)+G_b(u,gv,gv)]}, \\ \frac{G_b(u,gv,gv)G_b(u,v,w)}{1+sG_b(u,gu,gu)+s^2[G_b(v,gv,gv)+G_b(v,gu,gu)]} \end{array} \right\}.$$

Also, presume that the following assertions hold:

(i) there occurs $a_0 \in X$ s.t, $\alpha(a_0, ga_0, ga_0) \geq 1$;

(ii) for any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$.

Then g has a FP;

(iii) moreover, if $\forall\ u, v \in Fix(g)$, $\alpha(u, v, v) \geq 1$, then the FP is unique where $Fix(g) = \{u; gu = u\}$.

Now, we give an example to support Theorem 2.4.6.

2.4.9 Example

Assume that $X = [0, \infty)$ and $G_b : X \times X \times X \to R$ is a G_b-metric space defined by $G_b(u, v, w) = (|u - v| + |v - w| + |u - w|)^2$. Clearly (X, G_b) is a complete G_b-metric space with $s = 2$. Also let $r = \frac{3}{5}$ and define $g : X \to X$, $\alpha : X \times X \times X \to R$ and $\theta : [0, \infty) \to [1, \infty)$ by

$$g(x) = \begin{cases} \frac{x}{5}, & \text{if } x \in [0, 1] \\ x^2, & \text{otherwise.} \end{cases}$$

$$\alpha(u, v, w) = \begin{cases} 1, & \text{if } u, v, w \in [0, 1] \\ 0, & \text{otherwise,} \end{cases}$$

and $\theta(t) = e^t$.

Assume that $\frac{1}{12} G_b(u, gu, gu) \leq G_b(u, v, w)$. If one of $u, v, w \notin [0, 1]$, then $\alpha(u, v, w) = 0$ and so, the conclusion of (2.4.6) is satisfied. If $u, v, w \in [0, 1]$, then $gu, gv, gw \in [0, 1]$ and

$\alpha(u,v,w) \geq 1$ with $gu \neq gv \neq gw$. Hence,

$$\begin{aligned}
\alpha(u,v,w)\theta(4G_b(gu,gv,gw)) &= e^{4(\frac{1}{5}(|u-v|+|v-w|+|u-w|))^2} \\
&= e^{\frac{4}{25}(|u-v|+|v-w|+|u-w|)^2} \\
&\leq e^{(3/5)(|u-v|+|v-w|+|u-w|)^2} \\
&= \left(e^{(|u-v|+|v-w|+|u-w|)^2}\right)^{\frac{3}{5}} \\
&= \left(e^{G_b(u,v,w)}\right)^{\frac{3}{5}} \\
&= (\theta(G_b(u,v,w)))^{\frac{3}{5}}.
\end{aligned}$$

Thus all the conditions of Theorem 2.4.6 are satisfied and $x = 0$ is the unique FP of g.

2.4.10 Corollary

Presume that (X, G_b) is a complete G_b- metric space with $s > 1$. Suppose that $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there occur $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0,1)$ with $\delta + \beta + \gamma < 1$ s.t.

$$\begin{aligned}
&\frac{1}{3s^2}G_b(u,gu,gu) \leq G_b(u,v,w) \implies \alpha(u,v,w)\,\theta\left(s^2 G_b(gu,gv,gw)\right) \\
&\leq \left[\theta\left(\delta G_b(u,v,w) + \beta\frac{G_b(u,gu,gu)\,G_b(v,gv,gw)}{1+G_b(u,v,w)} + \gamma\frac{G_b(u,gu,gu)\,G_b(v,gv,gw)}{1+G_b(gu,gv,gw)}\right)\right]^r
\end{aligned}$$

$\forall\ u, v, w \in X$ with at least two of gu, gv and gw, are not equal. Also, suppose that the following assertions hold:

(i) there occurs $a_0 \in X$ s.t, $\alpha(a_0, ga_0, ga_0) \geq 1$;

(ii) for any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$, $\forall\ n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$.

Then g has a FP;

(iii) moreover, if $\forall\ u, v \in Fix(g)$ implies $\alpha(u,v,v) \geq 1$, then the FP is unique where $Fix(g) = \{u : gu = u\}$.

2.4.11 Corollary

Presume that (X, G_b) is a complete G_b-metric space with $s > 1$. Suppose that $\alpha : X \times X \times X \to (0, \infty)$ and g be a rectangular α-admissible mapping. Suppose that there occur $\theta \in \Omega$ and $r, \delta, \beta, \gamma \in (0,1)$ with $\delta + \beta + \gamma < 1$ s.t,

$$\frac{1}{3s^2} G_b(u, gu, gu) \leq G_b(u, v, w) \implies \alpha(u, v, w)\, \theta\left(s^2 G_b(gu, gv, gw)\right)$$
$$\leq \left[\theta\left(\begin{array}{c} \delta G_b(u, v, w) + \beta \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+s[G_b(u,gu,gw)+G_b(v,gv,gw)]} \\ +\gamma \frac{G_b(u,gu,gu)G_b(u,gv,gw)+G_b(v,gv,gw)G_b(v,gu,gu)}{1+G_b(u,gv,gw)+G_b(v,gu,gw)} \end{array}\right)\right]^r$$

$\forall\ u, v, w \in X$ with at least two of gu, gv and gw are not equal. Also, suppose that the following assertions hold:

(i) there occurs $a_0 \in X$ s.t, $\alpha(a_0, ga_0, ga_0) \geq 1$;

(ii) for any convergent sequence $\{a_n\}$ to a with $\alpha(a_n, a_{n+1}, a_{n+1}) \geq 1$, $\forall\ n \in \mathbb{N} \cup \{0\}$, we have $\alpha(a_n, a, a) \geq 1\ \forall\ n \in \mathbb{N} \cup \{0\}$.

Then g has a FP;

(iii) moreover, if $\forall\ u, v \in Fix(g)$, $\alpha(u, v, v) \geq 1$, then the FP is unique where $Fix(g) = \{u; gu = u\}$.

Chapter 3

Fixed Point and Fuzzy Fixed Point Results for $F-$Contraction

In 2012, Wardowski [103] introduced a new type of contraction called $F-$contraction and proved a new FP theorem concerning $F-$contraction. He generalized the BCP in a different aspect from the well-known results from the literature. Afterwards, Secelean [96] proved FP theorems consisting of $F-$contractions by Iterated function systems. Piri et al. [84] proved a FP result for $F-$Suzuki contractions for some weaker conditions on the self map of a complete metric space which generalizes the result of Wardowski. Lately, Acar et al.[8] introduced the concept of generalized multivalued $F-$contraction mappings. Further Altun et al. [7] extended multivalued mappings with $\delta-$distance and established FP results in complete metric space. Sgroi et al. [98] established FP theorems for multivalued $F-$contractions and obtained the solution of certain functional and integral equations, which was a proper generalization of some multivalued FP theorems including Nadler's. Recently Ahmad et al. [12, 18, 46] recalled the concept of $F-$contraction to obtain some FP, and common FP results in the context of complete metric spaces.

In 1981, Heilpern [41] used the concept of fuzzy set to introduce a class of fuzzy mappings, which is a generalization of the set-valued mapping, and proved a FP theorem for fuzzy contraction mappings in metric linear space. It is worth noting that the result announced by Heilpern [41] is a fuzzy extension of the Banach contraction principle. Subsequently, several other authors have studied occurence of FPs of fuzzy mappings, for examples, Azam et al. [23, 24], Bose et al. [27], Chang et al. [29], Cho et al. [31], Qiu et al. [85], Rashwan et al. [86], Shi-sheng [99].

In 1969, Nadler [78], introduced a study of FP theorems involving multivalued mappings and proved that every multivalued contraction on a complete metric space has a FP. Fisher [37] obtained different type of multivalued FP theorems defining δ-distance between two bounded subsets of a metric space. Then a lot of generalization of multivalued mappings have been given in the literature.

A fuzzy set in X is a function with domain X and values in $[0,1]$. I^X is the group of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function whose value is $A(x)$ is called the grade of membership of x in A. The α -level set of A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0,1].$$

$$[A]_0 = \overline{\{x : A(x) > 0\}}.$$

Here $\overline{B}$ denotes the closure of the set B. Suppose that $\mathcal{F}(X)$ be the group of all fuzzy sets in a metric space X. For $A, B \in \mathcal{F}(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. We signify the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where $\chi_{\{x\}}$ is the characteristic function of the crisp set A. If there occurs an $\alpha \in [0,1]$ s.t, $[A]_\alpha, [B]_\alpha \in CB(X)$, then define

$$p_\alpha(A,B) = \inf_{x\in[A]_\alpha, y\in[B]_\alpha} d(x,y).$$

$$D_\alpha(A,B) = H([A]_\alpha, [B]_\alpha).$$

If $[A]_\alpha, [B]_\alpha \in CB(X)$ for each $\alpha \in [0,1]$, then define

$$p(A,B) = \sup_\alpha p_\alpha(A,B),$$

$$d_\infty(A,B) = \sup_\alpha D_\alpha(A,B).$$

We write $p(x,B)$ instead of $p(\{x\},B)$. A fuzzy set A in a metric linear space V is said to be an approximate quantity iff $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0,1]$ and $\sup_{x \in V} A(x) = 1$.The collection of all approximate quantities in V is denoted by $W(V)$. Suppose that X be an arbitrary set, Y be a metric space. A mapping T is called fuzzy mapping if T is a mapping from X into $\mathcal{F}(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

In this chapter, we continue the study of generalized $F-$contraction for single valued and multivalued mapping in complete metric spaces. In Section 3.1, we extend the concept of F-contraction into generalized F-contraction for single valued mapping. In Section 3.2, we discuss this concept for multivalued mappings. Section 3.3 deals with the application of FP theorem which was proved in the previous section to Volterra type integral equation. In Section 3.4, we establish some common α-fuzzy FP theorems for generalized F-contraction in the setting of complete metric spaces.

3.1 Fixed point results for single valued mappings

The results given in this section have been published in [61].

3.1.1 Definition

Suppose that (X,d) is a metric space. A mapping $J : X \to X$ is said to be generalized $F-$contraction if there occurs $\tau > 0$ s.t,

$$\forall x,y \in X,\ d(Jx,Jy) > 0 \Rightarrow \tau + F\left(d(Jx,Jy)\right) \leq F\left(M(x,y)\right), \tag{3.1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Jx), d(y,Jy), \left(\frac{d(x,Jy)+d(y,Jx)}{d(x,Jx)+d(y,Jy)+1}\right) d(x,y)\right\}.$$

We denote by F, the set of all functions satisfying the conditions from $(F1)$ - $(F3)$.

3.1.2 Theorem

Presume that (X, d) be a complete metric space and $J : X \longrightarrow X$ be generalized $F-$contraction. If J or F is continuous, then J has a FP in X.

Proof. Suppose that x_0 in X, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ s.t. $x_1 = Jx_0$, $x_2 = Jx_1 = J^2x_0$. Continuing this process, $x_{n+1} = Jx_n = J^{n+1}x_0$, $\forall\ n \in \mathbb{N}$. If there occurs $n \in \mathbb{N}$ s.t. $d(x_n, Jx_n) = 0$, there is nothing to prove and the proof is complete. So, we assume that

$$d(Jx_{n-1}, Jx_n) = d(x_n, Jx_n) > 0, \forall n \in \mathbb{N}. \tag{3.2}$$

Now for any $n \in \mathbb{N}$, we have

$$\tau + F\left(d(Jx_{n-1}, Jx_n)\right) \leq F\left(M(x_{n-1}, x_n)\right).$$

Therefore

$$F\left(d(x_n, x_{n+1})\right) = F\left(d(Jx_{n-1}, Jx_n)\right) \leq F\left(M(x_{n-1}, x_n)\right) - \tau. \tag{3.3}$$

Now

$$\begin{aligned}
M(x_{n-1}, x_n) &= \max\left\{\begin{array}{c} d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ \left(\frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}\right) d(x_{n-1}, x_n) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ \left(\frac{d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}\right) d(x_{n-1}, x_n) \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ \left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n-1})}{d(x_{n-1}, x_n) + d(x_n, x_{n-1}) + 1}\right) d(x_{n-1}, x_n) \end{array}\right\} \\
&= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}.
\end{aligned}$$

So, we have

$$F\left(d(x_n, x_{n+1})\right) = F\left(d(Jx_{n-1}, Jx_n)\right) \leq F\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\right) - \tau.$$

In the case $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ is impossible.

$$F(d(x_n, x_{n+1})) = F(d(Jx_{n-1}, Jx_n)) \leq F(d(x_n, x_{n+1})) - \tau < F(d(x_n, x_{n+1})),$$

which is a contradiction. Otherwise, in other case

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n).$$

Thus from (3.3), we have

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau.$$

Continuing this process, we get

$$\begin{aligned} F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\ &= F(d(Jx_{n-2}, Jx_{n-1})) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &= F(d(Jx_{n-3}, Jx_{n-2})) - 2\tau \\ &\leq F(d(x_{n-3}, x_{n-2})) - 3\tau \\ &\vdots \\ &\leq F(d(x_0, x_1)) - n\tau. \end{aligned}$$

This implies that

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau. \tag{3.4}$$

From (3.4), we obtain $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$,

which together with (F2) gives $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, i.e.,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.5}$$

From $(F3)$, there occurs $k \in (0,1)$ s.t,

$$\lim_{n\to\infty}\left((d(x_n,x_{n+1}))^k F\left(d(x_n,x_{n+1})\right)\right)=0. \tag{3.6}$$

From (3.4), the following holds $\forall\ n \in \mathbb{N}$,

$$(d(x_n,x_{n+1}))^k\left(F\left(d(x_n,x_{n+1})\right)-F\left(d(x_0,x_1)\right)\right)\leq -\left(d(x_n,x_{n+1})\right)^k n\tau \leq 0. \tag{3.7}$$

By using (3.5), (3.6) and letting $n \to \infty$ in (3.7), we have

$$\lim_{n\to\infty}\left(n\left(d(x_n,x_{n+1})\right)^k\right)=0. \tag{3.8}$$

We observe that from (3.8), then there occurs $n_1 \in \mathbb{N}$. s.t, $n\left(d(x_n,x_{n+1})\right)^k \leq 1\ \forall\ n \geq n_1$

$$d(x_n,x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}\ \forall\ n \geq n_1. \tag{3.9}$$

To prove that $\{x_n\}$ is a C-seq. Consider $m,n \in \mathbb{N}$ s.t, $m > n \geq n_1$. Then by the triangle inequality and from (3.9), we have

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+\ldots \\ &\quad +d(x_{m-1},x_m) \\ &= \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i,x_{i+1}) \\ &\leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{3.10}$$

The series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{k}}}$ is convergent. By taking limit as $n \to \infty$, in (3.10), we have $\lim_{n,m\to\infty} d(x_n,x_m) = 0$. Hence $\{x_n\}$ is a C-seq. Since X is a complete metric space there occurs $x^* \in X$ s.t, $\lim_{n\to\infty} x_n = x^*$. Now if J is continuous. Then we have $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Jx_n = J\left(\lim_{n\to\infty} x_n\right) = Jx^*$ and so x^* is a FP of J.

Now, suppose F is continuous. In this case, we claim that $x^* = Jx^*$. Assume the contrary, i.e., $x^* \neq Jx^*$. In this case, there occurs an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t. $d(Jx_{n_k}, Jx^*) > 0 \ \forall \ n_k \geq n_0$. (Otherwise, there occur $n_1 \in \mathbb{N}$ s.t. $x_n = Jx^* \ \forall \ n \geq n_1$, which implies that $x_n \to Jx^*$. This is a contradiction, since $x^* \neq Jx^*$). Since $d(Jx_{n_k}, Jx^*) > 0 \ \forall \ n_k \geq n_0$, from (3.1), we have

$$\begin{aligned} \tau + F(d(x_{n_k+1}, Jx^*)) &= \tau + F(d(Jx_{n_k}, Jx^*)), \\ &\leq F(M(x_{n_k}, x^*)), \\ &= F\left(\max\left\{\begin{array}{l} d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Jx^*), \\ \left(\frac{d(x_{n_k}, Jx^*)+d(x^*, x_{n_k+1})}{d(x_{n_k}, x^*)+d(x_{n_k}, x_{n_k+1})+1}\right) d(x_{n_k}, x^*) \end{array}\right\}\right). \end{aligned} \tag{3.11}$$

Taking the limit $k \to \infty$ and using the continuity of F we have $\tau + F(d(x^*, Jx^*)) \leq F(d(x^*, Jx^*))$, which is a contradiction. Therefore our claim is true, i.e., $x^* = Jx^*$. ■

3.1.3 Example

Assume that $X = [0, 1]$. Define a mapping, $J : X \to X$ by,

$$Jx = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, 1), \\ \frac{1}{4}, & \text{if } x = 1. \end{cases}$$

Since, J is not continuous, J is not a F-contraction by Remark 1.4.3.

For $x \in \{0, 1)$ and $y = 1$, we have

$$d(Jx, J1) = d\left(\frac{1}{2}, \frac{1}{4}\right) = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4} > 0$$

and

$$\max\left\{d(x,1), d(x, Jx), d(1, J1), \left(\frac{d(x, J1) + d(1, Jx)}{d(x, Jx) + d(1, J1) + 1}\right) d(x, 1)\right\} < d(1, J1) = \frac{3}{4}.$$

Now by choosing, $F(\alpha) = \ln \alpha$, $\alpha \in (0, +\infty)$ and $\tau = \ln 3$, we see that J is a generalized F-contraction.

Recently, Piri and Kumam [84] generalized the result of Wardowski [103] by replacing the conditions (F2) and (F3) with the following one:

$(F2^{'})$ $Inf\ F = -\infty$.

$(F3^{'})$ F is continuous on $(0, \infty)$.

Here $\mathcal{F}$ denotes the family of all functions $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ which satisfy conditions $(F1)$, $(F2)$ and $\left(F3^{'}\right)$.

3.1.4 Definition [18]

Assume that (X, d) is a metric space and $J : X \to X$ be a self mapping, then J is said to be generalized $F-$contraction of rational type A if there occurs $\tau > 0$ s.t.

$$\forall\ x, y \in X,\ d(Jx, Jy) > 0 \Rightarrow \tau + F\left(d(Jx, Jy)\right) \leq F\left(M(x, y)\right).$$

where

$$M(x, y) = \max\left\{d(x, y), d(x, Jx), d(y, Jy), \left(\frac{d(x, Jy) + d(y, Jx)}{d(x, Jx) + d(y, Jy) + 1}\right) d(x, y)\right\}.$$

$F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing, i.e. $\forall\ x, y \in \mathbb{R}_+$ s.t. $x < y$, $F(x) < F(y)$;

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ iff $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3$^{'}$) F is continuous on $(0, \infty)$.

We denote by $\mathcal{F}$, the set of all functions satisfying the conditions (F1)-(F3$^{'}$).

The following theorem is a direct consequence of Theorem 3.1.2 which has been published in [18] by using Piri technique.

3.1.5 Theorem [18]

Assume that (X, d) is a complete metric space and $J : X \longrightarrow X$ be generalized F contraction of rational type then J has a FP in X.

3.2 Fixed point results for multivalued mappings

In this section, we present a FP theorem for multivalued mappings with δ-distance using Wardowski's technique on complete metric spaces.

3.2.1 Theorem

Suppose that $J : X \to B(X)$ is a multivalued generalized F-contraction on a complete metric space X. Suppose $F \in \mathcal{F}$, and there occurs $\tau > 0$ s.t,

$$\forall x, y \in X \text{ with } \min\{\delta(Jx, Jy)d(x, y)\} > 0 \Rightarrow \tau + F(\delta(Jx, Jy)) \leq F(M(x, y)). \tag{3.12}$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, Jx), D(y, Jy), \left(\frac{D(x, Jy) + D(y, Jx)}{D(x, Jx) + D(y, Jy) + 1}\right) d(x, y)\right\}.$$

If F is continuous and Jx is closed $\forall\, x \in X$, then J has a FP in X.

Proof. Suppose that $x_0 \in X$ be an erratic point and define a sequence $\{x_n\}$ in X s.t. $x_{n+1} \in Jx_n$, $\forall\, n \geq 0$. If there occurs $n_0 \in \mathbb{N} \cup \{0\}$ for which $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a FP of J and so the proof is completed. Thus, assume that, for every $n_0 \in \mathbb{N} \cup \{0\}$, $x_n \neq x_{n+1}$. So

$$d(x_n, x_{n+1}) > 0 \text{ and } \delta(Jx_{n-1}, Jx_n) > 0 \quad \forall\, n \in \mathbb{N}. \tag{3.13}$$

Then from (3.13), we have

$$\tau + F(d(x_n, x_{n+1})) \leq \tau + F(\delta(Jx_{n-1}, Jx_n)) \leq F(M(x_{n-1}, x_n)). \tag{3.14}$$

Now

$$\begin{aligned} F(M(x_{n-1}, x_n)) &= F\left(\max\left\{\begin{array}{c} d(x_{n-1}, x_n), D(x_{n-1}, Jx_{n-1}), D(x_n, Jx_n), \\ \left(\frac{D(x_{n-1}, Jx_n) + D(x_n, Jx_{n-1})}{D(x_{n-1}, Jx_{n-1}) + D(x_n, Jx_n) + 1}\right) d(x_{n-1}, x_n) \end{array}\right\}\right). \\ &\leq F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned}$$

We have

$$F\left(d(x_n, x_{n+1})\right) = F\left(d(Jx_{n-1}, Jx_n)\right) \leq F\left(\max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\}\right) - \tau,$$

In the case $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ is impossible.

$$F\left(d(x_n, x_{n+1})\right) = F\left(d(Jx_{n-1}, Jx_n)\right) \leq F\left(d(x_n, x_{n+1})\right) - \tau < F\left(d(x_n, x_{n+1})\right).$$

which is a contradiction. Otherwise in other case

$$M(x_{n-1}, x_n) = \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\right\} = d(x_{n-1}, x_n).$$

Thus from (3.14), we have

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_{n-1}, x_n)\right) - \tau.$$

Continuing this process, we get

$$\begin{aligned} F\left(d(x_n, x_{n+1})\right) &\leq F\left(d(x_{n-1}, x_n)\right) - \tau \\ &\leq F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau \\ &\leq F\left(d(x_{n-3}, x_{n-2})\right) - 3\tau \\ &\vdots \\ &\leq F\left(d(x_0, x_1)\right) - n\tau. \end{aligned}$$

This implies that

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_0, x_1)\right) - n\tau. \tag{3.15}$$

From (3.15), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$ and imposing (F2) gives $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. i.e.,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.16}$$

From $(F3)$, there occurs $k \in (0,1)$ s.t,

$$\lim_{n\to\infty}\left((d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1}))\right) = 0. \tag{3.17}$$

From (3.15), the following holds $\forall\ n \in \mathbb{N}$,

$$(d(x_n, x_{n+1}))^k (F(d(x_n, x_{n+1})) - F(d(x_0, x_1))) \leq -(d(x_n, x_{n+1}))^k n\tau \leq 0. \tag{3.18}$$

By using (3.16), (3.17) and letting $n \to \infty$ in (3.18), we have

$$\lim_{n\to\infty}\left(n\,(d(x_n, x_{n+1}))^k\right) = 0. \tag{3.19}$$

We observe that from (3.19), then there occurs $n_1 \in \mathbb{N}$, s.t, $n\,(d(x_n, x_{n+1}))^k \leq 1\ \forall\ n \geq n_1$,

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}}\ \forall\ n \geq n_1. \tag{3.20}$$

Now we prove that $\{x_n\}$ is a C-seq. Consider $m, n \in \mathbb{N}$ s.t, $m > n \geq n_1$. Then by the triangle inequality and from (3.20), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots \\ &\quad \ldots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{3.21}$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent. By taking limit as $n \to \infty$, in (3.21), we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a C-seq. Since (X, d) is a complete metric space the sequence $\{x_n\}$ converges to some point $x^* \in X$ s.t, $\lim_{n\to\infty} x_n = x^*$. Now suppose that F is continuous. In this case we claim $x^* \in Jx^*$. Suppose contrary that $x^* \notin Jx^*$. In this case there occur $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ s.t, $D(x_{n_k+1}, Jx^*) > 0\ \forall\ n_k \geq n_0$.(On the other hand, there occurs $n_1 \in \mathbb{N}$

s.t. $x_n \in Jx^* \ \forall \ n \geq n_1$, this implies that $x^* \in Jx^*$. which is a contradiction. Since $x^* \notin Jx^*$). Since $D(x_{n_k+1}, Jx^*) > 0 \ \forall \ n_k \geq n_0$, we obtain

$$\begin{aligned} & \tau + F\left(D(x_{n_k+1}, Jx^*)\right) \\ = \ & \tau + F\left(\delta(Jx_{n_k}, Jx^*)\right) \\ \leq \ & F\left(M(x_{n_k}, x^*)\right) \\ \leq \ & F\left(\max\left\{\begin{array}{c} d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), D(x^*, Jx^*), \\ \left(\frac{D(x_{n_k}, Jx^*)+d(x^*, x_{n_k+1})}{d(x_{n_k}, x_{n_k+1})+D(x^*, Jx^*)+1}\right) d(x_{n_k}, x^*) \end{array}\right\}\right). \end{aligned} \tag{3.22}$$

Since F is continuous, taking the limit as $k \to \infty$ in (3.22), we obtain

$$\tau + F\left(D(x^*, Jx^*)\right) \leq F\left(D(x^*, Jx^*)\right). \tag{3.23}$$

which is a contradiction. Therefore, we have $x^* \in Jx^*$. Hence x^* is a FP of J. ∎

3.3 Application to integral equation

In this section, we discuss the application of FP theorem which was proved in the previous section to the following Volterra type integral equation.

$$u(\varrho) = \int_0^{\varrho} K(\varrho, s, w(s))ds + f(\varrho). \tag{3.24}$$

for $\varrho \in [0, a]$, where $a > 0$. We find the solution of (3.24). Suppose that $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{\varrho \in [0,a]} \{u(\varrho)e^{-\tau(\varrho)\varrho}\}$, where $\tau > 0$ is taken arbitrary. Suppose that $C([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{\varrho \in [0,a]} \|\,|u(\varrho) - v(\varrho)|\, e^{-\tau\varrho}\|_\tau \tag{3.25}$$

$\forall \ u, v \in C([0, a], \mathbb{R})$. With these setting $C([0, a], \mathbb{R}, \|\cdot\|_\tau)$ becomes a Banach space.

Now we prove the following theorem to ensure the occurence of solution of integral equation.

For more details on such applications, we refer the readers to [20, 83].

3.3.1 Theorem

Assume the following conditions are satisfied:

(i) $K : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ and $f : [0, a] \to \mathbb{R}$ are continuous:

(ii) Define

$$Ju(\varrho) = \int_0^{\varrho} K(\varrho, s, u(s))ds + f(\varrho).$$

Suppose there occurs $\tau > 1$, s.t,

$$|K(\varrho, s, u) - K(\varrho, s, v)| \leq \tau e^{-\tau}[M(u, v)]$$

$\forall\ \varrho, s \in [0, a]$ and $u, v \in C([0, a], \mathbb{R})$, where

$$\begin{aligned} M(u, v) &= \max\{|u(\varrho) - v(\varrho)|, |u(\varrho) - Ju(\varrho)|, |v(\varrho) - Jv(\varrho)|, \\ &\left(\frac{|u(\varrho) - Jv(\varrho)| + |v(\varrho) - Ju(\varrho)|}{|u(\varrho) - Ju(\varrho)| + |v(\varrho) - Jv(\varrho)| + 1}\right)|u(\varrho) - v(\varrho)|\}. \end{aligned}$$

Then integral equation given in (3.24) has a solution.

Proof. By assumption (ii)

$$\begin{aligned} |Ju(\varrho) - Jv(\varrho)| &= \int_0^{\varrho} |K(\varrho, s, u(s) - K(\varrho, s, v(s)))| ds \\ &\leq \int_0^{\varrho} \tau e^{-\tau}([M(u, v)]e^{-\tau s})e^{\tau s} ds \\ &\leq \int_0^{\varrho} \tau e^{-\tau}\|M(u, v)\|_{\tau} e^{\tau s} ds \\ &\leq \tau e^{-\tau}\|M(u, v)\|_{\tau} \int_0^{\varrho} e^{\tau s} ds \\ &\leq \tau e^{-\tau}\|M(u, v)\|_{\tau} \frac{1}{\tau} e^{\tau \varrho} \\ &\leq e^{-\tau}\|M(u, v)\|_{\tau} e^{\tau \varrho}. \end{aligned}$$

This implies

$$|Ju(\varrho) - Jv(\varrho)|\, e^{-\tau\varrho} \leq e^{-\tau} \|M(u,v)\|_{\tau}.$$

That is,

$$\|Ju(\varrho) - Jv(\varrho)\|_{\tau} \leq e^{-\tau} \|M(u,v)\|_{\tau},$$

which further implies

$$\tau + \ln \|Ju(\varrho) - Jv(\varrho)\|_{\tau} \leq \ln \|M(u,v)\|_{\tau}.$$

So all the conditions of Theorem 3.1.2 are satisfied. Hence integral equations given in (3.24) has a unique solution. ■

3.4 Fuzzy FP results for generalized contractions

In this section, we establish some common α-fuzzy FP theorems for generalized F-contraction in the setting of complete metric spaces. In this way, we unify, generalize and complement various known comparable results in the literature. We also provide an example to show the significance of the investigation of our results. As applications of our main results we derive some multi-valued FP theorems from our fuzzy FP theorems.

For the sake of convenience, we first state some known results for subsequent use in our main results.

3.4.1 Lemma

Assume that (X,d) is a metric space and $A, B \in CB(X)$, then for each $a \in A$,

$$d(a,B) \leq H(A,B).$$

3.4.2 Lemma [19]

Suppose that V is a metric linear space, $J : X \to W(V)$ be a fuzzy mapping and $x_0 \in V$. Then there occurs $x_1 \in V$ s.t, $\{x_1\} \subset J(x_0)$.

Now we state our main theorem of this section.

3.4.3 Theorem

Presume that (X, d) is a complete metric space and let S, J be fuzzy mappings from X into $\mathcal{F}(X)$ and for each $x \in X$, there occur $\alpha_S(x), \alpha_J(x) \in (0, 1]$ s.t. $[Sx]_{\alpha_S(x)}$, $[Jy]_{\alpha_J(x)}$ are nonempty closed bounded subsets of X. If there occur some $F \in \digamma$ and $\tau > 0$ s.t.

$$\tau + F\left(H\left([Sx]_{\alpha_S(x)}, [Jy]_{\alpha_J(x)}\right)\right) \leq F(M(x, y)) \tag{3.26}$$

$\forall\ x, y \in X$ with $H\left([Sx]_{\alpha_S(x)}, [Jy]_{\alpha_J(x)}\right) > 0$, where

$$M(x, y) = \max\left\{d(x, y), d\left(x, [Sx]_{\alpha_S(x)}\right), d\left(y, [Jy]_{\alpha_J(x)}\right), \frac{1}{2}[d\left(x, [Jy]_{\alpha_J(x)}\right) + d\left(y, [Sx]_{\alpha_S(x)}\right)]\right\}. \tag{3.27}$$

Then there occurs some $u \in [Su]_{\alpha_S(u)} \cap [Ju]_{\alpha_J(u)}$.

Proof. Suppose that x_0 be an erratic point in X, then by hypotheses there occurs $\alpha_S(x_0) \in (0, 1]$ s.t. $[Sx_0]_{\alpha_S(x_0)}$ is a nonempty closed bounded subset of X. For convenience, we denote $\alpha_S(x_0)$ by α_1. Suppose that $x_1 \in [Sx_0]_{\alpha_S(x_0)}$. For this x_1 there occurs $\alpha_J(x_1) \in (0, 1]$ s.t. $[Jx_1]_{\alpha_J(x_1)}$ is a nonempty, closed and bounded subset of X. By Lemma 3.4.1, ($F1$) and (3.26), we have

$$\begin{aligned} \tau + F(d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right) &\leq \tau + F\left(H\left([Sx_0]_{\alpha_S(x_0)}, [Jx_1]_{\alpha_J(x_1)}\right)\right) \\ &\leq F(M(x_0, x_1)) \\ &= F\left(\max\left\{\begin{array}{c} d(x_0, x_1), d\left(x_0, [Sx_0]_{\alpha_S(x_0)}\right), d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right), \\ \frac{1}{2}[d\left(x_0, [Jx_1]_{\alpha_J(x_1)}\right) + d\left(x_1, [Sx_0]_{\alpha_S(x_0)}\right)] \end{array}\right\}\right) \\ &\leq F\left(\max\left\{d(x_0, x_1), d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right)\right\}\right). \end{aligned} \tag{3.28}$$

If $\max\left\{d(x_0, x_1), d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right)\right\} = d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right)$, then from (3.28), we get

$$\tau + F\left(d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right)\right) \leq F\left(d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right)\right),$$

which is a contradiction. So, $\max\left\{d(x_0, x_1), d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right)\right\} = d(x_0, x_1)$. Then

$$\tau + F\left(d\left(x_1, [Jx_1]_{\alpha_J(x_1)}\right)\right) \leq F(d(x_0, x_1)). \tag{3.29}$$

From ($F4$), we know that

$$F\left(d\left(x_1,[Jx_1]_{\alpha_J(x_1)}\right)\right)=\inf_{y\in[Jx_1]_{\alpha_J(x_1)}}F(d(x_1,y)).$$

Thus from (3.29), we get

$$\tau+\inf_{y\in[Jx_1]_{\alpha_J(x_1)}}F(d(x_1,y))\leq[F(d(x_0,x_1)].\tag{3.30}$$

Then, from (3.30), there occurs $x_2\in[Jx_1]_{\alpha_J(x_1)}$ s.t,

$$\tau+F(d(x_1,x_2))\leq[F(d(x_0,x_1)].\tag{3.31}$$

For this x_2 there occurs $\alpha_S(x_2)\in(0,1]$ s.t, $[Sx2]_{\alpha_S(x_2)}$ is a nonempty closed bounded subset of X. By Lemma 3.4.1, ($F1$) and (3.26), we have

$$\begin{aligned}\tau+F\left(d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right) &\leq \tau+F(H\left([Jx1]_{\alpha_J(x_1)},[Sx2]_{\alpha_S(x_2)}\right)\\ &= F(H\left([Sx2]_{\alpha_S(x_2)},[Jx1]_{\alpha_J(x_1)}\right)\\ &\leq F(M(x_2,x_1))\\ &= F\left(\max\left\{\begin{array}{c} d(x_2,x_1),d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right),d\left(x_1,[Jx1]_{\alpha_J(x_1)}\right)\\ ,\frac{1}{2}[d\left(x_2,[Jx1]_{\alpha_J(x_1)}\right)+d\left(x_1,[Sx2]_{\alpha_S(x_2)}\right)]\end{array}\right\}\right)\\ &\leq F\left(\max\left\{d(x_2,x_1),d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right\}\right)\\ &= F\left(\max\left\{d(x_1,x_2),d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right\}\right).\end{aligned}\tag{3.32}$$

If $\max\left\{d(x_1,x_2),d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right\}=d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)$, then from (3.32), we get

$$\tau+F\left[d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right]\leq F\left[d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right],$$

which is a contradiction. So, $\max\left\{d(x_1,x_2),d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right\}=d(x_1,x_2)$. Then

$$\tau+F\left[d\left(x_2,[Sx2]_{\alpha_S(x_2)}\right)\right]\leq Fd(x_1,x_2).\tag{3.33}$$

From $(F4)$, we know that

$$F\left[d\left(x_2, [Sx_2]_{\alpha_S(x_2)}\right)\right] = \inf_{y_1 \in [Sx_2]_{\alpha_S(x_2)}} F(d(x_2, y_1)).$$

Thus

$$\tau + \inf_{y_1 \in [Sx_2]_{\alpha_S(x_2)}} F(d(x_2, y_1)) \leq F(d(x_1, x_2)). \tag{3.34}$$

Then, from (3.34), there occurs $x_3 \in [Sx_2]_{\alpha_S(x_2)}$ s.t.

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)). \tag{3.35}$$

So, continuing recursively, we obtain a sequence $\{x_n\}$ in X s.t. $x_{2n+1} \in [Sx_{2n}]_{\alpha_S(x_{2n})}$ and $x_{2n+2} \in [Jx_{2n+1}]_{\alpha_J(x_{2n+1})}$ and

$$\tau + F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1}) \tag{3.36}$$

and

$$\tau + F(d(x_{2n+2}, x_{2n+3})) \leq F(d(x_{2n+1}, x_{2n+2}) \tag{3.37}$$

$\forall\, n \in \mathbb{N}$. From (3.36) and (3.37), we have

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)). \tag{3.38}$$

Therefore

$$\begin{aligned} F\left(d(x_n, x_{n+1})\right) &\leq F\left(d(x_{n-1}, x_n)\right) - \tau \leq F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau \\ &\leq \ldots \leq F\left(d(x_0, x_1)\right) - n\tau. \end{aligned} \tag{3.39}$$

Suppose thatting $n \to \infty$ in above inequality, we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$ that together with $(F2)$ gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now by $(F3)$, there occurs $h \in (0,1)$ s.t.

$$\lim_{n\to\infty} [d(x_n, x_{n+1})]^h F(d(x_n, x_{n+1})) = 0.$$

From (3.39) we have

$$\begin{aligned} & [d(x_n, x_{n+1})]^h F(d(x_n, x_{n+1})) - [d(x_n, x_{n+1})]^h F(d(x_0, x_{n+1})) \\ \leq \; & -n\tau [d(x_n, x_{n+1})]^h \leq 0. \end{aligned}$$

On taking limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} n[d(x_n, x_{n+1})]^h = 0.$$

Hence $\lim_{n\to\infty} n^{\frac{1}{h}} d(x_n, x_{n+1}) = 0$ and there occurs $n_1 \in \mathbb{N}$ s.t. $n^{\frac{1}{h}} d(x_n, x_{n+1}) \leq 1 \; \forall \; n \geq n_1$. So we have

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/h}} \tag{3.40}$$

$\forall \; n \geq n_1$. Now consider $m, n \in \mathbb{N}$ s.t, $m > n \geq n_1$, we have

$$\begin{aligned} d(x_n, x_m) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ & \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/h}}$, we get $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is a C-seq in X. The completeness of (X, d) ensures that there occurs $u \in X$ s.t., $\lim_{n\to\infty} x_n \to u$. Now, we prove that $u \in [Ju]_{\alpha_J(u)}$. We suppose on the contrary that $u \notin [Ju]_{\alpha_J(u)}$, then there occur a $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t. $d(x_{2n_k+1}, [Ju]_{\alpha_J(u)}) > 0 \; \forall \; n_k \geq n_0$. Since

$d(x_{2n_k+1}, [Ju]_{\alpha_J(u)}) > 0 \ \forall \ n_k \geq n_0$, by Lemma 3.4.1, ($F1$) and (3.26), we have

$$\begin{aligned}
\tau + F\left[d(x_{2n_k+1}, [Ju]_{\alpha_J(u)})\right] &\leq \tau + F\left[H([Sx_{2n_k}]_{\alpha_S(x_{2n_k})}, [Ju]_{\alpha_J(u)})\right] \\
&\leq F(M(x_{2n_k}, u)) \\
&= F\left(\max\left\{\begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, [Sx_{2n_k}]_{\alpha_S(x_{2n_k})}\right), d\left(u, [Ju]_{\alpha_J(u)}\right), \\ \frac{1}{2}[d\left(x_{2n_k}, [Ju]_{\alpha_J(u)}\right) + d\left(u, [Sx_{2n_k}]_{\alpha_S(x_{2n_k})}\right)] \end{array}\right\}\right) \\
&\leq F\left(\max\left\{\begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, x_{2n_k+1}\right), d\left(u, [Ju]_{\alpha_J(u)}\right), \\ \frac{1}{2}[d\left(x_{2n_k}, [Ju]_{\alpha_J(u)}\right) + d\left(u, x_{2n_k+1}\right)] \end{array}\right\}\right).
\end{aligned}$$

which further implies that

$$\begin{aligned}
F\left[d(x_{2n_k+1}, [Ju]_{\alpha_J(u)})\right] &\leq F\left(\max\left\{\begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, x_{2n_k+1}\right), d\left(u, [Ju]_{\alpha_J(u)}\right), \\ \frac{1}{2}[d\left(x_{2n_k}, [Ju]_{\alpha_J(u)}\right) + d\left(u, x_{2n_k+1}\right)] \end{array}\right\}\right) - \tau \\
&< F\left(\max\left\{\begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, x_{2n_k+1}\right), d\left(u, [Ju]_{\alpha_J(u)}\right), \\ \frac{1}{2}[d\left(x_{2n_k}, [Ju]_{\alpha_J(u)}\right) + d\left(u, x_{2n_k+1}\right)] \end{array}\right\}\right).
\end{aligned}$$

Since F is strictly increasing, we have

$$d(x_{2n_k+1}, [Ju]_{\alpha_J(u)}) < \max\left\{\begin{array}{c} d(x_{2n_k}, u), d\left(x_{2n_k}, x_{2n_k+1}\right), d\left(u, [Ju]_{\alpha_J(u)}\right), \\ \frac{1}{2}[d\left(x_{2n_k}, [Ju]_{\alpha_J(u)}\right) + d\left(u, x_{2n_k+1}\right)] \end{array}\right\}.$$

Suppose thatting $n \to \infty$, we have

$$d(u, [Ju]_{\alpha_J(u)}) \leq d(u, [Ju]_{\alpha_J(u)}).$$

which is possible only if $u \in [Ju]_{\alpha_J(u)}$. Similarly, one can easily prove that $u \in [Su]_{\alpha_S(u)}$. Thus $u \in [Su]_{\alpha_S(u)} \cap [Ju]_{\alpha_J(u)}$. ■

The following theorem is a direct consequence.

3.4.4 Theorem

Suppose that (X, d) is a complete metric space and let S be fuzzy mapping from X into $\mathcal{F}(X)$ and for each $x \in X$, there occur $\alpha_S(x), \alpha_S(x) \in (0, 1]$ s.t, $[Sx]_{\alpha_S(x)}$, $[Sy]_{\alpha_S(y)}$ are nonempty closed bounded subsets of X. If there occur some $F \in \digamma$ and $\tau > 0$ s.t.

$$\tau + F\left(H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_S(y)}\right)\right) \leq F(M(x,y))$$

$\forall\ x, y \in X$ with $H\left([Sx]_{\alpha_S(x)}, [Sy]_{\alpha_S(y)}\right) > 0$, where

$$M(x,y) = \max\left\{d(x,y), d\left(x, [Sx]_{\alpha_S(x)}\right), d\left(y, [Sy]_{\alpha_S(y)}\right), \frac{1}{2}[d\left(x, [Sy]_{\alpha_S(y)}\right) + d\left(y, [Sx]_{\alpha_S(x)}\right)]\right\}.$$

Then there occurs some $u \in [Su]_{\alpha_S(u)}$.

3.4.5 Corollary

Presume that (X,d) be a complete metric space and let $F, G: X \to CB(X)$ be multivalued mappings. If there occur some $F \in \digamma$ and $\tau > 0$ s.t.

$$\tau + F(H(Fx, Gy)) \leq F(M(x,y))$$

$\forall\ x, y \in X$ with $H(Fx, Gy) > 0$, where

$$M(x,y) = \max\left\{d(x,y), d(x, Fx), d(y, Gy), \frac{1}{2}[d(x, Gy) + d(y, Fx)]\right\}.$$

Then there occurs some $u \in Fu \cap Gu$.

Proof. Consider a mapping $\alpha : X \to [0,1]$ and a pair of fuzzy mappings $S, J : X \to \mathcal{F}(X)$ defined by

$$S(x)(t) = \begin{cases} \alpha(x), & \text{if } t \in Fx, \\ 0, & \text{if } t \notin Fx \end{cases}$$

and

$$J(x)(t) = \begin{cases} \alpha(x), & \text{if } t \in Gx, \\ 0, & \text{if } t \notin Gx. \end{cases}$$

Then

$$[Sx]_{\alpha(x)} = \{t : S(x)(t) \geq \alpha(x)\} = Fx \quad \text{and} \quad [Jx]_{\alpha(x)} = \{t : J(x)(t) \geq \alpha(x)\} = Gx.$$

Thus, Theorem 3.4.4 can be applied to obtain $u \in X$ s.t.

$$u \in [Su]_{\alpha_S(u)} \cap [Ju]_{\alpha_J(u)} = Fu \cap Gu.$$

■

3.4.6 Corollary

Assume that (X, d) is a complete metric space and let $F : X \to CB(X)$ be multivalued mappings. If there occur some $F \in \digamma$ and $\tau > 0$ s.t.

$$\tau + F(H(Fx, Fy)) \leq F(M(x, y)$$

$\forall\ x, y \in X$ with $H(Fx, Fy) > 0$, where

$$M(x, y) = \max\left\{d(x, y), d(x, Fx), d(y, Fy), \frac{1}{2}[d(x, Fy) + d(y, Fx)]\right\}.$$

Then there occurs some $u \in Fu$.

3.4.7 Corollary

Assume that (X, d) is a complete metric linear space and let $S, J : X \to W(X)$ be fuzzy mappings. If there occur some $F \in \digamma$ and $\tau > 0$ s.t.

$$\tau + F(d_\infty(S(x), J(y))) \leq F(M(x, y))$$

$\forall\ x, y \in X$ with $d_\infty(S(x), J(y)) > 0$, where

$$M(x, y) = \max\left\{p(x, y), p(x, S(x)), p(y, J(y)), \frac{1}{2}[p(x, J(y)) + p(y, S(x))]\right\}.$$

Then there occurs some $u \in X$ s.t. $\{u\} \subset S(u)$ and $\{u\} \subset J(u)$.

Proof. Suppose that $x \in X$, then by Lemma 3.4.1 there occurs $y \in X$ s.t. $y \in [Sx]_1$. Similarly, we can find $z \in X$ s.t. $z \in [Jx]_1$. It follows that for each $x \in X$, $[Sx]_{\alpha(x)}$, $[Jx]_{\alpha(x)}$ are nonempty closed bounded subsets of X. As $\alpha(x) = \alpha(y) = 1$, by the definition of a d_∞-metric for fuzzy sets, we have

$$H\left([Sx]_{\alpha(x)}, [Jy]_{\alpha(x)}\right) \leq d_\infty(S(x), J(y))$$

$\forall\, x, y \in X$. It implies that

$$\begin{aligned}\tau + F\left(H\left([Sx]_{\alpha(x)}, [Jy]_{\alpha(x)}\right)\right) &\leq F\left(d_\infty(S(x), J(y))\right) \\ &\leq \left[F\left(\max\left\{\begin{array}{c} p(x, y), p(x, S(x)), p(y, J(y)), \\ \frac{1}{2}[p(x, J(y)) + p(y, S(x))] \end{array}\right\}\right)\right]\end{aligned}$$

$\forall\, x, y \in X$. Since $[Sx]_1 \subseteq [Sx]_\alpha$ for each $\alpha \in (0,1]$, $d(x, [Sx]_\alpha) \leq d(x, [Sx]_1)$ for each $\alpha \in (0, 1]$. It implies that $p(x, S(x)) \leq d(x, [Sx]_1)$. Similarly, $p(x, J(x)) \leq d(x, [Jx]_1)$. This further implies that $\forall\, x, y \in X$,

$$\tau + F\left(H\left([Sx]_1, [Jy]_1\right)\right) \leq F\left(\max\left\{d(x, y), d(x, [Sx]_1), d(y, [Jx]_1), \frac{1}{2}[d(x, [Jx]_1) + d(y, [Sx]_1)]\right\}\right).$$

Now, by Theorem 3.4.3, we obtain $u \in X$ s.t, $u \in [Su]_1 \cap [Ju]_1$, i.e., $\{u\} \subset J(u)$ and $\{u\} \subset S(u)$.

■

In the following, we suppose that $\widehat{J}$ (for details, see [93, 99]) is the set-valued mapping induced by fuzzy mappings $J : X \to \mathcal{F}(X)$, i.e.,

$$\widehat{J}x = \left\{y : J(x)(t) = \max_{t \in X} J(x)(t)\right\}.$$

3.4.8 Corollary

Assume that (X, d) is a complete metric space and let $S, J : X \to \mathcal{F}(X)$ be fuzzy mappings s.t, $\forall\, x \in X$, $\widehat{S}(x), \widehat{J}(x)$ are nonempty closed bounded subsets of X. If there occur some $F \in \mathcal{F}$ and $\tau > 0$ s.t,

$$\tau + F\left(H\left(\widehat{S}(x), \widehat{J}(y)\right)\right) \leq F(M(x, y))$$

$\forall\, x, y \in X$ with $H\left(\widehat{S}(x), \widehat{J}(y)\right) > 0$, where

$$M(x, y) = \max\left\{d(x, y), d\left(x, \widehat{S}(x)\right), d\left(y, \widehat{J}(y)\right), \frac{1}{2}[d\left(x, \widehat{J}(y)\right) + d\left(y, \widehat{S}(x)\right)]\right\}.$$

Then there occurs a point $x^* \in X$ s.t. $S(x^*)(x^*) \geq S(x^*)(x)$ and $J(x^*)(x^*) \geq J(x^*)(x)$ $\forall$ $x \in X$.

Proof. By Corollary 3.4.5, there occurs $x^* \in X$ s.t. $x^* \in \widehat{S}x^* \cap \widehat{J}x^*$. Then by Lemma 3.4.2, we have

$$S(x^*)(x^*) \geq S(x^*)(x) \qquad \text{and} \qquad J(x^*)(x^*) \geq J(x^*)(x)$$

$\forall$ $x \in X$. ■

3.4.9 Example

Suppose that $X = \{1, 2, 3\}, \{1\}, \{2\}, \{3\}$ be crisp sets and $d : X \times X \to \mathbb{R}$ be the metric defined by

$$d(x,y) = \begin{cases} 0 \text{ if } x = y, \\ \frac{4}{17} \text{ if } x \neq y \text{ and } x, y \in X - \{1\} \\ \frac{5}{17} \text{ if } x \neq y \text{ and } x, y \in X - \{2\} \\ 1 \text{ if } x \neq y \text{ and } x, y \in X - \{3\}. \end{cases}$$

Then (X, d) is a complete metric space. Suppose that $F(t) = \ln(t) \in F$ for $t > 0$. Define a pair of mappings $S : X \to \mathcal{F}(X)$ as follows:

$$S(1)(t) = S(2)(t) = S(3)(t) \begin{cases} \frac{3}{4} \text{ if } t = 1 \\ \frac{1}{19} \text{ if } t = 2 \\ 0 \text{ if } t = 3 \end{cases}.$$

and

$$J(1)(t) = J(3)(t) = \begin{cases} \frac{3}{4} \text{ if } t = 1 \\ \frac{1}{2} \text{ if } t = 2 \\ 0 \text{ if } t = 3 \end{cases}.$$

$$J(2)(t) = \begin{cases} 0 \text{ if } x = 1 \\ \frac{1}{3} \text{ if } t = 2 \\ \frac{3}{4} \text{ if } t = 3. \end{cases}.$$

Then, for $\alpha(x) = \frac{3}{4}$,we have

$$[Jx]_{\alpha(x)} = \{t : J(x)(t) = \alpha(x)\} = \begin{cases} \{1\} \text{ if } x \neq 2 \\ \{3\} \text{ if } x = 2 \end{cases} .$$

and

$$[Sx]_{\alpha(x)} = \{t : S(x)(t) = \alpha(x)\} = \{1\} .$$

Then

$$H\left([Sx]_{\alpha_S(x)}, [Jy]_{\alpha_J(y)}\right) = \begin{cases} H(\{1\},\{1\}) = 0 \text{ if } y \neq 2 \\ H(\{1\},\{3\}) = \frac{5}{17} \text{ if } y = 2. \end{cases}$$

Now we have the following three cases:

Case 1: If $x = 1$, $y = 2$, then

$$H\left([Sx]_{\alpha_S(x)}, [Jy]_{\alpha_J(y)}\right) = \frac{5}{17} > 0$$

and

$$\max\left\{d(x,y), d\left(x, [Sx]_{\alpha_S(x)}\right), d\left(y, [Jy]_{\alpha_J(x)}\right), \frac{1}{2}[d\left(x, [Jy]_{\alpha_J(x)}\right) + d\left(y, [Sx]_{\alpha_S(x)}\right)]\right\} = 1.$$

Then there occurs some $\tau \in (0, \ln(\frac{11}{5})]$ s.t. the inequalities (3.26) and (3.27) are satisfied.

Case 2: If $x = 2$, $y = 2$, then again

$$H\left([Sx]_{\alpha_S(x)}, [Jy]_{\alpha_J(y)}\right) = \frac{5}{17} > 0$$

and

$$\max\left\{d(x,y), d\left(x, [Sx]_{\alpha_S(x)}\right), d\left(y, [Jy]_{\alpha_J(x)}\right), \frac{1}{2}[d\left(x, [Jy]_{\alpha_J(x)}\right) + d\left(y, [Sx]_{\alpha_S(x)}\right)]\right\} = 1.$$

Then there occurs some $\tau \in (0, \ln(\frac{11}{5})]$ s.t. the inequalities (3.26) and (3.27) are satisfied.

Case 3: If $x = 3$, $y = 2$, then again

$$H\left([Sx]_{\alpha_S(x)}, [Jy]_{\alpha_J(y)}\right) = \frac{5}{17} > 0$$

and

$$\max\left\{d(x,y), d\left(x,[Sx]_{\alpha_S(x)}\right), d\left(y,[Jy]_{\alpha_J(x)}\right), \frac{1}{2}[d\left(x,[Jy]_{\alpha_J(x)}\right) + d\left(y,[Sx]_{\alpha_S(x)}\right)]\right\} = \frac{11}{17}.$$

Then there occurs some $\tau \in (0, \ln(\frac{11}{5})]$ s.t, the inequalities (3.26) and (3.27) are satisfied.

Hence all the conditions of Theorem 3.4.3 are satisfied to obtain $1 \in [S1]_{\frac{3}{4}} \cap [J1]_{\frac{3}{4}}$.

Chapter 4

Fixed Point Results In Partial Metric Spaces

In 1994, Matthews [65] introduced the notion of partial metric spaces (PMS) and obtained various FP theorems. In fact, he showed that the BCP can be generalized to the partial metric context for applications in program verification. Since then, many reserachers have investigated various results and generalizations in context of PMS.

Later on, Romaguera [91] introduced the notions of 0-C-seqs and 0-complete PMSs and proved some characterizations of PMS in terms of completeness and 0-completeness.

In this chapter, we continue these investigations and explore the FP and common FP results in PMSs. In Section 4.1, we give a brief introduction of PMSs. In Section 4.2, we introduce an F-rational cyclic contraction on PMSs and present new FP results for such cyclic contraction in 0-complete PMSs. Section 4.3 is devoted to a common FP theorem for a pair of multivalued $F-\Psi-$proximinal mappings satisfying Ciric-wardowski type contraction in PMSs. Examples are constructed to illustrate these result. In Section 4.4, applications to system of integral equations are presented to show the usability of our results.

In the sequel, $\mathbb{R}, \mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{N}^+$ will represent the set of all real numbers, non-negative real numbers, natural numbers and positive integers.

4.1 Introduction

First we recall some definitions and properties of PMSs.

4.1.1 Definition [65]

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+(\mathbb{R}^+$ stands for nonnegative s.t, $\forall\ x, y, z \in X$:

$(P1)$ $x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y)$;

$(P2)$ $p(x, x) \leq p(x, y)$;

$(P3)$ $p(x, y) = p(y, x)$;

$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A PMS is a pair (X, p) s.t. X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x, y)$ may not be 0. Also, every metric space is a PMS, with zero self distance.

4.1.2 Example [65]

If $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x, y) = \max\{x, y\}$, $\forall\ x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a PMS.

For more examples of PMSs, we refer the reader to [21] and the references therein.

Each partial metric p on X generates a T_0 topology $\tau(p)$ on X which has a base topology of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ and $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon + p(x, x)\}$.

A mapping $f : X \to X$ is continuous if and only if, whenever a sequence $\{x_n\}$ in X converging with respect to $\tau(p)$ to a point $x \in X$, the sequence $\{fx_n\}$ converges with respect to $\tau(p)$ to $fx \in X$.

Suppose that (X, p) be a PMS.

(i) A sequence $\{x_n\}$ in PMS (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$.

(ii) A sequence $\{x_n\}$ in PMS (X, p) is called C-seq if there occurs (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$. The space $(X,\ p)$ is said to be complete if every C-seq $\{x_n\}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ s.t. $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

(iii) A sequence $\{x_n\}$ in PMS (X, p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X, p) is said to be 0-complete if every 0-C-seq in X converges (in $\tau(p)$) to a point $x \in X$ s.t. $p(x, x) = 0$.

4.1.3 Lemma

Presume that (X, p) is a PMS.

(a) [5, 56] If $p(x_n, z) \to p(z, z) = 0$ as $n \to \infty$, then $p(x_n, y) \to p(z, y)$ as $n \to \infty$ for each $y \in X$.

(b) [91] If (X, p) is complete, then it is $0-$complete.

It is easy to see that every closed subset of a 0-complete PMS is 0-complete. The following example shows that the converse assertion of (b) need not hold.

4.1.4 Example [91]

The space $X = [0, +\infty) \cap \mathbb{Q}$ with the partial metric $p(x, y) = \max\{x, y\}$ is 0-complete, but is not complete. Moreover, the sequence $\{x_n\}$ with $x_n = 1$ for each $n \in \mathbb{N}$ is a C-seq in (X, p), but it is not a 0-C-seq.

4.1.5 Definition [52]

Presume that (X, d) is a metric space and $f : X \to X$ be a mapping. Then it is said that f satisfies the orbital condition if there occurs a constant $k \in (0, 1)$ s.t.

$$d\left(fx, f^2x\right) \leq k\ d(x, fx), \tag{4.1}$$

$\forall\ x \in X$.

4.1.6 Theorem [6]

Suppose that (X, p) is a $0-$complete PMS and $f : X \to X$ be continuous s.t,

$$p\left(fx, f^2x\right) \leq k\ p(x, fx) \tag{4.2}$$

holds $\forall\ x \in X$, where $k \in (0, 1)$. Then there occurs $z \in X$ s.t. $p(z, z) = 0$ and $p(fz, z) = p(fz, fz)$.

4.1.7 Definition [52]

Suppose that (X, p) is a PMS and $f : X \to X$ be a mapping with FP set $Fix(f) \neq \varphi$. Then f has property (P) if $Fix(f^n) = Fix(f)$, for each $n \in N$.

4.1.8 Lemma [52]

Presume that (X, p) is a PMS, $f : X \to X$ be a self map s.t, Fix $(f) \neq \phi$. Then f has the property (P) if (4.2) holds for some $k \in (0, 1)$ and either (i) $\forall\ x \in X$, or (ii) $\forall\ x \neq fx$.

One of the remarkable generalizations of BCP was reported by Kirk et al. [62] via cyclic contraction.

4.1.9 Theorem [62]

Suppose that $\{A_i\}_{i=1}^m$ is a nonempty closed subset of a complete metric space (X, d) and suppose $f : \cup_{i=1}^m A_i \to \cup_{i=1}^m A_i$ is a mapping satisfying the following conditions:

(1) $f(A_i) \subseteq A_{i+1}$ for $1 \leq i \leq m$, where $A_{m+1} = A_1$.

(2) $d(fx, fy) \leq \psi(d(x, y))$, for all $x \in A_i$, $y \in A_{i+1}$; $i \in \{1, 2, \cdots, m\}$,

where $A_{m+1} = A_1$ and $\psi : [0, 1) \to [0, 1)$ is a function, upper semi-continuous from the right and $0 \leq \psi(t) < t$ for $t > 0$. Then, f has a FP $z \in \cap_{i=1}^m A_i$.

4.1.10 Definition [25]

Assume that K be a nonempty set and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$d(x, K) = d_l(x, y_0), \text{ where } d(x, K) = \inf_{y \in K} d(x, y).$$

If each $x \in X$ has at least one best approximation in K, then K is called a proximinal set.

4.1.11 Definition [25]

The function $H_{d_l} : P(X) \times P(X) \to R^+$, defined by

$$H_p(A, B) = \max\{\sup_{a \in A} p(a, B), \ \sup_{b \in B} p(A, b)\}$$

is called a partial Hausdorff metric on $P(X)$.

4.1.12 Lemma [25]

Presume that $(P(X), H_p)$ be a partially Hausdorff metric space on $P(X)$. If $\forall\ A, B \in P(X)$ and for each $a \in A$ there occurs $b_a \in B$ satisfies $d_l(a, B) = d_l(a, b_a)$ then $H_p(A, B) \geq d_l(a, b_a)$.

4.2 Fixed point of $F-$rational cyclic mappings on $0-$complete PMS

The results given in this section have been presented in [72].

Suppose that (X, p) be a PMS, through out of this section we mean by Δ_p be the set of all nonempty closed subsets of X.

4.2.1 Definition

Assume that (X, p) is a PMS, $V_i \in \Delta_p$ for $i = 1, 2, \cdots, m$, $E = \bigcup_{i=1}^{m} V_i$ where $m \in N$. A mapping $f : E \to E$ is called an F-rational cyclic contraction if there occur $F \in \mathcal{F}$ and $\lambda \in \mathbb{R}_+$ s.t,

1. $f(V_i) \subseteq V_{i+1}, i = 1, 2, ..., m,$ where $V_{m+1} = V_1$,
2. for $x \in V_i$, $y \in V_{i+1}$, $i = 1, 2, ..., m$, with $p(fx, fy) > 0$, we have

$$\lambda + F(p(fx, fy)) \leq F(\mathcal{H}_f(x, y)). \tag{4.3}$$

where

$$\mathcal{H}_f(x, y) = ap(x, y) + bp(x, fx) + cp(y, fy) + dp(x, fy) + ep(y, fx) + l\frac{p(x, fx).p(y, fy)}{1 + p(x, y)}, \tag{4.4}$$

and

$$a, b, c, d, e, l \geq 0 \text{ with } a + b + c + d + e + l < 1. \tag{4.5}$$

The main result of this section is the following.

4.2.2 Theorem

Assume that (X,p) be a 0-complete PMS, $V_i \in \Delta_p$; $i = 1, 2, \cdots, m$ where $m \in \mathbb{N}$ and $E = \cup_{i=1}^{m} V_i$. Suppose that $f : E \to E$ is an F-rational cyclic contraction. Then,

1. f has a unique FP $z \in E$;
2. $p(z, z) = 0$ and $z \in \cap_{i=1}^{m} V_i$;
3. for any $x_0 \in E$, the sequence $x_n = f^n x_0$, converges to z in topology $\tau(p)$.

Proof. Suppose that $x_0 \in E$ be an erratic point. Then there occurs i_0 s.t. $x_0 \in V_{i_0}$, so there is $x_1 \in V_{i_0+1}$ where $x_1 = fx_0$. Continue in this process we can construct a sequence $x_n = fx_{n-1} = f^n x_0 \in V_{i_0+n}$. If $x_n = x_{n+1}$ for some for $n \in \mathbb{N}$, then x_n is a FP of f. From now on assume that $x_n \neq x_{n+1}$, $\forall\, n \in \mathbb{N}$ and let $p_n = p(x_n, x_{n+1})$, so $p_n > 0$ $\forall\, n \in \mathbb{N}$. Since $f : E \to E$ is an F-rational cyclic contractions, from (4.3) and (4.4) we have that

$$\begin{aligned}
\lambda + F(p_n) &= \lambda + F(p(x_n, x_{n+1})) \\
&= \lambda + F(p(fx_{n-1}, fx_n)) \\
&\leq F\left(\begin{array}{c} ap(x_{n-1}, x_n) + bp(x_{n-1}, x_n) + cp(x_n, x_{n+1}) + dp(x_{n-1}, x_{n+1}) \\ +ep(x_n, x_n) + l\frac{p(x_{n-1}, x_n).p(x_n, x_{n+1})}{1+p(x_{n-1}, x_n)} \end{array}\right).
\end{aligned}$$

Since $p(x_{n-1}, x_{n+1}) \leq p(x_{n-1}, x_n) + p(x_n, x_{n+1}) - p(x_n, x_n)$, F is strictly increasing and $\frac{p(x_{n-1}, x_n).p(x_n, x_{n+1})}{1+p(x_{n-1}, x_n)} < p(x_n, x_{n+1})$, the above inequality becomes

$$\lambda + F(p_n) \leq F((a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n, x_n)). \tag{4.6}$$

Since $\lambda > 0$,

$$F(p_n) \leq \lambda + F(p_n) \leq F((a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n, x_n)).$$

But, F is strictly increasing, so we deduce that

$$p_n \leq (a+b+d)p_{n-1} + (c+d+l)p_n + (e-d)p(x_n, x_n)). \tag{4.7}$$

By symmetry of $p(x_{n+1}, x_n) = p(x_n, x_{n+1})$, and using similar argument as above one can deduce that

$$\begin{aligned} \lambda + F\left(p\left(x_{n+1}, x_n\right)\right) &= \lambda + F\left(p\left(fx_n, fx_{n-1}\right)\right) \\ &\leq F\left((a+c+e)\, p_{n-1} + (b+e+l)\, p_n + (d-e)\, p\left(x_n, x_n\right)\right). \end{aligned}$$

Thus,

$$F\left(p_n\right) \leq \lambda + F\left(p_n\right) \leq F\left((a+c+e)\, p_{n-1} + (b+e+l)\, p_n + (d-e)\, p\left(x_n, x_n\right)\right),$$

which implies that

$$p_n \leq (a+c+e)p_{n-1} + (b+e+l)p_n + (d-e)p(x_n, x_n). \tag{4.8}$$

Adding up, equations (4.7) and (4.8) we get $p_n \leq \beta p_{n-1}$, where $\beta = \frac{2a+b+c+d+e}{2-b-c-d-e-2l} < 1$, which is a consequence of (4.5). Hence,

$$p_n < p_{n-1}, \ \forall\, n \in \mathbb{N}. \tag{4.9}$$

Using property(P2) of partial metric, equations (4.6), (4.9) and the property of strictly increasing of F we get

$$\begin{aligned} \lambda + F\left(p_n\right) &\leq F\left((a+b+d)\, p_{n-1} + (c+d+l)\, p_n + (e-d)\, p\left(x_n, x_n\right)\right) \\ &\leq F\left((a+b+d)\, p_{n-1} + (c+d+l)\, p_{n-1} + (e-d)\, p_{n-1}\right) \\ &= F\left((a+b+c+d+e+l)\, p_{n-1}\right) \\ &\leq F(p_{n-1}). \end{aligned}$$

Hence, $\lambda + F\left(p_n\right) \leq F\left(p_{n-1}\right) \ \forall\, n \in \mathbb{N}$. This implies

$$F\left(p_n\right) \leq F\left(p_{n-1}\right) - \lambda \leq \cdots \leq F\left(p_0\right) - n\lambda, \ \forall\, n \in \mathbb{N} \tag{4.10}$$

and so $\lim_{n \to +\infty} F\left(p_n\right) = -\infty$. By the property $(F2)$, we get that $p_n \to 0$ as $n \to +\infty$.

Now, by (F3) there occurs $k \in (0,1)$ s.t, $\lim_{n \to +\infty} p_n^k F\left(p_n\right) = 0$.

By (4.10), the following holds $\forall\ n \in \mathbb{N}$:

$$p_n^k F(p_n) - p_n^k F(p_0) \leq -n\lambda p_n^k \leq 0. \tag{4.11}$$

Suppose thatting $n \to +\infty$ in (4.11) we deduce that

$$lim_{n\to+\infty} np_n^k = 0. \tag{4.12}$$

By using the continuous function $g(x) = x^{\frac{1}{k}}$; $x \in (0, \infty)$, we get that

$$\lim_{n\to+\infty} n^{\frac{1}{k}} p_n = \lim_{n\to+\infty} g(np_n^k) = 0. \tag{4.13}$$

Now, by using the limit comparison test with $a_n = p_n$, $b_n = n^{\frac{-1}{k}}$ and equation (4.12) we ensure that the series $\sum_{n=1}^{+\infty} p_n$ is convergent. This implies that $\{x_n\}$ is a 0−C-seq. Since E is closed in a 0-complete partial metric (X, p). E is also 0-complete and there occurs $z \in E = \cup_{i=1}^m V_i$ s.t.

$$\lim_{n\to\infty} p(x_n, z) = 0 = p(z, z). \tag{4.14}$$

Notice that the iterative sequence $\{x_n\}$ has an infinite number of terms in V_i for each $i = 1, ..., m$. Hence, there is a subsequence of $\{x_n\}$ in each V_i, $i = 1, ..., m$, which converges to z. Using that each V_i, $i = 1, ..., m$, is closed, we conclude that $z \in \cap_{i=1}^m V_i$.

We shall prove that z is a FP of f. Using the triangle inequality (P4) of PMS and (4.4) (which is possible since z belongs to each V_i) to obtain

$$\begin{aligned} p(z, fz) &\leq p(z, x_{n+1}) + p(x_{n+1}, fz) - p(x_{n+1}, x_{n+1}) \\ &\leq p(z, x_{n+1}) + p(fx_n, fz) \\ &\leq p(z, x_{n+1}) + ap(x_n, z) + bp(x_n, x_{n+1}) + cp(z, fz) + dp(x_n, fz) \\ &\quad + ep(x_{n+1}, z) + l\ \frac{p(x_n, x_{n+1}) .p(z, fz)}{1 + p(x_n, z)}. \end{aligned} \tag{4.15}$$

Using Lemma 4.1.3 part (a) and passing to the limit when $n \to \infty$ in (4.15) we obtain that

$$(1 - c - d)\, p(z, fz) \leq 0,$$

and hence

$$p(z, fz) = 0. \tag{4.16}$$

Now by using triangle inequality (P4), (4.14) and (4.16) we deduce that $p(fz, fz) = 0$. Therefore, by (P1) we get $f(z) = z$.

Finally, we will prove the uniqueness, let u be another FP of f in E, with $p(u, z) \neq 0$. By the cyclic character of f, we have $u, z \in \cap_{i=1}^{m} V_i$. Since f is an F-rational cyclic contraction and using the property (P2) of partial metric, we have

$$\begin{aligned} \lambda + F(p(u, z)) &= \lambda + F(p(fu, fz)) \\ &\leq F\left(\begin{array}{c} ap(u, z) + bp(u, u) + cp(z, z) + dp(u, z) + ep(u, z) \\ +l\frac{p(u, fu).p(z, fz)}{1+p(u, z)} \end{array}\right) \\ &\leq F((a+b+c+d+e)p(u, z)), \end{aligned}$$

which is a contradiction deduced from the strictly increasing property of F and being $a + b + c + d + e < 1$, hence $z = u$. Thus z is a unique FP of f. ■

By taking $F(\alpha) = \alpha + \ln(\alpha)$ in Theorem 4.2.2 we get the following corollary.

4.2.3 Corollary

Suppose that (X, p) is a 0-complete PMS, $V_i \in \Delta_p$; $i = 1, 2, \cdots, m$ where $m \in \mathbb{N}$ and $E = \cup_{i=1}^{m} V_i$. Suppose that $f : E \to E$ and the following conditions hold:

1. $f(V_i) \subseteq V_{i+1}, i = 1, 2, ..., m,$ where $V_{m+1} = V_1$,
2. there occurs $\lambda > 0$ s.t, for $x \in V_i$, $y \in V_{i+1}$, $i = 1, 2, ..., m$, with $p(fx, fy) > 0$, we have

$$\begin{aligned} \lambda + \ln(p(fx, fy)) &\leq \left(\begin{array}{c} ap(x, y) + bp(x, fx) + cp(y, fy) + dp(x, fy) \\ +ep(y, fx) + l\frac{p(x, fx).p(y, fy)}{1+p(x, y)} \end{array}\right) \\ &+ ln\left(\begin{array}{c} ap(x, y) + bp(x, fx) + cp(y, fy) + dp(x, fy) \\ +ep(y, fx) + l\frac{p(x, fx).p(y, fy)}{1+p(x, y)} \end{array}\right) \end{aligned}$$

where $a, b, c, d, e, l \geq 0$ and $a + b + c + d + e + l < 1$. Then,

1. f has a unique FP $z \in E$;

2. $p(z,z) = 0$ and $z \in \cap_{i=1}^{m} V_i$;

3. for any $x_0 \in E$, the sequence $x_n = f^n x_0$, converges to z in topology $\tau(p)$.

By taking $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$ in Theorem 4.2.2 we get the following corollary.

4.2.4 Corollary

Suppose that (X,p) is a 0-complete PMS, $V_i \in \Delta_p$; $i = 1, 2, \cdots, m$ where $m \in \mathbb{N}$ and $E = \cup_{i=1}^{m} V_i$. Suppose that $f : E \to E$ and the following conditions hold:

1. $f(V_i) \subseteq V_{i+1}, i = 1, 2, ..., m,$ where $V_{m+1} = V_1$,
2. there occurs $\lambda > 0$ s.t. for $x \in V_i$, $y \in V_{i+1}$, $i = 1, 2, ..., m$, with $p(fx, fy) > 0$, we have

$$\lambda + \frac{-1}{\sqrt{p(fx,fy)}} \leq \frac{-1}{\sqrt{\begin{pmatrix} ap(x,y) + bp(x,fx) + cp(y,fy) + dp(x,fy) + ep(y,fx) \\ +l\frac{p(x,fx).p(y,fy)}{1+p(x,y)} \end{pmatrix}}}$$

where $a, b, c, d, e, l \geq 0$ and $a + b + c + d + e + l < 1$. Then,

1. f has a unique FP $z \in E$;
2. $p(z,z) = 0$ and $z \in \cap_{i=1}^{m} V_i$;
3. for any $x_0 \in E$, the sequence $x_n = f^n x_0$, converges to z in topology $\tau(p)$.

4.2.5 Example

Presume that $X = \mathbb{R}$ is equipped with the usual partial metric $p(x,y) = \max\{|x|, |y|\}$. Then, clearly (X,p) is 0−complete. Suppose $V_1 = \left[0, \frac{1}{2}\right]$, $V_2 = \left[\frac{-1}{6}, 0\right]$, $V_3 = \left[0, \frac{1}{18}\right]$, $V_4 = \left[\frac{-1}{54}, 0\right]$ and $E = \cup_{i=1}^{4} V_i$. Define $f : E \to E$ s.t. $fx = \frac{-x}{8}$ $\forall$ $x \in E$. It is clear that $f(V_i) \subset V_{i+1}$.

Take $\lambda = ln(4)$, $a = \frac{1}{2}$ and $b = c = d = e = l = \frac{1}{11}$. Suppose that $x \in V_i$ and $y \in V_{i+1}$ s.t.

either $x \neq 0$ or $y \neq 0$, then

$$\begin{aligned} p(fx, fy) &= \max\{|\frac{-x}{8}|, |\frac{-y}{8}|\} \\ &= \frac{1}{8}\max\{|x|, |y|\} \\ &= \frac{1}{8}p(x, y) \\ &= (\frac{1}{4})(\frac{1}{2})p(x, y). \end{aligned} \tag{4.17}$$

Now take ln for both sides of (4.17) we get

$$\begin{aligned} \ln(p(fx, fy)) &= \ln((\frac{1}{4})(\frac{1}{2})p(x, y)) \\ &= -ln(4) + \ln(\frac{1}{2}p(x, y)) \\ &\leq -ln(4) + \ln\Big(\frac{1}{2}p(x, y) + \frac{1}{11}p(x, fx) + \frac{1}{11}p(y, fy) + \frac{1}{11}p(x, fy) \\ &\quad + \frac{1}{11}p(y, fx) + \frac{1}{11}\frac{p(x, fx)p(y, fy)}{1 + p(x, y)}\Big). \end{aligned}$$

Hence,

$$ln(4) + F(p(x, y)) \leq F(\mathcal{H}_f(x, y)).$$

Therefore, all the conditions of Theorem 4.2.2 are satisfied and we deduce that f has a unique FP $z = 0 \in \bigcap_{i=1}^{4} V_i$ and $p(z, z) = 0$ holds true.

4.3 Ćirić-Wardowski type generalized multivalued maps in PMS

In this section, we prove common FP theorem for a pair of multivalued $F - \Psi-$proximinal mappings satisfying Ćirić-Wardowski type contraction in PMS.

Suppose that (X, p) be a PMS, $x_0 \in X$ and $S, T : X \to P(X)$ be the multifunctions on X. Suppose that $x_1 \in Sx_0$ be an element s.t, $p(x_0, Sx_0) = p(x_0, x_1)$. Suppose that $x_2 \in Tx_1$ be s.t, $p(x_1, Tx_1) = p(x_1, x_2)$. Suppose that $x_3 \in Sx_2$ be s.t, $p(x_2, Sx_2) = p(x_2, x_3)$. Continuing this process, we construct a sequence x_n of points in X s.t, $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$, where $n = 0, 1, 2, \ldots$. Also $p(x_{2n}, Sx_{2n}) = p(x_{2n}, x_{2n+1})$, $p(x_{2n+1}, Tx_{2n+1}) = p(x_{2n+1}, x_{2n+2})$. We denote this iterative sequence by $\{TS(x_n)\}$. We say that $\{TS(x_n)\}$ is a sequence in X

generated by x_0.

Suppose that Φ be the set of functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ s.t,

1. φ is upper semi-continuous;

2. $\varphi(t) < t$, for each $t > 0$.

Suppose that Ψ signify the set of all decreasing functions $\psi : (0, \infty) \rightarrow (0, \infty)$.

We begin with the following definition.

4.3.1 Definition

Assume that (X, p) is a complete PMS. The mappings $S, T : X \rightarrow P(X)$ are said to be a pair of Ćirić-Wardowski type generalized multivalued $F - \Psi-$proximinal contraction, if there occur $\psi \in \Psi$ and $\varphi \in \Phi$ s.t, $\forall\ x, y \in X$ with $p(Tx, Ty) > 0$,

$$\psi(p(x,y)) + F(H_p(Sx, Ty)) \leq F(\phi(M(x,y))) \tag{4.18}$$

where $F \in \triangle_F$and $\tau > 0$, and

$$M(x, y)) = \max\left\{p(x, y), \frac{p(x, Sx).p(y, Ty)}{1 + p(x, y)}, \frac{p(x, Sx).p(y, Ty)}{1 + p(Sx, Ty)}, p(x, Sx), p(y, Ty)\right\}. \tag{4.19}$$

The following theorem is one of our main results.

4.3.2 Theorem

Suppose that (X, p) is a PMS and $S, T : X \rightarrow P(X)$ are said to be a pair of multivalued mappings s.t.

(1). (S, T) are pair of continuous mappings,

(2). (S, T) are pair of Ćirić-Wardowski type generalized multivalued $F - \Psi-$proximinal contraction .

Then the pair (S, T) has a common FP u in X and $p(u, u) = 0$.

Proof: We begin with the following observation:

If, $M(y, x) = 0$, then clearly $x = y$ is a common FP of (S, T).and there is nothing to prove and our proof is complete. In ordered to find common FP of both S and T for the situation when $M(y, x) > 0\ \forall\ x, y \in X$ with $x \neq y$, we construct an iterative sequence $\{TS(x_n)\}$ generated by

x_0. Then from contractive condition (4.18) and Lemma 4.1.8, we get

$$\psi\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) + F(H_p(Sx_{2i}, Tx_{2i+1})) \leq F\left(\phi(M(x_{2i}, x_{2i+1}))\right) \tag{4.20}$$

$\forall\ i \in \mathbb{N} \cup \{0\}$, where

$$\begin{aligned}
M(x_{2i}, x_{2i+1}) &= \max\left\{\begin{array}{c} p(x_{2i}, x_{2i+1}), \frac{p(x_{2i}, Sx_{2i}).p(x_{2i+1}, Tx_{2i+1})}{1+p(x_{2i}, x_{2i+1})}, \frac{p(x_{2i}, Sx_{2i}).p(x_{2i+1}, Tx_{2i+1})}{1+p(Sx_{2i}, Tx_{2i+1})} \\ , p(x_{2i}, Sx_{2i}), p(x_{2i+1}, Tx_{2i+1}) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} p(x_{2i}, x_{2i+1}), \frac{p(x_{2i}, x_{2i+1}).p(x_{2i+1}, x_{2i+2})}{1+p(x_{2i}, x_{2i+1})}, \frac{p(x_{2i}, x_{2i+1}).p(x_{2i+1}, x_{2i+2})}{1+p(x_{2i+1}, x_{2i+2})} \\ , p(x_{2i}, x_{2i+1}), p(x_{2i+1}, x_{2i+2}) \end{array}\right\} \\
&= \max\{p(x_{2i}, x_{2i+1}), p(x_{2i+1}, x_{2i+2})\}.
\end{aligned}$$

If for some $i \in \mathbb{N}^+$, $M(x_{2i}, x_{2i+1}) = p(x_{2i+1}, x_{2i+2})$, then taking (4.20) into account, we get that

$$\psi\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) + F(H_p(Sx_{2i}, Tx_{2i+1})) \leq F\left(\phi(M(x_{2i+1}, x_{2i+2}))\right).$$

On using the property of ϕ and from $(F1)$, we get

$$\psi\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) + F(H_p(Sx_{2i}, Tx_{2i+1})) \leq F\left(p(x_{2i+1}, x_{2i+2})\right)$$

$\forall\ i \in \mathbb{N} \cup \{0\}$. Since, $\psi\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) > 0$, which give contradiction, yielding thereby

$$M(x_{2i+1}, x_{2i+2}) = p\left(x_{2i+1}, x_{2i+2}\right), \ \forall\ i \in \mathbb{N}^+.$$

Therefore from (4.20) and by the property of F, ϕ and ψ, we get

$$\begin{aligned}
F(p(x_{2i+1}, x_{2i+2})) &\leq F\left(\phi\left(p(x_{2i}, x_{2i+1})\right)\right) - \psi\left(p(x_{2i}, x_{2i+1})\right) \\
&\leq F\left(\phi\left(p(x_{2i}, x_{2i+1})\right)\right), \qquad (4.21) \\
F(p(x_{2i+1}, x_{2i+2})) &< F\left(p(x_{2i}, x_{2i+1})\right).
\end{aligned}$$

It follows from the above inequality and property of $(F1)$ that

$$p(x_{2i+1}, x_{2i+2}) < p(x_{2i}, x_{2i+1}) \quad \forall\, i \in \mathbb{N}^+.$$

Thus $\{p(x_{2i+1}, x_{2i+2})\}$ is a decreasing sequence of positive real numbers. Consequentely from (4.21), we have

$$\begin{aligned} F\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) &< F\left(p(x_{2i}, x_{2i+1})\right) - \psi\left(p(x_{2i}, x_{2i+1})\right) \\ &< F\left(p(x_{2i-1}, x_{2i})\right) - \psi\left(p(x_{2i-1}, x_{2i})\right) - \psi\left(p(x_{2i}, x_{2i+1})\right). \end{aligned}$$

As ψ is a decreasing function, we get

$$F\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) < F\left(p(x_{2i-1}, x_{2i})\right) - 2\psi\left(p(x_{2i-1}, x_{2i})\right).$$

Repeating the same process, we get

$$F\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) < F\left(p(x_0, x_1)\right) - n\psi\left(p(x_0, x_1)\right). \tag{4.22}$$

Since $F \in \triangle_F$, letting the limit as $i \to \infty$ in (4.22) we must have

$$\lim_{i\to\infty} F\left(p\left(x_{2i+1}, x_{2i+2}\right)\right) = -\infty \Longleftrightarrow \lim_{i\to\infty} p\left(x_{2i+1}, x_{2i+2}\right) = 0. \tag{4.23}$$

Further, by $(P2)$ we have the following equality

$$\lim_{i\to\infty} p\left(x_i, x_i\right) = 0. \tag{4.24}$$

Next, we will show that $\{x_i\}_{i=1}^{\infty}$ is a C-seq in X. Suppose, to contrary that, $\{x_i\}_{i=1}^{\infty}$ is not a C-seq in a complete PMS (X, p). Then there exisit $\varepsilon > 0$ and two sub-sequences $\{x_{i(k)}\}$ and $\{x_{j(k)}\}$ of $\{x_i\}_{i=1}^{\infty}$ s.t, $i(k) > j(k) \geq k$ and

$$p\left(x_{j(k)}, x_{i(k)}\right) \geq \varepsilon,$$

which yields

$$p\left(x_{j(k)}, x_{i(k)-1}\right) < \varepsilon. \tag{4.25}$$

Applying the property $(P4)$ and inequality (4.25), we get

$$\begin{aligned}
\varepsilon &\leq p\left(x_{j(k)}, x_{i(k)}\right) \leq p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{i(k)}\right) - p\left(x_{j(k)+1}, x_{j(k)+1}\right) \\
&\leq p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{j(k)}\right) + p\left(x_{j(k)}, x_{i(k)}\right) - p\left(x_{j(k)}, x_{j(k)}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)}, x_{i(k)}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)}, x_{i(k)-1}\right) + p\left(x_{i(k)-1}, x_{i(k)}\right) - p\left(x_{i(k)-1}, x_{i(k)-1}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + p\left(x_{j(k)}, x_{i(k)-1}\right) + p\left(x_{i(k)-1}, x_{i(k)}\right) \\
&\leq 2p\left(x_{j(k)}, x_{j(k)+1}\right) + \varepsilon + p\left(x_{i(k)-1}, x_{i(k)}\right),
\end{aligned}$$

which on making $k \to \infty$, yields

$$\lim_{k\to\infty} p\left(x_{j(k)}, x_{i(k)}\right) = \varepsilon. \tag{4.26}$$

Furthermore, from $(P4)$, (4.23), (4.24) and (4.26), we can get

$$\begin{aligned}
\lim_{k\to\infty} p\left(x_{j(k)}, x_{i(k)+1}\right) &= \varepsilon, \\
\lim_{k\to\infty} p\left(x_{j(k)+1}, x_{i(k)}\right) &= \varepsilon,
\end{aligned}$$

and

$$\lim_{k\to\infty} p\left(x_{j(k)+1}, x_{i(k)+1}\right) = \varepsilon. \tag{4.27}$$

Also from (4.24) there occurs a natural number $i_0 \in \mathbb{N}$ s.t,

$$p\left(x_{i(k)}, x_{i(k)+1}\right) = \frac{\varepsilon}{4} \text{ and } p\left(x_{j(k)}, x_{j(k)+1}\right) = \frac{\varepsilon}{4}.$$

$\forall\ i, k \geq i_0$. Now we claim that

$$p\left(Tx_{i(k)}, Tx_{j(k)}\right) = p\left(x_{i(k)+1}, x_{j(k)+1}\right) > 0. \tag{4.28}$$

Suppose on contrary that, $p\left(x_{i(k)+1}, x_{j(k)+1}\right) = 0$. Then

$$\begin{aligned}
\varepsilon &\leq p\left(x_{i(k)}, x_{j(k)}\right) \leq p\left(x_{i(k)}, x_{i(k)+1}\right) + p\left(x_{i(k)+1}, x_{j(k)}\right) - p\left(x_{i(k)+1}, x_{i(k)+1}\right) \\
&\leq p\left(x_{i(k)}, x_{i(k)+1}\right) + p\left(x_{i(k)+1}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{j(k)}\right) - p\left(x_{j(k)+1}, x_{j(k)+1}\right) \\
&\leq p\left(x_{i(k)}, x_{i(k)+1}\right) + p\left(x_{i(k)+1}, x_{j(k)+1}\right) + p\left(x_{j(k)+1}, x_{j(k)}\right) \\
&\leq \frac{\varepsilon}{4} + 0 + \frac{\varepsilon}{4} = \frac{\varepsilon}{2},
\end{aligned}$$

which yields a contradiction, thus (4.28) holds. Then it follows from the contractive condition (4.18) and the property of ψ that

$$\begin{aligned}
\psi\left(p\left(x_{i(k)}, x_{j(k)}\right)\right) + F\left(H_p\left(x_{i(k)+1}, x_{j(k)+1}\right)\right) &= \psi\left(p\left(x_{i(k)}, x_{j(k)}\right)\right) + F\left(H_p\left(Sx_{i(k)}, Tx_{j(k)}\right)\right) \\
&\leq F\left(\phi\left(M\left(x_{i(k)}, x_j(k)\right)\right)\right), \text{ i.e.,} \qquad (4.29) \\
F\left(H_p\left(x_{i(k)+1}, x_{j(k)+1}\right)\right) &\leq F\left(\phi\left(M\left(x_{i(k)}, x_j(k)\right)\right)\right).
\end{aligned}$$

By the definition of $M(x, y)$, (2.9), (2.10) and after repeating the same process, we get

$$\lim_{k\to\infty} M\left(x_{i(k)}, x_{j(k)}\right) = \varepsilon. \qquad (4.30)$$

Putting $k \to \infty$ in (4.29) and taking into account (4.23), (4.26), (4.27), (4.30), property $\left(F3'\right)$ and upper semi continuity of ϕ, we find that $F(\varepsilon) \leq F(\phi(\varepsilon)) \leq F(\varepsilon)$, which gives a contradiction. Thus, we conclude that

$$\lim_{i,j\to\infty} p(x_i, x_j) = 0,$$

i.e., the sequence $\{TS(x_i)\}_{i=1}^{\infty}$ is a 0-C-seq. Therefore, the 0-completeness of X ensures that there occurs a point $u \in X$ s.t, $\{TS(x_i)\} \to u$, i.e„

$$\lim_{i\to\infty} p(u, x_i) = 0. \qquad (4.31)$$

Now,

$$\begin{aligned} F(p(u,Tu)) &\leq F(p(u,x_{2i+1})+p(x_{2i+1},Tu)) \\ &\leq F(p(u,x_{2i+1})+H_p(Sx_{2i},Tu)) \text{ by Lemma 4.1.12} \\ &\leq F(p(u,x_{2i+1}))+F(H_p(Sx_{2i},Tu)). \end{aligned} \tag{4.32}$$

By using inequality (4.18), we have

$$\psi(p(u,Tu))+F(H_p(Sx_{2i},Tu)\leq F(\phi M(x_{2i},u)) \tag{4.33}$$

where,

$$\begin{aligned} M(x_{2i},u) &= \max\left\{ \begin{array}{c} p(x_{2i},u), \frac{p(x_{2i},Sx_{2i}).p(u,Tu)}{1+p(x_{2i},u)}, \frac{p(x_{2i},Sx_{2i}).p(u,Tu)}{1+p(Sx_{2i},Tu)}, \\ p(x_{2i},Sx_{2i}), p(u,Tu) \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} p(x_{2i},u), \frac{p(x_{2i},x_{2i+1}).p(u,Tu)}{1+p(x_{2i},x_{2i+1})}, \frac{p(x_{2i},x_{2i+1}).p(u,Tu)}{1+p(Sx_{2i},Tu)}, \\ p(x_{2i},x_{2i+1}), p(u,Tu) \end{array} \right\}. \end{aligned}$$

Taking limit $i\to\infty$, and by using (4.31), we get

$$p(x_{2i},u)=p(u,Tu). \tag{4.34}$$

It follows from the above inequality that

$$F(p(Tu,Tu))\leq F\left(\phi\left(p\left(u,Tu\right)\right)\right)-\psi\left(p\left(u,u\right)\right),$$

which implies that

$$p(u,Tu)<p(u,Tu).$$

That is a contradiction, hence $p(Tu,Tu)=p(u,Tu)=0$ or $u\in Tu$. Similarly by using (4.31), Lemma 4.1.12 and the inequality

$$\psi\left(p\left(u,Su\right)\right)+F(p(Su,u))\leq F(p(Su,x_{2n+2})+p(x_{2n+2},Su))$$

we can show that $p(Su,u)=0$ or $u\in Su$. Hence the pair (S,T) has a common FP u in (X,p). Now,

$$p(u,u)\leq p(u,Tu)+p(Tu,u)\leq 0.$$

This implies that $p(u,u)=0$.

4.3.3 Example

Presume that $X=\mathbb{R}$ is equipped with a usual partial metric $p(x,y)=\max\{(|x|,|y|)\}$. It is obvious that (X,p) is 0−complete PMS. Define the mappings $S,T:X\to P(X)$ as follows:

$$S(x)=\left[\frac{1}{3}x,\frac{2}{3}x\right] \text{ and } T(x)=\left[\frac{1}{5}x,\frac{2}{5}x\right] \ \forall\ x\in X.$$

Then S,T is a pair of continuous mappings. Define the function $F:R^{+}\to R$ by $F(x)=\ln(x)$ and for $x=2$ and $y=3$, we have

$$(H_p(S(2),T(3)))\leq\left\{p\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{2}{5},\frac{4}{5}\right]\right)\right\}=\frac{27}{15}$$

Define $\psi:(0,\infty)\to(0,\infty)$ by $\psi(t)=\frac{1}{30(t+1)}$ and let $\phi:[0,\infty)\to[0,\infty)$ be given by $\phi(t)=\frac{50t+1}{70}$. It is easy to see that S,T is a Ćirić-Wardowski type generalized multivalued $F-\Psi$−proximinal contraction on X. In short we proceed as follows:

L.H.S= $\psi(p(x,y))+F(H_p(Sx,Ty))=\frac{1}{30(x+1)}+F\left(\frac{27}{15}\right)=\frac{1}{30(x+1)}+\log(1.8)=\frac{1}{90}+0.25552=0.26666$.

R.H.S=$F(\phi(M(x,y)))$ where

$$M(x,y))=\max\left\{p(x,y),\frac{p(x,Sx).p(y,Ty)}{1+p(x,y)},\frac{p(x,Sx).p(y,Ty)}{1+p(Sx,Ty)},p(x,Sx),p(y,Ty)\right\}.$$

Now for $x=2$ and $y=3$, we have

$$
\begin{aligned}
M(2,3) &= \max\left\{\begin{array}{c} p(2,3), \frac{p(2,S(2)).p(3,T(3))}{1+p(2,3)}, \frac{p(2,S(2)).p(3,T(3))}{1+p(S(2),T(3))}, \\ p(2,S(2)), p(3,T(3)) \end{array}\right\} \\
&= \max\left\{\begin{array}{c} p(2,3), \frac{p(2,[\frac{2}{3},\frac{4}{3}]).p(3,[\frac{3}{5},\frac{6}{5}]))}{1+p(2,3)}, \frac{p(2,[\frac{2}{3},\frac{4}{3}]).p(3,[\frac{3}{5},\frac{6}{5}]))}{1+\left\{p\left(\left[\frac{2}{3},\frac{4}{3}\right],\left[\frac{2}{5},\frac{4}{5}\right]\right)\right\}}, \\ p(2,[\frac{2}{3},\frac{4}{3}]), p(3,[\frac{3}{5},\frac{6}{5}]) \end{array}\right\} \\
&= \max\left\{5, \frac{8}{5}, \frac{10}{7}, \frac{24}{7}, \frac{8}{3}, \frac{18}{5}\right\} = 5.
\end{aligned}
$$

Thus,

$$\ln(p(2,3)) = \ln(5) = 1.6094.$$

Hence

$$0.2666 \leq 1.6094.$$

Hence all the hypothesis of Theorem 4.3.2 are satisfied. So (S,T) has a common FP.

4.3.4 Corollary

Suppose that (X,p) is a complete PMS. The mappings $S,T : X \to P(X)$ be a pair of multivalued $F-\Psi-$proximinal contraction, if there occur $\psi \in \Psi$ and $\varphi \in \Phi$ s.t, $\forall\, x,y \in X$ with $p\,(Tx,Ty) >$ 0,

$$\psi\,(p\,(x,y)) + F(H_p(Sx,Ty)) \leq F\,(\phi\,(M\,(x,y)))$$

where $F \in \triangle_F$and $\tau > 0$, and

$$M(x,y)) = \max\left\{p(x,y), \frac{p\,(x,Sx)\,.p\,(y,Ty)}{1+p\,(x,y)}, \frac{p\,(x,Sx)\,.p\,(y,Ty)}{1+p\,(Sx,Ty)}, p(x,Sx)\right\}.$$

Then the pair (S,T) has a common FP u in X and $p(u,u) = 0$.

4.3.5 Corollary

Suppose that (X,p) be a PMS and $S,T : X \to P(X)$ are said to be a pair of multivalued mappings s.t,

(1). (S,T) is a pair of upper semi-continuous mappings, (2). (S,T) is a pair of Ciric-Wardowski type generalized multivalued $F-\Psi-$proximinal contraction .

Then the pair (S,T) have a common FP u in X and $p(u,u)=0$.

4.4 Applications to system of integral equations

In this section applications to system of integral equations are presented to show the usability of our previous results.

Consider the integral equation

$$u(t) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\,dr, \qquad \text{for all } t \in [0,1], \tag{4.35}$$

where, $\zeta:[0,1]\times\mathbb{R}\to\mathbb{R}$, $H:[0,1]\times[0,1]\to\mathbb{R}$ and $h:\mathbb{R}\to[0,\infty)$ are functions (see [72]).

Presume that $X=C([0,1])$ be the set of all real continuous functions on $[0,1]$, endowed with the partial metric

$$p(u,v) = \max\{\sup_{t\in[0,1]}|u(t)|, \sup_{t\in[0,1]}|v(t)|\}, \text{ for all } u,v\in X.$$

Clearly, (X,p) is a 0-complete PMS.

Suppose that $\kappa,\eta\in X$, $\kappa_0,\eta_0\in\mathbb{R}$ s.t. $\forall\ t\in[0,1]$ we have

$$\kappa_0 \le \kappa(t) \le \eta(t) \le \eta_0, \tag{4.36}$$

$$\kappa(t) \le h(u(t)) + \int_0^t H(t,r)\zeta(r,\eta(r))\,dr, \tag{4.37}$$

and

$$\eta(t) \ge h(u(t)) + \int_0^t H(t,r)\zeta(r,\kappa(r))\,dr. \tag{4.38}$$

Assume that $r\in[0,1]$, $\zeta(r,\cdot)$ and $h(.)$ be decreasing functions, i.e,,

$$x,y\in\mathbb{R}, x\ge y \text{ implies } \zeta(r,x)\le\zeta(r,y). \tag{4.39}$$

and

$$h(x) \leq h(y) \tag{4.40}$$

Assume that,

$$\max_{t\in[0,1]} \int_0^1 H(t,s)ds < e^{-\lambda}, \text{for some } \lambda \in (0,\infty) \tag{4.41}$$

and

$$\sup_{r\in[0,1]} |\zeta(r,u(r))| \leq \sup_{r\in[0,1]} |u(r)|. \tag{4.42}$$

Define a mapping $f : X \to X$ by

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))dr; \quad t \in [0,1]. \tag{4.43}$$

Also, suppose that $\forall\ x, y \in \mathbb{R}$ with ($x \leq \eta_0$ and $y \geq \kappa_0$) or ($x \geq \kappa_0$ and $y \leq \eta_0$) we have,

$$|h(u(t))| \leq \tfrac{1}{8}e^{-\lambda} \max\{\sup_{t\in[0,1]} |u(t)|, \sup_{t\in[0,1]} |f(u(t))|\}. \tag{4.44}$$

4.4.1 Theorem [72]

Under the assumptions (4.36)-(4.44), the integral equation (4.35) has a solution z s.t. $z \in C([0,1])$ with $\kappa(t) \leq z(t) \leq \eta(t)\ \forall\ t \in [0,1]$.

Proof. Define the closed subsets of X, U_1 and U_2 by

$$U_1 = \{u \in X : u \leq \eta\}$$

and

$$U_2 = \{u \in X : u \geq \kappa\}.$$

Also define the mapping $f : U_1 \cup U_2 \to U_1 \cup U_2$ by

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\ dr, \text{ for all } t \in [0,1].$$

Now we prove that,

$$f(U_1) \subseteq U_2 \text{ and } f(U_2) \subseteq U_1. \tag{4.45}$$

Suppose, $u \in U_1$, i.e,,

$$u(r) \leq \eta(r), \text{ for all } r \in [0,1].$$

Using condition (4.39) and (4.40) we obtain that

$$\zeta(r, u(r)) \geq \zeta(r, \eta(r)), \text{ for all } r \in [0,1]$$

and

$$h(u(r)) \geq h(\eta(r)), \text{ for all } r \in [0,1].$$

The above inequalities with condition (4.37) imply that

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\, dr \geq h(\eta(t)) + \int_0^t H(t,r)\zeta(r,\eta(r))\, dr = \eta(t) \geq \kappa(t).$$

$\forall\, t \in [0,1]$. Then we have $f(u(t)) \in U_2$. Similarly, let $u \in U_2$, i.e,,

$$u(r) \geq \kappa(r), \text{ for all } r \in [0,1].$$

Using condition (4.39) and (4.40) we obtain that $\zeta(r, u(r)) \leq \zeta(r, \kappa(r))$, for all $r \in [0,1]$

and

$$h(u(r)) \leq h(\kappa(r)), \text{ for all } r \in [0,1].$$

The above inequalities with condition (4.38) imply that

$$f(u(t)) = h(u(t)) + \int_0^t H(t,r)\zeta(r,u(r))\, dr \leq h(\kappa(t)) + \int_0^t H(t,r)\zeta(r,\kappa(r))\, dr = \kappa(t) \leq \eta(t).$$

$\forall\, t \in [0,1]$. Then we have $f(u(t)) \in U_1$. Also, we deduce that (4.45) holds.

Suppose that $x \in U_1$ and $y \in U_2$. Then from (4.43), $\forall\ t \in [0,1]$, we have

$$
\begin{aligned}
|f(x(t))| &= |h(x(t)) + \int_0^t H(t,r)\zeta(r,x(r))\ dr| \\
&\le |h(x(t))| + |\int_0^t H(t,r)\zeta(r,x(r))\ dr| \\
&\le |h(x(t))| + \int_0^t |H(t,r)||\zeta(r,x(r))|\ dr \\
&\le |h(x(t))| + \int_0^t |H(t,r)| \max\{\sup_{r\in[0,1]} |\zeta(r,x(r))|, \sup_{r\in[0,1]} |\zeta(r,y(r))|\}\ dr \\
&\le |h(x(t))| + \max_{t\in[0,1]} \int_0^t H(t,r)p(x,y)dr \\
&\le |h(x(t))| + \frac{1}{8}e^{-\lambda}p(x,y) \\
&\le \frac{1}{8}e^{-\lambda}p(x,fx) + \frac{1}{8}e^{-\lambda}p(x,y) \\
&= e^{-\lambda}(\frac{1}{8}p(x,fx) + \frac{1}{8}p(x,y)).
\end{aligned}
$$

Thus,

$$
\sup_{t\in[0,1]} |f(x(t))| \le e^{-\lambda}(\frac{1}{8}p(x,fx) + \frac{1}{8}p(x,y)). \tag{4.46}
$$

Similarly, we have

$$
\sup_{t\in[0,1]} |f(y(t))| \le e^{-\lambda}(\frac{1}{8}p(y,fy) + \frac{1}{8}p(x,y)). \tag{4.47}
$$

Hence, from (4.46) and (4.47) we have

$$
\begin{aligned}
\max\{\sup_{t\in[0,1]} |f(x(t))|, \sup_{t\in[0,1]} |f(y(t))|\} &\le e^{-\lambda}(\frac{1}{8}p(x,y) + \frac{1}{8}p(x,fx) + \frac{1}{8}p(y,fy)) \\
&\le e^{-\lambda}(\frac{1}{8}p(x,y) + \frac{1}{8}p(x,fx) + \frac{1}{8}p(y,fy) \\
&+ \frac{1}{8}p(x,fy) + \frac{1}{8}p(y,fx)).
\end{aligned}
$$

Therefore,

$$
p(fx,fy) \le e^{-\lambda}(\frac{1}{8}p(x,y) + \frac{1}{8}p(x,fx) + \frac{1}{8}p(y,fy) + \frac{1}{8}p(x,fy) + \frac{1}{8}p(y,fx))
$$

and so,

$$\ln(p(fx,fy)) \leq -\lambda + ln\Big(\frac{1}{8}p(x,y) + \frac{1}{8}p(x,fx) + \frac{1}{8}p(y,fy) + \frac{1}{8}p(x,fy) + \frac{1}{8}p(y,fx)\Big)$$

which implies that $\lambda + F(p(fx,fy)) \leq F(\mathcal{H}_f(x,y))$ is satisfied for $F(\alpha) = \ln(\alpha)\ \forall\ \alpha \in X$ with $a = b = c = d = e = \frac{1}{8}$ and $l = 0$. Hence, all conditions of Theorem 4.2.2 hold and f has a FP z s.t, $z \in C([0,1])$ with $\kappa \leq z(t) \leq \eta\ \forall\ t \in [0,1]\}$. That is, $z \in\ U_1 \cap U_2$ is a solution to (4.35).

■

Next we will discuss the application of Theorem 4.3.2 in form of following Volterra type integral equations

$$u(t) = \int_0^t K_1(t,s,u(s))ds + f(t), \tag{4.48}$$

$$v(t) = \int_0^t K_2(t,s,v(s))ds + g(t) \tag{4.49}$$

$\forall\ t \in [0,1]$. We find the solution of (4.48) and (4.49). Suppose that $X = C([0,1])$ be the set of all real continuous functions on $[0,1]$, endowed with the complete PMSs. For $u \in C([0,1],\mathbb{R})$, define supremum norm as:

$$\max \|u,v\|_\tau = \max\left\{\sup_{t\in[0,1]} \{u(t), v\,(t)\} e^{-\tau(t)t}\right\},$$

where $\tau > 0$ is taken arbitrary. Then

$$p_\tau(u,v) = \max\left\{\sup_{t\in[0,1]} \|\,|u(t),v(t)|\, e^{-\tau t}\|_\tau\right\}$$

$\forall\ u,v \in C([0,1],\mathbb{R})$. With these setting, $C([0,1],\mathbb{R},\|\cdot\|_\tau)$ becomes a complete PMS.

Now we prove the following theorem to guarantee the occurence of solution of integral equations.

4.4.2 Theorem

Assume the following conditions are satisfied:

(i) $K_1, K_2 : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ and $f, g : [0,1] \to \mathbb{R}$ are continuous;

(ii) Define

$$
\begin{aligned}
Su(t) &= \int_0^t K_1(t,s,u(s))ds + f(t), \\
Tv(t) &= \int_0^t K_2(t,s,v(s))ds + g(t).
\end{aligned}
$$

Suppose there occurs $\tau > 1$, s.t,

$$\max|K_1(t,s,u), K_2(t,s,v)| \leq \tau e^{-\tau}[M(u,v)]$$

$\forall\ t, s \in [0,1]$ and $u, v \in C([0,1], \mathbb{R})$, where

$$M(u,v) = \max\left\{ \begin{array}{c} \max|u(t),v(t)|, \frac{\max|u(t),Su(t)|.\max|v(t),Tv(t)|}{1+\max|u(t),v(t)|}, \frac{\max|u(t),Su(t)|.\max|v(t),Tv(t)|}{1+\max|Su(t),Tv(t)|} \\ \max|u(t),Su(t)|, \max|v(t),Tv(t)| \end{array} \right\}.$$

Then integral equations (4.48) and (4.49) have a solution.

Proof. By assumption (ii)

$$
\begin{aligned}
\max|Su(t),Tv(t)| &= \int_0^t \max|K_1(t,s,u(s), K_2(t,s,v(s)))|\, ds \\
&\leq \int_0^t \tau e^{-\tau}([M(u,v)]e^{-\tau s})e^{\tau s} ds \\
&\leq \int_0^t \tau e^{-\tau}\|M(u,v)\|_\tau e^{\tau s} ds \\
&\leq \tau e^{-\tau}\|M(u,v)\|_\tau \int_0^t e^{\tau s} ds \\
&\leq \tau e^{-\tau}\|M(u,v)\|_\tau \frac{1}{\tau} e^{\tau t} \\
&\leq e^{-\tau}\|M(u,v)\|_\tau e^{\tau t}.
\end{aligned}
$$

This implies

$$\max |Su(t), Tv(t)| \, e^{-\tau t} \leq e^{-\tau} \|M(u, v)\|_{\tau}.$$

That is

$$\max \|Su(t), Tv(t)\|_{\tau} \leq e^{-\tau} \|M(u, v)\|_{\tau},$$

which further implies

$$\tau + \ln\{\max \|Su(t), Tv(t)\|_{\tau}\} \leq \ln \|M(u, v)\|_{\tau}.$$

So all the conditions of Theorem 4.3.2 are satisfied. Hence integral equations given in (4.48) and (4.49) hava a unique common solution. ■

Chapter 5

Common Fixed Point Results in Dislocated Metric Spaces

In 1994, Matthews [65] intoduced the concept of PMSs and obtained various FP theorems. In particular, he established the precise relationship between PMSs and the so-called weightable quasi-metric spaces, and proved a partial metric generalization of Banach's contraction mapping theorem. Later on, Neill in [80] introduced the concept of dualistic PMSs (DPMS) by extending the range $R^+ \to R$. He developed several connections between partial metrics and the topological aspects of domain theory. In 2004, Oltra et al. [82] established Banach FP theorem for complete DPMS. Recently many authors developed some FP theorems using complete DPMS for Banach's contraction principle and partial order respectively.

Hitzler and Seda [42] introduced the concept of dislocated topologies and named their corresponding generalized metric a dislocated metric. They have also established a FP theorem in complete dislocated metric spaces to generalize the celebrated Banach contraction principle. The notion of dislocated topologies has useful applications in the context of logic programming semantics (see [43]).

In this chapter, we continue these inquaries to find FP and common FP results in DPMS and dislocated metric spaces. In Section 5.1, we use the notion of Hausdorff metric on the family of closed bounded subsets of a dualistic PMS (DPMS) and establish a common FP theorem of a pair of multivalued mappings satisfying Mizoguchi and Takahashi's contractive conditions. In

Section 5.2, we use the concept of dislocated metric spaces and obtain theorems asserting the occurance of common FPs for a pair of mappings satisfying new generalized rational contractions in such spaces. In Section 5.4, applications to system of integral equations are presented to show the usability of our previous result.

5.1 Common FP results in dualistic partial metric space (DPMS)

The results presented in this section have been published in [59].

Mizoguchi and Takahashi proved the following theorem on complete metric spaces in [67].

5.1.1 Theorem

Suppose that (X,d) is a complete metric space and let the mapping $S : X \to CB(X)$ be a multivalued map and $\varphi : [0,+\infty) \longrightarrow [0,1)$ be an MJ−function. Assume that

$$H(Sx,Sy) \leq \varphi\left(d\left(x,y\right)\right)d\left(x,y\right) \tag{5.1}$$

$\forall\ x,\ y \in X$. Then S has a FP in X.

We use the notion of Hausdorff metric on the family of closed bounded subsets of a dualistic partial metric space and establish a common FP theorem of a pair of multivalued mappings satisfying MT-function. Following is our main result.

5.1.2 Theorem [59]

Presume that (X,D) be a complete DPMS. $S,J : X \to CB^D(X)$ be multivalued mappings and $\varphi : [0,+\infty) \longrightarrow [0,1)$ be an MJ−function. Assume that

$$H_D(Sx,Jy) \leq \varphi\left(D\left(x,y\right)\right)D\left(x,y\right) \tag{5.2}$$

$\forall\ x,\ y \in X$, then there occurs $z \in X$ s.t, $z \in Sz$ and $z \in Jz$.

Proof. Suppose that $x_0 \in X$ and $x_1 \in Sx_0$. If $D\left(x_0,x_1\right) = 0$, then $x_0 = x_1$ and

$$H_D(Sx_0,Jx_1) \leq \varphi\left(D\left(x_0,x_1\right)\right)D\left(x_0,x_1\right) = 0.$$

■

Thus, $Sx_0 = Jx_1$, which implies that

$x_1 = x_0 \in Sx_0 = Jx_1 = Jx_0$, and we finish. Assume that $D(x_0, x_1) > 0$. By Lemma 1.3.16, we can take $x_2 \in Jx_1$ s.t,

$$|D(x_1, x_2)| \leq \frac{H_D(Sx_0, Jx_1) + |D(x_0, x_1)|}{2}. \tag{5.3}$$

If $D(x_1, x_2) = 0$, then $x_1 = x_2$ and

$$H_D(Jx_1, Sx_2) \leq \varphi(D(x_1, x_2)) D(x_1, x_2) = 0,$$

and so $Jx_1 = Sx_2$. That is, $x_2 = x_1 \in Jx_1 = Sx_2 = Sx_2$ and we finished.

Assume that $D(x_1, x_2) > 0$. Again By Lemma 1.3.16, we can take $x_3 \in Sx_2$ s.t.

$$|D(x_2, x_3)| \leq \frac{H_D(Jx_1, Sx_2) + |D(x_1, x_2)|}{2}. \tag{5.4}$$

By repeating this process, we can construct a sequence x_n of points in X and a sequence A_n of elements in $CB^D(X)$ s.t,

$$x_{j+1} \in A_j = \begin{cases} Sx_j, & j = 2k, k \geq 0 \\ Jx_j, & j = 2k+1, \ k \geq 0 \end{cases}, \tag{5.5}$$

and

$$|D(x_j, x_{j+1})| \leq \frac{H_D(A_{j-1}, A_j) + |D(x_{j-1}, x_j)|}{2}, \tag{5.6}$$

with $j \geq 0$, along with the assumption that $D(x_j, x_{j+1}) > 0$ for each $j \geq 0$. Now for

$j = 2k+1$, we have

$$\begin{aligned}
|D(x_j, x_{j+1})| &\leq \frac{H_D(A_{j-1}, A_j) + |D(x_{j-1}, x_j)|}{2} \\
&\leq \frac{H_D(Sx_{2k}, Jx_{2k+1}) + |D(x_{2k}, x_{2k+1})|}{2} \\
&\leq \frac{\varphi(D(x_{2k}, x_{2k+1}))(D(x_{2k}, x_{2k+1}) + |D(x_{2k}, x_{2k+1})|}{2} \\
&\leq \left(\frac{\varphi(D(x_{j-1}, x_j)) + 1}{2}\right)|D(x_{j-1}, x_j)| \\
&\leq D(x_{j-1}, x_j).
\end{aligned}$$

Similarly for $j = 2k+2$, we obtain

$$\begin{aligned}
|D(x_j, x_{j+1})| &\leq \frac{H_D(Jx_{2k+1}, Sx_{2k+2}) + |D(x_{j-1}, x_j)|}{2} \\
&\leq \left(\frac{\varphi(D(x_{j-1}, x_j)) + 1}{2}\right)|D(x_{j-1}, x_j)| \\
&\leq D(x_{j-1}, x_j).
\end{aligned}$$

It follows that the sequence $\{D(x_n, x_{n+1})\}$ is decreasing and converges to a nonnegative real number $t \geq 0$. Define a function $\psi : [0, \infty) \longrightarrow [0, 1)$ as follows:

$$\psi(\zeta) = \frac{\varphi(\zeta) + 1}{2}.$$

Then

$$\limsup_{\zeta \to t^+} \psi(\zeta) < 1.$$

Using Proposition 1.3.18, for $t \geq 0$, we can find $\delta(t) > 0$, $\lambda_t < 1$, s.t. $t \leq r \leq \delta(t) + t$ implies $\psi(r) < \lambda_t$ and there occurs a natural number N s.t, $t \leq D(x_n, x_{n+1}) \leq \delta(t) + t$, when ever $n > N$. Hence

$$\psi(D(x_n, x_{n+1})) < \lambda_t, \text{ whenever } n > N.$$

Then for $n = 1, 2, 3, \cdots$

$$\begin{aligned} |D(x_n, x_{n+1})| &\leq \left(\frac{\varphi(D(x_{n-1}, x_n)) + 1}{2}\right) |D(x_{n-1}, x_n)| \\ &\leq \psi(D(x_{n-1}, x_n)) |D(x_{n-1}, x_n)| \\ &\leq \max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\} |D(x_{n-1}, x_n)| \\ &\leq \left[\max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\}\right]^2 |D(x_{n-2}, x_{n-1})| \\ &\leq \left[\max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\}\right]^n |D(x_0, x_1)|. \end{aligned}$$

Put $\max\left\{\max_{n=1}^{N} \psi(D(x_{n-1}, x_n)), \lambda_t\right\} = \Phi$, then $\Phi < 1$.

$$|D(x_n, x_{n+1})| \leq \Phi^n |D(x_0, x_1)|. \tag{5.7}$$

Also we can deduce from the contraction that

$$|D(x_n, x_n)| \leq 2\Phi^{n-1} |D(x_0, x_1)|. \tag{5.8}$$

To prove that $\{x_n\}$ is a C-seq in (X, D), we will prove that $\{x_n\}$ is a C-seq in (X, d_p^s).

Since

$$d_p(x, y) = D(x, y) - D(x, x),$$

$$\begin{aligned} d_p(x_n, x_{n+1}) &= D(x_n, x_{n+1}) - D(x_n, x_n). \\ d_p(x_n, x_{n+1}) + D(x_n, x_n) &= D(x_n, x_{n+1}) \\ &\leq |D(x_n, x_{n+1})|. \end{aligned}$$

By (5.7), we have

$$\begin{aligned} d_p(x_n, x_{n+1}) + D(x_n, x_n) &\leq \Phi^n |D(x_0, x_1)|. \\ d_p(x_n, x_{n+1}) &\leq \Phi^n |D(x_0, x_1)| - D(x_n, x_n) \\ &\leq \Phi^n |D(x_0, x_1)| + |D(x_n, x_n)|. \end{aligned}$$

By using (5.8), we have

$$d_p(x_n, x_{n+1}) \leq \Phi^n |D(x_0, x_1)| + 2\Phi^{n-1} |D(x_0, x_1)|.$$

This implies that

$$d_p(x_n, x_{n+1}) \leq \Phi^n (3 - 2\varphi) |D(x_0, x_1)|. \tag{5.9}$$

and

$$d_p(x_{n+1}, x_{n+2}) \leq \Phi^{n+1} (3 - 2\varphi) |D(x_0, x_1)|. \tag{5.10}$$

Continuing in the same way, we have

$$d_p(x_{n+\gamma-1}, x_{n+\gamma}) \leq \Phi^{n+\gamma-1} (3 - 2\varphi) |D(x_0, x_1)|. \tag{5.11}$$

Now using the triangular inequality and equations (5.10)-(5.11), we have

$$\begin{aligned}
d_p(x_n, x_{n+\gamma}) &\leq d_p(x_n, x_{n+1}) + d_p(x_{n+1}, x_{n+2}) + \cdots + d_p(x_{n+\gamma-1}, x_{n+\gamma}) \\
&\leq \Phi^n (3 - 2\varphi) |D(x_0, x_1)| + \Phi^{n+1}(3 - 2\varphi) \ |D(x_0, x_1)| + \cdots + \\
&\quad \lambda^{n+\gamma-1} (3 - 2\varphi) |D(x_0, x_1)| \\
&\leq \frac{\Phi^n}{1 - \Phi} (3 - 2\varphi) |D(x_0, x_1)|.
\end{aligned}$$

Similarly, we can conclude that

$$d_p(x_{n+\gamma}, x_n) \leq \frac{\Phi^n}{1 - \Phi} (3 - 2\varphi) |D(x_0, x_1)|.$$

Now taking limit as $n \to \infty$ of last two inequalities, we obtain that

$$\lim_{n \to \infty} d_p(x_n, x_{n+\gamma}) = 0 = \lim_{n \to \infty} d_p(x_{n+\gamma}, x_n).$$

This implies

$$\lim_{n \to \infty} d_p^s(x_n, x_{n+\gamma}) = 0.$$

This implies that $\{x_n\}$ is a C-seq in (X, d_p^s). Since (X, d_p^s) is a complete metric space, there occurs $z \in X$ s.t, $x_n \longrightarrow z$ as $n \to \infty$. i.e.,

$$\lim_{n\to\infty} d_p^s(x_n, z) = 0.$$

Now from Lemma 1.3.12, we have $\lim_{n\to\infty} d_p^s(x_n, z) = 0$ iff

$$D(z,z) = \lim_{n\to\infty} D(x_n, z) = \lim_{n,m\to\infty} D(x_n, x_m).$$

So

$$\begin{aligned} \lim_{n,m\to\infty} d_p(x_n, x_m) &= 0, \\ \lim_{n,m\to\infty} [D(x_n, x_m) - D(x_n, x_n)] &= 0, \\ \lim_{n,m\to\infty} D(x_n, x_m) &= \lim_{n,m\to\infty} D(x_n, x_n). \end{aligned}$$

But (5.8) implies that

$$\lim_{n,m\to\infty} D(x_n, x_n) = 0.$$

It follows directly that

$$\lim_{n,m\to\infty} D(x_n, x_m) = 0.$$

This implies that

$$D(z,z) = \lim_{n\to\infty} D(x_n, z) = \lim_{n\to\infty} D(x_n, x_n) = 0. \tag{5.12}$$

Now, by (5.12), we have

$$\begin{aligned} d_p(z, Jz) &= D(z, Jz) - D(z,z) \\ &= D(z, Jz). \end{aligned} \tag{5.13}$$

So

$$D(z, Jz) \geq 0.$$

Now from ($P_{1.3.14}$) and (5.2), we get

$$
\begin{aligned}
D(Sz,z) &\leq D(Sz,x_{2n+2}) + D(x_{2n+2},z) - D(x_{2n+2},x_{2n+2}) \\
&\leq D(x_{2n+2},Sz) + D(x_{2n+2},z) + |D(x_{2n+2},x_{2n+2})| \\
&\leq \sup_{u\in Jx_{2n+1}} D(u,Sz) + D(x_{2n+2},z) + |D(x_{2n+2},x_{2n+2})| \\
&\leq \delta_D(Jx_{2n+1},Sz) + D(x_{2n+2},z) + |D(x_{2n+2},x_{2n+2})| \\
&\leq H_D(Jx_{2n+1},Sz) + D(x_{2n+2},z) + |D(x_{2n+2},x_{2n+2})| \\
&\leq \varphi(D(x_{2n+1},z))\, D(x_{2n+1},z) + D(x_{2n+2},z) + |D(x_{2n+2},x_{2n+2})| \\
&\leq D(x_{2n+1},z) + D(x_{2n+2},z) + |D(x_{2n+2},x_{2n+2})|.
\end{aligned}
$$

Taking limit as $n \longrightarrow \infty$, we get

$$
D(Sz,z) = 0. \tag{5.14}
$$

Thus from (5.12) and (5.14), we get

$D(z,z) = D(Sz,z)$.

Thus by Remark 1.3.15, we get that $z \in Sz$. It follows similarly that $z \in Jz$. This completes the proof of the theorem.

5.1.3 Example

Presume that $X = \mathbb{R}$ and $D(x,y) = \frac{1}{4}|x-y| + \frac{1}{2}\max\{x,y\}$, $\forall\ x,y \in X$. Note that if d_p is quasi metric on X, then $d_p^s(x,y) = \max\{d_p(x,y), d_p(y,x)\}$ is metric on X. Hence, $d_p^s(x,y) = |x-y|$ and so (X, d_p^s) is a complete metric space. Also define mappings $S, J : X \longrightarrow CB^D(X)$ by

$Sx = \overline{B\left(0,\frac{x}{4}\right)}$, $\quad Jy = \overline{B\left(0,\frac{x}{3}\right)}$.

Then

$H_D\left(\overline{B\left(0,\frac{x}{4}\right)}, \overline{B\left(0,\frac{x}{3}\right)}\right) = \max\left[\frac{x}{4},\frac{x}{3}\right]$ and

$$
\begin{aligned}
H_D(Sx,Jy) &= \max\left[\frac{x}{4},\frac{x}{3}\right] \\
&\leq \frac{1}{12}\max\{x,y\} \leq kD(x,y).
\end{aligned}
$$

Therefore, for $\varphi(t) = \frac{1}{12}$, all the conditions of Theorem 5.1.2 are satisfied. Also it is clear that $\forall\ x \in X$, the set Sx and Jx are bounded and closed with respect to the topology $\tau(D) = \tau(d_p)$. Hence, we can show that (5.2) holds $\forall\ x, y \in X$. i.e.,

$$H_D(Sx, Jy) = H_D\left(0, \left[\frac{y}{4}, \frac{y}{3}\right]\right) = \frac{y}{4}.$$

Now we deduce the result for single-valued self-mappings from Theorem 5.1.2.

5.1.4 Theorem

Assume that (X, d) is a complete DPMS, $S, J : X \to X$ be two self mappings and $\varphi : [0, +\infty) \longrightarrow [0, 1)$ be an $MJ-$function. Assume that

$$D(Sx, Jy) \leq \varphi(D(x, y))\, D(x, y)$$

$\forall\ x, y \in X$. Then S and J have a common FP.

5.1.5 Corollary

Suppose that (X, d) is a complete DPMS, $S, J : X \to CB^D(X)$ be multivalued mappings satisfying the following condition

$$H_D(Sx, Jy) \leq kD(x, y)$$

$\forall\ x, y \in X$, and $k \in [0, 1)$, then S and J have a common FP.

5.2 Common FPs of generalized rational contractive mappings

In this section, we will prove the occurence of common FPs of two self mappings involving rational expressions in dislocated metric space.

Results given in this section have been published in [60].

5.2.1 Theorem

Assume that (X,d) is a complete dislocated metric space and let the mappings $S, T : X \to X$ satisfy:

$$\begin{aligned} d_l(Sj,Tk) &\leq a_1\, d_l(j,k) + a_2 \frac{d_l(j,Sj).d_l(k,Tk)}{d_l(j,k)} + a_3 \frac{d_l(j,Tk).d_l(k,Sj)}{d_l(j,k)} + \\ & \quad a_4 \frac{d_l(j,Sj)\, d_l(k,Tk)}{d_l(j,Tk)+d_l(j,k)+d_l(k,Sj)} \end{aligned} \tag{5.15}$$

$\forall\, j,k \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + a_3 - a_4 < 1$. Then S, T have a unique common FP.

Proof: Suppose that j_0 be an erratic point in X and define $j_1 = Sj_0$ and $j_2 = Tj_1$ s.t,

$d_l(j_1,j_2) = d_l(Sj_0,Tj_1)$.

Then

$$\begin{aligned} d_l(j_1,j_2) &\leq a_1 d_l(j_0,j_1) + a_2 \frac{d_l(j_0,Sj_0).d_l(j_1,Tj_1)}{d_l(j_0,j_1)} + a_3 \frac{d_l(j_0,Tj_1).d_l(j_1,Sj_0)}{d_l(j_0,j_1)} + \\ & \quad a_4 \frac{d_l(j_0,Sj_0)\, d_l(j_1,Tj_1)}{d_l(j_0,Tj_1)+d_l(j_0,j_1)+d_l(j_1,Sj_0)} \\ &\leq a_1 d_l(j_0,j_1) + a_2 \frac{d_l(j_0,j_1).d_l(j_1,j_2)}{d_l(j_0,j_1)} + a_3 \frac{d_l(j_0,j_2).d_l(j_1,j_1)}{d_l(j_0,j_1)} + \\ & \quad a_4 \frac{d_l(j_0,j_1)\, d_l(j_1,j_2)}{d_l(j_0,j_2)+d_l(j_0,j_1)+d_l(j_1,j_1)} \\ &\leq a_1 d_l(j_0,j_1) + a_2 d_l(j_1,j_2) + a_4 \frac{d_l(j_0,j_1)\, d_l(j_1,j_2)}{d_l(j_0,j_2)+d_l(j_0,j_1)}. \end{aligned}$$

As (owing to triangular inequality),

$$d_l(j_1,j_2) < a_1 d_l(j_0,j_1) + a_2 d_l(j_1,j_2) + a_4 \frac{d_l(j_0,j_1)\, d_l(j_1,j_2)}{d_l(j_0,j_2)+d_l(j_0,j_1)},$$

where

$$d_l(j_1,j_2) \leq d_l(j_1,j_0) + d_l(j_0,j_2).$$

Hence

$$\begin{aligned} d_l(j_1, j_2) &< \left(\frac{a_1 + a_4}{1 - a_2}\right) |d_l(j_0, j_1)| \\ &< \lambda d_l(j_0, j_1), \end{aligned}$$

where $\lambda = \frac{a_1+a_4}{1-a_2}$. Similarly, by repeating the same process for

$$d_l(j_2, j_3) = d_l(Tj_1, Sj_2) = d_l(Sj_2, Tj_1)$$

we get

$$|d_l(j_2, j_3)| < \lambda^2 |d_l(j_0, j_1)|.$$

Consequently, we get

$$\begin{aligned} |d_l(j_{2n+1}, j_{2n+2})| &< \lambda d_l(j_{2n}, j_{2n+1}) \\ &< \lambda^2 d_l(j_{2n-1}, j_{2n}) \\ &< \lambda^{2n+1} d_l(j_0, j_1). \end{aligned}$$

Hence for any $m > n$,

$$\begin{aligned} d_l(j_n, j_m) &< d_l(j_n, j_{n+1}) + d_l(j_{n+1}, j_{n+2}) + \cdots + d_l(j_{m-1}, j_m) \\ &< \left(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right) d_l(j_0, j_1) \\ &< \frac{\lambda^n}{1-\lambda} d_l(j_0, j_1), \end{aligned}$$

and

$$d_l(j_n, j_m) < \frac{\lambda^n}{1-\lambda} d_l(j_0, j_1)$$

$$\longrightarrow 0, \text{ as } m, n \longrightarrow \infty.$$

This implies that $\{j_n\}$ is a C-seq. Since X is complete, there occurs $u \in X$ s.t. $j_n \longrightarrow u$. It follows that $u = Su$, otherwise $d(u, Su) = z > 0$ and we would then have

$$\begin{aligned}
d_l(u, Su) &\leq d_l(u, j_{2n+2}) + d_l(j_{2n+2}, Su) \\
d_l(u, Su) &\leq d_l(u, j_{2n+2}) + d_l(Tj_{2n+1}, Su) \\
d_l(u, Su) &\leq d_l(u, j_{2n+2}) + d_l(Su, Tj_{2n+1}) \\
d_l(u, Su) &\leq d_l(u, j_{2n+2}) + a_1 d_l(u, j_{2n+1}) + a_2 \frac{d_l(u, Su) . d_l(j_{2n+1}, Tj_{2n+1})}{d_l(u, j_{2n+1})} + \\
&\quad a_3 \frac{d_l(u, Tj_{2n+1}) . d_l(j_{2n+1}, Su)}{d_l(u, j_{2n+1})} + a_4 \frac{d_l(u, Su) d_l(j_{2n+1}, Tj_{2n+1})}{d_l(u, Tj_{2n+1}) + d_l(u, j_{2n+1}) + d_l(j_{2n+1}, Su)} \\
&\leq d_l(u, j_{2n+2}) + a_1 d_l(u, j_{2n+1}) + a_2 \frac{d_l(u, Su) . d_l(j_{2n+1}, j_{2n+2})}{d_l(u, j_{2n+1})} + \\
&\quad a_3 \frac{d_l(u, j_{2n+2}) . d_l(j_{2n+1}, Su)}{d_l(u, j_{2n+1})} + a_4 \frac{d_l(u, Su) d_l(j_{2n+1}, j_{2n+2})}{d_l(u, j_{2n+2}) + d_l(u, j_{2n+1}) + d_l(j_{2n+1}, Su)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
z &\leq d_l(u, j_{2n+2}) + a_1 \left| d_l(u, j_{2n+1}) \right| + a_2 \frac{z . d_l(j_{2n+1}, j_{2n+2})}{d_l(u, j_{2n+1})} + a_3 \frac{d_l(u, j_{2n+2}) . d_l(j_{2n+1}, Su)}{d_l(u, j_{2n+1})} + \\
&\quad a_4 \frac{z d_l(j_{2n+1}, j_{2n+2})}{d_l(u, j_{2n+2}) + d_l(u, j_{2n+1}) + d_l(j_{2n+1}, Su)},
\end{aligned}$$

which on making $n \to \infty$ gives rise to $d_l(u, Su) = 0$, which is a contradiction so that $u = Su$. Similarly, one can show that $u = Tu$ and its uniqueness.

5.2.2 Example

Suppose that $X = [0, 1]$ be a dislocate metirc space $d_l : X \times X \to X$ defined by

$$d_l(j, k) = \frac{j}{2} + \frac{k}{2}.$$

Suppose that $S : X \to X$ be defined by

$$Sj = \left\{ \frac{2j}{3} \right\}, j \in X.$$

And $T : X \to X$ is defined by

$$Tk = \left\{\frac{4k}{3}\right\}, k \in X.$$

Now

$$d_l(Sj, Sk) = \frac{j}{3} + \frac{2k}{3}.$$

Take $j = \frac{1}{2}$ and $k = \frac{1}{3}$ then we have

$$d_l(Sj, Sk) = \frac{j}{3} + \frac{2k}{3} = \frac{1}{6} + \frac{2}{9} = \frac{5}{18} = 0.2070.$$

Now by using the contractive condition we have

$$\begin{aligned} d_l(Sj, Tk) \leq \;& a_1\, d_l(j,k) + a_2 \frac{d_l\,(j, Sj)\,.d_l\,(k, Tk)}{d_l\,(j,k)} + a_3 \frac{d_l\,(j, Tk)\,.d_l\,(k, Sj)}{d_l\,(j,k)} + \\ & a_4 \frac{d_l\,(j, Sj)\, d_l\,(k, Tk)}{d_l\,(j, Tk) + d_l\,(j,k) + d_l\,(k, Sj)}. \end{aligned}$$

As given that $a_1 + a_2 + a_3 + a_4 < 1$. Select $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{4}$ $a_3 = \frac{1}{5}$ and $a_4 = \frac{1}{7}$, then clearly $a_1 + a_2 + a_3 + a_4 < 1$. Now putting $j = \frac{1}{2}$ and $k = \frac{1}{3}$,we get

$$\begin{aligned} \frac{5}{18} &\leq \frac{1}{3}(\frac{1}{4} + \frac{1}{6}) + \frac{1}{4}\frac{(\frac{1}{4} + \frac{1}{6}).(\frac{1}{6} + \frac{2}{9})}{(\frac{1}{4} + \frac{1}{6})} + \frac{1}{5}\frac{(\frac{1}{4} + \frac{2}{9}) + (\frac{1}{6} + \frac{1}{6})}{(\frac{1}{4} + \frac{1}{6})} + \frac{1}{7}\frac{(\frac{1}{4} + \frac{1}{6}).(\frac{1}{6} + \frac{2}{9})}{(\frac{1}{4} + \frac{2}{9}) + (\frac{1}{4} + \frac{1}{6}) + (\frac{1}{6} + \frac{1}{6})}, \\ 0.207 &\leq 0.1387 + 0.0972 + 0.2266 + 0.009044, \\ 0.207 &\leq 0.4715. \end{aligned}$$

Hence, all the contractive conditions of Theorem 5.2.1 are satisfied.

5.2.3 Corollary

Presume that (X, d_l) is a complete dislocated metric space and let the mappings $S, T : X \to X$ satisfy:

$$\begin{aligned} d_l(Sj, Tk) \leq \;& a_1\, d_l(j,k) + a_2 \frac{d_l\,(j, Sj)\,.d_l\,(k, Tk)}{d_l\,(j,k)} + \\ & a_3 \frac{d_l\,(j, Sj)\, d_l\,(k, Tk)}{d_l\,(j, Tk) + d_l\,(j,k) + d_l\,(k, Sj)} \end{aligned}$$

$\forall\ j, k \in X$, where a_1, a_2, a_3 are nonnegative reals with $a_1 + a_2 + a_3 < 1$. Then S, T have a unique common FP.

Proof: By putting $a_3 = 0$ in Theorem 5.2.1, we get the required result.

5.2.4 Theorem

Suppose that (X, d_l) is a complete dislocated metric space and let the mappings $S, T : X \to X$ satisfy:

$$d_l(S(j), T(k)) \leq a\ d_l(j, k) + b\frac{d_l(j, S(j))\, d_l(k, T(k))}{1 + d_l(j, k)}$$

$\forall\ j, k \in X$, where a, b are nonnegative reals with $a + b < 1$. Then S, T have a unique common FP.

Proof: Suppose that j_0 be an erratic point in X and define $j_1 = S(j_0)$ and $j_2 = T(j_1)$ s.t.

$$d_l(j_1, j_2) = d_l(S(j_0), T(j_1)).$$

Then

$$\begin{aligned} d_l(j_1, j_2) &\leq a d_l(j_0, j_1) + b\frac{d_l(j_0, S(j_0))\, d_l(j_1, T(j_1))}{1 + d_l(j_0, j_1)} \\ &\leq a d_l(j_0, j_1) + b\frac{d_l(j_0, j_1)\, d_l(j_1, j_2)}{1 + d_l(j_0, j_1)} \\ &\leq a d_l(j_0, j_1) + b d_l(j_1, j_2)\left(\frac{d_l(j_0, j_1)}{1 + d_l(j_0, j_1)}\right) \\ &\leq a d_l(j_0, j_1) + b d_l(j_1, j_2). \end{aligned}$$

This implies that

$$\begin{aligned} d_l(j_1, j_2) &\leq \left(\frac{a}{1-b}\right) d_l(j_0, j_1) \\ &\leq \lambda d_l(j_0, j_1). \end{aligned} \tag{5.16}$$

Similarly,

$$d_l(j_2, j_3) = d_l(j_3, j_2) = d_l(S(j_2), T(j_1)),$$

$$
\begin{aligned}
d_l\left(S\left(j_2\right), T\left(j_1\right)\right) &\leq a d_l\left(j_2, j_1\right)+b \frac{d_l\left(j_2, S\left(j_2\right)\right) d_l\left(j_1, T\left(j_1\right)\right)}{1+d_l\left(j_2, j_1\right)} \\
&\leq a d_l\left(j_2, j_1\right)+b \frac{d_l\left(j_2, j_3\right) d_l\left(j_1, j_2\right)}{1+d_l\left(j_2, j_1\right)} \\
&\leq a d_l\left(j_2, j_1\right)+b d_l\left(j_2, j_3\right)\left(\frac{d_l\left(j_1, j_2\right)}{1+d_l\left(j_2, j_1\right)}\right) \\
d_l\left(j_2, j_3\right) &\leq a d_l\left(j_1, j_2\right)+b d_l\left(j_2, j_3\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d_l\left(j_2, j_3\right) &\leq \left(\frac{a}{1-b}\right) d_l\left(j_1, j_2\right) \\
&\leq \lambda . \lambda d_l\left(j_0, j_1\right) \\
&\leq \lambda^2 d_l\left(j_0, j_1\right).
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
d_l\left(j_n, j_{n+1}\right) &\leq \lambda d_l\left(j_{n-1}, j_n\right) \\
&\leq \lambda^2 d_l\left(j_{n-2}, j_{n-1}\right) \\
&\vdots \\
&\leq \lambda^n d_l\left(j_0, j_1\right).
\end{aligned}
$$

To prove that $\{j_n\}$ is a C-seq, we have for any $m > n$,

$$
\begin{aligned}
d_l\left(j_n, j_m\right) &\leq d_l\left(j_n, j_{n+1}\right)+d_l\left(j_{n+1}, j_{n+2}\right)+\cdots+d_l\left(j_{m-1}, j_m\right) \\
&\leq \lambda^n d_l\left(j_0, j_1\right)+\lambda^{n+1} d_l\left(j_0, j_1\right)+\cdots+ \\
&\quad \lambda^{m-1} d_l\left(j_0, j_1\right) \\
&\leq \left(\lambda^n+\lambda^{n+1}+\cdots+\lambda^{m-1}\right) d_l\left(j_0, j_1\right) \\
&\leq \left(\frac{\lambda^n}{1-\lambda}\right) d_l\left(j_0, j_1\right) \\
&\longrightarrow 0 \text{ as } m, n \longrightarrow \infty.
\end{aligned}
$$

Hence $\{j_n\}$ is a C-seq. Since X is complete, for any $u \in X$ s.t. $j_n \longrightarrow u$ and suppose $\theta = d_l(u, Su)$. Therefore we have

$$\begin{aligned} d_l(u, Su) &\leq d_l(u, j_{2n+2}) + d_l(j_{2n+2}, Su) \\ &= d_l(u, j_{2n+2}) + d_l(T(j_{2n+1}), Su) \\ &= d_l(u, j_{2n+2}) + d_l(Su, T(j_{2n+1})) \end{aligned}$$

$$\begin{aligned} &\leq d_l(j_{2n+2}, u) + ad_l(u, j_{2n+1}) + b\frac{d_l(u, Su)\, d_l(j_{2n+1}, T(j_{2n+1}))}{1 + d_l(u, j_{2n+1})} \\ &\leq d_l(j_{2n+2}, u) + ad_l(u, j_{2n+1}) + b\frac{d_l(u, Su)\, d_l(j_{2n+1}, j_{2n+2})}{1 + d_l(u, j_{2n+1})} \\ \theta &\leq d_l(u, j_{2n+2}) + ad_l(u, j_{2n+1}) + b\frac{\theta + d_l(j_{2n+1}, j_{2n+2})}{1 + d_l(u, j_{2n+1})}. \end{aligned}$$

Putting $n \to \infty$, and $j_n \longrightarrow u$ we get,

$$\begin{aligned} (1-b)\theta &\leq 0, \\ (1-b) &\neq 0, \\ \theta &= d_l(u, Su) = 0. \end{aligned}$$

which implies that $u = Su$. It follows similarly that $u = Tu$. Now, we show that S and T have a unique common FP. For this, assume that v in X is a second common FP of S and T. Then

$$d_l(u, v) = d_l(Su, Tv)$$

$$\begin{aligned} &\leq a\ d_l(u, v) + b\frac{d_l(u, Su)\, d_l(v, Tv)}{1 + d_l(u, v)} \\ &\leq a\ d_l(u, v). \end{aligned}$$

This implies that

$$\begin{aligned} (1-a)\, d_l(u,v) &\le 0, \\ 1-a &\neq 0, \\ d_l(u,v) &= 0. \end{aligned}$$

This implies that $u = v$, completing the proof of the theorem.

5.3 Existence of a common solution for a system of integral equations

In this section, we show that Theorem 5.2.4 can be applied to the occurance of a common solution of the system of the integral equations.

5.3.1 Theorem [60]

Presume that $X = C([a,b], R)$, where $b > a \ge 0$ and $d_l : X \times X \to R$ be defined by

$$d_l(j,k) = \max_{t\in[a,b]} \|j(t) - k(t)\|_\infty \sqrt{1+a^2}e^{\cot^{-1} a}.$$

Consider the following system of integral equations:

$$\begin{aligned} j(t) &= \int_a^b k_1(t,r,j(r))\, dr + g(t), \\ j(t) &= \int_a^b k_2(t,r,j(r))\, dr + h(t), \end{aligned} \tag{5.17}$$

where, $X = C[a,b]$, $t \in [a,b] \subset R$ and $j, g, h \in X$.

Suppose that k_1, $k_2 : [a,b] \times [a,b] \times R \to R$ are continuous and s.t.

$$F_j(t) = \int_a^b k_1(t,r,j(r))\, dr \tag{5.18}$$

and

$$G_j(t) = \int_a^b k_2(t, r, j(r))\, dr \tag{5.19}$$

$\forall\ j \in X$ and $\forall\ t \in [a, b]$. Then the occurence of a solution to (5.17) is equaivalent to the occurence of common FP of S and T.

Consider

$$\|F_j(t) - G_k(t) + g(t) - h(t)\|_\infty \sqrt{1+a^2} e^{\cot^{-1} a} \leq A(j, k)(t) + B(j, k)(t),$$

where

$$A(j, k)(t) = \|j(t) - k(t)\|_\infty \sqrt{1 - a^2} e^{\cot^{-1} a}$$

and

$$B(j, k)(t) = \frac{\|F_j(t) + g(t) - j(t)\|_\infty \|G_k(t) + h(t) - y(t)\|_\infty}{1 + d(j, k)} \sqrt{1 + a^2} e^{\cot^{-1} a}.$$

Then the system of integral equations (5.18) and (5.19) has a unique common solution.

Proof: It is easily to check that (X, d_l) is a dislocated metric space. Define two mappings $S, T : X \times X \to X$ by $S_j = F_j + g$ and $T_j = G_j + h$. Then

$$d(S(j), T(k)) = \max_{t \in [a,b]} \|F_j(t) - G_k(t) + g(t) - h(t)\|_\infty \sqrt{1 + a^2} e^{\cot^{-1} a}.$$

$$d(j, S(j)) = \max_{t \in [a,b]} \|F_j(t) + g(t) - j(t)\|_\infty \sqrt{1 + a^2} e^{\cot^{-1} a}.$$

and

$$d(k, T(k)) = \max_{t \in [a,b]} \|G_k(t) + h(t) - k(t)\|_\infty \sqrt{1 + a^2} e^{\cot^{-1} a}.$$

Thus by Theorem 5.2.4, we get S and T have a common FP. Thus there occurs a unique point $v \in X$ s.t. $v = Sv = Tv$. Now, we have

$j = S(j) = F_j + g$

and

$j = T(j) = G_j + h,$

i.e.,

$$j(t) = \int_a^b k_1(t, r, j(r))\, dr + g(t),$$

and

$$j(t) = \int_a^b k_2(t, r, j(r))\, dr + h(t).$$

Therefore, we can conclude that the system of integral equations (5.17) has a unique common FP.

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