Generalizations of Neutrosophic Sets with Applications in Decision Making

By

Qaisar Khan

Reg. # 46-FBAS/PhDMA/F14

DEPARTMENT OF MATHEMATICS & STATISTICS

INTERNATIONAL ISLAMIC UNIVERSITY

ISLAMABAD, PAKISTAN 2018

Generalizations of Neutrosophic Sets with Applications in Decision Making

By

Qaisar Khan

Reg. # 46-FBAS/PhDMA/F14

Supervised By

Dr. Tahir Mahmood

DEPARTMENT OF MATHEMATICS & STATISTICS

INTERNATIONAL ISLAMIC UNIVERSITY

ISLAMABAD, PAKISTAN

2018

Generalizations of Neutrosophic Sets with Applications in Decision Making

By

Qaisar Khan

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF ***PhD IN MATHEMATICS*** AT THE DEPARTMENT OF MATHEMATICS & STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY ISLAMABAD.

Supervised By

Dr. Tahir Mahmood

DEPARTMENT OF MATHEMATICS & STATISTICS

INTERNATIONAL ISLAMIC UNIVERSITY

ISLAMABAD, PAKISTAN

2018

<u>Author's Declaration</u>

I, **Qaisar Khan** Reg. No. **46-FBAS/PHDMA/F14** hereby state that my Ph.D. thesis titled: **Generalizations of Neutrosophic Sets with Applications in Decision Making** is my own work and has not been submitted previously by me for taking any degree from this university, **International Islamic University, Sector H-10, Islamabad, Pakistan** or anywhere else in the country/world.

At any time if my statement is found to be incorrect even after my Graduation the university has the right to withdraw my Ph.D. degree.

Name of Student: ***(Qaisar Khan)***
Reg. No. **46-FBAS/PHDMA/F14**
Dated: **04/03/2019**

Plagiarism Undertaking

I solemnly declare that research work presented in the thesis titled: **Generalizations of Neutrosophic Sets with Applications in Decision Making** is solely my research work with no significant contribution from any other person. Small contribution/help wherever taken has been duly acknowledged and that complete thesis has been written by me.

I understand the zero tolerance policy of the HEC and University, **International Islamic University, Sector H-10, Islamabad, Pakistan** towards plagiarism. Therefore, I as an Author of the above titled thesis declare that no portion of my thesis has been plagiarized and any material used as reference is properly referred/cited.

I undertake that if I am found guilty of any formal plagiarism in the above titled thesis even after award of Ph.D. degree, the university reserves the rights to withdraw/revoke my Ph.D. degree and that HEC and the University has the right to publish my name on the HEC/University Website on which names of students are placed who submitted plagiarized thesis.

Student/Author Signature: ______________________________

Name: **(Qaisar Khan)**

Certificate of Approval

This is to certify that the research work presented in this thesis, entitled: **Generalizations of Neutrosophic Sets with Applications in Decision Making** was conducted by **Mr. Qaisar Khan,** Reg. No. **46-FBAS/PHDMA/F14** under the supervision of **Dr. Tahir Mahmood** no part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Department of Mathematics & Statistics, FBAS, IIU, Islamabad** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics, Department of Mathematics & Statistics, Faculty of Basic & Applied Science, International Islamic University, Sector H-10, Islamabad, Pakistan.**

Student Name: Qaisar Khan **Signature: ______________**

Examination Committee:

a) **External Examiner 1:**
Name/Designation/Office Address **Signature: ______________**

Prof. Dr. Muhammad Shabir
Professor of Mathematics,
Department of Mathematics,
QAU, Islamabad.

b) **External Examiner 2:**
Name/Designation/Office Address) **Signature: ______________**

Dr. Muhammad Ashiq
Associate Professor,
Department of Mathematics,
NUST, MCS, Rawalpindi.

c) **Internal Examiner:**
Name/Designation/Office Address) **Signature: ______________**

Prof. Dr. M. Arshad Zia
Professor

Supervisor Name:
Dr. Tahir Mahmood **Signature: ______________**

Name of Dean/HOD:
Prof. Dr. Muhammad Sajid, T.I **Signature: ______________**

DECLARATION

I, hereby declare that this thesis neither as a whole nor as a part there of has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor.

No portion of the work presented in this thesis has been submitted in support of an application for any degree or qualification of this or any other Institute of learning.

Qaisar Khan

PhD Mathematics

Reg. No 46-FBAS/PhDMA/F14

Department of Mathematics & Statistics

Faculty of Basic & Applied Sciences

International Islamic University, Islamabad, Pakistan.

DEDICATION

This work is dedicated

To

My Beloved Parents, my wife and sons, Valued Teacher

Dr. Tahir Mahmood, Professors of Department of Mathematics and Statistics, my friends and PhD fellow for Supporting and encouraging me throughout my PhD research work.

ACKNOWLEDGEMENTS

First and foremost, all praise to **'ALLAH'**, Lord of the world, the Almighty, and I find no words to thank **'ALLAH'**, who created me and all the universe around us. Who is the sole creator of all the things tinier than an electron to huge galaxies. I am thankful to the Prophet Muhammad (S.A.W.) whose teachings are a blessing for the whole mankind. May Allah guide us and the whole humanity to the right path.

Words of gratitude and appreciation do not always convey the depth of one's feelings, yet I wish to record my thanks to my most respected supervisor **Dr. Tahir Mahmood** who encouraged me very much in time, made me realize and enjoy the hard work. I am obliged to him for his able guidance, help, immense encouragement and limitless patience, which made massive task accomplished.

I pay my thanks to the whole faculty of Department of Mathematics and Statistics International Islamic University Islamabad. I also feel much pleasure in acknowledging nice company of my friends, class fellows who support and encourage me directly or indirectly in my research work.

In the end I want to pay my attribute to my parents whose love and guidance always gave me a ray of hope in the darkness of desperation. I am thankful to all of my family members specially my mother, wife and brothers for their support and well wishing, enormous love, support, encouragements and constant patience. I cannot forget their prayers for me throughout my life, without them this effort would have been nothing.

Qaisar Khan

(46-FBAS/PhDMA/F14)

0. Research Profile

1. Liu, P., Mahmood, T., & Khan, Q. (2018). Group decision making based on power Heronian aggregation operators under linguistic neutrosophic environment. *International Journal of Fuzzy Systems*, *20*(3), 970-985.
2. Liu, P., Khan, Q. and Mahmood, T. Application of Single-Valued Neutrosophic Power Muirhead Mean Operators to Multi-attribute Group Decision Making. (Submitted)
3. Khan, Q., Hassan, N., & Mahmood, T. (2018). Neutrosophic Cubic Power Muirhead Mean Operators with Uncertain Data for Multi-Attribute Decision-Making. *Symmetry*, *10*(10), 444.
4. Liu, P., Khan, Q. and Mahmood, T. Application of Interval Neutrosophic Power Hamy Mean Operators in MAGDM (Submitted).
5. Khan, Q., Liu, P., Mahmood, T., Smarandache, F., & Ullah, K. (2018). Some Interval Neutrosophic Dombi Power Bonferroni Mean Operators and Their Application in Multi–Attribute Decision–Making. *Symmetry*, *10*(10), 459.

6. Liu, P., Khan, Q., Ye, J., & Mahmood, T. Group Decision-Making Method Under Hesitant Interval Neutrosophic Uncertain Linguistic Environment. International Journal of Fuzzy Systems 20 (7), 2337-2353.
7. Liu, P., Khan, Q. and Mahmood, T. Multiple-Attribute Decision Making Based on Single-Valued Neutrosophic Schweizer-Sklar Prioritized aggregation Operators. (Accepted in press Cognitive Systems Research).

0.1 Literature review

Multiple-attribute decision making (MADM) and multiple attribute group decision making (MAGDM) are very important research topics in current decision making process. The main aim of MADM or MAGDM problems is to select the perfect alternative between limited alternatives conforming to the preference values specified by decision makers (DMs) with respect to the prescribed criteria. One of the difficulties in real MADM or MAGDM problems is how to express the attribute values in fuzzy and indeterminate decision making (DM) environments. Fuzzy set (FS) initially developed by Zadeh [1] is a better mechanism for describing and communicating uncertainty and vagueness. Since its initiation, FS has acquired a consequential attention from the scholars and they studied its practical and theoretical aspects. Various generalizations of FS have been proposed such as interval-valued FS (IVFS) [2], in which the truth-membership degree (TRD) is described by an interval value in the closed interval $[0,1]$, IFS [3], which can be expressed by the TRD and falsity-membership degree (FLD). Therefore, IFS can describe fuzziness and uncertainty more completely than FS. Atanassov and Gargov [4] further generalized IFS to interval valued IFS (IVIFS). However, neither FS nor IFS is capable to handle indeterminate and inconsistent information. For example, when we take a student opinion about the teaching skills of a professor with about 0.6 being the possibility that the teaching skills of a professor is good, 0.5 being the possibility that the teaching skills of the professor is not good and 0.3 is the possibility that he/she may not be sure about the teaching skill of the professor is not good or good. To handle such type of information, Smarandache [5] developed the notion of neutrosophic set (NS) in which a new component "indeterminacy-membership degree" (IMD) is added. NS described the uncertain information by TMD, IMD and FMD. These three functions are independent and are standard or non-standard subsets $]0^-,1^+[$. As the theory of NS has the IMD, therefore it can explain the uncertain information more accurately than FS and IFS and is more consistent with human natural feelings and judgement. But NS theory is hard to be utilized in real life problems expected to the constraint of non-standard subsets of $]0^-,1^+[$. To utilize NS in practical problems expediently, Wang et al. [6] developed the perception of single valued neutrosophic set (SVNS) which is subclass of NS by changing the $]0^-,1^+[$ into the $[0,1]$. SVNS was further generalized by Wang et al. [7] to developed interval neutrosophic set (INS) and Zhang et al. [8] developed various operational rules for interval neutrosophic numbers (INNs). Ye [9] developed the concept of simplified neutrosophic set (SNS), which was the extended form of SVNS and INS. Peng et al. [10] developed some improved operational rules for simplified neutrosophic numbers. Recently, Jun et al. [11], Ali et al. [12] developed the concept of neutrosophic cubic set (NCS), which is a hybrid structure that consist of SVNS and INS set. Jun et al. [13] further developed P-union, P-intersection, R-union, R-intersection and discussed some related properties. Zhang et al. [14] and Ye et al. [15] developed various operational rules for neutrosophic cubic numbers (NCNs) and they developed some aggregation operators which were further applied in MADM. Many researchers proposed distance and similarity measures [16-25], correlation coefficients [26-31], entropies [32-36] for these sets and applied them to various fields.

In actual decision making problems, there is an extensive arrangement of qualitative information which is simply articulated by linguistic variables (LVs). In addition, LVs can enlarge the reliability and flexibility of conventional decision models and they have been consistent with other theories in solving MADM or MAGDM problems [37, 38]. For instance, intuitionistic linguistic set (ILS) and interval valued intuitionistic uncertain linguistic sets (IVIULS) were proposed to solve MADM problems [39, 40]. The concept of ILSs and IVIULVs were further extended by Ye et al. [41-43] to IN uncertain linguistic set (INULS), IN linguistic set (INLS) and SVN linguistic set (SVNLS). Liu et al. [44] developed SVN uncertain linguistic set (SVNULs) and applied them to MADM problems. Hesitant fuzzy set (HFS) developed by Torre and Torre and Narukawa [45, 46] is another effective generalization of Zadeh's FS. Some authors combined linguistic variable with HFS and other extensions of HFS to develop some new hybrid structures and applied them to various fields [47-52].

Chen et al. [53] defined the notion of linguistic intuitionistic fuzzy numbers (LIFNs), and proposed various basic operational laws, score and accuracy functions, various aggregation operators, and are applied to MAGDM problems. After the introduction of LIFNs, many researchers classified various aggregation operators for LIFNs and they have been applied to various fields [54-56]. However, the shortcoming of LIFN is that it cannot handle inconsistent or indeterminate information. To overcome this shortcoming, Li et al. [57] proposed the concept of linguistic NSs (LNSs) and proposed various operational laws, score, certainty and accuracy functions, various aggregation operators and then applied these perceptions to MADM problems. LNS was further studied by Fang et al. [58, 59] and proposed some novel operational laws, score function, accuracy function, presented various aggregation operators and are applied to MAGDM problems. Obviously, LNS can deal with fuzzy, uncertain, inconsistent or indeterminate information by LVs and they are the generalizations of LV, LIFN and so on.

Aggregation operators (AOs) play a prevailing role in DM. subsequently, many authors developed distinct aggregation operators and their simplifications, such as power average (PA) operator [60], Bonferroni mean (BM) operator [61], Heronian mean (HrM) operator [62], Muirhead mean (MM) operator [63], Maclurin symmetric mean (MSM) operator [64], Hamy Mean (HM) operator [65] and so on. Definitely, different AOs have different functions. Some can remove the effect of awkward data given by prejudice DMs such as PA [60] operator developed by Yager has the capibality that it can aggregate the input information by giving the importance degree based on support degree among the input arguments, and attain this function. The PA operator was further extended by many researchers to deal with different fuzzy environments [66-71].

Some aggregation operator are competent to consider the interrelationship among two or more input arguments such as BM operator, HM operators, MSM operators and MM operators. All these AOs are extended into different environments such as BM operator was extended by Xu et al. [72] to deal with intuitionistic fuzzy information, Liu et al. [73] proposed various normalized BM for SVNNs and apply them to MADM. MSM operator was extended by Qin et al. [74] to deal with IF information and applied them to MADM. Wei et al. [75], Qin et al. [76] and Wang et al. [77] further generalized MM operator to pythagorean fuzzy environment, hesitant fuzzy environment and SVN linguistic environment and are applied to MADM. Recently, Liu et al. [78, 79] extended MM operator to deal with IF information and IN information and some advantages of MM operator over BM and HM operators were discussed, and applied them to the MAGDM and MADM.

Some authors developed some hybrid structures to take the full advantages of PA operator and other aggregation operators, such as BM, HM, MSM and MM operators such as He et al. [80] and Liu et al. [81] developed IF power Bonferroni mean and interval-valued IF power Bonferroni mean and applied these to MAGDM. Liu et al. [82] proposed interval-valued IF power HM operator and are applied to MAGDM. Recently, Liu et al. [83] proposed interval-valued IF power MSM operator and gave its application in MAGDM. Li et al. [84] introduced Pythagorean fuzzy power MM operator and applied these to MADM.

These existing AOs have not considered the situation in which the criteria have priority relationship among them. To solve this problem, Yager [85] developed prioritized aggregation (PrA) operator Moreover, Liu et al. [86] developed some Pr ordered weighted averaging/geometric operator to deal with neutrosophic information. Several other studies were conducted to extend PrA operators to some other fuzzy environment [87-92].

Some researchers developed AOs utilizing different T-norms (TN) and T-Conorm (TCN). For example, Ji et al. [93] combined PrA operators with BM operator and introduced some SVN prioritized BM operators by utilizing Frank operations. Wang et al. [94] developed some Frank Choquet Bonferroni mean operators of bipolar NSs and are applied to MAGDM. Recently, Wei et al. [95] proposed some PRA operators based on Dombi [96] TN and TCN and are applied to MADM. Several other AOs were developed by different authors on different TN and TCN in [97-105]

From the above stated AOs, most of the AOs for NS or SVNS are based on algebraic, Hamacher, Frank and Dombi operational laws, which are special cases of Archimedean TN (ATN) and TCN (ATCN). Certainly, ATN and ATCN are the extensions of many TNs and TCNs, which have various special cases selected to express the union and intersection of SVNS [106]. Schweizer-Sklar operations [107] are the special cases of ATN and ATCN. They are with a variable parameter, which makes them more pliable and better than the other operations. However, the majority of researchers mostly focused on the elementary theory and distinctiveness of Schweizer-Sklar TN (SSTN) and TCN (SSTCN) [108, 109]. Recently, Liu et al. [110] and Zhang [111] combined SS operations with IVIFS and IFS, and proposed power averaging/geometric operators along with weighted averaging operators for IVIFS and IFS respectively.

For better understanding of several concepts given in this thesis, the reader should have to study [112-121].

0. 2 Chapter wise Study

In chapter 1, some essential definitions of NS, SVNS, INSs, INULS, PA operator, PrA operator, Muirhead mean (MM) operator, BM operator, HM operators, HrM, operational laws of these sets, properties and associated theorems are given.

In chapter 2, the PA operator is combined with HrM operator and extended to process linguistic neutrosophic information, and presented the linguistic neutrosophic power Heronian AO, linguistic neutrosophic power weight Heronian AO. Further, some characteristics of these newly developed AOs are examined and various exacting cases are confered. A novel technique is developed based on these AOs for MAGDM. Lastly, an illustrative example was specified to exemplify the efficacy and compensation of the proposed method by contrasting with the existing methods.

In chapter 3, various newly AOs for aggregating SVN information and a novel technique for MAGDM are developed. To acquire full reward of MM operator and PA operator, the single-valued neutrosophic power Muirhead mean (SVNPMM) operator, its weighted form, single-valued neutrosophic power dual Muirhead mean (SVNPDMM) operator, its weighted form and discuss their basic properties along with particular cases with respect to the parameter vector. Moreover, based on the developed AOs, a novel approach to MAGDM problem is developed. Lastly, a numerical example is specified to explain the efficiency and practicality of the developed approach.

In chapter 4, some new aggregation operators for neutrosophic cubic numbers (NCNs), which is a fundamental member of NCS. Taking the advantages of MM operator and PA operator, we develop the power Muirhead mean (PMM) operator and examined it under NC information. Therefore, some new NC AOs, such as the NC power Muirhead mean operator, weighted NC power Muirhead mean operator, NC power dual Muirhead mean operator and weighted NC power dual Muirhead mean (WNCPDMM) operator are proposed and related properties of these proposed AOs are discussed. Furthermore, a novel MADM method initiated on the developed new aggregation operators. Lastly, a numerical example is given to show the effectiveness of the developed approach.

In chapter 5, the conventional HM operator is combined to the traditional PA operator in interval neutrosophic settings and presents the two novel IN AOs such as the IN power Hammy mean (INPHM) operator and its weighted form. Then, various preferable properties of the developed AOs are discussed. Moreover, based on these AOs, a new method for MAGDM is presented to deal with IN information. Lastly, an example is specified to explain the competence of the proposed method by comparing with other presented methods.

In chapter 6, some operational laws for INNs based on Dombi TN and TCN are developed. Several desirable characteristics of these operational rules are investigated. We extend PBM operator based on Dombi operations to developed IN Dombi PBM (INDPBM) operator, IN weighted Dombi PBM (INWDPBM) operator, IN Dombi power geometric Bonferroni mean (INDPGBM) operator, IN weighted Dombi power GBM (INWDPGBM) operator and discussed several properties of these aggregation operators. Then, we initiated a MADM method based on these AOs to deal with IN information. Lastly, a descriptive example is demonstrated to explain the competence and practicality of the developed MADM method.

In chapter 7, the notion of hesitant IN uncertain linguistic set (HINULS) and hesitant IN uncertain linguistic element (HINULE) are proposed, various basic operational laws, properties, the score, accuracy and certainty functions for HINULEs. Then, various AOs are presented to aggregate HINULEs. A group decision making based proposed AOs are initiated to handle MAGDM problems, in which criteria values acquire the form of HINULEs and there exist prioritized relations among the criteria. Lastly, a numerical example about investment alternatives is given to explain the efficiency of the proposed method.

In chapter 8, we enlarge SS TN and TCN to SVN numbers (SVNN) and gave the SS operational laws for SVNNs. Then, we merge PrA operator with SS operations, and develop the SVN Schweizer-Sklar prioritized weighted averaging (SVNSSPrA) operator, SVN Schweizer-Sklar prioritized ordered weighted averaging (SVNSSPrOWA) operator, SVN Schweizer-Sklar prioritized weighted geometric (SVNSSPrWG) operator, and SVN Schweizer-Sklar prioritized ordered weighted geometric (SVNSSPrOWG) operator. Moreover, we studied some useful characteristics of these proposed AOs and proposed two models on the basis of SVNSSPrWA and SVNSSPrWG operators. At the same time, we apply these two methods to deal with MADM problems under SVN information. Lastly, an illustrative example about talent introduction is specified to testify the effectiveness of the developed methods

Table of Contents

Chapter 1

Preliminaries

The aim of this chapter is to express various essential definitions of NS, SVNS, INS, INULS, PA operator, PrA operator, MM operators, BM, Hamy mean (HM) operators, HrM operator, linguistic scale function (LSF), operational laws of these sets, properties and related theorems.

1.1 Neutrosophic Sets and Their Generalizations

In this part, we present the definition of NSs and its generalizations such as SVNSs, INSs, NCSs, operational laws, score, accuracy and certainty functions of these sets and various related theorems are discussed.

1.1.1 Neutrosophic Sets

The idea of NS was first developed by Smarandache [5] from philospical point of view. The definition of NS is provided below. For deep understanding of the idea and notions of NS the reader should study [5].

1.1.1.1 Definition [5]

Let $\overline{\overline{UN}}$ be a universe set and $\overline{\overline{un}} \in \overline{\overline{UN}}$. Then, a NS $\overline{\overline{NE}}$ in $\overline{\overline{UN}}$ is expressed by,

$$\overline{\overline{NE}} = \left\{ \left\langle \overline{\overline{un}}, \overline{\overline{\Xi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right), \overline{\overline{\Psi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right), \overline{\overline{\Upsilon}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right) \right\rangle \overline{\overline{un}} \in \overline{\overline{UN}} \right\}, \tag{1.1}$$

where, $\overline{\overline{\Xi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right), \overline{\overline{\Psi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right)$ and $\overline{\overline{\Upsilon}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right)$ respectively expressing the truth-membership degree (TRD), indeterminacy-membership degree (IMD) and falsity-membership degree (FLD) such that $\overline{\overline{\Xi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right): \overline{\overline{UN}} \to \left]0^-, 1^+\right[, \overline{\overline{\Psi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right): \overline{\overline{UN}} \to \left]0^-, 1^+\right[$ and $\overline{\overline{\Upsilon}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right): \overline{\overline{UN}} \to \left]0^-, 1^+\right[$. The three functions must satisfy the condition that $0^- \leq \overline{\overline{\Xi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right) + \overline{\overline{\Psi}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right) + \overline{\overline{\Upsilon}}_{\overline{\overline{NE}}}\left(\overline{\overline{un}}\right) \leq 3^+.$

Smarandache [5] developed the concept of NS from philosophical point of view as a simplification of FS and IFS. But NS was difficult to apply in practical problems. To utilize NS easily in realistic problems, Wang et al. [6] developed the perception of SVNS which is the subclass of NS and is define as follows:

1.1.1.2 Definition [6]

Let $\overline{\overline{UN}}$ be a universe set and $\overline{\overline{un}} \in \overline{\overline{UN}}$. Then, a SVNS $\overline{\overline{SN}}$ in $\overline{\overline{UN}}$ is expressed by,

$$\overline{\overline{SN}} = \left\{\left\langle \overline{\overline{un}}, \Xi_{\overline{\overline{SN}}}(\overline{\overline{un}}), \Psi_{\overline{\overline{SN}}}(\overline{\overline{un}}), \Upsilon_{\overline{\overline{SN}}}(\overline{\overline{un}}) \right\rangle \mid \overline{\overline{un}} \in \overline{\overline{UN}}\right\}, \tag{1.2}$$

where, $\Xi_{\overline{\overline{SN}}}(\overline{\overline{un}}), \Psi_{\overline{\overline{SN}}}(\overline{\overline{un}})$ and $\Upsilon_{\overline{\overline{SN}}}(\overline{\overline{un}})$ respectively express TRD, IMD and FLD such that $\Xi_{\overline{\overline{SN}}}(\overline{\overline{un}}): \overline{\overline{UN}} \to [0,1], \Psi_{\overline{\overline{SN}}}(\overline{\overline{un}}): \overline{\overline{UN}} \to [0,1]$ and $\Upsilon_{\overline{\overline{SN}}}(\overline{\overline{un}}): \overline{\overline{UN}} \to [0,1]$. The sum of these three functions must be less or equal to 3. The triplet $\left\langle \Xi_{\overline{\overline{SN}}}(\overline{\overline{un}}), \Psi_{\overline{\overline{SN}}}(\overline{\overline{un}}), \Upsilon_{\overline{\overline{SN}}}(\overline{\overline{un}}) \right\rangle$ is said to be a SVN number (SVNN). For computational simplicity, we shall denote a SVNN by $\hbar = \langle \Xi, \Psi, \Upsilon \rangle$.

1.1.1.3 Definition [10]

Let $\hbar, \hbar_1$ and $\hbar_3$ be any three SVNNs and $\xi > 0$. Then, few operational laws for SVNNs are described as follows:

$$(1)\ \hbar_1 \oplus \hbar_2 = \left\langle \Xi_1 + \Xi_2 - \Xi_1\Xi_2, \Psi_1\Psi_2, \Upsilon_1\Upsilon_2 \right\rangle, \tag{1.3}$$

$$(2)\ \hbar_1 \otimes \hbar_2 = \left\langle \Xi_1\Xi_2, \Psi_1 + \Psi_2 - \Psi_1\Psi_2, \Upsilon_1 + \Upsilon_2 - \Upsilon_1\Upsilon_2 \right\rangle, \tag{1.4}$$

$$(3)\ \xi\hbar = \left\langle 1-\left(1-\Xi\right)^{\xi}, \Psi^{\xi}, \Upsilon^{\xi} \right\rangle, \tag{1.5}$$

$$(4)\ \hbar^{\xi} = \left\langle \Xi^{\xi}, 1-\left(1-\Psi\right)^{\xi}, 1-\left(1-\Upsilon\right)^{\xi} \right\rangle. \tag{1.6}$$

To compare two SVNNs, Ye [9] proposed the following cosine measure and comparison rules for SVNNs.

1.1.1.4 Definition [9]

For a SVNN $\hbar = \langle \Xi, \Psi, \Upsilon \rangle$, the cosine measure is identified as follows:

$CS(\hbar) = \dfrac{\Xi}{\sqrt{\Xi^2 + \Psi^2 + \Upsilon^2}}$. For any two SVNNs $\hbar_1 = \langle \Xi_1, \Psi_1, \Upsilon_1 \rangle$ and $\hbar_2 = \langle \Xi_2, \Psi_2, \Upsilon_2 \rangle$, if $CS(\hbar_1) < CS(\hbar_2)$ then $\hbar_1 < \hbar_2$.

1.1.1.5 Definition [20]

Let $\hbar_1 = \langle \Xi_1, \Psi_1, \Upsilon_1 \rangle$ and $\hbar_2 = \langle \Xi_2, \Psi_2, \Upsilon_2 \rangle$ be any two SVNNs. Then, the Hamming distance among $\hbar_1$ and $\hbar_2$ is identified as follows:

$$\overline{\overline{Ds}}\left(\hbar_1, \hbar_2\right) = \frac{1}{3}\left(\left|\Xi_1 - \Xi_2\right| + \left|\Psi_1 - \Psi_2\right| + \left|\Upsilon_1 - \Upsilon_2\right|\right). \tag{1.7}$$

1.1.1.6 Definition [10]

Let $\hbar = \langle \Xi, \Psi, \Upsilon \rangle$ be a SVNN. Then, a score function (SF) $\overline{\overline{SO}}$ can be expressed as follows:

$$\overline{\overline{SO}}\left(\hbar\right) = \frac{1}{3}\left(\Xi_{\hbar} + 2 - \Psi_{\hbar} - \Upsilon_{\hbar}\right), \overline{\overline{SO}}\left(\hbar\right) \in [0,1]. \tag{1.8}$$

1.1.1.7 Definition [10]

Let $\hbar = \langle \Xi, \Psi, \Upsilon \rangle$ be a SVNN. Then, an accuracy function (AC) $\overline{\overline{AR}}$ can be expressed as follows:

$$\overline{\overline{AR}}(\hbar) = (\Xi_{\hbar} - \Upsilon_{\hbar}), \overline{\overline{AR}}(\hbar) \in [-1,1]. \tag{1.9}$$

1.1.1.8 Definition [10]

Let $\hbar_1 = \langle \Xi_1, \Psi_1, \Upsilon_1 \rangle$ and $\hbar_2 = \langle \Xi_2, \Psi_2, \Upsilon_2 \rangle$ be two SVNNs. Then, the comparison rules for comparing SVNNs are described as follow:

(1) If $\overline{\overline{SO}}(\hbar_1) < \overline{\overline{SO}}(\hbar_2)$, then $\hbar_2$ is greater than $\hbar_1$, and is denoted as $\hbar_2 > \hbar_1$,

(2) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, and $\overline{\overline{AR}}(\hbar_1) < \overline{\overline{AR}}(\hbar_2)$, then $\hbar_2$ is greater than $\hbar_1$, and is denoted as $\hbar_2 > \hbar_1$,

(3) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, and $\overline{\overline{AR}}(\hbar_1) = \overline{\overline{AR}}(\hbar_2)$ then $\hbar_1$ is equal to $\hbar_2$, and is denoted by $\hbar_1 = \hbar_2$.

1.1.1.9 Definition [7]

Let $\overline{\overline{UN}}$ be a universe set and $\overline{\overline{un}} \in \overline{\overline{UN}}$. Then an INS $\overline{\overline{IN}}$ in $\overline{\overline{UN}}$ is expressed by,

$$\overline{\overline{IN}} = \left\{ \left\langle \overline{\overline{un}}, \Xi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right), \Psi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right), \Upsilon_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right) \right\rangle \mid \overline{\overline{un}} \in \overline{\overline{UN}} \right\}, \tag{1.10}$$

where, $\Xi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right), \Psi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right)$ and $\Upsilon_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right)$ respectively, signify the TRD, IMD and FLD of the element $\overline{\overline{un}} \in \overline{\overline{UN}}$ to the set $\overline{\overline{IN}}$. For each point $\overline{\overline{h}} \in \overline{\overline{\aleph}}$, we have, $\Xi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right), \Psi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right)$, $\Upsilon_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right) \subseteq [0,1]$, and $0 \le sup\Xi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right) + \sup \Psi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right) + \sup \Upsilon_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right) \le 3.$

For computational simplicity, we can utilize $\hbar = \left\langle \left[\Xi^L, \Xi^U \right], \left[\Psi^L, \Psi^U \right], \left[\Upsilon^L, \Upsilon^U \right] \right\rangle$ to express an element $\hbar$ in an INS, and the element $\hbar$ is called an interval neutrosophic number

(INN). Where, $\Xi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right)=\left[\Xi^{L},\Xi^{U}\right]\subseteq[0,1]$, $\Psi_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right)=\left[\Psi^{L},\Psi^{U}\right]\subseteq[0,1]$, $\Upsilon_{\overline{\overline{IN}}}\left(\overline{\overline{un}}\right)=\left[\Upsilon^{L},\Upsilon^{U}\right]\subseteq[0,1]$ and $0\le\Xi^{U}+\Psi^{U}+\Upsilon^{U}\le 3$.

1.1.1.10 Definition [8]

Let $\hbar_1=\left\langle\left[\Xi_1^L,\Xi_1^U\right],\left[\Psi_1^L,\Psi_1^U\right],\left[\Upsilon_1^L,\Upsilon_1^U\right]\right\rangle$ and $\hbar_2=\left\langle\left[\Xi_2^L,\Xi_2^U\right],\left[\Psi_2^L,\Psi_2^U\right],\left[\Upsilon_2^L,\Upsilon_2^U\right]\right\rangle$ be any two INNs, and $\zeta>0$. Then, various operational laws of INNs can be described as follows:

$$(1).\hbar_1\oplus\hbar_2=\left\langle\left[\Xi_1^L+\Xi_2^L-\Xi_1^L\Xi_2^L,\Xi_1^U+\Xi_2^U-\Xi_1^U\Xi_2^U\right],\left[\Psi_1^L\Psi_2^L,\Psi_1^U\Psi_2^U\right],\left[\Upsilon_1^L\Upsilon_2^L,\Upsilon_1^U\Upsilon_2^U\right]\right\rangle; \quad (1.11)$$

$$(2).\hbar_1\otimes\hbar_2=\left\langle\left[\Xi_1^L\Xi_2^L,\Xi_1^U\Xi_2^U\right],\left[\Psi_1^L+\Psi_2^L-\Psi_1^L\Psi_2^L,\Psi_1^U+\Psi_2^U-\Psi_1^U\Psi_2^U\right],\left[\Upsilon_1^L+\Upsilon_2^L-\Upsilon_1^L\Upsilon_2^L,\Upsilon_1^U+\Upsilon_2^U-\Upsilon_1^U\Upsilon_2^U\right]\right\rangle; \quad (1.12)$$

$$(3).\ \hbar_1^{\zeta}=\left\langle\left[\left(\Xi_1^L\right)^{\zeta},\left(\Xi_1^U\right)^{\zeta}\right],\left[1-\left(1-\Psi_1^L\right)^{\zeta},1-\left(1-\Psi_1^U\right)^{\zeta}\right],\left[1-\left(1-\Upsilon_1^L\right)^{\zeta},1-\left(1-\Upsilon_1^U\right)^{\zeta}\right]\right\rangle; \quad (1.13)$$

$$(4).\ \zeta\hbar_1=\left\langle\left[1-\left(1-\Xi_1^L\right)^{\zeta},1-\left(1-\Xi_1^U\right)^{\zeta}\right],\left[\left(\Psi_1^L\right)^{\zeta},\left(\Psi_1^U\right)^{\zeta}\right],\left[\left(\Upsilon_1^L\right)^{\zeta},\left(\Upsilon_1^U\right)^{\zeta}\right]\right\rangle. \quad (1.14)$$

1.1.1.11 Definition [79]

Let $\hbar=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle$, be an INN. Then, the SC $\overline{\overline{SO}}(\hbar)$ and AC $\overline{\overline{AR}}(\hbar)$ can be defined as follows:

$$(1)\ \overline{\overline{SO}}(\hbar)=\frac{\Xi^L+\Xi^U}{2}+1-\frac{\Psi^L+\Psi^U}{2}+1-\frac{\Upsilon^L+\Upsilon^U}{2}; \quad (1.15)$$

$$(2)\ \overline{\overline{AR}}(\hbar)=\frac{\Xi^L+\Xi^U}{2}+1-\frac{\Psi^L+\Psi^L}{2}+\frac{\Upsilon^L+\Upsilon^U}{2}. \quad (1.16)$$

For comparing with two INNs, the comparison rules were defined by Liu et al. [79], which can be stated as follows.

1.1.1.12 Definition [79]

Let $\hbar_1 = \left\langle \left[\Xi_1^L, \Xi_1^U\right], \left[\Psi_1^L, \Psi_1^U\right], \left[\Upsilon_1^L, \Upsilon_1^U\right] \right\rangle$ and $\hbar_2 = \left\langle \left[\Xi_2^L, \Xi_2^U\right], \left[\Psi_2^L, \Psi_2^U\right], \left[\Upsilon_2^L, \Upsilon_2^U\right] \right\rangle$ be any two INNs. Then we have

(1) If $\overline{\overline{SO}}(\hbar_1) > \overline{\overline{SO}}(\hbar_2)$, then $\hbar_1$ is better than $\hbar_2$, and denoted by $\hbar_1 > \hbar_2$;

(2) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, and $\overline{\overline{AR}}(\hbar_1) > \overline{\overline{AR}}(\hbar_2)$, then $\hbar_1$ is better than $\hbar_2$, and denoted by $\hbar_1 > \hbar_2$;

(3) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, and $\overline{\overline{AR}}(\hbar_1) = \overline{\overline{AR}}(\hbar_2)$, then $\hbar_1$ is equal to $\hbar_2$, and denoted by $\hbar_1 = \hbar_2$.

1.1.1.13 Definition [21]

Let $\hbar_1 = \left\langle \left[\Xi_1^L, \Xi_1^U\right], \left[\Psi_1^L, \Psi_1^U\right], \left[\Upsilon_1^L, \Upsilon_1^U\right] \right\rangle$ and $\hbar_2 = \left\langle \left[\Xi_2^L, \Xi_2^U\right], \left[\Psi_2^L, \Psi_2^U\right], \left[\Upsilon_2^L, \Upsilon_2^U\right] \right\rangle$ be any two INNs. Then, the normalized Hamming distance among $\hbar_1$ and $\hbar_2$ is described as follows;

$$\overline{\overline{D}}(\hbar_1, \hbar_2) = \frac{1}{6}\left(\left|\Xi_1^L - \Xi_2^L\right| + \left|\Xi_1^U - \Xi_2^U\right| + \left|\Psi_1^L - \Psi_2^L\right| + \left|\Psi_1^U - \Psi_2^U\right| + \left|\Upsilon_1^L - \Upsilon_2^L\right| + \left|\Upsilon_1^U - \Upsilon_2^U\right|\right). \tag{1.17}$$

1.1.1.14 Definition [11, 12]

Let $\overline{\overline{UN}}$ the universe set and $\overline{\overline{un}} \in \overline{\overline{UN}}$. Then, a neutrosophic cubic set (NCS) in $\overline{\overline{UN}}$ is a pair $\overline{\overline{\Im}} = \left\langle \overline{\overline{IN}}, \lambda \right\rangle$ where $\overline{\overline{IN}} = \left\{ \left\langle \overline{\overline{un}}, \Xi_{\overline{\overline{IN}}}(\overline{\overline{un}}), \Psi_{\overline{\overline{IN}}}(\overline{\overline{un}}), \Upsilon_{\overline{\overline{IN}}}(\overline{\overline{un}}) \right\rangle \mid \overline{\overline{un}} \in \overline{\overline{UN}} \right\}$ is an INS in $\overline{\overline{UN}}$ and $\lambda = \left\{ \left\langle \overline{\overline{un}}, \lambda_{\overline{T}}(\overline{\overline{un}}), \lambda_{\overline{I}}(\overline{\overline{un}}), \lambda_{\overline{F}}(\overline{\overline{un}}) \right\rangle \mid \overline{\overline{un}} \in \overline{\overline{UN}} \right\}$ is a SVNS in $\overline{\overline{UN}}$.

For simplicity, a basic element $\left\{\overline{\overline{un}}, \left\langle \Xi(\overline{\overline{un}}), \Psi(\overline{\overline{un}}), \Upsilon(\overline{\overline{un}}) \right\rangle, \left\langle \lambda_{\overline{\overline{T}}}(\overline{\overline{un}}), \lambda_{\overline{\overline{I}}}(\overline{\overline{un}}), \lambda_{\overline{\overline{F}}}(\overline{\overline{un}}) \right\rangle \right\}$ in a NCS can be expressed by $\hbar = \left(\langle \Xi, \Psi, \Upsilon \rangle, \left\langle \lambda_{\overline{\overline{T}}}, \lambda_{\overline{\overline{I}}}, \lambda_{\overline{\overline{F}}} \right\rangle \right)$, which is said to be NC number (NCN), where $\Xi, \Psi, \Upsilon \subseteq [0,1]$ and $\lambda_{\overline{\overline{T}}}, \lambda_{\overline{\overline{I}}}, \lambda_{\overline{\overline{F}}} \in [0,1]$, satisfying $0 \le \Xi^U + \Psi^U + \Upsilon^U \le 3$ and $0 \le \lambda_{\overline{\overline{T}}} + \lambda_{\overline{\overline{I}}} + \lambda_{\overline{\overline{F}}} \le 3$.

1.1.1.15 Definition [15]

Let $\hbar_1 = \left(\left\langle [\Xi_1^L, \Xi_1^U], [\Psi_1^L, \Psi_1^U], [\Upsilon_1^L, \Upsilon_1^U] \right\rangle, \left\langle \lambda_{\overline{\overline{T}}_1}, \lambda_{\overline{\overline{I}}_1}, \lambda_{\overline{\overline{F}}_1} \right\rangle \right)$ and $\hbar_2 = \left(\left\langle [\Xi_2^L, \Xi_2^U], [\Psi_2^L, \Psi_2^U], [\Upsilon_2^L, \Upsilon_2^U] \right\rangle, \left\langle \lambda_{\overline{\overline{T}}_2}, \lambda_{\overline{\overline{I}}_2}, \lambda_{\overline{\overline{F}}_2} \right\rangle \right)$ be any two NCNs and $\overline{\overline{\zeta}} > 0$. Then, the operational laws for NCNs defined by Ye [15] are as follows:

$$\begin{aligned}(1)\ \hbar_1 \oplus \hbar_2 = &\Big(\Big\langle \left[\Xi_1^L + \Xi_2^L - \Xi_1^L \Xi_2^L, \Xi_1^U + \Xi_2^U - \Xi_1^U \Xi_2^U \right], \left[\Psi_1^L \Psi_2^L, \Psi_1^U \Psi_2^U \right] \left[\Upsilon_1^L \Upsilon_2^L, \Upsilon_1^U \Upsilon_2^U \right] \Big\rangle, \\ &\left\langle \lambda_{\overline{\overline{T}}_1} + \lambda_{\overline{\overline{T}}_2} - \lambda_{\overline{\overline{T}}_1} \lambda_{\overline{\overline{T}}_2}, \lambda_{\overline{\overline{I}}_1} \lambda_{\overline{\overline{I}}_2}, \lambda_{\overline{\overline{F}}_2} \lambda_{\overline{\overline{F}}_2} \right\rangle \Big),\end{aligned} \tag{1.18}$$

$$\begin{aligned}(2)\ \hbar_1 \otimes \hbar_2 = &\Big(\Big\langle \left[\Xi_1^L \Xi_2^L, \Xi_1^U \Xi_2^U \right], \left[\Psi_1^L + \Psi_2^L - \Psi_1^L \Psi_2^L, \Psi_1^U + \Psi_2^U - \Psi_1^U \Psi_2^U \right] \left[\Upsilon_1^L + \Upsilon_2^L - \Upsilon_1^L \Upsilon_2^L, \Upsilon_1^U + \Upsilon_2^U - \Upsilon_1^U \Upsilon_2^U \right] \Big\rangle, \\ &\left\langle \lambda_{\overline{\overline{T}}_1} \lambda_{\overline{\overline{T}}_2}, \lambda_{\overline{\overline{I}}_1} + \lambda_{\overline{\overline{I}}_2} - \lambda_{\overline{\overline{I}}_1} \lambda_{\overline{\overline{I}}_2}, \lambda_{\overline{\overline{F}}_2} + \lambda_{\overline{\overline{F}}_2} - \lambda_{\overline{\overline{F}}_2} \lambda_{\overline{\overline{F}}_2} \right\rangle \Big),\end{aligned} \tag{1.19}$$

$$\begin{aligned}(3)\ \overline{\overline{\zeta}} \hbar_1 = &\left(\left\langle \left[1 - \left(1 - \left(\Xi_1^L\right)\right)^{\overline{\overline{\zeta}}}, 1 - \left(1 - \left(\Xi_1^U\right)\right)^{\overline{\overline{\zeta}}} \right], \left[\left(\Psi_1^L\right)^{\overline{\overline{\zeta}}}, \left(\Psi_1^U\right)^{\overline{\overline{\zeta}}} \right], \left[\left(\Upsilon_1^L\right)^{\overline{\overline{\zeta}}}, \left(\Upsilon_1^U\right)^{\overline{\overline{\zeta}}} \right] \right\rangle, \right. \\ &\left. \left\langle 1 - \left(1 - \lambda_{\overline{\overline{T}}_1}\right)^{\overline{\overline{\zeta}}}, \left(\lambda_{\overline{\overline{I}}_1}\right)^{\overline{\overline{\zeta}}}, \left(\lambda_{\overline{\overline{F}}_1}\right)^{\overline{\overline{\zeta}}} \right\rangle \right),\end{aligned} \tag{1.20}$$

$$\begin{aligned}(4)\ \hbar_1^{\overline{\overline{\zeta}}} = &\left(\left\langle \left[\left(\Xi_1^L\right)^{\overline{\overline{\zeta}}}, \left(\Xi_1^U\right)^{\overline{\overline{\zeta}}} \right], \left[1 - \left(1 - \Psi_1^L\right)^{\overline{\overline{\zeta}}}, 1 - \left(1 - \Psi_1^U\right)^{\overline{\overline{\zeta}}} \right], \left[1 - \left(1 - \Upsilon_1^L\right)^{\overline{\overline{\zeta}}}, 1 - \left(1 - \Upsilon_1^U\right)^{\overline{\overline{\zeta}}} \right] \right\rangle, \right. \\ &\left. \left\langle \left(\lambda_{\overline{\overline{T}}_1}\right)^{\overline{\overline{\zeta}}}, 1 - \left(1 - \lambda_{\overline{\overline{I}}_1}\right)^{\overline{\overline{\zeta}}}, 1 - \left(1 - \lambda_{\overline{\overline{F}}_1}\right)^{\overline{\overline{\zeta}}} \right\rangle \right).\end{aligned} \tag{1.21}$$

1.1.1.16 Definition [14]

Let $\hbar_1 = \left(\left\langle \left[\Xi_1^L, \Xi_1^U \right], \left[\Psi_1^L, \Psi_1^U \right], \left[\Upsilon_1^L, \Upsilon_1^U \right] \right\rangle, \left\langle \lambda_{\overline{\overline{T}}_1}, \lambda_{\overline{\overline{I}}_1}, \lambda_{\overline{\overline{F}}_1} \right\rangle \right)$ be a NCN. Then, the score, accuracy and certainty functions of NCN are described as follows:

$$\overline{\overline{SO}}(\hbar_1) = \frac{4 + \Xi_1^L - \Psi_1^L - \Upsilon_1^L + \Xi_1^U - \Psi_1^U - \Upsilon_1^U + \lambda_{\overline{\overline{T}}_1} + 2 - \lambda_{\overline{\overline{I}}_1} - \lambda_{\overline{\overline{F}}_1}}{9}; \tag{1.22}$$

$$\overline{\overline{AR}}(\hbar_1) = \frac{\Xi_1^L - \Upsilon_1^L + \Xi_1^U - \Upsilon_1^U + \lambda_{T_1} - \lambda_{\overline{\overline{F}}_1}}{3}; \text{ and } \overline{\overline{CR}}(z_1) = \frac{\Xi_1^L + \Xi_1^U + \lambda_{\overline{\overline{T}}_1}}{3}. \tag{1.23}$$

1.1.1.17 Definition [14]

Let $\hbar_1 = \left(\left\langle \left[\Xi_1^L, \Xi_1^U \right], \left[\Psi_1^L, \Psi_1^U \right], \left[\Upsilon_1^L, \Upsilon_1^U \right] \right\rangle, \left\langle \lambda_{\overline{\overline{T}}_1}, \lambda_{\overline{\overline{I}}_1}, \lambda_{\overline{\overline{F}}_1} \right\rangle \right)$ and $\hbar_2 = \left(\left\langle \left[\Xi_2^L, \Xi_2^U \right], \left[\Psi_2^L, \Psi_2^U \right], \left[\Upsilon_2^L, \Upsilon_2^U \right] \right\rangle, \left\langle \lambda_{\overline{\overline{T}}_2}, \lambda_{\overline{\overline{I}}_2}, \lambda_{\overline{\overline{F}}_2} \right\rangle \right)$.

Then, the comparison rules for NCNs can be described as follows:

(1) If $\overline{\overline{SO}}(\hbar_1) > \overline{\overline{SO}}(\hbar_2)$, then $\hbar_1$ is larger than $\hbar_2$, and is indicated by $\hbar_1 > \hbar_2$;

(2) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, and $\overline{\overline{AR}}(\hbar_1) > \overline{\overline{AR}}(\hbar_2)$, then $\hbar_1$ is larger than $\hbar_2$, and is indicated by $\hbar_1 > \hbar_2$;

(3) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, $\overline{\overline{AO}}(\hbar_1) = \overline{\overline{AO}}(\hbar_2)$, and $\overline{\overline{CR}}(\hbar_1) > \overline{\overline{CR}}(\hbar_2)$, then $\hbar_1$ is larger than $\hbar_2$, and is indicated by $\hbar_1 > \hbar_2$;

(4) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, $\overline{\overline{AR}}(\hbar_1) = \overline{\overline{AR}}(\hbar_2)$, and $\overline{\overline{CR}}(\hbar_1) = \overline{\overline{CR}}(\hbar_2)$, then $\hbar_1$ is equal to $\hbar_2$, and is indicated by $\hbar_1 = \hbar_2$.

1.2 The Linguistic Set and Uncertain Linguistic Numbers

The linguistic set (LS) is thought out as a very good , of highest quality tool to put in a given form these qualitative information. we can articulate the linguistic set by $\underline{\underline{S}} = \{s_0, s_1, \ldots, s_{l-1}\}$, and $s_\vartheta\,(\vartheta = 1, 2, \ldots, l-1)$ can be called a linguistic element (LE), and l is identified as the cardinality of the linguistic term set (LTS), usually odd values such as 3,5,7, etc. For example, when$l = 7$, the linguistic term set $\underline{\underline{S}} = \{s_0, s_1, \ldots, s_6\}$ $= (extremely\ poor, very\ poor, poor, medium,\ good, very\ good, extremely\ good)$.

Let s_i and s_j be any two LEs in LS $\underline{\underline{S}}$, then, they have the following characteristics [116, 117]:

(1) If $i > j$, then $s_i > s_j$,

(2) There exists a negative operator: neg (s_i)=s_j, where $j = l - 1 - i$,

(3) If $s_i \geq s_j$, $\max(s_i, s_j) = s_i$;

(4) If $s_i \leq s_j$, $\min(s_i, s_j) = s_i$.

In the process of calculation plenty of information is lost, to surmount the lost of information the discrete LS is enlarged to a continuous LTS $\underline{\underline{S}} = \{s_\vartheta \mid \vartheta \in R^+\}$ which also satisfy the properties of the original linguistic term set. The basic operational rules are described as follows [118, 119]:

(a) $\rho s_i = s_{\rho \times i}; \quad \rho \geq 0$,

(b) $s_i + s_j = s_{i+j}$,

(c) $s_i \times s_j = s_{i \times j}$,

(d) $(s_i)^n = s_{i^n}; n \geq 0$.

1.2.1.1 Definition [118, 119]

Assume that $\tilde{s} = [s_u, s_v]$, $s_u, s_v \in \underline{\underline{S}}$ with $u \leq v$ are respectively the inferior and superior limits of $\tilde{s}$, then $\tilde{s}$ is said to be an uncertain linguistic variable (ULV).

Let $\bar{S}$ represents the set of all ULVs and $\tilde{s}_1 = [s_{u_1}, s_{v_1}]$ and $\tilde{s}_2 = [s_{u_2}, s_{v_2}]$ be any two ULVs, then the basic operations are described as:

(1) $\tilde{s}_1 + \tilde{s}_2 = [s_{u_1}, s_{v_1}] + [s_{u_2}, s_{v_2}] = [s_{u_1+u_2}, s_{v_1+v_2}]$,

(2) $\tilde{s}_1 \times \tilde{s}_2 = [s_{u_1}, s_{v_1}] \times [s_{u_2}, s_{v_2}] = [s_{u_1 \times u_2}, s_{v_1 \times v_2}]$,

(3) $\delta\tilde{s}_1 = \delta[s_{u_1}, s_{v_1}] = \left[s_{\delta * u_1}, s_{\delta * v_1}\right], \delta \geq 0$,

(4) $\tilde{s}_1{}^{\delta} = [s_{u_1}, s_{v_1}]^{\delta} = \left[s_{u_1{}^{\delta}}, s_{v_1{}^{\delta}}\right], \delta \geq 0$.

1.2.2 The Interval Neutrosophic Uncertain Linguistic Sets (INULSs)

1.2.2.1 Definition [43]

Let $\overline{\overline{UN}}$ be the universe of discourse set and $[s_{\vartheta(\overline{\overline{un}})}, s_{\sigma(\overline{\overline{un}})}] \in \underline{\underline{S}}$ be an ULV. An INULS INU in $\overline{\overline{UN}}$ is described as:

$$INU = \left\{ \overline{\overline{un}}, \left[s_{\vartheta\left(\overline{\overline{un}}\right)}, s_{\sigma\left(\overline{\overline{un}}\right)} \right], \left(\Xi_{INU}\left(\overline{\overline{un}}\right), \Psi_{INU}\left(\overline{\overline{un}}\right), \Upsilon_{INU}\left(\overline{\overline{un}}\right) \right) \middle| \overline{\overline{un}} \in \overline{\overline{UN}} \right\}, \tag{1.24}$$

where, $s_\vartheta, s_\sigma \in \underline{\underline{S}}$, $\Xi_{INU}\left(\overline{\overline{un}}\right)=\left[inf\,\Xi_{INU}\left(\overline{\overline{un}}\right), sup\Xi_{INU}\left(\overline{\overline{un}}\right)\right]\subseteq[0,1], \Psi_{INU}\left(\overline{\overline{un}}\right)=\left[inf\,\Psi_{INU}\left(\overline{\overline{un}}\right), sup\Psi_{INU}\left(\overline{\overline{un}}\right)\right]\subseteq[0,1]$

and $\Upsilon_{INU}\left(\overline{\overline{un}}\right)=\left[inf\,\Upsilon_{INU}\left(\overline{\overline{un}}\right), sup\Upsilon_{INU}\left(\overline{\overline{un}}\right)\right]\subseteq[0,1]$ with the condition, $0\le sup\Xi_{INU}\left(\overline{\overline{un}}\right)+sup\Psi_{INU}\left(\overline{\overline{un}}\right)+ sup\Upsilon_{INU}\left(\overline{\overline{un}}\right)\le 3$ for any $\overline{\overline{un}}\in\overline{\overline{UN}}$. The functions, $\Xi_{INU}\left(\overline{\overline{un}}\right), \Psi_{INU}\left(\overline{\overline{un}}\right), \Upsilon_{INU}(\overline{\overline{un}})$ denote respectively TRM interval, IDM interval, and FLM interval of the element $\overline{\overline{un}}\in\overline{\overline{UN}}$.

1.2.3 Hesitant Fuzzy Set

1.2.3.1 Definition [45, 46]

Let $\overline{\overline{UN}}$ be a predetermined set, a HFS $\overline{\overline{HF}}$ on $\overline{\overline{UN}}$ is described in terms of a mapping $\overline{\overline{a}}_{\overline{\overline{HF}}}(\overline{\overline{un}})$, that when applied to $\overline{\overline{UN}}$ returns a finite subset of $[0,1]$, which can be mathematically represented as follows:

$$\overline{\overline{HF}}=\left\{\left\langle \overline{\overline{un}}, \overline{\overline{a}}_{\overline{\overline{HF}}}(\overline{\overline{un}})\right\rangle \mid \overline{\overline{un}}\in\overline{\overline{UN}}\right\}, \tag{1.25}$$

where, $\overline{\overline{a}}_{\overline{\overline{HF}}}(\overline{\overline{un}})=\bigcup_{\beta_{\overline{\overline{HF}}}(\overline{\overline{un}})\in\overline{\overline{a}}_{\overline{\overline{HF}}}(\overline{\overline{un}})}\left\{\beta_{\overline{\overline{HF}}}(\overline{\overline{un}})\right\}$ is a set of few different values in [0, 1], denoting the possible membership degrees of the element $\overline{\overline{un}}\in\overline{\overline{UN}}$ to $\overline{\overline{HF}}$. For simplicity, we shall write $\overline{\overline{a}}$ instead of $\overline{\overline{a}}_{\overline{\overline{HF}}}(\overline{\overline{un}})=\bigcup_{\beta_{\overline{\overline{HF}}}(\overline{\overline{un}})\in\overline{\overline{a}}_{\overline{\overline{HF}}}(\overline{\overline{un}})}\left\{\beta_{\overline{\overline{HF}}}(\overline{\overline{un}})\right\}$ and is said to be a hesitant fuzzy element.

Let $\overline{\overline{a}}, \overline{\overline{a}}_1$ and $\overline{\overline{a}}_2$ be any three HFEs, then the operational rules for HFEs are described below:

$$(1)\quad \overline{\overline{a}}^{\beta}=\bigcup_{\gamma\in\overline{\overline{a}}}\{\gamma^{\beta}\}, \tag{1.26}$$

(2) $\beta \overset{=}{a} = \bigcup_{\gamma \in \overset{=}{a}} \{1-(1-\gamma)^{\beta}\},$ (1.27)

(3) $\overset{=}{a}_1 + \overset{=}{a}_2 = \bigcup_{\gamma_1 \in \overset{=}{a}_1, \gamma_2 \in \overset{=}{a}_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\},$ (1.28)

(4) $\overset{=}{a}_1 \times \overset{=}{a}_2 = \bigcup_{\gamma_1 \in \overset{=}{a}_1, \gamma_2 \in \overset{=}{a}_2} \{\gamma_1\gamma_2\}.$ (1.29)

1.2.4 Linguistic Neutrosophic Set (LNS) and Their Operations

1.2.4.1 Definition [57]

Let $\overline{\overline{UN}}$ be the domain set and $\underline{\underline{S}} = \{s_\eta \mid s_0 \le s_\eta \le s_{2r}\}$, then a LNS is an object of the form:

$$LN = \left\{\left\langle \overline{\overline{un}}, s_\Xi, s_\Psi, s_\Upsilon \right\rangle \mid \overline{\overline{un}} \in \overline{\overline{UN}}\right\}, \tag{1.30}$$

where, s_Ξ, s_Ψ and s_Υ represent the TRM, IM and FLM functions of the element $\overline{\overline{un}} \in \overline{\overline{UN}}$ to the set LN, respectively and must satisfy condition that $0 \le s_\Xi + s_\Psi + s_\Upsilon \le 6r$. Furthermore, $\hbar = \langle s_\Xi, s_\Psi, s_\Upsilon \rangle$ is said to be a LNN and LN consists a group of LNNs. Moreover, when $s_\Xi, s_\Psi, s_\Upsilon \in \underline{\underline{S}}$, then $\langle s_\Xi, s_\Psi, s_\Upsilon \rangle$ is an original LNN; otherwise, we call it virtual LNN.

1.2.4.2 Definition [57]

Let $\hbar_1 = \langle s_{\Xi_1}, s_{\Psi_1}, s_{\Upsilon_1} \rangle$ and $\hbar_2 = \langle s_{\Xi_2}, s_{\Psi_2}, s_{\Upsilon_2} \rangle$ be any two LNNs. Then based on LSF, few operational rules for LNNs are described as follows:

$$(1)\ \hbar_1 \oplus \hbar_2 = \left\langle \overline{\overline{LS}}^{*-1}\left(\overline{\overline{LS}}^{*}(s_{\Xi_1}) + \overline{\overline{LS}}^{*}(s_{\Xi_2}) - \overline{\overline{LS}}^{*}(s_{\Xi_1})\overline{\overline{LS}}^{*}(s_{\Xi_2})\right), \overline{\overline{LS}}^{*-1}\left(\overline{\overline{LS}}^{*}(s_{\Psi_1})\overline{\overline{LS}}^{*}(s_{\Psi_2})\right), \overline{\overline{LS}}^{*-1}\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_1})\overline{\overline{LS}}^{*}(s_{\Upsilon_2})\right)\right\rangle, \tag{1.31}$$

$$(2)\ \hbar_1 \otimes \hbar_2 = \left\langle \overline{\overline{LS}}^{*-1}\left(\overline{\overline{LS}}^{*}(s_{\Xi_1})\overline{\overline{LS}}^{*}(s_{\Xi_2})\right), \overline{\overline{LS}}^{*-1}\left(\overline{\overline{LS}}^{*}(s_{\Psi_1}) + \overline{\overline{LS}}^{*}(s_{\Psi_2}) - \overline{\overline{LS}}^{*}(s_{\Psi_1})\overline{\overline{LS}}^{*}(s_{\Psi_2})\right), \overline{\overline{LS}}^{*-1}\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_1}) + \overline{\overline{LS}}^{*}(s_{\Upsilon_2}) - \overline{\overline{LS}}^{*}(s_{\Upsilon_1})\overline{\overline{LS}}^{*}(s_{\Upsilon_2})\right)\right\rangle, \tag{1.32}$$

$$(3)\ \xi\hbar_1 = \left\langle \overline{\overline{LS}}^{*-1}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_1})\right)^{\xi}\right), \overline{\overline{LS}}^{*-1}\left(\left(\overline{\overline{LS}}^{*}(s_{\Psi_1})\right)^{\xi}\right), \overline{\overline{LS}}^{*-1}\left(\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_1})\right)^{\xi}\right)\right\rangle, \tag{1.33}$$

$$(4)\ \hbar_1^{\xi} = \left\langle \overline{\overline{LS}}^{*-1}\left(\left(\overline{\overline{LS}}^{*}(s_{\Xi_1})\right)^{\xi}\right), \overline{\overline{LS}}^{*-1}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Psi_1})\right)^{\xi}\right), \overline{\overline{LS}}^{*-1}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Upsilon_1})\right)^{\xi}\right)\right\rangle. \tag{1.34}$$

1.2.4.3 Definition [57]

Let $\hbar_1 = \langle s_{\Xi_1}, s_{\Psi_1}, s_{\Upsilon_1} \rangle$ and $\hbar_2 = \langle s_{\Xi_2}, s_{\Psi_2}, s_{\Upsilon_2} \rangle$ be any two LNNs. Then

(1) $\hbar_1 = \hbar_2$ if and only if $s_{\Xi_1} = s_{\Xi_2}, s_{\Psi_1} = s_{\Psi_2}$ and $s_{\Upsilon_1} = s_{\Upsilon_2}$;

(2) $Neg(\hbar_1) = \langle s_{\Xi_1}, s_{\Psi_1}, s_{\Upsilon_1} \rangle$, where $Neg(\hbar_1)$ is the negation operator of $\hbar_1$.

1.2.4.4 Definition [57]

Let $\hbar = \langle s_{\Xi}, s_{\Psi}, s_{\Upsilon} \rangle$ be a LNN. Then the expected value, accuracy and certainty functions are denoted and defined as follows:

$$\overline{\overline{SO}}(\hbar) = \frac{1}{3}\left(\overline{\overline{LS}}^{*}(s_{\Xi}) + 2 - \overline{\overline{LS}}^{*}(s_{\Psi}) - \overline{\overline{LS}}^{*}(s_{\Upsilon})\right), \tag{1.35}$$

$$\overline{\overline{AR}}(\hbar) = \overline{\overline{LS}}^{*}(s_{\Xi}) - \overline{\overline{LS}}^{*}(s_{\Upsilon}), \tag{1.36}$$

$$\overline{\overline{CR}}(\hbar) = \overline{\overline{LS}}^{*}(s_{\Xi}). \quad (1.37)$$

1.2.4.5 Definition [57]

Let $\hbar_1$ and $\hbar_2$ be any two LNNs; then the comparison rules are described as follows:

(1) If $\overline{\overline{SO}}(\hbar_1) > \overline{\overline{SO}}(\hbar_2)$, then $\hbar_1 > \hbar_2$;

(2) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, and $\overline{\overline{AR}}(\hbar_1) > \overline{\overline{AR}}(\hbar_2)$, then $\hbar_1 > \hbar_2$;

(3) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, $\overline{\overline{AR}}(\hbar_1) = \overline{\overline{AR}}(\hbar_2)$ and $\overline{\overline{CR}}(\hbar_1) > \overline{\overline{CR}}(\hbar_2)$, then $\hbar_1 > \hbar_2$;

(4) If $\overline{\overline{SO}}(\hbar_1) = \overline{\overline{SO}}(\hbar_2)$, $\overline{\overline{AR}}(\hbar_1) = \overline{\overline{AR}}(\hbar_2)$ and $\overline{\overline{CR}}(\hbar_1) = \overline{\overline{CR}}(\hbar_2)$, then $\hbar_1 = \hbar_2$.

1.2.4.6 Definition [57]

Let $\hbar_1 = \langle s_{\Xi_1}, s_{\Psi_1}, s_{\Upsilon_1} \rangle$ and $\hbar_2 = \langle s_{\Xi_2}, s_{\Psi_2}, s_{\Upsilon_2} \rangle$ be any two LNNs, and $s_{\Xi_1}, s_{\Psi_1}, s_{\Upsilon_1}, s_{\Xi_2}, s_{\Psi_2}, s_{\Upsilon_2} \in S_{[0,2r]}$. When $\overline{\overline{LS}}^{*}(s_j)$ is LSF, $\overline{\overline{D}}$ is a mapping, and $\overline{\overline{Ds}} : \hbar \times \hbar \to R^{+}$, the hamming distance between $\hbar_1$ and $\hbar_2$ can be defined as

$$\overline{\overline{D}}(\hbar_1, \hbar_2) = \frac{1}{3}\left(|\overline{\overline{LS}}^{*}(s_{\Xi_1}) - \overline{\overline{LS}}^{*}(s_{\Xi_2})| + |\overline{\overline{LS}}^{*}(s_{\Psi_1}) - \overline{\overline{LS}}^{*}(s_{\Psi_2})| + |\overline{\overline{LS}}^{*}(s_{\Upsilon_1}) - \overline{\overline{LS}}^{*}(s_{\Upsilon_2})| \right). \quad (1.38)$$

1.3 Different Aggregation Operators

In this part, the definition and properties of different aggregation operators are discussed.

1.3.1 The Bonferroni Mean (BM) operator

The BM operator was first presented by Bonferroni [61], and was explained as follows:

1.3.1.1 Definition [61]

Let $\overline{\overline{NR}}_{\bar{\bar{z}}}(\bar{\bar{z}}=1,\ldots,m)$ be a group of non-negative real numbers, $x,y\geq 0,$ then the BM operator is a function $BM:R^m\rightarrow R,$ such that

$$BM^{x,y}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,\ldots.,\overline{\overline{NR}}_m\right)=\left(\frac{1}{m^2-m}\sum_{\bar{\bar{z}}=1}^{m}\sum_{\substack{\bar{\bar{s}}=1\\ \bar{\bar{z}}\neq\bar{\bar{s}}}}^{m}\overline{\overline{NR}}_{\bar{\bar{z}}}^{\,x}\,\overline{\overline{NR}}_{s}^{\,y}\right)^{\frac{1}{x+y}}. \tag{1.39}$$

The BM operator ignores the importance degree of each input argument, which can be given by decision makers according to their interest. To overcome this shortcoming of BM operator, He et al. [80] defined the weighted Bonferroni mean (WBM) operators which can be explained as follows:

1.3.1.2 Definition [80]

Let $\overline{\overline{NR}}_{\bar{\bar{z}}}(\bar{\bar{z}}=1,\ldots,m)$ be a group of non-negative real numbers, $x,y\geq 0,$ then a weighted BM operator (WBM) is a function $BM:R^m\rightarrow R,$ such that:

$$WBM^{x,y}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,\ldots.,\overline{\overline{NR}}_m\right)=\left(\frac{1}{m^2-m}\sum_{\bar{\bar{z}}=1}^{m}\sum_{\substack{\bar{\bar{s}}=1\\ \bar{\bar{z}}\neq\bar{\bar{s}}}}^{m}\frac{\bar{\bar{w}}_{\bar{\bar{z}}}\,\bar{\bar{w}}_s}{1-\bar{\bar{w}}_{\bar{\bar{z}}}}\overline{\overline{NR}}_{\bar{\bar{z}}}^{\,x}\,\overline{\overline{NR}}_{s}^{\,y}\right)^{\frac{1}{x+y}}, \tag{1.40}$$

where, $\bar{\bar{w}}=\left(\bar{\bar{w}}_1,\bar{\bar{w}}_2,\ldots,\bar{\bar{w}}_m\right)^T$ is the importance degree of every $\overline{\overline{NR}}_{\bar{\bar{z}}}(\bar{\bar{z}}=1,\ldots.,m).$

The WBM operator has the following characteristics:

1.3.1.3 Theorem (Reducibility)

If the importance degree of each $\overline{\overline{NR}}_z$ is $\bar{\bar{w}}=\left(\frac{1}{m},\frac{1}{m},\ldots.,\frac{1}{m}\right)^T$, then

$$WBM^{x,y}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,....,\overline{\overline{NR}}_m\right)=\left(\frac{1}{m^2-m}\sum_{\bar{\bar{z}}=1}^{m}\sum_{\substack{\bar{\bar{s}}=1\\ \bar{\bar{z}}\neq s}}^{m}\overline{\overline{NR}}_{\bar{\bar{z}}}^{x}\,\overline{\overline{NR}}_{s}^{y}\right)^{\frac{1}{x+y}} \tag{1.41}$$
$$= BM^{x,y}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,....,\overline{\overline{NR}}_m\right).$$

1.3.1.4 Theorem (Idempotency)

Let $\overline{\overline{NR}}_{\bar{\bar{z}}}=\overline{\overline{NR}},(\bar{\bar{z}}=1,...,m)$, then

$$BM^{x,y}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,....,\overline{\overline{NR}}_m\right)=\overline{\overline{NR}}. \tag{1.42}$$

1.3.1.5 Theorem (Permutation)

Let $\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_m\right)$ be any permutation of $\left(\overline{\overline{NR}}_1{}',\overline{\overline{NR}}'_2,...,\overline{\overline{NR}}'_m\right)$, then

$$WMB^{x,y}\left(\overline{\overline{NR}}'_1,\overline{\overline{NR}}'_2,...,\overline{\overline{NR}}'_m\right)=WBM\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_m\right). \tag{1.43}$$

1.3.1.6 Theorem (Monotonicity)

Let $\overline{\overline{NR}}_{\bar{\bar{z}}}\geq\overline{\overline{NR}}''_{\bar{\bar{z}}}(\bar{\bar{z}}=1,...,m)$, then

$$WBM^{x,y}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_m\right)\geq WBM^{x,y}\left(\overline{\overline{NR}}''_1,\overline{\overline{NR}}''_2,...,\overline{\overline{NR}}''_m\right). \tag{1.44}$$

1.3.1.7 Theorem (Boundedness)

The $WBM^{x,y}$ lies among the min and max operators, that is

$$\min\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_m\right)\leq WBM^{x,y}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_m\right)\leq\max\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_m\right). \tag{1.45}$$

Similar to BM operator, the geometric BM operator also considers the correlation among the input arguments. It can be explained as follows:

1.3.1.8 Definition [120]

Let $\overline{\overline{NR}}_{\bar{\bar{z}}}(\bar{\bar{z}} = 1,...,m)$ be a group of positive real numbers, $x, y \geq 0,$ then a geometric BM operator (GBM) is a function $GBM : R^m \to R,$ such that

$$GBM^{x,y}\left(\overline{\overline{NR}}_1, \overline{\overline{NR}}_2, ..., \overline{\overline{NR}}_m\right) = \frac{1}{x+y} \prod_{\bar{\bar{z}}=1}^{m} \prod_{\substack{s=1 \\ \bar{\bar{z}} \neq s}}^{m} \left(x\overline{\overline{NR}}_{\bar{\bar{z}}} + y\overline{\overline{NR}}_s\right)^{\frac{1}{m^2-m}}. \tag{1.46}$$

The GBM operator ignores the importance degree of each input argument, which can be given by decision makers according to their interest. In a similar way to WBM, the weighted geometric BM (WGBM) operator was also presented. The extension process is same as that of WBM, so it is omitted here.

1.3.2 Heronian Mean (HM) operator

HM [62] is also an essential tool, which can process the interrelationships of the input values, and is defined as follows:

1.3.2.1 Definition [62]

Let $I = [0,1], a, b \geq 0, H^{a,b} : I^m \to I,$ if $H^{x,y}$ satisfies;

$$H^{a,b}\left(\overline{\overline{NR}}_1, \overline{\overline{NR}}_2, ..., \overline{\overline{NR}}_m\right) = \left(\frac{2}{m^2+m} \sum_{i=1}^{m} \sum_{j=i}^{m} \overline{\overline{NR}}_i^{\,a} \, \overline{\overline{NR}}_j^{\,b}\right)^{\frac{1}{a+b}}. \tag{1.47}$$

Then, the mapping $H^{x,y}$ is said to be HM operator with parameters. The HM operator must have the properties of idempotency, boundedness and monotonicity.

1.3.3 Power Average (PA) operator

The PA operator was firstly introduced by Yager [60] for classical number. The dominant edge of PA operator is its capacity to diminish the inadequate effect of unreasonably high and low arguments on the results.

1.3.3.1 Definition [60]

Let $\overline{\overline{NR}}_{\bar{\bar{z}}}(\bar{\bar{z}}=1,..,a)$ be a set of non-negative real numbers. The PA operator is then represented as follows:

$$PA\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_a\right)=\sum_{z=1}^{a}\left(\frac{\left(1+T\left(\overline{\overline{NR}}_{\bar{\bar{z}}}\right)\right)}{\sum_{o=1}^{a}\left(1+T\left(\overline{\overline{NR}}_o\right)\right)}\overline{\overline{NR}}_{\bar{\bar{z}}}\right), \tag{1.48}$$

where, $\bar{\bar{T}}\left(\overline{\overline{NR}}_{\bar{\bar{z}}}\right)=\sum_{o=1}^{a}Sup\left(\overline{\overline{NR}}_{\bar{\bar{z}}},\overline{\overline{NR}}_o\right)$ and $Sup\left(\overline{\overline{NR}}_{\bar{\bar{z}}},\overline{\overline{NR}}_x\right)$ is the support degree for $\overline{\overline{NR}}_{\bar{\bar{z}}}$ and $\overline{\overline{NR}}_o$, which must gratify the following condition:

(1) $Sup\left(\overline{\overline{NR}}_{\bar{\bar{z}}},\overline{\overline{NR}}_o\right)\in[0,1]$;

(2) $Sup\left(\overline{\overline{NR}}_{\bar{\bar{z}}},\overline{\overline{NR}}_o\right)=Sup\left(\overline{\overline{NR}}_o,\overline{\overline{NR}}_z\right)$;

(3) If $\overline{\overline{Ds}}\left(\overline{\overline{NR}}_{\bar{\bar{z}}},\bar{\bar{\alpha}}_o\right)<\overline{\overline{Ds}}\left(\overline{\overline{NR}}_l,\overline{\overline{NR}}_m\right)$, then $Sup\left(\overline{\overline{NR}}_{\bar{\bar{z}}},\overline{\overline{NR}}_o\right)>Sup\left(\overline{\overline{NR}}_l,\overline{\overline{NR}}_m\right)$, where $\overline{\overline{Ds}}\left(\overline{\overline{NR}}_{\bar{\bar{z}}},\overline{\overline{NR}}_o\right)$ is the distance measure among $\overline{\overline{NR}}_{\bar{\bar{z}}}$ and $\overline{\overline{NR}}_o$.

1.3.4 Muirhead mean (MM) operator

The MM operator was first introduced by Muirhead [63] for classical numbers, which has the advantage of considering the interrelationship among any multiple aggregated arguments.

1.3.4.1 Definition [63]

Let $\overline{\overline{NR}}_{\overline{\overline{z}}}\ (\overline{\overline{z}}=1,...,o)$ be a set of real numbers and $\overline{\overline{Q}}=(q_1,q_2,...,q_o)\in R^o$ be a vector of parameters. Then, the MM operator is described as:

$$MM^{\overline{\overline{Q}}}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_o\right)=\left(\frac{1}{a!}\sum_{\theta\in S_o}\prod_{\overline{\overline{z}}=1}^{o}\overline{\overline{NR}}_{\theta(\overline{\overline{z}})}^{q_{\overline{\overline{z}}}}\right)^{\frac{1}{\sum_{z=1}^{o}q_{\overline{\overline{z}}}}}, \tag{1.49}$$

where, S_o is the group of permutation of $(1,2,...,o)$ and $\theta(z)$ is any permutation of $(1,2,...,o)$.

Now we can give some particular cases with respect to the parameter vector Q of MM operator, which are shown as follows:

(1) If $\overline{\overline{Q}}=(1,0,0,...,0)$, then MM operator deteriorates to the following form:

$$MM^{(1,0,...,0)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_o\right)=\frac{1}{o}\sum_{\overline{\overline{z}}=1}^{o}\overline{\overline{NR}}_z. \tag{1.50}$$

That is, the MM operator degenerates into arithmetic averaging operator.

(2) If $\overline{\overline{Q}}=\left(\frac{1}{o},\frac{1}{o},...,\frac{1}{o}\right)$, then MM operator degenerates into the following form:

$$MM^{\left(\frac{1}{o},\frac{1}{o},...,\frac{1}{o}\right)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_o\right)=\prod_{z=1}^{o}\left(\overline{\overline{NR}}_z\right)^{\frac{1}{o}}. \tag{1.51}$$

That is, the MM operator degenerates into geometric averaging operator.

(3) If $\overline{\overline{Q}}=(1,1,0,...,0)$, then MM operator degenerates to the following form:

$$MM^{(1,1,0,...,0)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_o\right)=\left(\frac{1}{o(o+1)}\sum_{\substack{z,g=1\\z\neq g}}^{o}\overline{\overline{NR}}_z\,\overline{\overline{NR}}_g\right)^{\frac{1}{2}}. \tag{1.52}$$

That is, the MM operator degenerates into BM operator (p=q=1).

(4) If $\overline{\overline{Q}}=\left(\overbrace{1,...,1}^{c},\overbrace{0,...,0}^{o-c}\right)$, then MM operator degenerates to the following form:

$$MM^{\left(\overbrace{1,...,1}^{c},\overbrace{0,...,0}^{o-c}\right)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_o\right)=\left(\frac{\sum_{1\le x_1<x_2<...<x_c\le a}\prod_{y=1}^{c}\overline{\overline{NR}}_{z_y}}{C_o^c}\right)^{\frac{1}{c}}. \tag{1.53}$$

That is, the MM operator degenerates into MSM operator.

1.3.5 The Hamy mean (HM) Operator

1.3.5.1 Definition [65]

The HM operator is described as follows:

$$HM^{(k)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_z\right)=\frac{\sum_{1\le i_1<i_2<...<i_k\le z}\left(\prod_{j=1}^{z}\overline{\overline{NR}}_{i_j}\right)^{\frac{1}{k}}}{C_z^k}, \tag{1.54}$$

where, $k,(1,2,...,z)$ are a parameter and $i_1,i_2,...,i_k$ are k integer values taken from the set of $\{1,2,...,z\}$ of z integer values, C_z^k express the binomial co-efficient and $C_z^k=\frac{z!}{k!(z-k)!}$.

The HM operator has the following properties, which are described below:

(1) When $\overline{\overline{NR}}_o=\overline{\overline{NR}}(o=1,2,...,z)$, then $HM^{(k)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_z\right)=\overline{\overline{NR}}$;

(2) When $\overline{\overline{NR}}_o\le\Im_o\left(o=1,2,...,z\right)$, then $HM^{(k)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_z\right)\le HM^{(k)}\left(\Im_1,\Im_2,...,\Im_z\right)$;

(3) $\min_o\overline{\overline{NR}}_o\le HM^{(k)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_z\right)\le\max\overline{\overline{NR}}_o\left(o=1,2,...,z\right)$.

The HM operator has two specific cases, which are defined below:

(1) When $k=1,HM^{(1)}\left(\overline{\overline{NR}}_1,\overline{\overline{NR}}_2,...,\overline{\overline{NR}}_z\right)=\frac{1}{k}\sum_{o=1}^{z}\overline{\overline{NR}}_o$, the HM operator degenerate into arithmetic mean operator.

(2) When $k = z, HM^{(z)}\left(\overline{\overline{NR}}_1, \overline{\overline{NR}}_2, ..., \overline{\overline{NR}}_z\right) = \left(\prod_{o=1}^{z} \overline{\overline{NR}}_o\right)^{\frac{1}{k}}$, the HM operator degenerate into geometric mean operator.

1.3.6 Prioritized Aggregation (PRA) Operator

PRA operator was first developed by Yager [85], which can consider the prioritization among the aggregated parameters. Let $\overline{\overline{O}} = \{\overline{\overline{O}}_1, \overline{\overline{O}}_2, ..., \overline{\overline{O}}_l\}$ be a family of attributes and ensure that there is a prioritization among the attributes represented by a linear ordering $\overline{\overline{O}}_1 > \overline{\overline{O}}_2 > ... > \overline{\overline{O}}_l$ which denote that the attribute $\overline{\overline{O}}_d$ has a high precedence then $\overline{\overline{O}}_f$, if $d < f$. $\overline{\overline{O}}_d(u)$ is an evaluation value expressing the execution of the alternative u under the attribute $\overline{\overline{O}}_d$ and satisfies $\overline{\overline{O}}_d(u) \in [0,1]$. if

$$PRA(\overline{\overline{O}}_d(u)) = \sum_{d=1}^{l} w_d \overline{\overline{O}}_d(u). \tag{1.55}$$

PA operators have been effectively applied in a condition where the input arguments are exact values.

1.3.7 Linguistic Scale Functions (LSFs)

To utilize data more capably and to articulate the semantics more pliably, LSFs give diverse semantic values to linguistic scales under diverse situations [48]. They are superior in practice since these functions are pliable and can give more settled results according to diverse semantics.

1.3.7.1 Definition [48]

Suppose $\vartheta_z \in [0,1]$ is a numeric value, then the LSF $\overline{\overline{LS}}^*$ that conducts the mapping from S_z to $\vartheta_z (z = 0,1,2,...,2r)$ which is defined as follows:

$$\overline{\overline{LS}}^{*} : s_z \to \vartheta_z \ \ (z = 1, 2, ..., 2r), \tag{1.56}$$

where, $0 \le \vartheta_0 < \vartheta_1 < .... < \vartheta_{2r}$. Clearly, the symbol $\vartheta_z (z = 0,1, 2, ..., 2r)$ imitates the preference of the DMs when they are utilizing the LT $s_z \in S(z = 0,1, 2, ..., 2r)$. Therefore, the function value in fact indicates the semantics of the LTs.

(1) Consider

$$\overline{\overline{LS}}^{*}_{1}(s_z) = \vartheta_z = \frac{z}{2r}. \tag{1.57}$$

The assessment scale of the linguistic information specified above is divided on average.

(2) Consider

$$\overline{\overline{LS}}^{*}_{2}(s_z) = \vartheta_z = \begin{cases} \frac{\phi^r - \phi^{r-z}}{2\phi^r - 2} & (z = 0,1, 2, ..., r) \\ \frac{\phi^r - \phi^{z-r} - 2}{2\phi^r - 2} & (z = r+1, r+2..., 2r) \end{cases}, \tag{1.58}$$

with the expansion from the center of the specified LTS to both ends, the absolute deviation among adjoining linguistic subscripts also amplifies.

(3) Consider

$$\overline{\overline{LS}}^{*}_{3}(s_z) = \vartheta_z = \begin{cases} \frac{r^\alpha - (r-z)^\alpha}{2r^\alpha} & (z = 0,1, 2, ..., r) \\ \frac{r^\beta - (z-r)^\beta}{2r^\beta} & (z = r+1, r+2..., 2r) \end{cases}, \tag{1.59}$$

with the expansion from the center of the specified LTS to both ends, the absolute deviation among adjoining linguistic subscripts will reduce.

To conserve all the specified information and make the calculation easy, the above function can be enlarged to $\overline{\overline{LS}}^{*} : \overline{S} \to R^{+} (R^{+} = \{c \mid c \ge 0, c \in R\})$, which satisfies $\overline{\overline{LS}}^{*}(s_z) = \vartheta_z$ and is a strictly monotonically increasing and continuous function.

Therefore, the function from $\overline{S}\ to\ R^{+}$ is one-to-one because of its monotonicity, and the inverse function of $\overline{\overline{LS}}^{*}$ exists and is denoted by $\overline{\overline{LS}}^{*-1}$.

Chapter 2

Group Decision Making Based on Power Heronian Aggregation Operators under Linguistic Neutrosophic Environment

In this chapter, we merged the PA operator with HM operator and enlarged them to process linguistic neutrosophic information, and presented the linguistic neutrosophic power Heronian aggregation (LNPHA) operator, linguistic neutrosophic power weight Heronian aggregation (LNPWHA) operator. Moreover, some characteristics of these new developed aggregation operators are examined and some particular cases are discussed. Furthermore, we propose a new technique based on these developed aggregation operators for MAGDM. Lastly, some illustrative examples were given to illustrate the efficiency and advantages of the developed method by comparing with some existing methods.

2.1 The Linguistic Neutrosophic Power Heronian Mean Operators

In this part, we propose the LNPHA operator and LNPWHA operators based on the operational laws for LNNs.

2.1.1 The Linguistic Neutrosophic PHA Operator

2.1.1.1 Definition

Let $\hbar_h = \langle s_{\Xi_h}, s_{\Psi_h}, s_{\Upsilon_h} \rangle (h = 1, 2, \cdots, r)$ be a group of LNNs, $p, q > 0,$ and LNPHA is a map $LNPHA : \Omega^r \to \Omega,$ if

$$LNPHA^{p,q}(\hbar_1, \hbar_2, ..., \hbar_r) = \left(\frac{2}{r(r+1)} \sum_{h=1}^{r} \sum_{g=i}^{r} \left(r \frac{\left(1 + \overline{\overline{T}}(\hbar_h)\right)}{\sum_{l=1}^{r}\left(1 + \overline{\overline{T}}(\hbar_l)\right)} \hbar_h \right)^P \otimes \left(r \frac{\left(1 + \overline{\overline{T}}(\hbar_g)\right)}{\sum_{l=1}^{r}\left(1 + \overline{\overline{T}}(\hbar_l)\right)} \hbar_g \right)^q \right)^{\frac{1}{p+q}}, \quad (2.1)$$

where, $\overline{\overline{T}}(\hbar_l) = \sum_{\substack{h=1 \\ h \neq l}}^{r} \sup(\hbar_l, \hbar_h),$ and $\sup(\hbar_l, \hbar_h)$ is the support degree (SD) for $\hbar_l$ from $\hbar_h,$ which must satisfy the following characteristics.

1) $\sup(\hbar_l, \hbar_h) \in [0,1]$; 2) $\sup(\hbar_l, \hbar_h) = \sup(\hbar_h, \hbar_l)$; 3) $\sup(\hbar, \hbar) \geq \sup(\overline{\hbar}, \hbar),$ if $\overline{\overline{Ds}}(\hbar, \hbar) < \overline{\overline{Ds}}(\overline{\hbar}, \hbar)$, in which $\overline{\overline{Ds}}(\hbar, \hbar)$ is the distance between LNNs $\hbar$ and $\hbar$.

In order to write expression (2.1) in a more simplified form, we can define

$$\omega_k = \frac{\left(1 + \overline{\overline{T}}(\hbar_k)\right)}{\sum_{k=1}^{r}\left(1 + \overline{\overline{T}}(\hbar_k)\right)}, \quad (2.2)$$

and call $(\omega_1, \omega_2, ..., \omega_r)^T$ as the power weighting vector (PWV) with $\omega_k \geq 0, \sum_{k=1}^{r} \omega_k = 1.$

Then, expression (2.1) can be represented as follows:

$$LNPHA^{p,q}(\hbar_1, \hbar_2, ..., \hbar_r) = \left(\frac{2}{r(r+1)} \sum_{h=1}^{r} \sum_{g=i}^{r} \left(r\omega_k \hbar_h\right)^P \otimes \left(r\omega_k \hbar_g\right)^q \right)^{\frac{1}{p+q}}. \quad (2.3)$$

2.1.1.2 Theorem

Let $\hbar_h = \left\langle s_{\Xi_h}, s_{\Psi_h}, s_{\Upsilon_h} \right\rangle (h = 1, 2, \cdots, r)$ be a group of LNNs, and $p, q > 0$, then, the result aggregated employing Equation (2.3) is still a LNN, and even

$$LNPHA^{p,q}\left(\hbar_1, \hbar_2, ..., \hbar_r\right) = \left\langle \left(\overline{\overline{LS}}^{*-1} \left(1 - \left(\prod_{h=1}^{r} \prod_{g=i}^{r} \left(1 - (1 - (1 - \overline{\overline{LS}}^{*}(s_{\Xi_h}))^{r\omega_h})^{p} (1 - (1 - \overline{\overline{LS}}^{*}(s_{\Xi_h}))^{r\omega_h})^{q} \right) \right)^{\frac{2}{r(r+1)}} \right)^{\frac{1}{p+q}} \right), \right.$$

$$\overline{\overline{LS}}^{*-1} \left(1 - \left(1 - \left(\prod_{h=1}^{r} \prod_{g=i}^{r} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Psi_h}) \right)^{r\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Psi_g}) \right)^{r\omega_g} \right)^{q} \right) \right)^{\frac{2}{r(r+1)}} \right)^{\frac{1}{p+q}} \right), \quad (2.4)$$

$$\left. \overline{\overline{LS}}^{*-1} \left(1 - \left(1 - \left(\prod_{h=1}^{r} \prod_{g=i}^{r} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h}) \right)^{r\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g}) \right)^{r\omega_g} \right)^{q} \right) \right)^{\frac{2}{r(r+1)}} \right)^{\frac{1}{p+q}} \right) \right) \right\rangle.$$

Proof. Firstly, we need to prove the following Equation.

$$\sum_{h=1}^{r} \sum_{g=i}^{r} \left(r\omega_h \hbar_h \right)^{p} \otimes \left(r\omega_g \hbar_g \right)^{q} = \left\langle \left(\overline{\overline{LS}}^{*-1} \left(1 - \left(\prod_{h=1}^{r} \prod_{g=i}^{r} \left(1 - (1 - (1 - \overline{\overline{LS}}^{*}(s_{\Xi_h}))^{r\omega_h})^{p} (1 - (1 - \overline{\overline{LS}}^{*}(s_{\Xi_g}))^{r\omega_g})^{q} \right) \right) \right) \right), \right.$$

$$\overline{\overline{LS}}^{*-1} \left(\left(\prod_{h=1}^{r} \prod_{g=i}^{r} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Psi_h}) \right)^{r\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Psi_g}) \right)^{r\omega_g} \right)^{q} \right) \right) \right), \qquad (2.5)$$

$$\left. \overline{\overline{LS}}^{*-1} \left(\left(\prod_{h=1}^{r} \prod_{g=i}^{r} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h}) \right)^{r\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g}) \right)^{r\omega_g} \right)^{q} \right) \right) \right) \right\rangle.$$

By the operational rules of LNNs defined in (1.31)-(1.34), we have

$$\left(r\omega_h \hbar_h \right)^{p} = \left\langle \left(\overline{\overline{LS}}^{*-1} \left(\left(1 - \left(1 - \overline{\overline{LS}}^{*}(s_{\Xi_h}) \right)^{r\omega_h} \right)^{p} \right), \overline{\overline{LS}}^{*-1} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Psi_h}) \right)^{r\omega_h} \right)^{p} \right), \overline{\overline{LS}}^{*-1} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h}) \right)^{r\omega_h} \right)^{p} \right) \right) \right\rangle,$$

$$\left(r\omega_g \hbar_g \right)^{q} =$$

$$\left\langle \left(\overline{\overline{\partial}}^{*-1} \left(\left(1 - \left(1 - \overline{\overline{\partial}}^{*}(s_{\Xi_g}) \right)^{r\omega_g} \right)^{q} \right), \overline{\overline{\partial}}^{*-1} \left(1 - \left(1 - \left(\overline{\overline{\partial}}^{*}(s_{\Psi_g}) \right)^{r\omega_g} \right)^{q} \right), \overline{\overline{\partial}}^{*-1} \left(1 - \left(1 - \left(\overline{\overline{\partial}}^{*}(s_{\Upsilon_g}) \right)^{r\omega_g} \right)^{q} \right) \right) \right\rangle.$$

$$
\begin{aligned}
\left(r\omega_h\hbar_h\right)^p \otimes \left(r\omega_g\hbar_g\right)^q = \Bigg\langle \Bigg(& \overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_h})\right)^{r\omega_h}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_g})\right)^{r\omega_g}\right)^q\right), \\
& \overline{\overline{LS}}^{*-1}\left(\left(1-\left(\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{r\omega_h}\right)^p\right)\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_g})\right)^{r\omega_g}\right)^q\right)\right), \\
& \overline{\overline{LS}}^{*-1}\left(\left(1-\left(\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{r\omega_h}\right)^p\right)\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g})\right)^{r\omega_g}\right)^q\right)\right)\Bigg)\Bigg\rangle,
\end{aligned}
\tag{2.6}
$$

(1) When $r=2,$ by Equation (1.31) and Equation (2.6), we have

$$
\begin{aligned}
\sum_{h=1}^{2}\sum_{g=i}^{2}\left(2\omega_h\hbar_h\right)^P \otimes \left(2\omega_g\hbar_g\right)^q &= \left(\left(2\omega_1\hbar_1\right)^P \otimes \left(2\omega_1\hbar_1\right)^q\right) \oplus \left(\left(2\omega_1\hbar_1\right)^P \otimes \left(2\omega_2\hbar_2\right)^q\right) \\
& \oplus \left(\left(2\omega_2\hbar_2\right)^P \otimes \left(2\omega_2\hbar_2\right)^q\right),
\end{aligned}
$$

$$
\begin{aligned}
= \Bigg\langle \Bigg(& \overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_1})\right)^{2\omega_1}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_1})\right)^{2\omega_1}\right)^q\right), \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_1})\right)^{2\omega_1}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_1})\right)^{2\omega_1}\right)^q\right) \\
& , \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_1})\right)^{2\omega_1}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_1})\right)^{2\omega_1}\right)^q\right)\Bigg)\Bigg\rangle \oplus \Bigg\langle\Bigg(\overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_1})\right)^{2\omega_1}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_2})\right)^{2\omega_2}\right)^q\right), \\
& \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_1})\right)^{2\omega_1}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_2})\right)^{2\omega_2}\right)^q\right), \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_1})\right)^{2\omega_1}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_2})\right)^{2\omega_2}\right)^q\right)\Bigg)\Bigg\rangle \oplus \\
& \Bigg\langle\Bigg(\overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_2})\right)^{2\omega_2}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_2})\right)^{2\omega_2}\right)^q\right), \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_2})\right)^{2\omega_2}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_2})\right)^{2\omega_2}\right)^q\right) \\
& , \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_2})\right)^{2\omega_2}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_2})\right)^{2\omega_2}\right)^q\right)\Bigg)\Bigg\rangle.
\end{aligned}
$$

By using Equation (1.31), we get

$$
\begin{aligned}
\sum_{h=1}^{2}\sum_{g=i}^{2}\left(r\omega_h\hbar_h\right)^p \otimes \left(r\omega_g\hbar_g\right)^q = \Bigg\langle\Bigg(& \overline{\overline{LS}}^{*-1}\left(1-\prod_{h=1}^{2}\prod_{g=1}^{2}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_h})\right)^{2\omega_h}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_g})\right)^{2\omega_g}\right)^q\right)\right), \\
& \overline{\overline{LS}}^{*-1}\left(\prod_{h=1}^{2}\prod_{g=1}^{2}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{2\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_g})\right)^{2\omega_g}\right)^q\right)\right), \\
& \overline{\overline{LS}}^{*-1}\left(\prod_{h=1}^{2}\prod_{g=1}^{2}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{2\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g})\right)^{2\omega_g}\right)^q\right)\right)\Bigg)\Bigg\rangle.
\end{aligned}
\tag{2.7}
$$

That is, Equation (2.5) holds for $r=2.$

(2) Let us assume that Equation (2.5) holds for $r = z$.

$$\sum_{\hbar=1}^{z}\sum_{g=i}^{z}\left(r\omega_h\hbar_h\right)^p \otimes \left(r\omega_g\hbar_g\right)^q = \left\langle \left(\overline{\overline{LS}}^{*-1}\left(1-\prod_{\hbar=1}^{z}\prod_{g=1}^{z}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_h})\right)^{z\omega_h}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_g})\right)^{z\omega_g}\right)^q\right)\right)\right), \right.$$
$$\overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{z}\prod_{g=1}^{z}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{z\omega_h}\right)^p,\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi g})\right)^{z\omega_g}\right)^q\right)\right), \tag{2.8}$$
$$\left. \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{z}\prod_{g=1}^{z}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{z\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g})\right)^{z\omega_g}\right)^q\right)\right)\right\rangle.$$

Furthermore, when $m = z+1$, we have

$$\sum_{\hbar=1}^{z+1}\sum_{g=i}^{z+1}\left((z+1)\omega_h\hbar_h\right)^p \otimes \left((z+1)\omega_g\hbar_g\right)^q = \sum_{\hbar=1}^{z}\sum_{g=i}^{z}\left((z+1)\omega_h\hbar_h\right)^p \otimes \left((z+1)\omega_g\hbar_g\right)^q \oplus \sum_{\hbar=1}^{z}\left((z+1)\omega_h\hbar_h\right)^p \otimes \tag{2.9}$$
$$\left((z+1)\omega_{z+1}\hbar_{z+1}\right)^q \oplus \left((z+1)\omega_{z+1}\hbar_{z+1}\right)^p \otimes \left((z+1)\omega_{z+1}\hbar_{z+1}\right)^q.$$

Firstly, we prove that

$$\sum_{\hbar=1}^{z}\left((z+1)\omega_h\hbar_h\right)^p \otimes \left((z+1)\omega_{z+1}\hbar_{z+1}\right)^q = \left\langle \left(\overline{\overline{LS}}^{*-1}\left(1-\prod_{\hbar=1}^{z}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_h})\right)^{(z+1)\omega_h}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_{z+1}})\right)^{(z+1)\omega_{z+1}}\right)^q\right)\right)\right), \right.$$
$$\overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{z}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{(z+1)\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{z+1}})\right)^{(z+1)\omega_{z+1}}\right)^q\right)\right), \tag{2.10}$$
$$\left. \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{z}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{(z+1)\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{z+1}})\right)^{(z+1)\omega_{z+1}}\right)^q\right)\right)\right\rangle.$$

We shall prove Equation (2.10) on mathematical induction on z.

(a) For $z = 2,$ we have

$$\sum_{\hbar=1}^{2}\left((z+1)\omega_h\hbar_h\right)^p \otimes \left((z+1)\omega_{z+1}\hbar_{z+1}\right)^q = \left((3\omega_1\hbar_1)^p \otimes (3\omega_3\hbar_3)^q\right) \oplus \left((3\omega_2\hbar_2)^p \otimes (3\omega_3\hbar_3)^q\right)$$

$$= \left\langle \left(\overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_1})\right)^{3\omega_1}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_3})\right)^{3\omega_3}\right)^q\right), \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_1})\right)^{3\omega_1}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_3})\right)^{3\omega_3}\right)^q\right)\right.\right.$$
$$\left.\left., \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_1})\right)^{3\omega_1}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_3})\right)^{3\omega_3}\right)^q\right)\right)\right\rangle \oplus \left\langle \left(\overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_2})\right)^{3\omega_2}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_3})\right)^{3\omega_3}\right)^q\right), \right.\right.$$
$$\left.\left. \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_2})\right)^{3\omega_2}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_3})\right)^{3\omega_3}\right)^q\right), \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_2})\right)^{3\omega_2}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_3})\right)^{3\omega_3}\right)^q\right)\right)\right\rangle.$$

$$
= \left\langle \left(\overline{\overline{LS}}^{*-1}\left(1-\prod_{\hbar=1}^{2}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_{\hbar}})\right)^{3\omega_{\hbar}}\right)^{p}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_{3}})\right)^{3\omega_{3}}\right)^{q}\right)\right), \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{2}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{\hbar}})\right)^{3\omega_{\hbar}}\right)^{p} \right.\right.\right.
$$

$$
\left.\left.\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{3}})\right)^{3\omega_{3}}\right)^{q}\right)\right), \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{2}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{\hbar}})\right)^{3\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{3}})\right)^{3\omega_{3}}\right)^{q}\right)\right)\right\rangle. \tag{2.11}
$$

(b) Let us assume that Equation (2.10) holds for $z=b$, that is;

$$
\sum_{\hbar=1}^{z}\left(\left((z+1)\omega_{\hbar}\hbar_{\hbar}\right)^{p}\otimes\left((z+1)\omega_{b+1}\hbar_{b+1}\right)^{q}\right)
$$

$$
\begin{aligned}
= \Bigg\langle \Bigg(& \overline{\overline{LS}}^{*-1}\left(1-\prod_{\hbar=1}^{b}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_{\hbar}})\right)^{(b+1)\omega_{\hbar}}\right)^{p}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_{b+1}})\right)^{(b+1)\omega_{b+1}}\right)^{q}\right)\right), \\
& \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{b}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{\hbar}})\right)^{(b+1)\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{b+1}})\right)^{(b+1)\omega_{b+1}}\right)^{q}\right)\right), \\
& \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{b}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{\hbar}})\right)^{(b+1)\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{b+1}})\right)^{(b+1)\omega_{b+1}}\right)^{q}\right)\right)\Bigg)\Bigg\rangle.
\end{aligned} \tag{2.12}
$$

Then, when $z=b+1$, we have

$$
\begin{aligned}
\sum_{\hbar=1}^{b+1}\left(\left((b+2)\omega_{\hbar}\hbar_{\hbar}\right)^{p}\otimes\left((b+2)\omega_{b+2}\hbar_{b+2}\right)^{q}\right) &= \sum_{\hbar=1}^{b}\left(\left((b+2)\omega_{\hbar}\hbar_{\hbar}\right)^{p}\otimes\left((b+2)\omega_{b+2}\hbar_{b+2}\right)^{q}\right) \\
&\oplus\left(\left((b+2)\omega_{b+1}\hbar_{b+1}\right)^{p}\otimes\left((b+2)\omega_{b+2}\hbar_{b+2}\right)^{q}\right),
\end{aligned}
$$

$$
= \left\langle \left(\overline{\overline{LS}}^{*-1}\left(1-\prod_{\hbar=1}^{b}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_{\hbar}})\right)^{(b+2)\omega_{\hbar}}\right)^{p}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_{b+2}})\right)^{(b+2)\omega_{b+2}}\right)^{q}\right)\right), \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{b}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{\hbar}})\right)^{(b+2)\omega_{\hbar}}\right)^{p} \right.\right.\right.
$$

$$
\left.\left.\left.\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{b+2}})\right)^{(b+2)\omega_{b+2}}\right)^{q}\right)\right), \overline{\overline{LS}}^{*-1}\left(\prod_{\hbar=1}^{b}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{\hbar}})\right)^{(b+2)\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{b+2}})\right)^{(b+2)\omega_{b+2}}\right)^{q}\right)\right)\right) \oplus
$$

$$
\left(\overline{\overline{\upsilon}}^{*-1}\left(1-\left(1-\left(1-\left(1-\overline{\overline{\upsilon}}^{*}(s_{\Xi_{b+1}})\right)^{(b+2)\omega_{b+1}}\right)^{p}\left(1-\left(1-\overline{\overline{\upsilon}}^{*}(s_{\Xi_{b+2}})\right)^{(b+2)\omega_{b+2}}\right)^{q}\right)\right)\right), \overline{\overline{\upsilon}}^{*-1}\left(1-\left(1-\left(\overline{\overline{\upsilon}}^{*}(s_{\Psi_{b+1}})\right)^{(b+2)\omega_{b+1}}\right)^{p} \right.
$$

$$
\left.\left(1-\left(\overline{\overline{\upsilon}}^{*}(s_{\Psi_{b+2}})\right)^{(b+2)\omega_{b+2}}\right)^{q}\right), \overline{\overline{\upsilon}}^{*-1}\left(\left(1-\left(1-\left(\overline{\overline{\upsilon}}^{*}(s_{\Upsilon_{b+1}})\right)^{(b+2)\omega_{b+1}}\right)^{p}\left(1-\left(\overline{\overline{\upsilon}}^{*}(s_{\Upsilon_{b+2}})\right)^{(b+2)\omega_{b+2}}\right)^{q}\right)\right)\right\rangle,
$$

$$= \left\langle \left(\overline{\overline{LS}}^{*-1} \left(1 - \prod_{h=1}^{b+1} \left(1 - \left(1 - \left(1 - \overline{\overline{LS}}^{*} (s_{\Xi_h}) \right)^{(b+2)\omega_h} \right)^{p} \left(1 - \left(1 - \overline{\overline{LS}}^{*} (s_{\Xi_{b+2}}) \right)^{(b+2)\omega_{b+2}} \right)^{q} \right) \right) \right), \right.$$

$$\overline{\overline{LS}}^{*-1} \left(\prod_{h=1}^{b+1} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Psi_h}) \right)^{(b+2)\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Psi_{b+2}}) \right)^{(b+2)\omega_{b+2}} \right)^{q} \right) \right),$$

$$\left. \overline{\overline{LS}}^{*-1} \left(\prod_{h=1}^{b+1} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Upsilon_h}) \right)^{(b+2)\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Upsilon_{b+2}}) \right)^{(b+2)\omega_{b+2}} \right)^{q} \right) \right) \right\rangle.$$

Therefore Equation (2.10) is true for $z = b+1$. Hence Equation (2.10), is also true for all z.

Similarly, we can prove the other parts of Equation (2.9).

So Equation (2.9) becomes

$$\sum_{h=1}^{z+1} \sum_{g=i}^{z+1} \left((z+1)\omega_h \hbar_h \right)^{p} \otimes \left((z+1)\omega_g \hbar_g \right)^{q} = \left\langle \left(\overline{\overline{LS}}^{*-1} \left(1 - \prod_{h=1}^{z+1} \prod_{g=1}^{z+1} \left(1 - \left(1 - \left(1 - \overline{\overline{LS}}^{*} (s_{\Xi_h}) \right)^{(z+1)\omega_h} \right)^{p} \left(1 - \left(1 - \overline{\overline{LS}}^{*} (s_{\Xi_g}) \right)^{(z+1)\omega_g} \right)^{q} \right) \right) \right), \right.$$

$$\overline{\overline{LS}}^{*-1} \left(\prod_{h=1}^{z+1} \prod_{g=1}^{z+1} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Psi_h}) \right)^{(z+1)\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Psi_g}) \right)^{(z+1)\omega_g} \right)^{q} \right) \right),$$

$$\left. \overline{\overline{LS}}^{*-1} \left(\prod_{h=1}^{z+1} \prod_{g=1}^{z+1} \left(1 - \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Upsilon_h}) \right)^{(z+1)\omega_h} \right)^{p} \left(1 - \left(\overline{\overline{LS}}^{*} (s_{\Upsilon_g}) \right)^{(z+1)\omega_g} \right)^{q} \right) \right) \right\rangle.$$

Therefore Equation (2.5) is true for $r = z+1$. Hence Equation (2.5) is true for all r.

By Equation (2.5), we can prove that Equation (2.4) is right. From Equation (2.5) and the operational laws defined for LNNs, we have

$$\frac{2}{r(r+1)}\sum_{h=1}^{r}\sum_{g=i}^{r}\left(r\omega_r\hbar_r\right)^p\otimes\left(r\omega_g\hbar_g\right)^q$$

$$=\left\langle\left(\overline{\overline{LS}}^{*-1}\left(1-\left(\prod_{h=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_h})\right)^{r\omega_h}\right)^p\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_g})\right)^{r\omega_g}\right)^q\right)\right)^{\frac{2}{r(r+1)}}\right)\right),\right.$$

$$\overline{\overline{LS}}^{*-1}\left(\left(\prod_{h=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{r\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_g})\right)^{r\omega_g}\right)^q\right)\right)^{\frac{2}{r(r+1)}}\right),$$

$$\left.\overline{\overline{LS}}^{*-1}\left(\left(\prod_{h=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{r\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g})\right)^{r\omega_g}\right)^q\right)\right)^{\frac{2}{r(r+1)}}\right)\right\rangle.$$

So,

$$\left(\frac{2}{r(r+1)}\sum_{h=1}^{r}\sum_{g=i}^{r}\left(r\omega_h\hbar_h\right)^p\otimes\left(r\omega_g\hbar_g\right)^q\right)^{\frac{1}{p+q}}$$

$$=\left\langle\left(\overline{\overline{LS}}^{*-1}\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_h}))^{r\omega_h})^p(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_g}))^{r\omega_g})^q\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{r\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_g})\right)^{r\omega_g}\right)^q\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right),$$

$$\left.\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{h=i}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{r\omega_h}\right)^p\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g})\right)^{r\omega_g}\right)^q\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)\right)\right\rangle.$$

This completes the proof of Theorem 2.1.1.2.

In order to calculate the PWV ω, we firstly need to calculate the SD between LNNs. In general, the support degree between LNNs can be replaced by the similarity degree between LNNs. That is,

$$Sup\left(\hbar_h,\hbar_l\right)=1-\overline{\overline{Ds}}\left(\hbar_h,\hbar_l\right)(h,l=1,2,....,r)\,. \tag{2.13}$$

2.1.1.3 Theorem

Let $\hbar_h=\left\langle s_{\Xi_h},s_{\Psi_h},s_{\Upsilon_h}\right\rangle(h=1,2,\cdots,r)$ be a group of LNNs, and $\hbar_h=\hbar=\left\langle s_{\Xi},s_{\Psi},s_{\Upsilon}\right\rangle$ for all $i=1,2,...,m,$ then

$$LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right)=\hbar. \tag{3.14}$$

Proof. Since $\hbar_h=\hbar=\left\langle s_{\Xi},s_{\Psi},s_{\Upsilon}\right\rangle$ for all $h=1,2,...,r,$ we have

$Sup\left(\hbar_l,\hbar_g\right)=1$, for all $l,g=1,2,...,r$, so $\omega_l=\dfrac{1}{r}$ for all $l=1,2,...,r.$ Then

$LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right)=LNPHA^{p,q}\left(\hbar,\hbar,...,\hbar\right)$,

$$=\left\langle\left(\overline{\overline{LS}}^{*-1}\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=h}^{r}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_\Xi)\right)^{r\frac{1}{r}}\right)^{p}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_\Xi)\right)^{r\frac{1}{r}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right),\right.$$

$$\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_\Psi)\right)^{r\frac{1}{r}}\right)\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_\Psi)\right)^{r\frac{1}{r}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right),$$

$$\left.\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=h}^{r}\left(\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_\Upsilon)\right)^{r\frac{1}{r}}\right)\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_\Upsilon)\right)^{r\frac{1}{r}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)\right\rangle,$$

$$=\left\langle\left(\overline{\overline{LS}}^{*-1}\left(\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-\left(\overline{\overline{LS}}^{*}\left(s_\Xi\right)\right)^{p+q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right),\right.$$

$$\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{g=1}^{r}\prod_{\hbar=i}^{r}\left(1-\left(1-\overline{\overline{LS}}^{*}\left(s_\Psi\right)\right)^{p+q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right),$$

$$\left.\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\overline{\overline{LS}}^{*}\left(s_\Upsilon\right)\right)^{p+q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)\right\rangle,$$

$$=\left\langle \left(\overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\left(\overline{\overline{LS}}^{*}\left(s_{\Xi}\right)\right)^{p+q}\right)\right)^{\frac{1}{p+q}}\right), \overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}\left(s_{\Psi}\right)\right)^{p+q}\right)^{\frac{1}{p+q}}\right), \overline{\overline{LS}}^{*-1}\left(\left(1-\left(1-\overline{\overline{LS}}^{*}\left(s_{\Upsilon}\right)\right)^{p+q}\right)^{\frac{1}{p+q}}\right)\right)\right\rangle,$$

$$=\left\langle s_{\Xi}, s_{\Psi}, s_{\Upsilon}\right\rangle .$$

This completes the proof of Theorem 2.1.1.3.

2.1.1.4 Theorem

Let $\hbar_h=\left\langle s_{\Xi_h}, s_{\Psi_h}, s_{\Upsilon_h}\right\rangle (h=1,2,\cdots,r)$ be a group of LNNs, and $s_{\Xi^-}=\min\limits_{1\le h\le r} s_{\Xi_h}, s_{\Xi^+}=\max\limits_{1\le h\le r} s_{\Xi_h}, s_{\Psi^-}=\min\limits_{1\le h\le r} s_{\Psi_h},\ s_{\Psi^+}=\max\limits_{1\le h\le r} s_{\Psi_h}, s_{\Upsilon^-}=\min\limits_{1\le h\le r} s_{\Upsilon_h}, s_{\Upsilon^+}=\max\limits_{1\le h\le r} s_{\Upsilon_h}$, for all $h=1,2,...,r.$ Then, the LNPHA operator lies:

$$\left\langle s_{\Xi^-}, s_{\Psi^+}, s_{\Upsilon^+}\right\rangle \le LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right) \le \left\langle s_{\Xi^+}, s_{\Psi^-}, s_{\Upsilon^-}\right\rangle. \tag{2.15}$$

Proof. Since $s_{\Xi^-}=\min\limits_{1\le h\le r} s_{\Xi_h}, s_{\Xi^+}=\max\limits_{1\le h\le r} s_{\Xi_h}, s_{\Psi^-}=\min\limits_{1\le h\le r} s_{\Psi_h},\ s_{\Psi^+}=\max\limits_{1\le h\le r} s_{\Psi_h}, s_{\Upsilon^-}=\min\limits_{1\le h\le r} s_{\Upsilon_h}, s_{\Upsilon^+}=\max\limits_{1\le h\le r} s_{\Upsilon_h}$, for all $h=1,2,...,r.$ Then, there are $s_{\Xi^-}\le s_{\Xi_h}\le s_{\Xi^+}, s_{\Psi^-}\le s_{\Psi_h}\le s_{\Psi^+}, s_{\Upsilon^-}\le s_{\Upsilon_h}\le s_{\Upsilon^+}$. Further, we have

$$s_{\Xi}=\overline{\overline{LS}}^{*-1}\left(\left(1-\left(\prod_{h=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_h})\right)^{r\omega_h}\right)^{p}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi_g})\right)^{r\omega_g}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)$$

$$\ge \overline{\overline{LS}}^{*-1}\left(\left(1-\left(\prod_{h=1}^{r}\prod_{h=1}^{r}\left(1-\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi^-})\right)^{r\omega_h}\right)^{p}\left(1-\left(1-\overline{\overline{LS}}^{*}(s_{\Xi^-})\right)^{r\omega_g}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)=s_{\Xi^-}$$

$$s_{\Psi} = \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{\hbar}})\right)^{r\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_{g}})\right)^{r\omega_{g}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)$$

$$\leq \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi^{+}})\right)^{r\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi^{+}})\right)^{r\omega_{g}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right) = s_{\Psi^{+}}$$

$$s_{\Xi} = \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{\hbar}})\right)^{r\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_{g}})\right)^{r\omega_{g}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)$$

$$\leq \overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=1}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon^{+}})\right)^{r\omega_{\hbar}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon^{+}})\right)^{r\omega_{g}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right) = s_{\Upsilon^{+}}$$

.

Because $\overline{\overline{LS}}^{*}$ is a montone increasing function, then, there is the following comparison:

(1) For the expected value:

$$\overline{\overline{SO}}(\hbar) = \frac{1}{3}\left(\overline{\overline{LS}}^{*}(s_{\Xi}) + 2 - \overline{\overline{LS}}^{*}(s_{\Psi}) - \overline{\overline{LS}}^{*}(s_{\Upsilon})\right) \geq \frac{1}{3}\left(\overline{\overline{LS}}^{*}(s_{\Xi^{-}}) + 2 - \overline{\overline{LS}}^{*}(s_{\Psi^{+}}) - \overline{\overline{LS}}^{*}(s_{\Upsilon^{+}})\right) = \overline{\overline{SO}}\left(\left\langle s_{\Xi}, s_{\Psi}, s_{\Upsilon}\right\rangle\right).$$

If $\overline{\overline{SO}}(\hbar) > \overline{\overline{SO}}\left(\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle\right)$,

then, $\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle < LNPHA^{p,q}(\hbar_1, \hbar_2, ..., \hbar_r)$.

Else, $\overline{\overline{SO}}(\hbar) = \overline{\overline{SO}}\left(\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle\right)$, then, we have the score function,

(2) $\overline{\overline{AR}}(\hbar) = \overline{\overline{LS}}^{*}(s_{\Xi}) - \overline{\overline{LS}}^{*}(s_{\Upsilon}) \geq \overline{\overline{LS}}^{*}(s_{\Xi^{-}}) - \overline{\overline{LS}}^{*}(s_{\Upsilon^{+}}) = \overline{\overline{AR}}\left(\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle\right)$.

If $\overline{\overline{AR}}(\hbar) > \overline{\overline{AR}}\left(\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle\right)$,

then, $\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle < LNPHA^{p,q}(\hbar_1, \hbar_2, ..., \hbar_r)$.

Else, $\overline{\overline{AR}}(\hbar) = \overline{\overline{AR}}\left(\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle\right)$, then, we have the certainty function,

(3) $\overline{\overline{CR}}(\hbar) = \overline{\overline{LS}}^{*}(s_{\Xi}) \geq \overline{\overline{LS}}^{*}(s_{\Xi^{-}}) = \overline{\overline{CR}}\left(\left\langle s_{\Xi^{-}}, s_{\Psi^{+}}, s_{\Upsilon^{+}}\right\rangle\right)$.

then, $\left\langle s_{\Xi^-}, s_{\Psi^+}, s_{\Upsilon^+}\right\rangle = LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right)$.

$$\left\langle s_{\Xi^-}, s_{\Psi^+}, s_{\Upsilon^+}\right\rangle \le LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right).$$

In a analogous way, we can prove that $LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right) \le \left\langle s_{\Xi^+}, s_{\Psi^-}, s_{\Upsilon^-}\right\rangle$.

Hence, we have

$$\left\langle s_{\Xi^-}, s_{\Psi^+}, s_{\Upsilon^+}\right\rangle \le LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right) \le \left\langle s_{\Xi^+}, s_{\Psi^-}, s_{\Upsilon^-}\right\rangle.$$

However, the property of monotonicity, for $LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right)$ cannot be proved. The main reason is that the importance degrees can be calculated from the support degrees, for the two groups of LNNs, and there is no constant inequality relationship among them.

Now we can discuss some particular cases of the LNPHA operator by assigning different values to the parameters *p* and *q*.

(1) When, $p = q = 1$, then Eq. (2.4) degenerates to linguistic neutrosophic power line Heronian mean operator, that is

$$LNPHA^{1,1}\left(\hbar_1,\hbar_2,...,\hbar_r\right) = \left\langle \begin{array}{l} \left(\overset{=}{\upsilon}^{*-1}\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-(1-(1-\overset{=}{\upsilon}^{*}(s_{\Xi_\hbar}))^{r\omega_\hbar})(1-(1-\overset{=}{\upsilon}^{*}(s_{\Xi_g}))^{r\omega_g})\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{2}}\right), \\ \overset{=}{\upsilon}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\left(\overset{=}{\upsilon}^{*}(s_{\Psi_\hbar})\right)^{r\omega_\hbar}\right)\left(1-\left(\overset{=}{\upsilon}^{*}(s_{\Psi_g})\right)^{r\omega_g}\right)\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{2}}\right), \\ \overset{=}{\upsilon}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\left(\overset{=}{\upsilon}^{*}(s_{\Upsilon_\hbar})\right)^{r\omega_\hbar}\right)\left(1-\left(\overset{=}{\upsilon}^{*}(s_{\Upsilon_g})\right)^{r\omega_g}\right)\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{2}}\right) \end{array} \right\rangle. \quad (2.16)$$

(2) When, $p = q = \frac{1}{2}$, then Eq. (2.4) degenerates to linguistic neutrosophic power basic Heronian mean operator, that is

$$LNPHA^{\frac{1}{2},\frac{1}{2}}(\hbar_1,\hbar_2,...,\hbar_r)=\left\langle \overline{\overline{LS}}^{*-1}\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-\sqrt{(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_h}))^{r\omega_h})(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_g}))^{r\omega_g})}\right)\right)^{\frac{2}{r(r+1)}}\right),\right.$$

$$\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-\sqrt{\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{r\omega_h}\right)\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_g})\right)^{r\omega_g}\right)}\right)\right)^{\frac{2}{r(r+1)}}\right)\right), \quad (2.17)$$

$$\left.\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-\sqrt{\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{r\omega_h}\right)\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g})\right)^{r\omega_g}\right)}\right)\right)^{\frac{2}{r(r+1)}}\right)\right)\right\rangle.$$

(3) When, $p=0,\ and\ q\neq 0,$ then Eq. (2.4) degenerates to linguistic neutrosophic power generalized linear ascending weighted operator, that is

$$LNPHA^{0,q}(n_1,n_2,...,n_r)=\left\langle \overline{\overline{LS}}^{*-1}\left(1-\left(\prod_{h=1}^{r}\prod_{g=h}^{r}\left(1-(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_g}))^{r\omega_g})^{q}\right)^{h}\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{q}},\right.$$

$$\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{g=h}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_g})\right)^{r\omega_g}\right)^{q}\right)^{h}\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{q}}\right), \quad (2.18)$$

$$\left.\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{g=h}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_j})\right)^{r\omega_g}\right)^{q}\right)^{h}\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{q}}\right)\right\rangle.$$

(4) When $p\neq 0,\ and\ q=0,$ then Eq. (2.4) degenerates to linguistic neutrosophic power generalized linear descending weighted operator, that is

$$LNPHA^{p,0}(\hbar_1,\hbar_2,...,\hbar_r)=\left\langle\left(\overline{\overline{LS}}^{*-1}\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_\hbar}))^{r\omega_\hbar})^{p}\right)^{r+1-\hbar}\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p}},\right.\right.$$

$$\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_\hbar})\right)^{r\omega_\hbar}\right)^{p}\right)^{r+1-\hbar}\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p}}\right), \qquad (2.19)$$

$$\left.\left.\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{\hbar=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_\hbar})\right)^{r\omega_\hbar}\right)^{p}\right)^{r+1-\hbar}\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p}}\right)\right)\right\rangle.$$

So, from Eq. (2.18) and Eq. (2.19), we can see that $LNPHA^{0,q}(\hbar_1,\hbar_2,...,\hbar_r)$ and $LNPHA^{p,0}(\hbar_1,\hbar_2,...,\hbar_r)$ weigh the information $\left((r\omega_1\hbar_1)^q,(r\omega_2\hbar_2)^q,...,(r\omega_r\hbar_r)^q\right)$ and $\left((r\omega_1\hbar_1)^p,(r\omega_2\hbar_2)^p,...,(r\omega_r\hbar_r)^p\right)$ with heavy weight vectors $(1,2,...,r)$ and $(r,r-1,...,1)$. Hence, whenever $p=0\,or\,q=0, LNPHA^{p,q}(\hbar_1,\hbar_2,...,\hbar_r)$ have the linear weighted function and also the parameters $p\,and\,q$ are not interchangeable.

In LNPHA operators, only the PWV and the interrelationship among LNNs are considered and the weight of every LNN is not taken under consideration. However, in real decision making problems, the weight vector of input arguments is also a necessary parameter. So, to overcome this limitation of the LNPHA operator, we will propose the linguistic neutrosophic power weighted Heronian aggregation (LNPWHA) operator.

2.1.2 The Linguistic Neutrosophic Power Weighted Heronian Aggregation Operators

2.1.2.1 Definition

Let $n_h = \left\langle s_{\Xi_h}, s_{\Psi_h}, s_{\Upsilon_h} \right\rangle (h = 1, 2, \cdots, r)$ be a group of LNNs, $p, q > 0,$ and $LNPHA : \Omega^r \to \Omega,$ if

$$LNPWHA^{p,q}\left(\hbar_1, \hbar_2, ..., \hbar_r\right) = \left(\frac{2}{r(r+1)} \sum_{h=1}^{r} \sum_{g=i}^{r} \left(\frac{r\omega_h \varphi_h}{\sum_{z=1}^{r} \omega_z \varphi_z} \hbar_i \right)^P \otimes \left(\frac{r\omega_g \varphi_g}{\sum_{z=1}^{r} \omega_z \varphi_z} \hbar_g \right)^q \right)^{\frac{1}{p+q}}, \qquad (2.20)$$

where, $\omega_z = \frac{\left(1 + \overline{\overline{T}}(\hbar_z)\right)}{\sum_{z=1}^{r}\left(1 + \overline{\overline{T}}(\hbar_z)\right)}$ and $\sum_{z=1}^{r} \omega_z = 1.$ $\overline{\overline{T}}(\hbar_l) = \sum_{\substack{\hbar=1 \\ h \neq l}}^{r} \sup(\hbar_l, \hbar_h),$ and $\sup(\hbar_l, \hbar_h)$ is the support degree for $\hbar_l$ from $\hbar_h,$ which has the following properties.

1) $\sup(\hbar_l, \hbar_h) \in [0,1]$; 2) $\sup(\hbar_l, \hbar_h) = \sup(\hbar_h, \hbar_l)$; 3) $\sup(\hbar, \hbar) \geq \sup(\overline{\overline{\hbar}}, \hbar),$ if $\overline{\overline{Ds}}(\hbar, \hbar) < \overline{\overline{Ds}}(\overline{\overline{\hbar}}, \hbar)$, in which $\overline{\overline{D}}(\hbar, \hbar)$ is the distance between LNNs $\hbar$ and $\hbar$. Then LNPWHA is called the linguistic neutrosophic power weighted Heronian aggregation operator.

2.1.2.2 Theorem

Let $\hbar_h = \left\langle s_{\Xi_h}, s_{\Psi_h}, s_{\Upsilon_h} \right\rangle (h = 1, 2, \cdots, r)$ be a group of LNNs, and $p, q > 0,$ then the result aggregated from (2.20) is still a LNN, and even

$$LNPHA^{p,q}\left(\hbar_1,\hbar_2,...,\hbar_r\right)=\left\langle \left(\overline{\overline{LS}}^{*-1}\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_h}))^{\frac{r\omega_i\varphi_i}{\sum_{z=1}^{r}\omega_z\varphi_z}})^{p}(1-(1-\overline{\overline{LS}}^{*}(s_{\Xi_g}))^{\frac{r\omega_g\varphi_g}{\sum_{z=1}^{r}\omega_z\varphi_z}})^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right),\right.$$

$$\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_h})\right)^{\frac{r\omega_h\varphi_h}{\sum_{z=1}^{r}\omega_z\varphi_z}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Psi_g})\right)^{\frac{r\omega_g\varphi_g}{\sum_{z=1}^{r}\omega_z\varphi_z}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right),$$

$$\left.\overline{\overline{LS}}^{*-1}\left(1-\left(1-\left(\prod_{h=1}^{r}\prod_{g=i}^{r}\left(1-\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_h})\right)^{\frac{r\omega_h\varphi_h}{\sum_{z=1}^{r}\omega_z\varphi_z}}\right)^{p}\left(1-\left(\overline{\overline{LS}}^{*}(s_{\Upsilon_g})\right)^{\frac{r\omega_g\varphi_g}{\sum_{z=1}^{r}\omega_z\varphi_z}}\right)^{q}\right)\right)^{\frac{2}{r(r+1)}}\right)^{\frac{1}{p+q}}\right)\right\rangle. \quad (2.21)$$

The proof of this theorem is similar to Theorem 2.1.1.2, therefore, omitted here.

Similar to the LNPHA operator, the LNPWHA operator has the characteristics of boundedness, however, it does not have the characteristics of idempotency and monotonicity.

2.2 Multi-criteria Group Decision Making Based on Linguistic Neutrosophic Power Weighted Heronian Mean operator

For a MAGDM problem with LNNs in which the weights of experts and attributes are known, Let $\overline{\overline{N}}=\left\{\overline{\overline{N}}_1,\overline{\overline{N}}_2,....,\overline{\overline{N}}_m\right\}$, $\overline{\overline{O}}=\left\{\overline{\overline{O}}_1,\overline{\overline{O}}_2,....,\overline{\overline{O}}_n\right\}$ represent the set of alternatives and attributes respectively, and the experts set can be sapcified by $e=\{e_1,e_2,...,e_z\}$. Suppose that $n_{ab}^{l}=\left\langle s_{\Xi_{ab}}^{l},s_{\Psi_{ab}}^{l},s_{\Upsilon_{ab}}^{l}\right\rangle$ is the attribute assessment value for the alternative $\overline{\overline{N}}_a$ about the attributes $\overline{\overline{O}}_b$ given by the expert e_l. Let the importance degree of the attributes $\left\{\overline{\overline{O}}_1,\overline{\overline{O}}_2,...,\overline{\overline{O}}_n\right\}$ can be denoted by $\varphi=(\varphi_1,\varphi_2,...,\varphi_n)$ and the importance degree of the experts $\{e_1, e_2, ..., e_z\}$ can be denoted by $\delta = (\delta_1, \delta_2, ..., \delta_z)$, both the importance degrees satisfy the condition that $\varphi_g, \delta_h \in [0,1]$, $\sum_{g=1}^{n} \varphi_g = 1$ and $\sum_{h=1}^{z} \delta_h = 1$. Then the aspire of this MAGDM problem is to rank the alternatives.

The MAGDM method consists of the following main steps by using LNPWHA operator.

Step 1. Normalize the decision matrix.

Generally, there are two types of attributes. One is of cost type and the other is of benefit type. In MAGDM method, we need to convert the attributes in cost type to ones in benefit type. The following method is used to convert the cost type into benefit type.

$$n_{ab}^{l} = \left\langle s_{\Xi_{ab}}^{l}, s_{\Psi_{ab}}^{l}, s_{\Upsilon_{ab}}^{l} \right\rangle = \begin{cases} \left\langle s_{\Xi_{ab}}^{l}, s_{\Psi_{ab}}^{l}, s_{\Upsilon_{ab}}^{l} \right\rangle & \text{for benefit attribute } \overline{\overline{O}}_b \\ \left\langle s_{\Upsilon_{ab}}^{l}, s_{2r-\Psi_{ab}}^{l}, s_{\Xi_{ab}}^{l} \right\rangle & \text{for cost attribute } \overline{\overline{O}}_b \end{cases}. \tag{2.22}$$

So, the decision matrices $A = \left[a_{ab}^{l} \right]_{m \times n}$ can be changed into matrices $R = \left[n_{ab}^{l} \right]_{m \times n}$.

Step 2. Determine the supports $Sup\left(n_{ab}^{l}, n_{ac}^{l}\right)\left(a = 1, 2, ..., m, l = 1, 2, ..., z, b, c = 1, 2, ..., n\right)$ by

$$Sup\left(n_{ab}^{l}, n_{ac}^{l}\right) = 1 - \overline{\overline{D}}\left(n_{ab}^{l}, n_{ac}^{l}\right), \tag{2.23}$$

where, $\overline{\overline{D}}\left(n_{ab}^{l}, n_{ac}^{l}\right)$ is the Hamming distance between two LNNs n_{ab}^{l} *and* n_{ac}^{l}, which is given in Definition 1.2.3.6.

Step 3. Calculate $T\left(n_{ab}^{l}\right)$ by

$$T\left(n_{ab}^{l}\right) = \sum_{\substack{c=1 \\ c \neq b}}^{n} Sup\left(n_{ab}^{l}, n_{ac}^{l}\right), (a = 1, 2, ..., m, l = 1, 2, ..., z, b, c = 1, 2, ..., n). \tag{2.24}$$

Step 4. Calculate

$$\kappa_{ab}^{l} = \frac{n\omega_b\left(1+T\left(n_{ab}^{l}\right)\right)}{\sum_{z=1}^{n}\omega_b\left(1+T\left(n_{az}^{l}\right)\right)}, (a=1,2,...,m, l=1,2,...,z, b=1,2,...,n) . \tag{2.25}$$

Step 5. Utilize the LNPHA operator

$$n_a^l = \left\langle s_{\Xi_a}, s_{\Psi_a}, s_{\Upsilon_a} \right\rangle = LNPHA\left(n_{a1}^l, n_{a2}^l, ..., n_{an}^l\right). \tag{2.26}$$

To determine the overall LNNs n_a^l $(a=1,2,...,m, l=1,2,...,z)$.

Step 6. Determine the supports $Sup\left(n_a^l, n_a^g\right)(a=1,2,....,m, l, g=1,2,....,z)$ by

$$Sup\left(n_a^l, n_a^g\right) = 1 - \overline{\overline{D}}\left(n_a^l, n_a^g\right), \tag{2.27}$$

where, $\overline{\overline{D}}\left(n_a^l, n_a^g\right)$ is the Hamming distance among two LNNs n_a^l and n_a^g, which is given in Definition 1.2.3.6.

Step 7. Calculate $T\left(n_a^l\right)$ by

$$T\left(n_a^l\right) = \sum_{\substack{g=1 \\ g\neq l}}^{z} Sup\left(n_a^l, n_a^g\right), (a=1,2,...,m, l=1,2,...,z) , \tag{2.28}$$

Step 8. Calculate

$$\kappa_a^l = \frac{l\delta_l\left(1+T\left(n_a^l\right)\right)}{\sum_{z=1}^{n}\delta_l\left(1+T\left(n_a^l\right)\right)}, (a=1,2,...,m, l=1,2,...,z) , \tag{2.29}$$

Step 9. Utilize the LNPHA operator.

$$n_a = \left\langle s_{\Xi_a}, s_{\Psi_a}, s_{\Upsilon_a} \right\rangle = LNPWHA\left(n_a^1, n_a^2, ..., n_a^z\right). \tag{2.30}$$

To determine the overall LNNs n_a $(a=1,2,...,m)$.

Step 10. Calculate the score values by using Definition 1.2.3.4 of the overall LNNs

Step 11. Rank the alternatives and select the best alternative according to their score values.   .

Step 12. End.

2.3 An illustrative example

In real applications, we can get the LNNs by questionnaire investigation or by Web comment data. (1) We should firstly give the LTS, then some customers are invited to give the TRM, IM and FLM by selecting a LT from LTS. So we can give a LNN. (2) There are a lot of evaluation data from the customers by Web based on the LT, then we can produce the LNNs by statistical method. In order to conveniently compare with the existing methods, we can cite an example from Liu [8] by changing the types of original evaluation data.

2.3.1 Example

Assume that there are four alternatives $\left(\overline{\overline{N}}_1, \overline{\overline{N}}_2, \overline{\overline{N}}_3, \overline{\overline{N}}_4\right)$ showing the air quality of Guangzhou City in 2006, 2007, 2008 and 2009. Three attributes were taken into account which consists of the $SO2(\overline{\overline{O}}_1)$, the $NO2(\overline{\overline{O}}_2)$, and the $PM10\left(\overline{\overline{O}}_3\right)$. Importance degree of attributes is $\varphi = (0.5, 0.3, 0.2)^T$. The possible four alternatives $\overline{\overline{N}}_a\ (a = 1,2,3,4)$ are evaluated by three air-quality monitoring stations assessed as experts $e = (e_1, e_2, e_3)$ under the LTS $\{s_0 = very\,low,\ s_1 = low, s_2 = slightly\,low, s_3 = medium, s_4 = slightly\,good$ $, s_5 = good, s_6 = very\,good\}$. The importance degree of the experts is $\delta = (0.4, 0.3, 0.3)^T$. The evaluation values are represented by the LNNs, which are given in Tables 2.1,2.2 and 2.3.

Table 2.1**.** Air quality data from station e_1

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_5, s_2, s_2 \rangle$	$\langle s_4, s_1, s_2 \rangle$	$\langle s_4, s_2, s_3 \rangle$

$\overline{\overline{N}}_2$	$\langle s_4, s_2, s_1 \rangle$	$\langle s_3, s_2, s_3 \rangle$	$\langle s_5, s_2, s_2 \rangle$
$\overline{\overline{N}}_3$	$\langle s_6, s_3, s_3 \rangle$	$\langle s_4, s_2, s_3 \rangle$	$\langle s_5, s_3, s_3 \rangle$
$\overline{\overline{N}}_4$	$\langle s_6, s_2, s_2 \rangle$	$\langle s_3, s_2, s_2 \rangle$	$\langle s_4, s_2, s_2 \rangle$

Table 2.2. Air quality data from station e_2

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_4, s_2, s_3 \rangle$	$\langle s_5, s_2, s_3 \rangle$	$\langle s_4, s_2, s_6 \rangle$
$\overline{\overline{N}}_2$	$\langle s_5, s_3, s_3 \rangle$	$\langle s_4, s_3, s_1 \rangle$	$\langle s_3, s_1, s_3 \rangle$
$\overline{\overline{N}}_3$	$\langle s_4, s_1, s_2 \rangle$	$\langle s_4, s_2, s_1 \rangle$	$\langle s_4, s_1, s_2 \rangle$
$\overline{\overline{N}}_4$	$\langle s_5, s_2, s_1 \rangle$	$\langle s_4, s_2, s_2 \rangle$	$\langle s_3, s_1, s_2 \rangle$

Table 2.3**.** Air quality data from station e_3

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_3, s_2, s_3 \rangle$	$\langle s_2, s_1, s_3 \rangle$	$\langle s_5, s_1, s_2 \rangle$
$\overline{\overline{N}}_2$	$\langle s_4, s_2, s_3 \rangle$	$\langle s_3, s_1, s_2 \rangle$	$\langle s_2, s_3, s_3 \rangle$
$\overline{\overline{N}}_3$	$\langle s_3, s_1, s_1 \rangle$	$\langle s_4, s_2, s_1 \rangle$	$\langle s_2, s_4, s_2 \rangle$
$\overline{\overline{N}}_4$	$\langle s_4, s_3, s_1 \rangle$	$\langle s_3, s_1, s_2 \rangle$	$\langle s_5, s_1, s_2 \rangle$

Rank the Alternatives by the Proposed Method.

Step 1. Convert the decision matrices $A = \left[a_{ab}^{l} \right]_{m \times n}$ into normalized matrices $R = \left[n_{ab}^{l} \right]_{m \times n}$.

Since all the attribute are the benefit type, so there is no need to normalize it.

Step 2. Calculate the supports $Sup\left(n_{ab}^{l}, n_{ac}^{l}\right)\left(a=1,2,3,4, l=1,2,3, b, c=1,2,3\right)$ by for formula (2.23). To be easily understood we shall denote $Sup\left(n_{ab}^{l}, n_{ac}^{l}\right)$ with $S_{ab,ac}^{l}$ $\left(a=1,2,3,4, l=1,2,3, b, c=1,2,3\right)$, we have

$S_{11,12}^{1}=S_{12,11}^{1}=0.8889, S_{12,13}^{1}=S_{13,12}^{1}=0.8889, S_{11,13}^{1}=S_{13,11}^{1}=0.7778$,

$S_{21,22}^{1}=S_{22,21}^{1}=0.8334, S_{22,23}^{1}=S_{23,22}^{1}=0.8334, S_{21,23}^{1}=S_{23,11}^{1}=0.8889$,

$S_{31,32}^{1}=S_{32,31}^{1}=0.8334, S_{32,33}^{1}=S_{33,32}^{1}=0.8889, S_{31,33}^{1}=S_{33,31}^{1}=0.9445$,

$S_{41,42}^{1}=S_{42,41}^{1}=0.8334, S_{42,43}^{1}=S_{43,42}^{1}=0.9445, S_{41,43}^{1}=S_{43,41}^{1}=0.8889$,

$S_{11,12}^{2}=S_{12,11}^{2}=0.9445, S_{12,13}^{2}=S_{13,12}^{2}=0.7778, S_{11,13}^{2}=S_{13,11}^{2}=0.8334$,

$S_{21,22}^{2}=S_{22,21}^{2}=0.8334, S_{22,23}^{2}=S_{23,22}^{2}=0.7778, S_{21,23}^{2}=S_{23,21}^{2}=0.7223$,

$S_{31,32}^{2}=S_{32,31}^{2}=0.8889, S_{32,33}^{2}=S_{33,32}^{2}=1.0000, S_{31,33}^{2}=S_{33,31}^{2}=0.8889$,

$S_{41,42}^{2}=S_{42,41}^{2}=0.8889, S_{42,43}^{2}=S_{43,42}^{2}=0.8889, S_{41,43}^{2}=S_{43,41}^{2}=0.7778$,

$S_{11,12}^{3}=S_{12,11}^{3}=0.8889, S_{12,13}^{3}=S_{13,12}^{3}=0.7778, S_{11,13}^{3}=S_{13,11}^{3}=0.7778$,

$S_{21,22}^{3}=S_{22,21}^{3}=0.8334, S_{22,23}^{3}=S_{23,22}^{3}=0.7778, S_{21,23}^{3}=S_{23,21}^{3}=0.8334$,

$S_{31,32}^{3}=S_{32,31}^{3}=0.8889, S_{32,33}^{3}=S_{33,32}^{3}=0.7223, S_{31,33}^{3}=S_{33,31}^{3}=0.7223$,

$S_{41,42}^{3}=S_{42,41}^{3}=0.7778, S_{42,43}^{3}=S_{43,42}^{3}=0.8889, S_{41,43}^{3}=S_{43,41}^{3}=0.7778$.

Step 3. Calculate $T\left(n_{ab}^{l}\right)\left(b=1,2,3; a=1,2,3,4; l=1,2,3\right)$ by formula (2.24) (for simplicity, we denote $T\left(n_{ab}^{l}\right)$ with T_{ab}^{l}).

$T_{11}^{1}=1.7778, T_{12}^{1}=1.7778, T_{13}^{1}=1.7778, T_{21}^{1}=1.7223, T_{22}^{1}=1.6667, T_{23}^{1}=1.7223,$

$T_{31}^{1}=1.7778, T_{32}^{1}=1.7223, T_{33}^{1}=1.8334, T_{41}^{1}=1.7223, T_{42}^{1}=1.7778, T_{43}^{1}=1.8334,$

$T_{11}^{2}=1.7778, T_{12}^{2}=1.7223, T_{13}^{2}=1.6112, T_{21}^{2}=1.6112, T_{22}^{2}=1.5556, T_{23}^{2}=1.5001,$

$T_{31}^2 = 1.8889, T_{32}^2 = 1.7778, T_{33}^2 = 1.8889, T_{41}^2 = 1.6667, T_{42}^2 = 1.7778, T_{43}^2 = 1.6667,$

$T_{11}^3 = 1.6667, T_{12}^3 = 1.6667, T_{13}^3 = 1.5556, T_{21}^3 = 1.6667, T_{22}^3 = 1.6112, T_{23}^3 = 1.6112,$

$T_{31}^3 = 1.6112, T_{32}^3 = 1.6112, T_{33}^3 = 1.4445, T_{41}^3 = 1.5556, T_{42}^3 = 1.6667, T_{43}^3 = 1.6667.$

Step 4. Calculate $\kappa_{ab}^l (a = 1,2,3,4; b = 1,2,3; l = 1,2,3)$, we have

$\kappa_{11}^1 = 1.5000, \kappa_{12}^1 = 0.9000, \kappa_{13}^1 = 0.6000, \kappa_{21}^1 = 1.5092, \kappa_{22}^1 = 0.8871, \kappa_{23}^1 = 0.6037,$

$\kappa_{31}^1 = 1.5030, \kappa_{32}^1 = 0.8838, \kappa_{33}^1 = 0.6132, \kappa_{41}^1 = 1.4789, \kappa_{42}^1 = 0.9054, \kappa_{43}^1 = 0.6157,$

$\kappa_{11}^2 = 1.5275, \kappa_{12}^2 = 0.8982, \kappa_{13}^2 = 0.5743, \kappa_{21}^2 = 1.5227, \kappa_{22}^2 = 0.8942, \kappa_{23}^2 = 0.5832,$

$\kappa_{31}^2 = 1.5175, \kappa_{32}^2 = 0.8755, \kappa_{33}^2 = 0.6070, \kappa_{41}^2 = 1.4815, \kappa_{42}^2 = 0.9259, \kappa_{43}^2 = 0.5926,$

$\kappa_{11}^3 = 1.5126, \kappa_{12}^3 = 0.9076, \kappa_{13}^3 = 0.5798, \kappa_{21}^3 = 1.5158, \kappa_{22}^3 = 0.8905, \kappa_{23}^3 = 0.5937,$

$\kappa_{31}^3 = 1.5194, \kappa_{32}^3 = 0.9116, \kappa_{33}^3 = 0.5690, \kappa_{41}^3 = 1.4681, \kappa_{42}^3 = 0.9191, \kappa_{43}^3 = 0.6128.$

Step 5. Now use the LNWPHA operator to calculate the overall LNNs n_a^l, and the results are given in Table 2.4 (assume that $p = 1, q = 2$).

Step 6. Calculate the supports $Sup\left(n_a^l, n_a^i\right)$ based on formula (2.26) (For simplicity, we denote $Sup\left(n_a^l, n_a^i\right)$ with $S_{l,i}^a (a = 1,2,3,4, l, i = 1,2,3)$. we have

$S_{1,2}^1 = S_{2,1}^1 = 0.9269, S_{2,3}^1 = S_{3,2}^1 = 0.9136, S_{1,3}^1 = S_{3,1}^1 = 0.8861, S_{1,2}^2 = S_{2,1}^2 = 0.9539,$

$S_{2,3}^2 = S_{3,2}^2 = 0.8840, S_{1,3}^2 = S_{3,1}^2 = 0.9040, S_{1,2}^3 = S_{2,1}^3 = 0.7442, S_{2,3}^3 = S_{3,2}^3 = 0.9026,$
$S_{1,3}^3 = S_{3,1}^3 = 0.6906, S_{1,2}^4 = S_{2,1}^4 = 0.8534, S_{2,3}^4 = S_{3,2}^4 = 0.9785, S_{1,3}^4 = S_{3,1}^4 = 0.8336.$

Table 2.4. The overall assessment values of four alternatives

	e_1	e_2	e_3
$\overline{\overline{N}}_1$	$\langle s_{4.3674}, s_{1.7092}, s_{2.4271} \rangle$	$\langle s_{4.2866}, s_{2.1191}, s_{3.2530} \rangle$	$\langle s_{3.3802}, s_{1.4735}, s_{3.2556} \rangle$
$\overline{\overline{N}}_2$	$\langle s_{3.9670}, s_{2.1155}, s_{2.0525} \rangle$	$\langle s_{4.2568}, s_{2.4202}, s_{2.2870} \rangle$	$\langle s_{3.3278}, s_{1.8438}, s_{2.8691} \rangle$

$\overline{\overline{N}}_3$	$\langle s_{6.0000}, s_{2.7123}, s_{3.3065} \rangle$	$\langle s_{4.2485}, s_{1.4020}, s_{1.7630} \rangle$	$\langle s_{3.1942}, s_{1.7968}, s_{1.4585} \rangle$
$\overline{\overline{N}}_4$	$\langle s_{6.0000}, s_{2.1098}, s_{2.4354} \rangle$	$\langle s_{4.2483}, s_{1.8095}, s_{1.8492} \rangle$	$\langle s_{3.9699}, s_{1.7157}, s_{1.8644} \rangle$

Step 7. Calculate $T\left(n_a^l\right)(a=1,2,3,4; l=1,2,3)$ by Eq. (2.27) (for simplicity, we denote $T\left(n_a^l\right)$ with T_l^a).

$T_1^1=1.8129, T_2^1=1.8405, T_3^1=1.7997, T_1^2=1.8580, T_2^2=1.8380, T_3^2=1.7881,$

$T_1^3=1.4348, T_2^3=1.6468, T_3^3=1.5932, T_1^4=1.6871, T_2^4=1.8319, T_3^4=1.8121.$

Step 8.Calculate $\kappa_l^a (a=1,2,34; l=1,2,3)$, we have

$\kappa_1^1=1.1982, \kappa_2^1=0.9074, \kappa_3^1=0.8944, \kappa_1^2=1.2114, \kappa_2^2=0.9022, \kappa_3^2=0.8864,$

$\kappa_1^3=1.1476, \kappa_2^3=0.9357, \kappa_3^3=0.9167, \kappa_1^4=1.1649, \kappa_2^4=0.9208, \kappa_3^4=0.9143.$

Step 9. Using LNWPHA operator to calculate collective LNNs, we can get (assume that $p=1, q=2$)

$n_1=\langle s_{4.0023}, s_{1.7753}, s_{2.9639} \rangle$, $n_2=\langle s_{3.8294}, s_{2.1430}, s_{2.4214} \rangle$, $n_3=\langle s_6, s_{1.9311}, s_{2.0548} \rangle$,

$n_4=\langle s_6, s_{1.8975}, s_{2.0608} \rangle$.

Step 10. Determine the score values of every alternative by using Definition (1.2.3.4). We get

$\overline{\overline{SO}}(n_1)=0.6257, \overline{\overline{SO}}(n_2)=0.6258, \overline{\overline{SO}}(n_3)=0.7785, \overline{\overline{SO}}(n_4)=0.7800.$

Step 11. According to the score values the ranking order of the alternatives is

$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1.$

So the best alternative is $\overline{\overline{N}}_4$, while the worst one is $\overline{\overline{N}}_1$.

2.3.2 Influence of Linguistic Scale Function on The Decision Results

In order to show the effect of the LSF $\overset{=}{\upsilon}^{*}$ on this MAGDM problem, this subsection uses different LSFs in the proposed MAGDM method to obtain the ranking results of all alternatives. The ranking results are shown in Table 2.5. (In general, $LM = 1.4$ and $\alpha, \beta = 0.8$).

From Table 2.5, we can see that the ranking orders obtained by different LSFs for LNWPHA operator are different. The reason is that these three LSFs represent the different semantics which are described in subsection 1.3.7. The first one is one balanced LTS which is divided on average, and the others are unbalanced LTSs. We can construct some new LSFs according to the semantics of real applications. Therefore, under different situation, the DMs may select different or re-define LSFs according to their actual need.

Table 2.5. Effect for different LSFs $\overset{=}{\upsilon}^{*}$ in Example 2.3.1

linguistic scale function $\Re^{*}$	**Score values**	**Ranking order**
$\overset{=}{\upsilon}^{*}(s_z) = \frac{z}{2g}(0 \le z \le 2g)$	$\overline{\overline{SO}}(n_1) = 0.6257, \overline{\overline{SO}}(n_2) = 0.6258,$ $\overline{\overline{SO}}(n_3) = 0.7785, \overline{\overline{SO}}(n_4) = 0.7800$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1$
$\overset{=}{\upsilon}^{*}(s_z) = \begin{cases} \frac{LM^g - LM^{g-z}}{2LM^g - 2}(0 \le z \le g) \\ \frac{LM^g + LM^{z-g} - 2}{2LM^g - 2}(g \le z \le 2g) \end{cases}$	$\overline{\overline{SO}}(n_1) = 0.5855, \overline{\overline{SO}}(n_2) = 0.5852,$ $\overline{\overline{SO}}(n_3) = 0.7476, \overline{\overline{SO}}(n_4) = 0.7460$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$
$\overset{=}{\upsilon}^{*}(s_z) = \begin{cases} \frac{g^\alpha - (g-z)^\alpha}{2g^\alpha}(0 \le z \le g) \\ \frac{g^\beta - (z-g)^\beta}{2g^\beta}(g \le z \le 2g) \end{cases}$	$\overline{\overline{SO}}(n_1) = 0.6413, \overline{\overline{SO}}(n_2) = 0.6412,$ $\overline{\overline{SO}}(n_3) = 0.7892, \overline{\overline{SO}}(n_4) = 0.7928$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$

2.3.3 Effect of the Parameters *p, q* on Ranking Results

In this subsection, dissimilar values for the parameters p and q are taken into account and the LSF takes $\left(\overline{\overline{\upsilon}}^{*}(s_z)=z/2g\right)$. The score values and ranking orders obtained for different values of parameters p and q were given in Table 2.6.

The results in Table 2.6 show that the ranking orders obtained for different values of the parameters p and q are also different. In fact, when the values of the parameters p and q are larger, more prominent interactions between different attribute values are. If either $p=0$ or $q=0$, the proposed operator cannot confine the interrelationship of the individual arguments. In actual decision making problem, for computational simplicity, one can select $p=q=1$ *or* $p=q=1/2$, which is not only simple and straightforward, but also takes the interrelationships of the input arguments into account.

Table 2.6. Ranking order using different parameters p and q

	Score values	Ranking order
$p=q=1$	$\overline{\overline{SO}}(n_1)=0.6281, \overline{\overline{SO}}(n_2)=0.6232,$ $\overline{\overline{SO}}(n_3)=0.7668, \overline{\overline{SO}}(n_4)=0.7730.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$
$p=0.5,\ q=0.5$	$\overline{\overline{SO}}(n_1)=0.6089, \overline{\overline{SO}}(n_2)=0.6002,$ $\overline{\overline{SO}}(n_3)=0.7489, \overline{\overline{SO}}(n_4)=0.7550.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$

$p=0,q=1$	$\overline{\overline{SO}}(n_1)=0.5497,\overline{\overline{SO}}(n_2)=0.5358,$ $\overline{\overline{SO}}(n_3)=0.7365,\overline{\overline{SO}}(n_4)=0.7353.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$
$p=1,q=0$	$\overline{\overline{SO}}(n_1)=0.7427,\overline{\overline{SO}}(n_2)=0.7421,$ $\overline{\overline{SO}}(n_3)=0.8097,\overline{\overline{SO}}(n_4)=0.8334.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$
$p=2,q=3$	$\overline{\overline{SC}}(n_1)=0.6640,\overline{\overline{SC}}(n_2)=0.6691,$ $\overline{\overline{SC}}(n_3)=0.8007,\overline{\overline{SC}}(n_4)=0.8002.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1$
$p=5,q=7$	$\overline{\overline{SO}}(n_1)=0.7366,\overline{\overline{SO}}(n_2)=0.7481,$ $\overline{\overline{SO}}(n_3)=0.8413,\overline{\overline{SO}}(n_4)=0.8340.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1$

2.3.4 Comparison and Discussion

In this subpart, we compare the developed approach with some existing approaches.

2.3.4.1 Example

A panel is gathered in order to select a desirable low-carbon supplier for manufacturer. The panel receives LN information by accumulating linguistic evaluations from a dozens of DMs and calculating the mean values of the subscripts of the LVs. The DMs provided the information independently by giving equal rights. The LTS $S=\{s_0=extremely\ low,\ s_1=very\ low, s_2=low, s_3=slightly\ low, s_4=medium, s_5=slightly\ high, s_6=high, s_7=very\ high, s_8=extremely\ high\}$ is employed here. The four potential suppliers $\overline{\overline{N}}_i(i=1,2,3,4)$ are evaluated by DMs based on the LTS S according to the following three attributes: $\overline{\overline{O}}_1$ represents low-carbon technology, $\overline{\overline{O}}_2$ represents cost, $\overline{\overline{O}}_3$ represent capacity. The method of acquiring mean values is used for integration the linguistic information provided by DMs and the final assessment values are represented by LNNs, and the decision matrix $R=[r_{ij}]_{4\times3}$ is given in Table 2.7.

Table 3.7. Assessment values provided by DMs

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_4,s_4,s_3\rangle$	$\langle s_3,s_1,s_5\rangle$	$\langle s_2,s_3,s_5\rangle$
$\overline{\overline{N}}_2$	$\langle s_4,s_3,s_2\rangle$	$\langle s_1,s_2,s_3\rangle$	$\langle s_5,s_1,s_3\rangle$
$\overline{\overline{N}}_3$	$\langle s_5,s_1,s_3\rangle$	$\langle s_2,s_2,s_4\rangle$	$\langle s_2,s_4,s_3\rangle$
$\overline{\overline{N}}_4$	$\langle s_3,s_5,s_1\rangle$	$\langle s_2,s_2,s_2\rangle$	$\langle s_3,s_1,s_2\rangle$

Since the attribute $\overline{\overline{O}}_2$ is cost type and the other two attributes are benefit type, we need to convert $\overline{\overline{O}}_2$ to benefit type by using the formula defined in Li et al. [57], and the normalized decision matrix is given in Table 3.8.

Table 3.8. The normalized assessment values provided by DMs

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_4,s_4,s_3\rangle$	$\langle s_5,s_1,s_3\rangle$	$\langle s_2,s_3,s_5\rangle$
$\overline{\overline{N}}_2$	$\langle s_4,s_3,s_2\rangle$	$\langle s_3,s_2,s_1\rangle$	$\langle s_5,s_1,s_3\rangle$
$\overline{\overline{N}}_3$	$\langle s_5,s_1,s_3\rangle$	$\langle s_4,s_2,s_2\rangle$	$\langle s_2,s_4,s_3\rangle$
$\overline{\overline{N}}_4$	$\langle s_3,s_5,s_1\rangle$	$\langle s_2,s_2,s_2\rangle$	$\langle s_3,s_1,s_2\rangle$

Then we use the Li et al.' method [57] based on the LNGHM, LNPGHM operators and the proposed method in this paper based on the LNPHA, LNWPHA operators (assume $p=q=2$) to solve this problem, and the scores values of collective LNNs can be shown in Table 3.9.

From Table 3.9, we can know there are the same ranking orders of alternatives, this can prove the validity of the proposed method in this paper.

The following Example is adapted from Z. Fang et al.[58,59] to show effectiveness of the proposed method.

2.3.4.2 Example

An investment company plans to invest a sum of money in the available four companies as a set of alternatives denoted by $\overline{\overline{M}} = \left(\overline{\overline{N}}_1, \overline{\overline{N}}_2, \overline{\overline{N}}_3, \overline{\overline{N}}_4\right)$, where $\overline{\overline{N}}_1, \overline{\overline{N}}_2, \overline{\overline{N}}_3$ and $\overline{\overline{N}}_4$ respectively, represent a car company, a food company, a computer company and an arm company. To evaluate these companies, they invite a group of three experts $e_z (z = 1,2,3)$ to select the best company for investment among these companies. For the evaluation process the following three attributes are considered, denoted by $\overline{\overline{O}} = \left(\overline{\overline{O}}_1, \overline{\overline{O}}_2, \overline{\overline{O}}_3\right)$. Those attributes represent respectively, risk ($\overline{\overline{O}}_1$), growth ($\overline{\overline{O}}_2$), environmental impact $\overline{\overline{O}}_3$, which importance degree is $\varphi = (0.35, 0.25, 0.4)^T$. The possible four alternatives are evaluated by three decision makers $e_z (z = 1,2,3)$ under the above three attributes based on a predefined LTS $S = \{s_0 = extremely\, bad, s_1 = very\, bad, s_2 = bad, s_3 = slightly\, bad, s_4 = fair, s_5 = slightly\ good, s_6 = good, s_7 = very\, good,$

$s_8 = extremely\, good\}$. The importance degree of the three DMs is $\delta = (0.37, 0.33, 0.3)^T$. The decision matrices $U_z = \left(n_{ij}^{\,z}\right)_{4\times5}$ are listed in Tables 2.10-2.12.

Table 2.9. Score values and ranking orders by different aggregation operators for Example 2.3.4.1

Methods	Score values	Ranking orders
Base on LNGHM $(p = q = 2)$ [57]	$\overline{\overline{SO}}(n_1) = 0.4142, \overline{\overline{SO}}(n_2) = 0.5307,$ $\overline{\overline{SO}}(n_3) = 0.4724, \overline{\overline{SO}}(n_4) = 0.4703.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1$

Base on LNPHA $(p=q=2)$ in this paper	$\overline{\overline{SO}}(n_1)=0.5913, \overline{\overline{SO}}(n_2)=0.6853,$ $\overline{\overline{SO}}(n_3)=0.6419, \overline{\overline{SO}}(n_4)=0.6249.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1$
Base on LNPGHM $(p=q=2)$ [59]	$\overline{\overline{SO}}(n_1)=0.4762, \overline{\overline{SO}}(n_2)=0.5779,$ $\overline{\overline{SO}}(n_3)=0.5608, \overline{\overline{SO}}(n_4)=0.4995.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1$
Base on LNWPHA $(p=q=2)$ in this paper	$\overline{\overline{SO}}(n_1)=0.6053, \overline{\overline{SO}}(n_2)=0.6728,$ $\overline{\overline{SO}}(n_3)=0.6705, \overline{\overline{SO}}(n_4)=0.6075$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1$

Table 2.10 Decision matrices $U_z(z=1,2,3)$ by DM e_1 for Example 2.3.4.2

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_6, s_1, s_2 \rangle$	$\langle s_7, s_2, s_1 \rangle$	$\langle s_6, s_2, s_2 \rangle$
$\overline{\overline{N}}_2$	$\langle s_7, s_1, s_1 \rangle$	$\langle s_7, s_3, s_2 \rangle$	$\langle s_7, s_2, s_1 \rangle$
$\overline{\overline{N}}_3$	$\langle s_6, s_2, s_2 \rangle$	$\langle s_7, s_1, s_1 \rangle$	$\langle s_6, s_2, s_2 \rangle$
$\overline{\overline{N}}_4$	$\langle s_7, s_1, s_2 \rangle$	$\langle s_7, s_2, s_3 \rangle$	$\langle s_7, s_2, s_1 \rangle$

Table 2.11 Decision matrices $U_z(z=1,2,3)$ by DM e_2 for Example 2.3.4.2

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_6, s_1, s_2 \rangle$	$\langle s_6, s_1, s_1 \rangle$	$\langle s_4, s_2, s_3 \rangle$
$\overline{\overline{N}}_2$	$\langle s_7, s_2, s_3 \rangle$	$\langle s_6, s_1, s_1 \rangle$	$\langle s_4, s_2, s_3 \rangle$
$\overline{\overline{N}}_3$	$\langle s_5, s_1, s_2 \rangle$	$\langle s_5, s_1, s_2 \rangle$	$\langle s_5, s_4, s_2 \rangle$
$\overline{\overline{N}}_4$	$\langle s_6, s_1, s_1 \rangle$	$\langle s_5, s_1, s_1 \rangle$	$\langle s_5, s_2, s_3 \rangle$

Table 2.12 Decision matrices $U_z(z=1,2,3,4)$ by DM e_3 for Example 2.3.4.2.

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle s_7, s_3, s_4\rangle$	$\langle s_7, s_3, s_3\rangle$	$\langle s_5, s_2, s_5\rangle$
$\overline{\overline{N}}_2$	$\langle s_7, s_2, s_3\rangle$	$\langle s_5, s_1, s_2\rangle$	$\langle s_6, s_2, s_3\rangle$
$\overline{\overline{N}}_3$	$\langle s_7, s_2, s_4\rangle$	$\langle s_6, s_1, s_2\rangle$	$\langle s_7, s_2, s_4\rangle$
$\overline{\overline{N}}_4$	$\langle s_7, s_2, s_3\rangle$	$\langle s_5, s_2, s_1\rangle$	$\langle s_6, s_1, s_1\rangle$

Now we use the MAGDM methods given in Z. Fang et al.[58,59], and the aggregation operator defined in this chapter to solve this MAGDM problem. LSF takes$\left(\overline{\overline{\Re}}^*(s_z)=z/2g\right)$.

Table 3.13 Ranking order using different aggregation operator

Aggregation operator	Parameter values	Score values	Ranking order
LNNWAA operators [30]	No	$\overline{\overline{SO}}(n_1)=0.7528, \overline{\overline{SO}}(n_2)=0.7777,$ $\overline{\overline{SO}}(n_3)=0.7613, \overline{\overline{SO}}(n_4)=0.8060.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
LNNWGA operators [30]	No	$\overline{\overline{SO}}(n_1)=0.7143, \overline{\overline{SO}}(n_2)=0.7408,$ $\overline{\overline{SO}}(n_3)=0.7293, \overline{\overline{SO}}(n_4)=0.7789.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
LNNNWBM operators [31]	Yes $(p=q=1)$	$\overline{\overline{SO}}(n_1)=0.7284, \overline{\overline{SO}}(n_2)=0.7461,$ $\overline{\overline{SO}}(n_3)=0.7424, \overline{\overline{SO}}(n_4)=0.7864.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
LNNNWGB	Yes $(p=q=1)$	$\overline{\overline{SO}}(n_1)=0.7808, \overline{\overline{SO}}(n_2)=0.7627,$ $\overline{\overline{SO}}(n_3)=0.7510, \overline{\overline{SO}}(n_4)=0.7948.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

M operators [31]			
LNWPHA operators in this article	Yes $(p=q=1)$	$\overline{\overline{SO}}(n_1)=0.7515, \overline{\overline{SO}}(n_2)=0.7713,$ $\overline{\overline{SO}}(n_3)=0.7534, \overline{\overline{SO}}(n_4)=0.7984.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

Obviously, there are the same ranking results, it can further prove the effectiveness of the developed method. In the following, we will explain the advantage of the developed method.

From the above analysis, we can know the developed method has the advantages, i.e., it can relieve the influence of the awkward data by power weights and it can also consider the relationships among the attributes, and it can give more accurate ranking order then the existing methods. Of course, because the proposed method considered the PA and HM operators simultaneously, it is a bit complex in calculations.

2.3.5 Conclusion

In this Chapter, we merged the PA operator with HM operator and developed the LNPHA operator and LNPWHA operator. The developed aggregation operators can take full advantages of PA operator and HM, i.e., they can consider the relationships of the aggregated arguments and can reduce the influences of the awkward data by the power weighting. Further, we investigated some properties of these new aggregation operators and argued some particular cases, and developed a new method for MAGDM problems with LNNs based on these operators. Lastly, we gave some examples to explain its advantages by comparing with the existing methods. In the future research, we will extend PHM operator to some new extension of linguistic variables, such as linguistic double valued neutrosophic sets, linguistic picture fuzzy numbers, and so on.

Chapter 3

Application of Single-Valued Neutrosophic Power Muirhead Mean Operators to Multi-attribute Group Decision Making

In this chapter, we develop few new operators for aggregating SVN information and apply them to MAGDM. To acquire complete advantages of both MM operator and PA operator, we develop the SVN power MM (SVNPMM) operator, weighted SVN power MM (WSVNPMM) operator, SVN power dual MM (SVNPDMM) operator and weighted SVN power dual MM (WSVNPDMM) operator, and discuss their basic properties, special cases with respect to the parameter vector. The important advantages of the developed AOs are that it can eliminate the effect of awkward data and can consider the interrelationship among aggregated data at the same time. Moreover, based on the developed AOs, a novel approach to MAGDM problem is developed. Lastly, a numerical example is presented to confirm the efficacy and practicality of the developed approach.

3.1 Single-valued Neutrosophic Power Muirhead Mean Operators

In this part, we developed the SVNPMM operator, the WSVNPMM operator and discussed some basic properties and related results.

3.1.1 The Single-valued Neutrosophic Power Muirhead Mean operator

3.1.1.1 Definition

Let $\hbar_{\overline{\overline{g}}}(\overline{\overline{g}}=1,...,\overline{\overline{u}})$ be a set of SVNNs and $\overline{\overline{Q}}=\left(q_1,q_2,...,q_{\overline{\overline{u}}}\right)\in R^{\overline{\overline{u}}}$ be a vector of parameters. If

$$SVNPMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,....,\hbar_{\overline{\overline{u}}}\right)=\left(\frac{1}{\overline{\overline{u}}!}\sum_{\vartheta\in S_{\overline{\overline{u}}}}\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(\frac{\overline{\overline{u}}\left(1+\overline{\overline{T}}\left(\hbar_{\vartheta(\overline{\overline{g}})}\right)\right)}{\sum_{x=1}^{\overline{\overline{u}}}\left(1+\overline{\overline{T}}\left(\hbar_x\right)\right)}\hbar_{\vartheta(\overline{\overline{g}})}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}. \tag{3.1}$$

Then, we call $SVNPMM^{\overline{\overline{Q}}}$ the SVN power Muirhead mean operator, where $S_{\overline{\overline{u}}}$ is the set of all permutation, $\vartheta\left(\overline{\overline{g}}\right)$ is any permutation of $\left(1,...,\overline{\overline{u}}\right)$, $\overline{\overline{T}}\left(\hbar_{\overline{\overline{h}}}\right)=\sum_{\overline{\overline{h}}=1,\overline{\overline{h}}\neq\overline{\overline{g}}}^{\overline{\overline{u}}}Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)$, and $Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)$ is the support degree for $\hbar_{\overline{\overline{g}}}$ and $\hbar_{\overline{\overline{h}}}$, satisfying the following axioms:

(1) $Sup\left(\hbar_{\overline{\overline{z}}},\hbar_{\overline{\overline{h}}}\right)\in[0,1]$;

(2) $Sup\left(\hbar_{\overline{\overline{z}}},\hbar_{\overline{\overline{h}}}\right)=Sup\left(\hbar_{\overline{\overline{h}}},\hbar_{\overline{\overline{z}}}\right)$;

(3) If $\overline{\overline{Ds}}(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}})<\overline{\overline{Ds}}(\hbar_u,\hbar_v)$, then $Sup(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}})>Sup(\hbar_u,\hbar_v)$, where $\overline{\overline{Ds}}(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}})$ is distance among $\hbar_{\overline{\overline{z}}}$ and $\hbar_{\overline{\overline{h}}}$.

In order to write expression (3.1) in a more simplified form, we can suppose

$$\overline{\overline{\Theta}}_{\overline{\overline{z}}}=\frac{\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{z}}}\right)\right)}{\sum_{\overline{\overline{h}}=1}^{\overline{\overline{u}}}\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{h}}}\right)\right)}. \tag{3.2}$$

For appropriateness, we can identify $\left(\overline{\overline{\Theta}}_1, \overline{\overline{\Theta}}_2, ..., \overline{\overline{\Theta}}_{\bar{\bar{u}}}\right)^T$ the power weight vector (PWV), with $\overline{\overline{\Theta}}_{\bar{\bar{g}}} \in [0,1]$ and $\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \Theta_{\bar{\bar{g}}} = 1.$ Based on Equation (3.2), Equation (3.1) can be expressed as,

$$SVNPMM^{\overline{\overline{Q}}}\left(\hbar_1, \hbar_2, ..., \hbar_{\bar{\bar{u}}}\right) = \left(\frac{1}{\bar{\bar{u}}!} \sum_{\vartheta \in S_{\bar{\bar{u}}}} \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}} \hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}. \tag{3.3}$$

Based on the operational rules given in Definition (1.1.1.3) for SVNNs, and Definition (3.1.1.1), we can have the following Theorem 3.1.1.2.

3.1.1.2 Theorem

Let $\hbar_{\bar{\bar{g}}} (\overline{\overline{g}} = 1, ..., \overline{\overline{u}})$ be a set of SVNNs and $\overline{\overline{Q}} = \left(q_1, ..., q_{\bar{\bar{u}}}\right) \in R^{\bar{\bar{u}}}$ be a vector of parameters. Then, the aggregated value obtained by using Equation (3.1), is still a SVNN and

$$\begin{aligned} SVNPMM^{\overline{\overline{Q}}}\left(\hbar_1, \hbar_2, ..., \hbar_{\bar{\bar{u}}}\right) = \Bigg\langle & \left(1 - \left(\prod_{\vartheta \in S_{\bar{\bar{u}}}} \left(1 - \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(1 - \left(1 - \Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}} \\ & , 1 - \left(1 - \prod_{\vartheta \in S_{\bar{\bar{u}}}} \left(1 - \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(1 - \Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}, 1 - \left(1 - \prod_{\vartheta \in S_{\bar{\bar{u}}}} \left(1 - \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(1 - \Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}} \Bigg\rangle. \end{aligned} \tag{3.4}$$

Proof: According to operational laws for SVNNs, we have

$$\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}} \hbar_{\vartheta(\bar{\bar{g}})} = \left\langle 1 - \left(1 - \Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}, \Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}, \Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}} \right\rangle.$$

Therefore,

$$\left(\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}} \hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}} = \left\langle \left(1 - \left(1 - \Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1 - \left(1 - \Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1 - \left(1 - \Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}} \right\rangle.$$

So,

$$\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}} \hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}} = \left\langle \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(1 - \left(1 - \Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1 - \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(1 - \Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1 - \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} \left(1 - \Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}} \right\rangle,$$

$$\sum_{\vartheta\in S_{\overline{\overline{u}}}}\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(\overline{\overline{u\Theta}}_{\overline{\overline{g}}}\hbar_{\vartheta(\overline{\overline{g}})}\right)^{q_{\overline{\overline{g}}}}=\left\langle 1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Xi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right),\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Psi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right),\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right\rangle,$$

Furthermore,

$$\frac{1}{\overline{\overline{u}}!}\sum_{\theta\in S_{\overline{\overline{u}}}}\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(\overline{\overline{u\Theta}}_{\overline{\overline{g}}}\hbar_{\vartheta(\overline{\overline{g}})}\right)^{q_{\overline{\overline{g}}}}$$

$$=\left\langle 1-\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Xi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}},\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Psi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}},\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}}\right\rangle.$$

Hence,

$$\left(\frac{1}{\overline{\overline{u}}!}\sum_{\vartheta\in S_{\overline{\overline{u}}}}\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(\overline{\overline{u\Theta}}_{\overline{\overline{g}}}\hbar_{\vartheta(\overline{\overline{g}})}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}=\left\langle\left(1-\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Xi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},1-\left(1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Psi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},\right.$$

$$\left.1-\left(1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}\right\rangle.$$

$$SVNPMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=\left\langle\left(1-\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Xi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},1-\left(1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Psi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}\right.$$

$$\left.,1-\left(1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}\right\rangle.$$

On the top of equations, first of all we need to find out the PWV $\overline{\overline{\Theta}}$, we have to find out the support degree $Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)$. As stated by Equation (1.1.1.5), we can acquire $Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)$ employing

$$Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)=1-\overline{\overline{Ds}}\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right). \tag{3.5}$$

Therefore, we exploit the equation

$$\overline{\overline{T}}\left(\hbar_{\overline{\overline{g}}}\right)=\sum_{\substack{\overline{\overline{g}}=1\\ \overline{\overline{g}}\neq\overline{\overline{h}}}}^{\overline{\overline{u}}}Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right). \tag{3.6}$$

To determine $T\left(\hbar_z\right)(z=1,..,a)$. Then, as stated by Equation (3.2), we can acquire the PWV. We provide an example to demonstrate the calculation process.

3.1.1.3 Example

Let $\hbar_1=\langle 0.4,0.3,0.5\rangle,\hbar_2=\langle 0.6,0.2,0.1\rangle$ and $\hbar_3=\langle 0.7,0.2,0.3\rangle$ be any three SVNNs and assume that $Q=(0.2,0.3,0.4)$. To utilize SVNPMM operator and aggregate these three SVNNs to get the comprehensive SVNN $\hbar=\langle\Xi,\Psi,\Upsilon\rangle$, the following steps can be followed:

Step 1. Firstly, we determine the support degree $Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)$, where $\bar{\bar{g}},\bar{\bar{h}}=1,2,3$. According to Equation (1.1.1.5) and Equation (3.5), we have

$Sup(\hbar_1,\hbar_2)=Sup(\hbar_2,\hbar_1)=0.77, Sup(\hbar_1,\hbar_3)=Sup(\hbar_3,\hbar_1)=0.8,$

$Sup(\hbar_2,\hbar_3)=Sup(\hbar_3,\hbar_2)=0.8333.$

Step 2. We determine the PWV by utilizing Equation (3.2) to get,

$\bar{\bar{T}}(\hbar_1)=Sup(\hbar_1,\hbar_2)+Sup(\hbar_1,\hbar_3)=0.77+0.8=1.5667,\ \bar{\bar{T}}(\hbar_2)=1.600, \bar{\bar{T}}(\hbar_3)=1.633.$

Therefore,

$$\bar{\bar{\Theta}}_1=\frac{\left(1+\bar{\bar{T}}(\hbar_1)\right)}{\left(1+\bar{\bar{T}}(\hbar_1)\right)+\left(1+\bar{\bar{T}}(\hbar_2)\right)+\left(1+\bar{\bar{T}}(\hbar_3)\right)}=0.3291, \bar{\bar{\Theta}}_2=0.3333, \bar{\bar{\Theta}}_3=0.3376.$$

$$\Xi=\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{3}\left(1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{3\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.3+0.4}}$$

$$=\left(1-\left(\left(1-\left(1-(1-0.4)^{3\bar{\bar{\Theta}}_1}\right)^{0.2}\times\left(1-(1-0.6)^{3\bar{\bar{\Theta}}_2}\right)^{0.3}\times\left(1-(1-0.7)^{3\bar{\bar{\Theta}}_3}\right)^{0.4}\right)\times\right.\right.$$
$$\left(1-\left(1-(1-0.4)^{3\bar{\bar{\Theta}}_1}\right)^{0.2}\times\left(1-(1-0.7)^{3\bar{\bar{\Theta}}_3}\right)^{0.3}\times\left(1-(1-0.6)^{3\bar{\bar{\Theta}}_2}\right)^{0.4}\right)\times$$
$$\left(1-\left(1-(1-0.6)^{3\bar{\bar{\Theta}}_2}\right)^{0.2}\times\left(1-(1-0.4)^{3\bar{\bar{\Theta}}_1}\right)^{0.3}\times\left(1-(1-0.7)^{3\bar{\bar{\Theta}}_3}\right)^{0.4}\right)\times$$
$$\left(1-\left(1-(1-0.6)^{3\bar{\bar{\Theta}}_2}\right)^{0.2}\times\left(1-(1-0.7)^{3\bar{\bar{\Theta}}_3}\right)^{0.3}\times\left(1-(1-0.4)^{3\bar{\bar{\Theta}}_1}\right)^{0.4}\right)\times \qquad =0.5525,$$
$$\left(1-\left(1-(1-0.7)^{3\bar{\bar{\Theta}}_3}\right)^{0.2}\times\left(1-(1-0.4)^{3\bar{\bar{\Theta}}_1}\right)^{0.3}\times\left(1-(1-0.6)^{3\bar{\bar{\Theta}}_2}\right)^{0.4}\right)\times$$
$$\left.\left.\left(1-\left(1-(1-0.7)^{3\bar{\bar{\Theta}}_3}\right)^{0.2}\times\left(1-(1-0.6)^{3\bar{\bar{\Theta}}_2}\right)^{0.3}\times\left(1-(1-0.4)^{3\bar{\bar{\Theta}}_1}\right)^{0.4}\right)\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.9}}$$

$$\Psi=1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{3}\left(1-\Psi_{\vartheta(\bar{\bar{g}})}^{3\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.2+0.3+0.4}}$$

$$=1-\left(1-\left(\left(\left(1-\left(1-0.3^{3\overline{\overline{\Theta}}_1}\right)^{0.2}\times\left(1-0.2^{3\overline{\overline{\Theta}}_2}\right)^{0.3}\times\left(1-0.4^{3\overline{\overline{\Theta}}_3}\right)^{0.4}\right)\times\left(1-\left(1-0.3^{3\overline{\overline{\Theta}}_1}\right)^{0.2}\times\left(1-0.4^{3\overline{\overline{\Theta}}_3}\right)^{0.3}\times\left(1-0.2^{3\overline{\overline{\Theta}}_2}\right)^{0.4}\right)\times\right.\right.\right.$$
$$\left(1-\left(1-0.2^{3\overline{\overline{\Theta}}_2}\right)^{0.2}\times\left(1-0.3^{3\overline{\overline{\Theta}}_1}\right)^{0.3}\times\left(1-0.4^{3\overline{\overline{\Theta}}_3}\right)^{0.4}\right)\times\left(1-\left(1-0.2^{3\overline{\overline{\Theta}}_2}\right)^{0.2}\times\left(1-0.4^{3\overline{\overline{\Theta}}_3}\right)^{0.3}\times\left(1-0.3^{3\overline{\overline{\Theta}}_2}\right)^{0.4}\right)\times$$
$$\left.\left.\left.\left(1-\left(1-0.4^{3\overline{\overline{\Theta}}_3}\right)^{0.2}\times\left(1-0.3^{3\overline{\overline{\Theta}}_1}\right)^{0.3}\times\left(1-0.2^{3\overline{\overline{\Theta}}_2}\right)^{0.4}\right)\times\left(1-\left(1-0.4^{3\overline{\overline{\Theta}}_3}\right)^{0.2}\times\left(1-0.2^{3\overline{\overline{\Theta}}_2}\right)^{0.3}\times\left(1-0.3^{3\overline{\overline{\Theta}}_1}\right)^{0.4}\right)\right)\right)^{\frac{1}{3!}}\right)^{\frac{1}{0.9}}$$

So, $\Psi = 0.2804,$

Similarly,

$\Upsilon = 0.3178.$

Hence, $\hbar = \langle 0.5525, 0.2804, 0.3178\rangle.$

3.1.1.4 Theorem (Idempotency)

Let $\hbar_{\overline{\overline{g}}}(\overline{\overline{g}}=1,\ldots,\overline{\overline{u}})$ be a group of SVNNs, and $\hbar_{\overline{\overline{g}}}=\hbar$ for all $\overline{\overline{g}}=1,\ldots,\overline{\overline{u}}.$ Then

$$SVNPMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=\hbar. \tag{3.7}$$

Proof. Since $\hbar_{\overline{\overline{g}}}=\hbar$ for all $\overline{\overline{g}}=1,\ldots,\overline{\overline{u}},$ we acquire $Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)=1$ for all $\overline{\overline{g}},\overline{\overline{h}}=1,\ldots,\overline{\overline{u}}.$ As a result, we can obtain $\overline{\overline{\Theta}}_{\overline{\overline{g}}}=\frac{1}{\overline{\overline{u}}}$ for all $\overline{\overline{g}}.$ In addition,

$$SVNPMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=SVNPMM^{\overline{\overline{Q}}}\left(\hbar,\hbar,\ldots,\hbar\right),$$

$$=\left\langle\left(1-\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-(1-\Xi)^{\overline{\overline{u}}\frac{1}{\overline{\overline{u}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},1-\left(1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Psi^{\overline{\overline{u}}\frac{1}{\overline{\overline{u}}}}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},\right.$$
$$\left.1-\left(1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Upsilon^{\overline{\overline{u}}\frac{1}{\overline{\overline{u}}}}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}\right\rangle,$$

$$=\left\langle\left(1-\left(\left(1-\left(1-(1-\Xi)\right)^{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}\right)^{\overline{\overline{u}}!}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},1-\left(1-\left(1-(1-\Psi)^{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}\right)^{\overline{\overline{u}}!\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},1-\left(1-\left(1-(1-\Upsilon)^{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}\right)^{\overline{\overline{u}}!\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}\right\rangle,$$

$$=\langle\Xi,\Psi,\Upsilon\rangle=\hbar.$$

3.1.1.5 Theorem (Boundedness)

Let $\hbar_{\overline{\overline{g}}}(\overline{\overline{g}}=1,2,\ldots,\overline{\overline{u}})$ be a set of SVNNs, $\bar{\hbar}=\min\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=\left(\Xi^-,\Psi^+,\Upsilon^+\right)$ and $\overset{+}{\hbar}=\max\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=\left(\Xi^+,\Psi^-,\Upsilon^-\right).$ Then

$$g \leq SVNPMM^{\bar{\bar{Q}}}\left(\hbar_1, \hbar_2, ..., \hbar_{\bar{\bar{u}}}\right) \leq h. \tag{3.8}$$

Where,

$$g = \left\langle \left(1-\left(\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\bar{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}, 1-\left(1-\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}},\right.$$

$$\left. 1-\left(1-\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}} \right\rangle,$$

and

$$h = \left\langle \left(1-\left(\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\overset{+}{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}, 1-\left(1-\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\bar{\Psi}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}},\right.$$

$$\left. 1-\left(1-\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\bar{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}} \right\rangle.$$

Proof. $\overline{\overline{u\Theta}}_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})} = \left\langle 1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}, \Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}, \Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}} \right\rangle \geq \left\langle 1-\left(1-\bar{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}, \overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}, \overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}} \right\rangle,$

Therefore,

$$\left(\overline{\overline{u\Theta}}_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}} = \left\langle \left(1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\left(1-\Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}} \right\rangle \geq$$

$$\left\langle \left(1-\left(1-\bar{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\left(1-\overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\left(1-\overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}} \right\rangle,$$

So,

$$\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(\overline{u\Theta}_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}} = \left\langle \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}} \right\rangle \geq$$

$$\left\langle \prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\bar{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}, 1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}} \right\rangle,$$

$$\sum_{\vartheta \in S_{\bar{\bar{u}}}}\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(\overline{\overline{u\Theta}}_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}} = \left\langle 1-\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right), \prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Psi_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right), \prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right) \right\rangle$$

$$\geq \left\langle 1-\prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\bar{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right), \prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{z}}\right), \prod_{\vartheta \in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\overline{\overline{u\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right) \right\rangle.$$

Furthermore,

$$\frac{1}{\bar{\bar{u}}!}\sum_{\vartheta\in S_{\bar{\bar{u}}}}\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}} =$$

$$\left\langle 1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}},\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Psi_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}},\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right\rangle \geq$$

$$\left\langle 1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\overset{-}{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}},\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}},\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right\rangle.$$

Hence,

$$\left(\frac{1}{\bar{\bar{u}}!}\sum_{\vartheta\in S_{\bar{\bar{u}}}}\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}} =$$

$$\left\langle\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Psi_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\right.$$

$$\left.1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}}\right\rangle \geq \left\langle\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\overset{-}{\Xi}_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\right.$$

$$\left.1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}}\right\rangle = g.$$

This implies that $g \leq SVNPMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,...,\hbar_{\bar{\bar{u}}}\right)$.

In a similar technique we can also prove that $SVNPMM^{Q}\left(\hbar_1,\hbar_2,...,\hbar_o\right) \leq h$. So $g \leq SVNPMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,...,\hbar_{\bar{\bar{u}}}\right) \leq h$.

In addition, then property of monotonicity is not satisfied by SVNPMM operator.

One of the most important advantages of SVNPMM is its capability to express the interrelationship among SVNNs. Besides, SVNPMM operator is more pliable in aggregation process due to parameter vector. Now, we will examine various particular cases of SVNPMM operators by conveying diverse values to the parameter vector.

Case 1. If $\bar{\bar{Q}} = (1,0,...,0)$, then SVNPMM operator degenerates into the following form:

$$SVNPMM^{(1,0,....,0)}\left(\hbar_1,\hbar_2,....,\hbar_{\bar{\bar{u}}}\right) = \left(\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\frac{\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{g}}}\right)\right)}{\sum_{\bar{\bar{h}}=1}^{\bar{\bar{u}}}\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{h}}}\right)\right)}\hbar_{\bar{\bar{g}}}\right). \quad (3.9)$$

This is the SVN power averaging operator.

Case 2. If $\bar{\bar{Q}} = \left(\frac{1}{\bar{\bar{a}}},\frac{1}{\bar{\bar{a}}},....,\frac{1}{\bar{\bar{a}}}\right)$, then SVNPMM operator degenerates into the following form:

$$SVNPMM^{\left(\frac{1}{\overline{\overline{u}}},\frac{1}{\overline{\overline{u}}},\ldots,\frac{1}{\overline{\overline{u}}}\right)}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\hbar_{\overline{\overline{g}}}^{\frac{\left(1+\overline{\overline{T}}\left(\hbar_{\vartheta(\overline{\overline{g}})}\right)\right)}{\sum_{\overline{\overline{h}}=1}^{\overline{\overline{u}}}\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{h}}}\right)\right)}}. \quad (3.10)$$

This is SVN power geometric operator.

Case 3. If $\overline{\overline{Q}}=(1,1,\ldots,0)$, then SVNPMM operator degenerates into the following form:

$$SVNPMM^{(1,1,0,\ldots,0)}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=\left\langle\left(1-\left(\prod_{\substack{\overline{\overline{g}},\overline{\overline{h}}=1\\ \overline{\overline{g}}\neq\overline{\overline{h}}}}^{\overline{\overline{u}}}\left(1-\left(1-\left(1-\Xi_{\overline{\overline{g}}}\right)^{\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)\left(1-\left(1-\Xi_{\overline{\overline{h}}}\right)^{\overline{\overline{\Theta}}_{\overline{\overline{h}}}}\right)\right)\right)^{\frac{1}{\overline{\overline{u}}^2-\overline{\overline{u}}}}\right)^{\frac{1}{2}},\right.$$

$$\left.1-\left(1-\left(\prod_{\substack{\overline{\overline{g}},\overline{\overline{h}}=1\\ \overline{\overline{g}}\neq\overline{\overline{h}}}}^{\overline{\overline{u}}}\left(1-\left(1-\Psi_{\overline{\overline{g}}}^{\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)\left(1-\Psi_{\overline{\overline{h}}}^{\overline{\overline{\Theta}}_{\overline{\overline{h}}}}\right)\right)\right)^{\frac{1}{\overline{\overline{u}}^2-\overline{\overline{u}}}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{\substack{\overline{\overline{g}},\overline{\overline{h}}=1\\ \overline{\overline{g}}\neq\overline{\overline{h}}}}^{\overline{\overline{u}}}\left(1-\left(1-\Upsilon_{\overline{\overline{g}}}^{\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)\left(1-\Upsilon_{\overline{\overline{h}}}^{\overline{\overline{\Theta}}_{\overline{\overline{h}}}}\right)\right)\right)^{\frac{1}{\overline{\overline{u}}^2-\overline{\overline{u}}}}\right)^{\frac{1}{2}}\right\rangle. \quad (3.11)$$

This is the SVN power BM operator $(p=q=1)$.

Case 4. . If $\overline{\overline{Q}}=\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)$, then SVNPMM operator degenerates into the following form:

$$SVNPMM^{\left(\overbrace{1,1,..,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\overline{\overline{u}}}\right)=\left\langle\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le\overline{\overline{u}}}\left(1-\prod_{\overline{\overline{h}}=1}^{i}\left(1-\left(1-\Xi_{\overline{\overline{g}}_{\overline{\overline{h}}}}\right)^{\overline{\overline{\Theta}}_{\overline{\overline{g}}_{\overline{\overline{h}}}}}\right)\right)^{\frac{1}{C_{\overline{\overline{u}}}^{i}}}\right)^{\frac{1}{k}},\right.$$

$$\left.1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le\overline{\overline{u}}}\left(1-\prod_{\overline{\overline{h}}=1}^{i}\left(1-\Psi_{\overline{\overline{g}}_{\overline{\overline{h}}}}^{\overline{\overline{\Theta}}_{\overline{\overline{g}}_{\overline{\overline{h}}}}}\right)\right)^{\frac{1}{C_{\overline{\overline{u}}}^{i}}}\right)^{\frac{1}{k}},1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le\overline{\overline{u}}}\left(1-\prod_{\overline{\overline{h}}=1}^{i}\left(1-\Upsilon_{\overline{\overline{g}}_{\overline{\overline{h}}}}^{\overline{\overline{\Theta}}_{\overline{\overline{g}}_{\overline{\overline{h}}}}}\right)\right)^{\frac{1}{C_{\overline{\overline{u}}}^{i}}}\right)^{\frac{1}{k}}\right\rangle. \quad (3.12)$$

This is the SVN power MSM operator.

3.1.2 Weighted Single-Valued Neutrosophic Power Muirhead Mean (WSVNMM) Operator

As we can notice that the SVNPMM operator does not judge the importance degree of the aggregated SVNNs. In this subsection, we propose the weighted single-valued neutrosophic power MM (WSVNPMM) operator, which has the capacity of taking the weights of SVNNs.

3.1.2.1 Definition

Let $\hbar_{\overline{\overline{g}}}(\overline{\overline{g}}=1,...,\overline{\overline{u}})$ be a set of SVNNs and $\overline{\overline{Q}}=\left(q_1,q_2,...,q_{\overline{\overline{u}}}\right)\in R^{\overline{\overline{u}}}$ be a vector of parameters. If

$$WSVNPMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,....,\hbar_{\overline{\overline{u}}}\right)=\left(\frac{1}{\overline{\overline{u}}!}\sum_{\vartheta\in S_{\overline{\overline{u}}}}\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(\frac{\overline{\overline{u}}\overline{\overline{\Phi}}_{\vartheta(\overline{\overline{g}})}\overline{\overline{\Theta}}_{\vartheta(\overline{\overline{g}})}}{\sum_{\overline{\overline{h}}=1}^{\overline{\overline{u}}}\overline{\overline{\Phi}}_{\overline{\overline{h}}}\overline{\overline{\Theta}}_{\overline{\overline{h}}}}\hbar_{\vartheta(\overline{\overline{g}})}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}. \quad (3.13)$$

Then, we call $WSVNPMM^{\overline{\overline{Q}}}$ the weighted single valued neutrosophic power Muirhead mean operator, where $\overline{\overline{\Phi}}=\left(\overline{\overline{\Phi}}_1,\overline{\overline{\Phi}}_2,...,\overline{\overline{\Phi}}_{\overline{\overline{u}}}\right)^T$ is the importance degree of $\hbar_{\overline{\overline{g}}}\left(\overline{\overline{g}}=1,2,...,\overline{\overline{u}}\right)$ with $\overline{\overline{\Phi}}_{\overline{\overline{g}}}\in[0,1],\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\overline{\overline{\Phi}}_{\overline{\overline{g}}}=1,$ $S_{\overline{\overline{u}}}$ is the set of all permutation, $\vartheta\left(\overline{\overline{g}}\right)$ is any permutation of $\left(1,2,...,\overline{\overline{u}}\right)$ and $\overline{\overline{\Theta}}_{\overline{\overline{g}}}$ is PVW fulfilling $\overline{\overline{\Theta}}_{\overline{\overline{g}}}=\frac{\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{g}}}\right)\right)}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{g}}}\right)\right)},\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\overline{\overline{\Theta}}_{\overline{\overline{z}}}=1,$ $\overline{\overline{T}}\left(\hbar_{\overline{\overline{h}}}\right)=\sum_{\substack{\overline{\overline{h}}=1\\ \overline{\overline{h}}\neq\overline{\overline{g}}}}^{\overline{\overline{u}}}Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)$, $Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)$ is the support degree for $\hbar_{\overline{\overline{g}}}$ and $\hbar_{\overline{\overline{h}}}$, fulfilling the following conditions:

(1) $Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)\in[0,1]$; (2) $Sup\left(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}\right)=Sup\left(\hbar_{\overline{\overline{h}}},\hbar_{\overline{\overline{g}}}\right)$;

(3) If $\overline{\overline{Ds}}(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}})<\overline{\overline{Ds}}(\hbar_u,\hbar_v)$, then $Sup(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}})>Sup(\hbar_u,\hbar_v)$, where $\overline{\overline{Ds}}(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}})$ is distance among $\hbar_g$ and $\hbar_{\overline{\overline{h}}}$.

From Definition (3.1.2.1), we have the following Theorem (3.1.2.2).

3.1.2.2 Theorem

Let $\hbar_{\overline{\overline{g}}}(\overline{\overline{g}}=1,2,...,\overline{\overline{u}})$ be a set of SVNNs and $\overline{\overline{Q}}=\left(q_1,q_2,...,q_{\overline{\overline{u}}}\right)\in R^{\overline{\overline{u}}}$ be a vector of parameters. Then, the result aggregated by exploiting Equation (3.13) is still a SVNN and

$$WSVNPMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,...,\hbar_{\bar{\bar{u}}}\right)=\left\langle \left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Xi_{\vartheta(\bar{\bar{g}})}\right)^{\frac{\bar{\bar{u}}\bar{\bar{\Theta}}_{\vartheta(\bar{\bar{g}})}\bar{\bar{\Phi}}_{\theta(\bar{\bar{g}})}}{\sum_{\bar{\bar{\hbar}}=1}^{\bar{\bar{u}}}\bar{\bar{\Theta}}_{\bar{\bar{\hbar}}}\bar{\bar{\Phi}}_{\bar{\bar{\hbar}}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}}\right.$$

$$\left.,1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Psi_{\vartheta(\bar{\bar{g}})}^{\frac{\bar{\bar{u}}\bar{\bar{\Theta}}_{\vartheta(\bar{\bar{g}})}\bar{\bar{\Phi}}_{\theta(\bar{\bar{g}})}}{\sum_{\bar{\bar{\hbar}}=1}^{\bar{\bar{u}}}\bar{\bar{\Theta}}_{\bar{\bar{\hbar}}}\bar{\bar{\Phi}}_{\bar{\bar{\hbar}}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}^{\frac{\bar{\bar{u}}\bar{\bar{\Theta}}_{\vartheta(\bar{\bar{g}})}\bar{\bar{\Phi}}_{\theta(\bar{\bar{g}})}}{\sum_{\bar{\bar{\hbar}}=1}^{\bar{\bar{u}}}\bar{\bar{\Theta}}_{\bar{\bar{\hbar}}}\bar{\bar{\Phi}}_{\bar{\bar{\hbar}}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}}\right\rangle. \tag{3.14}$$

Proof: Proof of Theorem (3.1.2.2) is same as Theorem (3.1.1.2).

3.1.3 The Single-Valued Neutrosophic Power Dual MM (SVNPDMM) Operator

In this sub-part, we develop the SVNPDMM operator and discuss some related properties.

3.1.3.1 Definition

Let $\hbar_{\bar{\bar{g}}}(\bar{\bar{g}}=1,...,\bar{\bar{u}})$ be a set of SVNNs and $\bar{\bar{Q}}=\left(q_1,q_2,...,q_{\bar{\bar{u}}}\right)\in R^{\bar{\bar{u}}}$ be a vector of parameters. If

$$SVNPDMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,....,\hbar_{\bar{\bar{u}}}\right)=\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(q_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}^{\frac{\bar{\bar{u}}\left(1+\bar{\bar{T}}\left(\hbar_{\vartheta(\bar{\bar{z}})}\right)\right)}{\sum_{\bar{\bar{\hbar}}=1}^{\bar{\bar{u}}}\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{\hbar}}}\right)\right)}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}. \tag{3.15}$$

Then, we call $SVNPDMM^{\bar{\bar{Q}}}$ the single valued neutrosophic power dual MM operator, where $S_{\bar{\bar{a}}}$ is the set of all combination, and $\vartheta\left(\bar{\bar{g}}\right)$ is any combination of $\left(1,2,...,\bar{\bar{u}}\right)$, $\bar{\bar{T}}\left(\hbar_{\bar{\bar{h}}}\right)=\sum_{\substack{\bar{\bar{\hbar}}=1\\ \bar{\bar{\hbar}}\neq\bar{\bar{g}}}}^{\bar{\bar{u}}}Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)$, and $Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)$ is the support degree for $\hbar_{\bar{\bar{g}}}$ and $\hbar_{\bar{\bar{h}}}$, fulfilling the following conditions:

(1) $Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)\in[0,1]$; (2) $Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)=Sup\left(\hbar_{\bar{\bar{h}}},\hbar_{\bar{\bar{g}}}\right)$;

(3) If $\overline{\overline{Ds}}(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}) < \overline{\overline{Ds}}(\hbar_u,\hbar_v)$, then $Sup(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}}) > Sup(\hbar_u,\hbar_v)$, where $\overline{\overline{Ds}}(\hbar_{\overline{\overline{g}}},\hbar_{\overline{\overline{h}}})$ is distance among $\hbar_{\overline{\overline{g}}}$ and $\hbar_{\overline{\overline{h}}}$.

In order to inscribe Equation (3.15) in an uncomplicated form, we can identify it as

$$\overline{\overline{\Theta}}_{\overline{\overline{g}}} = \frac{\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{g}}}\right)\right)}{\sum_{\overline{\overline{h}}=1}^{\overline{\overline{u}}}\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{h}}}\right)\right)}. \tag{3.16}$$

For appropriateness, we can identify $\left(\overline{\overline{\Theta}}_1,\overline{\overline{\Theta}}_2,...,\overline{\overline{\Theta}}_{\overline{\overline{u}}}\right)^T$ the PMV, with $\overline{\overline{\Theta}}_{\overline{\overline{g}}} \in [0,1]$ and $\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\overline{\overline{\Theta}}_{\overline{\overline{g}}} = 1$. Based on Equation (3.16), Equation (3.15) can be indicated as

$$SVNPDMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,....,\hbar_{\overline{\overline{u}}}\right) = \frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}} q_{\overline{\overline{g}}}}\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(q_{\overline{\overline{g}}}\hbar_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\vartheta(\overline{\overline{g}})}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}}. \tag{3.17}$$

3.1.3.2 Theorem

Let $\hbar_{\overline{\overline{g}}}(\overline{\overline{g}}=1,...,\overline{\overline{u}})$ be a set of SVNNs and the parameter vectors is indicated by $\overline{\overline{Q}} = \left(q_1,q_2,...,q_{\overline{\overline{u}}}\right) \in R^{\overline{\overline{u}}}$. Then, the result aggregated by employing Equation (3.15) is still a SVNN and

$$SVNPDMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,...,\hbar_{\overline{\overline{u}}}\right) = \left\langle 1-\left(1-\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Xi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}},\right.$$

$$\left.\left(1-\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Psi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}, 1-\left(\prod_{\vartheta\in S_{\overline{\overline{u}}}}\left(1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right)\right)^{\frac{1}{\overline{\overline{u}}!}}\right)^{\frac{1}{\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}q_{\overline{\overline{g}}}}}\right\rangle. \tag{3.18}$$

Proof: According to operational laws for SVNNs, we have

$$\hbar_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}} = \left\langle \Xi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}, 1-\left(1-\Psi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}, 1-\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right\rangle.$$

Therefore,

$$q_{\overline{\overline{g}}}\hbar_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}} = \left\langle 1-\left(1-\Xi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}, \left(1-\left(1-\Psi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}, \left(1-\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right\rangle.$$

So,

$$\sum_{\overline{\overline{g}}=1}^{\overline{\overline{u}}} q_{\overline{\overline{g}}}\hbar_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}} = \left\langle 1-\prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\Xi_{\vartheta(\overline{\overline{g}})}^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}, \prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Psi_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}, \prod_{\overline{\overline{g}}=1}^{\overline{\overline{u}}}\left(1-\left(1-\Upsilon_{\vartheta(\overline{\overline{g}})}\right)^{\overline{\overline{u}}\overline{\overline{\Theta}}_{\overline{\overline{g}}}}\right)^{q_{\overline{\overline{g}}}}\right\rangle,$$

$$\prod_{\vartheta\in S_{\bar{\bar{u}}}}\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}} = \left\langle \prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Xi_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right), 1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Psi_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right), 1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right\rangle.$$

Furthermore,

$$\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}$$

$$= \left\langle \left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Xi_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}, 1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Psi_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}, 1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right\rangle.$$

Hence,

$$\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}\hbar_{\theta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}} = \left\langle 1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Xi_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}, \left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Psi_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}},\right.$$

$$\left.\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}\right\rangle,$$

$$SVNPDMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\bar{\bar{u}}}\right) = \left\langle 1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Xi_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}},\right.$$

$$\left.\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Psi_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}, \left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\Theta_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}}\right\rangle.$$

3.1.3.3 Theorem (Idempotency)

Let $\hbar_{\bar{\bar{g}}}\,(\bar{\bar{g}}=1,2,\ldots,\bar{\bar{u}})$ be a set of SVNNs, and $\hbar_{\bar{\bar{g}}}=\hbar$ for all $\bar{\bar{g}}=1,\ldots,\bar{\bar{u}}.$ Then,

$$SVNPDMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\bar{\bar{u}}}\right)=\hbar. \tag{3.19}$$

3.1.3.4 Theorem (Boundedness)

Let $\hbar_{\bar{\bar{g}}}\,(\bar{\bar{g}}=1,\ldots,\bar{\bar{u}})$ be a set of SVNNs, $\bar{\hbar}=\min\left(\hbar_1,\hbar_2,\ldots,\hbar_{\bar{\bar{g}}}\right)=\left(\min\Xi,\max\Psi,\max\Upsilon\right)$ and $\overset{+}{\hbar}=\max\left(\hbar_1,\hbar_2,\ldots,\hbar_{\bar{\bar{u}}}\right)=\left(\max\Xi,\min\Psi,\min\Upsilon\right).$ Then

$$g\le SVNPDMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\bar{\bar{u}}}\right)\le h. \tag{3.20}$$

Where,

$$g=\left\langle 1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\bar{\Xi}_{\vartheta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\overset{+}{\Psi}_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\right.$$

$$\left.\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\overset{+}{\Upsilon}_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}}\right\rangle,$$

and

$$h=\left\langle 1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\overset{+}{\Xi}_{\theta(\bar{\bar{g}})}^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\bar{\Psi}_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\right.$$

$$\left.\left(1-\left(\prod_{\theta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\bar{\Upsilon}_{\vartheta(\bar{\bar{g}})}\right)^{\bar{\bar{u}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}}\right\rangle.$$

Now we will discuss some special cases of SVNPDMM operator with respect to the parameter vector $\bar{\bar{Q}}$.

Case 1. If $\bar{\bar{Q}}=(1,0,...,0)$, then SVNPDMM operator degenerates into the following equation:

$$SVNPMM^{(1,0,...,0)}\left(\hbar_1,\hbar_2,....,\hbar_{\bar{\bar{u}}}\right)=\left(\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\hbar_{\bar{\bar{g}}}^{\frac{\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{z}}}\right)\right)}{\sum_{\bar{\bar{h}}=1}^{\bar{\bar{u}}}\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{h}}}\right)\right)}}\right). \tag{3.21}$$

This is the SVN power geometric averaging operator.

Case 2. If $\bar{\bar{Q}}=\left(\frac{1}{\bar{\bar{u}}},\frac{1}{\bar{\bar{u}}},....,\frac{1}{\bar{\bar{u}}}\right)$, then SVNPMM operator degenerates into the following equation:

$$SVNPMM^{\left(\frac{1}{\bar{\bar{u}}},\frac{1}{\bar{\bar{u}}},....,\frac{1}{\bar{\bar{u}}}\right)}\left(\hbar_1,\hbar_2,....,\hbar_{\bar{\bar{u}}}\right)=\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\frac{\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{g}}}\right)\right)}{\sum_{\bar{\bar{h}}=1}^{\bar{\bar{u}}}\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{h}}}\right)\right)}\hbar_{\bar{\bar{g}}}. \tag{3.22}$$

This is SVN power arithmetic averaging operator.

Case 3. If $\bar{\bar{Q}}=(1,1,0,...,0)$, then SVNPDMM operator degenerates into the following equation:

$$SVNPDMM^{(1,1,0,\ldots,0)}\left(\hbar_1,\hbar_2,\ldots,\hbar_{\bar{\bar{u}}}\right)=\left\langle 1-\left(1-\left(\prod_{\substack{\bar{\bar{g}},\bar{\bar{h}}=1\\ \bar{\bar{g}}\neq\bar{\bar{h}}}}^{\bar{\bar{u}}}\left(1-\left(1-\Xi_{\bar{\bar{g}}}^{\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)\left(1-\Xi_{\bar{\bar{h}}}^{\bar{\bar{\Theta}}_{\bar{\bar{h}}}}\right)\right)\right)^{\frac{1}{\bar{\bar{u}}^2-\bar{\bar{u}}}}\right)^{\frac{1}{2}},\right.$$

$$\left.\left(1-\left(\prod_{\substack{\bar{\bar{g}},\bar{\bar{h}}=1\\ \bar{\bar{g}}\neq\bar{\bar{h}}}}^{\bar{\bar{u}}}\left(1-\left(1-\left(1-\Psi_{\bar{\bar{g}}}\right)^{\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)\left(1-\left(1-\Psi_{\bar{\bar{h}}}\right)^{\bar{\bar{\Theta}}_{\bar{\bar{h}}}}\right)\right)\right)^{\frac{1}{\bar{\bar{u}}^2-\bar{\bar{u}}}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{\bar{\bar{g}},\bar{\bar{h}}=1\\ \bar{\bar{g}}\neq\bar{\bar{h}}}}^{\bar{\bar{u}}}\left(1-\left(1-\left(1-\Upsilon_{\bar{\bar{g}}}\right)^{\bar{\bar{\Theta}}_{\bar{\bar{g}}}}\right)\left(1-\left(1-\Upsilon_{\bar{\bar{h}}}\right)^{\bar{\bar{\Theta}}_{\bar{\bar{h}}}}\right)\right)\right)^{\frac{1}{\bar{\bar{u}}^2-\bar{\bar{u}}}}\right)^{\frac{1}{2}}\right\rangle. \tag{3.23}$$

This is the SVN power geometric BM operator $(p=q=1)$.

Case 4. If $\bar{\bar{Q}}=\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{\bar{\bar{z}}-i}\right)$, then SVNPDMM operator degenerates into the following equation:

$$SVNPDMM^{\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{\bar{\bar{z}}-i}\right)}=\left(\hbar_1,\hbar_2,\ldots,\hbar_{\bar{\bar{u}}}\right)=\left\langle 1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le\bar{\bar{u}}}\left(1-\prod_{\bar{\bar{h}}=1}^{i}\left(1-\Xi_{\bar{\bar{g}}_{\bar{h}}}^{\bar{\bar{\Theta}}_{\bar{\bar{g}}_{\bar{h}}}}\right)\right)^{\frac{1}{C_{\bar{\bar{u}}}^{i}}}\right)^{\frac{1}{k}},\right.$$

$$\left.\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le\bar{\bar{u}}}\left(1-\prod_{\bar{\bar{h}}=1}^{i}\left(1-\left(1-\Psi_{\bar{\bar{g}}_{\bar{h}}}\right)^{\bar{\bar{\Theta}}_{\bar{\bar{g}}_{\bar{h}}}}\right)\right)^{\frac{1}{C_{\bar{\bar{u}}}^{i}}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le\bar{\bar{u}}}\left(1-\prod_{\bar{\bar{h}}=1}^{i}\left(1-\left(1-\Upsilon_{\bar{\bar{g}}_{\bar{h}}}\right)^{\bar{\bar{\Theta}}_{\bar{\bar{g}}_{\bar{h}}}}\right)\right)^{\frac{1}{C_{\bar{\bar{u}}}^{i}}}\right)^{\frac{1}{k}}\right\rangle. \tag{3.24}$$

This is the SVN power Dual Maclaurin symmetric mean operator.

3.1.4 Weighted Single-Valued Neutrosophic Power Dual MM (WSVNMM) Operator

The SVNPMM operator does not judge the importance degree of the aggregated SVNNs. In this subsection, we propose the weighted SVN power MM (WSVNPMM) operator, which has the capacity of taking the weights of SVNNs.

3.1.4.1 Definition

Let $\hbar_{\bar{\bar{g}}}(\bar{\bar{g}}=1,\ldots,\bar{\bar{u}})$ is a set of SVNNs and the parameter vector is denoted by $\bar{\bar{Q}}=\left(q_1,q_2,\ldots,q_{\bar{\bar{u}}}\right)\in R^{\bar{\bar{u}}}$. If

$$WSVNPDMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,....,\hbar_{\bar{\bar{u}}}\right)=\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}} q_{\bar{\bar{g}}}}\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(q_{\bar{\bar{g}}}\hbar_{\vartheta(\bar{\bar{g}})}^{\frac{\bar{\bar{u}}\bar{\bar{\Phi}}_{\theta(\bar{\bar{g}})}\bar{\bar{\Theta}}_{\vartheta(\bar{\bar{g}})}}{\sum_{\bar{\bar{h}}=1}^{\bar{\bar{u}}}\bar{\bar{\Phi}}_{\bar{\bar{h}}}\bar{\bar{\Theta}}_{\bar{\bar{h}}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}. \tag{3.25}$$

Then, we call $WSVNPDMM^{\bar{\bar{Q}}}$ the weighted single valued neutrosophic power dual Muirhead mean operator, where $\bar{\bar{\Phi}}=\left(\bar{\bar{\Phi}}_1,\bar{\bar{\Phi}}_2,...,\bar{\bar{\Phi}}_{\bar{\bar{a}}}\right)^T$ is the importance dgree of $\hbar_{\bar{\bar{g}}}\left(\bar{\bar{g}}=1,...,\bar{\bar{u}}\right)$ with $\bar{\bar{\Phi}}_{\bar{\bar{g}}}\in[0,1],\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\bar{\bar{\Phi}}_{\bar{\bar{g}}}=1,$ $S_{\bar{\bar{u}}}$ is the set of all permutation, $\vartheta\left(\bar{\bar{g}}\right)$ is any permutation of $\left(1,2,...,\bar{\bar{u}}\right)$ and $\bar{\bar{\Theta}}_{\bar{\bar{g}}}$ is PVW satisfying $\bar{\bar{\Theta}}_{\bar{\bar{g}}}=\frac{\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{g}}}\right)\right)}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1+\bar{\bar{T}}\left(\hbar_{\bar{\bar{g}}}\right)\right)},\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\bar{\bar{\Theta}}_{\bar{\bar{g}}}=1,$ $\bar{\bar{T}}\left(\hbar_{\bar{\bar{h}}}\right)=\sum_{\substack{\bar{\bar{h}}=1\\ \bar{\bar{h}}\neq\bar{\bar{g}}}}^{\bar{\bar{u}}}Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)$, $Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)$ is the support degree for $\hbar_{\bar{\bar{g}}}$ and $\hbar_{\bar{\bar{h}}}$, satisfying the following axioms:

(1) $Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)\in[0,1]$; (2) $Sup\left(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}}\right)=Sup\left(\hbar_{\bar{\bar{h}}},\hbar_{\bar{\bar{g}}}\right)$;

(3) If $\bar{\bar{Ds}}(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}})<\bar{\bar{Ds}}(\hbar_u,\hbar_v),$ then $Sup(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}})>Sup(\hbar_u,\hbar_v),$ where $\bar{\bar{Ds}}(\hbar_{\bar{\bar{g}}},\hbar_{\bar{\bar{h}}})$ is distance among $\hbar_{\bar{\bar{g}}}$ and $\hbar_{\bar{\bar{h}}}$.

From Definition (3.1.4.1), we have the following Theorem (3.1.4.2).

3.1.4.2 Theorem

Let $\hbar_{\bar{\bar{g}}}(\bar{\bar{g}}=1,...,\bar{\bar{u}})$ be a set of SVNNs and the parameter vector is denoted by $\bar{\bar{Q}}=\left(q_1,q_2,...,q_{\bar{\bar{u}}}\right)\in R^{\bar{\bar{u}}}$. Then, the result aggregated by employing Equation (3.25) is still a SVNN, and

$$WSVNPDMM^{\bar{\bar{Q}}}\left(\hbar_1,\hbar_2,...,\hbar_{\bar{\bar{u}}}\right)=\left\langle 1-\left(1-\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\Xi_{\vartheta(\bar{\bar{g}})}^{\frac{\bar{\bar{u}}\bar{\bar{\Theta}}_{\vartheta(\bar{\bar{g}})}\bar{\bar{\Phi}}_{\theta(\bar{\bar{g}})}}{\sum_{\bar{\bar{h}}=1}^{\bar{\bar{u}}}\bar{\bar{\Theta}}_{\bar{\bar{h}}}\bar{\bar{\Phi}}_{\bar{\bar{h}}}}}\right)^{q_{\bar{\bar{g}}}}\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\right. \tag{3.26}$$

$$\left.\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Psi_{\vartheta(\bar{\bar{g}})}\right)^{\frac{\bar{\bar{u}}\bar{\bar{\Theta}}_{\vartheta(\bar{\bar{g}})}\bar{\bar{\Phi}}_{\theta(\bar{\bar{g}})}}{\sum_{\bar{\bar{h}}=1}^{\bar{\bar{u}}}\bar{\bar{\Theta}}_{\bar{\bar{h}}}\bar{\bar{\Phi}}_{\bar{\bar{h}}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}},\left(1-\left(\prod_{\vartheta\in S_{\bar{\bar{u}}}}\left(1-\prod_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}\left(1-\left(1-\Upsilon_{\vartheta(\bar{\bar{g}})}\right)^{\frac{\bar{\bar{u}}\bar{\bar{\Theta}}_{\vartheta(\bar{\bar{g}})}\bar{\bar{\Phi}}_{\theta(\bar{\bar{g}})}}{\sum_{\bar{\bar{h}}=1}^{\bar{\bar{u}}}\bar{\bar{\Theta}}_{\bar{\bar{h}}}\bar{\bar{\Phi}}_{\bar{\bar{h}}}}}\right)^{q_{\bar{\bar{g}}}}\right)\right)^{\frac{1}{\bar{\bar{u}}!}}\right)^{\frac{1}{\sum_{\bar{\bar{g}}=1}^{\bar{\bar{u}}}q_{\bar{\bar{g}}}}}\right\rangle.$$

Proof: Proof of Theorem (3.1.4.2) is same as Theorem (3.1.3.2).

3.2 The MAGDM Approach Based on WSVNPMM Operator and WSVNPDMM Operator

In this part, we give a novel method to MAGDM with SVNNs, in which the importance degrees of the decision maker's and criterion are known. Let $\overline{\overline{Al}} = \left\{ \overline{\overline{Al}}_1, \overline{\overline{Al}}_2, ..., \overline{\overline{Al}}_a \right\}, \overline{\overline{Cr}} = \left\{ \overline{\overline{Cr}}_1, \overline{\overline{Cr}}_2, ..., \overline{\overline{Cr}}_b \right\}$ respectively, specified the set of alternatives and criterion and the set of decision makers be specified by $e = \{e_1, e_2, ..., e_c\}$. Presume that the evaluation value for the alternative $\overline{\overline{Al}}_g$ specified by the decision maker e_k about the criteria $\overline{\overline{Cr}}_h$ is specified by the form $\overline{\overline{Al}}_{gh}^k = \left\langle \Xi_{gh}^k, \Psi_{gh}^k, \Upsilon_{gh}^k \right\rangle$. The importance degree of the criterion $\overline{\overline{Cr}} = \left\{ \overline{\overline{Cr}}_1, \overline{\overline{Cr}}_2, ..., \overline{\overline{Cr}}_b \right\}$ is designated by $\varpi = (\varpi_1, \varpi_2, ..., \varpi_b)^T$ with $\varpi_h \in [0,1], \sum_{h=1}^{b} \varpi_h = 1$. $\Lambda = (\Lambda_1, \Lambda_2, ..., \Lambda_c)^T$ symbolize the importance degree of the decision maker's with $\Lambda_k \in [0,1], \sum_{k=1}^{c} \Lambda_k = 1$. Then the aspire of this MAGDM problem is to order the alternatives. To accomplish this, the subsequent steps are pursued.

Step 1. Homogenize the decision matrix. Normally, there are two kinds of criterion, 1) cost type and 2) benefit type. We necessitate exchanging the cost type of criterion into benefit types of criterion by utilizing the following Equation (3.27):

$$\overline{\overline{N}}_{gh}^k = \left\langle \Xi_{gh}^k, \Psi_{gh}^k, \Upsilon_{gh}^k \right\rangle = \begin{cases} \left\langle \Xi_{gh}^k, \Psi_{gh}^k, \Upsilon_{gh}^k \right\rangle & \text{for benefit attribute } \overline{\overline{O}}_h \\ \left\langle \Upsilon_{gh}^k, 1-\Psi_{gh}^k, \Xi_{gh}^k \right\rangle & \text{for cost attribute } \overline{\overline{O}}_h. \end{cases} \tag{3.27}$$

Consequently, the decision matrix $M = \left[\overset{=k}{n}_{gh} \right]_{a \times b}$ can be altered to homogenize matrix $N = \left[\delta_{gh}^{k} \right]_{a \times b}$.

Step 2. Establish the supports $Sup\left(\rho_{gh}^{k}, \rho_{gl}^{k}\right)(1,...,a; h,l = 1,...,b, k = 1,...,c)$ by

$$Sup\left(\rho_{gh}^{k}, \rho_{gl}^{k}\right) = 1 - \overline{\overline{Ds}}\left(\rho_{gh}^{k}, \rho_{gl}^{k}\right), \tag{3.28}$$

where, $\overline{\overline{Ds}}\left(\rho_{gh}^{k}, \rho_{gl}^{k}\right)$ represents the distance measure between any two SVNNs ρ_{gh}^{k} and ρ_{gl}^{k} given in Definition (1.1.1.5).

Step 3. Establish $\overline{\overline{T}}(\rho_{gh}^{k})$ by

$$\overline{\overline{T}}\left(\rho_{gh}^{k}\right) = \sum_{\substack{l=1 \\ l \neq h}}^{b} Sup\left(\delta_{gh}^{k}, \delta_{gl}^{k}\right)(1,...,a; h,l = 1,...,b, k = 1,...,c). \tag{3.29}$$

Step 4. Establish

$$\overline{\overline{\Phi}}_{gh}^{k} = \frac{b\varpi_{h}\left(1 + \overline{\overline{T}}\left(\rho_{gh}^{k}\right)\right)}{\sum_{d=1}^{b} \varpi_{d}\left(1 + \overline{\overline{T}}\left(\rho_{gd}^{k}\right)\right)}\left(g = 1,...,a; h, d = 1,...,b, k = 1,...,c\right). \tag{3.30}$$

Step 5. Utilize the WSVNPMM or WSVNPDMM operators

$$\rho_{g}^{k} = \left\langle \Xi_{g}^{k}, \Psi_{g}^{k}, \Upsilon_{g}^{k} \right\rangle = WSVNPMM^{\overline{\overline{Q}}}\left(\rho_{g1}^{k}, \rho_{g2}^{k}, ..., \rho_{gb}^{k}\right), \tag{3.31}$$

or

$$= WSVNPDMM^{\overline{\overline{Q}}}\left(\rho_{g1}^{k}, \rho_{g2}^{k}, ..., \rho_{gb}^{k}\right). \tag{3.32}$$

To established the overall SVNNs $\delta_{g}^{k}\ (g = 1,2,...,a; k = 1,2,...,c)$.

Step 6. Find out the supports $Sup\left(\rho_{g}^{k}, \rho_{g}^{m}\right)(g = 1,...,a; m,k = 1,...,c)$ by

$$Sup\left(\rho_{g}^{k}, \rho_{g}^{m}\right) = 1 - \overline{\overline{Ds}}\left(\rho_{g}^{k}, \rho_{g}^{m}\right), \tag{3.33}$$

where, $\overline{\overline{Ds}}\left(\rho_{g}^{k}, \rho_{g}^{m}\right)$ represents, the distance measure between any two SVNNs ρ_{g}^{k} and ρ_{g}^{m} given in Definition (1.1.1.5).

Step 7. Find out $\overline{\overline{T}}\left(\rho_g^k\right)$ by

$$\overline{\overline{T}}\left(\rho_g^k\right) = \sum_{\substack{m=1 \\ m \neq g}}^{b} Sup\left(\rho_g^k, \rho_g^m\right)\left(g = 1,\ldots,a; h,m,k = 1,\ldots,c\right). \tag{3.34}$$

Step 8. Find out

$$\overline{\overline{\Psi}}_g^k = \frac{c\Lambda_k\left(1+\overline{\overline{T}}\left(\rho_g^k\right)\right)}{\sum_{k=1}^{c}\Lambda_c\left(1+\overline{\overline{T}}\left(\rho_g^k\right)\right)}\left(g = 1,\ldots,a; h,k = 1,\ldots,c\right). \tag{3.35}$$

Step 9. Make use of WSVNPMM or WSVNPDMM operators

$$\rho_g = \left\langle \Xi_g, \Psi_g, \Upsilon_g \right\rangle = WSVNPMM^{\overline{\overline{Q}}}\left(\rho_g^1, \rho_g^2, \ldots, \rho_g^c\right), \tag{3.36}$$

or

$$= WSVNPDMM^{\overline{\overline{Q}}}\left(\rho_g^1, \rho_g^2, \ldots, \rho_g^c\right). \tag{3.37}$$

To acquire collective overall SVNNs $\rho_g\left(g = 1,\ldots,a\right)$.

Step 10. Make use of Definition (1.1.1.4), to analyze the cosine measure of the overall SVNN δ_g.

Step 11. Order all the alternatives, and exploit the comparison rules given in Definition (1.1.1.4) and the select the best one.

Step 12. End.

3.3 An illustrative Example

In this section, we give some numerical examples to confirm the efficacy and realism of the anticipated aggregation operators and anticipated decision making approach. The following example is adapted for Liu et al. [73].

3.3.1 Example

Let there are four alternatives $\{\overline{\overline{N}}_1, \overline{\overline{N}}_2, \overline{\overline{N}}_3, \overline{\overline{N}}_4\}$ respectively, confirming the quality of air in Guangzhou city for November of 2006, 2007, 2008, 2009. The experts considered three attributes into account, which are $\overline{\overline{SO}}_2(\overline{\overline{O}}_1), \overline{\overline{NO}}_2(\overline{\overline{O}}_2)$ and $\overline{\overline{PM}}_{10}(\overline{\overline{O}}_3)$. The importance degree of the attributes is $\varpi = (0.314, 0.355, 0.331)^T$. Let us presume that there are three decision makers that is, three air quality monitoring stations expressed by $\{e_1, e_2, e_3\}$ and the importance of these monitoring stations is $\Lambda = (0.40, 0.20, 0.40)^T$. The valuation values of the three air quality monitoring stations under the above three defined attributes are provided in the form of SVNNs, which are given in Tables 3.1, 3.2 and 3.3.

3.3.1.1 The evaluation steps by utilizing WSVNPMM or WSVNPDMM operators

The assessment steps by exploiting WSVNPMM operator or WSVNPDMM are as follows.

Table 3.1. Air quality data from station e_1

	$\overline{\overline{Cr}}_1$	$\overline{\overline{Cr}}_2$	$\overline{\overline{Cr}}_3$
$\overline{\overline{Al}}_1$	$\langle 0.265, 0.350, 0.385 \rangle$	$\langle 0.330, 0.390, 0.280 \rangle$	$\langle 0.245, 0.275, 0.480 \rangle$
$\overline{\overline{Al}}_2$	$\langle 0.345, 0.245, 0.410 \rangle$	$\langle 0.430, 0.290, 0.280 \rangle$	$\langle 0.245, 0.375, 0.380 \rangle$
$\overline{\overline{Al}}_3$	$\langle 0.365, 0.300, 0.335 \rangle$	$\langle 0.480, 0.315, 0.205 \rangle$	$\langle 0.440, 0.270, 0.290 \rangle$
$\overline{\overline{Al}}_4$	$\langle 0.430, 0.300, 0.270 \rangle$	$\langle 0.460, 0.245, 0.295 \rangle$	$\langle 0.310, 0.520, 0.170 \rangle$

Table 3.2. Air quality data from station e_2

	$\overline{\overline{Cr}}_1$	$\overline{\overline{Cr}}_2$	$\overline{\overline{Cr}}_3$
$\overline{\overline{Al}}_1$	$\langle 0.125,0.470,0.405 \rangle$	$\langle 0.220,0.420,0.36 \rangle$	$\langle 0.345,0.490,0.165 \rangle$
$\overline{\overline{Al}}_2$	$\langle 0.355,0.315,0.330 \rangle$	$\langle 0.300,0.370,0.330 \rangle$	$\langle 0.205,0.630,0.165 \rangle$
$\overline{\overline{Al}}_3$	$\langle 0.315,0.380,0.305 \rangle$	$\langle 0.330,0.565,0.105 \rangle$	$\langle 0.280,0.520,0.200 \rangle$
$\overline{\overline{Al}}_4$	$\langle 0.365,0.365,0.270 \rangle$	$\langle 0.355,0.320,0.325 \rangle$	$\langle 0.425,0.485,0.090 \rangle$

Table 3.3. Air quality data from station e_3

	$\overline{\overline{Cr}}_1$	$\overline{\overline{Cr}}_2$	$\overline{\overline{Cr}}_3$
$\overline{\overline{Al}}_1$	$\langle 0.260,0.425,0.315 \rangle$	$\langle 0.220,0.450,0.330 \rangle$	$\langle 0.255,0.500,0.245 \rangle$
$\overline{\overline{Al}}_2$	$\langle 0.270,0.370,0.360 \rangle$	$\langle 0.320,0.215,0.465 \rangle$	$\langle 0.135,0.575,0.290 \rangle$
$\overline{\overline{Al}}_3$	$\langle 0.445,0.265,0.290 \rangle$	$\langle 0.450,0.370,0.180 \rangle$	$\langle 0.2955,0.460,0.165 \rangle$
$\overline{\overline{Al}}_4$	$\langle 0.390,0.340,0.270 \rangle$	$\langle 0.305,0.475,0.220 \rangle$	$\langle 0.465,0.485,0.050 \rangle$

Step 1. Since all the attributes are benefit type, so there is no need to normalize them.

Step 2. Find out the supports $Sup\left(\overline{\overline{Al}}^{k}_{gh}, \overline{\overline{Al}}^{k}_{gl}\right)(g=1,..,4; h,l=1,...,3, k=1,...,3.)$ by utilizing formula (3.28). For simplicity, we shall denote $Sup\left(\overline{\overline{Al}}^{k}_{gh}, \overline{\overline{Al}}^{k}_{gl}\right)$ by $\overleftrightarrow{S}^{k}_{gh,gl}$ and are given below:

$$\overleftrightarrow{S}^{1}_{11,12} = \overleftrightarrow{S}^{1}_{12,11} = 0.9300, \overleftrightarrow{S}^{1}_{11,13} = \overleftrightarrow{S}^{1}_{13,11} = 0.9367, \overleftrightarrow{S}^{1}_{12,13} = \overleftrightarrow{S}^{1}_{13,12} = 0.8667, \overleftrightarrow{S}^{1}_{21,22} = \overleftrightarrow{S}^{1}_{22,21} = 0.9133,$$
$$\overleftrightarrow{S}^{1}_{21,23} = \overleftrightarrow{S}^{1}_{23,21} = 0.9133, \overleftrightarrow{S}^{1}_{22,23} = \overleftrightarrow{S}^{1}_{23,22} = 0.8767; \overleftrightarrow{S}^{1}_{31,32} = \overleftrightarrow{S}^{1}_{32,31} = 0.9133, \overleftrightarrow{S}^{1}_{31,33} = \overleftrightarrow{S}^{1}_{33,31} = 0.9500,$$
$$\overleftrightarrow{S}^{1}_{32,33} = \overleftrightarrow{S}^{1}_{33,32} = 0.9433, \overleftrightarrow{S}^{1}_{41,42} = \overleftrightarrow{S}^{1}_{42,41} = 0.9633, \overleftrightarrow{S}^{1}_{41,43} = \overleftrightarrow{S}^{1}_{43,41} = 0.8533, \overleftrightarrow{S}^{1}_{42,43} = \overleftrightarrow{S}^{1}_{43,42} = 0.8167;$$
$$\overleftrightarrow{S}^{2}_{11,12} = \overleftrightarrow{S}^{2}_{12,11} = 0.9367, \overleftrightarrow{S}^{2}_{11,13} = \overleftrightarrow{S}^{2}_{13,11} = 0.8400, \overleftrightarrow{S}^{2}_{12,13} = \overleftrightarrow{S}^{2}_{13,12} = 0.8700, \overleftrightarrow{S}^{2}_{21,22} = \overleftrightarrow{S}^{2}_{22,21} = 0.9633,$$
$$\overleftrightarrow{S}^{2}_{21,23} = \overleftrightarrow{S}^{2}_{23,21} = 0.7900, \overleftrightarrow{S}^{2}_{22,23} = \overleftrightarrow{S}^{2}_{23,22} = 0.8267; \overleftrightarrow{S}^{2}_{31,32} = \overleftrightarrow{S}^{2}_{32,31} = 0.8667, \overleftrightarrow{S}^{2}_{31,32} = \overleftrightarrow{S}^{2}_{32,31} = 0.9100,$$
$$\overleftrightarrow{S}^{2}_{32,33} = \overleftrightarrow{S}^{2}_{33,32} = 0.9233, \overleftrightarrow{S}^{2}_{41,42} = \overleftrightarrow{S}^{2}_{42,41} = 0.9633, \overleftrightarrow{S}^{2}_{41,43} = \overleftrightarrow{S}^{2}_{43,41} = 0.8800, \overleftrightarrow{S}^{2}_{42,43} = \overleftrightarrow{S}^{2}_{43,42} = 0.8433;$$

$\overleftrightarrow{S}^{3}_{11,12} = \overleftrightarrow{S}^{3}_{12,11} = 0.9733, \overleftrightarrow{S}^{3}_{11,13} = \overleftrightarrow{S}^{3}_{13,11} = 0.9500, \overleftrightarrow{S}^{3}_{12,13} = \overleftrightarrow{S}^{3}_{13,12} = 0.9433, \overleftrightarrow{S}^{3}_{21,22} = \overleftrightarrow{S}^{3}_{22,21} = 0.8967,$
$\overleftrightarrow{S}^{3}_{21,23} = \overleftrightarrow{S}^{3}_{23,21} = 0.8633, \overleftrightarrow{S}^{3}_{22,23} = \overleftrightarrow{S}^{3}_{23,22} = 0.7600; \overleftrightarrow{S}^{3}_{31,32} = \overleftrightarrow{S}^{3}_{32,31} = 0.9267, \overleftrightarrow{S}^{3}_{31,33} = \overleftrightarrow{S}^{3}_{33,31} = 0.8433,$
$\overleftrightarrow{S}^{3}_{32,33} = \overleftrightarrow{S}^{3}_{33,32} = 0.9133, \overleftrightarrow{S}^{3}_{41,42} = \overleftrightarrow{S}^{3}_{42,41} = 0.9100, \overleftrightarrow{S}^{3}_{41,43} = \overleftrightarrow{S}^{3}_{43,41} = 0.8533, \overleftrightarrow{S}^{3}_{42,43} = \overleftrightarrow{S}^{3}_{43,42} = 0.8867.$

Step 3. Find $\overline{\overline{T}}\left(\overline{\overline{Al}}^{k}_{gh}\right)(g = 1,...,4; h,k = 1,...,3)$ by employing formula (3.29). For ease, we shall indicate $\overline{\overline{T}}\left(\overline{\overline{Al}}^{k}_{gh}\right)(g = 1,...,4; h,k = 1,...,3)$ by T^{k}_{gh} and are provided below:

$\overline{\overline{T}}^{1}_{11} = 1.8667, \overline{\overline{T}}^{1}_{12} = 1.7967, \overline{\overline{T}}^{1}_{13} = 1.8033, \overline{\overline{T}}^{1}_{21} = 1.8267, \overline{\overline{T}}^{1}_{22} = 1.7900, \overline{\overline{T}}^{1}_{23} = 1.7900,$
$\overline{\overline{T}}^{1}_{31} = 1.8633, \overline{\overline{T}}^{1}_{32} = 1.8567, \overline{\overline{T}}^{1}_{33} = 1.8933, \overline{\overline{T}}^{1}_{41} = 1.8167, \overline{\overline{T}}^{1}_{42} = 1.7800, \overline{\overline{T}}^{1}_{43} = 1.67;$
$\overline{\overline{T}}^{2}_{11} = 1.7767, \overline{\overline{T}}^{2}_{12} = 1.8067, \overline{\overline{T}}^{2}_{13} = 1.7100, \overline{\overline{T}}^{2}_{21} = 1.7533, \overline{\overline{T}}^{2}_{22} = 1.7900, \overline{\overline{T}}^{2}_{23} = 1.6167,$
$\overline{\overline{T}}^{2}_{31} = 1.7767, \overline{\overline{T}}^{2}_{32} = 1.7900, \overline{\overline{T}}^{2}_{33} = 1.8333, \overline{\overline{T}}^{2}_{41} = 1.8433, \overline{\overline{T}}^{2}_{42} = 1.8067, \overline{\overline{T}}^{2}_{43} = 1.7233;$
$\overline{\overline{T}}^{3}_{11} = 1.9233, \overline{\overline{T}}^{3}_{12} = 1.9167, \overline{\overline{T}}^{3}_{13} = 1.8933, \overline{\overline{T}}^{3}_{21} = 1.7600, \overline{\overline{T}}^{3}_{22} = 1.6567, \overline{\overline{T}}^{3}_{23} = 1.6233,$
$\overline{\overline{T}}^{3}_{31} = 1.7700, \overline{\overline{T}}^{3}_{32} = 1.8400, \overline{\overline{T}}^{3}_{33} = 1.7567, \overline{\overline{T}}^{3}_{41} = 1.7633, \overline{\overline{T}}^{3}_{42} = 1.7967, \overline{\overline{T}}^{3}_{43} = 1.7400.$

Step 4. Determine $\overline{\overline{\Phi}}^{k}_{gh}$ by utilizing formula (3.30), and are given below:

$\overline{\overline{\Phi}}^{1}_{11} = 0.9573, \overline{\overline{\Phi}}^{1}_{12} = 1.0559, \overline{\overline{\Phi}}^{1}_{13} = 0.9868, \overline{\overline{\Phi}}^{1}_{21} = 0.9505, \overline{\overline{\Phi}}^{1}_{22} = 1.0606, \overline{\overline{\Phi}}^{1}_{23} = 0.9889,$
$\overline{\overline{\Phi}}^{1}_{31} = 0.9395, \overline{\overline{\Phi}}^{1}_{32} = 1.0597, \overline{\overline{\Phi}}^{1}_{33} = 1.0008, \overline{\overline{\Phi}}^{1}_{41} = 0.9631, \overline{\overline{\Phi}}^{1}_{42} = 1.0746, \overline{\overline{\Phi}}^{1}_{43} = 0.9623;$
$\overline{\overline{\Phi}}^{2}_{11} = 0.9459, \overline{\overline{\Phi}}^{2}_{12} = 1.0809, \overline{\overline{\Phi}}^{2}_{13} = 0.9732, \overline{\overline{\Phi}}^{2}_{21} = 0.9532, \overline{\overline{\Phi}}^{2}_{22} = 1.0920, \overline{\overline{\Phi}}^{2}_{23} = 0.9549,$
$\overline{\overline{\Phi}}^{2}_{31} = 0.9341, \overline{\overline{\Phi}}^{2}_{32} = 1.0611, \overline{\overline{\Phi}}^{2}_{33} = 1.0048, \overline{\overline{\Phi}}^{2}_{41} = 0.9598, \overline{\overline{\Phi}}^{2}_{42} = 1.0711, \overline{\overline{\Phi}}^{2}_{43} = 0.9691;$
$\overline{\overline{\Phi}}^{3}_{11} = 0.9460, \overline{\overline{\Phi}}^{3}_{12} = 1.0671, \overline{\overline{\Phi}}^{3}_{13} = 0.9870, \overline{\overline{\Phi}}^{3}_{21} = 0.9708, \overline{\overline{\Phi}}^{3}_{22} = 1.0565, \overline{\overline{\Phi}}^{3}_{23} = 0.9727,$
$\overline{\overline{\Phi}}^{3}_{31} = 0.9351, \overline{\overline{\Phi}}^{3}_{32} = 1.0839, \overline{\overline{\Phi}}^{3}_{33} = 0.9810, \overline{\overline{\Phi}}^{3}_{41} = 0.9406, \overline{\overline{\Phi}}^{3}_{42} = 1.0763, \overline{\overline{\Phi}}^{3}_{43} = 0.9832.$

Step 5. Exploit the WSVNPMM by formula (3.31) to acquire the overall $\overline{\overline{Al}}^{k}_{g}\ (g = 1,...,4, k = 1,...,3)$ and are given in Table 3.4. (assume $\left(\overline{\overline{Q}}(1,1,1)\right)$)

Table 3.4. Collective decision matrix M

	e_1	e_2	e_3

$\overline{\overline{Al}}_1$	$\langle 0.2773,0.3399,0.3889\rangle$	$\langle 0.2113,0.4623,0.3177\rangle$	$\langle 0.2442,0.4598,0.2980\rangle$
$\overline{\overline{Al}}_2$	$\langle 0.3304,0.3057,0.3611\rangle$	$\langle 0.2788,0.4597,0.2782\rangle$	$\langle 0.2263,0.4085,0.3749\rangle$
$\overline{\overline{Al}}_3$	$\langle 0.4244,0.2958,0.2814\rangle$	$\langle 0.3400,0.4795,0.2127\rangle$	$\langle 0.3885,0.3700,0.2170\rangle$
$\overline{\overline{Al}}_4$	$\langle 0.3932,0.3702,0.2461\rangle$	$\langle 0.3802,0.3961,0.2324\rangle$	$\langle 0.3811,0.4366,0.1860\rangle$

Step 6. Establish the supports $Sup\left(\overline{\overline{Al}}_g^k,\overline{\overline{Al}}_g^m\right)(g=1,\ldots,4;m,k=1,\ldots,3)$ by formula (2.33). For simplicity, we denote $Sup\left(\overline{\overline{Al}}_g^k,\overline{\overline{Al}}_g^m\right)(g=1,\ldots,4;m,k=1,\ldots,3)$ by $\vec{S}_{gk}$ and are given below.

$$\overleftrightarrow{S}_{11}=0.9134,\overleftrightarrow{S}_{12}=0.9187,\overleftrightarrow{S}_{13}=0.9816,\overleftrightarrow{S}_{21}=0.9038,\overleftrightarrow{S}_{22}=0.9264,\overleftrightarrow{S}_{23}=0.9332,$$
$$\overleftrightarrow{S}_{31}=0.8877,\overleftrightarrow{S}_{32}=0.9418,\overleftrightarrow{S}_{33}=0.9459,\overleftrightarrow{S}_{41}=0.9825,\overleftrightarrow{S}_{42}=0.9538,\overleftrightarrow{S}_{43}=0.9707.$$

Step 7. Find out $\overline{\overline{T}}\left(\overline{\overline{Al}}_g^k\right)(g=1,\ldots,4,k=1,\ldots,3)$ by formula (3.34), for ease, we signify $\overline{\overline{T}}\left(\overline{\overline{Al}}_g^k\right)(g=1,\ldots,4,k=1,\ldots,3)$ by $\overline{\overline{T}}_{gk}$ and are given below:

$$\overline{\overline{T}}_{11}=1.8321,\overline{\overline{T}}_{12}=1.8951,\overline{\overline{T}}_{13}=1.9003,\overline{\overline{T}}_{21}=1.8303,\overline{\overline{T}}_{22}=1.8370,\overline{\overline{T}}_{23}=1.8597,$$
$$\overline{\overline{T}}_{31}=1.8296,\overline{\overline{T}}_{32}=1.8336,\overline{\overline{T}}_{33}=1.8877,\overline{\overline{T}}_{41}=1.9363,\overline{\overline{T}}_{42}=1.9532,\overline{\overline{T}}_{43}=1.9245.$$

Step 8. Find out $\overline{\overline{\Phi}}_g^k$ by formula (3.35), for simplicity, we shall denote Φ_g^k by Φ_{gk} and are given below:

$$\overline{\overline{\Phi}}_{11}=1.1833,\overline{\overline{\Phi}}_{12}=0.6048,\overline{\overline{\Phi}}_{13}=1.2118,\overline{\overline{\Phi}}_{21}=1.1945,\overline{\overline{\Phi}}_{22}=0.5987,\overline{\overline{\Phi}}_{23}=1.2069,$$
$$\overline{\overline{\Phi}}_{31}=1.1899,\overline{\overline{\Phi}}_{32}=0.5958,\overline{\overline{\Phi}}_{33}=1.2143,\overline{\overline{\Phi}}_{41}=1.2005,\overline{\overline{\Phi}}_{42}=0.6037,\overline{\overline{\Phi}}_{43}=1.1957.$$

Step 9. Exploit the WSVNPMM specified in formula (3.36), to get $\overline{\overline{N}}_g\,(g=1,2,3,4)$.(assume $(Q(1,1,1))$).

$$\overline{\overline{Al}}_1=\langle 0.2307,0.4526,0.3626\rangle,\overline{\overline{Al}}_2=\langle 0.2622,0.4291,0.3607\rangle,$$
$$\overline{\overline{Al}}_3=\langle 0.3622,0.4248,0.2658\rangle,\overline{\overline{Al}}_4=\langle 0.3669,0.4275,0.2553\rangle.$$

Step 10. Utilizing Definition (1.1.1.4), to calculate the cosine measure for over all SVNNs $\overline{\overline{Al}}_g$.

$$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.0853,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1111,\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2121,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2177.$$

Step 11. Arrange all the alternatives descending order according to their cosine measure values, and utilizing comparison rules defined in Definition (1.1.1.4), and select the best one.

$$\overline{\overline{Al}}_4 > \overline{\overline{Al}}_3 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$$

Hence $\overline{\overline{Al}}_4$ is the optimal one and the worst one is $\overline{\overline{Al}}_1$.

Further, we exploit the WSVNPDMM operator to re-calculate this example.

Steps 1 to 4 are same.

Step 5. Employ the WSVNPDMM by operating formula (3.32), to acquire the overall $\overline{\overline{Al}}_g^k\,(g=1,...,4,k=1,...,3)$ and are given in Table 3.5. (assume $\left(Q(1,1,1)\right)$)

Table 3.5. Collective decision matrix M

	e_1	e_2	e_3
$\overline{\overline{Al}}_1$	$\langle 0.2805,0.3343,0.3743\rangle$	$\langle 0.2355,0.4588,0.2881\rangle$	$\langle 0.2468,0.4567,0.2937\rangle$
$\overline{\overline{Al}}_2$	$\langle 0.3437,0.2983,0.3523\rangle$	$\langle 0.2901,0.4190,0.2610\rangle$	$\langle 0.2442,0.3585,0.3639\rangle$
$\overline{\overline{Al}}_3$	$\langle 0.4298,0.2931,0.2712\rangle$	$\langle 0.3428,0.4668,0.1859\rangle$	$\langle 0.4012,0.3549,0.2049\rangle$
$\overline{\overline{Al}}_4$	$\langle 0.4029,0.3373,0.2378\rangle$	$\langle 0.3836,0.3842,0.1985\rangle$	$\langle 0.3924,0.4264,0.1435\rangle$

Step 6. Find out the supports $Sup\left(\overline{\overline{Al}}_g^k,\overline{\overline{Al}}_g^m\right)(g=1,...,4;m,k=1,...,3)$ by formula (3.33). For ease, we designate $Supp\left(\overline{\overline{Al}}_g^k,\overline{\overline{Al}}_g^m\right)(g=1,2,3,4;m,k=1,2,3)$ by S_{gk} and are provided below.

$$\overleftrightarrow{S}_{11}=0.9148,\overleftrightarrow{S}_{12}=0.9211,\overleftrightarrow{S}_{13}=0.9936,\overleftrightarrow{S}_{21}=0.9115,\overleftrightarrow{S}_{22}=0.9264,\overleftrightarrow{S}_{23}=0.9429,$$
$$\overleftrightarrow{S}_{31}=0.8850,\overleftrightarrow{S}_{32}=0.9481,\overleftrightarrow{S}_{33}=0.9369,\overleftrightarrow{S}_{41}=0.9649,\overleftrightarrow{S}_{42}=0.9354,\overleftrightarrow{S}_{43}=0.9646.$$

Step 7. Find out $\overline{\overline{T}}\left(\overline{\overline{Al}}_g^k\right)(g=1,...,4,k=1,...,3)$ by formula (3.34), for straightforwardness, we denote $\overline{\overline{T}}\left(\overline{\overline{Al}}_g^k\right)(g=1,...,4,k=1,...,3)$ by $\overline{\overline{T}}_{gk}$ and are provided below:

$$\overline{\overline{T}}_{11}=1.8359,\overline{\overline{T}}_{12}=1.9084,\overline{\overline{T}}_{13}=1.9148,\overline{\overline{T}}_{21}=1.8544,\overline{\overline{T}}_{22}=1.8418,\overline{\overline{T}}_{23}=1.8732,$$
$$\overline{\overline{T}}_{31}=1.8330,\overline{\overline{T}}_{32}=1.8219,\overline{\overline{T}}_{33}=1.8849,\overline{\overline{T}}_{41}=1.9003,\overline{\overline{T}}_{42}=1.9295,\overline{\overline{T}}_{43}=1.9001.$$

Step 8. Find out $\overline{\overline{\Phi}}_g^k$ by formula (3.35), for ease, we shall signify $\overline{\overline{\Phi}}_g^k$ by $\overline{\overline{\Phi}}_{gk}$ and are specified below:

$$\overline{\overline{\Phi}}_{11}=1.1808,\overline{\overline{\Phi}}_{12}=0.6055,\overline{\overline{\Phi}}_{13}=1.2137,\overline{\overline{\Phi}}_{21}=1.1979,\overline{\overline{\Phi}}_{22}=0.5963,\overline{\overline{\Phi}}_{23}=1.2058,$$
$$\overline{\overline{\Phi}}_{31}=1.1922,\overline{\overline{\Phi}}_{32}=0.5938,\overline{\overline{\Phi}}_{33}=1.2141,\overline{\overline{\Phi}}_{41}=1.1976,\overline{\overline{\Phi}}_{42}=0.6049,\overline{\overline{\Phi}}_{43}=1.1975.$$

Step 9. Exploit the WSVNPDMM specified in formula (3.37), to acquire $\overline{\overline{M}}_g\ (g=1,2,3,4)$.(assume $\left(Q(1,1,1)\right)$).

$$\overline{\overline{Al}}_1=\langle 0.2819,0.3957,0.3008\rangle,\overline{\overline{Al}}_2=\langle 0.3247.0.3410,0.3042\rangle,$$
$$\overline{\overline{Al}}_3=\langle 0.4152,0.3522,0.2063\rangle,\overline{\overline{Al}}_4=\langle 0.4186,0.3623,0.1799\rangle.$$

Step 10. Employing Definition (1.1.1.4), to calculate the cosine measure for over all SVNNs Γ_g.

$$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1390,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1881,\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2961,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.3010.$$

Step 11. Arrange all the alternatives descending order according to their cosine measure values, and utilizing comparison rules defined in Definition (1.1.1.4), and select the optimal one.

$$\overline{\overline{Al}}_4>\overline{\overline{Al}}_3>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$$

Hence $\overline{\overline{Al}}_4$ is the optimal one, and the worst one is $\overline{\overline{Al}}_1$.

3.3.1 Influence of the Parameter Vector $\overline{\overline{Q}}$ on Final Ranking Results

The developed method to MAGDM problems has two notable advantages. Firstly, it can eliminate the influence of the too high and low arguments on the final results. Secondly, it can consider the correlation among SVN attributes values. Furthermore, the developed aggregation operators have a parameter vector that makes the aggregation process more flexible. In simple words, when distinct parameters are given to the WSVNPMM operators and WSVNPDMM operators, different overall values can be derived, resulting in differing in cosine measures and ranking results. To show the effect of the parameter vector $\overline{\overline{Q}}$ on the ranking results, we give distinct parameter vectors $\overline{\overline{Q}}$ in the WSVNPMM operators and WSVNPDMM operators and discuss the ranking results in Table 3.6.

Table 3.6 shows that by utilizing distinct parameter vector $\overline{\overline{Q}}$, distinct ranking results are obtained. Furthermore, from Table 3.6, one can see that when the number of interrelationship attributes increases, the values of the cosine measures utilizing WSVNPMM operator decrease, while utilizing WSVNPDMM operator, the values of the cosine measures increase.

Table 3.6. Score values and ranking order for different values of parameter vector Q

Parameter values	Score values Utilizing WSVNPMM operator	Score values Utilizing WSVNPDMM operator	Ranking orders
$Q(1,0,0)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1150, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1562,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2811, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2679.$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1058, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1311,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2640, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2493.$	$\overline{\overline{Al}}_3 > \overline{\overline{Al}}_4 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$ $\overline{\overline{Al}}_3 > \overline{\overline{Al}}_4 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$

$Q(1,1,0)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.0981,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1279,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2403,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2356.$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1253,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1673,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2841,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2810.$	$\overline{\overline{Al}}_3>\overline{\overline{Al}}_4>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$ $\overline{\overline{Al}}_3>\overline{\overline{Al}}_4>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$
$Q(1,1,1)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.0853,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1111,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2121,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2177.$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1390,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1881,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2961,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.3010.$	$\overline{\overline{Al}}_4>\overline{\overline{Al}}_3>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$ $\overline{\overline{Al}}_4>\overline{\overline{Al}}_3>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$
$Q(5,0,0)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1577,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.2384,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.3597,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.3342.$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.0866,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.0926,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2244,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.1945.$	$\overline{\overline{Al}}_3>\overline{\overline{Al}}_4>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$ $\overline{\overline{Al}}_3>\overline{\overline{Al}}_4>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$

3.3.2 Comparison and Discussion

To confirm the efficacy and compensation of the proposeed method, we confer a relative analysis. We operate various presented methods to explain the same example and scrutinize the final results. We compare our method in this paper with the methods developed by Xu et al.[72] based on weighted SVNBM operator, developed by Liu et al. [71] based INPWA operator and developed by He et al. [80] based on SVN weighted power BM operator. The ranking results obtained by these four methods are listed in Table 3.7.

Table 3.7. Comparison with different approaches

Approach	Score Values	Ranking order
Weighted SVNBM operator [72]	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.00076,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.0010,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.0023,\overline{\overline{SO}}\left(Al_4\right)=0.0022.$	$\overline{\overline{Al}}_3>\overline{\overline{Al}}_4>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$
SVN power weighted averaging operator [71]	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1150,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1561,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2710,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2466.$	$\overline{\overline{Al}}_3>\overline{\overline{Al}}_4>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$
SVN weighted power BM operator [80]	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1266,\overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1291,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2390,\overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2199.$	$\overline{\overline{Al}}_3>\overline{\overline{Al}}_4>\overline{\overline{Al}}_2>\overline{\overline{Al}}_1.$

Proposed WSVNPMM for $Q(1,0,0)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1150, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1562,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2811, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2679.$	$\overline{\overline{Al}}_3 > \overline{\overline{Al}}_4 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$
Proposed WSVNPDMM for $Q(1,0,0)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1058, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1311,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2640, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2493.$	$\overline{\overline{Al}}_3 > \overline{\overline{Al}}_4 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$
Proposed WSVNPMM for $Q(1,1,0)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.0981, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1279,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2403, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2356.$	$\overline{\overline{Al}}_3 > \overline{\overline{Al}}_4 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$
Proposed WSVNPDMM for $Q(1,1,0)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1253, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1673,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2841, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2810.$	$\overline{\overline{Al}}_3 > \overline{\overline{Al}}_4 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$
Proposed WSVNPMM for $Q(1,1,1)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.0853, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1111,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2121, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.2177.$	$\overline{\overline{Al}}_4 > \overline{\overline{Al}}_3 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$
Proposed WSVNPDMM for $Q(1,1,1)$	$\overline{\overline{SO}}\left(\overline{\overline{Al}}_1\right)=0.1390, \overline{\overline{SO}}\left(\overline{\overline{Al}}_2\right)=0.1881,$ $\overline{\overline{SO}}\left(\overline{\overline{Al}}_3\right)=0.2961, \overline{\overline{SO}}\left(\overline{\overline{Al}}_4\right)=0.3010.$	$\overline{\overline{Al}}_4 > \overline{\overline{Al}}_3 > \overline{\overline{Al}}_2 > \overline{\overline{Al}}_1.$

From Table 3.7, we can see that methods in [71,72,80] produced the same ranking results as the proposed method in this paper when Q takes (1,0,0) and (1,1,0), and this can explain the validity of the proposed method in this paper. However, when Q takes (1,1,1), i.e., when we consider the interrelationship among three attributes, we get a different ranking result. Then we can give some explanations of the different existing methods as follows.

While our adopted method is supported on the WSVNPMM operator or WSVNPDMM operator, this can judge the correlation among SVNNs and also take away the cause of uncomfortable data at the same time.

The Xu et al. [72] method based on BM operator can judge the correlation between two SVNNs, but cannot eradicate the cause of uncomfortable data. While our proposed method can judge the correlation among any number of SVNNs and also eradicate the cause of uncomfortable data at the same time.

The Liu et al [71] method is based on PA operator, which can only remove the bad influence of too high or too low arguments. But this cannot consider the interrelationships among SVNNs. While our developed method can also remove the effect of awkward data, and can consider the interrelationship among any number of SVNNs at the same time.

The He et al. [80] method based on PBM operator, which can judge the correlation between two SVNNs and also eradicate the influence of too high and too low arguments by PA operator. While our propose method can judge the correlation among any number of SVNNs.

Thus, the developed method based on the developed aggregation operators is more effective and flexible for MAGDM problems.

3.3.3 Conclusion

In this article, we combined MM operator and PA operator and developed various AOs, such as SVNPMM operator, WSVNPMM operator, SVNPDMM operator and WSVNPDMM operator. The developed AOs take full advantage of MM operator and PA operator. In simple words, the developed AOs not only consider the interrelationship among SVNNs but also remove the influence of too high or too low arguments on the final results. Further, we inspected some analyzed several desirable properties and special cases of the developed AOs. We also proposed a novel approach to MAGDM with SVN information. Lastly, we provide a numerical example to confirm the efficacy and realism of the developed approach.

In future research, we will extend the developed AOs to different fuzzy environments such as double-valued NS, IFS, hesitant fuzzy sets, single valued neutrosophic hesitant fuzzy set.

Chapter 4

Neutrosophic Cubic Power Muirhead Mean Operators with Uncertain Data for Multi-Attribute Decision Making

In this chapter, we intend various AOs for NCNs, which is a basic member of NCS. Taking the full advantages of MM operator and PA operator, the PMM operator is developed and is examined under NC information. To handle the problems up stretched, various new NC AOs, such as the NCPMM operator, WNCPMM operator, NCPDMM operator and WNCPDMM operator are developed and allied characteristics of these developed AOs are granted. The significant advantage of the proposed AO is that it can eliminate the effect of uncomfortable data and it takes the interrelationship between aggregated values at the same time. Further, a novel MADM method is instituted over the developed AOs to bestow the effectiveness of these operators. Lastly, a numerical example is specified to show the efficiency of the proposed approach.

4.1 Some Power Muirhead Mean Operator for Neutrosophic Cubic Sets

4.1.1 The Neutrosophic Cubic Power Muirhead Mean (NCPMM) Operator

In this subsection, we extend the PMM operator to neutrosophic cubic environment and discuss some basic properties, and special cases of these developed aggregation operators with respect to the parameter $\overline{\overline{Q}}$.

4.1.1.1 Definition

Let $\hbar_g (g = 1,2,...,u)$ be a set of NCNs and the parameters vector is denoted by $\overline{\overline{Q}} = \left(q_1, q_2, ..., q_{\overline{\overline{a}}}\right) \in R^{\overline{\overline{a}}}$. If,

$$NCPMM^{\overline{\overline{Q}}}\left(\hbar_1, \hbar_2, ..., \hbar_{\overline{\overline{u}}}\right) = \left(\frac{1}{\overline{\overline{u}}!}\sum_{\theta \in S_{\overline{\overline{u}}}}\prod_{g=1}^{\overline{\overline{u}}}\left(\frac{\overline{\overline{u}}\left(1+\overline{\overline{T}}\left(\hbar_{\theta(g)}\right)\right)}{\sum_{\overline{\overline{m}}=1}^{\overline{\overline{u}}}\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{m}}}\right)\right)}\hbar_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum_{g=1}^{\overline{\overline{u}}} q_g}}. \quad (4.1)$$

Then, we call $NCPMM^{\overline{\overline{Q}}}$ the neutrosophic cubic power Muirhead mean operator, where $S_{\overline{\overline{a}}}$ is the set of all permutation, $\theta(g)$ signify any permutation of $\left(1,2,...,\overline{\overline{a}}\right)$ and $\overline{\overline{T}}\left(\hbar_{\overline{\overline{m}}}\right) = \sum_{\overline{\overline{m}}=1, x \neq g}^{\overline{\overline{a}}} Sup\left(\hbar_g, \hbar_{\overline{\overline{m}}}\right)$, $Sup\left(\hbar_g, \hbar_{\overline{\overline{m}}}\right)$ is the support degree for $\hbar_g$ and $\hbar_{\overline{\overline{m}}}$, satisfying the following axioms:

(1) $Sup\left(\hbar_g, \hbar_{\overline{\overline{m}}}\right) \in [0,1]$;

(2) $Sup\left(\hbar_g, \hbar_{\overline{\overline{m}}}\right) = Sup\left(\hbar_{\overline{\overline{m}}}, \hbar_z\right)$;

(3) If $\overline{\overline{Ds}}(\hbar_g, \hbar_{\overline{\overline{m}}}) < \overline{\overline{Ds}}(\hbar_u, \hbar_v)$, then $Sup(\hbar_g, \hbar_{\overline{\overline{m}}}) > Sup(\hbar_u, \hbar_v)$, where $\overline{\overline{Ds}}(\hbar_g, \hbar_{\overline{\overline{m}}})$ is the distance among $\hbar_g$ and $\hbar_x$.

In order to inscribe Equation (4.1) in an easy form, we can stipulate it as:

$$\Theta_g = \frac{\left(1+\overline{\overline{T}}\left(\hbar_g\right)\right)}{\sum_{\overline{\overline{m}}=1}^{a}\left(1+\overline{\overline{T}}\left(\hbar_{\overline{\overline{m}}}\right)\right)}. \quad (4.2)$$

For appropriateness, we can entitle $(\Theta_1, \Theta_2, ..., \Theta_a)^T$ the power weight vector (PMV), such that $\Theta_g \in [0,1]$ and $\sum_{g=1}^{a} \Theta_g = 1.$ From the exploit of Equation (4.2), Equation (4.1) can be articulated as:

$$NCPMM^{\overline{\overline{Q}}}\left(\hbar_1, \hbar_2, ..., \hbar_{\overline{\overline{a}}}\right) = \left(\frac{1}{\overline{\overline{a}}!} \sum_{\theta \in S_{\overline{\overline{a}}}} \prod_{g=1}^{\overline{\overline{a}}} \left(\overline{\overline{a}} \Theta_g \hbar_{\theta(g)} \right)^{q_g} \right)^{\frac{1}{\sum_{g=1}^{\overline{\overline{a}}} q_g}}. \quad (4.3)$$

Based on the operational rules given in Definition (1.1.1.15) for NCNs, and Definition (4.1.1.1), we can have the following Theorem (4.1.1. 2).

4.1.1.2 Theorem

Let $\hbar_g (g = 1, 2, ..., a)$ be a set of NCNs and $\overline{\overline{Q}} = (q_1, q_2, ..., q_a) \in R^a$ be a vector of parameters. Then, the result aggregated by employing Equation (4.1) is still an NCN and,

$$NCPMM^{\overline{\overline{Q}}}\left(\hbar_1, \hbar_2, ..., \hbar_a\right) =$$

$$\left(\left\langle \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\Xi^L \right)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\Xi^U \right)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \right.\right.$$

$$\left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\Psi^L \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\Psi^U \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right],$$

$$\left. \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\Upsilon^L \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\Upsilon^U \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right] \right\rangle,$$

$$\left\langle \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\lambda_T \right)_{\theta(g)} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\lambda_I \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right.$$

$$\left. \left. , 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\lambda_F \right)_{\theta(g)}^{a\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle \right). \quad (4.4)$$

Proof. According to the operational laws for NCNs, we have

$$a\Theta_g \hbar_{\theta(g)} = \left(\left\langle \left[1 - \left(1 - \left(\Xi^L \right)_{\theta(g)} \right)^{a\Theta_g}, 1 - \left(1 - \left(\Xi^U \right)_{\theta(g)} \right)^{a\Theta_g} \right], \left[\left(\Psi^L \right)_{\theta(g)}^{a\Theta_g}, \left(\Psi^U \right)_{\theta(g)}^{a\Theta_g} \right], \left[\left(\Upsilon^L \right)_{\theta(g)}^{a\Theta_g}, \left(\Upsilon^U \right)_{\theta(g)}^{a\Theta_g} \right] \right\rangle, \right.$$

$$\left. \left\langle 1 - \left(1 - \left(\lambda_T \right)_{\theta(g)} \right)^{a\Theta_g}, \left(\lambda_I \right)_{\theta(g)}^{a\Theta_g}, \left(\lambda_F \right)_{\theta(g)}^{a\Theta_g} \right\rangle \right).$$

So,

$$\left(a\Theta_g\hbar_{\theta(g)}\right)^{q_g}=\Bigg(\Bigg\langle\Bigg[\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g},\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\Bigg],\Bigg[1-\left(1-\left(\Psi^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g},$$

$$1-\left(1-\left(\Psi^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\Bigg],\Bigg[1-\left(1-\left(\Upsilon^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g},1-\left(1-\left(\Upsilon^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\Bigg]\Bigg\rangle,$$

$$\Bigg\langle\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g},1-\left(1-\left(\lambda_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g},1-\left(1-\left(\lambda_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\Bigg\rangle\Bigg).$$

Therefore,

$$\prod_{g=1}^{a}\left(a\Theta_g\hbar_{\theta(g)}\right)^{q_g}=\Bigg(\Bigg\langle\Bigg[\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g},\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\Bigg],$$

$$\Bigg[1-\prod_{g=1}^{a}\left(1-\left(\Psi^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g},1-\prod_{g=1}^{a}\left(1-\left(\Psi^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\Bigg],\Bigg[1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\Bigg]\Bigg\rangle,$$

$$\Bigg\langle\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g},1-\prod_{g=1}^{a}\left(1-\left(\lambda_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g},1-\prod_{g=1}^{a}\left(1-\left(\lambda_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\Bigg\rangle\Bigg),$$

and

$$\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g\hbar_{\theta(g)}\right)^{q_g}=\Bigg(\Bigg\langle\Bigg[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\Bigg],$$

$$\Bigg[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\Bigg],\Bigg[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\Bigg]\Bigg\rangle,$$

$$\Bigg\langle1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_f\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\Bigg\rangle\Bigg).$$

Furthermore,

$$\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g\hbar_{\theta(g)}\right)^{q_g}=\Bigg(\Bigg\langle\Bigg[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\Bigg],$$

$$\Bigg[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\Bigg],\Bigg[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^L\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^U\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\Bigg]\Bigg\rangle,$$

$$\Bigg\langle1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_I\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_F\right)^{a\Theta_g}_{\theta(g)}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\Bigg\rangle\Bigg).$$

Hence,

$$\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g\hbar_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}=\left\langle\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\Xi^L)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\Xi^U)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\right.\right.$$

$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\Psi^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\Psi^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],$$

$$\left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\Upsilon^L)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\Upsilon^U)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,$$

$$\left\langle\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\right.$$

$$\left.\left.1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right\rangle,$$

This is the required proof of Theorem (4.1.1.2).

In the above equations, we calculate the PWV Θ, after calculating the support degree $Sup\left(\hbar_g,\hbar_{\overline{\overline{m}}}\right)$. First, we determined the $Sup\left(\hbar_g,\hbar_{\overline{\overline{m}}}\right)$ utilizing

$$Sup\left(\hbar_g,\hbar_{\overline{\overline{m}}}\right)=1-\overline{\overline{Ds}}\left(\hbar_g,\hbar_{\overline{\overline{m}}}\right),\tag{4.5}$$

where,

$$\overline{\overline{Ds}}\left(\hbar_g,\hbar_x\right)=\sqrt{\frac{1}{9}\begin{pmatrix}\left(\Xi_g^L-\Xi_x^L\right)^2+\left(\Xi_g^U-\Xi_x^U\right)^2+\left(\Psi_g^L-\Psi_x^L\right)^2+\left(\Psi_g^U-\Psi_x^U\right)^2+\left(\Upsilon_g^L-\Upsilon_x^L\right)^2+\left(\Upsilon_g^U-\Upsilon_x^U\right)^2\\ \left(\lambda_{T_g}-\lambda_{T_x}\right)^2+\left(\lambda_{I_g}-\lambda_{I_x}\right)^2+\left(\lambda_{F_g}-\lambda_{F_x}\right)^2\end{pmatrix}}.\tag{4.6}$$

Therefore, we use the equation

$$\overline{\overline{T}}\left(\hbar_g\right)=\sum_{g=1,g\neq\overline{\overline{m}}}^{a}Sup\left(\hbar_g,\hbar_{\overline{\overline{m}}}\right).\tag{4.7}$$

To obtain values of $\overline{\overline{T}}\left(\hbar_g\right)(g=1,2...,a)$. Then using Equation (4.2) we can get the PWV.

4.1.1.3 Theorem (Idempotency)

Let $\hbar_g (g=1,2,...,a)$ be a set of NCNs, and $\hbar_g = \hbar,$ for all $g=1,2,...,a.$ Then

$$NCPMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,...,\hbar_a\right)=\hbar. \tag{4.8}$$

Proof. Since $\hbar_g = \hbar$ for all $g=1,2,...,a,$ we have $Supp\left(\hbar_g,\hbar_{\overline{\overline{m}}}\right)=1$ for all $g,\overline{\overline{m}}=1,2,...,a.$ Therefore, we can get $\Theta_g=\frac{1}{a}$ for all $g.$ Moreover,

$$NCPMM^{\overline{\overline{Q}}}\left(\hbar_1,\hbar_2,...,\hbar_a\right)=NCPMM^{\overline{\overline{Q}}}\left(\hbar,\hbar,...,\hbar\right)$$

$$=\left(\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\Xi^L\right)^{a\frac{1}{a}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\Xi^U\right)^{a\frac{1}{a}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\right.$$

$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\Psi^{La\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\Psi^{Ua\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],$$

$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\Upsilon^{La\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\Upsilon^{Ua\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]$$

$$\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\lambda_T\right)^{a\frac{1}{a}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\lambda_I{}^{a\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}$$

$$\left.,1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\lambda_F{}^{a\frac{1}{a}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\Bigg),$$

$$
= \left(\left\langle \left[\left(1 - \left(\left(1 - \left(1 - \left(1 - \Xi^{L} \right) \right)^{\sum_{g=1}^{a} q_g} \right)^{a!} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\left(1 - \left(1 - \left(1 - \Xi^{U} \right) \right)^{\sum_{g=1}^{a} q_g} \right)^{a!} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[1 - \left(1 - \left(1 - \left(1 - \Psi^{L} \right)^{\sum_{g=1}^{a} q_g} \right)^{a! \frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \right.\right.\right.
$$

$$
\left.\left. 1 - \left(1 - \left(1 - \left(1 - \Psi^{U} \right)^{\sum_{g=1}^{a} q_g} \right)^{a! \frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \left[1 - \left(1 - \left(1 - \left(1 - \Upsilon^{L} \right)^{\sum_{g=1}^{a} q_g} \right)^{a! \frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \left(1 - \left(1 - \Upsilon^{U} \right)^{\sum_{g=1}^{a} q_g} \right)^{a! \frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right] \right\rangle,
$$

$$
\left. \left\langle \left(1 - \left(\left(1 - \left(1 - \left(1 - \lambda_{T} \right) \right)^{\sum_{g=1}^{a} q_g} \right)^{a!} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \left(1 - \left(1 - \lambda_{I} \right)^{\sum_{g=1}^{a} q_g} \right)^{a! \frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \left(1 - \left(1 - \lambda_{F} \right)^{\sum_{g=1}^{a} q_g} \right)^{a! \frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle \right),
$$

$$
= \left\langle \left[\Xi^{L}, \Xi^{U} \right], \left[\Psi^{L}, \Psi^{U} \right], \left[\Upsilon^{L}, \Upsilon^{U} \right], \lambda_{T}, \lambda_{I}, \lambda_{F} \right\rangle = \hbar.
$$

which is the required proof of Theorem (4.1.1.3).

4.1.1.4 Theorem (Boundedness)

Let $\hbar_g \, (g = 1, 2, ..., a)$ be a group of NCNs, where

$\bar{\hbar} = \min\left(\hbar_1, \hbar_2, ..., \hbar_a\right) = \left\langle \left[\min \Xi^{L}, \min \Xi^{U} \right], \left[\max \Psi^{L}, \max \Psi^{U} \right], \left[\max \Upsilon^{L}, \max \Upsilon^{U} \right], \lambda_T^{-}, \lambda_I^{+}, \lambda_F^{+} \right\rangle$ and

$\overset{+}{\hbar} = \max\left(\hbar_1, \hbar_2, ..., \hbar_a\right) = \left\langle \left[\max \Xi^{L}, \max \Xi^{U} \right], \left[\min \Psi^{L}, \min \Psi^{U} \right], \left[\min \Upsilon^{L}, \min \Upsilon^{U} \right], \lambda_T^{+}, \lambda_I^{-}, \lambda_F^{-} \right\rangle.$

Then

$$
m \leq NCPMM^{Q}\left(\hbar_1, \hbar_2, ..., \hbar_a\right) \leq n. \tag{4.9}
$$

Where,

$$
m = \left(\left\langle \left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \min \Xi_{\theta(g)}^{L} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \min \Xi_{\theta(g)}^{U} \right)^{a\Theta_g} \right)^{q_g} \right) \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \right.\right.
$$

$$
\left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \max \Psi_{\theta(g)}^{La\Theta_g} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \max \Psi_{\theta(g)}^{La\Theta_z} \right)^{q_g} \right)^{\frac{1}{a!}} \right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right],
$$

$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\max\Upsilon_{\theta(g)}^{La\Theta_z}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\max\Upsilon_{\theta(g)}^{Ua\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]$$

$$\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\bar{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},$$

$$1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg\rangle\Bigg),$$

and

$$n=\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\overset{+}{\Xi}{}^{L}_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\overset{+}{\Xi}{}^{U}_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\right.\right.$$

$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\bar{\Psi}_{\theta(g)}^{La\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\bar{\Psi}_{\theta(g)}^{Ua\Theta_z}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],$$

$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\bar{\Upsilon}_{\theta(g)}^{La\Theta_z}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\bar{\Upsilon}_{\theta(g)}^{Ua\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]$$

$$\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{+}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\bar{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},$$

$$1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\bar{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg\rangle\Bigg).$$

Proof.

$$a\Theta_g\hbar_{\theta(g)}=\left\langle\left[1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g},1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right],\left[\left(\Psi^L\right)_{\theta(g)}^{a\Theta_g},\left(\Psi^U\right)_{\theta(g)}^{a\Theta_g}\right],\left[\left(\Upsilon^L\right)_{\theta(g)}^{a\Theta_g},\left(\Upsilon^U\right)_{\theta(g)}^{a\Theta_g}\right]\right\rangle,$$

$$\left\langle 1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g},\left(\lambda_I\right)_{\theta(g)}^{a\Theta_g},\left(\lambda_F\right)_{\theta(g)}^{a\Theta_g}\right\rangle\geq\left(\left\langle\left[1-\left(1-\left(\bar{\Xi}^{L}\right)_{\theta(g)}\right)^{a\Theta_g},1-\left(1-\left(\bar{\Xi}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right],\right.\right.$$

$$\left.\left.\left[\left(\overset{+}{\Psi}{}^{L}\right)_{\theta(g)}^{a\Theta_g},\left(\overset{+}{\Psi}{}^{U}\right)_{\theta(g)}^{a\Theta_g}\right],\left[\left(\overset{+}{\Upsilon}{}^{L}\right)_{\theta(g)}^{a\Theta_g},\left(\overset{+}{\Upsilon}{}^{U}\right)_{\theta(g)}^{a\Theta_g}\right]\right\rangle,\left\langle 1-\left(1-\left(\bar{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g},\left(\bar{\lambda}_I\right)_{\theta(g)}^{a\Theta_g},\left(\bar{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right\rangle\right),$$

and

$$\left(a\Theta_g \hbar_{\theta(g)}\right)^{q_g} =$$

$$\left(\left\langle\left[\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[,1-\left(1-\left(\Psi^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-\left(\Psi^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right],\right.$$

$$\left.\left[1-\left(1-\left(\Upsilon^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-\left(\Upsilon^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle, \left\langle\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\left(1-\left(\lambda_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\left(1-\left(\lambda_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right)$$

$$\geq \left(\left\langle\left[\left(1-\left(1-\left(\overset{-}{\Xi}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \left(1-\left(1-\left(\overset{-}{\Xi}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[,1-\left(1-\left(\overset{+}{\Psi}^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-\left(\overset{+}{\Psi}^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right],\right.$$

$$\left.\left[1-\left(1-\left(\overset{+}{\Upsilon}^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g} 1-\left(1-\left(\overset{+}{\Upsilon}^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle, \left\langle\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right).$$

Thus,

$$\prod_{g=1}^{a}\left(a\Theta_g \hbar_{\theta(g)}\right)^{q_g} = \left(\left\langle\left[\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[1-\prod_{g=1}^{a}\left(1-\left(\Psi^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g},\right.\right.\right.$$

$$\left.1-\prod_{g=1}^{a}\left(1-\left(\Psi^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right], \left.\left[1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle,$$

$$\left.\left\langle\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\lambda_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\lambda_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right) \geq$$

$$\left(\left\langle\left[\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\Xi}^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, \prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\Xi}^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right], \left[1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Psi}^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g},\right.\right.\right.$$

$$\left.1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Psi}^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right], \left.\left[1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Upsilon}^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Upsilon}^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right]\right\rangle,$$

$$\left.\left\langle\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}, 1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right\rangle\right),$$

and

$$\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Theta_g\hbar_{\theta(g)}\right)^{q_g}=\left(\left\langle\left[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right],\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\right.\right.\right.$$

$$\left.\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right],\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right]\Bigg\rangle,$$

$$\left.\left\langle 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right\rangle\right)$$

$$\geq\left(\left\langle\left[1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\bar{\Xi}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\bar{\Xi}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right],\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Psi}{}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\right.\right.\right.$$

$$\left.\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Psi}{}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right],\left[\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Upsilon}{}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Upsilon}{}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right]\Bigg\rangle,$$

$$\left.\left\langle 1-\prod_{\theta\in S_a}\left(\prod_{g=1}^{a}\left(1-\left(1-\left(\bar{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right),\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right\rangle\right).$$

Furthermore,

$$\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(a\Theta_g\hbar_{\theta(g)}\right)^{q_g}=\left(\left\langle\left[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\right.\right.$$

$$\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],$$

$$\left.\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^L\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^U\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right]\right\rangle,$$

$$\left.\left\langle 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\lambda_F\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}\right\rangle\right)$$

$$\geq\left(\left\langle\left[1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\bar{\Xi}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\bar{\Xi}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Psi}{}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\right.\right.\right.$$

$$\left.\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Psi}{}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right],\left[\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Upsilon}{}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Upsilon}{}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right]\Bigg\rangle,$$

$$\left.\left\langle 1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\bar{\lambda}_T\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\lambda}_I\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}},\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{+}{\lambda}_F\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}\right\rangle\right).$$

Hence,

$$\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Theta_z\hbar_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}} =$$

$$\Bigg(\Bigg\langle\Bigg[\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(\Xi^L)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(\Xi^U)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg],\Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\Psi^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},$$

$$1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\Psi^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg],\Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\Upsilon^L)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\Upsilon^U)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg]\Bigg\rangle$$

$$\Bigg\langle\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-(\lambda_T)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_I)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-(\lambda_F)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg\rangle\Bigg)\geq$$

$$\Bigg(\Bigg\langle\Bigg[\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-\big(\overset{-}{\Xi}^{L}\big)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-\big(\overset{-}{\Xi}^{U}\big)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg],\Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\Psi}^{L}\big)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},$$

$$1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\Psi}^{U}\big)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg],\Bigg[1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\Upsilon}^{L}\big)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\Upsilon}^{U}\big)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg]\Bigg\rangle$$

$$\Bigg\langle\Bigg(1-\Bigg(\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(1-\big(\overset{-}{\lambda}_T\big)_{\theta(g)}\big)^{a\Theta_g}\Big)^{q_g}\Bigg)\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\lambda}_I\big)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\Bigg(1-\prod_{\theta\in S_a}\Bigg(1-\prod_{g=1}^{a}\Big(1-\big(\overset{+}{\lambda}_F\big)_{\theta(g)}^{a\Theta_g}\Big)^{q_g}\Bigg)^{\frac{1}{a!}}\Bigg)^{\frac{1}{\sum_{g=1}^{a}q_g}}\Bigg\rangle\Bigg).$$

This implies that $m \leq NCPMM^{Q}(\hbar_1,\hbar_2,...,\hbar_a)$.

In a comparable technique we can prove that $NCPMM^{Q}(\hbar_1,\hbar_2,...,\hbar_a) \leq n$. Hence $m \leq NCPMM^{Q}(\hbar_1,\hbar_2,...,\hbar_a) \leq n$.

The NCPMM operator does not have the property of monotonicity.

One of the leading advantages of NCPMM is its capacity to represent the interrelationship among NCNs. Furthermore, the NCPMM operator is more flexible in aggregation process due to parameter vector. Now we discuss some special cases of NCPMM operators by assigning different values to the parameter vector.

Case 1. If $Q=(1,0,...,0)$, then the NCPMM operator degenerates into the following equation:

$$NCPMM^{(1,0,...,0)}(\hbar_1,\hbar_2,....,\hbar_u)=\left(\sum_{g=1}^{u}\frac{\left(1+\overline{\overline{T}}(\hbar_g)\right)}{\sum_{\overline{\overline{m}}=1}^{u}\left(1+\overline{\overline{T}}(\hbar_{\overline{\overline{m}}})\right)}\hbar_g\right). \qquad (4.10)$$

This is the NCPA operator.

Case 2. If $Q=\left(\frac{1}{a},\frac{1}{a},....,\frac{1}{a}\right)$, then the NCPMM operator degenerates into the following equation:

$$NCPMM^{\left(\frac{1}{u},\frac{1}{u},...,\frac{1}{u}\right)}\left(\hbar_1,\hbar_2,....,\hbar_u\right)=\prod_{g=1}^{u}\hbar_g^{\frac{\left(1+\overline{\overline{T}}\left(\hbar_{\theta(g)}\right)\right)}{\sum_{x=1}^{u}\left(1+\overline{\overline{T}}\left(\hbar_x\right)\right)}}. \tag{4.11}$$

This is the NCPG operator.

Case 3. If $Q=(1,1,...,0)$, then the NCPMM operator degenerates into the following equation:

$$\begin{aligned}
&NCPMM^{(1,1,0,...,0)}=\left(\hbar_1,\hbar_2,...,\hbar_a\right)=\\
&\left\langle\left[\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\left(1-\Xi_g^L\right)^{\Theta_g}\right)\left(1-\left(1-\Xi_x^L\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\left(1-\Xi_g^U\right)^{\Theta_g}\right)\left(1-\left(1-\Xi_x^U\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\right],\right.\\
&\left[1-\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\Psi_g^{L\Theta_g}\right)\left(1-\Psi_x^{L\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}1-\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\Psi_g^{U\Theta_g}\right)\left(1-\Psi_x^{U\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\right],\\
&\left[1-\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\Upsilon_g^{L\Theta_g}\right)\left(1-\Upsilon_x^{L\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\Upsilon_g^{U\Theta_g}\right)\left(1-\Upsilon_x^{U\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\right],\\
&\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\left(1-\left(\lambda_T\right)_g\right)^{\Theta_z}\right)\left(1-\left(1-T_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\left(\lambda_I\right)_g^{\Theta_g}\right)\left(1-\left(\lambda_T\right)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\\
&\left.1-\left(1-\left(\prod_{\substack{g,x=1\\g\neq x}}^{a}\left(1-\left(1-\left(\lambda_F\right)_z^{\Theta_z}\right)\left(1-\left(\lambda_F\right)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\right\rangle.
\end{aligned} \tag{4.12}$$

This is the NC power Bonferroni mean operator $(p=q=1)$.

Case 4. . If $Q=\left(\overbrace{1,1,...,1}^{i},\overbrace{0,0,...,0}^{z-i}\right)$, then the NCPMM operator degenerates into the following form:

$$NCPMM^{\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)}\left(\hbar_1,\hbar_2,\ldots,\hbar_a\right)=$$

$$\left\langle\left[\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-\Xi^L_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-\Xi^U_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right],\right.$$

$$\left[,1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\Psi^{L\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\Psi^{U\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right],$$

$$\left[1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\Upsilon^{L\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\Upsilon^{U\Theta_{g_x}}_{g_x}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right]$$

$$\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\lambda_T)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_I)_{g_x}^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},$$

$$\left.1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_F)_{g_x}^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right\rangle. \tag{4.13}$$

This is the NC power Maclaurin symmetric mean operator.

4.1.2 Weighted Neutrosophic Cubic Power Muirhead Mean (WNCPMM) Operator

The NCPMM operator does not consider the weight of the aggregated NCNs. In this subsection, we develop the weighted neutrosophic cubic power Muirhead mean (WNCPMM) operator, which has the capacity of taking the weights of NCNs.

4.1.2.1 Definition

Let $\hbar_g\,(g=1,2,\ldots,a)$ be a group of NCNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. If

$$WNCPMM^{Q}\left(\hbar_1,\hbar_2,\ldots,\hbar_a\right)=\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{g=1}^{a}\left(\frac{a\Phi_{\vartheta(g)}\Theta_{\theta(g)}}{\sum_{x=1}^{a}\Phi_x\Theta_x}\hbar_{\theta(g)}\right)^{q_g}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}. \tag{4.14}$$

Then, we call $WNCPMM^{Q}$ the weighted neutrosophic cubic power Muirhead mean operator, where $\Phi=(\Phi_1,\Phi_2,\ldots,\Phi_a)^T$ is the importance degree $\hbar_g\,(g=1,\ldots,a)$ such that $\Phi_z\in[0,1],\sum_{z=1}^{a}\Phi_z=1,$ S_a is the set of all permutation, $\theta(z)$ is any permutation of $(1,\ldots,a)$ and Θ_g

is PVW satisfying $\Theta_g = \frac{\left(1+\overline{\overline{T}}\left(\hbar_g\right)\right)}{\sum_{g=1}^{a}\left(1+\overline{\overline{T}}\left(\hbar_g\right)\right)}, \sum_{g=1}^{a}\Theta_g = 1,\ T\left(\hbar_m\right) = \sum_{m=1, x\neq g}^{a} Sup\left(\hbar_g, \hbar_m\right)$, $Sup\left(\hbar_g, \hbar_m\right)$ is the support degree for $\hbar_g$ and $\hbar_m$, fulfilling the following conditions:

(1) $Sup\left(\hbar_g, \hbar_m\right) \in [0,1]$;

(2) $Sup\left(\hbar_g, \hbar_m\right) = Sup\left(\hbar_m, \hbar_g\right)$;

(3) If $\overline{\overline{Ds}}(\hbar_g, \hbar_m) < \overline{\overline{Ds}}(\hbar_u, \hbar_v)$, then $Sup(\hbar_g, \hbar_m) > Sup(\hbar_u, \hbar_v)$, where $\overline{\overline{Ds}}(\hbar_g, \hbar_m)$ is distance among $\hbar_g$ and $\hbar_m$.

From Definition (4.1.2.1), we have the following Theorem (4.1.2.2).

4.1.2.2 Theorem

Let $\hbar_g (g = 1, 2, ..., a)$ is a set of NCNs and the parameters vector is denoted by $Q = (q_1, q_2, ..., q_a) \in R^a$. Then, the result aggregated by employing Equation (4.14) is still a NCN

$$WNCPMM^{Q}\left(\hbar_1, \hbar_2, ..., \hbar_a\right) =$$

$$\left(\left\langle\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^L\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Xi^U\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\right.\right.$$

$$\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^L\right)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Psi^U\right)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],$$

$$\left.\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^L\right)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Upsilon^U\right)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]\right\rangle,$$

$$\left\langle \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-(\lambda_T)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_I)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \right.$$

$$\left. 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-(\lambda_F)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \right\rangle\Bigg). \tag{4.15}$$

Proof. Proof of Theorem (4.1.2.2) is same as Theorem (4.1.1.2).

4.1.3 The Neutrosophic Cubic Power Dual Muirhead Mean (NCPDMM) Operator

In this subsection, we develop the NCPDMM operator and discuss some related properties.

4.1.3.1 Definition

Let $\hbar_g (g=1,2,...,a)$ is a set of NCNs and the parameters vector is denoted by $Q=(q_1,q_2,...,q_a)\in R^a$. If

$$NCPDMM^Q(\hbar_1,\hbar_2,....,\hbar_a)=\frac{1}{\sum_{g=1}^{a}q_g}\left(\prod_{\theta\in S_a}\sum_{g=1}^{a}\left(q_g\hbar_{\theta(g)}^{\frac{a\left(1+T\left(\hbar_{\theta(g)}\right)\right)}{\sum_{x=1}^{a}(1+T(\hbar_x))}}\right)\right)^{\frac{1}{a!}}. \tag{4.16}$$

Then, we call $NCPDMM^Q$ the neutrosophic cubic power dual Muirhead mean operator, where S_a is the set of all permutation, $\theta(g)$ is any permutation of $(1,...,a)$ and $\overline{\overline{T}}(\hbar_m)=\sum_{m=1,x\neq g}^{a}Sup(\hbar_g,\hbar_m)$, $Sup(\hbar_g,\hbar_m)$ is the support degree for $\hbar_g$ and $\hbar_m$, fulfilling the following conditions:

(1) $Sup(\hbar_g,\hbar_m)\in[0,1]$;

(2) $Sup(\hbar_g,\hbar_m)=Sup(\hbar_m,\hbar_g)$;

(3) If $\overline{\overline{Ds}}(\hbar_g,\hbar_m) < \overline{\overline{Ds}}(\hbar_u,\hbar_v)$, then $Sup(\hbar_g,\hbar_m) > Sup(\hbar_u,\hbar_v)$, where $\overline{\overline{Ds}}(\hbar_g,\hbar_m)$ is distance among $\hbar_g$ and $\hbar_m$.

In order to inscribe expression (4.16) in an easy structure, we can stipulate it as

$$\Theta_g = \frac{\left(1+\overline{\overline{T}}\left(\hbar_g\right)\right)}{\sum_{m=1}^{a}\left(1+\overline{\overline{T}}\left(\hbar_m\right)\right)}. \tag{4.17}$$

For appropriateness, we can label $(\Theta_1,\Theta_2,...,\Theta_a)^T$ the power weight vector (PMV), such that $\Theta_g \in [0,1]$ and $\sum_{g=1}^{a}\Theta_g = 1.$ From, the use of Equation (4.17), Equation (4.16) can be expressed as

$$NCPDMM^{Q}\left(\hbar_1,\hbar_2,....,\hbar_a\right) = \frac{1}{\sum_{g=1}^{a} q_g}\left(\prod_{\theta\in S_a}\sum_{g=1}^{a}\left(q_g \hbar_{\theta(g)}^{a\Theta_{\theta(g)}}\right)\right)^{\frac{1}{a!}}. \tag{4.18}$$

4.1.3.2 Theorem

Let $\hbar_g (g=1,2,...,a)$ is a set of SVNNs and the parameters vector is denoted by $Q=(q_1,q_2,...,q_a)\in R^a$. Then, the result aggregated by employing Equation (4.16) is still a NCN and

$$NCPDMM^{Q}\left(\hbar_1,\hbar_2,...,\hbar_a\right) = \left(\left\langle \begin{array}{l} \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Xi^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\Xi^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \\ \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Psi^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Psi^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right], \\ \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Upsilon^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\Upsilon^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right] \end{array}\right\rangle, \right. \tag{4.19}$$

$$\left. 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_T\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_I\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_F\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \right\rangle\right).$$

Proof. Proof of Theorem (4.1.3.2) same as Theorem (4.1.1.2).

4.1.3.3 Theorem (Idempotency)

Let $\hbar_g (g = 1,2,...,a)$ be a group of NCNs, and $\hbar_g = \hbar,$ for all $g = 1,2,...,a.$ Then

$$NCPDMM^{Q}\left(\hbar_1, \hbar_2, ..., \hbar_a\right) = \hbar. \tag{4.20}$$

4.1.3.4 Theorem (Boundedness)

Let $\hbar_g (g = 1,2,...,a)$ be a group of NCNs, $\bar{\hbar} = \min\left(\hbar_1, \hbar_2, ..., \hbar_a\right) = \left\langle \left[\overset{-}{\Xi}^{L}, \overset{-}{\Xi}^{U}\right], \left[\overset{+}{\Psi}^{L}, \overset{+}{\Psi}^{U}\right], \left[\overset{+}{\Upsilon}^{L}, \overset{+}{\Upsilon}^{U}\right], \lambda_T^{-}, \lambda_I^{+}, \lambda_F^{+} \right\rangle$ and

$$\hbar^{+} = \max\left(\hbar_1, \hbar_2, ..., \hbar_a\right) = \left\langle \left[\overset{+}{\Xi}^{L}, \overset{+}{\Xi}^{U}\right], \left[\overset{-}{\Psi}^{L}, \overset{-}{\Psi}^{U}\right], \left[\overset{-}{\Upsilon}^{L}, \overset{-}{\Upsilon}^{U}\right], \lambda_T^{+}, \lambda_I^{-}, \lambda_F^{-} \right\rangle.$$

Then,

$$m \le NCPDMM^{Q}\left(\hbar_1, \hbar_2, ..., \hbar_a\right) \le n. \tag{4.21}$$

Where,

$$m = \left(\left\langle \left[1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\overset{-}{\Xi}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, 1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(\overset{-}{\Xi}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right], \right.\right.$$

$$\left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\overset{+}{\Psi}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\overset{+}{\Psi}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right],$$

$$\left[\left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\overset{+}{\Upsilon}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\overset{+}{\Upsilon}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right]$$

$$1 - \left(1 - \prod_{\theta \in S_a} \left(1 - \prod_{z=1}^{a} \left(1 - \left(\overset{-}{\lambda}_T\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a} q_z}}, \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\overset{+}{\lambda}_I\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}},$$

$$\left.\left. \left(1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{g=1}^{a} \left(1 - \left(1 - \left(\overset{+}{\lambda}_F\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a} q_g}} \right\rangle \right),$$

and

$$n=\left(\left\langle\left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Xi}^{L}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{+}{\Xi}^{U}\right)_{\theta(g)}^{a\Theta_g}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],\right.\right.$$

$$\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\Psi}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\Psi}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right],$$

$$\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\Upsilon}^{L}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\Upsilon}^{U}\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right]$$

$$1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(\overset{+}{\lambda}_T\right)_{\theta(z)}^{a\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_I\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}},$$

$$\left.\left.\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{-}{\lambda}_F\right)_{\theta(g)}\right)^{a\Theta_g}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}\right\rangle\right).$$

Now we will discuss some special cases of NCPDMM operator with respect to the parameter vector Q.

Case 1. If $Q=(1,0,...,0)$, then NCPDMM operators disintegrates into the following equation:

$$NCPDMM^{(1,0,...,0)}\left(\hbar_1,\hbar_2,....,\hbar_a\right)=\left(\prod_{g=1}^{a}\hbar_g^{\frac{\left(1+T\left(\hbar_g\right)\right)}{\sum_{x=1}^{a}\left(1+T\left(\hbar_x\right)\right)}}\right). \tag{4.22}$$

This is the NC power geometric averaging operator.

Case 2. If $Q=\left(\frac{1}{a},\frac{1}{a},....,\frac{1}{a}\right)$, then NCPMM operators degenerate into the following form:

$$NCPDMM^{\left(\frac{1}{a},\frac{1}{a},....,\frac{1}{a}\right)}\left(\hbar_1,\hbar_2,....,\hbar_a\right)=\sum_{g=1}^{a}\frac{\left(1+T\left(\hbar_g\right)\right)}{\sum_{x=1}^{a}\left(1+T\left(\hbar_x\right)\right)}\hbar_g. \tag{4.23}$$

This is NC power arithmetic averaging operator.

Case 3. If $Q=(1,1,0,...,0)$, then NCPDMM operators degenerate into the following form:

$$
\begin{aligned}
&NCPDMM^{(1,1,0,\dots,0)}(\hbar_1,\hbar_2,\dots,\hbar_a)=\Bigg(\Bigg\langle\Bigg[1-\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-(\Xi^L)_g^{\Theta_g}\right)\left(1-(\Xi^L)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},1-\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-(\Xi^U)_g^{\Theta_g}\right)\left(1-(\Xi^U)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\Psi^L)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\Psi^L)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\Psi^U)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\Psi^U)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg],\\
&\Bigg[\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\Upsilon^L)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\Upsilon^L)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\Upsilon^U)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\Upsilon^U)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-(\lambda_T)_g^{\Theta_g}\right)\left(1-(\lambda_T)_x^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\lambda_I)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\lambda_I)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}},\\
&\left(1-\left(\prod_{\substack{g,x=1\\ g\neq x}}^{a}\left(1-\left(1-\left(1-(\lambda_F)_g\right)^{\Theta_g}\right)\left(1-\left(1-(\lambda_F)_x\right)^{\Theta_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2}}\Bigg\rangle\Bigg).
\end{aligned}
\tag{4.24}
$$

This is the NC power geometric Bonferroni mean operator $(p=q=1)$.

Case 4. . If $Q=\left(\overbrace{1,1,\dots,1}^{i},\overbrace{0,0,\dots,0}^{z-i}\right)$, then the NCPDMM operator degenerates into the following form:

$$
\begin{aligned}
&NCPDMM^{\left(\overbrace{1,1,\dots,1}^{i},\overbrace{0,0,\dots,0}^{z-i}\right)}(\hbar_1,\hbar_2,\dots,\hbar_a)=\\
&\Bigg(\Bigg\langle\Bigg[1-\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\Xi^L)_{g_x}^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},1-\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\Xi^U)_{g_x}^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg],\\
&\Bigg[\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\Psi^L)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\Psi^U)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg],\\
&\Bigg[\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\Upsilon^L)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\Upsilon^U)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg]\Bigg\rangle,\\
&\Bigg\langle 1-\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-(\lambda_T)_{g_x}^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\lambda_I)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}},\\
&\left(1-\prod_{1\le y_1<y_2<\dots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-(\lambda_F)_{g_x}\right)^{\Theta_{g_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\Bigg\rangle\Bigg).
\end{aligned}
\tag{4.25}
$$

This is the NC power dual Maclaurin symmetric mean operator.

4.1.4 Weighted Neutrosophic Cubic Power Dual Muirhead Mean (WNCPDMM) Operator

The NCPDMM operator does not judge the importance degree of the aggregated NCNs. In this subsection, we propose the weighted neutrosophic cubic power dual Murihead mean (WNCPDMM) operator, which has the ability of captivating the weights of NCNs.

4.1.4.1 Definition

Let $\hbar_g (g=1,2,...,a)$ is a set of NCNs and the parameters vector is denoted by $Q=(q_1,q_2,...,q_a)\in R^a$. If

$$WNCPDMM^{Q}(\hbar_1,\hbar_2,....,\hbar_a)=\frac{1}{\sum_{g=1}^{a}q_g}\left(\prod_{\theta\in S_a}\sum_{g=1}^{a}\left(q_g\hbar_{\theta(g)}^{\frac{a\Phi_{\theta(g)}\Theta_{\theta(g)}}{\sum_{x=1}^{a}\Phi_x\Theta_x}}\right)\right)^{\frac{1}{a!}}. \tag{4.26}$$

hen we call $WNCPDMM^{Q}$ the weighted neutrosophic cubic power dual Muirhead mean operator, where $\Phi=(\Phi_1,\Phi_2,...,\Phi_a)^T$ is the importance degree of $\hbar_g (g=1,2,...,a)$ with $\Phi_g\in[0,1], \sum_{g=1}^{a}\Phi_g=1,$ S_a is the group of all permutation, $\theta(g)$ is any permutation of $(1,2,...,a)$ and Θ_g is PVW fulfilling $\Theta_g=\frac{(1+T(\hbar_g))}{\sum_{g=1}^{a}(1+T(\hbar_g))}, \sum_{g=1}^{a}\Theta_g=1,$ $T(\hbar_m)=\sum_{m=1,x\neq g}^{a}Sup(\hbar_g,\hbar_m)$, $Sup(\hbar_g,\hbar_m)$ is the SPD for $\hbar_g$ and $\hbar_x$, gratifying the following conditions:

(1) $Sup(\hbar_g,\hbar_m)\in[0,1]$;

(2) $Sup(\hbar_g,\hbar_m)=Sup(\hbar_m,\hbar_g)$;

(3) If $\overline{\overline{Ds}}(\hbar_g,\hbar_m)<\overline{\overline{Ds}}(\hbar_u,\hbar_v),$ then $Sup(\hbar_g,\hbar_m)>Sup(\hbar_u,\hbar_v),$ where $\overline{\overline{Ds}}(\hbar_g,\hbar_m)$ is distance among $\hbar_g$ and $\hbar_m$.

From Definition (4.1.4.1), we have the following Theorem (4.1.4.2).

4.1.4.2 Theorem

Let $\hbar_g (g=1,2,...,a)$ be a group of NCNs and $Q=(q_1,q_2,...,q_a)\in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (4.26) is still a NCN and

$$
WNCPDMM^{Q}\left(\hbar_1,\hbar_2,...,\hbar_a\right)=
$$

$$
\left\langle \left[1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{L}{\Xi}\right)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\overset{U}{\Xi}\right)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \right], \right.
$$

$$
\left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{L}{\Psi}\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{U}{\Psi}\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \right],
$$

$$
\left. \left[\left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{L}{\Upsilon}\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\overset{U}{\Upsilon}\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \right] \right\rangle
$$

$$
\left\langle 1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(\lambda_T\right)_{\theta(g)}^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}, \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_I\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}}. \right.
$$

$$
\left. , \left(1-\left(\prod_{\theta\in S_a}\left(1-\prod_{g=1}^{a}\left(1-\left(1-\left(\lambda_F\right)_{\theta(g)}\right)^{\frac{a\Theta_{\theta(g)}\Phi_{\vartheta(g)}}{\sum_{x=1}^{a}\Theta_x\Phi_x}}\right)^{q_g}\right)\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{g=1}^{a}q_g}} \right\rangle \tag{4.27}
$$

Proof. Proof of Theorem (4.1.4.2) is similar to that of Theorem (4.1.1.2).

4.2 The MADM Approach Based on WNCPMM Operator and WNCPDMM Operator

In this section, we give a novel method to MADM with NCNs, in which the attributes values gain the form of NCNs. For a MADM problem, let the series of alternatives is represented by $\overline{\overline{N}} = \left\{ \overline{\overline{N}}_1, \overline{\overline{N}}_2, ..., \overline{\overline{N}}_a \right\}$ and the series of attributes is represented by $\overline{\overline{O}} = \left\{ \overline{\overline{O}}_1, \overline{\overline{O}}_2, ..., \overline{\overline{O}}_b \right\}$. The weight vector of the attributes is denoted by $\varpi = (\varpi_1, \varpi_2, ..., \varpi_b)^T$ such that $\varpi_z \in [0,1], \sum_{z=1}^{b} \varpi_z = 1.$ Assume that $z_{gh} = \left\langle \left[\Xi_{gh}^L, \Xi_{gh}^U \right], \left[\Psi_{gh}^L, \Psi_{gh}^U \right], \left[\Upsilon_{gh}^L, \Upsilon_{gh}^U \right], \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}} \right\rangle$ is the assessment values of the alternatives $\overline{\overline{N}}_g$ on the attribute $\overline{\overline{O}}_h$ which is expressed by the form of NCN. Then, the main aim is to rank the alternatives. The following decision steps are to be pursued.

Step 1. Homogenize the decision matrix. Normally, there are two kinds of criterion, 1) cost type and 2) benefit type. We necessitate exchanging the cost type of criterion into benefit types of criterion by utilizing the following equation:

$$z_{gh} = \left(\left\langle \left[\Xi_{gh}^L, \Xi_{gh}^U \right], \left[\Psi_{gh}^L, \Psi_{gh}^U \right], \left[\Upsilon_{gh}^L, \Upsilon_{gh}^U \right] \right\rangle, \left\langle \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}} \right\rangle \right)$$
$$= \begin{cases} \left(\left\langle \left[\Xi_{gh}^L, \Xi_{gh}^U \right], \left[\Psi_{gh}^L, \Psi_{gh}^U \right], \left[\Upsilon_{gh}^L, \Upsilon_{gh}^U \right] \right\rangle, \left\langle \lambda_{T_{gh}}, \lambda_{I_{gh}}, \lambda_{F_{gh}} \right\rangle \right) & \text{for benefit attribute } \overline{\overline{O}}_h \\ \left(\left\langle \left[\Upsilon_{gh}^L, \Upsilon_{gh}^U \right], \left[1-\Psi_{gh}^U, 1-\Psi_{gh}^L \right], \left[\Xi_{gh}^L, \Xi_{gh}^U \right] \right\rangle, \left\langle \lambda_{T_{gh}}, 1-\lambda_{I_{gh}}, \lambda_{F_{gh}} \right\rangle \right). & \text{for cost attribute } \overline{\overline{O}}_h \end{cases} \tag{4.28}$$

Therefore the decision matrix $M = \left[z_{gh} \right]_{a \times b}$ can be altered into regularized decision matrix $N = \left[\delta_{gh} \right]_{a \times b}$.

Step 2. Find out the supports $Supp\left(\delta_{gh}, \delta_{gl} \right) \left(1, 2, ..., a; h, l = 1, 2, ..., b \right)$ by

$$Supp\left(\delta_{gh}, \delta_{gl} \right) = 1 - \overline{\overline{D}}\left(\delta_{gh}, \delta_{gh} \right), \tag{4.29}$$

where, $\overline{\overline{D}}\left(\delta_{gh}, \delta_{gh} \right)$ is the distance measure among two NCNs δ_{gh} and δ_{gl} defined in Equation (4.6).

Step 3. Find out $T\left(\delta_{gh} \right)$ by

$$T\left(\delta_{gh}\right)=\sum_{\substack{l=1\\l\neq h}}^{b}Supp\left(\delta_{gh},\delta_{gl}\right)\left(1,2,...,a;h,l=1,2,...,b\right). \tag{4.30}$$

Step 4. Find out

$$\Phi_{gh}=\frac{b\varpi_h\left(1+T\left(\delta_{gh}\right)\right)}{\sum_{d=1}^{b}\varpi_d\left(1+T\left(\delta_{gh}\right)\right)}\left(g=1,2,...,a;h,d=1,2,...,b\right). \tag{4.31}$$

Step 5. Exploit the WNCPMM or WNCPDMM operators

$$\delta_g=\left\langle\left[\Xi_g^L,\Xi_g^U\right],\left[\Psi_g^L,\Psi_g^U\right],\left[\Upsilon_g^L,\Upsilon_g^U\right],\lambda_{Tg},\lambda_{Ig},\lambda_{Fg}\right\rangle=WNCPMM^Q\left(\delta_{g1},\delta_{g2},...,\delta_{gb}\right), \tag{4.32}$$

or

$$\delta_g=\left\langle\left[\Xi_g^L,\Xi_g^U\right],\left[\Psi_g^L,\Psi_g^U\right],\left[\Upsilon_g^L,\Upsilon_g^U\right],\lambda_{Tg},\lambda_{Ig},\lambda_{Fg}\right\rangle=WNCPDMM^Q\left(\delta_{g1},\delta_{g2},...,\delta_{gb}\right). \tag{4.33}$$

to calculate the overall NCNs $\delta_g\left(g=1,2,...,a\right)$.

Step 6. Determine the score values of the collective NCNs $\delta_g\left(g=1,2,...,a\right)$, using Definition (1.1.1.16).

Step 7. Rank all the alternatives according to their score values, and the select the best one using Theorem (1.1.1.17).

Step 8. End.

4.3 An illustrative Example

In order to show the application of the developed MADM method, an illustrative example is embraced from [14] with NC information.

4.3.1 Example

A passenger wants to travel and select the best vans (alternatives) $\overline{\overline{N}}_g (g = 1,2,3,4)$ among the possible four vans. The customer takes the following four attributes into account to evaluate the four alternatives: (1) the facility $\overline{\overline{O}}_1$; (2) saving rent $\overline{\overline{O}}_2$; (3) comfort $\overline{\overline{O}}_3$; (4) safety $\overline{\overline{O}}_4$. The importance degree of the attributes is expressed by $\varpi = (0.5, 0.25, 0.125, 0.125)^T$. Therefore, the following decision matrix $M = \left[z_{gh} \right]_{4\times4}$ can be obtained in the form of NCNs shown in Table 4.1.

Table 4.1. The decision matrix $M = \left[CN_{gh} \right]_{4\times4}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	$(\langle[0.2,0.5],[0.3,0.7],[0.1,0.2],0.9,0.7,0.2\rangle)$	$(\langle[0.2,0.4],[0.4,0.5],[0.2,0.5],0.7,0.4,0.5\rangle)$	$(\langle[0.2,0.7],[0.4,0.9],[0.5,0.7],0.7,0.7,0.5\rangle)$	$(\langle[0.1,0.6],[0.3,0.4],[0.5,0.8],0.5,0.5,0.7\rangle)$
$\overline{\overline{N}}_2$	$(\langle[0.3,0.9],[0.2,0.7],[0.3,0.5],0.5,0.7,0.5\rangle)$	$(\langle[0.3,0.7],[0.6,0.8],[0.2,0.4],0.7,0.6,0.8\rangle)$	$(\langle[0.3,0.9],[0.4,0.6],[0.6,0.8],0.9,0.4,0.6\rangle)$	$(\langle[0.2,0.5],[0.4,0.9],[0.5,0.8],0.5,0.2,0.7\rangle)$
$\overline{\overline{N}}_3$	$(\langle[0.3,0.4],[0.4,0.8],[0.2,0.6],0.1,0.4,0.2\rangle)$	$(\langle[0.2,0.4],[0.2,0.3],[0.2,0.5],0.2,0.2,0.2\rangle)$	$(\langle[0.4,0.7],[0.1,0.2],[0.4,0.5],0.9,0.5,0.5\rangle)$	$(\langle[0.6,0.7],[0.3,0.6],[0.3,0.7],0.7,0.5,0.3\rangle)$
$\overline{\overline{N}}_4$	$(\langle[0.5,0.9],[0.1,0.8],[0.2,0.6],0.4,0.6,0.2\rangle)$	$(\langle[0.4,0.6],[0.5,0.7],[0.1,0.2],0.5,0.3,0.2\rangle)$	$(\langle[0.5,0.6],[0.2,0.4],[0.3,0.5],0.5,0.4,0.5\rangle)$	$(\langle[0.3,0.7],[0.7,0.8],[0.6,0.7],0.4,0.2,0.8\rangle)$

Step 1. Since all the attributes are the same, hence there is no need for conversion.

Step 2. Utilize Equation (4.29), to calculate the support degree $Supp\left(z_{gh}, z_{gl}\right)(1,2,...,4; h,l = 1,2,...,4)$. We denote $Supp\left(z_{gh}, z_{gl}\right)$ by $Supp_{gh,gl}$.

$$Supp_{11,12} = Supp_{12,11} = 0.79452, Supp_{11,13} = Supp_{13,11} = 0.735425, Supp_{11,14} = Supp_{14,11} = 0.65359,$$
$$Supp_{12,13} = Supp_{13,12} = 0.771478, Supp_{12,14} = Supp_{14,12} = 0.805635, Supp_{13,14} = Supp_{14,13} = 0.786563;$$

$$Supp_{21,22} = Supp_{22,21} = 0.7972, Supp_{21,23} = Supp_{23,21} = 0.7667, Supp_{21,24} = Supp_{24,21} = 0.727155,$$
$$Supp_{22,23} = Supp_{23,22} = 0.750556, Supp_{22,24} = Supp_{24,22} = 0.750556, Supp_{23,24} = Supp_{23,24} = 0.76906,$$

$$Supp_{31,32} = Supp_{32,31} = 0.8, Supp_{31,33} = Supp_{33,31} = 0.614139, Supp_{31,34} = Supp_{34,31} = 0.735425,$$
$$Supp_{32,33} = Supp_{33,32} = 0.690879, Supp_{32,34} = Supp_{34,32} = 0.711325, Supp_{33,34} = Supp_{33,34} = 0.797241,$$

$$Supp_{41,42} = Supp_{42,41} = 0.7551, Supp_{41,43} = Supp_{43,41} = 0.783975, Supp_{41,44} = Supp_{44,41} = 0.645662,$$
$$Supp_{42,43} = Supp_{43,42} = 0.783975, Supp_{42,44} = Supp_{44,42} = 0.675107, Supp_{43,44} = Supp_{44,43} = 0.7152.$$

Step 3. Utilize Equation (4.30), to get $T\left(\delta_{gh}\right)(g,h = 1 to 4)$. We denote $T\left(\delta_{gh}\right)$ by T_{gh}.

T_{11}=2.183534, T_{12}=2.371633, T_{13}=2.293466, T_{14}=2.245787;

T_{21}=2.291063, T_{22}=2.298354, T_{23}=2.286283, T_{24}=2.246771,

T_{31}=2.149564, T_{32}=2.202204, T_{33}=2.102259, T_{34}=2.243991,

T_{41}=2.184688, T_{42}=2.214133, T_{43}=2.28315, T_{44}=2.035969.

Step 4. Utilize Equation (4.31), to obtain $\Phi_{gh}(g,h = 1,2,3,4)$.

$$\Phi_{11} = 1.957844, \Phi_{12} = 1.036761, \Phi_{13} = 0.506363, \Phi_{14} = 0.499032,$$

$$\Phi_{21} = 2.002623, \Phi_{22} = 1.00353, \Phi_{23} = 0.499929, \Phi_{24} = 0.493918,$$

$$\Phi_{31} = 1.987975, \Phi_{32} = 1.010601, \Phi_{33} = 0.489529, \Phi_{34} = 0.511894,$$

$$\Phi_{41} = 1.999323, \Phi_{42} = 1.008904, \Phi_{43} = 0.515284, \Phi_{44} = 0.476489.$$

Step 5. Utilize the WNCPMM given in Equation (4.32)

$$z_g = \left(\left\langle \left[\Xi_g^L, \Xi_g^U \right], \left[\Psi_g^L, \Psi_g^U \right], \left[\Upsilon_g^L, \Upsilon_g^U \right] \right\rangle, \left\langle \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle \right) = WCNPMM^Q \left(z_{g1}, z_{g2}, ..., z_{g4} \right) (g = 1, 2, ..., 4).$$

To get the overall NCNs $z_g \left(g = 1, 2, ..., 4 \right)$. Assume that $Q = (1,1,1,1)$.

$z_1 = \langle [0.13993, 0.469959], [0.442103, 0.702693], [0.469048, 0.684711], 0.548309, 0.636754, 0.602874 \rangle$;

$z_2 = \langle [0.2238, 0.60211], [0.5236, 0.8162], [0.5122, 0.715], 0.561704, 0.55054, 0.729379 \rangle$;

$z_3 = \langle [0.3002, 0.4736], [0.3232, 0.5782], [0.3881, 0.6445], 0.3255, 0.4952, 0.415668 \rangle$;

$z_4 = \langle [0.3413, 0.5540], [0.5437, 0.7485], [0.4487, 0.5965], 0.376197, 0.445143, 0.597579 \rangle$.

Step 6. Using Definition (1.1.1.16), we calculate the score values of the collective NCNs $z_g \left(g = 1, 2, ..., a \right)$.

$$\overline{\overline{SO}}(z_1) = 0.4022, \overline{\overline{SO}}(z_2) = 0.393352, \overline{\overline{SO}}(z_3) = 0.472717, \overline{\overline{SO}}(z_4) = 0.4324.$$

Step 7. According to the score values, ranking order of the alternative is $\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$.

Hence using Theorem (1.1.1.17), the best alternative is $\overline{\overline{M}}_3$ and the worst is $\overline{\overline{M}}_2$.

Similarly, by utilizing WNCPDMM operators, the Steps 1 to 4 are similar to the WCNPMM operator.

Step 5. Utilize the WNCPDMM given in Equation (4.33)

$$z_g = \left\langle \left[\Xi_g^L, \Xi_g^U \right], \left[\Psi_g^L, \Psi_g^U \right], \left[\Upsilon_g^L, \Upsilon_g^U \right] \right\rangle, \left\langle \lambda_{Tg}, \lambda_{Ig}, \lambda_{Fg} \right\rangle = WNCPDMM^Q \left(z_{g1}, z_{g2}, ..., z_{g4} \right) (g = 1, 2, ..., 4).$$

To get the overall NCNs $z_g \left(g = 1, 2, ..., 4 \right)$. Assume that, $Q = (1,1,1,1)$.

$z_1 = \langle [0.2569, 0.6239], [0.2929, 0.5112], [0.2375, 0.4571] \rangle, \langle 0.7682, 0.4666, 0.3905 \rangle$;

$$z_2 = \langle [0.3642, 0.8179], [0.3110, 0.6479], [0.3194, 0.5430] \rangle, \langle 0.7416, 0.3336, 0.5561 \rangle;$$

$$z_3 = \langle [0.4935, 0.6438], [0.1794, 0.3224], [0.2248, 0.4812] \rangle, \langle 0.6502, 0.3206, 0.2330 \rangle;$$

$$z_4 = \langle [0.4995, 0.7691], [0.2570, 0.5332], [0.2130, 0.3815] \rangle, \langle 0.5355, 0.2744, 0.3248 \rangle.$$

Step 6. Using Definition (1.1.1.16), we calculate the score values of the collective NCNs $z_g \left(g = 1, 2, ..., a \right)$.

$$\overline{\overline{SO}}(z_1) = 0.5881, \overline{\overline{SO}}(z_2) = 0.5782, \overline{\overline{SO}}(z_3) = 0.6688, \overline{\overline{SO}}(z_4) = 0.6467.$$

Step 7. According to the score values, ranking order of the alternative is $\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2$.

Hence using Theorem (1.1.1.17), the best alternatives is $\overline{\overline{N}}_3$ and the worst is $\overline{\overline{N}}_2$.

4.3.2 Effect of the Parameter Q on the Decision Result.

In this subsection, different values to the parameter vector Q and the results obtained from these values are shown in Table 4.2 and Table 4.3. From Table 4.2 and Table 4.3, it can be seen that, when the parameter vector Q is $(1,0,0,0)$, that is, when the interrelationship among the attributes is not considered, then according to the score values the best alternative is $\overline{\overline{N}}_4$ while the worst is $\overline{\overline{N}}_2$. Similarly, when the parameter vector Q is $(1,1,0,0)$, that is, when WCNPMM operator and WNCPDMM operator degenerate into NCPBM operator and NCPGBM operator respectively, the best alternative is $\overline{\overline{N}}_3$ and $\overline{\overline{N}}_4$ while the worst for both cases is $\overline{\overline{N}}_2$. When the parameter vector Q is $(1,1,1,0)$, the best alternative is $\overline{\overline{N}}_3$ and the worst is $\overline{\overline{N}}_2$. When the parameter vector Q is $(1,1,1,1)$, the best alternative is $\overline{\overline{N}}_3$ and the worst is $\overline{\overline{N}}_2$. Similarly, for other values of the parameter vector the score values and ranking order vary. Thus, one can select the value of the parameter vector according to the needs of the situations.

Table 4.2. Score values and ranking orders for different parameter values in WCNPMM operator

Parameter Vector Q	Score Values	Ranking orders
$Q(1,0,0,0)$	$\overline{\overline{SO}}(z_1)=0.5671, \overline{\overline{SO}}(z_2)=0.5230,$ $\overline{\overline{SO}}(z_3)=0.5593, \overline{\overline{SO}}(z_4)=0.6031.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_2.$
$Q(1,1,0,0)$	$\overline{\overline{SO}}(z_1)=0.4579, \overline{\overline{SO}}(z_2)=0.4468,$ $\overline{\overline{SO}}(z_3)=0.5092, \overline{\overline{SO}}(z_4)=0.5027.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(1,1,1,0)$	$\overline{\overline{SO}}(z_1)=0.4227, \overline{\overline{SO}}(z_2)=0.4133,$ $\overline{\overline{SO}}(z_3)=0.4866, \overline{\overline{SO}}(z_4)=0.4607.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(1,1,1,1)$	$\overline{\overline{SO}}(z_1)=0.5881, \overline{\overline{SO}}(z_2)=0.5782,$ $\overline{\overline{SO}}(z_3)=0.6688, \overline{\overline{SO}}(z_4)=0.6467.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(0.5,0.5,0.5,0.5)$	$\overline{\overline{SO}}(z_1)=0.3988, \overline{\overline{SO}}(z_2)=0.3910,$ $\overline{\overline{SO}}(z_3)=0.4708, \overline{\overline{SO}}(z_4)=0.4306.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(5,0,0,0)$	$\overline{\overline{SO}}(z_1)=0.6608, \overline{\overline{SO}}(z_2)=0.6235,$ $\overline{\overline{SO}}(z_3)=0.6313, \overline{\overline{SO}}(z_4)=0.6854.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$

Table4.3. Score values and ranking orders for different parameter values in WCNPDMM operator

Parameter Vector	Score Values	Ranking orders

Q		
$Q(1,0,0,0)$	$\overline{\overline{SO}}(z_1)=0.5588, \overline{\overline{SO}}(z_2)=0.5346,$ $\overline{\overline{SO}}(z_3)=0.6040, \overline{\overline{SO}}(z_4)=0.6081.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_2.$
$Q(1,1,0,0)$	$\overline{\overline{SO}}(z_1)=0.5881, \overline{\overline{SO}}(z_2)=0.5782,$ $\overline{\overline{SO}}(z_3)=0.6688, \overline{\overline{SO}}(z_4)=0.6467.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(1,1,1,0)$	$\overline{\overline{SO}}(z_1)=0.5760, \overline{\overline{SO}}(z_2)=0.5582,$ $\overline{\overline{SO}}(z_3)=0.6478, \overline{\overline{SO}}(z_4)=0.6276.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(1,1,1,1)$	$\overline{\overline{SO}}(z_1)=0.5881, \overline{\overline{SO}}(z_2)=0.5782,$ $\overline{\overline{SO}}(z_3)=0.6688, \overline{\overline{SO}}(z_4)=0.6467.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(0.5,0.5,0.5,0.5)$	$\overline{\overline{SO}}(z_1)=0.5909, \overline{\overline{SO}}(z_2)=0.5817,$ $\overline{\overline{SO}}(z_3)=0.6741, \overline{\overline{SO}}(z_4)=0.6488.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
$Q(5,0,0,0)$	$\overline{\overline{SO}}(z_1)=0.4671, \overline{\overline{SO}}(z_2)=0.4073,$ $\overline{\overline{SO}}(z_3)=0.4022, \overline{\overline{SO}}(z_4)=0.4559.$	$\overline{\overline{N}}_1 > \overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3.$

4.3.3 Comparison with Existing Methods

To show the efficiency and advantages of the proposed method, we give a comparative analysis. We exploit some existing methods to solve the same example and examine the final results. We compare our method in this paper with the methods developed by Qin et al.[74] based on weighted IFMSM operator, and the one developed by Liu et al. [71] based generalized INPWA operator. We extend the IFMSM operator method [74] for intuitionistic fuzzy information to neutrosophic cubic Maclaurin symmetric mean operator. We also extend the GINPWA operator [71] for interval neutrosophic information to generalized neutrosophic cubic power average operator.

The method developed by Qin et al. [74], is based on MSM operator, which is able to consider the interrelationship among the attribute values, but unable to remove the effect of awkward data. The MSM operator is a special case of the proposed aggregation operator. Also the ranking result obtained using the method of Qin et al. [74], is different from the one obtained using the proposed method.

Similarly, the method developed by Liu et al. [71], is based on power weighted averaging operator, which can remove the effect of awkward data but cannot consider the interrelationship among the attributes values. From Table 9.4, it can be seen that the ranking result obtained using Liu et al. [71] is the same as the ranking order obtained from the proposed method, when $Q(1,0,0,0)$. That is, when the interrelationship between NCNs are not considered. This shows the validity of the proposed approach. The ranking order is different when $Q(1,1,1,1)$. That is, when the interrelationship among four attributes are considered, then the ranking order is different. The main reason behind the different ranking results is due to the existing aggregation operators, can only consider a single characteristic at a time while aggregating the NCNs, meaning that they can only either consider interrelationship among attributes or remove the effect of awkward data. Our proposed aggregation operator, however, can consider two characteristics at a time. It has the ability to consider the interrelationship among the attributes and also remove the effect of awkward data. In fact, these existing aggregation operators can be regarded as special cases to our proposed aggregation operator. Hence, our proposed aggregation operator is more practical and flexible to be used in decision making problems.

Table 4.4. Score values and ranking orders for different parameter values in WCNPDMM operator

Aggregation operator	Score Values	Ranking orders
NCMSM operator [74]	$\overline{\overline{SO}}(z_1)=0.6263, \overline{\overline{SO}}(z_2)=0.6153,$ $\overline{\overline{SO}}(z_3)=0.6355, \overline{\overline{SO}}(z_4)=0.6373.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
GNCPWA operator [72]	$\overline{\overline{SO}}(z_1)=0.5694, \overline{\overline{SO}}(z_2)=0.5266,$ $\overline{\overline{SO}}(z_3)=0.5646, \overline{\overline{SO}}(z_4)=0.6054.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_2.$
Proposed WNCPMM operator $Q(1,0,0,0)$	$\overline{\overline{SO}}(z_1)=0.5671, \overline{\overline{SO}}(z_2)=0.5230,$ $\overline{\overline{SO}}(z_3)=0.5593, \overline{\overline{SO}}(z_4)=0.6031.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_2.$
Proposed WNCPDMM operator $Q(1,0,0,0)$	$\overline{\overline{SO}}(z_1)=0.5588, \overline{\overline{SO}}(z_2)=0.5346,$ $\overline{\overline{SO}}(z_3)=0.6040, \overline{\overline{SO}}(z_4)=0.6081.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_2.$
Proposed WNCPMM operator $Q(1,1,1,1)$	$\overline{\overline{SO}}(z_1)=0.5881, \overline{\overline{SO}}(z_2)=0.5782,$ $\overline{\overline{SO}}(z_3)=0.6688, \overline{\overline{SO}}(z_4)=0.6467.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$
Proposed WNCPDMM operator $Q(1,1,1,1)$	$\overline{\overline{SO}}(z_1)=0.5881, \overline{\overline{SO}}(z_2)=0.5782,$ $\overline{\overline{SO}}(z_3)=0.6688, \overline{\overline{SO}}(z_4)=0.6467.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2.$

4.3.4 Conclusion

In this chapter, we incorporate both the PA operator and MM operator to form a few new aggregation operators to aggregate CNNs, such as, the CNPMM operator, WCNPMM operator, CNPDMM operator and WCNPDMM operator. We discussed several basic results and properties, along with a few special cases of the proposed aggregation operators. In other words, the developed aggregation operators do not only consider the interrelationship among the NCNs, but also remove the influence of too high or too low arguments in the final results. Based on these aggregation operators, a novel approach to MADM problem is developed. Finally, a numerical example is illustrated to prove the efficacy and realism of the proposed approach.

In future, we aim to enlarge the developed aggregation operators to deal with intuitionistic fuzzy information [2], interval neutrosophic information [7], and others.

Chapter 5

Application of Interval Neutrosophic Power Hamy Mean Operators in MAGDM

In this chapter, we merge the conventional HM operator to the traditional PA operator in interval neutrosophic settings and present the two novel interval neutrosophic aggregation operators, that is, the interval neutrosophic power Hammy mean (INPHM) operator and the weighted interval neutrosophic power Hammy mean (WINPHM) operators. Then, some preferable characteristics of the developed aggregation operators are discussed. Moreover, based on these introduced AOs, a novel technique for MAGDM under the IN information. Lastly, some examples are given to show the effectiveness of the developed method by comparing with other existing methods.

5.1 Interval Neutrosophic Power Hamy Mean Aggregation Operators

5.1.1The Interval Neutrosophic Power Hamy Mean Operator

In this subpart, we develop interval neutrosophic Hamy mean operator and discussed it related properties, and results.

5.1.1.1 Definition

Let $\hbar_r = \langle [\Xi_r^L, \Xi_r^U], [\Psi_r^L, \Psi_r^U], [\Upsilon_r^L, \Upsilon_r^U] \rangle (r = 1, 2, ..., o)$ be a group of INNs, and the parameter $k = 1, 2, ..., o$. Then an interval neutrosophic power HM aggregation operator is a function $INPHM : \Theta^o \to \Theta$ defined as follows.

$$INPHM^{(k)}(\hbar_1, \hbar_2, ..., \hbar_o) = \frac{\sum_{1 \le r_1 < r_2 < ... < r_k \le o} \left(\prod_{j=1}^{k} \frac{o\left(1 + T\left(\hbar_{r_j}\right)\right)\hbar_{r_j}}{\sum_{a=1}^{o}\left(1 + T\left(\hbar_a\right)\right)} \right)^{\frac{1}{k}}}{\binom{o}{k}}, \tag{5.1}$$

Where Θ is the set of all INNs, and $\Phi_e = \frac{\left(1 + T\left(\hbar_e\right)\right)}{\sum_{e=1}^{o}\left(1 + T\left(\hbar_e\right)\right)}$, and $\sum_{a=1}^{n} \Phi_a = 1$. $T\left(\hbar_j\right) = \sum_{\substack{e=1 \\ e \ne j}}^{o} Supp\left(\hbar_e, \hbar_j\right)$ is the support degree for $\hbar_e$ from $\hbar_j$, which satisfy the following characteristics:

1) $Sup\left(\hbar_e, \hbar_j\right) \in [0,1]$,

2) $Sup\left(\hbar_e, \hbar_j\right) = Sup\left(\hbar_j, \hbar_e\right)$,

3) if $\overline{\overline{D}}\left(\hbar_e, \hbar_j\right) \le \overline{\overline{D}}\left(\hbar_x, \hbar_y\right)$,then $Sup\left(\hbar_e, \hbar_j\right) \ge Sup\left(\hbar_x, \hbar_y\right)$, where $\overline{\overline{D}}\left(\hbar_z, \hbar_j\right)$represent the distance measure between any two INNs defined in Definition (1.1.1.13), $\left(r_1, r_2, ..., r_k\right)$ traversals all the k-tuple combination of $(1, 2, ..., o)$. The denominator $\binom{o}{k}$ in the above Equation (5.1) represents the binomial coefficient $\frac{o!}{k!(o-k)!}$ and o is the balancing coefficient.

In order to write Equation (5.1) in a simple form, we can define

$$\Phi_c = \frac{\left(1+T\left(\hbar_c\right)\right)}{\sum_{c=1}^{o}\left(1+T\left(\hbar_c\right)\right)}, \tag{5.2}$$

Then we call $(\Phi_1, \Phi_2, ..., \Phi_o)$ as the power weight vector. Therefore, the simplified form Equation (5.1) is as follows:

$$INPHM^{(k)}\left(\hbar_1, \hbar_2, ..., \hbar_o\right) = \frac{\sum_{1 \le r_1 < r_2 < ... < r_k \le o}\left(\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j}\right)^{\frac{1}{k}}}{\binom{o}{k}}. \tag{5.3}$$

5.1.1.2 Theorem

Let $\hbar_r = \left\langle \left[\Xi_r^L, \Xi_r^U\right], \left[\Psi_r^L, \Psi_r^U\right], \left[\Upsilon_r^L, \Upsilon_r^U\right] \right\rangle (r = 1, 2, ..., o)$ be a group of INNs, and the parameter $k = 1, 2, ..., o$. Then the value aggregated utilizing Equation (5.3) is still an INN, and

$$\begin{aligned} &INPHM^{(k)}\left(\hbar_1, \hbar_2, ..., \hbar_o\right) = \\ &\left\langle \left[1 - \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(1 - \Xi_{r_j}^L\right)^{o\Phi_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}}, 1 - \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(1 - \Xi_{r_j}^U\right)^{o\Phi_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}} \right], \right. \\ &\left[\left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Psi_{r_j}^L\right)^{o\Phi_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Psi_{r_j}^U\right)^{o\Phi_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}} \right], \\ &\left. \left[\left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Upsilon_{r_j}^L\right)^{o\Phi_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Upsilon_{r_j}^L\right)^{o\Phi_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}} \right] \right\rangle. \end{aligned} \tag{5.4}$$

Proof. Based on the operational rules for INNs, we have

$$o\Phi_{r_j}\hbar_{r_j} = \left\langle \left[1 - \left(1 - \Xi_{r_j}^L\right)^{o\Phi_{r_j}}, 1 - \left(1 - \Xi_{r_j}^U\right)^{o\Phi_{r_j}} \right], \left[\left(\Psi_{r_j}^L\right)^{o\Phi_{r_j}}, \left(\Psi_{r_j}^U\right)^{o\Phi_{r_j}} \right], \left[\left(\Upsilon_{r_j}^L\right)^{o\Phi_{r_j}}, \left(\Upsilon_{r_j}^U\right)^{o\Phi_{r_j}} \right] \right\rangle,$$

and

$$\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j} = \left\langle \left[\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right), \prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right], \left[1-\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right),\right.\right.$$

$$\left.\left. 1-\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right], \left[1-\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right), 1-\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right]\right\rangle,$$

So,

$$\left(\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j}\right)^{\frac{1}{k}} = \left\langle \left[\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}, \left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right], \left[1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}},\right.\right.$$

$$\left.\left. 1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right], \left[1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}, 1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right]\right\rangle.$$

Then,

$$\sum_{1\le r_1<r_2<....<r_k\le o}\left(\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j}\right)^{\frac{1}{k}} = \left\langle \left[1-\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), 1-\prod_{11\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right],\right.$$

$$\left[\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), \prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right],$$

$$\left.\left[\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), \prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right]\right\rangle,$$

Hence,

$$\frac{\sum_{1\le r_1<r_2<....<r_k\le o}\left(\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j}\right)^{\frac{1}{k}}}{\binom{o}{k}} =$$

$$\left\langle \left[1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, 1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],$$

$$\left.\left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle,$$

Therefore,

$$INPHM^{(k)}\left(\hbar_1,\hbar_2,...,\hbar_o\right)=$$

$$\left\langle\left[1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}},1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}},\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],$$

$$\left.\left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}},\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle.$$

Now, we shall discuss some basic properties of INPHM operator, which are stated below:

5.1.1.3 Theorem (Idempotency)

If all $\hbar_r=\hbar=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle$ for $\left(r=1,2,...,o\right)$, then

$$INPHM^{(k)}\left(\hbar,\hbar,...,\hbar\right)=\hbar. \tag{5.5}$$

Proof. Since, all $\hbar_r=\hbar=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle$ for $\left(r=1,2,...,o\right)$, then

$$o\Phi_{r_j}=\frac{o\left(1+T\left(\hbar_{r_j}\right)\right)}{\sum_{c=1}^{o}\left(1+T\left(\hbar_c\right)\right)}=1.$$

So, according to Theorem (5.1.1.2), we have

$$INPHM^{(k)}\left(\hbar,\hbar,...,\hbar\right)=\left\langle\left[1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\left(\Xi^L\right)^k\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}},1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\left(\Xi^U\right)^k\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\left(1-\Psi^L\right)^k\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}},\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\left(1-\Psi^U\right)^k\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],$$

$$\left.\left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\left(1-\Upsilon^L\right)^k\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}},\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\left(1-\Upsilon^U\right)^k\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle,$$

$$=\left\langle\left[1-\left(\left(1-\Xi^L\right)^{\frac{1}{\binom{o}{k}}}\right)^{\frac{1}{\binom{o}{k}}},1-\left(\left(1-\Xi^U\right)^{\frac{1}{\binom{o}{k}}}\right)^{\frac{1}{\binom{o}{k}}}\right],\left[\left(\left(1-\left(1-\Psi^L\right)\right)^{\frac{1}{\binom{o}{k}}}\right)^{\frac{1}{\binom{o}{k}}},\left(\left(1-\left(1-\Psi^U\right)\right)^{\frac{1}{\binom{o}{k}}}\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left.\left[\left(\left(1-\left(1-\Upsilon^L\right)\right)^{\frac{1}{\binom{o}{k}}}\right)^{\frac{1}{\binom{o}{k}}},\left(\left(1-\left(1-\Upsilon^U\right)\right)^{\frac{1}{\binom{o}{k}}}\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle,$$

$$=\left\langle\left[1-\left(1-\Xi^L\right),1-\left(1-\Xi^U\right)\right],\left[\left(1-\left(1-\Psi^L\right)\right),\left(1-\left(1-\Psi^U\right)\right)\right],\left[\left(1-\left(1-\Upsilon^L\right)\right),\left(1-\left(1-\Upsilon^U\right)\right)\right]\right\rangle,$$

$$=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle.$$

5.1.1.4 Theorem (Commutativity)

Let $\hbar_r\,(r=1,2,...,o)$ be a group of INNs, and $\hbar_r$ be any permutation of $\hbar_r$. Then

$$INPHM^{(k)}\left(\hbar_1,\hbar_2,...,\hbar_o\right)=INPHM^{(k)}\left(\hbar_1,\hbar_2,...,\hbar_o\right). \tag{5.6}$$

Proof. Since, $\left(\hbar_1,\hbar_2,...,\hbar_o\right)$ is any permutation of $\left(\hbar_1,\hbar_2,...,\hbar_o\right)$, therefore, according to Definition (5.1.1.1), it is obvious that

$$INPHM^{(k)}\left(\hbar_1,\hbar_2,....,\hbar_o\right)=\frac{\sum_{1\le r_1<r_2<....<r_k\le o}\left(\prod_{j=1}^{k}\frac{o\left(1+T\left(\hbar_{r_j}\right)\right)\hbar_{r_j}}{\sum_{c=1}^{o}\left(1+T\left(\hbar_c\right)\right)}\right)^{\frac{1}{k}}}{\binom{o}{k}},$$

$$= \frac{\sum_{1 \le r_1 < r_2 < \ldots < r_k \le o} \left(\prod_{j=1}^{k} \frac{o\left(1+T\left(\hbar_{r_j}\right)\right)\hbar_{r_j}}{\sum_{c=1}^{o}\left(1+T\left(\hbar_c\right)\right)} \right)^{\frac{1}{k}}}{\binom{o}{k}},$$

$$= INPHM^{(k)}\left(\hbar_1, \hbar_2, \ldots, \hbar_o\right).$$

5.1.1.5 Theorem (Boundedness)

Let $\hbar_r (i = 1, 2, \ldots, o)$ be a group of INNs, and $\hbar^- = \min\left(\hbar_1, \hbar_2, \ldots, \hbar_o\right) = \left\langle \left[\Xi^L, \Xi^U\right], \left[\Psi^L, \Psi^U\right], \left[\Upsilon^L, \Upsilon^U\right] \right\rangle, \hbar^+ = \max\left(\hbar_1, \hbar_2, \ldots, \hbar_o\right) = \left\langle \left[\Xi^L, \Xi^U\right], \right.$ $\left[\Psi^L, \Psi^U\right], \left[\Upsilon^L, \Upsilon^U\right]$. Then, the INPHM operator lies:

$$\hbar^- \le INPHM\left(\hbar_1, \hbar_2, \ldots, \hbar_o\right) \le \hbar^+. \tag{5.7}$$

Proof. Since

$$\begin{aligned} o\Phi_{r_j}\hbar_{r_j} &= \left\langle \left[1-\left(1-\Xi_{r_j}^L\right)^{o\Phi_{r_j}}, 1-\left(1-\Xi_{r_j}^U\right)^{o\Phi_{r_j}}\right], \left[\left(\Psi_{r_j}^L\right)^{o\Phi_{r_j}}, \left(\Psi_{r_j}^U\right)^{o\Phi_{r_j}}\right], \left[\left(\Upsilon_{r_j}^L\right)^{o\Xi_{r_j}}, \left(\Upsilon_{r_j}^U\right)^{o\Phi_{r_j}}\right] \right\rangle \\ &\ge \left\langle \left[1-\left(1-\Xi^L\right)^{o\Phi_{r_j}}, 1-\left(1-\Xi^U\right)^{o\Phi_{r_j}}\right], \left[\left(\Psi^L\right)^{o\Phi_{r_j}}, \left(\Psi^U\right)^{o\Phi_{r_j}}\right], \left[\left(\Upsilon^L\right)^{o\Phi_{r_j}}, \left(\Upsilon^U\right)^{o\Phi_{r_j}}\right] \right\rangle, \end{aligned}$$

and

$$\begin{aligned} \prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j} &= \left\langle \left[\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^L\right)^{o\Phi_{r_j}}\right), \prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^U\right)^{o\Phi_{r_j}}\right)\right], \left[1-\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^L\right)^{o\Phi_{r_j}}\right), 1-\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^U\right)^{o\Phi_{r_j}}\right)\right], \right. \\ &\left. \left[1-\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^L\right)^{o\Phi_{r_j}}\right), 1-\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^U\right)^{o\Phi_{r_j}}\right)\right] \right\rangle \ge \left\langle \left[\prod_{j=1}^{k}\left(1-\left(1-\Xi^L\right)^{o\Phi_{r_j}}\right), \prod_{j=1}^{k}\left(1-\left(1-\Xi^U\right)^{o\Phi_{r_j}}\right)\right], \right. \\ &\left. \left[1-\prod_{j=1}^{k}\left(1-\left(\Psi^L\right)^{o\Phi_{r_j}}\right), 1-\prod_{j=1}^{k}\left(1-\left(\Psi^U\right)^{o\Phi_{r_j}}\right)\right], \left[1-\prod_{j=1}^{k}\left(1-\left(\Upsilon^L\right)^{o\Phi_{r_j}}\right), 1-\prod_{j=1}^{k}\left(1-\left(\Upsilon^U\right)^{o\Phi_{r_j}}\right)\right] \right\rangle. \end{aligned}$$

So,

$$\left(\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j}\right)^{\frac{1}{k}} = \left\langle \left[\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}, \left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right], \left[1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}},\right.\right.$$

$$\left.\left.1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right], \left[1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}, 1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right]\right\rangle \geq$$

$$\left\langle \left[\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}, \left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right], \left[1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}},\right.\right.$$

$$\left.\left.1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right], \left[1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}, 1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right]\right\rangle.$$

Then,

$$\sum_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j}\right)^{\frac{1}{k}} = \left\langle \left[1-\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), 1-\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right],\right.$$

$$\left[\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), \prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right],$$

$$\left.\left[\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), \prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right]\right\rangle \geq$$

$$\left\langle \left[1-\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), 1-\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right],\right.$$

$$\left[\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), \prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right],$$

$$\left.\left[\prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right), \prod_{1\leq r_1<r_2<\ldots<r_k\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{U}\right)^{m\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right]\right\rangle.$$

Hence,

$$\frac{\sum_{1\le r_1<r_2<\ldots<r_k\le o}\left(\prod_{j=1}^{k} o\Phi_{r_j}\hbar_{r_j}\right)^{\frac{1}{k}}}{\binom{o}{k}} = \left\langle \left[1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, 1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{ij}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],$$

$$\left.\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle$$

$$\ge \left\langle \left[1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, 1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],$$

$$\left.\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle.$$

Therefore,

$$INPHM^{(k)}\left(\hbar_1,\hbar_2,\ldots,\hbar_o\right)=$$

$$\left\langle \left[1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, 1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],$$

$$\left.\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle$$

$$\ge \left\langle \left[1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, 1-\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi^{U}\right)^{o\Phi_{ij}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],\right.$$

$$\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right],$$

$$\left.\left[\left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{L}\right)^{o\Phi_{ij}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1\le r_1<r_2<\ldots<r_k\le o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{o}{k}}}\right]\right\rangle.$$

$$=\hbar^{-}.$$

Similarly, we can prove that $INPHM\left(\hbar_1,\hbar_2,...,\hbar_o\right)\leq\hbar^+$.

Hence $\hbar^-\leq INPHM\left(\hbar_1,\hbar_2,...,\hbar_o\right)\leq\hbar^+$.

In what follows, we shall discuss some special cases of INPHM operators with respect to the parameter k, which were stated below.

1. When $k=1$, the INPHM operator in Equation (5.4), will degenerate to the following form:

$$INPHM^{(1)}\left(\hbar_1,\hbar_2,....,\hbar_o\right)=\frac{\sum_{1\leq r_1\leq o}\left(\prod_{j=1}^{1}\frac{o\left(1+T\left(\hbar_{r_j}\right)\right)\hbar_{r_j}}{\sum_{c=1}^{o}\left(1+T\left(\hbar_c\right)\right)}\right)^{\frac{1}{1}}}{\binom{o}{1}},$$

$$=\left\langle\left[1-\left(\prod_{1\leq r_1\leq o}\left(1-\left(\prod_{j=1}^{1}\left(1-\left(1-\Xi_{r_j}^L\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{\binom{n}{1}}},1-\left(\prod_{1\leq r_1\leq o}\left(1-\left(\prod_{j=1}^{1}\left(1-\left(1-\Xi_{r_j}^U\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{\binom{o}{1}}}\right],\right.$$
$$\left[\left(\prod_{1\leq r_1\leq o}\left(1-\left(\prod_{j=1}^{1}\left(1-\left(\Psi_{r_j}^L\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{\binom{o}{1}}},\left(\prod_{1\leq r_1\leq o}\left(1-\left(\prod_{j=1}^{1}\left(1-\left(\Psi_{r_j}^U\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{\binom{o}{1}}}\right],$$
$$\left.\left[\left(\prod_{1\leq r_1\leq o}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^L\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{\binom{o}{1}}},\left(\prod_{1\leq r_1\leq o}\left(1-\left(\prod_{j=1}^{1}\left(1-\left(\Upsilon_{r_j}^U\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{1}}\right)\right)^{\frac{1}{\binom{o}{1}}}\right]\right\rangle,$$

$$=\left\langle\left[1-\left(\prod_{r=1}^{o}\left(1-\Xi_r^L\right)^{o\Phi_r}\right)^{\frac{1}{o}},1-\left(\prod_{r=1}^{o}\left(1-\Xi_r^U\right)^{o\Phi_r}\right)^{\frac{1}{o}}\right],\left[\left(\prod_{r=1}^{o}\left(\Psi_r^L\right)^{o\Phi_r}\right)^{\frac{1}{o}},\left(\prod_{r=1}^{o}\left(\Psi_r^U\right)^{o\Phi_r}\right)^{\frac{1}{o}}\right],\right.$$
$$\left.\left[\left(\prod_{r=1}^{o}\left(\Upsilon_r^L\right)^{o\Phi_r}\right)^{\frac{1}{o}},\left(\prod_{r=1}^{o}\left(\Upsilon_r^U\right)^{o\Phi_r}\right)^{\frac{1}{o}}\right]\right\rangle,\left(let\, r_1=r\right) \tag{5.8}$$
$$=\frac{1}{o}\sum_{r=1}^{o}o\Phi_r\hbar_r=INPA\left(\hbar_1,\hbar_2,....,\hbar_o\right).$$

i.e., when $k=1$, the INPHM operator degenerates into power averaging operator proposed by Liu [71].

2. When $k=o,$ then the INPHM operator degenerates into the following form:

$$INPHM^{(o)}\left(\hbar_1,\hbar_2,....,\hbar_o\right)=\frac{\sum_{1\le r_1<r_2<....<r_k\le o}\left(\prod_{j=1}^{o}\frac{o\left(1+T\left(\hbar_{r_j}\right)\right)\hbar_{r_j}}{\sum_{c=1}^{o}\left(1+T\left(\hbar_c\right)\right)}\right)^{\frac{1}{o}}}{\binom{o}{o}},$$

$$=\left\langle\begin{array}{l}\left[1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{o}\left(1-\left(1-\Xi_{o_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{\binom{o}{o}}},1-\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{o}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{\binom{o}{o}}}\right],\\ \left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{o}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{\binom{o}{o}}},\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{o}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{\binom{o}{o}}}\right],\\ \left[\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{o}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{\binom{o}{o}}},\left(\prod_{1\le r_1<r_2<....<r_k\le o}\left(1-\left(\prod_{j=1}^{o}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right)\right)^{\frac{1}{\binom{o}{o}}}\right]\end{array}\right\rangle,$$

$$=\left\langle\begin{array}{l}\left[\left(\prod_{j=1}^{o}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}},\left(\prod_{j=1}^{o}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right],\left[1-\left(\prod_{j=1}^{o}\left(1-\left(\Psi_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}},\right.\\ \left.1-\left(\prod_{j=1}^{o}\left(1-\left(\Psi_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right],\left[1-\left(\prod_{j=1}^{o}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}},1-\left(\prod_{j=1}^{o}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{o\Phi_{r_j}}\right)\right)^{\frac{1}{o}}\right]\end{array}\right\rangle. \tag{5.9}$$

Further, if we suppose that $Sup\left(\hbar_r,\hbar_j\right)=\beta$ for all $i\neq j$, then $o\Phi_{r_j}=\frac{o\left(1+T\left(\hbar_{r_j}\right)\right)}{\sum_{c=1}^{o}\left(1+T\left(\hbar_c\right)\right)}=1,$ and Equation (5.4) can further degenerate into the following form.

$$=\left\langle\begin{array}{l}\left[\left(\prod_{j=1}^{o}\left(\Xi_{r_j}^{L}\right)\right)^{\frac{1}{o}},\left(\prod_{j=1}^{o}\Xi_{r_j}^{U}\right)^{\frac{1}{o}}\right],\left[1-\left(\prod_{j=1}^{o}\left(1-\Psi_{r_j}^{L}\right)\right)^{\frac{1}{o}},1-\left(\prod_{j=1}^{o}\left(1-\Psi_{r_j}^{U}\right)\right)^{\frac{1}{o}}\right],\\ \left[1-\left(\prod_{j=1}^{o}\left(1-\Upsilon_{r_j}^{L}\right)\right)^{\frac{1}{o}},1-\left(\prod_{j=1}^{o}\left(1-\Upsilon_{r_j}^{U}\right)\right)^{\frac{1}{o}}\right]\end{array}\right\rangle. \tag{5.10}$$

That is, Equation (5.4) degenerates into ING operator.

In the INPHM operator, we can notice that only the interrelation among inputs arguments and the power weight vector are taken into consideration, the weight vector of the aggregated arguments is ignored. However, in some situation, the importance degree of the attributes is an important factor in the aggregation process, especially, in MAGDM. So in order to overcome this deficiency, the weighted form of the INPHM operator is defined as follows.

5.1.1.6 Definition

Let $\hbar_r = \left\langle \left[\Xi_r^L, \Xi_r^U\right], \left[\Psi_r^L, \Psi_r^U\right], \left[\Upsilon_r^L, \Upsilon_r^U\right] \right\rangle (r = 1, 2, ..., o)$ be a group of INNs, and the parameter $k = 1, 2, ..., o$. Then a weighted interval neutrosophic power HM operator is a function $WINPHM : \Theta^o \to \Theta$ defined as follows.

$$WINPHM^{(k)}\left(\hbar_1, \hbar_2, ..., \hbar_o\right) = \frac{\sum_{1 \le r_1 < r_2 < ... < r_k \le o} \left(\prod_{j=1}^{k} o\omega_{r_j} \hbar_{r_j} \right)^{\frac{1}{k}}}{\binom{o}{k}}, \quad (5.11)$$

Where Θ is the set of all INNs, and $\omega_r = \frac{\varpi_r \left(1 + T\left(\hbar_r\right)\right)}{\sum_{c=1}^{o} \varpi_c \left(1 + T\left(\hbar_c\right)\right)}$, $T\left(\hbar_j\right) = \sum_{\substack{c=1 \\ c \ne j}}^{o} Sup\left(\hbar_c, \hbar_j\right)$ is the support degree for $\hbar_c$ from $\hbar_j$, which satisfies the following properties; 1) $Sup\left(\hbar_c, \hbar_j\right) \in [0,1]$, 2) $Sup\left(\hbar_c, \hbar_j\right) = Sup\left(\hbar_j, \hbar_c\right)$, 3) $\overline{\overline{D}}\left(\hbar_c, \hbar_j\right) \le \overline{\overline{D}}\left(\hbar_x, \hbar_y\right)$, then $Sup\left(\hbar_c, \hbar_j\right) \ge Sup\left(\hbar_x, \hbar_y\right)$, where $\overline{\overline{D}}\left(\hbar_c, \hbar_j\right)$ represents the distance measure between any two INNs defined in Definition (1.1.1.13), $\varpi = \left(\varpi_1, \varpi_2, ..., \varpi_o\right)^T$ is the weight vector of $\hbar_r (r = 1, 2, ..., o)$ such that $\varpi_r \in [0,1]$ *and* $\sum_{r=1}^{o} \varpi_r = 1$. $(r_1, r_2, ..., r_k)$ traversals all the k-tuple combination of $(1, 2, ..., o)$. The denominator $\binom{o}{k}$ in the above Equation (5.11) represents the binomial coefficient, $\frac{o!}{k!(o-k)!}$ and o are the balancing coefficients.

5.1.1.7 Theorem

Let $\hbar_r = \left\langle \left[\Xi_r^L, \Xi_r^U\right], \left[\Psi_r^L, \Psi_r^U\right], \left[\Upsilon_r^L, \Upsilon_r^U\right] \right\rangle (r = 1, 2, ..., o)$ be a group of INNs, and the parameter $k = 1, 2, ..., o$. Then, the value aggregated utilizing Equation (5.11) is still an INN, and

$$
\begin{aligned}
& WINPHM^{(k)}\left(\hbar_1, \hbar_2, ..., \hbar_o\right) = \\
& \left\langle \left[1 - \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(1 - \Xi_{r_j}^L \right)^{o\omega_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}}, 1 - \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(1 - \Xi_{r_j}^U \right)^{o\omega_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}} \right], \right. \\
& \left[\left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Psi_{r_j}^L \right)^{o\omega_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Psi_{r_j}^U \right)^{o\omega_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}} \right], \\
& \left. \left[\left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Upsilon_{r_j}^L \right)^{o\omega_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}}, \left(\prod_{1 \le r_1 < r_2 < .... < r_k \le o} \left(1 - \left(\prod_{j=1}^{k} \left(1 - \left(\Upsilon_{r_j}^U \right)^{o\omega_{r_j}} \right) \right)^{\frac{1}{k}} \right) \right)^{\frac{1}{\binom{o}{k}}} \right] \right\rangle .
\end{aligned}
\tag{5.12}
$$

Proof. Proof of this theorem is same as Theorem (5.1.1.2).

5.1.1.8 Theorem (Idempotency)

If all $\hbar_r = \hbar = \left\langle \left[\Xi^L, \Xi^U\right], \left[\Psi^L, \Psi^U\right], \left[\Upsilon^L, \Upsilon^U\right] \right\rangle$ for $(r = 1, 2, ..., o)$, then

$$WINPHM^{(k)}\left(\hbar, \hbar, ..., \hbar\right) = \hbar. \tag{5.13}$$

5.1.1.9 Theorem (Commutativity)

Let $\hbar_r (r = 1, 2, ..., o)$ be a group of INNs, and $\hbar_r$ be any permutation of $\hbar_r$. Then,

$$WINPHM^{(k)}\left(\hbar_1,\hbar_2,...,\hbar_o\right)=WINPHM^{(k)}\left(\hbar_1,\hbar_2,...,\hbar_o\right). \quad (5.14)$$

5.1.1.10 Theorem (Boundedness).

Let $\hbar_r\,(r=1,2,...,o)$ be a group of INNs, and $\hbar^- =\min\left(\hbar_1,\hbar_2,...,\hbar_o\right)=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle, \hbar^+ =\max\left(\hbar_1,\hbar_2,...,\hbar_o\right)=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle.$ Then, the INPHM operator lies:

$$\hbar^- \le WINPHM\left(\hbar_1,\hbar_2,...,\hbar_o\right)\le \hbar^+. \quad (5.15)$$

The proofs of the above theorems are same as the proofs of the theorems for INPHM operator, therefore omitted here.

5.2 MAGDM Approach Based on Developed WINPHM Operator

In this part, we will utilize the developed WINPHM operator to deal with MAGDM problem with the data presented in the form of INNs. Let the set of m alternatives be denoted by $\overline{\overline{N}}=\left\{\overline{\overline{N}}_1,\overline{\overline{N}}_2,...,\overline{\overline{N}}_m\right\}$ and the group of n attributes be denoted by $\overline{\overline{O}}=\left\{\overline{\overline{O}}_1,\overline{\overline{O}}_2,...,\overline{\overline{O}}_n\right\}$, the importance degree of n attributes be $\varpi=(\varpi_1,\varpi_2,...,\varpi_n)^T$, such that $\varpi_j\in[0,1], j=1,2,...,n, \sum_{j=1}^{n}\varpi_j=1.$ There is a set of z experts expressed by $e=\{e_1,e_2,...,e_z\}$ who are asked to provide the assessment information, and the importance degree of the experts is expressed by $\psi=(\psi_1,\psi_2,...,\psi_z)^T$, such that $\psi_a\in[0,1],(a=1,2,...,z),\sum_{a=1}^{z}\psi_a=1.$ The expert e_a assesses every attribute $\overline{\overline{O}}_j$ of every alternative $\overline{\overline{N}}_i$ by the form of INN $\hbar_{ij}^a=\left\langle\left[\Xi_{ij}^{aL},\Xi_{ij}^{aU}\right],\left[\Psi_{ij}^{aL},\Psi_{ij}^{aU}\right],\left[\Upsilon_{ij}^{aL},\Upsilon_{ij}^{aU}\right]\right\rangle\ (i=1,2,...,m;1,2,...,n),$ then the decision matrices

$DM_a = \left(\hbar_{ij}^{a}\right)(a=1,2,...,z)$ is established. The subsequent purpose is to execute a ranking of all alternatives.

Then, in order to solve this problem, we will execute the following steps:

Step 1. Firstly, the given decision matrices $DM_a = \left(\hbar_{ij}^{a}\right)_{m\times n}$ should be transformed into standardized decision matrices $SDM_a = \left(\hbar_{ij}^{a}\right)_{m\times n}$. We change the cost-type attribute into benefit-type attribute using the following formula.

$$\hbar_{ij}^{a} = \begin{cases} \hbar_{ij}^{a} = \left\langle \left[\Xi_{ij}^{aL}, \Xi_{ij}^{aU}\right], \left[\Psi_{ij}^{aL}, \Psi_{ij}^{aU}\right], \left[\Upsilon_{ij}^{aL}, \Upsilon_{ij}^{aU}\right] \right\rangle & for\, benefit-type\, attribute\, \overline{\overline{O}}_j, \\ i=1,2,...,m,\ j=1,2,...,n. & \\ \left(\hbar_{ij}^{a}\right)^{c} = \left\langle \left[\Upsilon_{ij}^{aL}, \Upsilon_{ij}^{aU}\right], \left[1-\Psi_{ij}^{aU}, 1-\Psi_{ij}^{aL}\right], \left[\Xi_{ij}^{aL}, \Xi_{ij}^{aU}\right] \right\rangle & for\, benefit-type\, attribute\, \overline{\overline{O}}_j, \\ i=1,2,...,m,\ j=1,2,...,n. & \end{cases} \tag{5.16}$$

Step 2. Determine the supports

$$Supp\left(\hbar_{ij}^{c}, \hbar_{ij}^{d}\right) = 1 - \overline{\overline{D}}\left(\hbar_{ij}^{c}, \hbar_{ij}^{d}\right), (c,d=1,2,...,z), \tag{5.17}$$

which fulfils the required axioms given in Definition (5.1.1.1), and $\overline{\overline{D}}\left(\hbar_{ij}^{c}, \hbar_{ij}^{d}\right)$ represents the distance measure given in Definition (1.1.1.13).

Step 3. Determine the supports $T\left(\hbar_{ij}^{c}\right)$ of the INN $\hbar_{ij}^{c}$ by other $\hbar_{ij}^{d}\,(d=1,2,...,z\ and\ c\neq d)$.

$$T\left(\hbar_{ij}^{c}\right) = \sum_{d=1;c\neq d}^{z} \psi_d Supp\left(\hbar_{ij}^{c}, \hbar_{ij}^{d}\right); c,d=1,2,...z; i=1,2,...,m,\ j=1,2,...,n. \tag{5.18}$$

Then use the importance degrees $\psi_c\,(c=1,2,...,z)$ of the DMs $e_a\,(a=1,2,...,z)$ to calculate the importance degrees

$$\varpi_{ij}^{(c)} = \frac{\psi_c\left(1+T\left(\hbar_{ij}^{c}\right)\right)}{\sum_{d}^{z}\psi_d\left(1+T\left(\hbar_{ij}^{d}\right)\right)}; c=1,2,...,z; i=1,2,....,m,\ j=1,2,...,n. \tag{5.19}$$

Where $\varpi_{ij} \geq 0$ and $\sum_{c=1}^{z} \varpi_{ij} = 1.$

Step 4. Utilize the WINPHM operator expressed by (5.12)

$$\hbar_{ij} = WSVNPHM^{(k)}\left(\hbar_{ij}^{(1)}, \hbar_{ij}^{(2)}, ..., \hbar_{ij}^{(z)}\right) =$$
$$\left\langle \begin{array}{l} \left[1-\left(\prod_{1\leq r_1<r_2<....<r_k\leq z}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{z\varpi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{z}{k}}}, 1-\left(\prod_{1\leq r_1<r_2<....<r_k\leq z}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{z\varpi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{z}{k}}}\right], \\ \left[\left(\prod_{1\leq r_1<r_2<....<r_k\leq z}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{z\varpi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{z}{k}}}, \left(\prod_{1\leq r_1<r_2<....<r_k\leq z}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{z\varpi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{z}{k}}}\right], \\ \left[\left(\prod_{1\leq r_1<r_2<....<r_k\leq z}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{z\varpi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{z}{k}}}, \left(\prod_{1\leq r_1<r_2<....<r_k\leq z}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{z\varpi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{z}{k}}}\right] \end{array} \right\rangle. \quad (5.20)$$

to aggregate all the decision matrices $DM_a = \left(\hbar_{ij}^{a}\right)_{m\times n} (a = 1, 2, ..., z)$ given by the DMs into the comprehensive decision matrix $CDM = \left(\hbar_{ij}\right)_{m\times n}$.

Step 5. Determine the supports:

$$Supp\left(\hbar_{ij}, \hbar_{iq}\right) = 1 - \overline{\overline{D}}\left(\hbar_{ij}, \hbar_{iq}\right), i = 1, 2, ..., m; q = 1, 2, ..., n. \quad (5.21)$$

which fulfils the required axioms given in Definition (5.1.1.1), and $\overline{\overline{D}}\left(\hbar_{ij}, \hbar_{iq}\right)$ represents the distance measure given in Definition (1.1.1.13).

Step 6. Determine the supports $T\left(\hbar_{ij}\right)$ of the INN $\hbar_{ij}\left(i = 1, 2, ..., m; j = 1, 2, ..., n\right)$ by the importance degrees ω_j of the attributes $\overline{\overline{O}}_j$ and the importance degrees ϕ_{ij} that are associated with the INN $\hbar_{ij}$ by the importance degree ω_j of the attributes $\overline{\overline{O}}_j$.

$$T\left(\hbar_{ij}\right) = \sum_{q=1; q\neq j}^{z} \omega_j Supp\left(\hbar_{ij}, \hbar_{iq}\right), i = 1, 2, ..., m, j, q = 1, 2, ..., n. \quad (5.22)$$

$$\phi_{ij} = \frac{\omega_c\left(1+T\left(\hbar_{ij}\right)\right)}{\sum_{j=1}^{n}\omega_j\left(1+T\left(\hbar_{ij}\right)\right)}; i=1,2,....,m,\ j=1,2,...,n. \tag{5.23}$$

Where $\phi_{ij} \geq 0$ and $\sum_{c=1}^{z}\phi_{ij} = 1.$

Step 7. Utilize the WINPHM operator (Equation (5.12))

$$\hbar_i = WSVNPHM^{(k)}\left(\hbar_{i1},\hbar_{i2},...,\hbar_{in}\right) = \\ \left\langle \left[1-\left(\prod_{1\le r_1<r_2<....<r_k\le n}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{L}\right)^{n\phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, 1-\left(\prod_{1\le r_1<r_2<....<r_k\le n}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(1-\Xi_{r_j}^{U}\right)^{n\phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}\right], \right. \\ \left[\left(\prod_{1\le r_1<r_2<....<r_k\le n}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{L}\right)^{n\phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \left(\prod_{1\le r_1<r_2<....<r_k\le n}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Psi_{r_j}^{U}\right)^{n\phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}\right], \\ \left. \left[\left(\prod_{1\le r_1<r_2<....<r_k\le n}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{L}\right)^{n\phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}, \left(\prod_{1\le r_1<r_2<....<r_k\le n}\left(1-\left(\prod_{j=1}^{k}\left(1-\left(\Upsilon_{r_j}^{U}\right)^{n\phi_{r_j}}\right)\right)^{\frac{1}{k}}\right)\right)^{\frac{1}{\binom{n}{k}}}\right] \right\rangle. \tag{5.24}$$

to get the comprehensive evaluation value.

Step 8. Determine the score and accuracy value of each INN $\hbar_r (r=1,2,...,n)$ using Definition (1.1.1.11).

Step 9. Rank all the alternatives and select the best one using Definition (1.1.1.12).

5.3 Numerical Example

In this part, two numerical examples will be provided to show the application and advantages of proposed approach. The first example is about the selection of emerging technology enterprises (ETEs), cited from [112].

5.3.1 An illustrative example

Let us assume that there are five ETEs represented by $\overline{\overline{N}}_i\left(i=1,2,3,4,5\right)$, which are selected. These five ETEs are evaluated with respect to the following four attributes $\overline{\overline{O}}_j\,(1,2,3,4)$, which are (1) $\overline{\overline{O}}_1$: the employment formation, (2) $\overline{\overline{O}}_2$: the progress of science and technology, (3) $\overline{\overline{O}}_3$: technical improvement, (4) $\overline{\overline{O}}_4$: the industrialization configuration. There are three experts $e_a\,(a=1,2,3)$ with importance degree $\left(0.25,0.4,0.35\right)^T$, who evaluate the five ETEs with respect to the four attributes with importance degree $\left(0.15,0.2,0.25,0.4\right)^T$, and provide their information in the form of INNs, which are listed in Tables 5.1-5.3.

In the following, we need to select the best alternatives. The précised steps are illustrated as follows:

Step 1. Normalize the decision matrices using Equation (5.16). Since all the attributes are of benefit type so there is no need to normalize it.

Step 2. Determine the supports $Supp\left(\hbar_{ij}^{c},\hbar_{ij}^{d}\right)\left(i=1,2,...,5,\ j=1,2,3,4,c,d=1,2,3,c\neq d\right)$ using Equation (5.17). In order to define the supports between $\hbar_{ij}^{c}$ and $\hbar_{ij}^{d}$, we denote $\left(Supp\left(\hbar_{ij}^{c},\hbar_{ij}^{d}\right)\right)_{5\times 4}$ as $Supp^{(cd)}$, which are given as follows:

Table 5.1. The decision matrix e_1

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	$\langle[0.3,0.4],[0.6,0.7],[0.30.5]\rangle$	$\langle[0.4,0.5],[0.2,0.3],[0.1,0.2]\rangle$	$\langle[0.1,0.2],[0.4,0.5],[0.1,0.2]\rangle$	$\langle[0.3,0.4],[0.5,0.6],[0.2,0.3]\rangle$
$\overline{\overline{N}}_2$	$\langle[0.5,0.7],[0.6,0.8],[0.2,0.4]\rangle$	$\langle[0.5,0.6],[0.3,0.5],[0.2,0.3]\rangle$	$\langle[0.5,0.7],[0.4,0.6],[0.2,0.3]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.2,0.3]\rangle$
$\overline{\overline{N}}_3$	$\langle[0.4,0.5],[0.5,0.6],[0.2,0.3]\rangle$	$\langle[0.3,0.4],[0.5,0.6],[0.1,0.2]\rangle$	$\langle[0.3,0.4],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.4,0.5],[0.1,0.2],[0.3,0.4]\rangle$
$\overline{\overline{N}}_4$	$\langle[0.6,0.7],[0.2,0.3],[0.1,0.2]\rangle$	$\langle[0.4,0.5],[0.1,0.2],[0.2,0.3]\rangle$	$\langle[0.4,0.5],[0.2,0.3],[0.1,0.2]\rangle$	$\langle[0.3,0.4],[0.4,0.5],[0.2,0.3]\rangle$
$\overline{\overline{N}}_5$	$\langle[0.40.5],[0.2,0.3],[0.2,0.3]\rangle$	$\langle[0.2,0.3],[0.6,0.7],[0.2,0.3]\rangle$	$\langle[0.5,0.6],[0.4,0.5],[0.2,0.3]\rangle$	$\langle[0.3,0.4],[0.6,0.7],[0.3,0.4]\rangle$

Table 5.2. The decision matrix e_2

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	$\langle[0.4,0.6],[0.5,0.7],[0.3,0.4]\rangle$	$\langle[0.6,0.7],[0.5,0.6],[0.5,0.6]\rangle$	$\langle[0.5,0.6],[0.4,0.5],[0.3,0.4]\rangle$	$\langle[0.6,0.7],[0.4,0.5],[0.3,0.4]\rangle$
$\overline{\overline{N}}_2$	$\langle[0.6,0.9],[0.4,0.5],[0.3,0.4]\rangle$	$\langle[0.7,0.8],[0.6,0.7],[0.4,0.5]\rangle$	$\langle[0.7,0.8],[0.3,0.4],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.4,0.5],[0.3,0.4]\rangle$
$\overline{\overline{N}}_3$	$\langle[0.8,0.9],[0.8,0.9],[0.4,0.5]\rangle$	$\langle[0.7,0.8],[0.5,0.6],[0.5,0.6]\rangle$	$\langle[0.7,0.8],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.5,0.6],[0.2,0.3]\rangle$
$\overline{\overline{N}}_4$	$\langle[0.6,0.7],[0.3,0.4],[0.5,0.6]\rangle$	$\langle[0.8,0.9],[0.5,0.6],[0.6,0.7]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.4,0.5]\rangle$	$\langle[0.5,0.6],[0.7,0.9],[0.3,0.4]\rangle$
$\overline{\overline{N}}_5$	$\langle[0.4,0.5],[0.6,0.7],[0.6,0.7]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.3,0.4]\rangle$	$\langle[0.9,1],[0.4,0.5],[0.3,0.4]\rangle$	$\langle[0.7,0.8],[0.8,0.9],[0.1,0.2]\rangle$

Table 5.3. The decision matrix e_3

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	$\langle[0.7,0.8],[0.4,0.5],[0.4,0.5]\rangle$	$\langle[0.7,0.8],[0.3,0.4],[0.6,0.7]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.4,0.5]\rangle$	$\langle[0.5,0.6],[0.4,0.5],[0.4,0.5]\rangle$
$\overline{\overline{N}}_2$	$\langle[0.6,0.7],[0.5,0.6],[0.4,0.5]\rangle$	$\langle[0.7,0.8],[0.6,0.7],[0.5,0.6]\rangle$	$\langle[0.8,0.9],[0.2,0.3],[0.7,0.8]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.4,0.6]\rangle$
$\overline{\overline{N}}_3$	$\langle[0.7,0.8],[0.3,0.4],[0.5,0.6]\rangle$	$\langle[0.8,0.9],[0.2,0.4],[0.6,0.7]\rangle$	$\langle[0.8,0.9],[0.2,0.4],[0.4,0.5]\rangle$	$\langle[0.9,1],[0.1,0.2],[0.5,0.6]\rangle$
$\overline{\overline{N}}_4$	$\langle[0.7,0.8],[0.4,0.5],[0.6,0.7]\rangle$	$\langle[0.6,0.9],[0.1,0.2],[0.7,0.8]\rangle$	$\langle[0.6,0.7],[0.1,0.2],[0.5,0.6]\rangle$	$\langle[0.6,0.7],[0.3,0.4],[0.4,0.5]\rangle$

$\overline{\overline{N}}_5$	$\langle[0.6,0.7],[0.7,0.8],[0.2,0.3]\rangle$	$\langle[0.7,0.8],[0.3,0.5],[0.4,0.5]\rangle$	$\langle[0.7,0.9],[0.3,0.4],[0.4,0.5]\rangle$	$\langle[0.8,0.9],[0.5,0.6],[0.5,0.6]\rangle$

$$Supp^{12} = Supp^{21} = \begin{bmatrix} 0.9167 & 0.7 & 0.8 & 0.8333 \\ 0.85 & 0.7833 & 0.867 & 0.8667 \\ 0.7 & 0.7333 & 0.8333 & 0.7 \\ 0.8333 & 0.6 & 0.867 & 0.7833 \\ 0.7333 & 0.7333 & 0.8333 & 0.7333 \end{bmatrix}, Supp^{13} = Supp^{31} = \begin{bmatrix} 0.7833 & 0.7 & 0.7 & 0.8333 \\ 0.8833 & 0.75 & 0.6667 & 0.9167 \\ 0.7333 & 0.5833 & 0.7167 & 0.7667 \\ 0.7333 & 0.7333 & 0.7667 & 0.8 \\ 0.7667 & 0.6833 & 0.8167 & 0.7333 \end{bmatrix}$$

$$Supp^{23} = Supp^{32} = \begin{bmatrix} 0.8333 & 0.8667 & 0.9 & 0.9333 \\ 0.9 & 0.9667 & 0.8 & 0.85 \\ 0.7667 & 0.85 & 0.8833 & 0.7333 \\ 0.9 & 0.8 & 0.9 & 0.7833 \\ 0.7667 & 0.9167 & 0.8833 & 0.7333 \end{bmatrix}$$

Step 3. Determine the weighted supports $T\left(\hbar_{ij}^{c}\right)$ of INN $\hbar_{ij}^{c}$ by other INNs $\hbar_{ij}^{d}\left(d = 1,2,3\, and\, c \neq d\right)$ by utilizing Equation (5.18), and determine the weight $\varpi_{ij}^{(c)}\left(i = 1,2,3,4,5,\ j = 1,2,3,4, c = 1,2,3\right)$ of INN $\hbar_{ij}^{c}$ $\left(i = 1,2,3,4,5,\ j = 1,2,3,4, c = 1,2,3\right)$ by utilizing Equation (5.19). In order to represent $\left(T\left(\hbar_{ij}^{c}\right)\right)_{5\times 4}$ as $T_c\left(c = 1,2,3\right)$ and $\left(\varpi_{ij}^{(c)}\right)_{5\times 4}$ as $U_c(c = 1,2,3)$, which are given as follows:

$$T_1 = \begin{bmatrix} 0.6408 & 0.525 & 0.565 & 0.625 \\ 0.6492 & 0.5758 & 0.58 & 0.6675 \\ 0.5367 & 0.4975 & 0.5842 & 0.5483 \\ 0.59 & 0.4967 & 0.615 & 0.5933 \\ 0.5617 & 0.5325 & 0.6192 & 0.55 \end{bmatrix}; \quad T_2 = \begin{bmatrix} 0.5208 & 0.4783 & 0.515 & 0.535 \\ 0.5275 & 0.5342 & 0.4967 & 0.5142 \\ 0.4433 & 0.4808 & 0.5175 & 0.4317 \\ 0.5233 & 0.43 & 0.5317 & 0.47 \\ 0.4517 & 0.5042 & 0.5175 & 0.44 \end{bmatrix};$$

$$\widetilde{T_3} = \begin{bmatrix} 0.5292 & 0.5217 & 0.535 & 0.5817 \\ 0.5808 & 0.5742 & 0.4867 & 0.5692 \\ 0.49 & 0.4858 & 0.5325 & 0.485 \\ 0.5433 & 0.5033 & 0.5517 & 0.5133 \\ 0.4983 & 0.5375 & 0.5575 & 0.4767 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} 0.2640 & 0.2533 & 0.2550 & 0.2581 \\ 0.26151 & 0.2528 & 0.2609 & 0.2652 \\ 0.2591 & 0.2518 & 0.2573 & 0.2616 \\ 0.2570 & 0.2541 & 0.2589 & 0.2628 \\ 0.2611 & 0.2516 & 0.2600 & 0.2618 \end{bmatrix}; \quad U_2 = \begin{bmatrix} 0.3915 & 0.3929 & 0.3949 & 0.3901 \\ 0.3875 & 0.3937 & 0.3954 & 0.3853 \\ 0.3893 & 0.3984 & 0.3943 & 0.3871 \\ 0.3939 & 0.3885 & 0.3928 & 0.3879 \\ 0.3883 & 0.3951 & 0.3898 & 0.3891 \end{bmatrix};$$

$$U_3 = \begin{bmatrix} 0.3445 & 0.3538 & 0.3501 & 0.3517 \\ 0.3509 & 0.3535 & 0.3437 & 0.3494 \\ 0.3517 & 0.3498 & 0.3484 & 0.3513 \\ 0.3492 & 0.3574 & 0.3482 & 0.3494 \\ 0.3507 & 0.3534 & 0.3501 & 0.3491 \end{bmatrix}$$

Step 4. Utilize the WINPHM operator (Equation (5.20)) to get the overall decision matrix (and assume that $k = 2$), which are given in Table 5.4.

Table 5.4. The overall decision matrix CDM

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	⟨[0.4482,0.5903],[0.5066,0.6434],[0.3399,0.47	⟨[0.5577,0.6593],[0.3391,0.4442],[0.4088,0.523	⟨[0.3662,0.4810],[0.3740,0.4734],[0.2646,0.372	⟨[0.4553,0.5563],[0.4397,0.5387],[0.3044,0.4067
$\overline{\overline{N}}_2$	⟨[0.5608,0.7529],[0.5077,0.6470],[0.3039,0.43	⟨[0.6251,0.7255],[0.5132,0.6455],[0.3726,0.477	⟨[0.6603,0.7947],[0.3067,0.4417],[0.4156,0.529	⟨[0.6553,0.7538],[0.3382,0.4386],[0.3034,0.4425
$\overline{\overline{N}}_3$	⟨[0.6234,0.7285],[0.5705,0.6868],[0.3717,0.47	⟨[0.5913,0.7040],[0.4143,0.5437],[0.4088,0.523	⟨[0.5914,0.7042],[0.1392,0.2745],[0.3048,0.407	⟨[0.7040,0.8332],[0.2277,0.3427],[0.3462,0.447
$\overline{\overline{N}}_4$	⟨[0.6264,0.7261],[0.3048,0.4071],[0.4087,0.52	⟨[0.5848,0.7756],[0.2294,0.3443],[0.5230,0.636	⟨[0.4928,0.5928],[0.1731,0.2735],[0.3336,0.444	⟨[0.4567,0.5586],[0.4874,0.6544],[0.3038,0.4061
$\overline{\overline{N}}_5$	⟨[0.4574,0.5579],[0.5225,0.6354],[0.3428,0.45	⟨[0.4810,0.5912],[0.4091,0.5453],[0.3055,0.407	⟨[0.6863,0.8395],[0.3731,0.4727],[0.3042,0.406	⟨[0.5914,0.7041],[0.6535,0.7623],[0.3205,0.4214

Step 5. Determine the supports $Supp\left(\hbar_{ij}, \hbar_{iq}\right)(i = 1, 2, ...5,\ j = 1, 2.., 4; q = 1, 2, ..., 4)$ by using Equation (5.21). For simplicity, $Supp\left(\hbar_{ij}, \hbar_{iq}\right)_{5\times1}$ is denoted by $Supp_{jq}$ to define the supports among the jth and qth column of CDM.

$$Supp_{12} = Supp_{12} = \begin{bmatrix} 0.8894 \\ 0.9659 \\ 0.9267 \\ 0.9239 \\ 0.9428 \end{bmatrix}, Supp_{13} = Supp_{31} = \begin{bmatrix} 0.8883 \\ 0.8752 \\ 0.8273 \\ 0.8857 \\ 0.8487 \end{bmatrix}, Supp_{14} = Supp_{41} = \begin{bmatrix} 0.9475 \\ 0.9206 \\ 0.8456 \\ 0.8352 \\ 0.9013 \end{bmatrix},$$

$$Supp_{23} = Supp_{32} = \begin{bmatrix} 0.8909 \\ 0.8969 \\ 0.8862 \\ 0.8582 \\ 0.9016 \end{bmatrix}, Supp_{24} = Supp_{42} = \begin{bmatrix} 0.8965 \\ 0.9093 \\ 0.8720 \\ 0.7729 \\ 0.8811 \end{bmatrix}, Supp_{34} = Supp_{43} = \begin{bmatrix} 0.9384 \\ 0.9534 \\ 0.9200 \\ 0.8611 \\ 0.8614 \end{bmatrix}.$$

Step 6. Determine the weighted supports $T(\hbar_{ij})$ of INN $\hbar_{ij}$ by utilizing Equation (5.22) and determine the weighs $\phi_{ij}\ (j = 1,2,3,4)$ of the INNs $\hbar_{ij}$ by utilizing Equation (5.23). For computational clarity, we denote $\left(T(\hbar_{ij})\right)_{5\times4}$ as T and $\left(\phi_{ij}\right)_{5\times4}$ as U, which are given as follows:

$$T = \begin{bmatrix} 0.7790 & 0.7147 & 0.6868 & 0.5560 \\ 0.7802 & 0.7328 & 0.6920 & 0.5583 \\ 0.7304 & 0.7094 & 0.6693 & 0.5312 \\ 0.7403 & 0.6623 & 0.6489 & 0.4951 \\ 0.7613 & 0.7193 & 0.6522 & 0.5268 \end{bmatrix}, U = \begin{bmatrix} 0.1613 & 0.2074 & 0.2550 & 0.3763 \\ 0.1609 & 0.2088 & 0.2548 & 0.3755 \\ 0.1591 & 0.2096 & 0.2558 & 0.3755 \\ 0.1628 & 0.2073 & 0.2570 & 0.3729 \\ 0.1619 & 0.2107 & 0.2531 & 0.3743 \end{bmatrix}$$

Step 7. Using the WINPHM operator in Equation (5.24), to aggregate all the execution values $\hbar_{ij}\ (j = 1,2,3,4)$ in the ith line of CDM and get the comprehensive execution values $\overline{\overline{N}}_i\ (i = 1,2,...,5)$ (assume that $k = 2$).

$\overline{\overline{N}}_1 = \langle [0.4401, 0.5531], [0.4305, 0.5385], [0.3498, 0.4614] \rangle$;

$\overline{\overline{N}}_2 = \langle [0.6068, 0.7359], [0.4345, 0.5587], [0.3712, 0.4904] \rangle$;

$\overline{\overline{N}}_3 = \langle [0.6045, 0.7137], [0.3484, 0.4767], [0.3774, 0.4805] \rangle$;

$\overline{\overline{N}}_4 = \langle [0.5221, 0.6468], [0.3153, 0.4443], [0.4129, 0.5198] \rangle$;

$\overline{\overline{N}}_5 = \langle [0.5221, 0.6468], [0.5101, 0.6262], [0.3373, 0.4390] \rangle$.

Step 8. Determine the score values of $\overline{\overline{N}}_i \, (i = 1, 2, ..., 5)$ by using Definition (1.1.1.11), we have

$$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right) = 1.6065, \overline{\overline{SO}}\left(\overline{\overline{N}}_2\right) = 1.7439, \overline{\overline{SO}}\left(\overline{\overline{N}}_3\right) = 1.8176, \overline{\overline{SO}}\left(\overline{\overline{N}}_4\right) = 1.7383, \overline{\overline{SO}}\left(\overline{\overline{N}}_5\right) = 1.6281.$$

Then the alternatives can be arranged in decreasing order according to their score values:

$$\overline{\overline{N}}_3 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_5 > \overline{\overline{N}}_1.$$

Step 9. Based on Definition (1.1.1.12), and the best ETEs is $\overline{\overline{N}}_3$ while the worst one is $\overline{\overline{N}}_1$.

5.3.2 Effect of the parameter k

In this subsection, we take different values of the parameter *k* in the WINPHM operator to observe the ranking results, hence we can determine the score values produced for different values of the parameter *k*, and the ranking results are given in Table 5.5.

Table 5.5. Scores and ranking of the alternatives for different parameter values

k	**Score values** Θ_i	**Ranking order**
$k = 1$	$\overline{\overline{SC}}\left(\overline{\overline{N}}_1\right) = 1.7512, \overline{\overline{SC}}\left(\overline{\overline{N}}_2\right) = 1.9127, \overline{\overline{SC}}\left(\overline{\overline{N}}_3\right) = 2.1819,$ $\overline{\overline{SC}}\left(\overline{\overline{N}}_4\right) = 1.9061, \overline{\overline{SC}}\left(\overline{\overline{N}}_5\right) = 1.9670.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_5 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1.$
$k = 2$	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right) = 1.6065, \overline{\overline{SO}}\left(\overline{\overline{N}}_2\right) = 1.7439, \overline{\overline{SO}}\left(\overline{\overline{N}}_3\right) = 1.8176,$ $\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right) = 1.7383, \overline{\overline{SO}}\left(\overline{\overline{N}}_5\right) = 1.6281.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_5 > \overline{\overline{N}}_1.$
$k = 3$	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right) = 2.9781, \overline{\overline{SO}}\left(\overline{\overline{N}}_2\right) = 2.9816, \overline{\overline{SO}}\left(\overline{\overline{N}}_3\right) = 2.9869,$ $\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right) = 2.9854, \overline{\overline{SO}}\left(\overline{\overline{N}}_5\right) = 2.9773.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$

From Table 5, we can see that when the value of the parameter $k=1$, the ranking order are slightly different, but the best and worst alternative remain the same as for the parameter value $k=2$. When the value of the parameter $k=3$, then the ranking order are different from the ones obtained for the parameter value $k=1,2$. The best choice remains the same, but the worst alternative is changed. That is, for $k=1,2$ the worst alternative is $\overline{\overline{N}}_3$, while for $k=3$ the worst alternative is $\overline{\overline{N}}_5$, these results are reasonable, as we can consider the interrelationship for different number of attributes, when *k*=1, we don't consider the interrelationship of the attributes; when *k*=2, we can take into account the interrelationship between any two attributes, and when *k*=3, we consider the interrelationship among any three attributes. These results show that the proposed AO is more flexible and practical.

5.3.3 Comparison with Other Approaches

In the following, we will utilize the other two approaches to solve the same example, and compare and examine the decision results obtained by these methods. The first approach is based on INBM operator proposed by Ji et al. [113], and the second approach is based on INPWA operator proposed by Liu et al. [71]. The score values and ranking order on these different approaches are shown in Table 5.6.

Table 5.6. Score values and ranking order of different approaches

Approach	Score values of	Ranking order
Based on INWBM operator ($p=q=1$) by Ji et al. [113]	$\overline{\overline{SO}}(\overline{\overline{N}}_1)=0.2004, \overline{\overline{SO}}(\overline{\overline{N}}_2)=0.2326, \overline{\overline{SO}}(\overline{\overline{N}}_3)=0.2633, \overline{\overline{SO}}(\overline{\overline{N}}_4)=0.2250, \overline{\overline{SO}}(\overline{\overline{N}}_5)=0.2205.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_5 > \overline{\overline{N}}_1.$
Based on INWPA operator ($\lambda=1$) by Liu et al. [71]	$\overline{\overline{SO}}(\overline{\overline{N}}_1)=1.7623, \overline{\overline{SO}}(\overline{\overline{N}}_2)=1.9254, \overline{\overline{SO}}(\overline{\overline{N}}_3)=2.1943, \overline{\overline{SO}}(\overline{\overline{N}}_4)=1.8964, \overline{\overline{SO}}(\overline{\overline{N}}_5)=1.9794.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_5 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1.$

Base on the proposed operator in this article $(k=2)$	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=1.6065, \overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=1.7439, \overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=1.8176,$ $\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=1.7383, \overline{\overline{SO}}\left(\overline{\overline{N}}_5\right)=1.6281.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_5 > \overline{\overline{N}}_1.$
Base on the proposed operator in this article $(k=1)$	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=1.7512, \overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=1.9127, \overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=2.1819,$ $\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=1.9061, \overline{\overline{SO}}\left(\overline{\overline{N}}_5\right)=1.9670.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_5 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1$

From Table 5.6, we can observe that when the value of the parameter gets $k=1$, there are the same ranking results of our method in this paper with the method in Liu et al. [71], while when the value of the parameter gets $k=2$, we get the same ranking results of our method in this paper with Ji et al' method [113]. However, they are different in ranking results from the methods [71] and [113]. We think these results are reasonable and can explain them as follows.

(1) When *k*=1, our method proposed in this paper can reduce into PA operator for INNs, and it is similar to method in [71], so these two methods produced the same ranking results. Obviously, this can explain the validity of our proposed method.

(2) When *k*=2, our method proposed in this paper can reduce into BM operator for INNs, and it is similar to method in [113], so these two methods produced the same ranking results. Obviously, this can further explain the validity of our proposed method.

(3) There are the deferent ranking results of our method when *k*=1 and method in [71] with our method when *k*=2 and method in [113], and the reason is that our method when *k*=1 and method in [71] cannot consider the interrelationship of the attributes while our method when *k*=2 and method in [113] can do.

Further, we can compare the existing two methods [113] and [71] with our method in this paper as follows.

(1) Ji et al. [113] developed the method based on INWBM operator, and the developed aggregation operators only consider the interrelationship between two attributes and cannot eliminate the effect of awkward data. While the proposed aggregation operator has the properties that it can consider the interrelationship among more than two attributes $(k \geq 2)$ or doesn't consider the interrelationship of the attributes (when k=1), and also remove the effect of awkward data. Obviously, our method is more flexible and practical then the method in [8].

(2) Liu et al. [71] developed the method based on INPWA operator. The developed operator can only eliminate the effect of awkward data given by the DMs and cannot consider the correlation among attributes. Obviously, our method is also more flexible and practical then the method in [71].

In practical MAGDM or MADM problems, our proposed approach is superior to the existing two approaches.

5.3.4 Conclusion

The HM operator is an aggregation tool that can consider the interrelationship between multiple input parameters, and the PA operator has the property that it can reduce the potency of awkward assessment values in the decision consequences. The INSs are a more powerful tool to handle uncertain information that exists in real life problems. Therefore, for some complex decision-making situations in this article, we combine the conventional HM operator to the traditional PA operator in interval neutrosophic settings and present the two novel interval neutrosophic aggregation operators, that is, the interval neutrosophic power Hammy mean (INPHM) operator and the weighted interval neutrosophic power Hammy mean (WINPHM) operators. Then, some preferable properties and special cases of the developed aggregation operators are discussed. Moreover, based on these developed aggregation operators, we propose a new method to MAGDM. Lastly, the developed approach is applied to some practical problems and shows that the proposed aggregation operators are better and flexible then some existing aggregation operators. The other feature of the developed aggregation operator is generalization of some existing aggregation operators.

Chapter 6

Multi-attribute Decision Making Method Based Interval Neutrosophic Dombi Power Bonferroni Mean Operator

In this chapter, firstly, we describe some operational laws for INNs over Dombi TN and TCN and examined numerous enviable properties of these newly developed operational laws. Secondly, we enlarged PBM operator over Dombi operations to develop INDPBM operator, INWDPBM operator, INDPGBM operator, INWDPGBM operator and discussed some properties of these aggregation operators. Then, we develop a MADM method over these aggregation operators to deal with IN information. Lastly, an illustrative example is demonstrated to show the effectiveness and practicality of the developed MADM method.

6.1 Some operations of INSs based on Dombi TN and TCN

6.1.1 Dombi TN and TCN

Dombi operations consist of the Dombi sum and Dombi product.

6.1.1.1 Definition [96]

Let ψ and ξ be any two real number. Then, the Dombi TN and TCN among $\Im$ and $\aleph$ are explain as follows:

$$T_D(\psi,\xi)=\frac{1}{1+\left\{\left(\frac{1-\psi}{\psi}\right)^{\gamma}+\left(\frac{1-\xi}{\xi}\right)^{\gamma}\right\}^{\frac{1}{\gamma}}}; \tag{6.1}$$

$$T_D^*(\psi,\xi)=\frac{1}{1+\left\{\left(\frac{\psi}{1-\psi}\right)^{\gamma}+\left(\frac{\xi}{1-\xi}\right)^{\gamma}\right\}^{\frac{1}{\gamma}}}; \tag{6.2}$$

Where $\gamma \geq 1,$ and $(\psi,\xi)\in[0,1]\times[0,1]$.

According to the Dombi TN and TCN, we develop few operational rules for INNs.

6.1.1.2 Definition

Let $\hbar=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle$, $\hbar_1=\left\langle\left[\Xi_1^L,\Xi_1^U\right],\left[\Psi_1^L,\Psi_1^U\right],\left[\Upsilon_1^L,\Upsilon_1^U\right]\right\rangle$ and $\hbar_2=\left\langle\left[\Xi_2^L,\Xi_2^U\right],\left[\Psi_2^L,\Psi_2^U\right],\left[\Upsilon_2^L,\Upsilon_2^U\right]\right\rangle$ be any INNs and $\mathfrak{A}>0$. Then, the operational rules based on Dombi TN and TCN for INNs are expressed as follows;

$$(1)\ \hbar_1\oplus\hbar_2=\left\langle\left[1-\frac{1}{1+\left(\left(\frac{\Xi_1^L}{1-\Xi_1^L}\right)^{\gamma}+\left(\frac{\Xi_2^L}{1-\Xi_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}},1-\frac{1}{1+\left(\left(\frac{\Xi_1^U}{1-\Xi_1^U}\right)^{\gamma}+\left(\frac{\Xi_2^U}{1-\Xi_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right],\right.$$
$$\left[\frac{1}{1+\left(\left(\frac{1-\Psi_1^L}{\Psi_1^L}\right)^{\gamma}+\left(\frac{1-\Psi_2^L}{\Psi_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{1}{1+\left(\left(\frac{1-\Psi_1^U}{\Psi_1^U}\right)^{\gamma}+\left(\frac{1-\Psi_2^U}{\Psi_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right],$$
$$\left.\left[\frac{1}{1+\left(\left(\frac{1-\Upsilon_1^L}{\Upsilon_1^L}\right)^{\gamma}+\left(\frac{1-\Upsilon_2^L}{\Upsilon_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{1}{1+\left(\left(\frac{1-\Upsilon_1^U}{\Upsilon_1^U}\right)^{\gamma}+\left(\frac{1-\Upsilon_2^U}{\Upsilon_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right]\right\rangle; \tag{6.3}$$

$$(2)\,\hbar_1 \otimes \hbar_2 = \left\langle \left[\frac{1}{1+\left(\left(\frac{1-\Xi_1^L}{\Xi_1^L}\right)^{\gamma}+\left(\frac{1-\Xi_2^L}{\Xi_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\left(\frac{1-\Xi_1^U}{\Xi_1^U}\right)^{\gamma}+\left(\frac{1-\Xi_2^U}{\Xi_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \right.$$

$$\left[1-\frac{1}{1+\left(\left(\frac{\Psi_1^L}{1-\Psi_1^L}\right)^{\gamma}+\left(\frac{\Psi_2^L}{1-\Psi_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\left(\frac{\Psi_1^U}{1-\Psi_1^U}\right)^{\gamma}+\left(\frac{\Psi_2^U}{1-\nu_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right],$$

$$\left. \left[1-\frac{1}{1+\left(\left(\frac{\Upsilon_1^L}{1-\Upsilon_1^L}\right)^{\gamma}+\left(\frac{\Upsilon_2^L}{1-\Upsilon_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\left(\frac{\Upsilon_1^U}{1-\Upsilon_1^U}\right)^{\gamma}+\left(\frac{\Upsilon_2^U}{1-\Upsilon_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \right\rangle; \tag{6.4}$$

$$(3)\,\mathfrak{A}\hbar = \left\langle \left[1-\frac{1}{1+\left(\mathfrak{A}\left(\frac{\Xi^L}{1-\Xi^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\mathfrak{A}\left(\frac{\Xi^U}{1-\Xi^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1+\left(\mathfrak{A}\left(\frac{1-\Psi^L}{\Psi^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\mathfrak{A}\left(\frac{1-\Psi^U}{\Psi^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \right. \tag{6.5}$$

$$\left. \left[\frac{1}{1+\left(\mathfrak{A}\left(\frac{1-\Upsilon^L}{\Upsilon^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\mathfrak{A}\left(\frac{1-\Upsilon^U}{\Upsilon^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \right\rangle;$$

$$(4)\,\hbar^{\mathfrak{A}} = \left\langle \left[\frac{1}{1+\left(\mathfrak{A}\left(\frac{1-\Xi^L}{\Xi^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\mathfrak{A}\left(\frac{1-\Xi^U}{\Xi^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[1-\frac{1}{1+\left(\mathfrak{A}\left(\frac{\Psi^L}{1-\Psi^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\mathfrak{A}\left(\frac{\Psi^U}{1-\Psi^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \right. \tag{6.6}$$

$$\left. \left[1-\frac{1}{1+\left(\mathfrak{A}\left(\frac{\Upsilon^L}{1-\Upsilon^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\mathfrak{A}\left(\frac{\Upsilon^U}{1-\Upsilon^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \right\rangle.$$

6.2 The INPBM operator based on Dombi TN and Dombi TCN

In this part, based on the Dombi operational laws for INNs, we combine PA operator and BM to introduced INDPBM, INWDPBM, INDPGBM, and INWDPGBM and discussed some related properties.

6.2.1 The INDPBM operator and INWDPBM operator

In this subpart, based on the Dombi operational laws for INNs, we combine PA operator and BM to introduced INDPBM, INWDPBM and discussed some related properties.

6.2.1.1 Definition

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right]\right\rangle, (u = 1, 2, ..., s)$, be a group of INNs, and $g, h \geq 0$. If

$$INDPBM^{x,y}\left(\hbar_1, \hbar_2, ..., \hbar_s\right) = \left(\frac{1}{s^2 - s}\left(\underset{\substack{u,j=1\\u\neq j}}{\overset{s}{\oplus}}\left(\left(\frac{s\left(1+T(\hbar_u)\right)}{\overset{s}{\underset{a=1}{\oplus}}\left(1+T(\hbar_a)\right)}\hbar_u\right)^g \otimes_D \left(\frac{s\left(1+T(\hbar_j)\right)}{\overset{s}{\underset{a=1}{\oplus}}\left(1+T(\hbar_a)\right)}\hbar_j\right)^h\right)\right)\right)^{\frac{1}{g+h}}. \quad (6.7)$$

Then, $INDPBM^{x,y}$ is said to be IN Dombi power Bonferroni mean (INDPBM) operator, where $T\left(\hbar_f\right) = \underset{f=1, f\neq a}{\overset{s}{\oplus}} Sup\left(\hbar_a, \hbar_f\right)$. $Sup\left(\hbar_a, \hbar_f\right)$ is the SPD for $\hbar_a$ from $\hbar_f$, which must assures the following characteristics: (1) $Sup\left(\hbar_a, \hbar_f\right) \in [0,1]$; (2) $Sup\left(\hbar_a, \hbar_f\right) = Sup\left(\hbar_f, \hbar_a\right)$; (3) $Sup\left(\hbar_a, \hbar_f\right) \geq Sup\left(\hbar_c, \hbar_b\right)$, if $D\left(\hbar_a, \hbar_f\right) < D\left(\hbar_c, \hbar_b\right)$, in which $\overline{\overline{D}}\left(\hbar_c, \hbar_b\right)$ is the distance measure among INNs $\hbar_c$ and $\hbar_b$ defined in Definition (1.1.1.13).

In order to simplify Equation (6.7), we can define

$$\Phi_z = \frac{\left(1+T\left(\hbar_z\right)\right)}{\overset{s}{\underset{z=1}{\oplus}}\left(1+T\left(\hbar_z\right)\right)}, \tag{6.8}$$

and call $\Phi = \left(\Phi_1, \Phi_2, ..., \Phi_s\right)^T$ is the power weight vector (PWV), such that $\Phi_a \geq 0, \overset{s}{\underset{a=1}{\oplus}} \Phi_a = 1.$ Then, the simplified form of Equation (6.7) can be written as follows:

$$INDPBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right) = \left(\frac{1}{s^2 - s} \overset{s}{\underset{\substack{u,j=1 \\ u \neq j}}{\oplus}} \left(v\Phi_u \hbar_u\right)^g \otimes_D \left(v\Phi_j \hbar_j\right)^h\right)^{\frac{1}{g+h}}. \tag{6.9}$$

6.2.1.2 Theorem

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right] \right\rangle, (u = 1, 2, ..., s)$, be a group of INNs. Then, the result obtained utilizing Eq. (6.7), is expressed as;

$$INDPBM^{x,y}\left(\hbar_1, \hbar_2, ..., \hbar_s\right)$$

$$= \left\langle \left[1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{s\Phi_u \left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^{\gamma}} + \frac{h}{s\Phi_j \left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right), 1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{s\Phi_i \left(\frac{\Xi_u^U}{1-\Xi_u^U}\right)^{\gamma}} + \frac{h}{s\Phi_j \left(\frac{\Xi_j^U}{1-\Xi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right) \right],\right.$$

$$\left[1 - 1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{s\Phi_i \left(\frac{1-\Psi_u^L}{\Psi_u^L}\right)^{\gamma}} + \frac{h}{s\Phi_j \left(\frac{1-\Psi_j^L}{\Psi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right), 1 - 1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{s\Phi_u \left(\frac{1-\Psi_u^U}{\Psi_u^U}\right)^{\gamma}} + \frac{h}{s\Phi_j \left(\frac{1-\Psi_j^U}{\Psi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right) \right],$$

$$\left[1 - 1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{s\Phi_u \left(\frac{1-\Upsilon_u^L}{\Upsilon_u^L}\right)^{\gamma}} + \frac{h}{s\Phi_j \left(\frac{1-\Upsilon_j^L}{\Upsilon_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right), 1 - 1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{s\Phi_i \left(\frac{1-\Upsilon_u^U}{\Upsilon_u^U}\right)^{\gamma}} + \frac{h}{s\Phi_i \left(\frac{1-\Upsilon_j^U}{\Upsilon_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right) \right].$$

(6.10)

Proof: Since

$$s\Phi_u \hbar_u = \left\langle \left[1-\frac{1}{1+\left(s\Phi_i\left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(s\Phi_u\left(\frac{\Xi_u^U}{1-\Xi_u^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1+\left(s\Phi_u\left(\frac{1-\Psi_u^L}{\Psi_u^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(s\Phi_u\left(\frac{1-\Psi_u^U}{\Psi_u^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1+\left(s\Phi_u\left(\frac{1-\Upsilon_u^L}{\Upsilon_u^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(s\Phi_u\left(\frac{1-\Upsilon_u^U}{\Upsilon_u^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \right\rangle;$$

and

$$s\Phi_j \hbar_j = \left\langle \left[1-\frac{1}{1+\left(s\Phi_j\left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(s\Phi_j\left(\frac{\Xi_j^U}{1-\Xi_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1+\left(s\Phi_j\left(\frac{1-\Psi_j^L}{\Psi_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(s\Phi_j\left(\frac{1-\Psi_j^U}{\Psi_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1+\left(s\Phi_j\left(\frac{1-\Upsilon_j^L}{\Upsilon_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(s\Phi_j\left(\frac{1-\Upsilon_j^U}{\Upsilon_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \right\rangle.$$

And let $a_u = \frac{\Xi_u^L}{1-\Xi_u^L}, b_u = \frac{\Xi_i^U}{1-\Xi_i^U}, c_u = \frac{1-\Psi_u^L}{\Psi_u^L}, d_u = \frac{1-\Psi_u^U}{\Psi_u^U}, g_u = \frac{1-\Upsilon_u^L}{\Upsilon_u^L}, h_u = \frac{1-\Upsilon_u^U}{\Upsilon_u^U}, a_j = \frac{\Xi_j^L}{1-\Xi_j^L},$ $b_j = \frac{\Xi_j^U}{1-\Xi_j^U}, c_j = \frac{1-\Psi_j^L}{\Psi_j^L}, d_j = \frac{1-\Psi_j^U}{\Psi_j^U}, g_j = \frac{1-\Upsilon_j^L}{\Upsilon_j^L}, h_j = \frac{1-\Upsilon_j^U}{\Upsilon_j^U}.$ Then, we have

$$s\Phi_u \hbar_u = \left\langle \left[1-\frac{1}{1+(s\Phi_u)^{\frac{1}{\gamma}} a_u}, 1-\frac{1}{1+(s\Phi_u)^{\frac{1}{\gamma}} b_u} \right], \left[\frac{1}{1+(s\Phi_u)^{\frac{1}{\gamma}} c_u}, \frac{1}{1+(s\Phi_u)^{\frac{1}{\gamma}} d_u} \right], \left[\frac{1}{1+(s\Phi_u)^{\frac{1}{\gamma}} g_u}, \frac{1}{1+(s\Phi_u)^{\frac{1}{\gamma}} h_u} \right] \right\rangle;$$

$$s\Phi_j \hbar_j = \left\langle \left[1-\frac{1}{1+(s\Phi_j)^{\frac{1}{\gamma}} a_j}, 1-\frac{1}{1+(s\Phi_j)^{\frac{1}{\gamma}} b_j} \right], \left[\frac{1}{1+(s\Phi_j)^{\frac{1}{\gamma}} c_j}, \frac{1}{1+(s\Phi_j)^{\frac{1}{\gamma}} d_j} \right], \left[\frac{1}{1+(s\Phi_j)^{\frac{1}{\gamma}} g_j}, \frac{1}{1+(s\Phi_j)^{\frac{1}{\gamma}} h_j} \right] \right\rangle.$$

and

$$\left(s\Phi_u \hbar_u\right)^g = \left\langle \begin{array}{l} \left[\frac{1}{1+g^{\frac{1}{\gamma}}\Big/\left(s\Phi_u\right)^{\frac{1}{\gamma}} a_u}, \frac{1}{1+g^{\frac{1}{\gamma}}\Big/\left(s\Phi_u\right)^{\frac{1}{\gamma}} b_u}\right], \left[1-\frac{1}{1+g^{\frac{1}{\gamma}}\Big/\left(s\Phi_u\right)^{\frac{1}{\gamma}} c_u}, 1-\frac{1}{1+g^{\frac{1}{\gamma}}\Big/\left(s\Phi_u\right)^{\frac{1}{\gamma}} d_u}\right], \\ \left[1-\frac{1}{1+g^{\frac{1}{\gamma}}\Big/\left(s\Phi_u\right)^{\frac{1}{\gamma}} g_u}, 1-\frac{1}{1+g^{\frac{1}{\gamma}}\Big/\left(s\Phi_u\right)^{\frac{1}{\gamma}} h_u}\right] \end{array} \right\rangle;$$

$$\left(s\Phi_j \hbar_j\right)^y = \left\langle \begin{array}{l} \left[\frac{1}{1+h^{\frac{1}{\gamma}}\Big/\left(s\Phi_j\right)^{\frac{1}{\gamma}} a_j}, \frac{1}{1+h^{\frac{1}{\gamma}}\Big/\left(s\Phi_j\right)^{\frac{1}{\gamma}} b_j}\right], \left[1-\frac{1}{1+h^{\frac{1}{\gamma}}\Big/\left(s\Phi_j\right)^{\frac{1}{\gamma}} c_j}, 1-\frac{1}{1+h^{\frac{1}{\gamma}}\Big/\left(s\Phi_j\right)^{\frac{1}{\gamma}} d_j}\right], \\ \left[1-\frac{1}{1+h^{\frac{1}{\gamma}}\Big/\left(s\Phi_j\right)^{\frac{1}{\gamma}} g_j}, 1-\frac{1}{1+h^{\frac{1}{\gamma}}\Big/\left(s\Phi_j\right)^{\frac{1}{\gamma}} h_j}\right] \end{array} \right\rangle.$$

Moreover, we have

$$\left(s\Phi_u \hbar_u\right)^g \otimes_D \left(s\Phi_j \hbar_j\right)^h$$

$$= \left\langle \left[\frac{1}{1+\left(g\big/s\Phi_u a_u^\gamma + h\big/s\Phi_j a_j^\gamma\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(g\big/s\Phi_u b_u^\gamma + h\big/s\Phi_j b_j^\gamma\right)^{\frac{1}{\gamma}}}\right], \left[1-\frac{1}{1+\left(g\big/s\Phi_u c_u^\gamma + h\big/s\Phi_j c_j^\gamma\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(g\big/s\Phi_u d_u^\gamma + h\big/s\Phi_j d_j^\gamma\right)^{\frac{1}{\gamma}}}\right],\right.$$
$$\left.\left[1-\frac{1}{1+\left(g\big/s\Phi_u g_u^\gamma + h\big/s\Phi_j g_j^\gamma\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(g\big/s\Phi_u h_u^\gamma + h\big/s\Phi_j h_j^\gamma\right)^{\frac{1}{\gamma}}}\right]\right\rangle,$$

and

$$\sum_{\substack{u,j=1 \\ u\neq j}}^{s} \left(s\Phi_u \hbar_u\right)^g \otimes_D \left(s\Phi_j \hbar_j\right)^h$$

$$= \left\langle \left[1- 1\Bigg/\left(1+\left(\sum_{\substack{u,j=1 \\ u\neq j}}^{s}\left(\frac{1}{1+\left(g\big/s\Phi_u a_u^\gamma + h\big/s\Phi_j a_j^\gamma\right)^{\frac{1}{\gamma}}}\Bigg/\left(1-\frac{1}{1+\left(g\big/s\Phi_u a_u^\gamma + h\big/s\Phi_j a_j^\gamma\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right],\right.$$
$$1- 1\Bigg/\left(1+\left(\sum_{\substack{u,j=1 \\ u\neq j}}^{s}\left(\frac{1}{1+\left(g\big/s\Phi_u b_u^\gamma + h\big/s\Phi_j b_j^\gamma\right)^{\frac{1}{\gamma}}}\Bigg/\left(1-\frac{1}{1+\left(g\big/s\Phi_u b_u^\gamma + h\big/s\Phi_j b_j^\gamma\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\Bigg]$$

$$\left[\begin{array}{l} 1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(\left(1-1+\frac{1}{1+\left(g/s\Phi_u c_u^{\gamma}+h/s\Phi_j c_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Big/\left(1-\frac{1}{1+\left(g/s\Phi_i c_i^{\gamma}+h/s\Phi_j c_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right), \\ 1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(\left(1-1+\frac{1}{1+\left(g/s\Phi_u d_u^{\gamma}+h/s\Phi_j d_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Big/\left(1-\frac{1}{1+\left(g/s\Phi_u d_u^{\gamma}+h/s\Phi_j d_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right) \end{array}\right]$$

$$\left.\left[\begin{array}{l} 1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{v}\left(\left(1-1+\frac{1}{1+\left(g/s\Phi_u g_u^{\gamma}+h/s\Phi_j g_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Big/\left(1-\frac{1}{1+\left(g/s\Phi_u g_u^{\gamma}+h/s\Phi_j g_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right), \\ 1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(\left(1-1+\frac{1}{1+\left(g/s\Phi_u h_u^{\gamma}+h/s\Phi_j h_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Big/\left(1-\frac{1}{1+\left(g/s\Phi_u h_u^{\gamma}+h/v\Phi_j h_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right) \end{array}\right]\right\rangle,$$

$$=\left\langle\left[1-1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u a_u^{\gamma}}+\frac{h}{s\Phi_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1-1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u b_u^{\gamma}}+\frac{h}{s\Phi_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{x}{s\Phi_u c_u^{\gamma}}+\frac{y}{s\Phi_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),\right.\right.$$

$$\left.\left.1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u d_u^{\gamma}}+\frac{h}{s\Phi_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u g_u^{\gamma}}+\frac{h}{s\Phi_j g_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1\Big/\left(1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u h_u^{\gamma}}+\frac{h}{s\Phi_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right]\right\rangle.$$

So, we can have

$$\frac{1}{s^2-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(s\Phi_u\hbar_u\right)^{x}\otimes_D\left(s\Phi_j\hbar_j\right)^{y}$$

$$=\left\langle\left[\left(1-1\Big/\left(1+\left(\frac{1}{s^2-s}\left(\left(1-1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u a_u^{\gamma}}+\frac{h}{s\Phi_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1-1+1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u a_u^{\gamma}}+\frac{h}{s\Phi_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right),\right.\right.$$

$$\left.\left(1-1\Big/\left(1+\left(\frac{1}{s^2-s}\left(\left(1-1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u b_u^{\gamma}}+\frac{h}{s\Phi_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1-1+1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u b_u^{\gamma}}+\frac{h}{s\Phi_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right)\right],$$

$$\left[\left(1\Big/\left(1+\left(\frac{1}{s^{2}-s}\left(\left(1-1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}c_{u}^{\gamma}}+\frac{h}{s\Phi_{j}c_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}c_{u}^{\gamma}}+\frac{h}{s\Phi_{j}c_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}\right),\right.$$

$$\left.\left(1\Big/\left(1+\left(\frac{1}{s^{2}-s}\left(\left(1-1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}d_{u}^{\gamma}}+\frac{h}{v\Phi_{j}d_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}d_{u}^{\gamma}}+\frac{h}{v\Phi_{j}d_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}\right)\right].$$

$$\left[\left(1\Big/\left(1+\left(\frac{1}{s^{2}-s}\left(\left(1-1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}g_{u}^{\gamma}}+\frac{h}{s\Phi_{j}g_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}g_{u}^{\gamma}}+\frac{h}{s\Phi_{j}g_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}\right),\right.$$

$$\left.\left(1\Big/\left(1+\left(\frac{1}{s^{2}-s}\left(\left(1-1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}h_{u}^{\gamma}}+\frac{h}{s\Phi_{j}h_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}h_{u}^{\gamma}}+\frac{h}{s\Phi_{j}h_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}\right)\right].$$

$$=\left\langle\left[\left(1-1\Big/\left(1+\left(\frac{1}{s^{2}-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}a_{u}^{\gamma}}+\frac{h}{s\Phi_{j}a_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1-1\Big/\left(1+\left(\frac{1}{s^{2}-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}b_{u}^{\gamma}}+\frac{h}{s\Phi_{j}b_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\right.$$

$$\left[\left(1\Big/\left(1+\left(\frac{1}{s^{2}-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}c_{u}^{\gamma}}+\frac{h}{s\Phi_{j}c_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1-1\Big/\left(1+\left(\frac{1}{s^{2}-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}d_{u}^{\gamma}}+\frac{h}{s\Phi_{j}d_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],$$

$$\left.\left[\left(1\Big/\left(1+\left(\frac{1}{s^{2}-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}g_{u}^{\gamma}}+\frac{h}{s\Phi_{j}g_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\Big/\left(1+\left(\frac{1}{s^{2}-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_{u}h_{u}^{\gamma}}+\frac{h}{s\Phi_{j}h_{j}^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\right\rangle.$$

Then,

$$\left(\frac{1}{s^{2}-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(s\Phi_{u}\hbar_{u}\right)^{g}\otimes_{D}\left(s\Phi_{j}\hbar_{j}\right)^{h}\right)^{\frac{1}{g+h}}$$

$$=\left\langle\left[\left[\left(1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u a_u^{\gamma}}+\frac{h}{s\Phi_j a_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u b_u^{\gamma}}+\frac{h}{s\Phi_j b_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\right.$$

$$\left[\left(1-1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u c_i^{\gamma}}+\frac{h}{s\Phi_j c_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1-1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u d_u^{\gamma}}+\frac{h}{s\Phi_j d_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \quad (6.11)$$

$$\left.\left[\left(1-1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u g_u^{\gamma}}+\frac{h}{s\Phi_j g_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\left(1-1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u h_u^{\gamma}}+\frac{h}{s\Phi_j h_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\right\rangle.$$

Now, put

$a_u=\frac{\Xi_u^L}{1-\Xi_u^L}, b_u=\frac{\Xi_u^U}{1-\Xi_u^U}, c_u=\frac{1-\Psi_u^L}{\Psi_u^L}, d_u=\frac{1-\Psi_u^U}{\Psi_u^U}, g_u=\frac{1-\Upsilon_u^L}{\Upsilon_u^L}, h_u=\frac{1-\Upsilon_u^U}{\Upsilon_u^U}, a_j=\frac{\Xi_j^L}{1-\Xi_j^L}, b_j=\frac{\Xi_j^U}{1-\Xi_j^U}, c_j=\frac{1-\Psi_j^L}{\Psi_j^L}, d_j=\frac{1-\Psi_j^U}{\Psi_j^U}, g_j=\frac{1-\Upsilon_j^L}{\Upsilon_j^L}$,

$, h_j=\frac{1-\Upsilon_j^U}{\Upsilon_j^U}.$ in Equation (6.11), We can get

$$=\left\langle\left[\begin{array}{l}1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),\\ 1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^U}{1-\Xi_u^U}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^U}{1-\Xi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\end{array}\right],\right.$$

$$\left[\begin{array}{l}1-1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^L}{\Psi_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Psi_j^L}{\Psi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),\\ 1-1\Big/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Big/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Big/\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^U}{\Psi_u^U}\right)^{\gamma}}+\frac{y}{s\Phi_j\left(\frac{1-\Psi_j^U}{\Psi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\end{array}\right]$$

$$\left.\left[\begin{array}{l} 1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{l} 1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{1-\Upsilon_u^L}{\Upsilon_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Upsilon_j^L}{\Upsilon_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right), \\ 1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s} 1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{1-\Upsilon_u^U}{\Upsilon_u^U}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Upsilon_j^U}{\Upsilon_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right) \end{array}\right]\right\rangle.$$

This is the required proof of the Theorem (6.2.1.2).

In order to determine the PWV Φ, we firstly need to determine the support degree among INNs. In general, the similarity measure among INNs can replace the support degree among INNs. i.e;

$$Sup(\hbar_d,\hbar_l)=1-\overline{\overline{D}}(\hbar_d,\hbar_l)(d,l=1,2,....,s). \tag{6.12}$$

6.2.1.3 Example

Let $\hbar_1=\langle[0.3,0.7],[0.2,0.4],[0.3,0.5]\rangle, \hbar_2=\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$ and $\hbar_3=\langle[0.1,0.3],[0.4,0.6],[0.2,0.4]\rangle$ be any three INNs, $g=1, h=1, \gamma=3$, then, by Theorem (6.2.1.2), Equation (6.10), we can aggregate these three INNs and generate the comprehensive value $\hbar=\langle[\Xi^L,\Xi^U],[\Psi^L,\Psi^U],[\Upsilon^L,\Upsilon^U]\rangle$ which is calculated as follows.

Step 1. Determine the supports $Sup(\hbar_i,\hbar_j), i,j=1,2,3$ by using Eq. (6.12), and then we get

$Sup(\hbar_1,\hbar_2)=Sup(\hbar_2,\hbar_1)=0.9, Sup(\hbar_1,\hbar_3)=Sup(\hbar_3,\hbar_1)=0.933, Sup(\hbar_2,\hbar_3)=Sup(\hbar_3,\hbar_2)=1.$

Step 2. Determine the PWV $T(\hbar_z)=\sum_{s=1,s\neq z}^{3} Sup(\hbar_z,\hbar_s),$, and we have

$T(\hbar_1)=Sup(\hbar_1,\hbar_2)+Sup(\hbar_1,\hbar_3)=1.833, T(\hbar_2)=Sup(\hbar_2,\hbar_1)+Sup(\hbar_2,\hbar_3)=1.9,$

$T(\hbar_3)=Sup(\hbar_3,\hbar_1)+Sup(\hbar_3,\hbar_2)=1.933,$

and

$$\Phi_1=\frac{\left(T\left(\hbar_1\right)+1\right)}{\left(T\left(\hbar_1\right)+1\right)+\left(T\left(\hbar_2\right)+1\right)+\left(T\left(\hbar_3\right)+1\right)}=0.3269,\ \Phi_2=\frac{\left(T\left(\hbar_2\right)+1\right)}{\left(T\left(\hbar_1\right)+1\right)+\left(T\left(\hbar_2\right)+1\right)+\left(T\left(\hbar_3\right)+1\right)}=0.3346,$$

$$\Phi_3=\frac{\left(T\left(\hbar_3\right)+1\right)}{\left(T\left(\hbar_1\right)+1\right)+\left(T\left(\hbar_2\right)+1\right)+\left(T\left(\hbar_3\right)+1\right)}=0.3385.$$

Step 3. Determine the comprehensive value $\hbar=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle$ by using Eq. (6.10), we have

$$\sum_{\substack{i,j=1\\i\neq j}}^{3} 1\Big/\left(\frac{x}{3\Phi_i\left(\frac{\Xi_i^L}{1-\Xi_i^L}\right)^3}+\frac{y}{3\Phi_j\left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^3}\right)=1\Big/\left(\frac{1}{3\Phi_1\left(\frac{\Xi_1^L}{1-\Xi_1^L}\right)^3}+\frac{2}{3\Phi_2\left(\frac{\Xi_2^L}{1-\Xi_2^L}\right)^3}\right)+1\Big/\left(\frac{1}{3\Phi_1\left(\frac{\Xi_1^L}{1-\Xi_1^L}\right)^3}+\frac{2}{3\Phi_3\left(\frac{\Xi_3^L}{1-\Xi_3^L}\right)^3}\right)+1\Big/\left(\frac{1}{3\Phi_2\left(\frac{\Xi_2^L}{1-\Xi_2^L}\right)^3}+\frac{2}{3\Phi_1\left(\frac{\Xi_1^L}{1-\Xi_1^L}\right)^3}\right)$$

$$+1\Big/\left(\frac{1}{3\Phi_2\left(\frac{\Xi_2^L}{1-\Xi_2^L}\right)^3}+\frac{2}{3\Phi_3\left(\frac{\Xi_3^L}{1-\Xi_3^L}\right)^3}\right)+1\Big/\left(\frac{1}{3\Phi_3\left(\frac{\Xi_3^L}{1-\Xi_3^L}\right)^3}+\frac{2}{3\Phi_1\left(\frac{\Xi_1^L}{1-\Xi_1^L}\right)^3}\right)+1\Big/\left(\frac{1}{3\Phi_3\left(\frac{\Xi_3^L}{1-\Xi_3^L}\right)^3}+\frac{2}{3\Phi_2\left(\frac{\Xi_2^L}{1-\Xi_2^L}\right)^3}\right)=0.1281$$

$$1\Bigg/\left(1+\left(\frac{3^2-3}{1+1}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{3}1\Bigg/\left(\frac{1}{3\Phi_i\left(\frac{\Xi_i^L}{1-\Xi_i^L}\right)^{\gamma}}+\frac{1}{3\Phi_j\left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{3}}\right)=0.2590$$

Similarly, we calculate

$$\hbar=\left\langle[0.2590,0.5525],[0.2221,0.4373],[0.2334,0.4365]\right\rangle.$$

6.2.1.4 Theorem (Idempotency)

Let $\hbar_u=\left\langle\left[\Xi_u^L,\Xi_u^U\right],\left[\Psi_u^L,\Psi_u^U\right],\left[\Upsilon_u^L,\Upsilon_u^L\right]\right\rangle,(u=1,2,...,s)$, be a group of INNs, if all $\hbar_u\left(u=1,2,...,s\right)$ are equal, that is $\hbar_u=\hbar=\left\langle\left[\Xi^L,\Xi^U\right],\left[\Psi^L,\Psi^U\right],\left[\Upsilon^L,\Upsilon^U\right]\right\rangle,\left(u=1,2,...,s\right)$, then

$$INDPBM^{g,h}\left(\hbar_1,\hbar_2,...,\hbar_s\right)=\hbar\ . \tag{6.13}$$

Proof. Since all $\hbar_u = \hbar = \left\langle \left[\Xi^L, \Xi^U\right], \left[\Psi^L, \Psi^U\right], \left[\Upsilon^L, \Upsilon^U\right] \right\rangle, (u = 1, 2, \ldots, s),$ so we have

$Sup(\hbar_d, \hbar_a) = 1,$ for all $d, a = 1, 2, \ldots, s$, so $\Phi_d = \dfrac{1}{s}$, for all $d = 1, 2, \ldots, s.$ Then

$$INDPBM^{g,h}(\hbar_1, \hbar_2, \ldots, \hbar_s) = INDPBM^{g,h}(\hbar, \hbar, \ldots, \hbar)$$

$$= \left\langle \left[\left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g}{s \frac{1}{s}\left(\frac{\Xi^L}{1-\Xi^L}\right)^{\gamma}} + \frac{h}{s \frac{1}{s}\left(\frac{\Xi^L}{1-\Xi^L}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right), \left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g}{s \frac{1}{s}\left(\frac{\Xi^U}{1-\Xi^U}\right)^{\gamma}} + \frac{h}{s \frac{1}{s}\left(\frac{\Xi^U}{1-\Xi^U}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \right], \right.$$

$$\left[\left(1 - 1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g}{s \frac{1}{s}\left(\frac{1-\Psi^L}{\Psi^L}\right)^{\gamma}} + \frac{h}{s \frac{1}{s}\left(\frac{1-\Psi^L}{\Psi^L}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right), \left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g}{s \frac{1}{s}\left(\frac{1-\Psi^U}{\Psi^U}\right)^{\gamma}} + \frac{h}{s \frac{1}{s}\left(\frac{1-\Psi^U}{\Psi^U}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \right],$$

$$\left. \left[\left(1 - 1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g}{s \frac{1}{s}\left(\frac{1-\Upsilon^L}{\Upsilon^L}\right)^{\gamma}} + \frac{h}{s \frac{1}{s}\left(\frac{1-\Upsilon^L}{\Upsilon^L}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right), \left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g}{s \frac{1}{s}\left(\frac{1-\Upsilon^U}{\Upsilon^U}\right)^{\gamma}} + \frac{h}{s \frac{1}{s}\left(\frac{1-\Upsilon^U}{\Upsilon^U}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \right] \right\rangle,$$

$$= \left\langle \left[\left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g+h}{\left(\frac{\Xi^L}{1-\Xi^L}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right), \left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g+h}{\left(\frac{\Xi^U}{1-\Xi^U}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \right], \right.$$

$$\left[\left(1 - 1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g+h}{\left(\frac{1-\Psi^L}{\Psi^L}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right), \left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g+h}{\left(\frac{1-\Psi^U}{\Psi^U}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \right],$$

$$\left. \left[\left(1 - 1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g+h}{\left(\frac{1-\Upsilon^L}{\Upsilon^L}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right), \left(1 \Bigg/ \left(1 + \left(\frac{s^2 - s}{g+h} \times 1 \Bigg/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Bigg/ \left(\frac{g+h}{\left(\frac{1-\Upsilon^U}{\Upsilon^U}\right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right) \right] \right\rangle,$$

$$= \left\langle \left[\left(1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(1 \Big/ \left(\frac{g+h}{\left(\frac{\Xi^L}{1-\Xi^L}\right)^{\gamma}} \right)\right)\right)\right)^{\frac{1}{\gamma}} \right), \left(1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(1 \Big/ \left(\frac{g+h}{\left(\frac{\Xi^U}{1-\Xi^U}\right)^{\gamma}} \right)\right)\right)\right)^{\frac{1}{\gamma}} \right) \right], \right.$$

$$\left[\left(1 - 1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(1 \Big/ \left(\frac{g+h}{\left(\frac{1-\Psi^L}{\Psi^L}\right)^{\gamma}} \right)\right)\right)\right)^{\frac{1}{\gamma}} \right), \left(1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(1 \Big/ \left(\frac{x+y}{\left(\frac{1-\Psi^U}{\Psi^U}\right)^{\gamma}} \right)\right)\right)\right)^{\frac{1}{\gamma}} \right) \right],$$

$$\left. \left[\left(1 - 1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(1 \Big/ \left(\frac{g+h}{\left(\frac{1-\Upsilon^L}{\Upsilon^L}\right)^{\gamma}} \right)\right)\right)\right)^{\frac{1}{\gamma}} \right), \left(1 \Big/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \Big/ \left(1 \Big/ \left(\frac{g+h}{\left(\frac{1-\Upsilon^U}{\Upsilon^U}\right)^{\gamma}} \right)\right)\right)\right)^{\frac{1}{\gamma}} \right) \right] \right\rangle,$$

$$\left\langle \left[\Xi^L, \Xi^U\right], \left[\Psi^L, \Psi^U\right], \left[\Upsilon^L, \Upsilon^U\right] \right\rangle = \hbar. .$$

6.2.1.5 Theorem (Commutativity)

Assume that $\hbar'_u$ is any permutation of $\hbar_u \left(u = 1, 2, ..., s\right)$, then

$$INDPBM^{g,h}\left(\hbar'_1, \hbar'_2, ..., \hbar'_s\right) = INDPBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right). \quad (6.14)$$

Proof. From Definition (6.2.1.1), we have

$$INDPBM^{x,y}\left(\hbar'_1, \hbar'_2, ..., \hbar'_s\right) = \left(\frac{1}{s^2-s} \sum_{\substack{u,j=1 \\ u \neq j}}^{s} \left(s\Phi'_u \hbar'_u\right)^g \otimes_D \left(s\Phi'_j \hbar'_j\right)^h \right)^{\frac{1}{g+h}},$$

and

$$INDPBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right) = \left(\frac{1}{s^2-s} \sum_{\substack{u,j=1 \\ u \neq j}}^{s} \left(s\Phi_u \hbar_u\right)^g \otimes_D \left(s\Phi_j \hbar_j\right)^h \right)^{\frac{1}{g+h}}.$$

Because, $\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(s\Phi'_u\hbar'_u\right)^g\otimes_D\left(s\Phi'_j\hbar'_j\right)^h=\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(s\Phi_u\hbar_u\right)^g\otimes_D\left(s\Phi_j\hbar_j\right)^h$.

Hence, $INDPBM^{g,h}\left(\hbar'_1,\hbar'_2,...,\hbar'_s\right)=INDPBM^{g,h}\left(\hbar_1,\hbar_2,...,\hbar_s\right)$.

6.2.1.6 Theorem (Boundedness)

Let $\hbar_u=\left\langle\left[\Xi_u^L,\Xi_u^U\right],\left[\Psi_u^L,\Psi_u^U\right],\left[\Upsilon_u^L,\Upsilon_u^L\right]\right\rangle,(u=1,2,...,s)$, be a group of INNs, and $n^+=\left\langle\max_{u=1}^{s}\left[\Xi_u^L,\Xi_u^U\right],\min_{u=1}^{s}\left[\Psi_u^L,\Psi_u^U\right],\min_{u=1}^{s}\left[\Upsilon_u^L,\Upsilon_u^U\right]\right\rangle,n^-=\left\langle\min_{u=1}^{s}\left[\Xi_u^L,\Xi_u^U\right],\max_{u=1}^{s}\left[\Psi_u^L,\Psi_u^U\right],\max_{u=1}^{s}\left[\Upsilon_u^L,\Upsilon_u^U\right]\right\rangle$, then

$$\hbar^-\leq INDPBM\left(\hbar_1,\hbar_2,...,\hbar_s\right)\leq\hbar^+. \tag{6.15}$$

Proof. Since $\hbar^+=\left\langle\max_{u=1}^{s}\left[\Xi_u^L,\Xi_u^U\right],\min_{u=1}^{s}\left[\Psi_u^L,\Psi_u^U\right],\min_{u=1}^{s}\left[\Upsilon_u^L,\Upsilon_u^U\right]\right\rangle,\hbar^-=\left\langle\min_{u=1}^{s}\left[\Xi_u^L,\Xi_u^U\right],\max_{u=1}^{s}\left[\Psi_u^L,\Psi_u^U\right],\max_{u=1}^{s}\left[\Upsilon_u^L,\Upsilon_u^U\right]\right\rangle$.

Then, there are

$\Xi^{L-}\leq\Xi^L(n_i)\leq\Xi^{L+},\Xi^{U-}\leq\Xi^L(n_i)\leq\Xi^{U+},\Psi^{L-}\leq\Psi^L(n_i)\leq\Psi^{L+},\Psi^{U-}\leq\Psi^U(n_i)\leq\Psi^{U+},\Upsilon^{L-}\leq\Upsilon^L(n_i)\leq\Upsilon^{L+}$, $\Upsilon^{U-}\leq\Upsilon^U(n_i)\leq\Upsilon^{U+}$, for all $u=1,2,...,s.$ We have

$$\Xi^L(\hbar_u)=\left(1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^\gamma}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^\gamma}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\geq$$

$$\left(1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^{L-}}{1-\Xi_u^{L-}}\right)^\gamma}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^{L-}}{1-\Xi_j^{L-}}\right)^\gamma}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\Xi^{L-},$$

$$\Xi^{U}(\hbar_u)=\left(1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_i\left(\frac{\Xi_u^{U}}{1-\Xi_u^{U}}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^{U}}{1-\Xi_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\geq$$

$$\left(1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^{U-}}{1-\Xi_u^{U-}}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^{U-}}{1-\Xi_j^{U-}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\Xi^{U-},$$

$$\Psi^{L}(\hbar_u)=\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_i\left(\frac{1-\Psi_u^{L}}{\Psi_u^{L}}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Psi_j^{L}}{\Psi_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\leq$$

$$\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{x}{s\Phi_i\left(\frac{1-\Psi_u^{L+}}{\Psi_u^{L+}}\right)^{\gamma}}+\frac{y}{s\Phi_j\left(\frac{1-\Psi_j^{L+}}{\Psi_j^{L+}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\Psi^{L+}$$

$$\Psi^{U}(\hbar_u)=\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_i\left(\frac{1-\Psi_u^{U}}{\Psi_u^{U}}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Psi_j^{U}}{\Psi_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\leq$$

$$\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{x}{s\Phi_i\left(\frac{1-\Psi_u^{U+}}{\Psi_u^{U+}}\right)^{\gamma}}+\frac{y}{s\Phi_j\left(\frac{1-\Psi_j^{U+}}{\Psi_j^{U+}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\Psi^{U+}$$

$$\Upsilon^{L}(\hbar_u)=\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_i\left(\frac{1-\Upsilon_u^{L}}{\Upsilon_u^{L}}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Upsilon_j^{L}}{\Upsilon_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\leq$$

$$\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{x}{s\Phi_i\left(\frac{1-\Upsilon_u^{L+}}{\Upsilon_u^{L+}}\right)^{\gamma}}+\frac{y}{s\Phi_j\left(\frac{1-\Upsilon_j^{L+}}{\Upsilon_j^{L+}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\Upsilon^{L+}$$

$$\Upsilon^{U}(\hbar_u)=\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_i\left(\frac{1-\Upsilon_u^{U}}{\Upsilon_u^{U}}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Upsilon_j^{U}}{\Upsilon_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\leq$$

$$\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{x}{s\Phi_i\left(\frac{1-\Upsilon_u^{U+}}{\Upsilon_u^{U+}}\right)^{\gamma}}+\frac{y}{s\Phi_j\left(\frac{1-\Upsilon_j^{U+}}{\Upsilon_j^{U+}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\Upsilon^{U+}$$

Then, there are the following scores,

$$\frac{\Xi^L+\Xi^U}{2}+1-\frac{\Psi^L+\Psi^U}{2}+1-\frac{\Upsilon^L+\Upsilon^U}{2}=$$

$$\left(\left(1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)+1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^U}{1-\Xi_u^U}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^U}{1-\Xi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Bigg/2\right.$$

$$+1-\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^L}{\Psi_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Psi_j^L}{\Psi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)+1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^U}{\Psi_u^U}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Psi_j^U}{\Psi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Bigg/2$$

$$\left.+1-\left(1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{1-\Upsilon_u^L}{\Upsilon_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Upsilon_j^L}{\Upsilon_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)+1-1\Bigg/\left(1+\left(\frac{s^2-s}{g+h}\times 1\Bigg/\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}1\Bigg/\left(\frac{g}{s\Phi_u\left(\frac{1-\Upsilon_u^U}{\Upsilon_u^U}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Upsilon_j^U}{\Upsilon_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Bigg/2\right)$$

Therefore according to the Definition (1.1.1.12), we have

$$\hbar^- \leq INDPBM\left(\hbar_1,\hbar_2,...,\hbar_s\right).$$

In a similar way , we can show that $INDPBM\left(\hbar_1,\hbar_2,...,\hbar_s\right)\leq \hbar^+$. Hence $\hbar^- \leq INDPBM\left(\hbar_1,\hbar_2,...,\hbar_s\right)\leq \hbar^+$.

Now, we shall study few special cases of the $INDPBM^{g,h}$, with respect to g and h.

(1) When $h\to 0,\gamma>0,$ then we can get

$$INDPBM^{g,0}\left(\hbar_1,\hbar_2,....,\hbar_s\right)$$

$$= \lim_{h\to 0}\left\langle \left[\left(1\Big/\left(1+\left(\frac{s^2-s}{g}\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right), \left(1\Big/\left(1+\left(\frac{s^2-s}{g}\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{g}{s\Phi_i\left(\frac{\Xi_u^U}{1-\Xi_u^U}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\right.$$

$$\left[\left(1- 1\Big/\left(1+\left(\frac{s^2-s}{g}\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^L}{\Psi_u^L}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right), \left(1\Big/\left(1+\left(\frac{s^2-s}{g}\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^U}{\Psi_u^U}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left.\left[\left(1- 1\Big/\left(1+\left(\frac{s^2-s}{g}\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{g}{s\Phi_u\left(\frac{1-\Upsilon_u^L}{\Upsilon_u^L}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right), \left(1\Big/\left(1+\left(\frac{s^2-s}{g}\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{g}{s\Phi_i\left(\frac{1-\Upsilon_i^U}{\Upsilon_i^U}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right]\right\rangle.$$

(6.16)

(2) When $g=1, h\to 0, \gamma>0,$ then we can get

$INDPBM^{1,0}\left(\hbar_1,\hbar_2,....,\hbar_s\right)$

$$= \lim_{y\to 0}\left\langle \left[\left(1\Big/\left(1+\left(s^2-s\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{1}{s\Phi_i\left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right), \left(1\Big/\left(1+\left(s^2-s\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{1}{s\Phi_i\left(\frac{\Xi_i^U}{1-\Xi_i^U}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\right.$$

$$\left[\left(1- 1\Big/\left(1+\left(s^2-s\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{1}{s\Phi_u\left(\frac{1-\Psi_u^L}{\Psi_u^L}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right), \left(1\Big/\left(1+\left(s^2-s\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{1}{s\Phi_u\left(\frac{1-\Psi_u^U}{\Psi_u^U}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right], \quad (6.17)$$

$$\left.\left[\left(1- 1\Big/\left(1+\left(s^2-s\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{1}{s\Phi_u\left(\frac{1-\Upsilon_u^L}{\Upsilon_u^L}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right), \left(1\Big/\left(1+\left(s^2-s\times 1\Big/\left(\sum_{\substack{u,j=1\\ u\neq j}}^{s} 1\Big/\left(\frac{1}{s\Phi_i\left(\frac{1-\Upsilon_u^U}{\Upsilon_u^U}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right]\right\rangle.$$

(3) When $g=h=1, \gamma>0,$ then we can get

$INDPBM^{1,1}\left(\hbar_1,\hbar_2,....,\hbar_s\right)$

$$
=\left\langle\left[\frac{1}{\left(1+\left(\frac{s^2-s}{2}\times\frac{1}{\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\frac{1}{\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^L}{1-\Xi_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^L}{1-\Xi_j^L}\right)^{\gamma}}\right)}\right)}\right)^{\frac{1}{\gamma}}\right)},\ \frac{1}{\left(1+\left(\frac{s^2-s}{2}\times\frac{1}{\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\frac{1}{\left(\frac{g}{s\Phi_u\left(\frac{\Xi_u^U}{1-\Xi_u^U}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{\Xi_j^U}{1-\Xi_j^U}\right)^{\gamma}}\right)}\right)}\right)^{\frac{1}{\gamma}}\right)}\right],\right.
$$

$$
\left[1-\frac{1}{\left(1+\left(\frac{s^2-s}{2}\times\frac{1}{\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\frac{1}{\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^L}{\Psi_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Psi_j^L}{\Psi_j^L}\right)^{\gamma}}\right)}\right)}\right)^{\frac{1}{\gamma}}\right)},1-\frac{1}{\left(1+\left(\frac{s^2-s}{2}\times\frac{1}{\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\frac{1}{\left(\frac{g}{s\Phi_u\left(\frac{1-\Psi_u^U}{\Psi_u^U}\right)^{\gamma}}+\frac{h}{l\Lambda_j\left(\frac{1-\Psi_j^U}{\Psi_j^U}\right)^{\gamma}}\right)}\right)}\right)^{\frac{1}{\gamma}}\right)}\right], \quad (6.18)
$$

$$
\left.\left[1-\frac{1}{\left(1+\left(\frac{s^2-s}{2}\times\frac{1}{\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\frac{1}{\left(\frac{g}{s\Phi_u\left(\frac{1-\Upsilon_u^L}{\Upsilon_u^L}\right)^{\gamma}}+\frac{h}{s\Phi_j\left(\frac{1-\Upsilon_j^L}{\Upsilon_j^L}\right)^{\gamma}}\right)}\right)}\right)^{\frac{1}{\gamma}}\right)},1-\frac{1}{\left(1+\left(\frac{s^2-s}{2}\times\frac{1}{\left(\sum_{\substack{u,j=1\\u\neq j}}^{s}\frac{1}{\left(\frac{g}{s\Phi_u\left(\frac{1-\Upsilon_u^U}{\Upsilon_u^U}\right)^{\gamma}}+\frac{h}{l\Lambda_i\left(\frac{1-\Upsilon_j^U}{\Upsilon_j^U}\right)^{\gamma}}\right)}\right)}\right)^{\frac{1}{\gamma}}\right)}\right]\right\rangle.
$$

In the INDPBM operator, we can only take the correlation among the input arguments and the PWV, and cannot consider the importance degree of input arguments. In what follows, the INWPDBM operator shall be proposed to overcome the shortcoming of the INDPBM operator.

6.2.1.7 Definition

Let $\hbar_u = \left\langle\left[\Xi_u^L,\Xi_u^U\right],\left[\Psi_u^L,\Psi_u^U\right],\left[\Upsilon_u^L,\Upsilon_u^L\right]\right\rangle,(u=1,2,...,s)$, be a group of INNs, then the INWDPBM operator is described as

$$
INWDPBM^{g,h}\left(\hbar_1,\hbar_2,...,\hbar_s\right)=\left(\frac{1}{s^2-s}\sum_{\substack{u,j=1\\u\neq j}}^{s}\left(\left(\frac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z+1)\right)}\hbar_u\right)^{g}\otimes_D\left(\frac{sw_j\left(T(\hbar_j)+1\right)}{\sum_{z=1}^{g}w_z\left(T(\hbar_z+1)\right)}\hbar_j\right)^{h}\right)\right)^{\frac{1}{g+h}}. \quad (6.19)
$$

Where $T\left(\hbar_i\right)=\sum_{j=1,i\neq j}^{l}Sup\left(\hbar_i,\hbar_j\right),x,y>0,w=\left(w_1,w_2,....,w_l\right)^T$ is the importance degree of the INNs, such that $0\leq w_z\leq 1\left(z=1,2,...,l\right)$ and $\sum_{z=1}^{l}w_k=1.$

6.2.1.8 Theorem

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U \right], \left[\Psi_u^L, \Psi_u^U \right], \left[\Upsilon_u^L, \Upsilon_u^L \right] \right\rangle, (u = 1, 2, ..., s)$, be a group of INNs, then the result obtained using Definition (6.2.1.7), is represented by

$INDPBM^{x,y}\left(\hbar_1, \hbar_2, ..., \hbar_s\right)$

$$
= \left\langle \begin{array}{l}
\left[\begin{array}{l}
\left(1 \Big/ \left(1 + \left(\frac{s^2 - s}{g+h} \times \right. 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{\frac{s w_u \left(T(\hbar_u) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{\Xi_u^L}{1 - \Xi_u^L} \right)^{\gamma}} + \frac{h}{\frac{s w_j \left(T(\hbar_j) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{\Xi_j^L}{1 - \Xi_j^L} \right)^{\gamma}} \right) \right) \right) \right)^{\frac{1}{\gamma}} \right), \\
\left(1 \Big/ \left(1 + \left(\frac{s^2 - s}{g+h} \times \right. 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{\frac{s w_u \left(T(\hbar_u) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{\Xi_u^U}{1 - \Xi_u^U} \right)^{\gamma}} + \frac{h}{\frac{s w_j \left(T(\hbar_i) + 1 \right)}{\sum_{z=1}^{g} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{\Xi_j^U}{1 - \Xi_j^U} \right)^{\gamma}} \right) \right) \right) \right)^{\frac{1}{\gamma}} \right)
\end{array} \right], \\
\left[\begin{array}{l}
\left(1 - 1 \Big/ \left(1 + \left(\frac{s^2 - s}{g+h} \times \right. 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{\frac{s w_u \left(T(\hbar_u) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Psi_u^L}{\Psi_u^L} \right)^{\gamma}} + \frac{h}{\frac{s w_j \left(T(\hbar_i) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Psi_j^L}{\Psi_j^L} \right)^{\gamma}} \right) \right) \right) \right)^{\frac{1}{\gamma}} \right), \\
\left(1 \Big/ \left(1 + \left(\frac{s^2 - s}{g+h} \times \right. 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{\frac{s w_u \left(T(\hbar_u) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Psi_u^U}{\Psi_u^U} \right)^{\gamma}} + \frac{h}{\frac{s w_j \left(T(\hbar_i) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Psi_j^U}{\Psi_j^U} \right)^{\gamma}} \right) \right) \right) \right)^{\frac{1}{\gamma}} \right)
\end{array} \right], \\
\left[\begin{array}{l}
\left(1 - 1 \Big/ \left(1 + \left(\frac{s^2 - s}{g+h} \times \right. 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{\frac{s w_u \left(T(\hbar_u) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Upsilon_u^L}{\Upsilon_u^L} \right)^{\gamma}} + \frac{h}{\frac{s w_j \left(T(\hbar_i) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Upsilon_j^L}{\Upsilon_j^L} \right)^{\gamma}} \right) \right) \right) \right)^{\frac{1}{\gamma}} \right), \\
\left(1 \Big/ \left(1 + \left(\frac{s^2 - s}{g+h} \times \right. 1 \Big/ \left(\sum_{\substack{u,j=1 \\ u \neq j}}^{s} 1 \Big/ \left(\frac{g}{\frac{s w_u \left(T(\hbar_u) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Upsilon_u^U}{\Upsilon_u^U} \right)^{\gamma}} + \frac{h}{\frac{s w_j \left(T(\hbar_i) + 1 \right)}{\sum_{z=1}^{s} w_z \left(T(\hbar_z + 1) \right)} \left(\frac{1 - \Upsilon_j^U}{\Upsilon_j^U} \right)^{\gamma}} \right) \right) \right) \right)^{\frac{1}{\gamma}} \right)
\end{array} \right]
\end{array} \right\rangle. \qquad (6.20)
$$

Proof: Similar to Theorem (6.2.1.2).

Similar to the INDPBM operator, the INWDPBM operator has the properties of boundedness, idempotency and commutativity.

6.2.2 The INDPGBM Operator and INWDPGBM Operator

In this subsection, we develop INDPGBM and INWDPGBM operators and discussed related properties and special cases with respect to the parameter.

6.2.2.1 Definition

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right] \right\rangle, (u = 1, 2, ..., s)$, be a group of INNs, then the INDPGBM operator is defined as;

$$INDPGBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right) = \frac{1}{g+h}\left(\prod_{\substack{u,j=1 \\ u \neq j}}^{s}\left(g\hbar_u^{\frac{s(T(\hbar_u)+1)}{\sum_{z=1}^{s}(T(\hbar_z)+1)}} + h\hbar_j^{\frac{s(T(\hbar_i)+1)}{\sum_{z=1}^{s}(T(\hbar_z)+1)}}\right)\right)^{\frac{1}{s^2-s}} \tag{6.21}$$

Then, $INDPGBM^{g,h}$ is said to be an INDPGBM operator. Where $T(\hbar_z) = \sum_{s=1, s \neq z}^{l} Sup(\hbar_z, \hbar_s)$, $Sup(\hbar_z, \hbar_s)$ is the support degree for $\hbar_z$ from $\hbar_s$, which satisfy the following axioms: (1) $Sup(\hbar_z, \hbar_s) \in [0,1]$; (2) $Sup(\hbar_z, \hbar_s) = Sup(\hbar_s, \hbar_z)$; (3) $Sup(\hbar_z, \hbar_s) \geq Sup(\hbar_a, \hbar_b)$, if $\overline{\overline{D}}(\hbar_z, \hbar_s) < \overline{\overline{D}}(\hbar_a, \hbar_b)$, in which $\overline{\overline{D}}(\hbar_a, \hbar_b)$ is the distance measure between INNs $\hbar_a$ and $\hbar_b$ defined in Definition (1.1.1.13).

In order to simplify Equation (6.21), we can define

$$\Phi_z = \frac{\left(1+T\left(\hbar_z\right)\right)}{\sum_{z=1}^{l}\left(1+T\left(\hbar_z\right)\right)}, \tag{6.22}$$

and call $\Phi = \left(\Phi_1, \Phi_2, ..., \Phi_s\right)^T$ is the power weight vector, such that $\Phi_z \geq 0, \sum_{z=1}^{s} \Phi_z = 1.$

Then, Equation (6.21) can be written as follows:

$$INDPGBM^{x,y}\left(\hbar_1, \hbar_2, ..., \hbar_s\right) = \frac{1}{g+h}\left(\prod_{\substack{u,j=1\\u\neq j}}^{s}\left(g\hbar_u^{s\Phi_i} + h\hbar_j^{s\Phi_j}\right)\right)^{\frac{1}{s^2-s}}. \tag{6.23}$$

6.2.2.2 Theorem

Let $\hbar_u = \left\langle\left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right]\right\rangle, (u = 1, 2, ..., s)$, be a group of INNs. Then, the result obtained from Eq. (6.21), is expressed as

$$INDPGBM^{x,y}\left(\hbar_1, \hbar_2, ..., \hbar_s\right)$$

$$= \left\langle \left[\left(1 - 1 \middle/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \middle/ \left(\sum_{\substack{u,j=1\\u\neq j}}^{s} 1 \middle/ \left(\frac{g}{s\Phi_u\left(\frac{1-\Xi_u^L}{\Xi_u^L}\right)^{\gamma}} + \frac{h}{s\Phi_j\left(\frac{1-\Xi_j^L}{\Xi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1 \middle/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \middle/ \left(\sum_{\substack{u,j=1\\u\neq j}}^{s} 1 \middle/ \left(\frac{g}{s\Phi_u\left(\frac{1-\Xi_u^U}{\Xi_u^U}\right)^{\gamma}} + \frac{h}{s\Phi_j\left(\frac{1-\Xi_j^U}{\Xi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\right.$$

$$\left[\left(1 \middle/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \middle/ \left(\sum_{\substack{u,j=1\\u\neq j}}^{s} 1 \middle/ \left(\frac{g}{s\Phi_u\left(\frac{\Psi_u^L}{1-\Psi_u^L}\right)^{\gamma}} + \frac{h}{s\Phi_j\left(\frac{\Psi_j^L}{1-\Psi_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1 \middle/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \middle/ \left(\sum_{\substack{u,j=1\\u\neq j}}^{s} 1 \middle/ \left(\frac{g}{s\Phi_u\left(\frac{\Psi_u^U}{1-\Psi_u^U}\right)^{\gamma}} + \frac{h}{s\Phi_j\left(\frac{\Psi_j^U}{1-\Psi_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \tag{6.25}$$

$$\left. \left[\left(1 \middle/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \middle/ \left(\sum_{\substack{u,j=1\\u\neq j}}^{s} 1 \middle/ \left(\frac{g}{s\Phi_u\left(\frac{\Upsilon_u^L}{1-\Upsilon_u^L}\right)^{\gamma}} + \frac{h}{s\Phi_j\left(\frac{\Upsilon_j^L}{1-\Upsilon_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1 \middle/ \left(1 + \left(\frac{s^2-s}{g+h} \times 1 \middle/ \left(\sum_{\substack{u,j=1\\u\neq j}}^{s} 1 \middle/ \left(\frac{g}{s\Phi_u\left(\frac{\Upsilon_u^U}{1-\Upsilon_u^U}\right)^{\gamma}} + \frac{h}{s\Phi_i\left(\frac{\Upsilon_j^U}{1-\Upsilon_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\right\rangle.$$

6.2.2.3 Theorem (Idempotency)

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right] \right\rangle, (u = 1, 2, ..., s)$, be a group of INNs, if all $\hbar_i \left(i = 1, 2, ..., l\right)$ are equal, that is $\hbar_u = \hbar = \left\langle \left[\Xi^L, \Xi^U\right], \left[\Psi^L, \Psi^U\right], \left[\Upsilon^L, \Upsilon^U\right] \right\rangle, \left(u = 1, 2, ..., s\right)$, then

$$INDPGBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right) = \hbar.. \tag{6.26}$$

6.2.2.4 Theorem (Commutativity)

Assume that $\hbar'_u$ is any permutation of $\hbar_u \left(u = 1, 2, ..., s\right)$, then

$$INDPGBM^{g,h}\left(\hbar'_1, \hbar'_2, ..., \hbar'_s\right) = INDPGBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right). \tag{6.27}$$

6.2.2.5 Theorem (Boundedness)

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right] \right\rangle, (u = 1, 2, ..., s)$, be a group of INNs, and $\hbar^+ = \left\langle \max_{u=1}^{s}\left[\Xi_u^L, \Xi_u^U\right], \min_{u=1}^{s}\left[\Psi_u^L, \Psi_u^U\right], \min_{u=1}^{s}\left[\Upsilon_u^L, \Upsilon_u^U\right] \right\rangle, \hbar^- = \left\langle \min_{u=1}^{s}\left[\Xi_u^L, \Xi_u^U\right], \max_{u=1}^{s}\left[\Psi_u^L, \Psi_u^U\right], \max_{u=1}^{s}\left[\Upsilon_u^L, \Upsilon_u^U\right] \right\rangle$, then

$$\hbar^- \le INDPGBM\left(\hbar_1, \hbar_2, ..., \hbar_s\right) \le \hbar^+. \tag{6.28}$$

6.2.2.6 Definition

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right] \right\rangle, (u = 1, 2, ..., s)$, be a group of INNs, then the INWDPGBM operator is defined as

$$INWDPGBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right) = \frac{1}{g+h}\left(\prod_{\substack{u,j=1\\u\ne j}}^{s}\left(g\hbar_u^{\frac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s} w_z\left(T(\hbar_z)+1\right)}} + h\hbar_j^{\frac{sw_j\left(T(\hbar_i)+1\right)}{\sum_{z=1}^{s} w_z\left(T(\hbar_z)+1\right)}}\right)\right)^{\frac{1}{s^2-s}}. \tag{6.29}$$

6.2.2.7 Theorem

Let $\hbar_u = \left\langle \left[\Xi_u^L, \Xi_u^U\right], \left[\Psi_u^L, \Psi_u^U\right], \left[\Upsilon_u^L, \Upsilon_u^L\right]\right\rangle, (u = 1, 2, ..., s)$, be a group of INNs. Then the aggregated result from Eq. (6.29), is expressed as;

$INWDPGBM^{g,h}\left(\hbar_1, \hbar_2, ..., \hbar_s\right)$

$$
= \left\langle \begin{bmatrix} \left(1 - \dfrac{1}{1+\left(\dfrac{s^2-s}{g+h}\times\dfrac{1}{\sum\limits_{\substack{u,j=1\\u\neq j}}^{s}\dfrac{1}{\dfrac{g}{\dfrac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{1-\Xi_u^L}{\Xi_u^L}\right)^{\gamma}}+\dfrac{h}{\dfrac{sw_j\left(T(\hbar_j)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{1-\Xi_j^L}{\Xi_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right), \\ \left(\dfrac{1}{1+\left(\dfrac{s^2-s}{g+h}\times\dfrac{1}{\sum\limits_{\substack{i,j=1\\i\neq j}}^{l}\dfrac{1}{\dfrac{g}{\dfrac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{1-\Xi_u^U}{\Xi_u^U}\right)^{\gamma}}+\dfrac{h}{\dfrac{swj\left(T(\hbar_j)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{1-\Xi_j^U}{\Xi_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right) \end{bmatrix},\right.
$$

$$
\begin{bmatrix} \left(\dfrac{1}{1+\left(\dfrac{s^2-s}{g+h}\times\dfrac{1}{\sum\limits_{\substack{u,j=1\\u\neq j}}^{s}\dfrac{1}{\dfrac{g}{\dfrac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Psi_u^L}{1-\Psi_u^L}\right)^{\gamma}}+\dfrac{h}{\dfrac{sw_j\left(T(\hbar_j)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Psi_j^L}{1-\Psi_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right), \\ \left(\dfrac{1}{1+\left(\dfrac{s^2-s}{g+h}\times\dfrac{1}{\sum\limits_{\substack{u,j=1\\u\neq j}}^{s}\dfrac{1}{\dfrac{g}{\dfrac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Psi_u^U}{1-\Psi_u^U}\right)^{\gamma}}+\dfrac{h}{\dfrac{lw_j\left(T(\hbar_j)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Psi_j^U}{1-\Psi_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right) \end{bmatrix},
$$

$$
\left. \begin{bmatrix} \left(\dfrac{1}{1+\left(\dfrac{s^2-s}{g+h}\times\dfrac{1}{\sum\limits_{\substack{u,j=1\\u\neq j}}^{s}\dfrac{1}{\dfrac{g}{\dfrac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Upsilon_u^L}{1-\Upsilon_u^L}\right)^{\gamma}}+\dfrac{h}{\dfrac{sw_j\left(T(\hbar_j)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Upsilon_j^L}{1-\Upsilon_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right), \\ \left(\dfrac{1}{1+\left(\dfrac{s^2-s}{g+h}\times\dfrac{1}{\sum\limits_{\substack{u,j=1\\u\neq j}}^{s}\dfrac{1}{\dfrac{g}{\dfrac{sw_u\left(T(\hbar_u)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Upsilon_u^U}{1-\Upsilon_u^U}\right)^{\gamma}}+\dfrac{h}{\dfrac{sw_j\left(T(\hbar_j)+1\right)}{\sum_{z=1}^{s}w_z\left(T(\hbar_z)+1\right)}\left(\dfrac{\Upsilon_j^U}{1-\Upsilon_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right) \end{bmatrix} \right\rangle. \qquad (6.30)
$$

6.3 MADM Approach Based on the Developed Aggregation Operators

In this section, based upon the developed INWDPBM and INWDPGBM operators, we will propose a novel MADM method, which is defined as follows;

Let us assume that, in a MADM problem, we need to evaluate u alternatives $\overline{\overline{N}} = \left\{\overline{\overline{N}}_1, \overline{\overline{N}}_2, ..., \overline{\overline{N}}_u\right\}$ with respect to v attributes $\overline{\overline{O}} = \left\{\overline{\overline{O}}_1, \overline{\overline{O}}_2, ..., \overline{\overline{O}}_v\right\}$, and the importance degree of the attributes is represented by $\varpi = (\varpi_1, \varpi_2, ..., \varpi_v)^T$, satisfying the condition $\varpi_h \in [0,1], \sum_{h=1}^{v} \varpi_h = 1.$ The decision matrix for this decision problem is denoted by $\overline{\overline{M}} = \left[\overline{\overline{d}}_{gh}\right]_{m\times n}$, where $\overline{\overline{d}}_{gh} = \left\langle \left[\Xi_{gh}^L, \Xi_{gh}^U\right], \left[\Psi_{gh}^L, \Psi_{gh}^U\right], \left[\Upsilon_{gh}^L, \Upsilon_{gh}^U\right] \right\rangle$ is an INN, provided by the DM for the alternative $\overline{\overline{N}}_g$ for the attribute $\overline{\overline{O}}_h, (g = 1, 2, ..., u; h = 1, 2, ..., v)$. Then the main purpose is to rank the alternative and select the best alternative.

In the following, we will use the proposed INWDPBM and INWDPGBM to solve this MADM problem, and the detailed decision steps are as follows:

Step 1. Standardize the attribute values. Normally, in real DM problems, the attributes are of two types, (1) cost type, (2) benefit type. To get better result, it is necessary to change cost type of attribute values to benefit type using the following formula:

$$\overline{\overline{d}}_{gh} = \left\langle \left[\Upsilon_{gh}^L, \Upsilon_{gh}^U\right], \left[1-\Psi_{gh}^U, 1-\Psi_{gh}^L\right], \left[\Xi_{gh}^L, \Xi_{gh}^U\right] \right\rangle \tag{6.31}$$

Step 2. Calculate the supports;

$$Supp\left(\overline{\overline{d}}_{gh},\overline{\overline{d}}_{gl}\right)=1-\overline{\overline{D}}\left(\overline{\overline{d}}_{gh},\overline{\overline{d}}_{gl}\right),\ (g=1,2,...,u;h,l=1,2,...,v), \tag{6.32}$$

where, $D\left(\overline{\overline{d}}_{gh},\overline{\overline{d}}_{gl}\right)$, is the distance measure given in Equation (1.17).

Step 3. Calculate $T\left(\overline{\overline{d}}_{gh}\right)$;

$$T\left(\overline{\overline{d}}_{gh}\right)=\sum_{\substack{l=1\\l\neq h}}^{u}Supp\left(\overline{\overline{d}}_{gh},\overline{\overline{d}}_{gl}\right),\left(g=1,2,...,u;h,l=1,2,...,v\right); \tag{6.33}$$

Step 4. Calculate all the attribute values $\overline{\overline{d}}_{gh}\ (h=1,2,...,v)$ to the comprehensive value R_g by using INWDPBM or INWDPGBM operators shown as follows;

$$R_g=INWDPBM\left(\overline{\overline{d}}_{g1},\overline{\overline{d}}_{g2},....,\overline{\overline{d}}_{gv}\right); \tag{6.34}$$

Or

$$R_g=INWDPGBM\left(\overline{\overline{d}}_{g1},\overline{\overline{d}}_{g2},....,\overline{\overline{d}}_{gv}\right); \tag{6.35}$$

Step 5. Determine the score values, accuracy values of $R_g\,(g=1,2,...,u)$, using Definition (1.1.1.11).

Step 6. Rank all the alternatives according to their score and accuracy values, and select the best alternative using Definition (1.1.1.12).

Step 7. End.

This decision steps are also described in Figure 1.

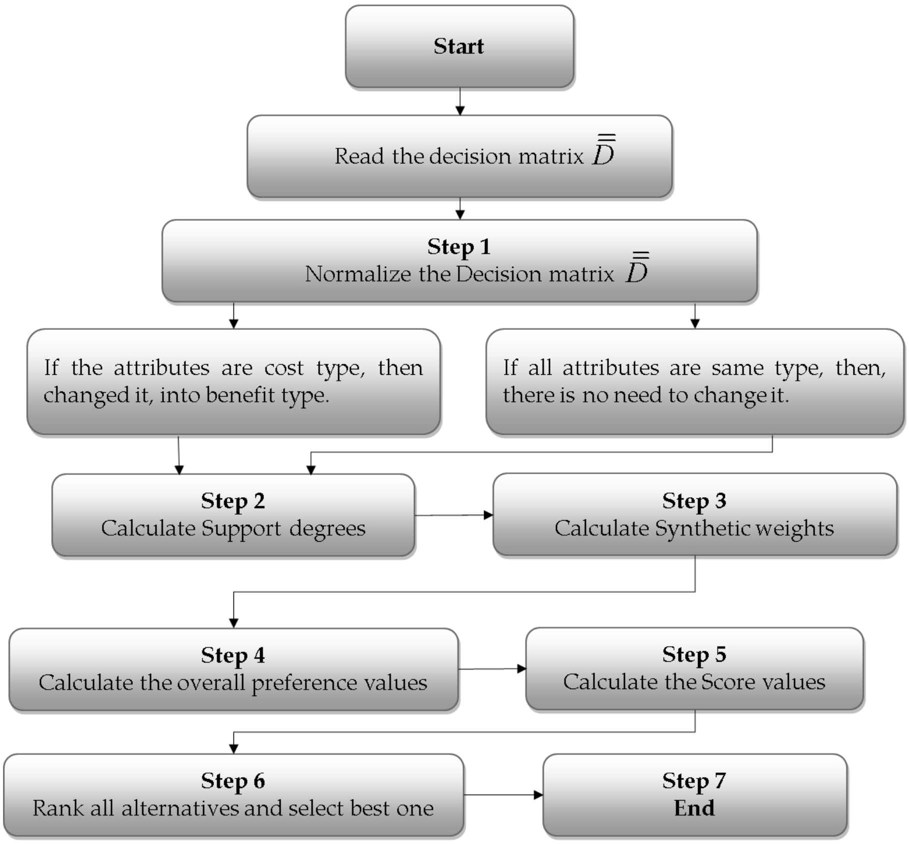

Figure 1. Flow chart for developed approach.

6.4 Illustrative Example

In this part, an example adapted from [79] is used to illustrate the application and effectiveness of the proposed method in MADM problem.

An investment company wants to invest a sum of money in the best option. The company must invest a sum of money in the following four possible companies (alternatives); (1) car company $\overline{\overline{N}}_1$; (2) food company $\overline{\overline{N}}_2$; (3) Computer company $\overline{\overline{N}}_3$; (4) An arm company $\overline{\overline{N}}_4$, and the attributes under consideration are (1) risk analysis $\overline{\overline{O}}_1$; (2) growth analysis $\overline{\overline{O}}_2$; (3) environmental impact analysis $\overline{\overline{O}}_3$. The importance degree of the attributes is $\varpi = (0.35, 0.4, 0.25)^T$. The four possible alternatives $\overline{\overline{N}}_g\,(g=1,...,4)$ are evaluated with to the three above attributes $\overline{\overline{O}}_h\,(h=1,...,4)$ by the form of INN, and the IN decision matrix $\overline{\overline{M}}$ is listed in Table 6.1. The purpose of this decision making problem is to rank the alternatives.

6.1 The Decision Making Steps.

Step 1. Since $\overline{\overline{O}}_1$, $\overline{\overline{O}}_2$ are of benefit type, and $\overline{\overline{O}}_3$ is of cost type. So $\overline{\overline{O}}_3$ will be change into benefit type utilizing Equation (6.31). So the normalize decision matrix $\overline{\overline{NM}}$ is given in Table 6.2.

Table 6.1. The IN decision matrix $\overline{\overline{M}}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$	$\langle[0.7,0.9],[0.2,0.3],[0.4,0.5]\rangle$
$\overline{\overline{N}}_2$	$\langle[0.6,0.8],[0.1,0.2],[0.1,0.2]\rangle$	$\langle[0.6,0.7],[0.15,0.25],[0.2,0.3]\rangle$	$\langle[0.3,0.6],[0.6,0.7],[0.8,0.9]\rangle$
$\overline{\overline{N}}_3$	$\langle[0.3,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.4,0.5],[0.6,0.8],[0.7,0.9]\rangle$
$\overline{\overline{N}}_4$	$\langle[0.7,0.8],[0.01,0.1],[0.2,0.3]\rangle$	$\langle[0.6,0.7],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.4,0.6],[0.4,0.5],[0.8,0.9]\rangle$

Step 2. Determine the supports $Supp\left(\overline{\overline{d}}_{gh},\overline{\overline{d}}_{gl}\right)$, $(g=1,2,3,4;h,l=1,2,3)$, using Equation (6.32),(for simplicity we denote, $Supp\left(\overline{\overline{d}}_{gh},\overline{\overline{d}}_{gl}\right)$ with $S^g_{gh,gl}$), we have

$S^1_{11,12} = S^1_{12,11} = 0.950; S^1_{12,13} = S^1_{13,12} = 0.800; S^1_{11,13} = S^1_{13,11} = 0.85; S^2_{11,12} = S^2_{12,11} = 0.933;$
$S^2_{12,13} = S^2_{13,12} = 0.717; S^2_{11,13} = S^2_{13,11} = 0.683; S^3_{11,12} = S^3_{12,11} = 0.967; S^3_{12,13} = S^3_{13,12} = 0.733;$
$S^3_{11,13} = S^3_{13,11} = 0.700; S^4_{11,12} = S^4_{12,11} = 0.902; S^4_{12,13} = S^4_{13,12} = 0.783; S^4_{11,13} = S^4_{13,11} = 0.752;$

Step 3. Determine $T\left(\overline{\overline{d}}_{gh}\right);(g = 1,2,3,4; h = 1,2,3)$ using Equation (6.33),

$$T^1_{11} = 1.800, T^1_{12} = 1.750, T^1_{13} = 1.650, T^2_{11} = 1.617, T^2_{12} = 1.650, T^2_{13} = 1.400,$$
$$T^3_{11} = 1.667, T^3_{12} = 1.700, T^3_{13} = 1.433, T^4_{11} = 1.653, T^4_{12} = 1.685, T^4_{13} = 1.535.$$

Step 4. (a) Determine the comprehensive value of every alternative using INWDPBM operator, that is Equation (6.34), $(Assume\, that\; g = h = 1; \gamma = 3)$, we have

$$R_1 = \langle [0.3974, 0.5195], [0.1823, 0.3023], [0.3353, 0.4796] \rangle; ;$$

$$R_2 = \langle [0.6457, 0.7954], [0.1700, 0.2885], [0.2044, 0.3265] \rangle;$$

$$R_3 = \langle [0.4846, 0.6503], [0.2556, 0.3711], [0.3376, 0.4394] \rangle;$$

$$R_4 = \langle [0.6938, 0.7953], [0.1062, 0.2154], [0.3069, 0.4278] \rangle.$$

(b) Determine the comprehensive value of every alternative using INWDPGBM operator, that is Equation (6.35), $(Assume\, that\; g = h = 1; \gamma = 3)$, we have

$$R_1 = \langle [0.4026, 0.5381], [0.1570, 0.2977], [0.2998, 0.4520] \rangle;$$

$$R_2 = \langle [0.6654, 0.8193], [0.1558, 0.2686], [0.1836, 0.3035] \rangle;$$

$$R_3 = \langle [0.5159, 0.6732], [0.2366, 0.3473], [0.3265, 0.4279] \rangle;$$

$$R_4 = \langle [0.5159, 0.8193], [0.0938, 0.1952], [0.2862, 0.4037] \rangle.$$

Table 6.2. The Normalize IN decision matrix $\overline{\overline{D}}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$	$\langle[0.4,0.5],[0.2,0.3],[0.7,0.9]\rangle$
$\overline{\overline{N}}_2$	$\langle[0.6,0.8],[0.1,0.2],[0.1,0.2]\rangle$	$\langle[0.6,0.7],[0.15,0.25],[0.2,0.3]\rangle$	$\langle[0.8,0.9],[0.6,0.7],[0.3,0.6]\rangle$
$\overline{\overline{N}}_3$	$\langle[0.3,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.7,0.9],[0.6,0.8],[0.4,0.5]\rangle$
$\overline{\overline{N}}_4$	$\langle[0.7,0.8],[0.01,0.1],[0.2,0.3]\rangle$	$\langle[0.6,0.7],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.4,0.5],[0.4,0.6]\rangle$

Step 5. (a) Determine the score values of $R_g\,(g=1,2,3,4),$ using Definition (1.1.1.11), we have

$$\overline{\overline{SO}}(R_1)=1.8087, \overline{\overline{SO}}(R_2)=2.2259, \overline{\overline{SO}}(R_3)=1.8656, \overline{\overline{SO}}(R_4)=2.2164;$$

(b) Determine the score values of $R_g\,(g=1,2,3,4),$ using Definition (1.1.1.11), we have

$$\overline{\overline{SO}}(R_1)=1.8671, \overline{\overline{SO}}(R_2)=2.2866, \overline{\overline{SO}}(R_3)=1.9254, \overline{\overline{SO}}(R_4)=2.1781;$$

Step 6.(a) According to their score and accuracy values, using Definition (1.1.1.12), the ranking order is $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1$. So the best alternative is $\overline{\overline{N}}_2$, while the worst alternative is $\overline{\overline{N}}_1$.

(b) According to their score and accuracy values, using Definition (1.1.1.12), the ranking order is $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1$. So the best alternative is $\overline{\overline{N}}_2$, while the worst alternative is $\overline{\overline{N}}_1$.

So using INWDPBM or INWDPGBM operators the best alternative is $\overline{\overline{N}}_2$ while the worst alternative is $\overline{\overline{N}}_1$.

6.4.1 Effect of Parameters γ g and h on DM Result of This Example

In order to show the effect of the parameter x and y on the DM result of this example, we set different parameter values for g and h, and $\gamma = 3$, is fixed, to show the ranking result of this example. The ranking results are given in Table 6.3.

As we know from Table 6.3, and Table 6.4 the score values and ranking order are different for different values of the parameter g and h, and $\gamma = 3$, is fixed, while using INWDPBM operator and INWDPGBM operator. We can see from Table 6.3, and Table 6.4, when the parameter . In some situation the ranking order may also be different, while using different parameter values $g = 1 or\ 0$ and $h = 0\ or\ 1$, then the best choice is $\overline{\overline{N}}_4$ and the worst one is $\overline{\overline{N}}_1$. In simple words, when the interrelationship among attributes are not considered the best choice is $\overline{\overline{N}}_4$ and the worst one is $\overline{\overline{N}}_1$. On the other hand when we use different values for the parameters g and h, while using INWPBM and INWDPGBM operators, the ranking result is changed. That is from Table 4, we can see that when the parameter values $g = 1, h = 1$, the ranking results are changed as the one obtained for $g = 1 or\ 0$ and $h = 0\ or\ 1$. In this case the best alternative is $\overline{\overline{N}}_2$ while the worst alternative remains the same.

Table 6.3. Ranking order of decision result using different values for g and h for INWDPBM

Parameter Values	INWDPBM Operator	Ranking Orders
$g=1,h=0,\gamma=3$	$\overline{\overline{SO}}(R_1)=1.9319,\overline{\overline{SO}}(R_2)=2.4172,$ $\overline{\overline{SO}}(R_3)=2.0936,\overline{\overline{SO}}(R_4)=2.4222;$	$\overline{\overline{N}}_4>\overline{\overline{N}}_2>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
$g=1,h=5,\gamma=3$	$\overline{\overline{SO}}(R_1)=1.8338,\overline{\overline{SO}}(R_2)=2.2684,$ $\overline{\overline{SO}}(R_3)=1.9049,\overline{\overline{SO}}(R_4)=2.2666;$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
$g=3,h=7,\gamma=3$	$\overline{\overline{SO}}(R_1)=1.8169,\overline{\overline{SO}}(R_2)=2.2398,$ $\overline{\overline{SO}}(R_3)=1.8777,\overline{\overline{SO}}(R_4)=2.2327;$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
$g=5,h=10,\gamma=3$	$\overline{\overline{SO}}(R_1)=1.8143,\overline{\overline{SO}}(R_2)=2.2354,$ $\overline{\overline{SO}}(R_3)=1.8738,\overline{\overline{SO}}(R_4)=2.2275;$	$\overline{\overline{N}}_4>\overline{\overline{N}}_2>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
$g=1,h=10,\gamma=3$	$\overline{\overline{SO}}(R_1)=1.8501,\overline{\overline{SO}}(R_2)=2.2966,$ $\overline{\overline{SO}}(R_3)=1.9355,\overline{\overline{SO}}(R_4)=2.3012;$	$\overline{\overline{N}}_4>\overline{\overline{N}}_2>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
$g=10,h=4,\gamma=3$	$\overline{\overline{SO}}(R_1)=1.8182,\overline{\overline{SO}}(R_2)=2.2419,$ $\overline{\overline{SO}}(R_3)=1.8796,\overline{\overline{SO}}(R_4)=2.2352;$	$\overline{\overline{N}}_4>\overline{\overline{N}}_2>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
$g=3,h=12,\gamma=3$	$\overline{\overline{SO}}(R_1)=1.8285,\overline{\overline{SO}}(R_2)=2.2592,$ $\overline{\overline{SO}}(R_3)=1.8958,\overline{\overline{SO}}(R_4)=2.2557;$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$

Table 6.4. Ranking order of decision result using different values for g and h for INWDPGBM

Parameter	INWDPGBM Operator	Ranking Orders

Values		
$g=1, h=0, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.5032, \overline{\overline{SO}}(R_2)=1.7934,$ $\overline{\overline{SO}}(R_3)=1.5136, \overline{\overline{SO}}(R_4)=1.8037;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=1, h=5, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.8220, \overline{\overline{SO}}(R_2)=2.2256,$ $\overline{\overline{SO}}(R_3)=1.8717, \overline{\overline{SO}}(R_4)=2.1140;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=3, h=7, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.8539, \overline{\overline{SO}}(R_2)=2.2686,$ $\overline{\overline{SO}}(R_3)=1.9094, \overline{\overline{SO}}(R_4)=2.1584;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=5, h=10, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.8583, \overline{\overline{SO}}(R_2)=2.2745,$ $\overline{\overline{SO}}(R_3)=1.9146, \overline{\overline{SO}}(R_4)=2.1647;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=1, h=10, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.7814, \overline{\overline{SO}}(R_2)=2.1710,$ $\overline{\overline{SO}}(R_3)=1.8248, \overline{\overline{SO}}(R_4)=2.0632;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=10, h=4, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.8087, \overline{\overline{SO}}(R_2)=2.2259,$ $\overline{\overline{SO}}(R_3)=1.8656, \overline{\overline{SO}}(R_4)=2.2164;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=3, h=12, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.8671, \overline{\overline{SO}}(R_2)=2.2866,$ $\overline{\overline{SO}}(R_3)=1.9254, \overline{\overline{SO}}(R_4)=2.1781;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

From Table 6.3 and Table 6.4, we can observe that, when the values of the parameter increases the score values obtained using INWDPBM decreases, while using INWDPGBM operator the score values increases, but the best choice is $\overline{\overline{N}}_2$ for $g=h\geq 1$.

From Table 6.5, we can see that different ranking orders are obtained for different values of γ. When, $\gamma = 0.5$ and $\gamma = 2$, the best choice is $\overline{\overline{N}}_4$ by the INWPBM operator; when we use the INWPGBM operator, it is $\overline{\overline{N}}_2$. Similarly, for other values of $\gamma > 2$, the best choice is $\overline{\overline{N}}_2$ while the worst is $\overline{\overline{N}}_1$.

Table 6.5. Ranking order of decision result using different values for γ

Parameter Values	**INWDPBM Operator**	**INWDPGBM Operator**	**Ranking Orders**
$g=1, h=1, \gamma=0$	$\overline{\overline{SO}}(R_1)=1.6662, \overline{\overline{SO}}(R_2)=2.1025,$ $\overline{\overline{SO}}(R_3)=1.7606, \overline{\overline{SO}}(R_4)=2.1972$	$\overline{\overline{SO}}(R_1)=1.7870, \overline{\overline{SO}}(R_2)=2.2347$ $\overline{\overline{SO}}(R_3)=1.9103, \overline{\overline{SO}}(R_4)=2.1812$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$ $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=1, h=1, \gamma=2$	$\overline{\overline{SO}}(R_1)=1.7783, \overline{\overline{SO}}(R_2)=2.2015,$ $\overline{\overline{SO}}(R_3)=1.8408, \overline{\overline{SO}}(R_4)=2.2091;$	$\overline{\overline{SO}}(R_1)=1.8491, \overline{\overline{SO}}(R_2)=2.2786,$ $\overline{\overline{SO}}(R_3)=1.9213, \overline{\overline{SO}}(R_4)=2.1799;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$ $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=1, h=1, \gamma=4$	$\overline{\overline{SO}}(R_1)=1.8229, \overline{\overline{SO}}(R_2)=2.2363,$ $\overline{\overline{SO}}(R_3)=1.8803, \overline{\overline{SO}}(R_4)=2.2219;$	$\overline{\overline{SO}}(R_1)=1.8740, \overline{\overline{SO}}(R_2)=2.2856$ $\overline{\overline{SO}}(R_3)=1.9275, \overline{\overline{SO}}(R_4)=2.1751$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$ $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=1, h=1, \gamma=7$	$\overline{\overline{SO}}(R_1)=1.8375, \overline{\overline{SO}}(R_2)=2.2455,$ $\overline{\overline{SO}}(R_3)=1.9037, \overline{\overline{SO}}(R_4)=2.2315;$	$\overline{\overline{SO}}(R_1)=1.8747, \overline{\overline{SO}}(R_2)=2.2763$ $\overline{\overline{SO}}(R_3)=1.9331, \overline{\overline{SO}}(R_4)=2.1669$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$ $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=1, h=1, \gamma=1$	$\overline{\overline{SO}}(R_1)=1.8418, \overline{\overline{SO}}(R_2)=2.2477,$ $\overline{\overline{SO}}(R_3)=1.9160, \overline{\overline{SO}}(R_4)=2.2365;$	$\overline{\overline{SO}}(R_1)=1.8701, \overline{\overline{SO}}(R_2)=2.2698,$ $\overline{\overline{SO}}(R_3)=1.9373, \overline{\overline{SO}}(R_4)=2.1622;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$ $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

$g=1, h=1, \gamma=1$	$\overline{\overline{SO}}(R_1)=1.8447, \overline{\overline{SO}}(R_2)=2.2488,$ $\overline{\overline{SO}}(R_3)=1.9270, \overline{\overline{SO}}(R_4)=2.2409;$	$\overline{\overline{SO}}(R_1)=1.8642, \overline{\overline{SO}}(R_2)=2.2637$ $\overline{\overline{SO}}(R_3)=1.9414, \overline{\overline{SO}}(R_4)=2.1582$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$ $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$g=1, h=1, \gamma=2$	$\overline{\overline{SO}}(R_1)=1.8460, \overline{\overline{SO}}(R_2)=2.2492,$ $\overline{\overline{SO}}(R_3)=1.9328, \overline{\overline{SO}}(R_4)=2.2432;$	$\overline{\overline{SO}}(R_1)=1.8608, \overline{\overline{SO}}(R_2)=2.2604,$ $\overline{\overline{SO}}(R_3)=1.9435, \overline{\overline{SO}}(R_4)=2.1562;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$ $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

6.3 Comparing With the Other Methods

To illustrate the advantages and effectiveness of the developed method in this article, we solve the above example by four existing MADM methods, including IN weighted averaging operator, IN weighted geometric operator [8], the similarity measure defined by Ye [21], Muirhead mean operators developed by Liu et al. [79], IN power aggregation operator developed by Liu et al [71].

From Table 6.6, we can see that the ranking orders are the same as the ones produced by the existing aggregation operators when the parameter values $x=1, y=0, \gamma=3$, but the ranking orders are different when the interrelationship among attributes are considered. That is why the developed method based on the proposed aggregation operators is more flexible due the parameter and practical as it can consider the interrelationship among input arguments.

Table 6. Ranking order of the alternatives using different aggregation operators.

Aggregation Operator	Parameter	Score Values	Ranking Order

INWA operator [8]	No	$\overline{\overline{SO}}(R_1) = 1.8430, \overline{\overline{SO}}(R_2) = 2.2497,$ $\overline{\overline{SO}}(R_3) = 1.9151, \overline{\overline{SO}}(R_4) = 2.2788;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
INWGA operator [8]	No	$\overline{\overline{SO}}(R_1) = 1.7286, \overline{\overline{SO}}(R_2) = 2.0991,$ $\overline{\overline{SO}}(R_3) = 1.7751, \overline{\overline{SO}}(R_4) = 2.1608;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
Similarity measure Hamming distance [21]	No	$D_1(R^*, R_1) = 0.7948, D_1(R^*, R_2) = 0.9581,$ $D_1(R^*, R_3) = 0.8805, D_1(R^*, R_4) = 0.9725;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
Generalized power Aggregation operator [71]	Yes $\lambda = 1$	$\overline{\overline{SO}}(R_1) = 1.8460, \overline{\overline{SO}}(R_2) = 2.2543,$ $\overline{\overline{SO}}(R_3) = 1.9163, \overline{\overline{SO}}(R_4) = 2.2799;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
INWMM operator [79]	Yes $P(1,1,1)$	$\overline{\overline{SO}}(R_1) = 1.8054, \overline{\overline{SO}}(R_2) = 2.2321,$ $\overline{\overline{SO}}(R_3) = 1.9172, \overline{\overline{SO}}(R_4) = 2.2773;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
INWDMM operator [79]	Yes $P(1,1,1)$	$\overline{\overline{SO}}(R_1) = 1.6260, \overline{\overline{SO}}(R_2) = 1.9202,$ $\overline{\overline{SO}}(R_3) = 1.7061, \overline{\overline{SO}}(R_4) = 2.0798;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
Proposed INWDPBM $x = 1, y = 0, \gamma = 3$	Yes	$\overline{\overline{SO}}(R_1) = 1.9319, \overline{\overline{SO}}(R_2) = 2.4172,$ $\overline{\overline{SO}}(R_3) = 2.0936, \overline{\overline{SO}}(R_4) = 2.4222;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
Proposed INWDPGBM	Yes	$\overline{\overline{SO}}(R_1) = 1.5032, \overline{\overline{SO}}(R_2) = 1.7934,$ $\overline{\overline{SO}}(R_3) = 1.5136, \overline{\overline{SO}}(R_4) = 1.8037;$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

$g=1, h=0, \gamma=3$			
INWDPBM operator in this article	Yes $g=h=1, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.8087, \overline{\overline{SO}}(R_2)=2.2259,$ $\overline{\overline{SO}}(R_3)=1.8656, \overline{\overline{SO}}(R_4)=2.2164;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
INWDPBM operator in this article	Yes $g=h=1, \gamma=3$	$\overline{\overline{SO}}(R_1)=1.8671, \overline{\overline{SO}}(R_2)=2.2866,$ $\overline{\overline{SO}}(R_3)=1.9254, \overline{\overline{SO}}(R_4)=2.1781;$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

From the above comparative analysis, we can know the proposed method has the following advantages, that is, it can consider the interrelationship among the input arguments and can relieve the effect of the awkward data by PWV at the same time, and it can permit more precise ranking order than the existing methods. The proposed method can take the advantages of PA operator and BM operator concurrently, these factors makes it a little complex in calculations.

The score values and ranking orders by these methods are shown in Table 6.6.

6.4.2 Conclusion

The PBM operator can take the advantage of PA operator, which can eliminate the impact of awkward data given by the predisposed DMs, and BM operator, which can consider the correlation between two attributes. The Dombi operations of TN and TCN proposed by Dombi have the edge of good flexibility with general parameter. In this chapter, we combined PBM with Dombi operation and proposed some aggregation operators to aggregate INNs. Firstly, we defined some operational laws for INSs based on Dombi TN and TCN and discussed some properties of these operations. Secondly, we extended PBM operator based on Dombi operations to introduce INDPBM operator, INWDPBM operator, INDPGBM operator, INWDPGBM operator and discussed some properties of these aggregation operators. The developed aggregation operators have the edge that they can take the correlation among the attributes by BM operator, and can also remove the effect of awkward data by PA operator at the same and due to general parameter, so they are more flexible in the aggregation process. Further, we developed a novel MADM method based on developed aggregation operators to deal with interval neutrosophic information. Finally, an illustrative example is used to show the effectiveness and practicality of the proposed MADM method and comparison were made with the existing methods. The proposed aggregation operators are very useful to solve MADM problems.

In future research, we shall define some distinct aggregation operators for SVHFSs, INHFSs, double valued neutrosophic sets and so on based on Dombi operations and apply them to MAGDM.

Chapter 7

Group Decision Making Method under Hesitant Interval Neutrosophic Uncertain Linguistic Environment

In this chapter, we propose the concept of HINULSs and HINULEs, then developed some basic operational rules, properties, score, accuracy and certainty functions for HINULEs. Then, based on these operational rules, we described some aggregation operators, such as HINULPWA operator, HINULPWG operator and GHINULPWA operator to aggregate HINUL information. Further, some desired characteristics of these developed operators are examined. A GDM method over GHINULPWA operator is initiated to handle MCGDM problems, in which criteria values take the form of HINULEs and there exist prioritized relations between the criteria. Lastly, a numerical example about investment alternatives is given to show the efficiency of the proposed method.

7.1 Hesitant Interval Neutrosophic Uncertain Linguistic Set

In this section, the concept of HINULS, some operational laws, and related properties for HINULSs are developed.

7.1.1 HINULS and their operational laws

7.1.1.1 Definition

Let $\aleph$ be the domain set, then a HINULS in $\overline{\overline{HI}}$ is represented by the following mathematical symbol:

$$\overline{\overline{HI}} = \left\{ \left\langle o, \hbar_{\overline{\overline{HI}}}(o) \right\rangle \mid o \in \aleph \right\}. \tag{7.1}$$

Where, $\hbar_{\overline{\overline{HI}}}(o) = \bigcup_{b_{\overline{\overline{HI}}}(o) \in \hbar_{\overline{\overline{HI}}}} \left\{ b_{\overline{\overline{HI}}}(o) \right\}$ is a set of INULNs, representing the possible INULNs of the element $o \in \aleph$ to the set $\overline{\overline{HI}}$, and $u_{\overline{\overline{HI}}}(o) = \left\langle \left[s_{\theta(o)}, s_{\tau(o)} \right], \left(\left[\Xi^{L}_{\overline{\overline{HI}}}(o), \Xi^{U}_{\overline{\overline{HI}}}(o) \right], \left[\Psi^{L}_{\overline{\overline{HI}}}(o), \Psi^{U}_{\overline{\overline{HI}}}(o) \right], \left[\Upsilon^{L}_{\overline{\overline{HI}}}(o), \Upsilon^{U}_{\overline{\overline{HI}}}(o) \right] \right) \right\rangle$ is an INULE. For simplicity, we shall write $\hbar = \bigcup_{b \in \hbar} \{b\}$ instead of $\hbar_{\overline{\overline{HI}}}(o) = \bigcup_{b_{\overline{\overline{HI}}}(o) \in \hbar_{\overline{\overline{HI}}}} \left\{ b_{\overline{\overline{HI}}}(o) \right\}$ in $\overline{\overline{HI}}$. Here we call $\hbar$ a HINULE and $b = \left\langle \left[s_{\theta(b)}, s_{\tau(b)} \right], \left(\left[\Xi^{L}(b), \Xi^{U}(b) \right], \left[\Psi^{L}(b), \Psi^{U}(b) \right], \left[\Upsilon^{L}(b), \Upsilon^{U}(b) \right] \right) \right\rangle$ is called an INULE. Then, $\overline{\overline{HI}}$ is the set of all INULEs.

7.1.1.2 Definition

Let $\hbar, \hbar_1$ and $\hbar_2$ be any three HINULEs and $\zeta > 0$. Then we presented some basic operations for HINULEs as follows:

$$\begin{aligned}(1)\ \hbar_1 \times \hbar_2 = \bigcup_{b_1 \in \hbar_1, b \in \hbar_2} \Bigg\langle & \left[\overline{\overline{\partial}}^{*-1} \left(\overline{\overline{\partial}}^{*} \left(s_{\vartheta(b_1)} \right) \overline{\overline{\partial}}^{*} \left(s_{\vartheta(b_1)} \right) \right), \overline{\overline{\partial}}^{*-1} \left(\overline{\overline{\partial}}^{*} \left(s_{\tau(b_1)} \right) \overline{\overline{\partial}}^{*} \left(s_{\tau(b_1)} \right) \right) \right], \\ & \Big(\left[\Xi^{L}(b_1) \Xi^{L}(b_2), \Xi^{U}(b_1) \Xi^{U}(b_2) \right], \left[\Psi^{L}(b_1) + \Psi^{L}(b_2) - \Psi^{L}(b_1) \Psi^{L}(b_2), \right. \\ & \left. \Psi^{U}(b_1) + \Psi^{U}(b_2) - \Psi^{U}(b_1) \Psi^{U}(b_2) \right], \left[\Upsilon^{L}(b_1) + \Upsilon^{L}(b_2) - \Upsilon^{L}(b_1) \Upsilon^{L}(b_2), \right. \\ & \left. \Upsilon^{U}(b_1) + \Upsilon^{U}(b_2) - \Upsilon^{U}(b_1) \Upsilon^{U}(b_2) \right] \Big) \Bigg\rangle, \end{aligned} \tag{7.2}$$

$$
(2)\,\hbar_1+\hbar_2=\bigcup_{b_1\in\hbar_1,b_2\in\hbar_2}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)\right),\overline{\overline{\partial}}^{*-1}\left(\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)\right],\right.
$$

$$
\left(\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)\Xi^{L}\left(b_1\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)t^{L}\left(b_2\right)}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)\Xi^{U}\left(b_1\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)\Xi^{U}\left(b_2\right)}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)}\right],\right.
$$

$$
\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)\Psi^{L}\left(b_1\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)\Psi^{L}\left(b_2\right)}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)\Psi^{U}\left(b_1\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)\Psi^{U}\left(b_2\right)}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)}\right],
$$

$$
\left.\left.\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)\Upsilon^{L}\left(b_1\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)\Upsilon^{L}\left(b_2\right)}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)\Upsilon^{U}\left(b_1\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)\Upsilon^{U}\left(b_2\right)}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_1)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(b_2)}\right)\right)}\right]\right)\right\rangle, \tag{7.3}
$$

$$
(3)\,\hbar^{\zeta}=\bigcup_{b\in\hbar}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b)}\right)\right)^{\zeta}\right),\overline{\overline{\partial}}^{*-1}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\tau(b)}\right)\right)^{\zeta}\right)\right],\left(\left[\left(\Xi^{L}\left(b\right)\right)^{\zeta},\left(\Xi^{U}\left(b\right)\right)^{\zeta}\right],\right.\right.
$$

$$
\left.\left.\left[1-\left(1-\Psi^{L}\left(b\right)\right)^{\zeta},1-\left(1-\Psi^{U}\left(b\right)\right)^{\zeta}\right],\left[1-\left(1-\Upsilon^{L}(b)\right)^{\zeta},1-\left(1-\Upsilon^{U}(b)\right)^{\zeta}\right]\right)\right\rangle, \tag{7.4}
$$

$$
(4)\,\zeta\hbar=\bigcup_{b\in\hbar}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\zeta\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b)}\right)\right)\right),\overline{\overline{\partial}}^{*-1}\left(\zeta\left(\overline{\overline{\partial}}^{*}\left(s_{\tau(b)}\right)\right)\right)\right],\right.
$$

$$
\left.\left(\left[\left(\Xi^{L}\left(b\right)\right),\left(\Xi^{U}\left(b\right)\right)\right],\left[\left(\Psi^{L}\left(b\right)\right),\left(\Psi^{U}\left(b\right)\right)\right],\left[\left(\Upsilon^{L}\left(b\right)\right),\left(\Upsilon^{U}\left(b\right)\right)\right]\right)\right\rangle, \tag{7.5}
$$

$$
(5)\,neg\left(\hbar\right)=\bigcup_{b\in\hbar}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\overline{\overline{\partial}}^{*}\left(s_{2t}\right)-\overline{\overline{\partial}}^{*}\left(s_{\tau(b)}\right)\right),\overline{\overline{\partial}}^{*-1}\left(\overline{\overline{\partial}}^{*}\left(s_{2t}\right)-\overline{\overline{\partial}}^{*}\left(s_{\vartheta(b)}\right)\right)\right],\right.
$$

$$
\left.\left(\left[\left(\Upsilon^{L}(b)\right),\left(\Upsilon^{U}\left(b\right)\right)\right],\left[1-\left(\Psi^{U}\left(b\right)\right),1-\left(\Psi^{L}\left(b\right)\right)\right],\left[\left(\Xi^{L}(b)\right),\left(\Xi^{U}\left(b\right)\right)\right]\right)\right\rangle. \tag{7.6}
$$

7.1.1.3 Theorem

Let $\hbar, \hbar_1$ and $\hbar_2$ be any three HINULEs and $\zeta, \zeta_1, \zeta_2 \geq 0$. Then the operational rules defined for HINULEs have the following characteristics.

$$(1)\ \hbar_1 + \hbar_2 = \hbar_2 + \hbar_1, \tag{7.7}$$

$$(2)\ \hbar_1 \times \hbar_2 = \hbar_2 \times \hbar_1, \tag{7.8}$$

$$(3)\ \zeta(\hbar_1 + \hbar_2) = \zeta\hbar_1 + \zeta\hbar_2, \tag{7.9}$$

$$(4)\ \zeta_1\hbar + \zeta_2\hbar = (\zeta_1 + \zeta_2)\hbar, \tag{7.10}$$

$$(5)\ \hbar^{\zeta_1} \times \hbar^{\zeta_2} = (\hbar)^{(\zeta_1+\zeta_2)}, \tag{7.11}$$

$$(6)\ \hbar_1^{\zeta} \times \hbar_2^{\zeta} = (\hbar_1 \times \hbar_2)^{\zeta}. \tag{7.12}$$

7.1.1.4 Example

Let $\hbar_1 = \{\langle [s_1, s_2], [0.2, 0.4], [0.1, 0.2], [0.3, 0.5] \rangle, \langle [s_1, s_2], [0.4, 0.5], [0.1, 0.2], [0.2, 0.3] \rangle\}$ and

$\hbar_2 = \langle [s_2, s_3], [0.3, 0.4], [0.2, 0.3], [0.1, 0.3] \rangle$ be two HINULNs and $\zeta = 2$. Then

(1) If we use LSF $\overset{=*}{\partial}(s_\vartheta) = \dfrac{i}{2t}$, then

(i) $\hbar_1 \times \hbar_2 = \left\langle \begin{matrix} ([s_{0.3334}, s_{0.9999}], [0.06, 0.16], [0.28, 0.44], [0.37, 0.65]), \\ ([s_{0.6665}, s_{0.9999}], [0.12, 0.2], [0.28, .44], [0.28, 0.51]) \end{matrix} \right\rangle;$

(ii) $\hbar_1 + \hbar_2 = \left\langle \begin{matrix} ([s_3, s_{4.999}], [0.2625, 0.4], [0.1625, 0.2625], [0.1750, 0.3750]), \\ ([s_{3.999}, s_{4.999}], [0.3444, 0.4444], [0.1556, 0.2556], [0.1444, 0.3]) \end{matrix} \right\rangle;$

(iii) $\hbar_1^{\zeta} = \langle [s_{0.6665}, s_{1.5}], [0.09, 0.16], [0.36, 0.51], [0.19, 0.51] \rangle;$

(iv) $\zeta\hbar_1 = \langle [s_{3.999}, s_6], [0.3, 0.4], [0.2, 0.3], [0.1, 0.3] \rangle.$

(2) If we use LSF $\overset{=*}{\partial}_2(s_z) = \vartheta_z = \begin{cases} \dfrac{\phi^r - \phi^{r-z}}{2\phi^r - 2} & (z = 0, 1, 2, ..., r) \\ \dfrac{\phi^r - \phi^{z-r} - 2}{2\phi^r - 2} & (z = r+1, r+2..., 2r) \end{cases}$. Then

(i) $\hbar_1 \times \hbar_2 = \left\langle \begin{matrix} \left(\left[s_{0.3467}, s_{0.8348}\right], [0.06, 0.16], [0.28, 0.44], [0.37, 0.65]\right), \\ \left(\left[s_{0.6215}, s_{0.8348}\right], [0.12, 0.2], [0.28, .44], [0.28, 0.51]\right) \end{matrix} \right\rangle$;

(ii) $\hbar_1 + \hbar_2 = \left\langle \begin{matrix} \left(\left[s_{3.9659}, s_{5.5316}\right], [0.2592, 0.4], [0.1592, 0.2592], [0.1815, 0.3815]\right), \\ \left(\left[s_{4.975}, s_{5.5316}\right], [0.3465, 0.4465], [0.1534, 0.2534], [0.1465, 0.3]\right) \end{matrix} \right\rangle$;

(iii) $\hbar_1^{\zeta} = \left\langle \left[s_{0.6215}, s_{1.1365}\right], [0.09, 0.16], [0.36, 0.51], [0.19, 0.51] \right\rangle$;

(iv) $\zeta \hbar_1 = \left\langle \left[s_{4.9754}, s_6\right], [0.3, 0.4], [0.2, 0.3], [0.1, 0.3] \right\rangle$.

(3) If we use LSF $\overset{=*}{\partial}_3(s_z) = \vartheta_z = \begin{cases} \frac{r^{\alpha} - (r-z)^{\alpha}}{2r^{\alpha}} & (z = 0, 1, 2, ..., r) \\ \frac{r^{\beta} - (z-r)^{\beta}}{2r^{\beta}} & (z = r+1, r+2..., 2r) \end{cases}$, then

(i) $\hbar_1 \times \hbar_2 = \left\langle \begin{matrix} \left(\left[s_{0.3148}, s_{1.031}\right], [0.06, 0.16], [0.28, 0.44], [0.37, 0.65]\right), \\ \left(\left[s_{0.6453}, s_{1.031}\right], [0.12, 0.2], [0.28, .44], [0.28, 0.51]\right) \end{matrix} \right\rangle$,

(ii) $\hbar_1 + \hbar_2 = \left\langle \begin{matrix} \left(\left[s_{2.829}, s_{4.741}\right], [0.2637, 0.4], [0.1638, 0.2638], [0.1724, 0.3724]\right), \\ \left(\left[s_{3.590}, s_{4.741}\right], [0.3433, 0.4433], [0.1567, 0.2567], [0.1433, 0.3]\right) \end{matrix} \right\rangle$;

(iii) $\hbar_1^{\zeta} = \left\langle \left[s_{0.6453}, s_{1.635}\right], [0.09, 0.16], [0.36, 0.51], [0.19, 0.51] \right\rangle$;

(iv) $\zeta \hbar_1 = \left\langle \left[s_{3.590}, s_6\right], [0.3, 0.4], [0.2, 0.3], [0.1, 0.3] \right\rangle$.

7.1.1.5 Definition

Let $[s_{\vartheta}, s_{\tau}]$ be ULV, and $\overset{=*}{\partial}$ be a LSF. Then, an aggregation expression of $[s_{\vartheta}, s_{\tau}]$ can be defined as

$$\begin{aligned} \mathrm{E}\left(\left[s_{\vartheta}, s_{\tau}\right]\right) &= f_{\zeta}\left(\left[s_{\vartheta}, s_{\tau}\right]\right), \\ &= \int_0^1 \left(d\zeta(x) \big/ dx \right) \left(\overset{=*}{\partial}(s_{\tau}) - x\left(\overset{=*}{\partial}(s_{\tau}) - \overset{=*}{\partial}(s_{\vartheta}) \right) \right) dx. \end{aligned} \tag{7.13}$$

where the function ζ is expressed by a basic unit interval monotonic function proposed by Yager [114] and f_{ζ} is the continuous ordered weighted averaging operator. If $(\sigma \geq 0)$, then $\zeta(x) = x^{\sigma}$. Moreover, $\mathrm{E}([s_{\vartheta}, s_{\tau}])$ is an increasing function with respect to s_{ϑ} and s_{τ}, and $\mathrm{E}([s_{\vartheta}, s_{\tau}])$ satisfies $0 \leq \mathrm{E}([s_{\vartheta}, s_{\tau}]) \leq 1$.

Hence, the score, accuracy and certainty functions of HINULE $\hbar$ are defined as follows:

$$\overline{\overline{SO}}(\hbar) = \frac{1}{\Diamond\hbar} \sum_{b\in\hbar} \left(\frac{1}{6} \left(\mathrm{E}\left(\left[s_{\vartheta(b)}, s_{\tau(b)} \right]\right) \left(\left(4 + \Xi^{L}(b) - \Psi^{L}(b) - \Upsilon^{L}(b) + \Xi^{U}(b) - \Psi^{U}(b) - \Upsilon^{U}(b) \right) \right) \right) \right); \quad (7.14)$$

$$\overline{\overline{AR}}(\hbar) = \frac{1}{\Diamond\hbar} \sum_{b\in\hbar} \left(\left(\mathrm{E}\left(\left[s_{\vartheta(b)}, s_{\tau(b)} \right]\right) \left(\left(\Xi^{L}(b) - \Upsilon^{L}(b) + \Xi^{U}(b) - \Upsilon^{U}(b) \right) \right) \right) \right) \quad (7.15)$$

$$\overline{\overline{CR}}(\hbar) = \frac{1}{\Diamond\hbar} \sum_{b\in\hbar} \left(\left(\mathrm{E}\left(\left[s_{\vartheta(b)}, s_{\tau(b)} \right]\right) \left(\left(\Xi^{L}(b) + \Xi^{U}(b) \right) \right) \right) \right). \quad (7.16)$$

Where $\Diamond\hbar$ represents the number of INULEs in $\hbar$.

7.1.1.6 Definition

Let $\hbar$ and $\hbar_1$ be two HINULEs, then the comparison rules between two HINULEs can be defined as follows:

(1) If $\overline{\overline{SO}}(\hbar) > \overline{\overline{SO}}(\hbar_1)$, then $\hbar > \hbar_1$;

(2) If $\overline{\overline{SO}}(\hbar) = \overline{\overline{SO}}(\hbar_1)$ and $\overline{\overline{AR}}(\hbar) > \overline{\overline{AR}}(\hbar_1)$, then $\hbar > \hbar_1$;

(3) If $\overline{\overline{SC}}(\hbar) = \overline{\overline{SC}}(\hbar_1)$, $\overline{\overline{AR}}(\hbar) = \overline{\overline{AC}}(\hbar_1)$ and $\overline{\overline{CR}}(\hbar) > \overline{\overline{CR}}(\hbar_1)$, then $\hbar > \hbar_1$;

(4) If $\overline{\overline{SC}}(\hbar) = \overline{\overline{SC}}(\hbar_1)$, $\overline{\overline{AR}}(\hbar) = \overline{\overline{AC}}(\hbar_1)$ and $\overline{\overline{CR}}(\hbar) = \overline{\overline{CR}}(\hbar_1)$, then $\hbar = \hbar_1$.

7.2 The Hesitant Interval Neutrosophic Uncertain Linguistic Aggregation Operators

In this part, we present two HINUL aggregation operators to aggregate HINUL information. These aggregation operators are based on the operational rules of HINULNs.

7.2.1 The Hesitant Interval Neutrosophic Uncertain Linguistic Prioritized Weighted Averaging Operator

In this subpart, we propose HINULPWA operators to aggregate HINULEs, and related properties are discussed.

7.2.1.1 Definition .

Let $\hbar_v$ $(v = 1,2,\ldots,o)$ be a group of HINULNs. The HINUL prioritized weighted average operator (HINULPWA) is defined as

$$HINULPWA(\hbar_1,\hbar_2,\ldots,\hbar_o) = \frac{T_1}{\sum_{g=1}^{o} T_g}\hbar_1 \oplus \frac{T_2}{\sum_{g=1}^{o} T_g}\hbar_2 \oplus \ldots \oplus \frac{T_o}{\sum_{g=1}^{o} T_o}\hbar_o = \oplus_{v=1}^{o}\left(\frac{T_v \hbar_v}{\sum_{g=1}^{o} T_g}\right), \qquad (7.17)$$

where, $T_1 = 1, T_v = \prod_{z=1}^{v-1} \overline{\overline{SO}}(\hbar_z)(v = 2,\ldots,o)$, and $\overline{\overline{SO}}(\hbar_z)$ is the score function of $\hbar_v$.

Based on the operational laws for HIFULNs, and Definition (7.1.1.2), we have the following Theorem.

7.2.1.2 Theorem

Let $\hbar_v$ $(v = 1,2,\ldots,o)$ be a group of HINULEs. Then by Eq. (7.17) and the operational laws for HINULEs, we obtain the following result.

$$HINULPWA(\hbar_1,\hbar_2,\ldots,\hbar_o) = \frac{T_1}{\sum_{g=1}^{o} T_g}\hbar_1 \oplus \frac{T_2}{\sum_{g=1}^{o} T_g}\hbar_2 \oplus \ldots \oplus \frac{T_o}{\sum_{g=1}^{o} T_g}\hbar_o$$

$$
\begin{aligned}
&= \bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right],\right.\\
&\left(\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],\right.\\
&\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],\\
&\left.\left.\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right]\right)\right\rangle.
\end{aligned}
\tag{7.18}
$$

Where, $T_1 = 1, T_v = \prod_{z=1}^{v-1}\overline{\overline{SO}}(\hbar_z)(v = 2,\ldots,o)$, and $\overline{\overline{SC}}(\hbar_z)$ is the score function of $\hbar_v$.

Proof. Equation (7.18) can be proved by utilizing mathematical induction.

(1) For $v = 2$, we have

$$
\frac{T_1}{\sum_{g=1}^{2}T_g}\hbar_1 = \bigcup_{u_1\in\hbar_1}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\frac{T_1}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)\right), \overline{\overline{\partial}}^{*-1}\left(\frac{T_1}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)\right], \left(\Xi(u_1),\Psi(u_1),\Upsilon(u_1)\right)\right\rangle,
$$

$$
\frac{T_2}{\sum_{g=1}^{2}T_g}\hbar_2 = \bigcup_{u_2\in\hbar_2}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\frac{T_2}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)\right), \overline{\overline{\partial}}^{*-1}\left(\frac{T_2}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)\right], \left(\Xi(u_2),\Psi(u_2),\Upsilon(u_2)\right)\right\rangle.
$$

Then,

$$HINULPWA\left(\hbar_1,\hbar_2\right)=\frac{T_1}{\sum_{g=1}^{2}T_g}\hbar_1\oplus\frac{T_2}{\sum_{g=1}^{2}T_g}\hbar_2$$

$$\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\frac{T_1}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(a_1)}\right)+\frac{T_2}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(a_2)}\right)\right),\overline{\overline{\partial}}^{*-1}\left(\frac{T_1}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(a_1)}\right)+\frac{T_2}{\sum_{g=1}^{2}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(a_2)}\right)\right)\right],\right.$$

$$=\bigcup_{u_1\in\hbar_1,u_2\in\hbar_2}\left(\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1\Xi_1^L+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2\Xi_2^L}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1\Xi_1^U+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2\Xi_2^U}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2}\right],\right.$$

$$\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1\Psi_1^L+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2\Psi_2^L}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1\Psi_1^U+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2\Psi_2^U}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2}\right],$$

$$\left.\left.\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1\Upsilon_1^L+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2\Upsilon_2^L}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1\Upsilon_1^U+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2\Upsilon_2^U}{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_1)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_1)}\right)\right)T_1+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_2)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_2)}\right)\right)T_2}\right]\right)\right\rangle.$$

(2) Let us assume that Eq. (7.18) is true for $v=b$, then

$$HINULPWA\left(\hbar_1,\hbar_2,\ldots,\hbar_b\right)=\frac{T_1}{\sum_{g=1}^{o}T_g}\hbar_1\oplus\frac{T_2}{\sum_{g=1}^{o}T_g}\hbar_2\oplus\ldots\oplus\frac{T_b}{\sum_{g=1}^{o}T_g}\hbar_b,$$

$$=\bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_b\in\hbar_b}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{b}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right),\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{b}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right],\right.$$

$$\left(\left(\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^L}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^U}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],\right.\right.$$

$$\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^L}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^U}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],$$

$$\left.\left.\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^L}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^U}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right]\right)\right\rangle.$$

(3) When $v=b+1,$ then

$$HINULPWA\left(\hbar_1,\hbar_2,\ldots,\hbar_{b+1}\right)=\frac{T_1}{\sum_{g=1}^{o}T_j}\hbar_1\oplus\frac{T_2}{\sum_{g=1}^{o}T_g}\hbar_2\oplus\ldots\oplus\frac{T_b}{\sum_{g=1}^{o}T_g}\hbar_b\oplus\frac{T_{b+1}}{\sum_{g=1}^{o}T_g}\hbar_{b+1}$$

$$=\bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_b\in\hbar_b}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{b}\left(\frac{T_v}{\sum_{j=1}^{m}T_j}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right),\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{b}\left(\frac{T_v}{\sum_{j=1}^{m}T_j}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right]\right.,$$

$$\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^L}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^U}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],$$

$$\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^L}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^U}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],$$

$$\left.\left[\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^L}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^U}{\sum_{v=1}^{b}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right]\right\rangle$$

$$\oplus\bigcup_{u_{b+1}\in\hbar_{b+1}}\left\langle\left(\left(\left[\overline{\overline{\partial}}^{*-1}\left(\frac{T_{b+1}}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_{b+1})}\right)\right),\overline{\overline{\partial}}^{*-1}\left(\frac{T_{b+1}}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_{b+1})}\right)\right)\right]\right),\left[\Xi_{b+1}^L,\Xi_{b+1}^U\right],\left[\Psi_{b+1}^L,\Psi_{b+1}^U\right],\left[\Upsilon_{b+1}^L,\Upsilon_{b+1}^U\right]\right\rangle,$$

$$\left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{b+1}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{b+1}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right] \right.$$

$$= \bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_{b+1}\in\hbar_{b+1}} \left(\left[\frac{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^L}{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^U}{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right], \right.$$

$$\left[\frac{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^L}{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^U}{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right],$$

$$\left. \left[\frac{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^L}{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^U}{\sum_{v=1}^{b+1}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right] \right) \right\rangle.$$

Hence Eq. (7.18) is true for all o.

According to (1)-(3), Eq. (7.18) is kept.

7.2.1.3 Theorem

Let $\hbar_v$ $(v = 1, 2, \ldots, o)$ be a collection of HINULEs, $T_1 = 1, T_v = \prod_{z=1}^{v-1}\overline{\overline{SC}}\left(\hbar_z\right)\left(v = 2, \ldots, o\right),$ and $\overline{\overline{SC}}\left(\hbar_z\right)$ is the score function of $\hbar_v$. If $b > 0,$ then

$$HINULPWA\left(a\hbar_1, a\hbar_2, \ldots., a\hbar_o\right) = aHINULPWA\left(\hbar_1, \hbar_2, \ldots., \hbar_o\right). \tag{7.19}$$

Proof. According to the operational rules described for HINULNs, we have

$$\zeta\hbar_v = \bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_m\in\hbar_m} \begin{array}{l} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\zeta\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\zeta\left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right], \right. \\ \left. \left(\left[\left(\Xi_v^L\right),\left(\Xi_v^U\right)\right],\left[\left(\Psi_v^L\right),\left(\Psi_v^U\right)\right],\left[\left(\Upsilon_v^L\right),\left(\Upsilon_v^U\right)\right]\right)\right\rangle. \end{array}$$

According to Theorem (7.2.1.2), we have

$HINULPWA\left(a\hbar_1, a\hbar_2, \ldots., a\hbar_m\right)$

$$= \bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_m\in\hbar_m} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}a\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}a\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right], \right.$$

$$\left(\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],\right.$$

$$\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],$$

$$\left.\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right]\right)\right\rangle,$$

$$=\bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(a\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right),\overline{\overline{\partial}}^{*-1}\left(a\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right],\right.$$

$$\left(\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],\right.$$

$$\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],$$

$$\left.\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right]\right)\right\rangle.$$

$$bHINULPWA\left(\hbar_1,\hbar_2,\ldots,\hbar_o\right)=(a).\bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right),\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right],\right.$$

$$\left(\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],\right.$$

$$\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],$$

$$\left.\left.\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right]\right)\right\rangle,$$

$$=\bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_m\in\hbar_m}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(a\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{j=1}^{o}T_j}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right),\overline{\overline{\partial}}^{*-1}\left(a\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{j=1}^{o}T_j}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right],\right.$$

$$\left(\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],\right.$$

$$\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right],$$

$$\left.\left.\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v},\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}\right]\right)\right\rangle.$$

Hence, $HINULPWA\left(b\hbar_1,b\hbar_2,....,b\hbar_m\right)=bHINULPWA\left(\hbar_1,\hbar_2,....,\hbar_m\right)$.

7.2.1.4 Theorem

Let $\hbar_v\left(1,2,...,m\right)$ and $\rho_v(v=1,2,....,m)$ be two collections of HINULNs. Then

$$\begin{aligned}HINULPWA\left(\hbar_1\oplus\rho_1,\hbar_2\oplus\rho_2,...,\hbar_m\oplus\rho_m\right)=\\ HINULPWA\left(\hbar_1,\hbar_2,...,\hbar_m\right)\oplus HINULPWA\left(\rho_1,\rho_2,...,\rho_m\right).\end{aligned}\tag{7.20}$$

Proof. According to Definition (7.2.1.1)

$HINULPWA\left(\hbar_1\oplus\rho_1,\hbar_2\oplus\rho_2,...,\hbar_m\oplus\rho_m\right)$

$$
= \bigcup_{\substack{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o,\\ k_1\in\rho_1,k_2\in\rho_2,\ldots,k_o\in\rho_o,}} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right) + \left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right) + \left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\right)\right], \right.
$$

$$
\left(\left[\frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Xi_v^{L}+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Xi_v^{L}\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Xi_v^{U}+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Xi_v^{U}\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v} \right], \right.
$$

$$
\left[\frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Psi_v^{L}+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Psi_v^{L}\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Psi_v^{U}+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Psi_v^{U}\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v} \right],
$$

$$
\left.\left. \left[\frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Upsilon_v^{L}+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Upsilon_v^{L}\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Upsilon_v^{U}+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Upsilon_v^{U}\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v} \right]\right)\right\rangle.
$$

$HINULPWA\left(\hbar_1,\hbar_2,\ldots,\hbar_o\right)\oplus HINULPWA\left(\rho_1,\rho_2,\ldots,\rho_o\right)$

$$
= \bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\right)\right], \right.
$$

$$
\left(\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right], \right.
$$

$$
\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right],
$$

$$
\left.\left. \left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{L}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^{U}}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right]\right)\right\rangle
$$

$$\oplus \bigcup_{k_1\in\rho_1,k_2\in\rho_2,...,k_o\in\rho_o} \Bigg\langle \Bigg[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\right)\Bigg]$$

$$\Bigg(\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v\Xi_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v\Xi_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}\right],$$

$$\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v\Psi_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v\Psi_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}\right],$$

$$\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v\Upsilon_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v\Upsilon_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}\right]\Bigg)\Bigg\rangle,$$

$$= \bigcup_{\substack{u_1\in\hbar_1,u_2\in\hbar_2,...,u_o\in\hbar_o,\\ k_1\in\rho_1,k_2\in\rho_2,...,k_o\in\rho_o,}} \Bigg\langle \Bigg[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)+\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)\right)\right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o}\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\right)\Bigg],$$

$$\Bigg(\left[\frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Xi_v^L+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Xi_v^L\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Xi_v^U+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Xi_v^U\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(a_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(a_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}\right],$$

$$\left[\frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Psi_v^L+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Psi_v^L\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Psi_v^U+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Psi_v^U\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}\right],$$

$$\left[\frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Upsilon_v^L+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Upsilon_v^L\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)\Upsilon_v^U+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)\Upsilon_v^U\right)T_v}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)+\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(k_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(k_v)}\right)\right)T_v}\right]\Bigg)\Bigg\rangle.$$

Thus, $HINULPWA(\hbar_1\oplus\rho_1,\hbar_2\oplus\rho_2,...,\hbar_o\oplus\rho_o)=HINULPWA(\hbar_1,\hbar_2,...,\hbar_o)\oplus HINULPWA(\rho_1,\rho_2,...,\rho_o)$.

7.2.2 The Hesitant Interval Neutrosophic Uncertain Linguistic Prioritized Weighted Geometric (HINULPWG) Operator

In this subpart, we propose hesitant interval neutrosophic prioritized weighted geometric operator and related theorems, properties are investigated.

7.2.2.1 Definition

Let $\hbar_v$ $(v = 1, 2, \ldots, o)$ be a collection of HINULNs. The HINUL prioritized weighted geometric operator is defined as

$$HINULPWG\left(\hbar_1, \hbar_2, \ldots, \hbar_o\right) = \hbar_1^{\frac{T_1}{\sum_{g=1}^{o} T_g}} \otimes \hbar_2^{\frac{T_2}{\sum_{g=1}^{o} T_g}} \otimes \ldots \otimes \hbar_o^{\frac{T_g}{\sum_{g=1}^{g} T_g}} = \otimes_{v=1}^{o} \left(\hbar_o^{\frac{T_v}{\sum_{g=1}^{o} T_g}} \right) \tag{7.21}$$

Where $T_1 = 1, T_v = \prod_{z=1}^{v-1} \overline{\overline{SO}}\left(\hbar_z\right)\left(v = 2, ..., o\right)$, and $\overline{\overline{SO}}\left(\hbar_z\right)$ is the score function of $\hbar_v$.

Based on the operational laws for HIFULNs, and Definition (7.2.2.1), we have the following Theorem.

7.2.2.2 Theorem

Let $\hbar_v$ $(v = 1, 2, \ldots, o)$ be a group of HINULEs. Then by Eq. (7.21) and the operational rules for HINULNs, we obtain the following result:

$$HINULPWG(\hbar_1,\hbar_2,\ldots,\hbar_o)=\hbar_1^{\frac{T_1}{\sum_{g=1}^{o}T_g}}\otimes\hbar_2^{\frac{T_2}{\sum_{g=1}^{o}T_g}}\otimes\ldots\otimes\hbar_o^{\frac{T_o}{\sum_{g=1}^{o}T_g}}$$

$$=\bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o}\left\langle\left[\overline{\overline{\partial}}^{*-1}\left(\prod_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)^{\frac{T_v}{\sum_{g=1}^{o}T_g}}\right),\overline{\overline{\partial}}^{*-1}\left(\prod_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)^{\frac{T_v}{\sum_{g=1}^{o}T_g}}\right)\right],\right.$$

$$\left(\left[\prod_{v=1}^{o}\left(\Xi_v^L\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_j}\right)},\prod_{v=1}^{o}\left(\Xi_v^U\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)}\right],\left[1-\prod_{v=1}^{o}\left(1-\Psi_v^L\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)},1-\prod_{v=1}^{o}\left(1-\Psi_v^U\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)}\right],\right. \quad (7.22)$$

$$\left.\left.\left[1-\prod_{v=1}^{o}\left(1-\Upsilon_v^L\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)},1-\prod_{v=1}^{g}\left(1-\Upsilon_v^U\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)}\right]\right)\right\rangle.$$

Proof. Same as Theorem (7.2.1.2), omitted here.

7.2.2.3 Theorem

Let $\hbar_v$ $(v=1,2,\ldots,o)$ be a collection of HINULEs, $T_1=1,T_v=\prod_{z=1}^{v-1}\overline{\overline{SO}}(\hbar_z)(v=2,\ldots,o)$, and

$\overline{\overline{SO}}(\hbar_z)$ is the score function of $\hbar_v$. If $a>0$, then

$$HINULPWG\left((\hbar_1)^a,(\hbar_2)^a,\ldots,(\hbar_o)^a\right)=\left(HINULPWG(\hbar_1,\hbar_2,\ldots,\hbar_o)\right)^a. \quad (7.23)$$

Proof. Proof of this Theorem is same as Theorem (7.2.1.3), it is omitted here.

7.2.2.4 Theorem

Let $\hbar_v$ $(v=1,2,\ldots,o)$ and $\rho_v(v=1,2,\ldots,o)$ be two collections of HINULNs. Then

$$\begin{aligned}&HINULPWG(\hbar_1\otimes\rho_1,\hbar_2\otimes\rho_2,\ldots,\hbar_o\otimes\rho_o)=\\&\qquad HINULPWG(\hbar_1,\hbar_2,\ldots,\hbar_o)\otimes HINULPWG(\rho_1,\rho_2,\ldots,\rho_o).\end{aligned} \quad (7.24)$$

Proof. Proof of this Theorem is same as Theorem (7.2.1.4), omitted here.

7.2.3 Generalized HINUL Prioritized Weighted Aggregation Operators

In this subsection, we define some generalized prioritized weighted aggregation operators for HINULNs.

7.2.3.1 Definition

Let $\hbar_v$ $(v = 1, 2, \ldots, o)$ be a collection of HINULNs. The generalized HINUL prioritized weighted aggregation operator is defined as

$$GHINULPWA(\hbar_1, \hbar_2, \ldots, \hbar_o) = \left(\frac{T_1}{\sum_{g=1}^{o} T_g} \hbar_1^\eta \oplus \frac{T_2}{\sum_{g=1}^{o} T_g} \hbar_2^\eta \oplus \ldots \oplus \frac{T_o}{\sum_{g=1}^{o} T_g} \hbar_o^\eta \right)^{\frac{1}{\eta}} = \oplus_{v=1}^{o} \left(\frac{T_v \hbar_v^\eta}{\sum_{g=1}^{o} T_g} \right)^{\frac{1}{\eta}} \quad (7.25)$$

Where, $\eta > 0$, and $T_1 = 1, T_v = \prod_{z=1}^{v-1} \overline{\overline{SO}}(\hbar_z)(v = 2, \ldots, o)$, and $\overline{\overline{SO}}(\hbar_z)$ is the score function of $\hbar_v$.

7.2.3.2 Theorem

Let $\hbar_v$ $(v = 1, 2, \ldots, o)$ be a collection of HINULEs. Then by Eq. (7.25) and the operational rules for HINULEs, we obtain the following result.

$$GHINULPWA(\hbar_1, \hbar_2, \ldots, \hbar_o) = \left(\frac{T_1}{\sum_{g=1}^{o} T_g} \hbar_1^\eta \oplus \frac{T_2}{\sum_{g=1}^{o} T_g} \hbar_2^\eta \oplus \ldots \oplus \frac{T_o}{\sum_{g=1}^{o} T_g} \hbar_o^\eta \right)$$

$$= \bigcup_{u_1 \in \hbar_1, u_2 \in \hbar_2, \ldots, u_o \in \hbar_o} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\left(\left(\sum_{v=1}^{o} \left(\frac{T_v}{\sum_{g=1}^{o} T_g} \left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} \right) \right)^{\frac{1}{\eta}} \right) \right), \overline{\overline{\partial}}^{*-1}\left(\left(\left(\sum_{v=1}^{o} \left(\frac{T_v}{\sum_{g=1}^{o} T_g} \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) \right)^{\frac{1}{\eta}} \right) \right) \right], \right.$$

$$\left(\left[\left(\frac{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v \left(\Xi_v^L \right)^{\eta}}{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v} \right)^{\frac{1}{\eta}}, \left(\frac{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v \left(\Xi_v^U \right)^{\eta}}{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v} \right)^{\frac{1}{\eta}} \right], \right.$$

$$\left[\begin{array}{l} 1 - \left(1 - \frac{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v \left(1 - \left(1 - \Psi_v^L \right)^{\eta} \right)}{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v} \right)^{\frac{1}{\eta}}, \\ 1 - \left(1 - \frac{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v \left(1 - \left(1 - \Psi_v^U \right)^{\eta} \right)}{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v} \right)^{\frac{1}{\eta}} \end{array} \right],$$

$$\left. \left. \left[\begin{array}{l} 1 - \left(1 - \frac{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v \left(1 - \left(1 - \Upsilon_v^L \right)^{\eta} \right)}{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v} \right)^{\frac{1}{\eta}}, \\ 1 - \left(1 - \frac{\sum_{v=1}^{o} \left(\left(\Re^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v \left(1 - \left(1 - \Upsilon_v^U \right)^{\eta} \right)}{\sum_{v=1}^{o} \left(\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right)^{\eta} + \left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right)^{\eta} \right) T_v} \right)^{\frac{1}{\eta}} \end{array} \right] \right) \right\rangle. \quad (7.26)$$

Now, we discuss some special cases of HINULPWA operator.

1. If $\eta = 1,$ then the GHINULPWA reduce into the HINULPWA operator. i.e.,

$$GHINULPWA\left(\hbar_1, \hbar_2, \ldots, \hbar_o\right) = \oplus_{v=1}^{o} \left(\frac{T_v \hbar_v}{\sum_{g=1}^{o} T_g} \right) =$$

$$\bigcup_{u_1 \in \hbar_1, u_2 \in \hbar_2, \ldots, u_o \in \hbar_o} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o} \left(\frac{T_v}{\sum_{g=1}^{o} T_g} \overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)} \right) \right) \right), \overline{\overline{\partial}}^{*-1}\left(\sum_{v=1}^{o} \left(\frac{T_v}{\sum_{g=1}^{o} T_g} \overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)} \right) \right) \right) \right] \right.$$

$$\left\langle \left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Xi_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right], \right.$$

$$\left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Psi_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right],$$

$$\left. \left[\frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^L}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v}, \frac{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v\Upsilon_v^U}{\sum_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)+\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)T_v} \right] \right\rangle. \tag{7.27}$$

2. If $\eta \to 0$, then the GHINULPWA operator reduces into the HINULPWG operator.

$$\lim_{\eta\to 0} GHINULPWA(\hbar_1,\hbar_2,\ldots,\hbar_o) = \bigotimes_{v=1}^{o} \hbar_o^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)}$$

$$= \bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o} \left\langle \left[\overline{\overline{\partial}}^{*-1}\left(\prod_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\vartheta(u_v)}\right)\right)^{\frac{T_v}{\sum_{g=1}^{o}T_g}}\right), \overline{\overline{\partial}}^{*-1}\left(\prod_{v=1}^{o}\left(\overline{\overline{\partial}}^{*}\left(s_{\tau(u_v)}\right)\right)^{\frac{T_v}{\sum_{g=1}^{o}T_g}}\right) \right], \right.$$

$$\left(\left[\prod_{v=1}^{o}\left(\Xi_v^L\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)}, \prod_{v=1}^{o}\left(\Xi_v^U\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)} \right], \left[1-\prod_{v=1}^{o}\left(1-\Psi_v^L\right)^{\left(\frac{T_v}{\sum_{g=1}^{g}T_g}\right)}, 1-\prod_{v=1}^{o}\left(1-\Psi_v^U\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)} \right], \right.$$

$$\left. \left. \left[1-\prod_{v=1}^{o}\left(1-\Upsilon_v^L\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)}, 1-\prod_{v=1}^{o}\left(1-\Upsilon_v^U\right)^{\left(\frac{T_v}{\sum_{g=1}^{o}T_g}\right)} \right] \right) \right\rangle. \tag{7.28}$$

3. If $\eta = \infty$, then the GHINULPWA operator reduces into the following form.

$$\lim_{\eta\to\infty} GHINULPW(\hbar_1,\hbar_2,\ldots,\hbar_o) = \bigcup_{u_1\in\hbar_1,u_2\in\hbar_2,\ldots,u_o\in\hbar_o} \left\langle \left[\max_v s_{\vartheta(u_v)}, \max_v s_{\tau(u_v)}\right], \left[\max_v \Xi_v^L, \max_v \Xi_v^U\right], \right.$$
$$\left. \left[\min_v \Psi_v^L, \min_v \Psi_v^U\right], \left[\min_v \Upsilon_v^L, \min_v \Upsilon_v^U\right] \right\rangle. \tag{7.29}$$

7.3 Group Decision-making Method Based on GHINULPWA Operator

In this section, we proposed a group decision making method based on GHINULPWA operator and the score, accuracy and certainty functions of HINULNs under a HINUL environment.

Let us assume that $\overline{\overline{N}}=\left\{\overline{\overline{N}}_1,\overline{\overline{N}}_2,\ldots,\overline{\overline{N}}_m\right\}$ and $\overline{\overline{O}}=\left\{\overline{\overline{O}}_1,\overline{\overline{O}}_2,\ldots,\overline{\overline{O}}_n\right\}$ be the sets of m alternatives and n attributes respectively in a group decision making problem, and that there is a prioritization among the attributes represented by the linear ordering $\overline{\overline{O}}_1>\overline{\overline{O}}_2>\ldots>\overline{\overline{O}}_n$ specifies that the attribute $\overline{\overline{O}}_v$ has a higher priority than $\overline{\overline{O}}_u$, if $v<u$. Let the set of z DMs is denoted by $D=\{d_1,d_2,\ldots,d_z\}$, and that there is a prioritization among the DMs represented by the linear ordering $d_1>d_2>\ldots>d_z$ specifies that the DM d_p has a higher priority than d_q, if $p<q$. Let the decision matrix is represented by $\overline{\overline{M}}^g=\left(\overline{\overline{n}}_{ij}^g\right)_{m\times n}$. In a group decision process the evaluation information provided about the alternative $\overline{\overline{N}}_i$ $(i=1,2,\ldots,m)$ with respect to the attribute $\overline{\overline{O}}_j=\left\{\overline{\overline{O}}_1,\overline{\overline{O}}_2,\ldots,\overline{\overline{O}}_n\right\}(j=1,2,..,n)$ for every decision maker d^g is expressed by the form of HINULN: $\overline{\overline{a}}_i^g=\left\{\overline{\overline{O}}_j,\overline{\overline{a}}_{\overline{\overline{n}}_i^g}\left(\overline{\overline{O}}_j\right)\middle|\overline{\overline{n}}_j\in\overline{\overline{O}}\right\}$, where $\overline{\overline{a}}^g\left(\overline{\overline{n}}_j\right)=\bigcup_{u_{\overline{\overline{n}}_i^g}(\overline{\overline{O}}_j)\in\overline{\overline{a}}_{\overline{\overline{N}}_i}^g\left(\overline{\overline{O}}_j\right)}\left\{u_{\overline{\overline{n}}_i^z}(\overline{\overline{O}}_j)\right\}$ is a set of INULNs, representing the possible INULNs of the element $\overline{\overline{O}}_j\in\overline{\overline{O}}$ to the set $\overline{\overline{M}}_i^g$, and $u_{\overline{\overline{n}}_i^g}(\overline{\overline{O}}_j)=\left\langle\left[s^g_{\vartheta_i^z\left(\overline{\overline{O}}_j\right)},s^g_{\tau_i^z\left(\overline{\overline{O}}_j\right)}\right],\left(\left[\inf\Xi_{\overline{\overline{n}}_i^g}\left(\overline{\overline{O}}_j\right),\sup\Xi_{\overline{\overline{n}}_i^g}(\overline{\overline{O}}_j)\right],\left[\inf\Psi_{\overline{\overline{n}}_i^g}\left(\overline{\overline{O}}_j\right),\sup\Psi_{\overline{\overline{n}}_i^g}(\overline{\overline{O}}_j)\right],\left[\inf\Upsilon_{\overline{\overline{n}}_i^g}\left(\overline{\overline{O}}_j\right),\sup\Upsilon_{\overline{\overline{n}}_i^g}(\overline{\overline{O}}_j)\right]\right)\right\rangle$ is an INULN. For simplicity, we shall write $\overline{\overline{a}}_{\overline{\overline{n}}_i^g}=\{u_{\overline{\overline{n}}_i^g}\}$ instead of $\overline{\overline{a}}_{\overline{\overline{n}}_i^g}\left(\overline{\overline{O}}_j\right)=\bigcup_{u_{\overline{\overline{n}}_i^g}(\overline{\overline{O}}_j)\in\overline{\overline{n}}_{\overline{\overline{n}}_i^g}\left(\overline{\overline{n}}_j\right)}\left\{u_{\overline{\overline{n}}_i^g}(\overline{\overline{O}}_j)\right\}$

in $\overline{\overline{M}}_i^g$. Here we call $\bar{\bar{a}}_{\bar{\bar{n}}_i^g}$ a HINULN and $u^g{}_{ij} = \left[s_{\vartheta_{ij}}, s_{\tau_{ij}} \right], \left(\left[\Xi_{ij}{}^L, \Xi_{ij}{}^U \right], \left[\Psi_{ij}{}^L, \Psi_{ij}{}^U \right], \left[\Upsilon_{ij}{}^L, \Upsilon_{ij}{}^U \right] \right)$ is called an INULN. Hence, one can ascertain the $g-th$ hesitant interval neutrosophic uncertain lingujistic decision matrix $M^g = \left(\bar{\bar{n}}^g{}_{ij}\right)_{m\times n}$ $for\ \ g = (1,2,\ldots,z)$.

Generally, there are two types of attributes, one of which is of benefit type and the other is cost type. So, we must transform the cost attribute into benefit attribute by the following formula.

For cost attribute:

$$\bar{\bar{n}}_{ij} = \bigcup_{u_{ij}\in \bar{\bar{n}}_{ij}} \left\{ \left\langle \begin{array}{l} \left[\Re^{*-1}\left(\Re^*\left(s_{2t}\right) - \Re^*\left(s_{\tau(u_{ij})}\right)\right), \Re^{*-1}\left(\Re^*\left(s_{2t}\right) - \Re^*\left(s_{\vartheta(u_{ij})}\right)\right)\right], \\ \left(\left[\left(\Upsilon^L\left(u_{ij}\right)\right), \left(\Upsilon^U\left(u_{ij}\right)\right)\right], \left[1-\left(\Psi^U\left(u_{ij}\right)\right), 1-\left(\Psi^L\left(u_{ij}\right)\right)\right], \left[\left(\Xi^L\left(u_{ij}\right)\right), \left(\Xi^U\left(u_{ij}\right)\right)\right]\right) \end{array} \right\rangle \right\}. \quad (7.30)$$

For benefit criteria:

$$\bar{\bar{n}}_{ij} = \bigcup_{u_{ij}\in \bar{\bar{n}}_{ij}} \left\{ \left\langle \begin{array}{l} \left[\Re^{*-1}\left(\Re^*\left(s_{\vartheta(u_{ij})}\right)\right), \Re^{*-1}\left(\Re^*\left(s_{\tau(u_{ij})}\right)\right)\right], \\ \left(\left[\left(\Xi^L\left(u_{ij}\right)\right), \left(\Xi^U\left(u_{ij}\right)\right)\right], \left[\left(\Psi^U\left(u_{ij}\right)\right), \left(\Psi^L\left(u_{ij}\right)\right)\right], \left[\left(\Upsilon^L\left(u_{ij}\right)\right), \left(\Upsilon^U\left(u_{ij}\right)\right)\right]\right) \end{array} \right\rangle \right\}. \quad (7.31)$$

The decision steps are depicted as follows.

Step 1. Calculate the values of $T_{ij}^g\ (g = 1,2,\ldots,z)$ by using the following formula:

$$T_{ij} = \prod_{g=1}^{z} \overline{\overline{SO}}\left(\bar{\bar{n}}_{ij}^g\right)\left(g = 2,\ldots,z\right); T_{ij}^{(1)} = 1. \quad (7.32)$$

Step 2. Using GHINULPWA operator to aggregate all the individual HINUL decision matrices $M^g = \left(\bar{\bar{n}}^g{}_{ij}\right)_{m\times n}$ to a collective HINUL decision matrix $M = \left(\bar{\bar{n}}_{ij}\right)_{m\times n}$ $(i = 1,2,\ldots,m, j = 1,2,\ldots,n)$ by the following aggregation formula:

$$u_{ij}=\langle[s_{\vartheta_{ij}},s_{\tau_{ij}}],([\Xi_{ij}{}^{L},\Xi_{ij}{}^{U}],[\Psi_{ij}{}^{L},\Psi_{ij}{}^{U}],[\Upsilon_{ij}{}^{L},\Upsilon_{ij}{}^{U}])\rangle = GHINULPWA_{\eta}(\bar{\bar{n}}^{1}{}_{ij},\bar{\bar{n}}^{2}{}_{ij},\bar{\bar{n}}^{3}{}_{ij},\dots,\bar{\bar{n}}^{z}{}_{ij})$$

$$= \bigcup_{u_{ij}\in \bar{\bar{n}}_{ij}} \left\langle \left[\bar{\bar{\partial}}^{*-1}\left(\left(\sum_{g=1}^{z}\left(\frac{T_{ij}}{\sum_{g=1}^{z}T_j}\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}\right)\right)^{\frac{1}{\eta}}\right), \bar{\bar{\partial}}^{*-1}\left(\left(\sum_{g=1}^{z}\left(\frac{T_{ij}}{\sum_{g=1}^{z}T_j}\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)\right)^{\frac{1}{\eta}}\right)\right],\right.$$

$$\left(\left[\left(\frac{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}\left(\Xi_{ij}^{L}\right)^{\eta}}{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{v}}\right)^{\frac{1}{\eta}}, \left(\frac{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}\left(\Xi_{ij}^{U}\right)^{\eta}}{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{v}}\right)^{\frac{1}{\eta}}\right],\right.$$

$$\left[\begin{array}{l} 1-\left(1-\frac{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}\left(1-\left(1-\Psi_{ij}^{L}\right)^{\eta}\right)}{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{v})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{v}}\right)^{\frac{1}{\eta}}, \\ 1-\left(1-\frac{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}\left(1-\left(1-\Psi_{ij}^{U}\right)^{\eta}\right)}{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}}\right)^{\frac{1}{\eta}} \end{array}\right],$$

$$\left.\left.\left[\begin{array}{l} 1-\left(1-\frac{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}\left(1-\left(1-\Upsilon_{ij}^{L}\right)^{\eta}\right)}{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}}\right)^{\frac{1}{\eta}}, \\ 1-\left(1-\frac{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}\left(1-\left(1-\Upsilon_{ij}^{U}\right)^{\eta}\right)}{\sum_{g=1}^{z}\left(\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}+\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)T_{ij}}\right)^{\frac{1}{\eta}} \end{array}\right]\right)\right\rangle. \quad (7.33)$$

Step 3. Calculate the values of $T_{ij}\ (i=1,2,\dots,m,\ j=1,2,\dots,n)$ by using the following formula.

$$T_{ij}=\prod_{k=1}^{j-1}\overline{\overline{SO}}\left(\bar{\bar{n}}_{ij}\right)(g=2,\dots,z);T_{i1}=1. \quad (7.34)$$

Step 4. Aggregate the HINULN $\bar{\bar{n}}_i$ for each alternative $\bar{\bar{N}}_i\ (i=1,2,\dots,m)$ by the using the following aggregation formula:

$HINULPWA_{\eta}(\bar{\bar{n}}_{i1},\bar{\bar{n}}_{i2},\bar{\bar{n}}_{i3},\dots,\bar{\bar{n}}_{in})$

$$= \bigcup_{u_{ij}\in \bar{\bar{n}}_{ij}} \left\langle \left[\bar{\bar{\partial}}^{*-1}\left(\left(\sum_{j=1}^{n}\left(\frac{T_{ij}}{\sum_{j=1}^{n}T_j}\left(\bar{\bar{\partial}}^{*}\left(s_{\vartheta(u_{ij})}\right)\right)^{\eta}\right)\right)^{\frac{1}{\eta}}\right), \bar{\bar{\partial}}^{*-1}\left(\left(\sum_{j=1}^{n}\left(\frac{T_{ij}}{\sum_{j=1}^{n}T_j}\left(\bar{\bar{\partial}}^{*}\left(s_{\tau(u_{ij})}\right)\right)^{\eta}\right)\right)^{\frac{1}{\eta}}\right)\right],\right.$$

$$\left\langle \left[\left(\frac{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij} \left(\Xi_{ij}^{L} \right)^{\eta}}{\sum_{g=1}^{z} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{v}} \right)^{\frac{1}{\eta}}, \left(\frac{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij} \left(\Xi_{ij}^{U} \right)^{\eta}}{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{v}} \right)^{\frac{1}{\eta}} \right], \right.$$

$$\left[\begin{array}{l} 1 - \left(1 - \frac{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij} \left(1 - \left(1 - \Psi_{ij}^{L} \right)^{\eta} \right)}{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{v})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{v}} \right)^{\frac{1}{\eta}}, \\ 1 - \left(1 - \frac{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij} \left(1 - \left(1 - \Psi_{ij}^{U} \right)^{\eta} \right)}{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij}} \right)^{\frac{1}{\eta}} \end{array} \right],$$

$$\left. \left[\begin{array}{l} 1 - \left(1 - \frac{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij} \left(1 - \left(1 - \Upsilon_{ij}^{L} \right)^{\eta} \right)}{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij}} \right)^{\frac{1}{\eta}}, \\ 1 - \left(1 - \frac{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij} \left(1 - \left(1 - \Upsilon_{ij}^{U} \right)^{\eta} \right)}{\sum_{j=1}^{n} \left(\left(\Re^{*}\left(s_{\vartheta(u_{ij})} \right) \right)^{\eta} + \left(\Re^{*}\left(s_{\tau(u_{ij})} \right) \right)^{\eta} \right) T_{ij}} \right)^{\frac{1}{\eta}} \end{array} \right] \right\rangle. \tag{7.35}$$

Step 5. Utilize the score function, accuracy function and certainty function for $\overline{\overline{SC}}\left(\overline{\overline{n}}_i\right), \overline{\overline{CR}}\left(\overline{\overline{n}}_i\right)$ and $\overline{\overline{AC}}\left(\overline{\overline{n}}_i\right)$ by using Eqs. (7.14), (7.15) and (7.16).

Step 6. Rank all the alternatives according to their score values and select the best one.

Step 7. End..

7.4 Numerical Example

In this section we adapted a numerical example from Ye [43] about speculation alternatives is used to show the efficiency of the proposed decision-making method under HINUL environment.

There is an investment company. In the available options, the investment company wants to invest a sum of money in that option which is the best option for it. The company should invest money according to the following group of possible four alternatives: (a) $\overline{\overline{N}}_1$ represents the car company; (b) $\overline{\overline{N}}_2$ represents a food company (c)

$\overline{\overline{N}}_3$ represents a computer company (d) $\overline{\overline{N}}_4$ represents an arm company. The decision must be taken by the investment company according to the following three criteria: (a) $\overline{\overline{O}}_1$ represents the risk; (b) $\overline{\overline{O}}_2$ represents the growth; (a) $\overline{\overline{O}}_3$ represents the environmental impact. The priority among the criteria is $\overline{\overline{O}}_1 > \overline{\overline{O}}_2 > \overline{\overline{O}}_3$. The four possible alternatives $\overline{\overline{N}}_i (i = 1,2,3,4)$ with respect to the criteria $\overline{\overline{O}}_j (j = 1,2,3)$ is assessed by three experts, $E = \{e_1, e_2, e_3\}$, where the assessment information is represented in the form of HINULNs under the linguistic term set $\mathcal{S} = \{\mathfrak{s}_0 = extremely\ poor, \mathfrak{s}_1 = very\ poor,\ \mathfrak{s}_2 = poor,\ \mathfrak{s}_3 = medium,\ \mathfrak{s}_4 = good,\ \mathfrak{s}_5 = very\ good, \mathfrak{s}_6 = extremely\ good\ \}$. The priority among the experts is $e_1 > e_2 > e_3$. Then, the assessment information with respect to the criteria $\overline{\overline{C}}_j (j = 1,2,3)$ of the alternatives $\overline{\overline{M}}_i (i = 1,2,3,4)$ can be given by the three decision makers. For example, the assessment value about alternative $\overline{\overline{M}}_2$ with respect to the criterion $\overline{\overline{C}}_1$ provided by the first expert is $\{\langle [\mathfrak{s}_5, \mathfrak{s}_6], [0.7,0.8], [0.0,0.1], [0.1,0.2] \rangle\}$, which shows that the value of the alternative $\mathfrak{A}_2$ with respect to the criteria c_1 is the uncertain linguistic variable $[\mathfrak{s}_5, \mathfrak{s}_6]$ with the satisfaction degree is $[0.7,0.8]$, dissatisfaction value $[0.0,0.1]$ and the indeterminacy is $[0.1,0.2]$. Similarly, the three experts provide their assessment about the four alternatives with respect to the three criteria. So we have the following three HINUL decision matrices (see tables 7.1-7.3).

Table 7.1. The HINUL decision matrix $\overline{\overline{M}}^{(1)}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\{\langle[\mathfrak{s}_2,\mathfrak{s}_3],[0.3,0.4],[0.1,0.2],[0.3,0.4]\rangle$ $\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.4,0.5],[0.15,0.25],[0.35,0$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.4,0.5],[0.2,0.3],[0.1,0.3$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_3],[0.45,0.55],[0.15,0.25],[0.25,0.$ $\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.3,0.4],[0.2,0.3],[0.3,0.4]\rangle$
$\overline{\overline{N}}_2$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.7,0.8],[0.0,0.1],[0.1,0.2]\rangle\}$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.6,0.7],[0.1,0.2],[0.1,0.$ $\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.8,0.9],[0.0,0.1],[0.1,0.$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle\}$
$\overline{\overline{N}}_3$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.4,0.5],[0.1,0.2],[0.2,0.3]\rangle$ $\langle[\mathfrak{s}_4,\mathfrak{s}_4],[0.5,0.6],[0.1,0.15],[0.1,0.2$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.5,0.6],[0.1,0.2],[0.1,0.3$ $\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.4,0.5],[0.2,0.3],[0.1,0.2$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.6],[0.1,0.2],[0.2,0.3]\rangle\}$
$\overline{\overline{N}}_4$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.6],[0.1,0.2],[0.3,0.4]\rangle$ $\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.6,0.7],[0.0,0.1],[0.1,0.2]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.6],[0.15,0.25],[0.1,$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.6,0.7],[0.15,0.25],[0.25,0.35$ $\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.4,0.5],[0.2,0.3],[0.1,0.2]\rangle$

Table 7.2. The HINUL decision matrix $M^{(2)}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.6,0.7],[0.1,0.2],[0.2,0.3]\rangle$	$\{\langle[\mathfrak{s}_2,\mathfrak{s}_3],[0.2,0.4],[0.1,0.3],[0.2,0.4]\rangle$ $\langle[\mathfrak{s}_3,\mathfrak{s}_3],[0.3,0.5],[0.1,0.2],[0.1,0.2]\rangle$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_3],[0.3,0.4],[0.2,0.3],[0.3,0.4]\rangle\}$
$\overline{\overline{N}}_2$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_5],[0.5,0.6],[0.1,0.2],[0.1,0.2]\rangle$ $\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.7,0.8],[0.0,0.1],[0.0,0.1]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.4,0.6],[0.1,0.2],[0.1,0.2]\rangle$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_5],[0.6,0.8],[0.0,0.0],[0.0,0.1]\rangle,$ $\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.7],[0.1,0.15],[0.1,0.15]$
$\overline{\overline{N}}_3$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle$ $\langle[\mathfrak{s}_5,\mathfrak{s}_5],[0.4,0.5],[0.1,0.3],[0.2,0.4]\rangle$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.5,0.6],[0.1,0.2],[0.2,0.3]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.6],[0.1,0.2],[0.1,0.2]\rangle\}$
$\overline{\overline{N}}_4$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_5],[0.5,0.6],[0.1,0.2],[0.2,0.3]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.4,0.5],[0.1,0.2],[0.1,0.2]\rangle$ $\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.7,0.8],[0.0,0.1],[0.0,0.1]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.6],[0.1,0.2],[0.1,0.2]\rangle,$ $\langle[\mathfrak{s}_5,\mathfrak{s}_5],[0.6,0.7],[0.0,0.1],[0.1,0.15]\rangle$

Table 7.3. The HINUL decision matrix $M^{(3)}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$
$\overline{\overline{N}}_1$	$\{\langle[\mathfrak{s}_1,\mathfrak{s}_2],[0.1,0.2],[0.3,0.4],[0.4,0.5]\rangle$ $\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.4,0.5],[0.2,0.3],[0.2,0.3]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_4],[0.4,0.5],[0.1,0.2],[0.2,0.3]\rangle$ $\langle[\mathfrak{s}_5,\mathfrak{s}_5],[0.6,0.7],[0.1,0.2],[0.1,0.2]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.6,0.7],[0.1,0.2],[0.1,0.2]\rangle\}$
$\overline{\overline{N}}_2$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.7,0.9],[0.0,0.1],[0.1,0.15]$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.8,0.9],[0.0,0.0],[0.0,0.1]\rangle$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.4,0.5],[0.2,0.3],[0.2,0.3]\rangle$

$\overline{\overline{N}}_3$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.2,0.4],[0.2,0.3],[0.3,0.4]\rangle\}$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_4],[0.5,0.6],[0.0,0.15],[0.1,0.2]\rangle\}$	$\{\langle[\mathfrak{s}_3,\mathfrak{s}_4],[0.3,0.5],[0.1,0.2],[0.1,0.2]\rangle\}$
$\overline{\overline{N}}_4$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.3,0.4],[0.1,0.2],[0.2,0.3]\rangle\}$	$\{\langle[\mathfrak{s}_5,\mathfrak{s}_6],[0.7,0.8],[0.0,0.1],[0.0,0.1]\rangle\}$	$\{\langle[\mathfrak{s}_4,\mathfrak{s}_4],[0.4,0.5],[0.2,0.3],[0.1,0.3]\rangle,$ $\langle[\mathfrak{s}_4,\mathfrak{s}_5],[0.5,0.6],[0.1,0.2],[0.1,0.3]\rangle\}$

7.4.1 Decision Making Steps

Since all the criteria are of same type, so there is no need of normalization.

The GDM method presented in this article can handle such type of problems in the following way.

Step 1: Calculate the values of $T_{ij}^{g}\,(g=1,2,...,z)$ by using formula (7.32), we can get

$$T_{ij}^{(1)}=\begin{pmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix},T_{ij}^{(2)}=\begin{pmatrix} 0.5416 & 0.2916 & 0.2916\\ 0.7050 & 0.5296 & 0.6508\\ 0.5296 & 0.3981 & 0.5416\\ 0.5972 & 0.6514 & 0.6145\end{pmatrix},T_{ij}^{(1)}=\begin{pmatrix} 0.2496 & 0.1615 & 0.1649\\ 0.5483 & 0.4393 & 0.3054\\ 0.1667 & 0.2056 & 0.2105\\ 0.2803 & 0.5114 & 0.2992\end{pmatrix}$$

Step 2. Using GHINULPWA operator to aggregate all the individual HINUL decision matrices $M^{g}=\left(\bar{\bar{n}}^{g}{}_{ij}\right)_{m\times n}$ to a collective HINUL decision matrix $M=\left(\bar{\bar{n}}_{ij}\right)_{m\times n}(i=1{,}23{,}4,j=1{,}2{,}3)$ by the following formula (2.33) (assume $\eta=1,\Re^{*}(s_{g})=\dfrac{i}{2t}$), and have

$$\bar{\bar{n}}_1=\begin{bmatrix}\{\langle[s_{2.465},s_{3.465}],[0.4236,0.5236],[0.1141,0.2141],[0.26116,0.36116]\rangle,\\ \langle[s_{2.744},s_{3.744}],[0.4409,0.5409],[0.1150,0.2150],[0.2430,0.3430]\rangle,\\ \langle[s_{3.023},s_{4.023}],[0.4594,0.5594],[0.1396,0.2396],[0.3506,0.3950]\rangle,\\ \langle[s_{3.302},s_{4.302}],[0.4716,0.5716],[0.1385,0.2385],[0.2771,0.3771]\rangle\}\end{bmatrix}$$

$$\{\langle[s_{2.910},s_{3.799}],[0.3701,0.4850],[0.1718,0.2867],[0.1282,0.3150]\rangle$$
$$\langle[s_{3.0215},s_{3.9102}],[0.4031,0.5176],[0.1695,0.2840],[0.1145,0.2900]\rangle$$
$$\langle[s_{3.111},s_{3.799}],[0.3826,0.5],[0.1697,0.2697],[0.1129,0.2826]\rangle$$
$$\langle[s_{3.222},s_{3.910}],[0.4143,0.5312],[0.1675,0.2675],[0.1,0.2675]\rangle\},$$

$$\left\langle\left[s_{3.1131},s_{3.226}\right],\left[0.4457,0.5457\right],\left[0.1514,0.2514\right],\left[0.2354,0.3354\right]\right\rangle$$
$$\left.\left\langle\left[s_{3.113},s_{3.912}\right],\left[0.3435,0.4435\right],\left[0.1855,0.2855\right],\left[0.2710,0.2710\right]\right\rangle\right].$$

$$\overset{=}{n}_2=\left[\begin{array}{l}\left\langle\left[s_{5.000},s_{5.687}\right],\left[0.6414,0.7665\right],\left[0.0283,0.1283\right],\left[0.1,0.1928\right]\right\rangle,\\ \left\langle\left[s_{5.000},s_{6}\right],\left[0.7,0.8243\right],\left[0,0.1\right],\left[0.0698,0.1628\right]\right\rangle\end{array}\right.$$

$$\left[s_{4.223},s_{5.223}\right],\left[0.6007,0.7263\right],\left[0.0740,0.1480\right],\left[0.0740,0.1740\right]$$
$$\left[s_{4.731},s_{5.731}\right],\left[0.7074,0.8306\right],\left[0.0231,0.0997\right],\left[0.0765,0.1498\right]$$

$$\left\langle\left[s_{4.844},s_{5.511}\right],\left[0.5729,0.7050\right],\left[0.0765,0.1394\right],\left[0.1259,0.2209\right]\right\rangle$$
$$\left.\left\langle\left[s_{4.511},s_{5.511}\right],\left[0.5421,0.6720\right],\left[0.1089,0.1889\right],\left[0.1599,0.2399\right]\right\rangle\right]$$

$$\overset{=}{n}_3=\left[\begin{array}{l}\left\langle\left[s_{3.312},s_{4.312}\right],\left[0.4188,0.5278\right],\left[0.1459,0.2459\right],\left[0.1722,0.2631\right]\right\rangle\\ \left\langle\left[s_{3.624}s_{4.312}\right],\left[0.3827,0.4913\right],\left[0.1087,0.2480\right],\left[0.2087,0.3393\right]\right\rangle,\\ \left\langle\left[s_{3.901},s_{4.312}\right],\left[0.4749,0.5832\right]\left[0.1426,0.2139\right]\left[0.1168,0.2084\right]\right\rangle,\\ \left\langle\left[s_{4.214},s_{4.312}\right],\left[0.4392,0.5472\right]\left[0.1081,0.2170\right],\left[0.1528,0.2813\right]\right\rangle\end{array}\right.$$

$$\left\langle\left[s_{3.128},s_{4.000}\right],\left[0.5,0.6\right]\left[0.1100,0.2172\right],\left[0.1,0.2612\right]\right\rangle$$
$$\left\langle\left[s_{3.751},s_{4.375}\right],\left[0.4330,0.5330\right],\left[0.1755,0.2816\right]\left[0.1,0.2\right]\right\rangle$$

$$\left.\left\langle\left[s_{3.880},s_{4.880}\right],\left[0.4808,0.5904\right],\left[0.1,0.2\right],\left[0.1586,0.2586\right]\right\rangle\right]$$

$$\overset{=}{n}_4=\left[\begin{array}{l}\left\langle\left[s_{4.318},s_{5.000}\right]\left[0.4712,0.5712\right]\left[0.1,0.2\right]\left[0.2514,0.3514\right]\right\rangle\\ \left\langle\left[s_{4.850},s_{5.533}\right]\left[0.5305,0.6305\right]\left[0.0436,0.1436\right]\left[0.1436,0.2436\right]\right\rangle\end{array}\right.$$

$$\left\langle\left[s_{4.236},s_{5.236}\right],\left[0.5263,0.6263\right],\left[0.0930,0.1914\right],\left[0.0714,0.1698\right]\right\rangle$$
$$\left\langle\left[s_{4.537},s_{5.538}\right],\left[0.6174,0.7174\right],\left[0.0610,0.1595\right],\left[0.0407,0.1392\right]\right\rangle$$

$$\left\langle\left[s_{4.000},s_{4.843}\right]\left[0.5390,0.6390\right],\left[0.1407,0.2407\right],\left[0.1798,0.2939\right]\right\rangle$$
$$\left\langle\left[s_{4.00},s_{5.000}\right],\left[0.5523,0.6523\right],\left[0.1261,0.2418\right],\left[0.2227,0.2940\right]\right\rangle$$
$$\left\langle\left[s_{4.321},s_{4.843}\right],\left[0.5727,0.6727\right],\left[0.1043,0.2043\right]\left[0.1787,0.2782\right]\right\rangle$$
$$\left\langle\left[s_{4.321},s_{5.000}\right]\left[0.5849,0.6849\right]\left[0.0908,0.1908\right]\left[0.1757,0.2736\right]\right\rangle$$
$$\left\langle\left[s_{4.522},s_{5.366}\right],\left[0.4292,0.5292\right],\left[0.1708,0.2708\right],\left[0.1,0.2126\right]\right\rangle$$
$$\left\langle\left[s_{4.522},s_{5.522}\right],\left[0.4428,0.5428\right],\left[0.1572,0.2572\right],\left[0.1,0.2140\right]\right\rangle$$
$$\left\langle\left[s_{4.843},s_{5.366}\right],\left[0.4629,0.5629\right],\left[0.1371,0.2371\right],\left[0.10.1965\right]\right\rangle$$
$$\left.\left\langle\left[s_{4.843},s_{5.522}\right],\left[0.4755,0.5755\right],\left[0.1245,0.2245\right],\left[0.1,0.1981\right]\right\rangle\right].$$

Step 3. Calculate the values of $T_{ij}\left(i=1,2,3,4,\ j=1,2,3\right)$ by using formula (7.34), we have

$$T_{ij} = \begin{pmatrix} 1 & 0.40437 & 0.149235 \\ 1 & 0.718826 & 0.482424 \\ 1 & 0.470867 & 0.258899 \\ 1 & 0.671732 & 0.400866 \end{pmatrix}$$

Step 4. Aggregate the HINULN $\bar{\bar{n}}_i$ for each alternative $\overline{\overline{N}}_i\,(i=1,2,3,4)$ by the using the following aggregation formula (7.35) (assume $\eta = 1, \Re^*(s_g) = \frac{i}{2t}$), and have

$$\overline{\overline{N}}_1 = \begin{cases} \langle [s_{2.643}, s_{3.529}], [0.4106, 0.5148], [0.1341, 0.2383], [0.22100.3455] \rangle \\ \langle [s_{2.643}, s_{3.595}], [0.3999, 0.5041], [0.1380, 0.2422], [0.2250, 0.3385] \rangle \end{cases}$$

$\langle [s_{2.672}, s_{3.558}], [0.4198, 0.5240], [0.1338, 0.2380], [0.2162, 0.3380] \rangle$
$\langle [s_{2.672}, s_{3.624}], [0.409, 0.5133], [0.1376, 0.2418], [0.2202, 0.3311] \rangle$
$\langle [s_{2.695}, s_{3.529}], [0.4139, 0.5189], [0.1338, 0.2338], [0.2158, 0.3359] \rangle$
$\langle [s_{2.695}, s_{3.595}][0.4033, 0.5082], [0.1377, 0.2377], [0.2198, 0.3290] \rangle$
$\langle [s_{2.724}, s_{3.558}][0.4230, 0.5279], [0.1335, 0.2335], [0.2110, 0.3310] \rangle$

$\langle [s_{2.724}, s_{3.624}], [0.4123, 0.5173], [0.1373, 0.2373], [0.2151, 0.3242] \rangle$
$\langle [s_{2.823}, s_{3.709}], [0.4224, 0.5264], [0.13360.2376], [0.21160.3348] \rangle$
$\langle [s_{2.823}, s_{3.775}], [0.4122, 0.5264], [0.13730.2412], [0.21550.3195] \rangle$
$\langle [s_{2.852}, s_{3.738}], [0.4310, 0.5349], [0.1333, 0.2373], [0.20710.3278] \rangle$
$\langle [s_{2.852}, s_{3.803}], [0.4208, 0.5247], [0.1369, 0.2409][0.2110, 0.3214] \rangle$

$\langle [s_{2.875}, s_{3.709}], [0.4254, 0.5301], [0.1333, 0.2333], [0.2068, 0.3258] \rangle$
$\langle [s_{2.875}, s_{3.775}], [0.4152, 0.5199], [0.1370, 0.2370], [0.2107, 0.3194] \rangle$
$\langle [s_{2.904}, s_{3.738}], [0.4339, 0.5386], [0.1330, 0.2330], [0.2023, 0.3212] \rangle$
$\langle [s_{2.904}, s_{3.803}], [0.4237, 0.5284], [0.1367, 0.2367], [0.2063, 0.3149] \rangle$

$$\langle[s_{3.003},s_{3.888}],[0.4356,0.5394],[0.1488,0.2526],[0.2841,0.3695]\rangle$$
$$\langle[s_{3.003},s_{3.954}],[0.4258,0.5295],[0.1488,0.2526],[0.2836,0.3692]\rangle$$
$$\langle[s_{3.032},s_{3.917}],[0.4436,0.5474],[0.1484,0.2521],[0.2792,0.3626]\rangle$$
$$\langle[s_{3.032},s_{3.983}],[0.4338,0.5375],[0.1484,0.2554][0.2822,0.3561]\rangle$$
$$\langle[s_{3.055},s_{3.888}][0.43830.5428],[0.14840.2484],[0.27890.3607]\rangle$$
$$\langle[s_{3.055},s_{3.954}],[0.4286,0.5330],[0.15174,0.2517],[0.2819,0.3542]\rangle$$
$$\langle[s_{3.084},s_{3.917}][0.44630.5507],[0.14800.2480][0.27410.3560]\rangle$$
$$\langle[s_{3.084},s_{3.983}],[0.4365,0.5409][0.1513,0.2513][0.2772,0.3497]\rangle$$
$$\langle[s_{3.182},s_{4.068}],[0.4450,0.5486],[0.1476,0.2512],[0.2377,0.3586]\rangle$$
$$\langle[s_{3.182},s_{4.134}],[0.43550.5391],[0.1508,0.2544],[0.2410,0.3525]\rangle$$
$$\langle[s_{3.211},s_{4.097}][0.4525,0.5561][0.1472,0.2508][0.2335,0.3521]\rangle$$

$$\left.\begin{matrix}\langle[s_{3.211},s_{4.163}],[0.4431,0.5466],[0.1504,0.2539][0.2367,0.3461]\rangle,\\ \langle[s_{3.234},s_{4.068}],[0.4475,0.5518],[0.1473,0.2473],[0.23320.3503]\rangle,\\ \langle[s_{3.234},s_{4.134}],[0.4381,0.5424][0.1504,0.2504],[0.2364,0.3443]\rangle,\\ \langle[s_{3.263},s_{4.097}],[0.4550,0.5592],[0.1469,0.2469],[0.22900.3460]\rangle,\\ \langle[s_{3.263},s_{4.163}],[0.4456,0.5498],[0.1469,0.2469],[0.2290,0.34591]\rangle\end{matrix}\right\}.$$

$$\overline{\overline{N}}_2=\left\{\begin{matrix}\langle[s_{4.712},s_{5.497}],[0.6139,0.7407],[0.0528,0.1367],[0.0979,0.1934]\rangle\\ \langle[s_{4.639},s_{5.497}],[0.6075,0.7338],[0.0597,0.1474],[0.1051,0.1973]\rangle\\ \langle[s_{4.878},s_{5.663}][0.6481,0.7740][0.0370,0.1214],[0.0980,0.1849]\rangle\end{matrix}\right.$$
$$\langle[s_{4.805},s_{5.663}],[0.6421,0.7676],[0.0435,0.1317],[0.1049,0.1887]\rangle$$
$$\langle[s_{4.712},s_{5.639}],[0.6425,0.7690],[0.0388,0.1230],[0.0833,0.1789]\rangle$$
$$\left.\begin{matrix}\langle[s_{4.639},s_{5.639}],[0.6365,0.7624],[0.0455,0.1334],[0.0903,0.1826]\rangle\\ \langle[s_{4.878},s_{5.805}],[0.6754,0.7624],[0.0237,0.1083],[0.0838,0.1710]\rangle\\ \langle[s_{4.805},s_{5.805}],[0.6697,0.7948],[0.0300,0.1183],[0.0906,0.1746]\rangle\end{matrix}\right\}$$
$$\overline{\overline{N}}_3=\left\{\begin{matrix}\langle[s_{3.347},s_{4.312}],[0.4500,0.5568],[0.1289,0.2308],[0.1516,0.2619]\rangle\\ \langle[s_{3.527},s_{4.312}],[0.4281,0.5348],[0.1075,0.2323],[0.1734,0.3065]\rangle\end{matrix}\right.$$
$$\langle[s_{3.688},s_{4.312}],[0.4819,0.5885],[0.1277,0.2124],[0.1196,0.2294]\rangle$$
$$\langle[s_{3.868},s_{4.312}],[0.4603,0.5667],[0.1072,0.2143],[0.1412,0.2729]\rangle$$
$$\langle[s_{3.517},s_{4.414}],[0.4330,0.5396],[0.1466,0.2483],[0.1498,0.2448]\rangle$$
$$\langle[s_{3.697},s_{4.414}],[0.4123,0.5187],[0.1255,0.2494],[0.1709,0.2883]\rangle$$
$$\left.\begin{matrix}\langle[s_{3.857},s_{4.414}],[0.4646,0.5709],[0.1446,0.2298],[0.1189,0.2141]\rangle\\ \langle[s_{4.038},s_{4.414}][0.4440,0.5502],[0.1245,0.2313],[0.1399,0.2565]\rangle\end{matrix}\right\}$$

$$\overline{\overline{N}}_4 = \begin{Bmatrix} \langle [s_{4.230}, s_{5.046}], [0.5019, 0.6019], [0.1052, 0.2047], [0.1786, 0.2807] \rangle \\ \langle [s_{4.230}, s_{5.076}], [0.5045, 0.6045], [0.1026, 0.2050], [0.1867, 0.2808] \rangle \end{Bmatrix}$$

$\langle [s_{4.292}, s_{5.046}], [0.5086, 0.6086], [0.0985, 0.1980], [0.17840.2778] \rangle$

$\langle [s_{4.292}, s_{5.076}] [0.5111, 0.6111], [0.0959, 0.1954], [0.1779, 0.2769] \rangle$

$\langle [s_{4.331}, s_{5.147}], [0.4806, 0.5806], [0.1120, 0.2115], [0.1626, 0.2646] \rangle,$

$\langle [s_{4.331}, s_{5.178}], [0.4832, 0.5832], [0.1094, 0.2089], [0.1624, 0.2647] \rangle,$

$\langle [s_{4.393}, s_{5.147}], [0.4872, 0.5872], [0.10540.2049], [0.16220.2609] \rangle,$

$\langle [s_{3.456}, s_{5.178}], [0.4898, 0.5898], [0.1029, 0.2024], [0.16200.2611] \rangle,$

$\langle [s_{4.327}, s_{5.144}], [0.5338, 0.6338], [0.0939, 0.1934], [0.1658, 0.2679] \rangle$

$\langle [s_{4.327}, s_{5.174}], [0.5363, 0.6363], [0.0914, 0.1937], [0.1737, 0.2680] \rangle$

$\langle [s_{4.390}, s_{5.144}], [0.5401, 0.6401], [0.0874, 0.1869], [0.1657, 0.2651] \rangle$

$\langle [s_{4.390}, s_{5.174}], [0.5425, 0.6425], [0.0850, 0.1844], [0.1652, 0.2643] \rangle$

$\langle [s_{4.428}, s_{5.245}], [0.5122, 0.6122], [0.1008, 0.2003], [0.1504, 0.2524] \rangle$

$\langle [s_{4.428}, s_{5.275}], [0.5147, 0.6147], [0.0983, 0.1978], [0.1502, 0.2525] \rangle$

$\langle [s_{4.491}, s_{5.245}], [0.5185, 0.6185], [0.0945, 0.1940], [0.1500, 0.2488] \rangle$

$\langle [s_{4.491}, s_{5.275}], [0.52100.6210], [0.0920, 0.1915], [0.1499, 0.2490] \rangle$

$\langle [s_{4.487}, s_{5.303}], [0.5307, 0.6307], [0.0760, 0.1756], [0.1273, 0.2292] \rangle$

$\langle [s_{4.487}, s_{5.333}], [0.5331, 0.6331] [0.0737, 0.1759], [0.1350, 0.2295] \rangle$

$\langle [s_{4.549}, s_{5.303}], [0.5368, 0.6368], [0.0699, 0.1694], [0.1274, 0.2268] \rangle$

$\langle [s_{4.549}, s_{5.333}], [0.5391, 0.6391], [0.0675, 0.1670], [0.1270, 0.2261] \rangle$

$\langle [s_{4.588}, s_{5.404}], [0.5098, 0.6098], [0.0831, 0.1826], [0.1131, 0.2150] \rangle$

$\langle [s_{4.588}, s_{5.435}], [0.5122, 0.6122], [0.0807, 0.1803], [0.1130, 0.2152] \rangle$

$\langle [s_{4.650}, s_{5.404}], [0.5160, 0.6160], [0.0770, 0.1766], [0.1130, 0.2118] \rangle$

$\langle [s_{4.650}, s_{5.435}], [0.5183, 0.6183], [0.0747, 0.1742], [0.1129, 0.2121] \rangle$

$\langle [s_{4.584}, s_{5.401}], [0.5604, 0.6604], [0.0659, 0.1654], [0.1161, 0.2181] \rangle$

$\langle [s_{4.584}, s_{5.431}], [0.5626, 0.6626], [0.0636, 0.1658], [0.1238, 0.2183] \rangle$

$\langle [s_{4.6465}, s_{5.401}], [0.5662, 0.6662], [0.0600, 0.1595], [0.1163, 0.2158] \rangle$

$\langle [s_{4.647}, s_{5.431}], [0.5684, 0.6684], [0.0577, 0.1572], [0.1160, 0.2151] \rangle$

$\langle [s_{4.686}, s_{5.502}], [0.5394, 0.6394], [0.0730, 0.1726], [0.1024, 0.2043] \rangle$

$$\left. \begin{matrix} \langle [s_{4.686}, s_{5.532}] [0.5416, 0.6416], [0.0708, 0.1703] [0.1024, 0.2046] \rangle \\ \langle [s_{4.748}, s_{5.502}] [0.5452, 0.6452], [0.0672, 0.1667] [0.1024, 0.2013] \rangle \\ \langle [s_{4.748}, s_{5.532}] [0.5474, 0.6474], [0.0649, 0.1644] [0.1024, 0.2015] \rangle \end{matrix} \right\}$$

Step 5. Calculate the score values of the alternatives $\overline{\overline{N}}_i \,(i=1,2,3,4)$ by using Definition (2.1.1.5), and get

$$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.3941,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.7375,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.4873,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.6273..$$

Step 6. Ranking order of the alternatives according to their score values:

$$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1..$$

So $\overline{\overline{N}}_2$ is the best alternative.

7.4.2 Influence on the Ranking Results of the Generalized Parameter η

In this subpart, the effects of generalized parameter are discussed.

Table 7. 4. Influence of the parameter η on decision result

η	Score values	Ranking order
$\eta \to 0$,	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.3733,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.7277,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.4811,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.6139.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$\eta = 2$,	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.4142,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.7476,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.4934,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.6387.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$\eta = 5$,	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.4620,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.7740,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.5089,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.6715.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$
$\eta = 7$,		$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1.$

	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.4852,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.7887,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.5174,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.6915.$	
$\eta=15$,		$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.5292,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.8279,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.5424,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.7492.$	
$\eta=25$,		$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_1>\overline{\overline{N}}_3.$
	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.5757,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.8509,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.5613,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.7831.$	

In order to show the effects on the ranking results of the generalized parameter η for this example, we can use the different generalized parameter η in steps 2 and 4, and get the ranking results shown in Table 7.4.

From Table 7.4, we can see the ranking orders obtained by using different values of the parameter η are not always same, and when the value of $\eta=25$, they are changed, but the best alternative remains the same.

7.4.3 Influence on the Ranking Results of LSF

In order to show the effects on the ranking results of LSF for this example, we can use the different LSFs in steps 2 and 4, and get the ranking results shown in Table 7.5.

From Table 7.5, we can see the ranking orders obtained by using different LSFs are same.

Table 7.5. Score values and ranking order using different LSFs

$\Re_1^*(s_\vartheta)$	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.3941,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.7375,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.4873,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.62$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$
$\Re_2^*(s_\vartheta)$	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.3530,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.6604,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.4315,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.5$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1$
$\Re_3^*(s_\vartheta)$	$\overline{\overline{SO}}\left(\overline{\overline{N}}_1\right)=0.3665,\overline{\overline{SO}}\left(\overline{\overline{N}}_2\right)=0.7060,\overline{\overline{SO}}\left(\overline{\overline{N}}_3\right)=0.4712,\overline{\overline{SO}}\left(\overline{\overline{N}}_4\right)=0.609$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1.$

7.4.4 Compared With the Existing Method

Compared with the method proposed by Ye [43], the above ranking order is the same as the one obtained in Ye [43]. Compared with the relative decision making method based on INULSs, the decision making method in this chapter use HINUL information, while the decision making methods in [43] use INUL information and also the proposed aggregation operators in this chapter can handle decision making problems in the HINUL environment, where criteria takes different priority levels, and the criteria weights are obtained by using PA operators according to priority of the priority level and are more reasonable than a set of known one. Since HINULSs is a further extension of the concept of INULSs. HINUL information include INUL information. Therefore, the group decision making method proposed in this article can deal with not only HINUL information INUL information. To some extent, the group decision making method in this chapter, is more general and feasible than the existing method INUL setting [43].

7.4.5 Conclusion

The chapter presented the concept of HINULSs based on the combination of INULSs and HFSs, and as a further generalization of these fuzzy concepts and defined some basic operational rules for HINULEs and the score, accuracy, certainty functions of HINULEs respectively, some of their properties were investigated. Then, based on these operational rules we defined some aggregation operators, such as HINULPWA operator and a HINULPWG operator to aggregate HINUL information. Furthermore some desired properties of the two operators were investigated. Moreover, GHINULPWA operator and some special cases of GHINULPWA operator are investigated. After that GHINUPLWA operator was applied to a group decision making under HINUL environment, where values of the attributes with respect to the alternatives takes the form of HINULEs and the attributes and experts weight are known information. We utilize the score function, (accuracy and certainty functions) to rank the alternatives and select the best ones. Lastly, an illustrative example was provided to demonstrate the application of the proposed group decision making method. The main advantage of the developed method is that, it can defined the incomplete, indeterminate and inconsistent information by several INULNs in which the uncertain linguistic variable indicate whether attribute is good or bad in qualitative and INNs are adopted to demonstrate the satisfaction degree, dissatisfaction degree and indeterminacy degree to an uncertain linguistic variable in quantitative. Therefore, the proposed MAGDM method under HINUL environment is more suitable for real science and engineering applications. In future, we shall develop some more aggregation operators and apply them to MADM, medical diagnosis, and expert system.

Chapter 8

Multiple-Attribute Decision Making Based on Single-Valued Neutrosophic Schweizer-Sklar Prioritized aggregation Operators

In this chapter, we enlarge SS TN and TCN to SVNN and give the SS operational laws of SVNNs. Then, we merge prioritize aggregation (PRA) operator with SS operations, and develop the single-valued neutrosophic Schweizer-Sklar prioritized weighted averaging (SVNSSPRWA) operator, single-valued neutrosophic Schweizer-Sklar prioritized ordered weighted averaging (SVNSSPROWA) operator, single-valued neutrosophic Schweizer-Sklar prioritized weighted geometric (SVNSSPRWG) operator, and single-valued neutrosophic Schweizer-Sklar prioritized ordered weighted geometric (SVNSSPROWG) operator. Moreover, we study some useful characteristics of these proposed aggregation operators (AOs) and propose two models on the basis of SVNSSPRWA and SVNSSPRWG operators. At the same time, we apply these two methods to deal with multiple-attribute decision making (MADM) problems under SVN information. Lastly, an illustrative example about talent introduction is given to testify the effectiveness of the developed methods.

8.1 Schweizer-Sklar Operations For SVNNs

In this section, we develop some operational rules for SVNNs over SS T-norm and T-conorm.

8.1.1 Schweizer-Sklar product and sum

SS operations consist of the SS product and SS sum, which are special cases of ATT, respectively.

8.1.1.1 Definition [6]

Let $SV_1 = \langle \Xi_{SV_1}, \Psi_{SV_1}, \Upsilon_{SV_1} \rangle$ and $SV_2 = \langle \Xi_{SV_2}, \Psi_{SV_2}, \Upsilon_{SV_2} \rangle$ be two SVNSs. Then, the generalized union and intersection are described as follows:

$$SV_1 \bigcup_{T,T^*} SV_2 = \left\{ \left\langle \wp, T^*\left(\Xi_{SV_1}, \Xi_{SV_2}\right), T\left(\Psi_{SV_1}, \Psi_{SV_2}\right), T\left(\Upsilon_{SV_1}, \Upsilon_{SV_2}\right) \right\rangle \mid \wp \in \aleph \right\}, \tag{8.1}$$

$$SV_1 \bigcap_{T,T^*} SV_2 = \left\{ \left\langle \wp, T\left(\Xi_{SV_1}, \Xi_{SV_2}\right), T^*\left(\Psi_{SV_1}, \Psi_{SV_2}\right), T^*\left(\Upsilon_{SV_1}, \Upsilon_{SV_2}\right) \right\rangle \mid \wp \in \aleph \right\}. \tag{8.2}$$

Where, T and T^*, expressed T-norm (TN) and T-conorm (TCN) respectively.

The Schweizer-Sklar (SS) TN and TCN [107] are described as follows:

$$T_{SS}\left(\partial, \wp\right) = \left(\partial^{\Re} + \wp^{\Re} - 1\right)^{\frac{1}{\Re}}, \tag{8.3}$$

$$T^*_{SS}\left(\partial, \wp\right) = 1 - \left(\left(1-\partial\right)^{\Re} + \left(1-\wp\right)^{\Re} - 1\right)^{\frac{1}{\Re}}. \tag{8.4}$$

Where, $\Re < 0, \partial, \wp \in [0,1]$.

Additionally, when $\Re = 0,$ we have $T_\chi(\partial,\wp) = \partial\wp$ and $T_\chi^*(\partial,\wp) = \partial + \wp - \partial\wp$. That is, SS TN and TCN reduce to algebraic TN and TCN.

Now, in the next subsection, based on TN $T_\chi(\partial,\wp)$ and TCN $T_\chi^*(\partial,\wp)$ of SS operations, we can permit the following definition about SS operations of SVNNs.

8.1.2 Schweizer-Sklar operations for SVNNs

8.1.2.1 Definition

Assume $\hbar_1 = \langle \Xi_1, \Psi_1, \Upsilon_1 \rangle$ and $\hbar_2 = \langle \Xi_2, \Psi_2, \Upsilon_2 \rangle$ are any two SVNNs. Then based on SS operations, the generalized union and intersection are introduced as follows:

$$\hbar_1 \oplus_{T^*,T} \hbar_2 = \left\langle T^*(\Xi_1, \Xi_2), T(\Psi_1, \Psi_2), T(\Upsilon_1, \Upsilon_2) \right\rangle, \tag{8.5}$$

$$\hbar_1 \otimes_{T,T^*} \hbar_2 = \left\langle T(\Xi_1, \Xi_2), T^*(\Psi_1, \Psi_2), T^*(\Upsilon_1, \Upsilon_2) \right\rangle. \tag{8.6}$$

On the basis of Definition (1.1.1.3) and Equation (8.5), and Equation (8.6), we can introduce the SS operations of SVNNs are described as follows $(\chi < 0, ƛ > 0)$:

(1) $$\hbar_1 \otimes_{SS} \hbar_2 = \left\langle \left(\Xi_1^\Re + \Xi_2^\Re - 1\right)^{\frac{1}{\Re}}, 1 - \left(\left(1-\Psi_1\right)^\Re + \left(1-\Psi_2\right)^\Re - 1\right)^{\frac{1}{\Re}}, 1 - \left(\left(1-\Upsilon_1\right)^\Re + \left(1-\Upsilon_2\right)^\Re - 1\right)^{\frac{1}{\Re}} \right\rangle, \tag{8.7}$$

(2) $$\hbar_1 \otimes_{SS} \hbar_2 = \left\langle 1 - \left(\left(1-\Xi_1\right)^\Re + \left(1-\Xi_2\right)^\Re - 1\right)^{\frac{1}{\Re}}, \left(\Psi_1^\Re + \Psi_2^\Re - 1\right)^{\frac{1}{\Re}}, \left(\Upsilon_1^\Re + \Upsilon_2^\Re - 1\right)^{\frac{1}{\Re}} \right\rangle, \tag{8.8}$$

(3) $$\hbar_1^{ƛ} = \left\langle \left(ƛ\Xi_1^\Re - (ƛ - 1)\right)^{\frac{1}{\Re}}, 1 - \left(ƛ\left(1-\Psi_1\right)^\Re - (ƛ-1)\right)^{\frac{1}{\Re}}, 1 - \left(ƛ\left(1-\Upsilon_1\right)^\Re - (ƛ-1)\right)^{\frac{1}{\Re}} \right\rangle, \tag{8.9}$$

(4) $ƛ\hbar_1 = \left\langle 1-\left(ƛ\left(1-\Xi_1\right)^{\Re}-\left(ƛ-1\right)\right)^{\frac{1}{\Re}}, \left(ƛ\Psi_1^{\Re}-\left(ƛ-1\right)\right)^{\frac{1}{\Re}}, \left(ƛ\Upsilon_1^{\Re}-\left(ƛ-1\right)\right)^{\frac{1}{\Re}} \right\rangle.$ (8.10)

8.1.2.2 Theorem

Let $\hbar_1 = \langle \Xi_1, \Psi_1, \Upsilon_1 \rangle$ and $\hbar_2 = \langle \Xi_2, \Psi_2, \Upsilon_2 \rangle$ be any two SVNNs, then

(1) $\hbar_1 \oplus_{SS} \hbar_2 = \hbar_2 \oplus_{SS} \hbar_1$, (8.11)

(2) $\hbar_1 \otimes_{SS} \hbar_2 = \hbar_2 \otimes_{SS} \hbar_1$, (8.12)

(3) $ƛ\left(\hbar_1 \oplus_{SS} \hbar_2\right) = ƛ\hbar_1 \oplus_{SS} ƛ\hbar_2, \quad ƛ \geq 0;$ (8.13)

(4) $ƛ_1\hbar_1 \oplus_{SS} ƛ_2\hbar_1 = \left(ƛ_1 + ƛ_2\right)\hbar_1, \quad ƛ_1, ƛ_2 \geq 0;$ (8.14)

(5) $\hbar_1^{ƛ} \otimes_{SS} \hbar_2^{ƛ} = \left(\hbar_1 \otimes_{SS} \hbar_2\right)^{ƛ}, \quad ƛ \geq 0,$ (8.15)

(6) $\hbar_1^{ƛ_1} \otimes_{SS} \hbar_1^{ƛ_2} = \left(\hbar_1\right)^{\left(ƛ_1 + ƛ_2\right)}, ƛ_1, ƛ_2 \geq 0.$ (8.16)

8.1.3 Single-Valued Neutrosophic Scheweizer-Sklar Prioritized Weighted Operator

In this part, for a group of SVNNs $\hbar_g = \left\langle \Xi_g, \Psi_g, \Upsilon_g \right\rangle, (g = 1, 2, ..., s)$ symbolized by Ω, we introduce a few new PRAOs for SVNNs, namely SVNSSPRAWA operator, SVNSSPRAOWA operator, and discuss some characteristics of these developed aggregation operators.

8.1.4 Single valued neutrosophic Schweizer-Sklar prioritized weighted averaging (SVNSSPRAWA) operator

8.1.4.1 Definition

A SVN Schweizer-Sklar prioritized weighted averaging (SVNSSPRWA) operator is a function $SVNSSPRWA:\Omega^s \to \Omega$, which is described as:

$$SVNSSPRWA(\hbar_1,\hbar_2,...,\hbar_s)=\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\hbar_g. \tag{8.17}$$

Where, $\overline{\overline{T}}_1=1,$ and $\overline{\overline{T}}_l=\bigotimes_{l=1}^{s-1}\overline{\overline{SC}}(\hbar_l),(l=2,3,....,s)$. Here, $\overline{\overline{SC}}(\hbar_l)$ expresses the score value of SVNN $\hbar_l$.

8.1.4.2 Theorem

For a group of SVNNs $\hbar_g=\langle\Xi_g,\Psi_g,\Upsilon_g\rangle,(g=1,2,...,s)$, the value aggregated by utilizing developed SVNPRWA operator is still a SVNN and is specified by:

$$SVNSSPRWA(\hbar_1,\hbar_2,....,\hbar_s)=\left\langle 1-\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}(1-\Xi_g)^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\right.$$
$$\left.\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\Psi_g^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\Upsilon_g^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle. \tag{8.18}$$

Proof: Firstly, Equation (8.18) will be proved by utilizing mathematical induction (MI). The following steps of MI have been followed:

Step 1. For $g=2$, we have

$$SVNSSPRWA(\hbar_1,\hbar_2)=\bigoplus_{g=1}^{2}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\hbar_g,$$

$$=\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\hbar_1\oplus\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\hbar_2 \tag{8.19}$$

From the operational laws for SVNNs, proposed in Definition (8.1.2.1), we have

$$\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\hbar_1=$$

$$\left\langle 1-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}(1-\Xi_1)^{\frac{1}{\Re}}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Psi_1^{\ \Re}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Upsilon_1^{\ \Re}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right\rangle,$$

and

$$\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\hbar_2=$$

$$\left\langle 1-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}(1-\Xi_2)^{\frac{1}{\Re}}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Psi_2^{\ \Re}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Upsilon_2^{\ \Re}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right\rangle.$$

So, Equation (8.19) becomes

$$\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\hbar_1 \oplus \frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\hbar_2 =$$

$$\left\langle 1-\left(\left(1-\left(1-\left(\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\left(1-\Xi_1\right)^{\Re}\right)-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)\right)^{\Re}+\left(1-\left(1-\left(\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\left(1-\Xi_2\right)^{\Re}\right)-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)\right)^{\Re}-1\right)^{\frac{1}{\Re}},\right.$$

$$\left(\left(\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Psi_1^{\Re}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)^{\Re}+\left(\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Psi_2^{\chi}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)^{\Re}-1\right)^{\frac{1}{\Re}},$$

$$\left.\left(\left(\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Upsilon_1^{\Re}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)^{\Re}+\left(\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Upsilon_2^{\Re}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)^{\Re}-1\right)^{\frac{1}{\Re}}\right\rangle,$$

$$=\left\langle 1-\left(\bigoplus_{g=1}^{2}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\left(1-\Xi_g\right)^{\Re}-\bigoplus_{g=1}^{2}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_{\Re}}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{g=1}^{2}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Psi_g^{\Re}-\bigoplus_{g=1}^{2}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{g=1}^{2}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Upsilon_g^{\Re}-\bigoplus_{g=1}^{2}\frac{T_{=g}}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle.$$

i.e. when $g=2$, Equation (8.18) is true.

Step 2. Assume that for $g=r$, Equation (8.18) is true. i.e.,

$$SVNSSPRWA\left(\hbar_1,\hbar_2,....,\hbar_r\right)=\left\langle 1-\left(\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}\left(1-\Xi_g\right)^{\Re}-\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\right.$$

$$\left.\left(\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}\Psi_g^{\Re}-\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\chi}},\left(\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}\Upsilon_g^{\Re}-\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle. \quad (8.20)$$

Then, for $g = r+1$, according to the operational rules developed for SVNNs in Definition (8.1.2.1), we have

$$\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\hbar_{r+1} =$$

$$\left\langle 1-\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\left(1-\Xi_{r+1}\right)^{\frac{1}{\Re}}-\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}, \left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\Psi_{r+1}{}^{\Re}-\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\chi}}, \left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\Upsilon_{r+1}{}^{\Re}-\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right\rangle,$$

and

$$SVNSSPRWA\left(\hbar_1,\hbar_2,....,\hbar_r,\hbar_{r+1}\right) = SVNSSPRWA\left(\hbar_1,\hbar_2,....,\hbar_r\right) \oplus_{SS} \frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\hbar_{r+1},$$

$$= \left\langle 1-\left(\left(1-\left(1-\left(\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}\left(1-\Xi_g\right)^{\Re}-\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right)\right)^{\Re}+\left(1-\left(1-\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\left(1-\Xi_{r+1}\right)^{\Re}-\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)\right)^{\Re}-1\right)^{\frac{1}{\Re}},\right.$$

$$\left(\left(\left(\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_u}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}\Psi_g^{\Re}-\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right)^{\Re}+\left(\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\Xi_{r+1}^{\Re}-\left(\bigoplus_{g=1}^{r+1}\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)^{\Re}-1\right)^{\frac{1}{\Re}},$$

$$\left.\left(\left(\left(\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_u}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}\Psi_g^{\Re}-\bigoplus_{g=1}^{r}\frac{\overline{\overline{T}}_u}{\bigoplus_{g=1}^{r}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right)^{\Re}+\left(\left(\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}\Upsilon_{r+1}^{\Re}-\left(\bigoplus_{g=1}^{r+1}\frac{\overline{\overline{T}}_{r+1}}{\bigoplus_{g=1}^{r+1}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)^{\Re}-1\right)^{\frac{1}{\Re}}\right\rangle,$$

So, when $g = r+1$, Equation (8.18) is true. Therefore, Equation (8.18) is true for all s.

When $\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g} \geq 0$, such that $\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g} = 1$, then, Equation (8.18) degenerates into the following form:

$$SVNSSPRWA\left(\hbar_1,\hbar_2,....,\hbar_s\right)=\left\langle 1-\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\left(1-\Xi_g\right)^{\Re}\right)^{\frac{1}{\Re}},\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\Psi_g^{\Re}\right)^{\frac{1}{\Re}},\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\Upsilon_g^{\Re}\right)^{\frac{1}{\Re}}\right\rangle. \quad (8.21)$$

8.1.4.3 Example

Let $\hbar_1=\langle 0.3,0.4,0.5\rangle$, $\hbar_2=\langle 0.4,0.2,0.1\rangle$ and $\hbar_3=\langle 0.6,0.1,0.2\rangle$ be three SVNNs. Based on the score function of SVNNs, we get $\overline{\overline{SO}}(\hbar_1)=0.4667, \overline{\overline{SO}}(\hbar_2)=0.7$ and $\overline{\overline{SO}}(\hbar_3)=0.7667$, and hence $\overline{\overline{T}}_1=1, \overline{\overline{T}}_2=0.4667$ and $\overline{\overline{T}}_3=0.3267$. By using this information, $(\chi=-2)$

$$SVNSSPRAWA\left(\hbar_1,\hbar_2,\hbar_3\right)=$$

$$\left\langle 1-\left(\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_u}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}\left(1-\Xi_g\right)^{\Re}-\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}\Psi_g^{\Re}-\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}\Upsilon_g^{\Re}-\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle.$$

$$=\langle 0.4226,0.1883,0.1746\rangle.$$

8.1.4.4 Theorem

For a group of SVNNs $\hbar_u=\langle\Xi_u,\Psi_u,\Upsilon_u\rangle,(u=1,2,...,s)$, the SVNPRWA operator satisfies the following characteristics:

(1) **(Idempotency)** If all $\hbar_u\,(u=1,2,...,s)$ are equal, i.e., $\hbar_u=\hbar=\langle\Xi,\Psi,\Upsilon\rangle$; then

$$SVNSSPRWA\left(\hbar_1,\hbar_2,....,\hbar_s\right)=\hbar. \quad (8.22)$$

Proof. Since $\hbar_u=\hbar=\langle\Xi,\Psi,\Upsilon\rangle$; for all $\hbar.$ so,

$$SVNSSPRWA(\hbar_1,\hbar_2,....,\hbar_s)=$$

$$\left\langle 1-\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}(1-\Xi_u)^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}},\left(\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Psi_u^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Upsilon_u^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}}\right\rangle,$$

$$=\left\langle 1-\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}(1-\Xi)^{\Re}\right)^{\frac{1}{\Re}},\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Psi^{\Re}\right)^{\frac{1}{\Re}},\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Upsilon^{\Re}\right)^{\frac{1}{\Re}}\right\rangle,$$

$$=\left\langle 1-\left((1-\Xi)^{\Re}\right)^{\frac{1}{\Re}},\left(\Psi^{\Re}\right)^{\frac{1}{\Re}},\left(\Upsilon^{\Re}\right)^{\frac{1}{\Re}}\right\rangle,$$

$$=\langle\Xi,\Psi,\Upsilon\rangle.$$

(2) **(Monotonicity)** If $\hbar_u'=\left\langle\Xi'_u,\Psi'_u,\Upsilon'_u\right\rangle$ and $\hbar_u=\langle\Xi_u,\Psi_u,\Upsilon_u\rangle$ are two groups of SVNNs, such that $\hbar_u'\geq\hbar_u$. i.e., $\Xi'_u\geq\Xi_u,\Psi'_u\leq\Psi_u$ and $\Upsilon'_u\leq\Upsilon_u$ for all $h,$ then

$$SVNSSPRWA(\hbar_1',\hbar_2',....,\hbar_s')\geq SVNSSPRWA(\hbar_1,\hbar_2,....,\hbar_s). \tag{8.23}$$

Proof. Since $\hbar_u'\geq\hbar_u$, which implies $\Xi'_u\geq\Xi_u$ and so

$$1-\Xi'_u\leq 1-\Xi_u$$

$$0\leq\left(1-\Xi'_u\right)^{\Re}\leq(1-\Xi_u)^{\Re}\leq 1,$$

$$\Rightarrow\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}'_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}'_u}\left(1-\Xi'_u\right)^{\Re}\leq\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}(1-\Xi_u)^{\Re},$$

$$\Rightarrow\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}'_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}'_u}\left(1-\Xi'_u\right)^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}'_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}'_u}+1\leq\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\underline{\underline{T}}_u}(1-\Xi_u)^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1,$$

$$\Rightarrow\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}'_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}'_u}\left(1-\Xi'_u\right)^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}'_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}'_u}+1\right)^{\frac{1}{\Re}}\leq\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\left(1-\Xi_u\right)^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}},$$

$$\Rightarrow 1-\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}'_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}'_u}\left(1-\Xi'_u\right)^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}'_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}'_u}+1\right)^{\frac{1}{\Re}}\geq 1-\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\left(1-\Xi_u\right)^{\Re}-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}}. \quad (8.24)$$

and

Since $\Psi'^{\Re}_u \leq \Psi^{\Re}_u$

$$\bigoplus_{u=1}^{s}\frac{\overline{\overline{T'}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T'}}_u}\Psi'^{\Re}_u\leq\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Psi^{\Re}_u,$$

$$\Rightarrow\bigoplus_{u=1}^{s}\frac{\overline{\overline{T'}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T'}}_u}\Psi'^{\Re}_u-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T'}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T'}}_u}+1\leq\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Psi^{\Re}_u-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1,$$

$$\Rightarrow\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T'}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T'}}_u}\Psi'^{\Re}_u-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T'}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T'}}_u}+1\right)^{\frac{1}{\Re}}\leq\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Psi^{\Re}_u-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}}, \quad (8.25)$$

Similarly, we have

$$\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T'}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T'}}_u}\Upsilon'^{\Re}_u-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T'}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T'}}_u}+1\right)^{\frac{1}{\Re}}\leq\left(\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}\Upsilon^{\Re}_u-\bigoplus_{u=1}^{s}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}}. \quad (8.26)$$

From Equations (8.24),(8.25) and (8.26), we get

$$SVNSSPRWA\left(\hbar'_1,\hbar'_2,....,\hbar'_s\right)\geq SVNSSPRWA\left(\hbar_1,\hbar_2,....,\hbar_s\right).$$

(3) **(Boundedness)** Let $\hbar_u = \langle \Xi_u, \Psi_u, \Upsilon_u \rangle$ be a group of SVNNs, and $\hbar_u^+ = \left\langle \max_{u=1}^{s} \Xi_u, \min_{u=1}^{s} \Psi_u, \min_{u=1}^{s} \Upsilon_u \right\rangle$, , $\hbar_u^- = \left\langle \min_{u=1}^{s} \Xi_u, \max_{u=1}^{s} \Psi_u, \max_{u=1}^{s} \Upsilon_u \right\rangle$. Then,

$$\hbar^- \leq SVNSSPRWA(\hbar_1, \hbar_2, ..., \hbar_s) \leq \hbar^+. \tag{8.27}$$

Proof. Since $\min_{u=1}^{s} \Xi_u \leq \Xi_u \leq \max_{u=1}^{s} \Xi_u, \min_{u=1}^{s} \Psi_u \leq \Psi_u \leq \max_{u=1}^{s} \Psi_u$, and $\min_{u=1}^{s} \Upsilon_u \leq \Upsilon_u \leq \max_{u=1}^{s} \Upsilon_u$,

Hence, from property (2), we have

$SVNSSPRWA(\hbar_1', \hbar_2', ..., \hbar_s') \geq SVNSSPRWA(\hbar_1, \hbar_2, ..., \hbar_s)$.

When $\rho = 0$, then the SVNPRASSWA operator reduces to the PRA operator based on the algebraic operational laws for SVNNs. That is,

$$SVNSSPRWA_{\Re=0}(\hbar_1, \hbar_2, ..., \hbar_s) = \left\langle 1 - \bigotimes_{u=1}^{s} (1 - \Xi_u)^{\frac{\bar{\bar{T}}_u}{\oplus_{u=1}^{q} \bar{\bar{T}}_u}}, \bigotimes_{u=1}^{s} (\Psi_u)^{\frac{\bar{\bar{T}}_u}{\oplus_{u=1}^{s} \bar{\bar{T}}_u}}, \bigotimes_{u=1}^{s} (\Upsilon_u)^{\frac{\bar{\bar{T}}_u}{\oplus_{u=1}^{s} \bar{\bar{T}}_u}} \right\rangle. \tag{8.28}$$

8.1.5 Single-Valued Neutrosophic Schweizer-Sklar Prioritized ordered weighted averaging operator

In this subpart, we develop an AO which merges the conviction of PRWA with ordered weighted operator and called as SVNSSPR ordered weighted averaging (SVNSSOWA) operator.

8.1.5.1 Definition

A SVNPROWA operator is a function $SVNSSPROWA : \Omega^s \to \Omega$, which is described as follows:

$$SVNSSPRWA(\hbar_1, \hbar_2, ..., \hbar_s) = \bigoplus_{u=1}^{s} \frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s} \overline{\overline{T}}_u} \hbar_{\wp(u)}. \tag{8.29}$$

Where, $\overline{\overline{T}}_1 = 1$, and $\overline{\overline{T}}_l = \bigotimes_{l=1}^{s-1} \overline{\overline{SO}}(\hbar_l), (l = 2,3,....,s)$, $\wp$ is a permutation of $(1,2,...,s)$ such that $\wp(a) \geq \wp(a-1)$ for $a = 2,3,...,s.$

8.1.5.2 Theorem

For a group of SVNNs $\hbar_u = \langle \Xi_u, \Psi_u, \Upsilon_u \rangle, (u = 1,2,...,s)$, the value aggregated by utilizing the developed SVNPROWA operator is still a SVNN and is specified by:

$$SVNSSPROWA(\hbar_1, \hbar_2, ..., \hbar_s) = \left\langle 1 - \left(\bigoplus_{u=1}^{s} \frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s} \overline{\overline{T}}_u} (1 - \Xi_{\wp(u)})^{\Re} - \bigoplus_{u=1}^{s} \frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s} \overline{\overline{T}}_u} + 1 \right)^{\frac{1}{\Re}}, \right.$$
$$\left. \left(\bigoplus_{u=1}^{s} \frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s} \overline{\overline{T}}_u} \Psi_{\wp(u)}^{\Re} - \bigoplus_{u=1}^{s} \frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s} \overline{\overline{T}}_u} + 1 \right)^{\frac{1}{\Re}}, \left(\bigoplus_{u=1}^{s} \frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s} \overline{\overline{T}}_u} \Upsilon_{\wp(u)}^{\Re} - \bigoplus_{u=1}^{s} \frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{s} \overline{\overline{T}}_u} + 1 \right)^{\frac{1}{\Re}} \right\rangle. \tag{8.30}$$

Proof: Same as Theorem (8.1.4.2).

8.1.5.3 Example

Consider the SVNNs given in Example 8.1.4.3, we have $\overline{\overline{T}}_1 = 1, \overline{\overline{T}}_2 = 0.4667$ and $\overline{\overline{T}}_3 = 0.3267.$ the score values are $\overline{\overline{SO}}(\hbar_1) = 0.4667, \overline{\overline{SO}}(\hbar_2) = 0.7$ and $\overline{\overline{SO}}(\hbar_3) = 0.7667$. So, we have $\overline{\overline{SO}}(\hbar_3) > \overline{\overline{SO}}(\hbar_2) > \overline{\overline{SO}}(\hbar_1)$ and hence, $\hbar_{\wp(1)} = \hbar_3, \hbar_{\wp(2)} = \hbar_2, \hbar_{\wp(3)} = \hbar_1.$ By using this information $(\chi = -2)$, we can get

$SVNSSPROWA(\hbar_1,\hbar_2,\hbar_3)=$

$$\left\langle 1-\left(\bigoplus_{u=1}^{3}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{3}\overline{\overline{T}}_u}\left(1-\Xi_{\wp(u)}\right)^{\Re}-\bigoplus_{u=1}^{3}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{3}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{u=1}^{3}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{3}\overline{\overline{T}}_u}\Psi_{\wp(u)}^{\Re}-\bigoplus_{u=1}^{3}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{3}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}},\left(\bigoplus_{u=1}^{3}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{3}\overline{\overline{T}}_u}\Upsilon_{\wp(u)}^{\Re}-\bigoplus_{u=1}^{3}\frac{\overline{\overline{T}}_u}{\bigoplus_{u=1}^{3}\overline{\overline{T}}_u}+1\right)^{\frac{1}{\Re}}\right\rangle.$$

$=\langle 0.5327,0.1256,0.1568\rangle$.

8.1.5.4 Theorem

For a group of SVNNs $\hbar_g=\langle\Xi_g,\Psi_g,\Upsilon_g\rangle,(g=1,2,...,s)$, the SVNPROWA operator satisfies the following properties:

(1) **(Idempotency)** If all $\hbar_g\,(g=1,2,...,s)$ are equal, i.e., $\hbar_g=\hbar=\langle\Xi,\Psi,\Upsilon\rangle$; then

$$SVNSSPROWA(\hbar_1,\hbar_2,....,\hbar_s)=\hbar. \tag{8.31}$$

(2) **(Monotonicity)** If $\hbar'_g=\langle\Xi'_g,\Psi'_g,\Upsilon'_g\rangle$ and $\hbar_g=\langle\Xi_g,\Psi_g,\Upsilon_g\rangle$ are two groups of SVNNs, such that $\hbar'_g\geq\hbar_g$. i.e., $\Xi'_g\geq\Xi_g,\Psi'_g\leq\Psi_g$ and $\Upsilon'_g\leq\Upsilon_g$ for all $h,$ then

$$SVNSSPROWA(\hbar'_1,\hbar'_2,....,\hbar'_s)\geq SVNSSPROWA(\hbar_1,\hbar_2,....,\hbar_s). \tag{8.32}$$

(3) **(Boundedness)** Let $\hbar_g=\langle\Xi_g,\Psi_g,\Upsilon_g\rangle$ be a group of SVNNs, and $\hbar_g^+=\left\langle\max_{g=1}^{s}\Xi_g,\min_{g=1}^{s}\Psi_s,\min_{g=1}^{s}\Upsilon_g\right\rangle,,\hbar_g^-=\left\langle\min_{g=1}^{s}\Xi_g,\max_{g=1}^{s}\Psi_g,\max_{g=1}^{s}\Upsilon_g\right\rangle$. Then,

$$\hbar^-\leq SVNSSPROWA(\hbar_1,\hbar_2,....,\hbar_s)\leq\hbar^+. \tag{8.33}$$

8.2 Single-Valued Neutrosophic Schweizer-Sklar Prioritized Weighted Geometric Operator

In this subpart, we develop single-valued neutrosophic Schweizer-Sklar prioritized weighted geometric (SVNSSPRWG) and single-valued neutrosophic Schweizer-Sklar prioritized ordered weighted geometric (SVNSSPROWG) operators. We also discuss some characteristics of the developed aggregation operators.

8.2.1 Single-Valued Neutrosophic Schweizer-Sklar Prioritized Weighted Geometric (SVNSSPRWG) Operator

8.2.1.1 Definition

A single valued neutrosophic Schweizer-Sklar prioritized weighted geometric (SVNSSPRWG) operator is a function $SVNSSPRWG:\Omega^s \to \Omega$, which is described as:

$$SVNSSPRWG\left(\hbar_1,\hbar_2,...,\hbar_s\right)=\bigotimes_{g=1}^{s}\hbar_g^{\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}} \tag{8.34}$$

Where, $\overline{\overline{T}}_1=1,$ and $\overline{\overline{T}}_l=\bigotimes_{l=1}^{s-1}\overline{\overline{SO}}(\hbar_l),\left(l=2,3,....,s\right)$.

8.2.1.2 Theorem

Let $\hbar_g=\left\langle \Xi_g,\Psi_g,\Upsilon_g\right\rangle,\left(g=1,2,...,s\right)$, the value aggregated by utilizing developed SVNPRWG operator is still a SVNN and is specified by:

$$SVNSSPRWG\left(\hbar_1,\hbar_2,....,\hbar_s\right)=\left\langle\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\Xi_g^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\right.$$

$$\left.1-\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{q}\overline{\overline{T}}_g}\left(1-\Psi_g\right)^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_a}+1\right)^{\frac{1}{\Re}},1-\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}\left(1-\Upsilon_g\right)^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle. \quad (8.35)$$

Proof: Firstly, we will prove Equation (8.35), by utilizing MI. The following steps of MI have been followed:

Step 1. For $g=2$, we have

$$SVNSSPRWG\left(\hbar_1,\hbar_2\right)=\bigotimes_{g=1}^{2}\hbar_g^{\frac{\overline{\overline{T}}_a}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}},$$

$$=\hbar_1^{\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}}\otimes\hbar_2^{\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}} \quad (8.36)$$

From the operational laws for SVNNs, proposed in Definition (8.1.2.1), we have

$$\hbar_1^{\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}}=\left\langle\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Xi_1^{\Re}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},1-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\left(1-\Psi_1\right)^{\frac{1}{\Re}}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},1-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\left(1-\Upsilon_1\right)^{\frac{1}{\Re}}-\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right\rangle,$$

and

$$\hbar_2^{\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}}=\left\langle\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\Xi_2^{\Re}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},1-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}\left(1-\Psi_2\right)^{\frac{1}{\Re}}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}},1-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_a}\left(1-\Upsilon_2\right)^{\frac{1}{\Re}}-\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right\rangle,$$

So, Equation (8.36) becomes

$$\hbar_1^{\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}} \otimes \hbar_2^{\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g}} = \left\langle \left(\left(\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} \Xi_1^{\Re} - \left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}} \right)^{\Re} + \left(\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} \Xi_2^{\Re} - \left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}} \right)^{\Re} - 1 \right)^{\frac{1}{\Re}},\right.$$

$$1 - \left(\left(1 - \left(1 - \left(\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} (1-\Psi_1)^{\Re} \right) - \left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}} \right) \right)^{\Re} + \left(1 - \left(1 - \left(\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} (1-\Psi_2)^{\Re} \right) - \left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}} \right) \right)^{\Re} - 1 \right)^{\frac{1}{\Re}},$$

$$\left. ,1 - \left(\left(1 - \left(1 - \left(\left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} (1-\Upsilon_1)^{\Re} \right) - \left(\frac{\overline{\overline{T}}_1}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}} \right) \right)^{\Re} + \left(1 - \left(1 - \left(\left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} (1-\Upsilon_2)^{\Re} \right) - \left(\frac{\overline{\overline{T}}_2}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_a} - 1 \right) \right)^{\frac{1}{\Re}} \right) \right)^{\Re} - 1 \right)^{\frac{1}{\Re}} \right\rangle,$$

$$= \left\langle \left(\bigoplus_{g=1}^{2} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\underline{\underline{T}}_g} \Xi_g^{\Re} - \bigoplus_{g=1}^{2} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}} 1 - \left(\bigoplus_{g=1}^{2} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} (1-\Psi_g)^{\Re} - \bigoplus_{g=1}^{2} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}}, 1 - \left(\bigoplus_{g=1}^{2} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} (1-\Upsilon_g)^{\Re} - \bigoplus_{g=1}^{2} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{2}\overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}} \right\rangle.$$

i.e., when $g = 2$, Equation (8.35) is true.

Step 2. Assume that for $g = c$, Equation (8.35) is true, i.e.,

$$SVNSSPRWG(\hbar_1, \hbar_2, ..., \hbar_c) =$$
$$\left\langle \left(\bigoplus_{g=1}^{c} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g} \Xi_g^{\Re} - \bigoplus_{g=1}^{c} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}}, 1 - \left(\bigoplus_{g=1}^{c} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g} (1-\Psi_g)^{\Re} - \bigoplus_{g=1}^{c} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}}, 1 - \left(\bigoplus_{g=1}^{c} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g} (1-\Upsilon_g)^{\Re} - \bigoplus_{g=1}^{c} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}} \right\rangle. \tag{8.37}$$

Then, for $g = c+1$, according to the operational rules developed for SVNNs in Definition (8.1.2.1), we have

$$\hbar_{c+1}^{\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}} = \left\langle \left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g} \Xi_{c+1}{}^{\Re} - \left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}}, 1 - \left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g} (1-\Psi_{c+1})^{\frac{1}{\Re}} - \left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}}, 1 - \left(\frac{\underline{\underline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\underline{\underline{T}}_g} (1-\Upsilon_{c+1})^{\frac{1}{\Re}} - \left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g} - 1 \right) \right)^{\frac{1}{\Re}} \right\rangle,$$

and

$$SVNSSPRWG\left(\hbar_1,\hbar_2,....,\hbar_c,\hbar_{c+1}\right)=SVNSSPRWG\left(\hbar_1,\hbar_2,....,\hbar_c\right)\otimes_{SS}\hbar_{c+1}^{\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}}$$

$$=\left\langle\left(\left(\left(\left(\bigoplus_{g=1}^{c}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}\Xi_g^{\Re}-\bigoplus_{g=1}^{c}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right)^{\Re}+\left(\left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}\Xi_{c+1}^{\Re}-\left(\bigoplus_{g=1}^{c+1}\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)^{\Re}-1\right)^{\frac{1}{\Re}},\right.$$

$$1-\left(\left(1-\left(1-\left(\bigoplus_{g=1}^{c}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}\left(1-\Xi_c\right)^{\Re}-\bigoplus_{g=1}^{c}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right)\right)^{\Re}+\left(1-\left(1-\left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}\left(1-\Psi_{c+1}\right)^{\Re}-\left(\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)\right)^{\Re}-1\right)^{\frac{1}{\Re}}$$

$$\left.,1-\left(\left(1-\left(1-\left(\bigoplus_{g=1}^{c}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}\left(1-\Upsilon_c\right)^{\Re}-\bigoplus_{g=1}^{c}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right)\right)^{\Re}+\left(1-\left(1-\left(\frac{\overline{\overline{T}}_{c+1}}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}\left(1-\Upsilon_{c+1}\right)^{\Re}-\left(\frac{\overline{\overline{T}}_c}{\bigoplus_{g=1}^{c}\overline{\overline{T}}_g}-1\right)\right)^{\frac{1}{\Re}}\right)\right)^{\Re}-1\right)^{\frac{1}{\Re}}\right\rangle.$$

$$=\left\langle\left(\bigoplus_{g=1}^{c+1}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}\Xi_g^{\Re}-\bigoplus_{g=1}^{c+1}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},1-\left(\bigoplus_{g=1}^{c+1}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}\left(1-\Psi_g\right)^{\Re}-\bigoplus_{g=1}^{c+1}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},1-\left(\bigoplus_{g=1}^{c+1}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}\left(1-\Upsilon_g\right)^{\Re}-\bigoplus_{g=1}^{c+1}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{c+1}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle.$$

So, when $g=c+1$, Equation (8.35) is true. Therefore, Equation (8.35) is true for all h.

When $\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_a}\geq 0,$ such that $\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}=1,$ then, Equation (8.35) degenerates into the following form:

$$SVNSSPRWG(\hbar_1,\hbar_2,....,\hbar_s)=\left\langle \left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_s}\Xi_g^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},\right.$$

$$\left. 1-\left(\bigoplus_{g=1}^{g}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}(1-\Psi_g)^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},1-\left(\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}(1-\Upsilon_g)^{\Re}-\bigoplus_{g=1}^{s}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle. \tag{8.38}$$

8.2.1.3 Example

Let $\hbar_1=\langle 0.5,0.2,0.4\rangle$, $\hbar_2=\langle 0.7,0.3,0.4\rangle$ and $\hbar_3=\langle 0.3,0.4,0.6\rangle$ be three SVNNs. Based on the score function of SVNNs, we get $\overline{\overline{SO}}(\hbar_1)=0.6333, \overline{\overline{SO}}(\hbar_2)=0.6667$ and $\overline{\overline{SO}}(\hbar_3)=0.4333$, and hence $\overline{\overline{T}}_1=1, \overline{\overline{T}}_2=0.6333$ and $\overline{\overline{T}}_3=0.42222$. By using this information $(\Re=-2)$, we can obtain

$$SVNSSPRWG(\hbar_1,\hbar_2,\hbar_3)=\left\langle \left(\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_a}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_a}\Xi_a^{\Re}-\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right.$$

$$\left.,1-\left(\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}(1-\Psi_g)^{\Re}-\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}},1-\left(\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}(1-\Upsilon_g)^{\Re}-\bigoplus_{g=1}^{3}\frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3}\overline{\overline{T}}_g}+1\right)^{\frac{1}{\Re}}\right\rangle.$$

$$=\langle 0.4537,0.2856,0.4648\rangle.$$

8.2.1.4 Theorem

For a group of SVNNs $\hbar_g=\langle \Xi_g,\Psi_g,\Upsilon_g\rangle,(g=1,2,...,s)$, the SVNPROWA operator satisfies the following properties:

(1) **(Idempotency)** If all $\hbar_g\ (g=1,2,...,s)$ are equal, i.e., $\hbar_g=\hbar=\langle\Xi,\Psi,\Upsilon\rangle$; then

$$SVNSSPRWG(\hbar_1,\hbar_2,....,\hbar_s)=\hbar. \tag{8.39}$$

(2) **(Monotonicity)** If $\hbar'_g=\left\langle\Xi'_g,\Psi'_g,\Upsilon'_g\right\rangle$ and $\hbar_g=\left\langle\Xi_g,\Psi_g,\Upsilon_g\right\rangle$ are two groups of SVNNs, such that $\hbar'_g\geq\hbar_g$. i.e., $\Xi'_g\geq\Xi_g,\Psi'_g\leq\Psi_g$ and $\Upsilon'_g\leq\Upsilon_g$ for all $\hbar,$ then

$$SVNSSPRWG(\hbar'_1,\hbar'_2,....,\hbar'_s)\geq SVNSSPRWG(\hbar_1,\hbar_2,....,\hbar_s). \tag{8.40}$$

(3) **(Boundedness)** Let $\hbar_g=\left\langle\Xi_g,\Psi_g,\Upsilon_g\right\rangle$ be a group of SVNNs, and $\hbar_g^+=\left\langle\max_{g=1}^{s}\Xi_s,\min_{g=1}^{s}\Psi_g,\min_{g=1}^{s}\Upsilon_g\right\rangle,$, $\hbar_g^-=\left\langle\min_{g=1}^{s}\Xi_g,\max_{g=1}^{s}\Psi_g,\max_{g=1}^{s}\Upsilon_g\right\rangle$. Then,

$$\hbar^-\leq SVNSSPRWG(\hbar_1,\hbar_2,....,\hbar_s)\leq\hbar^+. \tag{8.41}$$

When $\Re=0,$ the SVNPRASSWG operator reduces to the PG operator based on the algebraic operational laws for SVNNs. That is,

$$SVNSSPRWG_{\Re=0}(\hbar_1,\hbar_2,....,\hbar_s)=\left\langle\bigotimes_{g=1}^{s}(\Xi_g)^{\frac{\bar{\bar{T}}_g}{\bigoplus_{g=1}^{s}\bar{\bar{T}}_g}},1-\bigotimes_{g=1}^{s}(1-\Psi_g)^{\frac{\bar{\bar{T}}_g}{\bigoplus_{g=1}^{s}\bar{\bar{T}}_g}},1-\bigotimes_{g=1}^{s}(1-\Upsilon_g)^{\frac{\bar{\bar{T}}_g}{\bigoplus_{g=1}^{s}\bar{\bar{T}}_g}}\right\rangle. \tag{8.42}$$

8.2.2 Single-Valued Neutrosophic Schweizer-Sklar Prioritized ordered weighted Geometric operator

8.2.2.1 Definition

A SVNPROWG operator is a function $SVNPROWG:\Omega^s\rightarrow\Omega,$ described as follows:

$$SVNSSPROWG(\hbar_1,\hbar_2,....,\hbar_s)=\bigotimes_{g=1}^{s}\hbar_{\wp(g)}^{\frac{\bar{\bar{T}}_g}{\bigoplus_{g=1}^{s}\bar{\bar{T}}_g}}. \tag{8.43}$$

Where, $\overline{\overline{T}}_1 = 1$, and $\overline{\overline{T}}_l = \bigotimes_{l=1}^{s-1} \overline{\overline{SO}}(\hbar_l), (l = 2,3,....,s)$, $\wp$ is a permutation of $(1,2,...,s)$ such that $\wp(a) \geq \wp(a-1)$ for $a = 2,3,...,s.$

8.2.2.2 Theorem

For a group of SVNNs $\hbar_g = \langle \Xi_g, \Psi_g, \Upsilon_g \rangle, (g = 1,2,...,s)$, the value aggregated by the developed SVNPROWG operator is still a SVNN and is specified by:

$$SVNSSPROWG(\hbar_1, \hbar_2, ..., \hbar_s) = \left\langle \left(\bigoplus_{g=1}^{s} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s} \overline{\overline{T}}_g} \Xi_{\wp(g)}^{\Re} - \bigoplus_{g=1}^{s} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s} \overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}}, \right.$$
$$\left. 1 - \left(\bigoplus_{g=1}^{s} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s} \overline{\overline{T}}_g} \left(1 - \Psi_{\wp(g)}\right)^{\Re} - \bigoplus_{g=1}^{s} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s} \overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}}, 1 - \left(\bigoplus_{g=1}^{s} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s} \overline{\overline{T}}_g} \left(1 - \Upsilon_{\wp(g)}\right)^{\Re} - \bigoplus_{g=1}^{s} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{s} \overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}} \right\rangle. \tag{8.44}$$

Proof: Same as Theorem (8.2.1.2).

8.2.2.3 Example

Consider the SVNNs given in Example (8.2.1.2), , we have $\overline{\overline{T}}_1 = 1, \overline{\overline{T}}_2 = 0.6333$ and $\underline{\underline{T}}_3 = 0.42222.$ the score values are $\overline{\overline{SO}}(\hbar_1) = 0.6333, \overline{\overline{SC}}(\hbar_2) = 0.6667$ and $\overline{\overline{SC}}(\hbar_3) = 0.4333$. So, we have $\overline{\overline{SC}}(\hbar_2) > \overline{\overline{SC}}(\hbar_1) > \overline{\overline{SC}}(\hbar_3)$ and hence, $\hbar_{\wp(1)} = \hbar_2, \hbar_{\wp(2)} = \hbar_1, \hbar_{\wp(3)} = \hbar_3$. By using this information $(\Re = -2)$, we can obtain

$$SVNSSPROWG(\hbar_1, \hbar_2, \hbar_3) =$$
$$\left\langle \left(\bigoplus_{g=1}^{3} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3} \overline{\overline{T}}_g} \Xi_{\wp(g)}^{\Re} - \bigoplus_{g=1}^{3} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3} \overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}}, 1 - \left(\bigoplus_{g=1}^{3} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3} \overline{\overline{T}}_g} \left(1 - \Psi_{\wp(g)}\right)^{\Re} - \bigoplus_{g=1}^{3} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3} \overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}}, 1 - \left(\bigoplus_{g=1}^{3} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3} \overline{\overline{T}}_g} \left(1 - \Upsilon_{\wp(g)}\right)^{\rho} - \bigoplus_{g=1}^{3} \frac{\overline{\overline{T}}_g}{\bigoplus_{g=1}^{3} \overline{\overline{T}}_g} + 1 \right)^{\frac{1}{\Re}} \right\rangle,$$

$$= \langle 0.4710, 0.3007, 0.4648 \rangle.$$

8.2.2.4 Theorem

For a group of SVNNs $\hbar_g = \langle \Xi_g, \Psi_g, \Upsilon_g \rangle, (g = 1, 2, ..., s)$, the SVNPROWA operator satisfies the following properties:

(1) **(Idempotency)** If all $\hbar_g\,(g = 1, 2, ..., s)$ are equal, i.e., $\hbar_s = \hbar = \langle \Xi, \Psi, \Upsilon \rangle$; then

$$SVNSSPROWG(\hbar_1, \hbar_2, ..., \hbar_s) = \hbar. \tag{8.45}$$

(2) **(Monotonicity)** If $\hbar'_g = \langle \Xi'_g, \Psi'_g, \Upsilon'_g \rangle$ and $\hbar_g = \langle \Xi_g, \Psi_g, \Upsilon_g \rangle$ are two groups of SVNNs, such that $\hbar'_g \geq \hbar_g$. i.e., $\Xi'_g \geq \Xi_g, \Psi'_g \leq \Psi_g$ and $\Upsilon'_g \leq \Upsilon_g$ for all $h,$ then

$$SVNSSPROWG(\hbar'_1, \hbar'_2, ..., \hbar'_s) \geq SVNSSPROWG(\hbar_1, \hbar_2, ..., \hbar_s). \tag{8.46}$$

(3) **(Boundedness)** Let $\hbar_g = \langle \Xi_g, \Psi_g, \Upsilon_g \rangle, (g = 1, 2, ..., s)$ be a group of SVNNs, and $\hbar_g^+ = \left\langle \max_{g=1}^{s} \Xi_g, \min_{g=1}^{s} \Psi_g, \min_{g=1}^{s} \Upsilon_g \right\rangle, , \hbar_g^- = \left\langle \min_{g=1}^{s} \Xi_g, \max_{g=1}^{s} \Psi_g, \max_{g=1}^{s} \Upsilon_g \right\rangle$. Then,

$$\hbar^- \leq SVNSSPROWG(\hbar_1, \hbar_2, ..., \hbar_s) \leq \hbar^+. \tag{8.47}$$

8.3 The MADM Methods Based on the Proposed Aggregation Operators

In this part, we shall use the SVNPRWA and SVNPRWG operators with SVNNs to solve the MADM problem. The following presumptions or notations are utilized to express the MADM problems. Let the discrete set of alternatives be expressed by $\overline{\overline{N}} = \left\{ \overline{\overline{N}}_1, \overline{\overline{N}}_2, ..., \overline{\overline{N}}_h \right\}$, and the set of attributes be expressed by $\overline{\overline{O}} = \left\{ \overline{\overline{O}}_1, \overline{\overline{O}}_2, ..., \overline{\overline{O}}_h \right\}$, and that there is a prioritization among the attributes represented by the linear-ordering

$\overline{\overline{O}}_1 > \overline{\overline{O}}_2 > .... > \overline{\overline{O}}_{g-1} > \overline{\overline{O}}_g$, the specified attribute $\overline{\overline{O}}_n$ has a higher priority than $\overline{\overline{O}}_m$ if $n < m$. Assume that $\overline{\overline{M}} = \left(\overline{\overline{m}}_{rs}\right)_{h\times g} = (\Xi_{rs}, \Psi_{rs}, \Upsilon_{rs})_{h\times g}$ is the SVNN decision matrix, where Ξ_{rs}, Ψ_{rs} and Υ_{rs} express the TM function, IM function and FM function respectively, such that $\Xi_{rs} \in [0,1], \Psi_{rs} \in [0,1], \Upsilon_{rs} \in [0,1], 0 \le \Xi_{rs} + \Psi_{rs} + \Upsilon_{rs} \le 3, (r = 1,2,..., g,\ s = 1,2,..., h)$. The goal of this problem is to rank the alternatives.

8.3.1 The Method Based on SVNSSPRWA Operator

In the following, a process for ranking and selecting the most preferable alternative(s) is provided as follows.

Step 1. Standardize the decision matrix.

First, the decision making information $\overline{\overline{m}}_{rs}$ in the matrix $\overline{\overline{M}} = \left(\overline{\overline{m}}_{rs}\right)_{h\times g}$ must be standardized. Consequently, the attribute can be grouped into the cost and benefit types. For benefit type attribute, the assessment information does not need to changed, but for cost type attribute, it must be modified with the complement set.

The decision matrix can be standardized by the following formula:

$$\overline{\overline{m}}_{rs} = \begin{cases} \langle \Xi_{rs}, \Psi_{rs}, \Upsilon_{rs} \rangle & \textit{for benefit type attribute } \overline{\overline{O}}_{rs} \\ \langle \Upsilon_{rs}, 1-\Psi_{rs}, \Xi_{rs} \rangle & \textit{for cost type attribute } \overline{\overline{O}}_{rs} \end{cases} \tag{8.48}$$

Step 2. Determine the values of $\underline{\underline{T}}_{rs} (r = 1,2,...,h; s = 1,2,...,g)$ by using the following formula:

$$\underline{\underline{T}}_{rs} = \prod_{l=1}^{s-1} \overline{\overline{SO}}\left(\overline{\overline{m}}_{rl}\right)\left(r = 1,2,...,h; s = 2,3,....,g\right). \tag{8.49}$$

Where, $\underline{\underline{T}}_{r1} = 1$ for $r = 1, 2, ..., h.$

Step 3. Use the decision information from decision matrix $\overline{\overline{M}} = \left(\overline{\overline{m}}_{rs}\right)_{h\times g}$ and the SVNSSPRWA operator given in Equation (8.18),

$$\overline{\overline{m}}_{r} = \langle \Xi_r, \Psi_r, \Upsilon_r \rangle = SVNSSPRWA\left(\overline{\overline{m}}_{r1}, \overline{\overline{m}}_{r2}, ..., \overline{\overline{m}}_{rg}\right). \quad (8.50)$$

To get the overall SVNN, $\overline{\overline{m}}_{r} (r = 1, 2, ..., h)$.

Step 4. Determine the score values $\overline{\overline{SO}}\left(\overline{\overline{m}}_{r}\right)(r = 1, 2, ..., h)$ of the overall SVNNs $\overline{\overline{m}}_{r} (r = 1, 2, ..., h)$ by Definition (1.1.1.6) to rank all the alternatives $\overline{\overline{N}}_{r} (r = 1, 2, ..., h)$.

Step 5. Rank all the alternatives $\overline{\overline{N}}_{r} (r = 1, 2, ..., h)$ and select best one utilizing Theorem (1.1.1.5).

Step 6. End.

8.3.2 The Method Based on SVNSSPRWA Operator

Steps 1 and 2 are same.

Step 3. Use the decision information permitted decision matrix $\overline{\overline{M}} = \left(\overline{\overline{m}}_{rs}\right)_{h\times g}$ and the SVNSSPRWG operator given in Equation (8.34)

$$\overline{\overline{m}}_{r} = \langle \Xi_r, \Psi_r, \Upsilon_r \rangle = SVNSSPRWA\left(\overline{\overline{m}}_{r1}, \overline{\overline{m}}_{r2}, ..., \overline{\overline{m}}_{rg}\right) \quad (8.51)$$

To get the overall SVNN $\overline{\overline{m}}_{r} (r = 1, 2, ..., h)$.

Step 4. Determine the score values $\overline{\overline{SO}}\left(\overline{\overline{m}}_r\right)(r=1,2,...,h)$ of the overall SVNNs $\overline{\overline{m}}_r(r=1,2,....,h)$ by Definition (1.1.1.6) to rank all the alternatives $\overline{\overline{N}}_r(r=1,2,...,h)$.

Step 5. Rank all the alternatives $\overline{\overline{N}}_r(r=1,2,...,h)$ and select best one utilizing Theorem (1.1.1.5).

Step 6. End.

8.4 An Illustrative Examples

In this part, we use a numerical example of selecting third-party logistics (TPL) providers with SVNNs [39] to show the effectiveness and advantages of the developed approach.

8.4.1 Example

An electronic commerce distributer expects to select a suitable TPL provider. Initially, four providers (alternatives) $\overline{\overline{N}}_r(r=1,2,...,4)$ are available for selection and are evaluated by experts with respect to the following four attributes (1) customer satisfaction $\overline{\overline{O}}_1$, (2) service cost $\overline{\overline{O}}_2$, (3) market reputation $\overline{\overline{O}}_3$, and (4) operational experience in the industry $\overline{\overline{O}}_4$. The following priority relationship $\overline{\overline{O}}_1 > \overline{\overline{O}}_2 > \overline{\overline{O}}_3 > \overline{\overline{O}}_4$ among the four attributes is considered by the electronic commerce distributer. The assessment values of the four providers with respect to the four attributes are provided by expert in the form of SVNNs and listed in Table 8.1.

Table 8.1. Decision matrix $\overline{\overline{M}}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	$\langle 0.7,0.1,0.2\rangle$	$\langle 0.3,0.9,0.5\rangle$	$\langle 0.3,0.2,0.1\rangle$	$\langle 0.5,0.1,0.4\rangle$
$\overline{\overline{N}}_2$	$\langle 0.9,0.1,0.1\rangle$	$\langle 0.3,0.8,0.4\rangle$	$\langle 0.5,0.3,0.2\rangle$	$\langle 0.3,0.2,0.4\rangle$
$\overline{\overline{N}}_3$	$\langle 0.5,0.1,0.4\rangle$	$\langle 0.1,0.8,0.7\rangle$	$\langle 0.6,0.2,0.2\rangle$	$\langle 0.8,0.1,0.3\rangle$
$\overline{\overline{N}}_4$	$\langle 0.4,0.3,0.2\rangle$	$\langle 0.2,0.9,0.6\rangle$	$\langle 0.7,0.2,0.1\rangle$	$\langle 0.2,0.2,0.5\rangle$

Table 2. Normalize decision matrix $\overline{\overline{M}}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	$\langle 0.7,0.1,0.2\rangle$	$\langle 0.5,0.1,0.3\rangle$	$\langle 0.3,0.2,0.1\rangle$	$\langle 0.5,0.1,0.4\rangle$
$\overline{\overline{N}}_2$	$\langle 0.9,0.1,0.1\rangle$	$\langle 0.4,0.2,0.3\rangle$	$\langle 0.5,0.3,0.2\rangle$	$\langle 0.3,0.2,0.4\rangle$
$\overline{\overline{N}}_3$	$\langle 0.5,0.1,0.4\rangle$	$\langle 0.7,0.2,0.1\rangle$	$\langle 0.6,0.2,0.2\rangle$	$\langle 0.8,0.1,0.3\rangle$
$\overline{\overline{N}}_4$	$\langle 0.4,0.3,0.2\rangle$	$\langle 0.6,0.1,0.2\rangle$	$\langle 0.7,0.2,0.1\rangle$	$\langle 0.2,0.2,0.5\rangle$

Step 1. Normalize the decision matrix. Since $\overline{\overline{O}}_1$**,** $\overline{\overline{O}}_3$ and $\overline{\overline{O}}_4$ are of benefit type, and $\overline{\overline{O}}_2$ is of cost type attribute. Hence, by using Equation (8.48), the normalized decision matrix is given in Table 8.2.

Step 2. Determine the values of $\underline{\underline{T}}_{rs}\,(r=1,2,...,4; s=1,2,...,4)$ by using the formula (8.49), and get

$$\underline{\underline{T}}_{rs}=\begin{bmatrix}1 & 0.800 & 0.5600 & 0.2987\\ 1 & 0.900 & 0.5700 & 0.3420\\ 1 & 0.6667 & 0.5333 & 0.2607\\ 1 & 0.6333 & 0.4856 & 0.2460\end{bmatrix}$$

Step 3. Use the SVNSSPRWA given in Equation (8.50) to get the overall SVNN $\overline{\overline{m}}_r\,(r=1,2,...,4)$ (assume $\rho=-2$), and obtain

$$\overline{\overline{m}}_1 = \langle 0.6004, 0.1090, 0.1702 \rangle, \overline{\overline{m}}_2 = \langle 0.8367, 0.1431, 0.3204 \rangle,$$
$$\overline{\overline{m}}_3 = \langle 0.6598, 0.1256, 0.1661 \rangle, \overline{\overline{m}}_4 = \langle 0.5653, 0.1597, 0.1618 \rangle.$$

Step 4. Determine the score values $\overline{\overline{SO}}\left(\overline{\overline{m}}_r\right)(r=1,2,...,4)$ of the overall SVNNs $\overline{\overline{m}}_r\,(r=1,2,....,4)$ by Definition (1.1.1.6), and have

$$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.7737, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7911, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7894, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.7479.$$

So, we get $\overline{\overline{m}}_2 > \overline{\overline{m}}_3 > \overline{\overline{m}}_1 > \overline{\overline{m}}_4$.

Step 5. According to score values, ranking order of alternatives is $\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_4$. So the best provider is $\overline{\overline{N}}_2$, while the worst one is $\overline{\overline{N}}_4$.

Similarly, we solve the above Example (8.4.1) by utilizing SVNSSPWG operator:

Step 1 and step 2 are same.

Step 3. Use the SVNSSPRWG operator to get the overall SVNN $\overline{\overline{m}}_r\,(r=1,2,...,4)$. (assume $\rho = -2$), and have

$$\overline{\overline{m}}_1 = \langle 0.4583, 0.1242, 0.2492 \rangle, \overline{\overline{m}}_2 = \langle 0.4662, 0.1949, 0.2434 \rangle,$$
$$\overline{\overline{m}}_3 = \langle 0.5826, 0.1532, 0.2961 \rangle, \overline{\overline{m}}_4 = \langle 0.3951, 0.2278, 0.2438 \rangle.$$

Step 4. Determine the score values $\overline{\overline{SO}}\left(\overline{\overline{m}}_r\right)(r=1,2,...,4)$ of the overall SVNNs $\overline{\overline{m}}_r\,(r=1,2,....,4)$, and have

$$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.6950, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.6760, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7111, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.6412.$$

So, $\overline{\overline{m}}_3 > \overline{\overline{m}}_1 > \overline{\overline{m}}_2 > \overline{\overline{m}}_4$.

Step 5. According to score values, ranking order of alternatives is $\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4$. So, the best provider is $\overline{\overline{N}}_3$, while the worst one is $\overline{\overline{N}}_4$.

8.4.2 Effect of the Parameter ρ on Decision Result of This Example

In order to see the effect of the parameter ρ on the decision-making result, we set the distinct values for the parameter ρ in step 3, to rank the alternatives. The score values and ranking order are described in Table 8.3 and Table 8.4.

As from Table 8.3, we can notice that the ranking orders by utilizing SVNSSPWA operator are slightly different when the parameter ρ takes the distinct values. When the value of the parameter ρ tends to zero, the best choice is $\overline{\overline{N}}_3$ and the worst choice is $\overline{\overline{N}}_2$. When the value of the parameter ρ decreases from -2 then the best choice is $\overline{\overline{N}}_2$ while the worst one is $\overline{\overline{N}}_4$. We can also see from Table 8.3, when the value of the parameter decreases the score values become bigger and bigger.

From Table 8.4, we can see that the ranking orders by utilizing SVNSSPWG operator do not change for different values of the parameter ρ, the best choice is $\overline{\overline{N}}_3$, while the worst one is $\overline{\overline{N}}_4$. We can also notice from Table 4, when the value of the parameter ρ decreases, the score values become smaller and smaller. Generally, different DMs can set different values of the parameter ρ according to their actual need.

Table 8.3. Score values and ranking order for different values of ρ utilizing SVNSSPWA operator for example 5

ρ	Score values	Ranking order
$\rho \to 0$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.7434, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.6932,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7513, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.7103.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_4 > \overline{\overline{N}}_2.$

$\rho = -1$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.7595, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7552,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7718, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.7302.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$
$\rho = -2$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.7682, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7858,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7984, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.7430.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1 > \overline{\overline{N}}_4.$
$\rho = -7$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.8095, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.8401,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.8334, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.7959.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_4.$
$\rho = -20$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.8252, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.8576,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.8554, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.8201.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_4.$
$\rho = -100$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.8317, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.8649,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.8645, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.8308.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_4.$
$\rho = -200$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.8325, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.8658,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.8656, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.8321.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_4.$

Table 8.4. Score values and ranking order for different values of ρ utilizing SVNSSPWG operator

ρ	**Score values**	**Ranking order**
$\rho \to 0$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.7168, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7077,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7244, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.6754.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$
$\rho = -1$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.7059, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.6904,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7176, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.6591.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$
$\rho = -2$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.6853, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.6580,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.7218, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.5998.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$

$\rho = -7$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.6541, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.6315,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.6857, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.5656.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$
$\rho = -20$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.6137, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.5828,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.6584, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.5042.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$
$\rho = -100$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.5768, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.5435,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.6386, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.4740.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$
$\rho = -200$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.5717, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.5384,$ $\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.6359, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.4703.$	$\overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_2 > \overline{\overline{N}}_4.$

8.4.3 Example [40]

In order to reinforce the academic education, the school of management in a Chinese university wants to introduce excellent overseas teachers. This introduction caught much attention from the school, university president, dean of management school and human resource officer sets of a panel of decision makers who will take the whole responsibility for this introduction. The panel made strict assessment for five alternatives (candidates) $\overline{\overline{N}}_r (r = 1, 2, ..., 5)$ from four characteristics (attributes) namely, morality $\overline{\overline{O}}_1$, research potential $\overline{\overline{O}}_2$, skill of teaching $\overline{\overline{O}}_3$, education background $\overline{\overline{O}}_4$. The president of the university has absolute priority in decision making, and the dean of the school of management is next. In addition, this introduction will be in a strict accordance with the principle of combining ability with political integrity. The prioritization among the attributes is as follow, $\overline{\overline{O}}_1 > \overline{\overline{O}}_2 > \overline{\overline{O}}_3 > \overline{\overline{O}}_4$. The decision makers assess possible 5 alternatives $\overline{\overline{N}}_r (r = 1, 2, ..., 5)$ with respect to the 4 attributes $\overline{\overline{O}}_s (s = 1, 2, ..., 4)$ and construct the following SVN decision matrix given in Table 5.

Table 8.5. Decision matrix $\overline{\overline{M}}$

	$\overline{\overline{O}}_1$	$\overline{\overline{O}}_2$	$\overline{\overline{O}}_3$	$\overline{\overline{O}}_4$
$\overline{\overline{N}}_1$	$\langle 0.5,0.8,0.1 \rangle$	$\langle 0.6,0.3,0.3 \rangle$	$\langle 0.3,0.6,0.1 \rangle$	$\langle 0.5,0.7,0.2 \rangle$
$\overline{\overline{N}}_2$	$\langle 0.7,0.2,0.1 \rangle$	$\langle 0.7,0.2,0.2 \rangle$	$\langle 0.7,0.2,0.4 \rangle$	$\langle 0.8,0.2,0.1 \rangle$
$\overline{\overline{N}}_3$	$\langle 0.6,0.5,0.2 \rangle$	$\langle 0.5,0.7,0.3 \rangle$	$\langle 0.5,0.3,0.1 \rangle$	$\langle 0.6,0.3,0.2 \rangle$
$\overline{\overline{N}}_4$	$\langle 0.8,0.1,0.3 \rangle$	$\langle 0.6,0.3,0.2 \rangle$	$\langle 0.6,0.2,0.1 \rangle$	$\langle 0.6,0.2,0.2 \rangle$
$\overline{\overline{N}}_5$	$\langle 0.5,0.5,0.4 \rangle$	$\langle 0.4,0.8,0.1 \rangle$	$\langle 0.7,0.6,0.1 \rangle$	$\langle 0.5,0.8,0.2 \rangle$

Step 1. Normalize the decision matrices by using Equation (8.49). Since all the attributes are of benefit type so there is no need to normalize it.

Step 2. Determine the values of $\underline{\underline{T}}_{rs} (r = 1, 2, ..., 5; s = 1, 2, ..., 4)$ by using the formula (8.50), and get

$$\underline{\underline{T}}_{rs} = \begin{bmatrix} 1 & 0.5333 & 0.3556 & 0.1011 \\ 1 & 0.8000 & 0.6133 & 0.3435 \\ 1 & 0.6333 & 0.3167 & 0.1404 \\ 1 & 0.8000 & 0.5600 & 0.3435 \\ 1 & 0.5333 & 0.2667 & 0.0948 \end{bmatrix}$$

Step 3. Use the SVNSSPRWA given in Equation (8.51) to get the overall SVNN $\overline{\overline{m}}_r (r = 1, 2, ..., 5)$ (assume $\rho = -2$), and obtain

$$\overline{\overline{m}}_1 = \langle 0.5151, 0.4787, 0.1176 \rangle, \overline{\overline{m}}_2 = \langle 0.7209, 0.2000, 0.1320 \rangle,$$
$$\overline{\overline{m}}_3 = \langle 0.5626, 0.4490, 0.1764 \rangle, \overline{\overline{m}}_4 = \langle 0.7246, 0.1434, 0.1681 \rangle,$$
$$\overline{\overline{m}}_5 = \langle 0.5366, 0.5754, 0.1462 \rangle.$$

Step 4. Determine the score values $\overline{\overline{SO}}\left(\overline{\overline{m}}_r\right)(r=1,2,...,5)$ of the overall SVNNs $\overline{\overline{m}}_r\,(r=1,2,....,5)$ by using Definition (1.1.1.6), and have

$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.6396, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7963, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6458, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8044, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6050.$

So, $\overline{\overline{m}}_4 > \overline{\overline{m}}_2 > \overline{\overline{m}}_3 > \overline{\overline{m}}_1 > \overline{\overline{m}}_5$.

Step 5. According to score values, ranking order of alternatives is $\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5$. So, the best candidate is $\overline{\overline{N}}_4$, while the worst one is $\overline{\overline{N}}_5$.

Similarly, we solve the above Example (8.4.3) by the SVNSSPWG operator:

Step 1 and step 2 are same.

Step 3. Use the SVNSSPRWG operator given Equation (Step 3) to get the overall SVNN $\overline{\overline{m}}_r\,(r=1,2,...,5)$ (assume $\rho=-2$), and have

$\overline{\overline{m}}_1=\langle 0.4498, 0.7400, 0.1745\rangle, \overline{\overline{m}}_2=\langle 0.7104, 0.2000, 0.2269\rangle, \overline{\overline{m}}_3=\langle 0.5477, 0.5821, 0.2233\rangle,$
$\overline{\overline{m}}_4=\langle 0.6554, 0.2051, 0.2265\rangle, \overline{\overline{m}}_5=\langle 0.4790, 0.7022, 0.3042\rangle.$

Step 4. Determine the score values $\overline{\overline{SO}}\left(\overline{\overline{m}}_r\right)(r=1,2,...,5)$ of the overall SVNNs $\overline{\overline{m}}_r\,(r=1,2,....,5)$ by using Definition (1.1.1.6), and get

$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5118, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7612, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.5808, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7412, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.4909.$

So, $\overline{\overline{m}}_2 > \overline{\overline{m}}_4 > \overline{\overline{m}}_3 > \overline{\overline{m}}_1 > \overline{\overline{m}}_5$.

Step 5. According to score values ranking order of alternatives is $\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5$. So, the best candidate is $\overline{\overline{N}}_2$, while the worst one is $\overline{\overline{N}}_5$.

8.4.4 Effect of The Parameter ρ on Decision Result of This Example

In order to see the effect of the parameter ρ on the decision-making result, we set the distinct values for the parameter ρ in step 3, to rank the alternatives. The score values and ranking order are described in Table 8.6, Table 8.7, and Fig.8.1, Fig.8.2. In Fig. 8.1, Fig. 8.2, $\overline{\overline{N}}_r (r=1,2,...,5)$ are expressed by $G_i(1,...,5)$.

Table 8.6. Score values and ranking order for different values of ρ utilizing SVNSSPWA operator for example 8.4.3

ρ	**Score values**	**Ranking order**
$\rho \to 0$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5935, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7828, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6195$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7716, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5652.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
$\rho=-1$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.6179, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7908, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6325$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7887, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5877.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
$\rho=-2$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.6396, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7963, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6458$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8044, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6050.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
$\rho=-7$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.6916, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.8110, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6913$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8433, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6521.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_5.$
$\rho=-20$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.7170, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.8248, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.7181$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8588, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6829.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
$\rho=-100$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.7301, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.8317, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.7304$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8651, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6967.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$

$\rho=-200$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.7317, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.8325, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.7318$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8659, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6983.$	$\overline{\overline{N}}_4>\overline{\overline{N}}_2>\overline{\overline{N}}_3>\overline{\overline{N}}_1>\overline{\overline{N}}_5.$

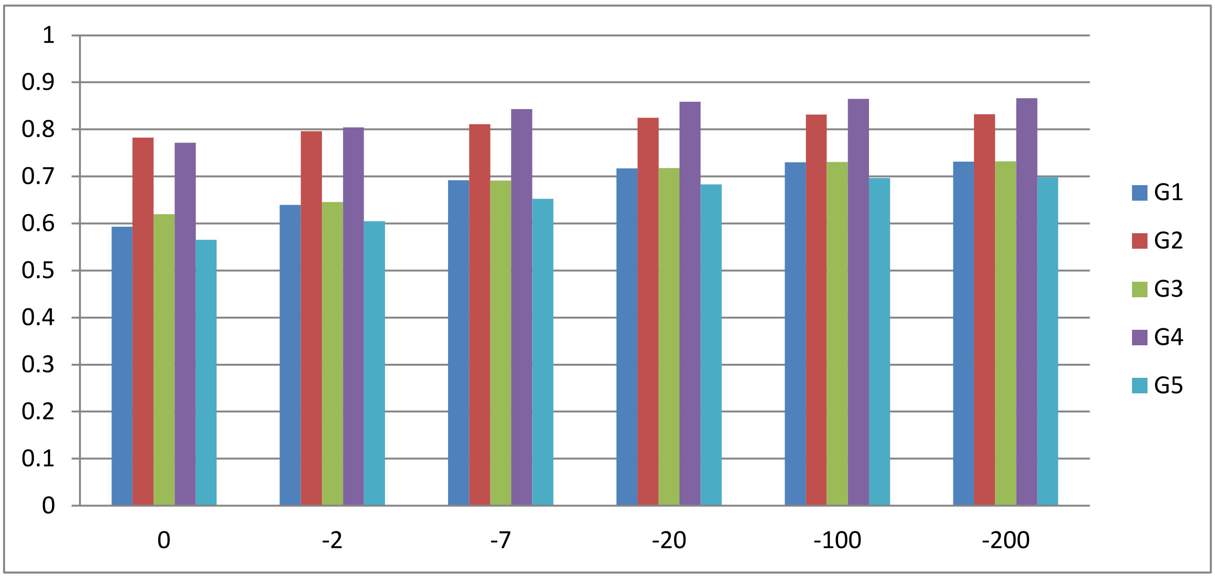

Fig 1. Chart for different values of parameter ρ utilizing SVNSSPWA operator in Example 8.4.3

Table 8.7. Score values and ranking order for different values of ρ utilizing SVNSSPWG operator for example 6

ρ	**Score values**	**Ranking order**
$\rho\to 0$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5463, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7688, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.5985$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7503, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5241.$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1>\overline{\overline{N}}_5.$
$\rho=-1$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5271, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7651, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.5894$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7457, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5067.$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1>\overline{\overline{N}}_5.$
$\rho=-2$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5118, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7612, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.5808$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7412, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.4909.$	$\overline{\overline{N}}_2>\overline{\overline{N}}_4>\overline{\overline{N}}_3>\overline{\overline{N}}_1>\overline{\overline{N}}_5.$

$\rho = -7$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.4651, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7414, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.5514$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.7224, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right) = 0.4470.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
$\rho = -20$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.4268, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7170, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.5255$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.6959, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right) = 0.4188.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
$\rho = -100$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.4053, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7033, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.5053$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.6728, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right) = 0.4037.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
$\rho = -200$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right) = 0.4026, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right) = 0.7017, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right) = 0.5027$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right) = 0.6697, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right) = 0.4019.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$

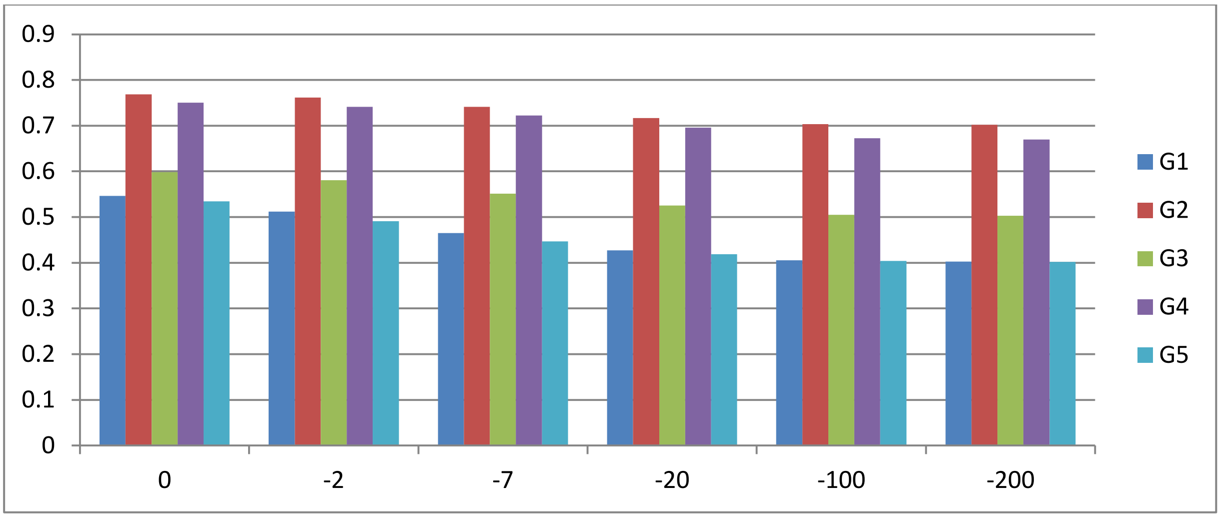

Fig 2. Chart for different values of parameter ρ utilizing SVNSSPWG operator in Example 8.4.3

From Table 8.6, we can notice that the ranking orders by utilizing SVNSSPWA operator are slightly different when the parameter ρ takes the distinct values. When the value of the parameter ρ is -1 and tends to zero, the best choice is $\overline{\overline{N}}_2$. When the value of the parameter ρ decreases from -1 then the best choice is $\overline{\overline{N}}_4$. We can also see from Table 8.6, when the value of the parameter decreases the score values become bigger and bigger.

From Table 8.7, we can see that the ranking orders by utilizing SVNSSPWG operator do not change for different values of the parameter ρ, the best choice is $\overline{\overline{N}}_2$. We can also notice from Table 7, when the value of the parameter ρ decreases, the score values become smaller and smaller. Generally, different DMs can set different values of the parameter ρ according to their actual need.

8.4.5 Comparison With the Other Methods

In order to further show the effectiveness of the proposed methods based on the proposed AOs, in this article, we solve Example (8.4.3) by seven existing methods based on different aggregation operators under SVN environment. SVN weighted averaging (SVNWA) operator proposed by Ye [8], SVNWA operator proposed by Peng et al. [9] based on improved operational laws for SVNNs, SVN-MABAC [21], SVN-TOPSIS [21], SVN prioritized weighted averaging (PRWA) operator developed by Wu et al. [36], SVN Dombi prioritized weighted averaging (PRWA) operator developed by Wei et al. [40] and SVNN normalized BM (SVNNBM) operator developed by Liu et al. [30]. The score values and ranking order are given in Table 8.8.

The weight vector of attributes for these methods is obtained using the PRA operator.

Table 8.8. Score values and ranking orders with different methods

Methods	Score values	Ranking order
SVNWA operator [8]	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5532, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7698, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6001$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7580, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5301.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
SVNWA operator [9]	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5934, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7828, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6195$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7716, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5652.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
SVNDPAW [40] $\rho=2$	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.6540, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7973, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6529$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8080, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6136.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_5.$
SVN-TOPSIS [21]	$\overline{\overline{C}}\left(\overline{\overline{m}}_1\right)=-3.3557, \overline{\overline{C}}\left(\overline{\overline{m}}_2\right)=-0.8123, \overline{\overline{C}}\left(\overline{\overline{m}}_3\right)=-2.7509,$ $\overline{\overline{C}}\left(\overline{\overline{m}}_4\right)=-0.4144, \overline{\overline{C}}\left(\overline{\overline{m}}_5\right)=-3.8097.$	$\overline{\overline{N}}_4 > \overline{\overline{N}}_2 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
SVN-MABAC [21]	$\overline{\overline{Q}}\left(\overline{\overline{m}}_1\right)=0.2637, \overline{\overline{Q}}\left(\overline{\overline{m}}_2\right)=0.6122, \overline{\overline{Q}}\left(\overline{\overline{m}}_3\right)=0.2176,$ $\overline{\overline{Q}}\left(\overline{\overline{m}}_4\right)=0.5891, \overline{\overline{Q}}\left(\overline{\overline{m}}_5\right)=0.1903.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_1 > \overline{\overline{N}}_3 > \overline{\overline{N}}_5.$
SVNPWA operator [36]	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5934, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7828, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6195$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7716, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5652.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
SVNNBM operator (p=q=1) [30]	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.565597, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.774729, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6083$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.754169, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5391.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$
SVNSSPRWA operator (in this article) ($\rho \to 0$)	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.5935, \overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7828, \overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6195$ $, \overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.7716, \overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.5652.$	$\overline{\overline{N}}_2 > \overline{\overline{N}}_4 > \overline{\overline{N}}_3 > \overline{\overline{N}}_1 > \overline{\overline{N}}_5.$

SVNSSPRWA operator (in this article) ($\rho=-2$)	$\overline{\overline{SO}}\left(\overline{\overline{m}}_1\right)=0.6396,\overline{\overline{SO}}\left(\overline{\overline{m}}_2\right)=0.7963,\overline{\overline{SO}}\left(\overline{\overline{m}}_3\right)=0.6458$ $,\overline{\overline{SO}}\left(\overline{\overline{m}}_4\right)=0.8044,\overline{\overline{SO}}\left(\overline{\overline{m}}_5\right)=0.6050.$	$\overline{\overline{N}}_4>\overline{\overline{N}}_2>\overline{\overline{N}}_3>\overline{\overline{N}}_1>\overline{\overline{N}}_5.$

From Table 8, we can see that when value of the parameter ρ tends to zero, the ranking orders obtained by the proposed method based on the proposed aggregation operators are same with the other five methods. This shows that our method is valid. Further, when we set the parameter value $\rho=-2$, then, the ranking order is same as that obtained from the methods developed in [21] and [40] based on SVN-TOPSIS and SVN Dombi prioritized averaging operators.

Moreover, the comparison among our method with the existing seven methods can be pointed out as follows:

(1) The methods developed by Ye [8] and Peng et al. [9] are based on SVNWA operators. These aggregation operators are based on algebraic operations, while the aggregation operators in this article are based on Schweizer-Sklar operations. Although the best alternative is same, however, when we change the value of the parameter ρ the best alternative changed. That's why our method is more flexible and effective than Ye [8] and Peng [9].

(2) The method of Wu et al. [36] is based on SVN prioritized weighted averaging operator. This is a special case of the developed aggregation operators, when the value of the parameter tends to Zero.

(3) The method developed by Liu et al. [30] is based on the SVNNNWBM operator, to solve the same example, we set $\mathrm{p}=\mathrm{q}=1$, then the ranking order is same as the one obtained by the developed aggregation operators, when the value of the parameter tends to zero. This shows the effectiveness of the proposed approach based on the

developed aggregation operator. But the advantage of the developed method in this article is that it can deal with the situation in which the attributes are with the prioritized relationship.

(4) The methods developed by Peng et al. [21] is based on SVN-TOPSIS and SVN-MABAC method in which the weights of the attributes are obtained via gray system theory and cannot consider the prioritized relationship among the attributes.

(5) The method developed by Wei et al. [40] is based on Dombi prioritized aggregation for SVNSs. The Dombi prioritized aggregation operator also consists of parameter, but the decision makers can considered the parameter greater than zero, while in the proposed aggregation operators in this article the decision makers can considered the parameter values less than zero.

Certainly, the developed methods in this article are more general and flexible by the parameter, and are more advanced to be used in practical decision-making problems.

8.4.6 Conclusion

Since SVNNs are a better mathematical tool and can define uncertain information more accurately than the FS and IFS. In this chapter, we investigated some Schweizer-Sklar prioritized aggregation operator based on SVNNs and proposed two methods to deal with single-valued neutrosophic information. First, we have developed some new aggregation operators and studied their desirable properties such as idempotency, monotonicity and boundedness. Moreover, we have analyzed some special cases of the developed operators, and have presented two MADM methods based on the proposed aggregation operators to deal with SVN information. Lastly, a practical example about talent introduction is given to show the verification of the developed methods and to demonstrate the effectiveness and practicality of the developed approaches and a comparison analysis is also given to verify the developed methods.

BiBLIOGRAPY

[1] L. A. Zadeh. (1965). Fuzzy sets, *Information and Control*, 8, 338- 356.

[2] I. B. Turksen. (1986) . Interval valued fuzzy sets based on normal forms. *Fuzzy sets and systems*, *20*(2), 191-210.

[3] K.T. Atanassov. (1986). Intuitionistic fuzzy sets, *Fuzzy sets and Systems 20*(1), 87-96.

[4] K. T. Atanassov, and G. Gargov. (1989). Interval valued intuitionistic fuzzy sets. *Fuzzy sets and systems*, *31*(3), , 343-349.

[5] F. Smarandache. (1998). Neutrosophy: Neutrosophic Probability, Set, and Logic; American Research Press: Rehoboth, DE, USA.

[6] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman. (2005). Single valued neutrosophic sets, Proceedings of 10th 476 International Conference on Fuzzy Theory and Technology, Salt Lake City, 477 Utah.

[7] H. Wang, F. Smarandache, R. Sunderraman, and Y. Q. Zhang. (2005). Interval neutrosophic sets and logic: Theory and applications in computing. Hexis, Phoenix.

[8] H.Y. Zhang, J.Q. Wang, and X. H. Chen. (2014). Interval neutrosophic sets and their application in multicriteria decision making problems, *The Scientific World Journal 1-25*.

[9] J. Ye. (2014). A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets, *Journal of Intelligent and Fuzzy Systems 26*(5), 2459-2466.

[10] J. J. Peng, J. Q. Wang, J. Wang, H. Y. Zhang, and X. H. Chen. (2016). Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems, *International journal of systems science 47*(10), 2342-2358.

[11] M. Ali, I. Deli and F. Smarandache. (2016). The theory of neutrosophic cubic sets and their applications in pattern recognition. *Journal of Intelligent & Fuzzy Systems*, *30*(4), 1957-1963.

[12] Y. B. Jun, F. Smarandache and C. S. Kim. (2017). Neutrosophic cubic sets. *New mathematics and natural computation*, *13*(01), 41-54.

[13] Y. B. Jun, F. Smarandache and C. S. Kim. (2017). P-union and P-intersection of neutrosophic cubic sets. *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, *25*(1), 99-115.

[14] J. Zhan, M. Khan, M. Gulistan and A. Ali. (2017). Applications of neutrosophic cubic sets in multi-criteria decision-making. *International Journal for Uncertainty Quantification*, *7*(5).

[15] J. Ye. (2018). Operations and aggregation method of neutrosophic cubic numbers for multiple attribute decision-making. *Soft Computing*, 1-10.

[16] W. L. Hung and M. S. Yang. (2004). Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. *Pattern Recognition Letters*, *25*(14), 1603-1611.

[17] Z. Xu. and M. Xia. (2011). Distance and similarity measures for hesitant fuzzy sets. *Information Sciences*, *181*(11), 2128-2138.

[18] Z. Xu. (2007). Some similarity measures of intuitionistic fuzzy sets and their applications to multiple attribute decision making. *Fuzzy Optimization and Decision Making*, *6*(2), 109-121.

[19] P. Majumdar and S. K. Samanta. (2014). On similarity and entropy of neutrosophic sets. *Journal of Intelligent & Fuzzy Systems*, *26*(3), 1245-1252.

[20] J. Ye. (2014). Clustering methods using distance-based similarity measures of single-valued neutrosophic sets. *Journal of Intelligent Systems*, *23*(4), 379-389.

[21] J. Ye. (2014). Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. *Journal of Intelligent & Fuzzy Systems*, *26*(1), 165-172.

[22] J. Ye. (2015). Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. *Artificial intelligence in medicine*, *63*(3), 171-179.

[24] Z. Lu. and J. Ye. (2017). Cosine measures of neutrosophic cubic sets for multiple attribute decision-making. *Symmetry*, *9*(7), 121.

[25] A. Tu, J. Ye and B. Wang. (2018). Multiple attribute decision-making method using similarity measures of neutrosophic cubic sets. *Symmetry*, *10*(6), 215.

[26] H. B. Mitchell. (2004). A correlation coefficient for intuitionistic fuzzy sets. *International Journal of Intelligent Systems*, *19*(5), 483-490.

[27] D. G. Park, Y. C. Kwun, J. H. Park and I. Y. Park. (2009). Correlation coefficient of interval-valued intuitionistic fuzzy sets and its application to multiple attribute group decision making problems. *Mathematical and Computer Modelling*, *50*(9-10), 1279-1293.

[28] W. Zeng and J. Wang. (2011). Correlation coefficient of interval-valued intuitionistic fuzzy sets. In *Fuzzy Systems and Knowledge Discovery (FSKD), 2011 Eighth International Conference on* (Vol. 1, pp. 98-102). IEEE.

[29] J. Ye. (2013). Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment. *International Journal of General Systems*, *42*(4), 386-394.

[30] H. Y. Zhang, P. Ji, J. Q. Wang and X. H. Chen. (2015). An improved weighted correlation coefficient based on integrated weight for interval neutrosophic sets and its application in multi-criteria decision-making problems. *International Journal of Computational Intelligence Systems*, *8*(6), 1027-1043.

[31] R. Şahin and P. Liu. (2017). Correlation coefficient of single-valued neutrosophic hesitant fuzzy sets and its applications in decision making. *Neural Computing and Applications*, *28*(6), 1387-1395.

[32] M. Xia and Z. Xu. (2012). Entropy/cross entropy-based group decision making under intuitionistic fuzzy environment. *Information Fusion*, *13*(1), 31-47.

[33] J. Ye. (2011). Fuzzy cross entropy of interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives. *Expert Systems with Applications*, *38*(5), 6179-6183.

[34] J. Ye. (2014). Single valued neutrosophic cross-entropy for multicriteria decision making problems. *Applied Mathematical Modelling*, *38*(3), 1170-1175.

[35] Z. P. Tian, H. Y. Zhang, J. Wang, J. Q. Wang and X. H. Chen. (2016). Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets. *International Journal of Systems Science*, *47*(15), 3598-3608.

[36] R. Şahin. (2017). Cross-entropy measure on interval neutrosophic sets and its applications in multicriteria decision making. *Neural Computing and Applications*, *28*(5), 1177-1187.

[37] P. Liu and F. Teng, (2016). An extended TODIM method for multiple attribute group decision making based on 2-dimension uncertain linguistic Variable, *Complexity 21*(5), 20-30.

[38] P. Liu and F. Jin. (2012). Methods for aggregating intuitionistic uncertain linguistic variables and their application to group decision making, *Information Sciences*, *205*, 58-71.

[39] P. Liu and Y. Wang. (2014). Multiple attribute group decision making methods based on intuitionistic linguistic power generalized aggregation operators, *Applied soft computing 17*, 90-104.

[40] J. Q. Wang, P. Wang, J. Wang, H.Y. Zhang and X.H. Chen. (2015). Atanassov's interval-valued intuitionistic linguistic multicriteria group decision-making method based on the trapezium cloud model, *IEEE Transactions on Fuzzy Systems 23*(3), 542-554.

[41] J. Ye. (2014). Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *27*(5), 2231-2241.

[42] J. Ye. (2015). An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers. *Journal of Intelligent & Fuzzy Systems*, *28*(1), 247-255.

[43] J. Ye. (2017). Multiple attribute group decision making based on interval neutrosophic uncertain linguistic variables. *International Journal of Machine Learning and Cybernetics*, *8*(3), 837-848.

[44] P. Liu and L. Shi. (2017). Some neutrosophic uncertain linguistic number Heronian mean operators and their application to multi-attribute group decision making. *Neural Computing and Applications*, *28*(5), 1079-1093.

[45] V. Torra and Y. Narukawa. (2009, August). On hesitant fuzzy sets and decision. In *Fuzzy systems, 2009. FUZZ-IEEE 2009. IEEE international conference on* (pp. 1378-1382). IEEE.

[46] V. Torra (2010). Hesitant fuzzy sets. *International Journal of Intelligent Systems*, *25*(6), 529-539.

[47] Z. Wu and J. Xu. (2016). Possibility distribution-based approach for MAGDM with hesitant fuzzy linguistic information. *IEEE Transactions on Cybernetics* 46(3). 694-705.

[48] J. Q. Wang, J. T. Wu, J. Wang, H. Y. Zhang and X. H. Chen. (2014). Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems. *Information Sciences*, *288*, 55-72.

[49] G. Wei, F. E. Alsaadi, T. Hayat and A. Alsaedi (2016). Hesitant fuzzy linguistic arithmetic aggregation operators in multiple attribute decision making. *Iranian Journal of Fuzzy Systems*, *13*(4), 1-16.

[50] H. Liao and Z. Xu. (2015). Approaches to manage hesitant fuzzy linguistic information based on the cosine distance and similarity measures for HFLTSs and their application in qualitative decision making. *Expert Systems with Applications*, *42*(12), 5328-5336.

[51] X. Liu, Y. Ju and S. Yang. (2014). Hesitant intuitionistic fuzzy linguistic aggregation operators and their applications to multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *27*(3), 1187-1201.

[52] Zhang, Z., and Wu, C. (2014). Hesitant fuzzy linguistic aggregation operators and their applications to multiple attribute group decision making. *Journal of Intelligent & Fuzzy Systems*, *26*(5), 2185-2202.

[53] Z. Chen, P. Liu and Z. Pei. (2015). An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers, *International Journal of Computational Intelligence Systems 8*(4), 747-760

[54] P. Liu and X. Qin. (2017). Power average operators of linguistic intuitionistic fuzzy numbers and their application to multiple-attribute decision making, *Journal of Intelligent & Fuzzy Systems 32*(1), 1029-1043.

[55] P. Liu and X. Liu. (2017). Multiattribute Group Decision Making Methods Based on Linguistic Intuitionistic Fuzzy Power Bonferroni Mean Operators. *Complexity* 1-15.

[56] P. Liu and P. Wang. (2017). Some improved linguistic intuitionistic fuzzy aggregation operators and their applications to multiple-attribute decision

making, *International Journal of Information Technology & Decision Making 16*(03), 817-850.

[57] Y.Y. Li, H.Y. Zhang, J.Q. Wang. (2017). Linguistic neutrosophic sets and its application to multi-criteria decision-making problems, *International Journal for Uncertainty Quantification* 7(2), 135-154.

[58] Z. Fang, J. Ye. (2017). Multiple attribute group decision-making method based on linguistic neutrosophic numbers, Symmetry, 9(7), 111, p. 1-12.

[59] C. Fan, J. Ye, K. Hu and E. Fan. (2017). Bonferroni Mean Operators of Linguistic Neutrosophic Numbers and Their Multiple Attribute Group Decision-Making Methods, Information, 8(3), 107.

[60] R. R. Yager. (2001). The power average operator, *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans 31*(6), 724-731.

[61] C. Bonferroni. (1950). Sulle medie multiple di potenze. *Bollettino dell'Unione Matematica Italiana 5*(3-4), 267-270.

[62] S. Sykora. (2009) *Mathematical means and averages: generalized Heronian means. Stan's Library*.

[63] R. F. Muirhead. (1902). Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters, *Proceedings of the Edinburgh Mathematical Society 21*, 144-162.

[64] C. Maclaurin. (1729). A second letter to Martin Folkes, Esq.; concerning the roots of equations, with the demonstartion of other rules in algebra, Phil. *Transactions 36*, 59-96.

[65] T. Hara, M. Uchiyama, and S. E. Takahasi. (1998). A refinement of various mean inequalities, *Journal of Inequalities and Applications* 2(4), 387-395.

[66] Z. Xu. (2011). Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators. *Knowledge-Based Systems*, *24*(6), 749-760.

[67] Z. Zhang. (2013). Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making. *Information Sciences*, *234*, 150-181.

[68] Y. Xu and H. Wang. (2011). Approaches based on 2-tuple linguistic power aggregation operators for multiple attribute group decision making under linguistic environment. *Applied Soft Computing*, *11*(5), 3988-3997.

[69] L. Wang, Q. Shen and L. Zhu. (2016). Dual hesitant fuzzy power aggregation operators based on Archimedean t-conorm and t-norm and their application to multiple attribute group decision making. *Applied Soft Computing*, *38*, 23-50.

[70] Liu, P., and Teng, F. (2018). Multiple attribute decision making method based on normal neutrosophic generalized weighted power averaging operator. *International Journal of Machine Learning and Cybernetics*, *9*(2), 281-293.

[71] Liu, P., and Tang, G. (2016). Some power generalized aggregation operators based on the interval neutrosophic sets and their application to decision making. *Journal of Intelligent & Fuzzy Systems*, *30*(5), 2517-2528.

[72] Z. Xu, and R. R. Yager. (2011). Intuitionistic fuzzy Bonferroni means, *IEEE Transactions on Systems, Man and Cybernetics, Part B: Cybernetics 41*(2), 568-578.

[73] P. Liu, and Y. Wang. (2014). Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean, *Neural Computing and Applications 25*(7-8), 2001-2010.

[74] J. Qin, and X. Liu. (2014). An approach to intuitionistic fuzzy multiple attribute decision making based on Maclaurin symmetric mean operators, *Journal of Intelligent & Fuzzy Systems 27*(5), 2177-2190.

[75] G. Wei, and M. Lu. (2018). Pythagorean fuzzy Maclaurin symmetric mean operators in multiple attribute decision making. *International Journal of Intelligent Systems 33*(5), 1043-1070.

[76] J. Qin, X. Liu, and W. Pedrycz. (2015). Hesitant fuzzy Maclaurin symmetric mean operators and its application to multiple-attribute decision making, *International Journal of Fuzzy Systems 17*(4), 509-520.

[77] J. Q. Wang, Y. Yang, and L. Li. (2016). Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators. *Neural Computing and Applications,* 1-19.

[78] P. Liu, and D. Li. (2017). Some muirhead mean operators for intuitionistic fuzzy numbers and their applications to group decision making, *PloS one 12*(1), e0168767.

[79] P. Liu, and X. You. (2017). Interval neutrosophic Muirhead mean operators and their application in multiple attribute group decision-making, *International Journal for Uncertainty Quantification 7*(4), 303-334.

[80] Y. He, Z. He, Y. Deng, and P. Zhou. (2016). IFPBM s and their application to multiple attribute group decision making, *Journal of the Operational Research Society 67*(1), 127-147.

[81] P. Liu, and H. Li. (2017). Interval-valued intuitionistic fuzzy power Bonferroni aggregation operators and their application to group decision making, *Cognitive Computation 9*(4), 494-512.

[82] P. Liu. (2017), Multiple attribute group decision making method based on interval-valued intuitionistic fuzzy power Heronian aggregation operators, *Computers & Industrial Engineering 108*, 199-212.

[83] Z. Liu, F. Teng, P. Liu, and Q. Ge. (2018) . interval-valued intuitionistic fuzzy power maclaurin symmetric mean aggregation operators and their application to multiple attribute group decision-making, *International Journal for Uncertainty Quantification 8*(3), 211-232.

[84] L. Li, R. Zhang, J. Wang, X. Zhu, and Y. Xing. (2018) . Pythagorean fuzzy power Muirhead mean operators with their application to multi-attribute decision making, *Journal of Intelligent & Fuzzy Systems*, In Press, DOI: 10.3233/JIFS-171907.

[85] R. R Yager. (2008). Prioritized aggregation operators. *International Journal of Approximate Reasoning*, *48*(1) 263-274.

[86] P. Liu, and Y. Wang. (2016). Interval neutrosophic prioritized OWA operator and its application to multiple attribute decision making. *Journal of Systems Science and Complexity*, *29*(3), 681-697.

[87] Z. P. Tian, J. Wang, H. Y. Zhang, and J. Q. Wang. (2018). Multi-criteria decision-making based on generalized prioritized aggregation operators under simplified neutrosophic uncertain linguistic environment. *International Journal of Machine Learning and Cybernetics*, *9*(3), 523-539.

[88] C. Liang, S. Zhao, and J. Zhang. (2017). Multi-criteria group decision making method based on generalized intuitionistic trapezoidal fuzzy prioritized aggregation operators. *International Journal of Machine Learning and Cybernetics*, *8*(2), 597-610.

[89] H. Garg. (2018). Linguistic single-valued neutrosophic prioritized aggregation operators and their applications to multiple-attribute group decision-making. *Journal of Ambient Intelligence and Humanized Computing*, 1-23.

[90] T. Y. Chen. (2014). A prioritized aggregation operator-based approach to multiple criteria decision making using interval-valued intuitionistic fuzzy sets: A comparative perspective. *Information Sciences*, *281*, 97-112.

[91] Z, Liu and P. Liu. (2016). Intuitionistic normal fuzzy prioritized aggregation operators and their application to group decision making. *Systems Engineering-Theory & Practice*, *36*(2), 494-504.

[92] R. Arora and H. Garg. (2018). Prioritized averaging/geometric aggregation operators under the intuitionistic fuzzy soft set environment. *Scientia Iranica*, *25*(1), 466-482.

[93] P. Ji, J. Q. Wang, and H. Y. Zhang. (2018). Frank prioritized Bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers. *Neural Computing and Applications*, *30*(3), 799-823.

[94] L. Wang, H. Y. Zhang, and J. Q. Wang. (2018). Frank Choquet Bonferroni mean operators of bipolar neutrosophic sets and their application to multi-criteria decision-making problems. *International Journal of Fuzzy Systems*, *20*(1), 13-28.

[95] G. Wei, and Y. Wei. (2018). Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems* , DOI: 10.3233/JIFS-171741.

[96] J. Dombi. (1982). A general class of fuzzy operators, the DeMorgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. *Fuzzy sets and systems*, *8*(2), 149-163.

[97] J. Qin, X. Liu and W. Pedrycz. (2016). Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making. *Applied Soft Computing*, *41*, 428-452.

[98] X. Tang, S. Yang and W. Pedrycz, (2018). Multiple attribute decision-making approach based on dual hesitant fuzzy Frank aggregation operators. *Applied Soft Computing*, *68*, 525-547.

[99] Z. Zhang. (2017). Interval-valued intuitionistic fuzzy Frank aggregation operators and their applications to multiple attribute group decision making. *Neural Computing and Applications*, *28*(6), 1471-1501.

[100] P. Liu. (2014). Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making. *IEEE Transactions on Fuzzy systems*, *22*(1), 83-97.

[101] M. Lu, G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi. (2017). Hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, *33*(2), 1105-1117.

[102] Liu, P., Chu, Y., Li, Y., and Chen, Y. (2014). Some Generalized Neutrosophic Number Hamacher Aggregation Operators and Their Application to Group Decision Making. *International Journal of Fuzzy Systems*, *16*(2).

[103] Wei, G, and Lu, M. (2017). Dual hesitant Pythagorean fuzzy Hamacher aggregation operators in multiple attribute decision making. *Archives of Control Sciences*, *27*(3), 365-395.

[104] Garg, H. (2016). Generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein t-norm and t-conorm and their application to decision making. *Computers & Industrial Engineering*, *101*, 53-69.

[105] Garg, H. (2016). A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. *International Journal of Intelligent Systems*, *31*(9), 886-920.

[106] P. Liu. (2016). The aggregation operators based on archimedean t-Conorm and t-Norm for single-valued neutrosophic numbers and their application to decision making. *International Journal of Fuzzy Systems*, *18*(5), 849-863.

[107] G. Deschrijver, and E. E. Kerre. (2002) . A generalization of operators on intuitionistic fuzzy sets using triangular norms and conorms. *Notes on IFS*, *8*(1), 19-27.

[108] G. Deschrijver. (2009). Generalized arithmetic operators and their relationship to t-norms in interval-valued fuzzy set theory. *Fuzzy Sets and Systems*, *160*(21), 3080-3102.

[109] X. Zhang, H. He, and Y. Xu. (2006). A fuzzy logic system based on Schweizer-Sklar t-norm. *Science in China Series F: Information Sciences*, *49*(2), 175-188.

[110] P. Liu, and P. Wang. (2018). Some interval-valued intuitionistic fuzzy Schweizer–Sklar power aggregation operators and their application to supplier selection. *International Journal of Systems Science*, https://doi.org/10.1080/00207721.2018.1442510.

[111] L. Zhang, Intuitionistic fuzzy averaging Schweizer-Sklar operators based on interval-valued intuitionistic fuzzy numbers and its applications. In *2018 Chinese Control and Decision Conference (CCDC)*. IEEE (2018).

[112] Y. H. Huang, G. W. Wei, and C. Wei. (2017). VIKOR method for interval neutrosophic multiple attribute group decision-making, *Information 8*(4), 144.

[113] P. Ji, P. F. Cheng, H. Y. Zhang, and J. Q. Wang. (2018). Interval-Valued Neutrosophic Bonferroni Mean Operators and the Application in the Selection of Renewable Energy, New Trends in Neutrosophic Theory and Applications, 2, 12-39.

[114] R. R. Yager. (2004). Choquet aggregation using order inducing variables, *Int J Uncertain Fuzziness Knowl Based Syst* 12(1), 69–88.

[115] X. H. Wu, J. Q. Wang, J. J. Peng, and X. H. Chen. (2016). Cross-entropy and prioritized aggregation operator with simplified neutrosophic sets and their application in multi-criteria decision-making problems. *International Journal of Fuzzy Systems*, *18*(6), 1104-1116.

[116] F. Herrera and E. Herrera-Viedma. (1996). A model of consensus in group decision making under linguistic assessments, *Fuzzy sets and Systems 78*(1), 73-87.

[117] F. Herrera and E. Herrera-Viedma. (2000). Linguistic decision analysis: steps for solving decision problems under linguistic information, *Fuzzy Sets and systems 115*(1), 67-82.

[118] Z. Xu. (2004). Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment, *Information Sciences 168*(1), 171-184.

[119] Z. Xu. (2006). A note on linguistic hybrid arithmetic averaging operator in multiple attribute group decision making with linguistic information, *Group Decision and Negotiation 15*(6), 593-604.

[120] Y. He, Z. He, C. Jin, H. Chen. (2015). Intuitionistic fuzzy power geometric Bonferroni means and their application to multiple attribute group decision making. *Int. J. Uncertain. Fuzz. Knowl.-Based Syst*, *23*, 285–315.

[121] X. Peng and D. Jingguo (2018). Approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function, *Neural Computing and Applications* 29(10), 939-954.