

# Generalized Schauder and Krasnoselskii Type Fixed Point Results with Applications



By:

**Niaz Ahmad**

Reg. No. 65-FBAS/PHDMA/F16

**Department of Mathematics and Statistics  
Faculty of Sciences  
International Islamic University Islamabad  
Pakistan  
2022**

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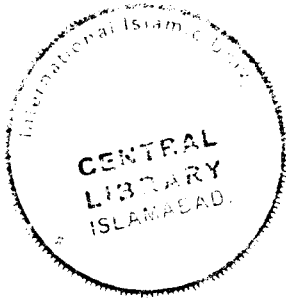
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Supervised By:

**Dr. Nayyar Mehmood**

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Faculty of Sciences  
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Pakistan  
2022**

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**Niaz Ahmad**

Reg. No. 65-FBAS/PHDMA/F16

*A Dissertation*

*Submitted in Partial Fulfillment*

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**IN**

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Supervised By:

**Dr. Nayyar Mehmood**

**Department of Mathematics and Statistics  
Faculty of Sciences  
International Islamic University Islamabad  
Pakistan  
2022**

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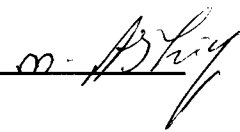
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Student Name: Niaz Ahmad

Signature: 

### Examination Committee:

a) **External Examiner 1:**  
Name/Designation/Office Address

Signature: 

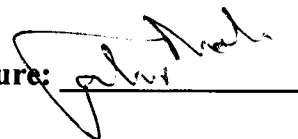
**Dr. Muhammad Ashiq**  
Professor of Mathematics,  
Dean/Principal,  
NUTECH, Islamabad.

b) **External Examiner 2:**  
Name/Designation/Office Address)

Signature: 

**Dr. Muhammad Asad Zaigham**  
Associate Professor  
Department of Mathematics and Statistics,  
Riphah International University, I-14, Islamabad.

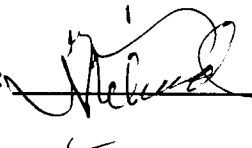
c) **Internal Examiner:**  
Name/Designation/Office Address)

Signature: 

**Dr. Tahir Mahmood**  
Assistant Professor

Supervisor Name:

**Dr. Nayyar Mehmood**

Signature: 

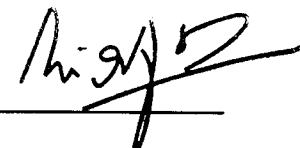
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# Preface

Different physical phenomena of physics, chemistry, biology and economics such as surface waves of fluids, shock waves in physics, processes in nuclear reactors, stability of scientific processes, models of game theory in economics, optimization of production processes, study of many particle systems, gravitational effects in general relativity, study of variation of electrical potential in the heart etc., lead to nonlinear problems. Many of these problems reduce mathematically to operator equation  $F = 0$ , where  $F$  depends on  $x$  and  $y$ . This operator equation may include the following forms.

(1)  $\xi x = x$ .

(2) Inhomogeneous problem  $Tx = y$ ,  $y$  is known and  $x$  is to be determined.

Consider a mapping  $\xi$  of  $\Omega$  into  $\Omega$  and  $\xi x = x$ . Then every solution of the operator equation is a fixed point of  $\xi$ .

We investigate the existence results for differential and integral equations (DEs and IEs) by fixed point (FP) analysis. For this purpose, we generalize existing fixed point results and apply them to different types of problems. Our main focus is to obtain generalized Krasnoselskii and Schauder type fixed point theorems (FPTs) and apply them to DEs to give a comparison of the present results to new one. So we improve the existing work of fixed point analysis so that broad class of differential equations might be discussed for the existence of solutions. Our work is motivated by compactness, convexity, boundedness, measure of noncompactness (MNC), continuity, contraction, monotonicity, and the analysis to be presented reveal key features of Krasnoselskii and Schauder type FPTs in the framework of topology, analysis and algebra.

The thesis contains generalized Schauder and Krasnoselski type fixed point results in metric, normed and Banach spaces (BS) with or without ordered hypothesis (total ordered or partial ordered topological spaces (TS) with monotonicity condition on operators) for the operator equations  $\xi v + \zeta v = v$  or  $\Gamma(v, v) = v$ . Generalized equiexpansive (GEE) and equicontractive conditions are used to find the fixed point results. The concept of MNC is generalized to investigate fixed point results by weak assumptions on the space and operator. The main results are applied to find the existence of solutions of differential and integral equations. The thesis is divided into eight chapters.

Chapter one is compiled for basic definitions and theorems, literature review, motivation

and research methodology.

Chapter two contains some variant results of Krasnoselskii type fixed point theorems. The Krasnoselskii type results are established in a setting consist of a map  $\Gamma$  depending upon two variables. The FP results for the operator equation  $\Gamma(v, v) = v$  are discussed under the assumption of equiexpansive mappings. The results are generalization of some important results in the literature. An existence result for controlled problem with less conditions is discussed which is generalized type of Cauchy problem for evolution differential equations. The contents of this chapter can be seen in **Journal of Function Spaces. 2021 Sep 11;2021.**

Chapter three discusses the fixed point results for the equations of types  $\Gamma(v, v) = v$  using the tool of MNC and the assumption of equiexpansive (EE) and equicontractive (EC) family. Noncompact operators,  $k$ -set contraction (KSC) and condensing, which is more general class than compact operators, are assumed for the investigation of fixed point results. The results are generalized form of the some variants of Krasnoselskii type fixed point theorems and generalization of Sadovskii theorem, which is generalized form of Schauder FPT. Application is given to apply the main result.

Chapter four consists of some fixed point theorems with newly defined generalized equiexpansive mapping. Fixed point theorems are proved for operator equation  $\Gamma(v, v) = v$ . Contraction principle and tool of measure of noncompactness to prove the theorems. Some important results in literature are special case of these theorems which are extended forms of some Krasnoselskii type theorems. As an application, the existence solution of a general class of integral is discussed which presents many types of evolutions. The contents of this chapter are published in **Advances in Continuous and Discrete Models: Theory and Modern Applications. 2022 Dec;2022(1):1-9.**

Chapter five presents the concept of partial E-MNC in ordered E-metric space (MS). The fixed point result are proved using monotone operators in Banach space with partial order. The concept of partially continuous, partially bounded and partially compact sets are introduced using E-valued function "E metric" on Banach space. FP results are established for the sum of two monotone operators. Application is given for the illustration of results. The investigation of this chapter is published in **Vietnam Journal of Mathematics. 2022 Jan;50(1):275-85.**

Chapter six illustrates the concept of fuzzy MNC, fuzzy condensing and fuzzy KSC and fixed

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# Chapter 1

## Preliminaries

### 1.1 Introduction

This chapter consists of some preliminary results and basic knowledge of topology and FPs that are used in proving the new theorems. Literature review, motivation and research methodology are also included.

**Notations:** Let  $\mathcal{D}$  be a Banach space.

(1)  $\bar{\mathcal{D}}_n(\mathcal{D}), \bar{\mathcal{D}}_{cl}(\mathcal{D}), \bar{\mathcal{D}}_{cv}(\mathcal{D}), \bar{\mathcal{D}}_{prc}(\mathcal{D}), \bar{\mathcal{D}}_{cp}(\mathcal{D}), \bar{\mathcal{D}}_b(\mathcal{D})$  are the sets in which there are all nonempty, nonempty closed, nonempty convex, nonempty precompact, nonempty compact and nonempty bounded subsets of  $\mathcal{D}$  respectively.

(2)  $\check{S}_b, \check{S}_{ct}, \check{S}_{cp}, \check{S}_{cl}, \check{S}_{cv}$  are the sets in which there are bounded, continuous, compact and convex mappings respectively and if, for example, there is a mapping  $\xi \in \check{S}_{cp}$ , then it means that  $\xi$  is compact on its domain.

However one can combine any of these notations, for example,  $\bar{\mathcal{D}}_{cl,b}(\mathcal{D})$  is the set in which there are all nonempty closed and bounded subsets of  $\mathcal{D}$ .

(3) If  $\mathcal{D}$  is a Banach space then the the operation of addition is defined by  $\oplus(\nu, v) = \nu \oplus v$  for  $\nu, v \in \mathcal{D}$  and for  $\nu \in \mathcal{D}$ ,  $\oplus'\nu$  represents the inverse of  $\nu$ . Moreover, for a field  $F = R$ , the same notation for the operation of addition is used.

#### 1.1.1 Basic definitions

Here are some basic useful definitions.

**Definition 1.1.** [119] (contraction mapping) Consider a MS  $\mathcal{D}$  and  $\xi : \mathcal{D} \rightarrow \mathcal{D}$ . Then  $\xi$  is contraction mapping if there is a number  $\delta$  such that  $0 < \delta < 1$  and

$$d(\xi\nu, \xi v) \leq \delta d(\nu, v)$$

for all  $\nu, v \in \mathcal{D}$ .

**Definition 1.2.** [128] (expansive mapping) Let  $\Omega$  be the set contained in a MS  $\mathcal{D}$ . A mapping  $\xi : \Omega \rightarrow \mathcal{D}$  is known to be an expansive if for some constant  $h > 1$  it satisfies the inequality

$$d(\xi\nu, \xi v) \geq h d(\nu, v)$$

for all  $\nu, v \in \Omega$ .

**Definition 1.3.** [119] (FP property) A TS  $\mathcal{D}$  is known to possess the FP property if "every continuous mapping of  $\mathcal{D}$  into  $\mathcal{D}$  has a FP".

**Definition 1.4.** [72] (equicontractive mapping) Suppose  $\mathcal{D}$  is a BS and  $\bar{\Pi} : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ . The family  $\{\bar{\Pi}(\cdot, \nu) : \nu \in \mathcal{D}\}$  is equicontractive if there is a  $k \in [0, 1)$  such that

$$\|\bar{\Pi}(\nu_1, \nu) \otimes' \bar{\Pi}(\nu_2, \nu)\| \leq k \|\nu_1 \otimes' \nu_2\|,$$

for all  $(\nu_1, \nu), (\nu_2, \nu)$  in the domain of  $\bar{\Pi}$ .

**Definition 1.5.** [21] Let  $B(\mathcal{D})$  be the family of all bounded subset of BS  $\mathcal{D}$ . A mapping  $\wp : B(\mathcal{D}) \rightarrow [0, +\infty)$  is called MNC defined on  $\mathcal{D}$  if:

- (1) Regularity: for any  $\Omega \in B(\mathcal{D})$ ,  $\wp(\Omega) = 0$ ,  $\Leftrightarrow \Omega$  is pre-compact;
- (2) Invariant under closure: for any  $\Omega \in B(\mathcal{D})$ ,  $\wp(\Omega) = \wp(\bar{\Omega})$ ;
- (3) Semi-additivity: for  $\Omega, \Omega^* \in B(\mathcal{D})$ ,  $\wp(\Omega \cup \Omega^*) = \max\{\wp(\Omega), \wp(\Omega^*)\}$ .

we can also deduce the following;

- (4) Monotonicity: for  $\Omega, \Omega^* \in B(\mathcal{D})$ , with  $\Omega \subseteq \Omega^*$  implies  $\wp(\Omega) \leq \wp(\Omega^*)$ ;
- (5) Algebraic semi-additivity:  $\Omega, \Omega^* \in B(\mathcal{D})$ ,  $\wp(\Omega \otimes \Omega^*) \leq \wp(\Omega) \otimes \wp(\Omega^*)$ .

**Definition 1.6.** [128] (KSC mapping) Consider a set  $\Omega$  in the BS  $\mathcal{D}$ . A mapping  $\xi : \Omega \rightarrow \mathcal{D}$  is known as KSC if

- (1)  $\xi$  is bounded and continuous

(2)  $\wp(\xi(\mathcal{U})) \leq k\wp(\mathcal{U})$  for any bounded subset  $\mathcal{U}$  of  $\Omega$ .

$\xi$  is called strictly KSC if it is KSC and for all bounded subsets  $\mathcal{U}$  of  $\Omega$  with  $\wp(\mathcal{U}) \neq 0$ , we have  $\wp(\xi(\mathcal{U})) < k\wp(\mathcal{U})$ . Every 1-set contraction map is called condensing map.

**Definition 1.7.** [79] (Kuratowski MNC) The ball MNC  $\alpha$  can be written as

$$\alpha(\Omega) = \inf \left\{ \lambda > 0 : \Omega \subset \bigcup_{i=0}^n \zeta_i, \delta(\zeta_i) \leq \lambda \right\},$$

where  $\zeta_i$  are nonempty sets in  $\mathcal{D}$  and  $\delta$  denotes the diameter of a set.

**Definition 1.8.** (Hausdorff MNC) Sadovskii [117] defined the Hausdorff MNC  $\beta$ ,

$$\beta(\Omega) = \inf \left\{ \lambda > 0 : \Omega \subset \bigcup_{i=0}^n B(\kappa_i, \lambda) \text{ for some } \kappa_i \in M \right\},$$

where  $B(\kappa_i, \lambda)$  are open balls whose center at  $\kappa_i$  and  $\lambda$  is their radius.

**Definition 1.9.** [19] (ordered space) Let  $\mathbb{Y}$  be a VS and  $\preceq$  be a partial order relation,  $\mathbb{Y}$  is called an ordered space if

(O1)  $\forall \kappa, v$ , and  $\omega \in \mathbb{Y}$ ,  $\kappa \preceq v$  implies  $\kappa \oplus \omega \preceq v \oplus \omega$ ,

(O2)  $\forall a \in \mathbb{R}^{\otimes}$  and  $\kappa \in \mathbb{Y}$  such that  $\kappa \succeq 0_{\mathbb{Y}}$ , implies that  $a\kappa \succeq 0_{\mathbb{Y}}$ .

If  $\mathbb{Y}$  is normed space with  $\|\cdot\|_{\mathbb{Y}}$  such that

(O3) there exists  $\lambda > 0$  and  $\forall \kappa, v \in \mathbb{Y}$ , with  $0_{\mathbb{Y}} \preceq \kappa \preceq v$  implies  $\|\kappa\|_{\mathbb{Y}} \leq \lambda \|v\|_{\mathbb{Y}}$ , in this case  $\mathbb{Y}$  is known as a normed ordered space.

Let  $\mathbb{Y}$  be a space having order. The element  $v \in \mathbb{Y}$  is known as positive if  $v \succeq 0_{\mathbb{Y}}$ . We represent the set of all positive elements (also called a cone) of  $\mathbb{Y}$  by  $\mathbb{Y}^*$ . For  $x \succ y$  clearly indicate that  $x \succeq y$  but  $x \neq y$ .

**Definition 1.10.** [19] ( $\mathbb{Y}$ -MS) Let  $\mathbb{Y}$  be a real ordered VS, and  $\mathcal{D}$  be a set which is nonempty. An  $\mathbb{Y}$ -valued function  $d^{\mathbb{Y}} : \mathcal{D} \times \mathcal{D} \rightarrow \mathbb{Y}$  is  $\mathbb{Y}$ -metric on  $\mathcal{D}$ , if for all  $\kappa, v$  and  $\omega \in \mathcal{D}$ , there are following conditions

( $\mathbb{Y}$ 1)  $d^{\mathbb{Y}}(\kappa, v) \succeq 0_{\mathbb{Y}}$  and  $d^{\mathbb{Y}}(\kappa, v) = 0_{\mathbb{Y}} \Leftrightarrow \kappa = v$ ,

( $\mathbb{Y}$ 2)  $d^{\mathbb{Y}}(\kappa, v) = d^{\mathbb{Y}}(v, \kappa)$ ,

( $\mathbb{Y}$ 3)  $d^{\mathbb{Y}}(\kappa, v) \preceq d^{\mathbb{Y}}(\kappa, \omega) \oplus d^{\mathbb{Y}}(\omega, v)$ .

Then  $(\mathcal{D}, d^{\mathbb{Y}})$  is known as an  $\mathbb{Y}$ -MS.

**Definition 1.11.** A mapping  $\xi : \mathcal{D} \rightarrow \overline{\mathbb{I}}_n(\mathcal{D})$  is known as set-valued/multivalued mapping

and a point  $\hat{u} \in \mathcal{D}$  is known as a FP of  $\xi$  if  $\hat{u} \in \xi\hat{u}$ .

We set  $Fix_\xi$ , the collection of all FPs of the map  $\xi$ , and  $\xi(A) = \bigcup_{\hat{u} \in A} \xi\hat{u}$  for any  $A \subset \mathcal{D}$ .

**Definition 1.12.** [45] Let  $\mathcal{D}$  and  $\acute{E}$  be BSs, an operator  $\xi : \mathcal{D} \longrightarrow \bar{\Pi}_n(\acute{E})$  is said to be an upper semi-continuous USC, ( resp, lower semi-continuous LSC) if the set  $\{\hat{u} \in \mathcal{D} : \xi\hat{u} \subset A\}$  ( $\{\hat{u} \in \mathcal{D} : \xi\hat{u} \cap A \neq \emptyset\}$  for the case of LSC ) is an open set in  $\mathcal{D}$  for each open subset  $A$  of  $\acute{E}$ .

**Definition 1.13.** [45] A multivalued operator  $\xi : \mathcal{D} \longrightarrow \bar{\Pi}_{cp}(\acute{E})$  is compact if

- (i)  $\overline{\xi(\mathcal{D})}$  is compact subset of  $\acute{E}$ , and the operator  $\xi$  is called totally bounded if
- (ii) the image of each bounded set under  $\xi$  is totally bounded subset of  $\acute{E}$ .

**Remark 1.1.** [45] It is well known that "compact multivalued operator implies totally bounded operator".

**Definition 1.14.** [45] The multivalued operator  $\xi : \mathcal{D} \longrightarrow \bar{\Pi}_{cp}(\acute{E})$  is completely continuous if it is USC and TB

**Definition 1.15.** [45] The graph of a multivalued operator  $\xi : \mathcal{D} \longrightarrow \bar{\Pi}_n(\acute{E})$  is defined as

$$\mathcal{G}\tau(\xi) = \left\{ (\hat{u}, v) \in \mathcal{D} \times \acute{E} : v \in \xi\hat{u} \right\},$$

and the graph is closed if for  $\{(\hat{u}_n, v_n)\}_{n=0}^\infty \subset \mathcal{G}\tau(\xi)$ , with condition that  $(\hat{u}_n, v_n) \longrightarrow (\hat{u}, v)$  we have  $(\hat{u}, v) \in \mathcal{G}\tau(\xi)$ . If the graph of  $\xi$  is closed then it is closed.

**Definition 1.16.** [16] A multivalued operator  $\xi : \mathcal{D} \longrightarrow \bar{\Pi}_{cl,b}(\mathcal{D})$  is called a  $\wp$ -condensing, if for any  $B \in \bar{\Pi}_b(\mathcal{D})$ , the set  $\xi(B) \in \bar{\Pi}_b(\mathcal{D})$  and

$$\wp(\xi(B)) < \wp(B)$$

for  $\wp(B) > 0$ .

**Definition 1.17.** [16] A multivalued operator  $\xi : \mathcal{D} \longrightarrow \bar{\Pi}_{cl,b}(\mathcal{D})$  is known as  $\wp$ -KSC, if for any  $B \in \bar{\Pi}_b(\mathcal{D})$ , the set  $\xi(B) \in \bar{\Pi}_b(\mathcal{D})$  and

$$\wp(\xi(B)) \leq k \cdot \wp(B),$$

for some  $k \in (0, 1)$ .

**Definition 1.18.** [38] Suppose  $f \in H^1(a, b)$ , and  $\alpha \in (0, 1)$ , then the left FD (in the sense

of Caputo and Fabrizio) is defined by

$$({}^{CF}D_a^\alpha f)(t) = \frac{B(\alpha)}{1 \otimes' \alpha} \int_a^t \left( f'(x) \exp \left[ \otimes' \alpha \frac{(t \otimes' x)}{1 \otimes' \alpha} \right] \right) dx$$

and the right FD is

$$({}^{CF}D_b^\alpha f)(t) = \frac{\otimes' B(\alpha)}{1 \otimes' \alpha} \int_t^b \left( f'(x) \exp \left[ \otimes' \alpha \frac{(x \otimes' t)}{1 \otimes' \alpha} \right] \right) dx.$$

Here  $f'$  is an ordinary derivative of  $f$ . The corresponding integrals are

$$({}^{CF}I_a^\alpha f)(t) = \frac{1 \otimes' \alpha}{B(\alpha)} f(t) \otimes \frac{\alpha}{B(\alpha)} \int_a^t f(\kappa) d\kappa$$

and

$$({}^{CF}I_b^\alpha f)(t) = \frac{1 \otimes' \alpha}{B(\alpha)} f(t) \otimes \frac{\alpha}{B(\alpha)} \int_t^b f(\kappa) d\kappa$$

respectively.

The higher order FD and integrals are given below in the following definitions.

**Definition 1.19.** [3] Let  $n < \alpha \leq n \otimes 1$  and  $f$  be such that  $f^n \in H^1(a, b)$ . Set  $\beta = \alpha \otimes' n$ . Then  $\beta \in (0, 1]$  and we define

$$({}^{CF}D_a^\alpha f)(t) = \left( {}^{CF}D_a^\beta f^n \right)(t),$$

the associated fractional integral is

$$({}^{CF}I_a^\alpha f)(t) = \left( {}_a I_a^{nCF} I^\beta f \right)(t).$$

**Definition 1.20.** [3] Let  $n < \alpha \leq n \otimes 1$  and  $f$  be such that  $f^n \in H^1(a, b)$ . Set  $\beta = \alpha \otimes' n$ . Then  $\beta \in (0, 1]$  and we define

$$({}^{CF}D_b^\alpha f)(t) = \left( {}^{CF}D_b^\beta (\otimes' 1)^n f^n \right)(t),$$

the associated fractional integral is

$$({}^{CF}I_b^\alpha f)(\iota) = \left( I_b^{nCF} I_b^\beta f \right) (\iota).$$

Let  $H$  be the space in which there are all continuous functions from  $[0, c]$  into  $\mathcal{D}$ .

**Definition 1.21.** [5] A function  $f : J \times H \rightarrow H$  is Carathéodory if

- (a)  $\iota \rightarrow f(\iota, v)$  is measurable for every second variable.
- (b)  $v \rightarrow f(\iota, v)$  is continuous for almost all  $\iota \in J$ .

### 1.1.2 Basic theorems

These are basic theorems that are useful for proving new results.

**Theorem 1.1.** [119] (Banach) If a self map is contraction and the domain of the map is complete, then it has a unique FP.

**Theorem 1.2.** [119] (Brouwer) Every  $\Omega \in \overline{\mathbb{I}}_{cp,cv}(\mathbb{R}^n)$  has FP property.

**Theorem 1.3.** [45] (Schauder) Let  $\mathcal{D}$  be a normed space and  $\Omega \in \overline{\mathbb{I}}_{cl,cv}(\mathcal{D})$ . Then every compact and continuous function  $\xi : \Omega \rightarrow \Omega$  possesses at least one FP.

**Theorem 1.4.** [94] (Krasnoselskii) Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\mathbb{I}}_{cl,cv}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  map  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi\nu \otimes \zeta v \in \Omega$  for all  $\nu, v$  in  $\Omega$ ;
- (2)  $\xi \in \check{S}_{cl,cp}$
- (3)  $\zeta$  is a contraction.

Then there is a  $v$  in  $\Omega$  such that  $\xi v \otimes \zeta v = v$ .

**Theorem 1.5.** [73] (Schaefer) Let  $\mathcal{D}$  be a BS and  $Q : \mathcal{D} \rightarrow \mathcal{D}$  be a completely continuous mapping. Then either  $y = \lambda Qy$  has a solution for  $\lambda = 1$ , or the set  $\{y \in \mathcal{D} : y = \lambda Qy \text{ for some } \lambda \in (0, 1)\}$  is unbounded.

**Theorem 1.6.** [133] (Arzela-Ascoli) Consider a subset  $F$  of the space  $C(J, \mathbb{R})$ . If  $F$  is relatively compact then it is UB and equicontinuous on  $J$ . The converse is true also.

**Lemma 1.1** [128] Suppose  $\Omega \in \overline{\mathbb{I}}_n(\mathcal{D})$  and  $\xi : \Omega \rightarrow \mathcal{D}$  Lipschitzean with

$$\|\xi\nu \otimes' \xi v\| \leq k \|\nu \otimes' v\|$$

for  $\nu, v \in \Omega$ . Then we have  $\wp(\xi(S)) \leq k\wp(S)$  for every  $\Omega \in \overline{\Pi}_b(\varnothing)$ .

**Theorem 1.7.** [50] (Dhage) Let  $\varnothing$  be a BS and  $\Omega \in \overline{\Pi}_{cl}(\varnothing)$  and let  $\xi : \Omega \longrightarrow \overline{\Pi}_{cv,cp}(\varnothing)$  be an USC operator. Then the operator has a FP.

**Lemma 1.2.** [45] If a multivalued operator  $\xi : \varnothing \longrightarrow \overline{\Pi}_{cl}(\dot{E})$  is compact and its graph is closed, then it is USC. The converse is true also.

**Theorem 1.7.** [69] Let  $\Omega$  be a subset of a BS  $\mathbb{Y}$ . Also  $\Omega$  is closed and convex and let  $T : \Omega \rightarrow \Gamma_{cp,cv}(\mathbb{Y})$  be an upper semicontinuous compact map. Then it has a FP.

**Theorem 1.8.** [104] For a given BS  $\varnothing$  and  $\Omega \in \overline{\Pi}_{cl,cv}(\varnothing)$  and let  $\xi : \Omega \longrightarrow \overline{\Pi}_{cp,cv}(\varnothing)$  be an USC compact map. Then it has a FP.

The following theorem is more general case of the above theorem and is stated as follows.

**Theorem 1.9.** [100] For a given BS  $\varnothing$  and  $\Omega \in \overline{\Pi}_{cl,cv,b}(\varnothing)$  and let  $\xi : \Omega \longrightarrow \overline{\Pi}_{cp}(\varnothing)$  be a map, where  $\xi \in \check{S}_{cl,cp}$ . Then it has a FP.

**Theorem 1.10.** [49] For a given BS  $\varnothing$  and  $\Omega \in \overline{\Pi}_{cl,b}(\varnothing)$ . Let  $\xi : \Omega \longrightarrow \overline{\Pi}_{cl,cv}(\varnothing)$  be  $\beta$ -condensing and USC operator. Then  $\xi$  has a FP.

**Lemma 1.3.** [3] For  $0 < \alpha < 1$ , we have

$$({}^{CF}I_a^\alpha {}^{CF}D_a^\alpha f)(\iota) = f(\iota) \otimes' f(a)$$

and

$$({}^{CF}I_b^\alpha {}^{CF}D_b^\alpha f)(\iota) = f(b) \otimes' f(\iota)$$

The next proposition will be crucial for our main results.

**Proposition 1.1.** [3] For  $y(\iota)$  defined on  $[a, b]$  and  $\alpha \in (n, n \otimes 1]$  for some  $n \in \mathbb{N}$ ,

$$({}^{CF}I_a^\alpha {}^{CF}D_a^\alpha y)(\iota) = y(\iota) \otimes' \sum_{k=0}^n \frac{y^k(a)}{k!} (\iota \otimes' a)^k$$

and

$$({}^{CF}I_b^\alpha {}^{CF}D_b^\alpha y)(\iota) = y(\iota) \otimes' \sum_{k=0}^n \frac{(\otimes' 1)^k y^k(b)}{k!} (b \otimes' \iota)^k.$$

**Lemma 1.4.** [5] Let  $\varnothing$  be a BS and  $\Omega \in \overline{\Pi}_{cl,cv,b}(\varnothing)$  and  $G$  be a continuous map whose domain is  $J \times J$  and  $f$  be a Carathéodory on  $J \times \varnothing$ . If there exists  $q \in L^1(J, \mathbb{R}^{\otimes})$  and for every

subset  $\beta$  of  $\mathcal{D}$  which are bounded, we have

$$\lim_{\kappa \rightarrow 0^{\circ}} (\mathfrak{f}(J \cap [\iota \otimes' h, \iota] \times \beta)) \leq q(\iota) \alpha(\beta).$$

If  $U$  is equicontinuous subset of  $\Omega$ , then

$$\alpha \left( \left\{ \int_J G(\kappa, \iota) \mathfrak{f}(\kappa, v(\kappa)) d\kappa : v \in U \right\} \right) \leq \int_J \|G(\kappa, \iota)\| q(\kappa) \alpha(U(\kappa)) d\kappa.$$

## 1.2 Literature Review

FP theory play very important part in the ES of many kinds of differential, integral and fractional differential equations. In 1910, Brouwer [119] proved a basic FP theorem such that a continuous self mapping of convex and compact subset of a euclidean space has FP. The most simple form of Brouwer result is from closed interval to itself and closed disk into itself. In his proof, he used the definitions of degree of mappings (rotation of a vector fields). Another proof using classical methods (calculus and determinants) was constructed by Birkhoff and Kellogg (1922) and by Dunford and Schwartz in 1958. In 1922, Banach [119] constructed a FP result that every self contraction on a MS which is also complete has a unique FP. Schauder [119], in 1930, extended Brouwer theorem to infinite dimensional BSs. These two results have fundamental importance in FP theory. Darbo [119], in 1955, generalized Schauder FP theorem by introducing k-set contraction and relaxing the compactness condition on the space. Later on Sadovskii [119], in 1967, used more general class of functions than k-set contraction of Darbo theorem, called condensing mapping, to get FP theorems

In 1955, Krasnoselskii [75] made a useful attempt and proved a FP result for the sum of two operators. During the analysis of solutions of neutral and delayed differential equations, Krasnoselskii observed that, in most of the cases, the solutions of these equations might be expressed as a sum of contractive and compact operators. Therefore Krasnoselskii [75] generalized both results of Banach and Schauder, for finding the FPs of sum of contractive and compact operators. This theorem of Krasnoselskii is applied in constructing ES for the equations with perturbed operators. The perturbed operators may be split into compact and contraction operators. The application in the existence theory of DEs and IEs can be seen in the following

monographs [73, 105, 133].

Considering the importance of Krasnoselskii FP theorem, many generalizations and modifications are being established. One of the most impressive extensions was given by Hoa and Schmitt [64] in 1994. They studied FPTs of Krasnoselskii type (KT) in locally convex spaces. In 2005, Barroso and Teixeira [28] investigated FP results through topological and geometric approach on sum of two functions. Burton [33] in 1998, replaced Krasnoselskii assumption  $(\xi x \oplus \zeta y \in \Omega$  for all  $x$  and  $y$  in  $\Omega$ ) by  $([x = \zeta x \oplus \xi y, y \text{ in } \Omega]$  implies  $x$  is in  $\Omega$ ) and find FP result which is improvement in the Krasnoselskii's theorem. Burton also [34] investigated the stability results by FPTs of Krasnoselskii type in 2002. Ok [98] in 2009, explored the FP problem for  $\xi \oplus \zeta$ , where  $\xi$  is continuous and  $\zeta$  is compact. The domain of these functions was not necessarily convex. He assumed the compact operator to be multivalued while the other operator is multivalued or a suitable nonexpansive map. Park [103] also generalized Krasnoselskii FPTs. He assumed Hausdorff topological VSs with fundamental system  $V$  of neighborhoods about origin. In 2009, Xiang and Yuan [128] studied FP results of KT which complements Krasnoselskii FPT. They applied the results to IEs having periodic solutions.

O'Regan and Taudi [101] obtained FPTs of Krasnoselskii-Leray-Schauder type FP results in weak topology. In 2010, Xiang and Yuan [129] used MNC for the sum of two operators where one is noncompact and the other is not necessarily continuous. They applied the results to neutral DEs having periodic solutions with delay in critical cases. Ben and Xiang [29] explored the FP results by measure of weak noncompactness to have FPT for sum  $\xi \oplus \zeta$  of a metrizable locally convex space in weak topology where  $1 \oplus' \xi$  may not be invertible,  $\xi$  may not be continuous and  $\zeta$  is not weakly compact.

The Krasnoselskii type FPT assuming order properties on the space was considered by Dhage [51]. He made an extension of Tarski FP theorem. In 1992, he studied FP analysis in BSs having order properties. Ran and Reurings [107] investigated hybrid FPTs using partially ordered (PO) sets. Nieto and Lopez [95] explored FP analysis in PO sets and applied it to ordinary DEs. Hybrid FP analysis in normed spaces with PO assumption on the space was done by Dhage [50] in 2013 and applied the results to fractional integral equations. He used mixed arguments from topology, algebra and analysis. He also proved analogous results in complete normed linear spaces with partial order. These results are useful for applications of

perturbed DEs and IEs.

Fractal fractional calculus is an emerging topic from last few decades. Fractal derivative is a natural generalization of Leibniz's derivative for discontinuous fractal media. It can be considered as a local FD and used in modeling of many physical phenomena, for instance one can see in details the articles [36, 38, 57] and the references cited therein. Many FDs has been defined in the literature [2, 20, 36, 38, 57, 83]. The same argument will work for the case of fractal derivatives as well [57]. Every new definition improves some aspects of derivative but till now no perfect definition of fractional and fractal derivative have been defined. It was started in 18th century when Liouville tried for defining FD. The working most known FDs are Riemman-Liouville, Hadamard, Caputo and Grunwald-Letnikov, Caputo Fabrizio and Atangana-Baleanu. Beside these definitions, every derivative have had been applied in many physical problems and more accuracy has been obtained [36, 38, 57]. Here people got some realistic and natural questions regarding their physical problems. Whether the problem has any solution? These types of questions allow us to study the qualitative theory of fractional differential equations. The existence can be guaranteed by means of many metric and topological FP results. Since most of the metric FP results are concerned with contractive conditions. There are very small classes of differential equation whose kernels are contractive, so that we choose topological FP theory which is applicable for most of the existing and new problems.

### 1.3 Motivation

Keeping the importance of the above work, to generalize its results and apply them to differential and fractional differential equations, it is appropriate to give new definitions and generalized FP results for the operator depending upon two variables in general setting and sum of two operators. Schauder, Krasnoselskii, Darbo and Sadovskii type FPTs may be generalized by using monotone functions, partial MNC and fuzzy MNC and weakening the condition of norm on the space. Krasnoselskii type results can be improved by generalizing contractive, condensing and expansive conditions on operators. This work will be motivated by compactness, convexity, boundedness, MNC, continuity, contraction, monotonicity, and the analysis to be presented reveal key features of Krasnoselskii and Schauder type FPTs in the framework of topology,

analysis and algebra. As a result the ERs for broad class of differential equations may be achieved.

## 1.4 Research Methodology

For analysis in FP theory, there is an operator from a set into itself with different structure such as BS, normed space, MS, TS or these types of spaces with algebraic arguments. So we use general continuity and convergence criteria, convexity properties, metric properties, normed properties, ordered properties and topological results to make analysis of the problems. New definitions and lemmas are constructed to solve our main problems and sometimes there are already available results to be applied. Properties of MNC,  $k$ -set contraction and condensing operator are used to weak the condition of compactness. For TSs with algebraic structure having partial order or total order, monotonicity condition, partial continuity and partial convergence criteria are used to prove the ERs.

## Chapter 2

# Some Variants of Krasnoselskii-type Fixed Point Results for Equiexpansive Mappings with Applications

### 2.1 Introduction

Krasnoselskii FP theorem [119] is a generalized form of Schauder and Banach FP theorems. While studying the solutions of delay and neutral differential/integral equations, it has been noticed that the solution can be expressed as a sum of contractive and compact operators. In Krasnoselskii FP theorem there are two operators in which one is compact and continuous and the other is contraction. Therefore the theorem plays an important role in the ESs of delay integral equations and neutral functional equations. Many generalizations and modifications of the Krasnoselskii FP theorem have appeared; for example, see [22, 33, 41, 72, 89, 94, 106, 103, 126, 128, 129] and the references therein. It is important to note that there is a very restrict and small class of operators that are contractions, due to this reason many theorems involving contractive operators are less applicable. Therefore it is required to cover more applications with different types of operators. In this regard, one of suitable choices is the class of expansive

operators. Also there are number of equations in which we cannot decompose the operator as a sum of two or more operators. To overcome this situation the authors in [72, 89, 94] established Krasnoselskii type results in more general setting consists of a single map  $\bar{\Pi}$  depending upon two variables. The following theorem is in [94].

**Theorem 2.1** Let  $\Omega \in \bar{\Pi}_{cl,cv}(\mathcal{D})$ , where  $\mathcal{D}$  be a BS and  $\bar{\Pi} : \Omega \times \Omega \longrightarrow \Omega$  be a mapping with

$$(1) \quad \|\bar{\Pi}(\nu, \nu_1) \otimes' \bar{\Pi}(\nu, \nu_2)\| \leq \gamma \|\nu_1 \otimes' \nu_2\| \quad \text{for some } \gamma \in [0, 1);$$

and for all  $\nu \in \mathcal{D}$ ,

$$(2) \quad \|\bar{\Pi}(\nu_1, \nu) \otimes' \bar{\Pi}(\nu_2, \nu)\| \leq \|\xi\nu_1 \otimes' \xi\nu_2\|;$$

where  $\xi$  is completely continuous and maps  $\Omega$  into  $\Omega$ . Then there is an element  $\nu \in \Omega$  such that  $\bar{\Pi}(\nu, \nu) = \nu$ .

In most FP results there is a condition such that  $T(\Omega) \subseteq \Omega$ , where  $\Omega \subseteq \mathcal{D}$ ,  $\mathcal{D}$  is a BS and  $T$  is an operator on  $\Omega$  into  $\Omega$ . No much attention has been given to the case  $\Omega \subseteq T(\Omega)$ . Xiang and Yuan [128] investigated the case  $\Omega \subseteq T(\Omega)$  and obtained the following important results in this direction.

**Theorem 2.2.** [128]. Let  $\Omega \neq \phi$ , which is closed subset of a complete MS  $\mathcal{D}$  and  $\zeta$  is an expansive mapping of  $\Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \zeta(\Omega)$ . Then there is a unique  $\kappa \in \Omega$  such that  $\zeta\kappa = \kappa$ .

**Theorem 2.3.** [128]. For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\Pi}_{cl,cv}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that

$$(1) \quad \xi \text{ is continuous and } \xi(\Omega) \subset C \in \bar{\Pi}_{cp}(\mathcal{D});$$

$$(2) \quad \zeta \text{ is an expansive mapping;}$$

$$(3) \quad \Omega \subseteq \nu \otimes \zeta(\Omega) \text{ for } \nu \in \xi(\Omega).$$

Then there is  $\kappa \in \Omega$  with  $\xi\kappa \otimes \zeta\kappa = \kappa$ .

Inspired by the above results we study the Krasnoselskii type FP results for the operator  $\bar{\Pi}$  with conditions:

$$(i) \quad \text{The family } \{\bar{\Pi}(\nu, \cdot) : \nu \in \mathcal{D}\} \text{ is equiexpansive and;}$$

$$(ii) \quad \Omega \subseteq \bar{\Pi}(\nu, \Omega) \text{ for } \nu \in \Omega \text{ or } \Omega \subseteq \bar{\Pi}(\nu, \Omega) \text{ for } \nu \in \xi(\Omega);$$

where  $\Omega \subseteq \mathcal{D}$  and  $\xi$  is an operator of  $\Omega$  into  $\Omega$  or  $\mathcal{D}$  and, motivated by [72], the equiexpansive family can be defined here.

**Definition 2.1.** Let  $\mathcal{D}$  be a BS and  $\bar{\Pi} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$ . The family  $\{\bar{\Pi}(\nu, \cdot) : \nu \in \mathcal{D}\}$  is called

equiexpansive if there is  $h > 1$  such that

$$\|\overline{\mathbb{I}}(\nu, v_1) \otimes' \overline{\mathbb{I}}(\nu, v_2)\| \geq h \|v_1 \otimes' v_2\|,$$

for all  $(\nu, v_1), (\nu, v_2)$  in the domain of  $\overline{\mathbb{I}}$ .

**Remark 2.1.** In Theorem 2.1, the mapping  $\overline{\mathbb{I}}$  is equicontractive and  $\overline{\mathbb{I}}(\nu, v) \in \Omega$  for  $\nu, v$  in  $\Omega$  while in (i) and (ii) the family  $\{\overline{\mathbb{I}}(\nu, \cdot) : \nu\}$  is equiexpansive and  $\Omega \subseteq \overline{\mathbb{I}}(v, \Omega)$  for  $v \in \xi(\Omega)$ .

## 2.2 Main results

Here some some FPTs which are generalization of some well known results.

**Theorem 2.4.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\mathbb{I}}_{cl, cv}(\mathcal{D})$  and  $\xi$  be a continuous mapping of  $\Omega$  into  $\mathcal{D}$  such that  $\xi(\Omega) \in \overline{\mathbb{I}}_{cp}(\mathcal{D})$ . Let  $\overline{\mathbb{I}} : \xi(\Omega) \times \Omega \rightarrow \mathcal{D}$  be a mapping such that the family  $\{\overline{\mathbb{I}}(v, \cdot) : v \in \xi(\Omega)\}$  is equiexpansive and  $\Omega \subseteq \overline{\mathbb{I}}(v, \Omega)$  for  $v \in \xi(\Omega)$  with

$$\|\overline{\mathbb{I}}(v_1, v) \otimes' \overline{\mathbb{I}}(v_2, v)\| \leq \|v_1 \otimes' v_2\|.$$

Then there is  $\kappa \in \Omega$  such that  $\overline{\mathbb{I}}(\xi\kappa, \kappa) = \kappa$ .

**Proof.** For  $\hat{u} \in \xi(\Omega)$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  by  $H(v) = \overline{\mathbb{I}}(\hat{u}, v)$ . The mapping  $H$  is expansive because

$$\|H(v_1) \otimes' H(v_2)\| = \|\overline{\mathbb{I}}(\hat{u}, v_1) \otimes' \overline{\mathbb{I}}(\hat{u}, v_2)\| \geq h \|v_1 \otimes' v_2\|,$$

and  $\Omega \subseteq \overline{\mathbb{I}}(v, \Omega) = H(\Omega)$ . By Theorem 2.2, there exists at most one point in  $\kappa^* \in \Omega$  with  $H(\kappa^*) = \kappa^*$ . Since for  $\hat{u} \in \xi(\Omega)$  there is a unique  $\kappa^* \in \Omega$ , so we can define  $\mathbb{P} : \xi(\Omega) \rightarrow \Omega$  such that  $\mathbb{P}\hat{u} = \kappa^* = H(\kappa^*) = \overline{\mathbb{I}}(\hat{u}, \kappa^*) = \overline{\mathbb{I}}(\hat{u}, \mathbb{P}\hat{u})$  for  $\hat{u} \in \xi(\Omega)$ . Therefore, for  $v_1, v_2 \in \xi(\Omega)$  we

have

$$\begin{aligned}
& \| \mathbf{P}(v_1) \otimes' \mathbf{P}(v_2) \| \\
&= \| \overline{\mathbf{P}}(v_1, \mathbf{P}v_1) \otimes' \overline{\mathbf{P}}(v_2, \mathbf{P}v_2) \| \\
&= \| (\overline{\mathbf{P}}(v_2, \mathbf{P}v_1) \otimes' \overline{\mathbf{P}}(v_2, \mathbf{P}v_2)) \otimes' (\overline{\mathbf{P}}(v_2, \mathbf{P}v_1) \otimes' \overline{\mathbf{P}}(v_1, \mathbf{P}v_1)) \| \\
&\geq \| \overline{\mathbf{P}}(v_2, \mathbf{P}v_1) \otimes' \overline{\mathbf{P}}(v_2, \mathbf{P}v_2) \| \otimes' \| \overline{\mathbf{P}}(v_2, \mathbf{P}v_1) \otimes' \overline{\mathbf{P}}(v_1, \mathbf{P}v_1) \| \\
&\geq h \| \mathbf{P}v_1 \otimes' \mathbf{P}v_2 \| \otimes' \| v_1 \otimes' v_2 \|.
\end{aligned}$$

This means that

$$\| \mathbf{P}v_1 \otimes' \mathbf{P}v_2 \| \leq \frac{1}{h \otimes' 1} \| v_1 \otimes' v_2 \| \quad h > 1,$$

which shows that  $\mathbf{P}$  is continuous function of  $\xi(\Omega)$  into  $\Omega$ . Also  $\mathbf{P} \circ \xi$  is a continuous function of  $\Omega$  into  $\Omega$ . Since  $\xi(\Omega) \subseteq C$ , where  $C$  is any compact set, therefore  $\overline{\xi(\Omega)} \subseteq \overline{C} = C$  and  $\overline{\xi(\Omega)}$  is compact by [[114],page 37]. By [[77],page 412]  $\overline{\mathbf{P}(\xi(\Omega))}$  is compact because  $\xi\Omega$  is relatively compact, and  $\mathbf{P}(\xi(\Omega))$  resides in a compact set  $\overline{\mathbf{P}(\xi(\Omega))}$ . Since  $\mathbf{P}(\xi(\Omega)) \subseteq \Omega$ , therefore  $\overline{\mathbf{P}(\xi(\Omega))} \subseteq \overline{\Omega} = \Omega$ . Hence  $\mathbf{P} \circ \xi(\Omega) = \mathbf{P}(\xi(\Omega)) \subseteq \overline{\mathbf{P}(\xi(\Omega))} = \overline{\mathbf{P} \circ \xi(\Omega)} \subseteq \Omega$  and  $\mathbf{P} \circ \xi$  is a continuous function of  $\Omega$  into a compact set  $\overline{\mathbf{P} \circ \xi(\Omega)} \subseteq \Omega$ . By Schauder second theorem [119] there is  $\kappa \in \Omega$  such that  $\mathbf{P} \circ \xi(\kappa) = \kappa$  or  $\mathbf{P}(\xi\kappa) = \kappa$ . Since for  $\xi\kappa \in \xi(\Omega)$  there is unique  $\mathbf{P}(\xi\kappa) \in \Omega$  such that  $\overline{\mathbf{P}}(\xi\kappa, \mathbf{P}(\xi\kappa)) = \mathbf{P}(\xi\kappa)$  and also  $\mathbf{P}(\xi\kappa) = \kappa$ , therefore  $\overline{\mathbf{P}}(\xi\kappa, \kappa) = \kappa$ . ■

**Corollary 2.1.** [128] Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\mathbf{P}}_{cl,cv}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi$  is continuous and  $\xi(\Omega) \subset C \in \overline{\mathbf{P}}_{cp}(\mathcal{D})$ ;
- (2)  $\zeta$  is an expansive mapping;
- (3)  $\Omega \subseteq \nu \otimes' \zeta(\Omega)$  for  $\nu \in \xi(\Omega)$ .

Then there is  $\kappa \in \Omega$  with  $\xi\kappa \otimes' \zeta\kappa = \kappa$ .

**Proof.** Define  $\overline{\mathbf{P}}(\nu, v) = \nu \otimes' \zeta v$  with  $\nu \in \xi(\Omega)$  (Theorem 2.4). Since  $\overline{\mathbf{P}}(\nu, \Omega) = \nu \otimes' \zeta(\Omega)$  and from (3)  $\Omega \subseteq \nu \otimes' \zeta(\Omega)$ , therefore  $\Omega \subseteq \overline{\mathbf{P}}(\nu, \Omega)$ . Now since  $\zeta$  is expansive mapping therefore

$$\| \overline{\mathbf{P}}(\nu, v_1) \otimes' \overline{\mathbf{P}}(\nu, v_2) \| = \| \zeta v_1 \otimes' \zeta v_2 \| \geq h \| v_1 \otimes' v_2 \|.$$

Also

$$\|\overline{\mathbb{I}}(\nu_1, v) \otimes' \overline{\mathbb{I}}(\nu_2, v)\| = \|\nu_1 \otimes' \nu_2\| \leq \|\nu_1 \otimes' \nu_2\|.$$

Hence there is  $\kappa \in \Omega$  such that  $\kappa = \overline{\mathbb{I}}(\xi\kappa, \kappa) = \xi\kappa \otimes \zeta\kappa$ . ■

**Remark 2.2.** The above corollary is the Theorem 2.2 in [128]. If we take  $\xi = 0$ , zero operator, then we obtain Theorem 2.1 in [128].

**Corollary 2.2.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\mathbb{I}}_{cl,cv}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi$  nonexpansive and  $\xi(\Omega) \subset C \in \overline{\mathbb{I}}_{cp}(\mathcal{D})$ ;
- (2)  $\zeta$  is expansive mapping;
- (3)  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for  $\nu \in \xi(\Omega)$ .

Then there is  $\kappa \in \Omega$  with  $(\xi \circ \xi)\kappa \otimes \zeta\kappa = \kappa$ .

**Proof.** Define  $\overline{\mathbb{I}}(\nu, v) = \xi\nu \otimes \zeta v$  with  $\nu \in \xi(\Omega)$  (Theorem 2.4). Since  $\overline{\mathbb{I}}(\nu, \Omega) = \xi\nu \otimes \zeta(\Omega)$  and from (3)  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for  $\nu \in \xi(\Omega)$ , therefore  $\Omega \subseteq \overline{\mathbb{I}}(\nu, \Omega)$ . Now since  $\zeta$  is expansive mapping therefore

$$\|\overline{\mathbb{I}}(\nu, v_1) \otimes' \overline{\mathbb{I}}(\nu, v_2)\| = \|\zeta v_1 \otimes' \zeta v_2\| \geq h \|\nu_1 \otimes' \nu_2\|.$$

Also from (1)  $\xi$  is continuous. Also

$$\|\overline{\mathbb{I}}(\nu_1, v) \otimes' \overline{\mathbb{I}}(\nu_2, v)\| = \|\xi\nu_1 \otimes' \xi\nu_2\| \leq \|\nu_1 \otimes' \nu_2\|.$$

Hence there is  $\kappa \in \Omega$  such that  $\kappa = \overline{\mathbb{I}}(\xi\kappa, \kappa) = (\xi \circ \xi)\kappa \otimes \zeta\kappa$ . ■

**Theorem 2.5.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\mathbb{I}}_{cl,cv}(\mathcal{D})$ , such that  $\Omega \subset C \in \overline{\mathbb{I}}_{cp}(\mathcal{D})$ .  $\overline{\mathbb{I}}$  be a mapping of  $\Omega \times \Omega$  into  $\mathcal{D}$  such that the family  $\{\overline{\mathbb{I}}(\nu, \cdot) : \nu \in \Omega\}$  is equiexpansive and  $\Omega \subseteq \overline{\mathbb{I}}(\nu, \Omega)$  for  $\nu \in \Omega$ . Suppose  $\xi$  is a continuous operator of  $\Omega$  into a MS  $Y$  with

$$\|\overline{\mathbb{I}}(\nu_1, v) \otimes' \overline{\mathbb{I}}(\nu_2, v)\| \leq d_Y(\xi\nu_1, \xi\nu_2).$$

Then there is  $\kappa \in \Omega$  such that  $\overline{\mathbb{I}}(\kappa, \kappa) = \kappa$ .

**Proof.** For  $\nu \in \Omega$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  by  $H(v) = \overline{\mathbb{I}}(\nu, v)$ .  $H$  is expansive

and  $\Omega \subseteq \bar{\mathbb{P}}(\nu, \Omega) = H(\Omega)$ . Hence there is a point in  $\Omega$  say  $\mathbb{P}\nu$  which is unique such that  $\mathbb{P}\nu = \bar{\mathbb{P}}(\nu, \mathbb{P}\nu)$ . Now

$$\begin{aligned}
\|\mathbb{P}(\nu_1) \otimes' \mathbb{P}(\nu_2)\| &= \|\bar{\mathbb{P}}(\nu_1, \mathbb{P}\nu_1) \otimes' \bar{\mathbb{P}}(\nu_2, \mathbb{P}\nu_2)\| \\
&= \|(\bar{\mathbb{P}}(\nu_2, \mathbb{P}\nu_1) \otimes' \bar{\mathbb{P}}(\nu_2, \mathbb{P}\nu_2)) \otimes' (\bar{\mathbb{P}}(\nu_2, \mathbb{P}\nu_1) \otimes' \bar{\mathbb{P}}(\nu_1, \mathbb{P}\nu_1))\| \\
&\geq \|\bar{\mathbb{P}}(\nu_2, \mathbb{P}\nu_1) \otimes' \bar{\mathbb{P}}(\nu_2, \mathbb{P}\nu_2)\| \otimes' \|\bar{\mathbb{P}}(\nu_2, \mathbb{P}\nu_1) \otimes' \bar{\mathbb{P}}(\nu_1, \mathbb{P}\nu_1)\| \\
&\geq h \|\mathbb{P}\nu_1 \otimes' \mathbb{P}\nu_2\| \otimes' d_v(\xi\nu_1, \xi\nu_2)
\end{aligned}$$

This means that

$$\|\mathbb{P}\nu_1 \otimes' \mathbb{P}\nu_2\| \leq \frac{1}{h \otimes' 1} d_Y(\xi\nu_1, \xi\nu_2) \quad h > 1.$$

In the above expression since  $\xi$  is continuous so  $\mathbb{P}$  is continuous. Since  $\Omega$  lies in a compact subset of  $\mathcal{D}$  and  $\mathbb{P}$  is continuous, therefore  $\mathbb{P}(\Omega) \in \check{S}_{cp}$ . So by Schauder Second Theorem [119] there is  $\kappa \in \Omega$  such that  $\mathbb{P}\kappa = \kappa$ . Since for  $\kappa \in \Omega$  there is unique  $\mathbb{P}\kappa \in \Omega$  such that  $\bar{\mathbb{P}}(\kappa, \mathbb{P}\kappa) = \mathbb{P}\kappa$  and also  $\mathbb{P}\kappa = \kappa$ , therefore  $\bar{\mathbb{P}}(\kappa, \kappa) = \kappa$ . ■

**Corollary 2.3.** For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\mathbb{P}}_{cl,cv}(\mathcal{D})$ , such that  $\Omega \subset C \in \bar{\mathbb{P}}_{cp}(\mathcal{D})$ .  $\bar{\mathbb{P}}$  be a mapping of  $\Omega \times \Omega$  into  $\mathcal{D}$  such that

- (1)  $\Omega \subseteq \bar{\mathbb{P}}(\nu, \Omega)$  for  $\nu$  in  $\Omega$ ;
- (2)  $\|\bar{\mathbb{P}}(\nu, v_1) \otimes' \bar{\mathbb{P}}(\nu, v_2)\| \geq h \|v_1 \otimes' v_2\| \quad h > 1$ ;
- (3)  $\|\bar{\mathbb{P}}(\nu_1, v) \otimes' \bar{\mathbb{P}}(\nu_2, v)\| \leq \|\xi\nu_1 \otimes' \xi\nu_2\|$ ;

where  $\xi$  be a continuous mapping of  $\Omega$  into  $\mathcal{D}$ , then there is  $\kappa \in \Omega$  such that  $\bar{\mathbb{P}}(\kappa, \kappa) = \kappa$ .

**Proof.** Take  $Y = \mathcal{D}$  (where  $\mathcal{D}$  is a BS) in the above theorem. ■

**Remark 2.3.** The above result is similiar to the Theorem 8 in [94].

**Corollary 2.4.** For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\mathbb{P}}_{cl,cv}(\mathcal{D})$ , such that  $\Omega \subset C \in \bar{\mathbb{P}}_{cp}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi$  is continuous mapping;
- (2)  $\zeta$  is an expansive mapping;
- (3)  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for  $\nu \in \Omega$ .

Then there is  $\kappa \in \Omega$  such that  $\xi\kappa \otimes \zeta\kappa = \kappa$ .

**Proof.** Define  $\bar{\mathbb{P}}(\nu, v) = \xi\nu \otimes \zeta v$  (in the above Corollary 2.3). Since  $\bar{\mathbb{P}}(\nu, \Omega) = \xi\nu \otimes \zeta(\Omega)$

and from (3)  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  therefore  $\Omega \subseteq \bar{\Pi}(\nu, \Omega)$ . Now since  $\zeta$  is expansive mapping therefore

$$\|\bar{\Pi}(\nu, \nu_1) \otimes' \bar{\Pi}(\nu, \nu_2)\| = \|\zeta\nu_1 \otimes' \zeta\nu_2\| \geq h \|\nu_1 \otimes' \nu_2\|.$$

Also

$$\|\bar{\Pi}(\nu_1, \nu) \otimes' \bar{\Pi}(\nu_2, \nu)\| = \|\xi\nu_1 \otimes' \xi\nu_2\| \leq \|\xi\nu_1 \otimes' \xi\nu_2\|.$$

Hence there is  $\kappa \in \Omega$  such that  $\kappa = \bar{\Pi}(\kappa, \kappa) = \xi\kappa \otimes \zeta\kappa$ . ■

**Theorem 2.6.** For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\Pi}_{cl,cv}(\mathcal{D})$ , such that  $\Omega \subset C \in \bar{\Pi}_{cp}(\mathcal{D})$ , and  $\xi$  be a continuous mapping of  $\Omega$  into  $\Omega$ . Let  $\bar{\Pi}$  be a mapping of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$  such that the family  $\{\bar{\Pi}(v, \cdot) : v \in \xi(\Omega)\}$  equiexpansive and  $\Omega \subseteq \bar{\Pi}(v, \Omega)$  for  $v \in \xi(\Omega)$  with

$$\|\bar{\Pi}(v_1, v) \otimes' \bar{\Pi}(v_2, v)\| \leq \|\xi v_1 \otimes' \xi v_2\|.$$

Then there exists  $\kappa \in \Omega$  such that  $\bar{\Pi}(\xi\kappa, \kappa) = \kappa$ .

**Proof.** For  $\hat{u} \in \xi(\Omega)$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  by  $H(v) = \bar{\Pi}(\hat{u}, v)$ . The mapping  $H$  is expansive because

$$\|H(v_1) \otimes' H(v_2)\| = \|\bar{\Pi}(\hat{u}, v_1) \otimes' \bar{\Pi}(\hat{u}, v_2)\| \geq h \|\nu_1 \otimes' \nu_2\|.$$

Also  $\Omega \subseteq \bar{\Pi}(v, \Omega) = H(\Omega)$ . Hence (Theorem 2.2) there is point say  $\mathcal{P}\hat{u}$  which is unique with  $\mathcal{P}\hat{u} = \bar{\Pi}(\hat{u}, \mathcal{P}\hat{u})$ . Now

$$\begin{aligned} \|\mathcal{P}(v_1) \otimes' \mathcal{P}(v_2)\| &= \|\bar{\Pi}(v_1, \mathcal{P}v_1) \otimes' \bar{\Pi}(v_2, \mathcal{P}v_2)\| \\ &= \|(\bar{\Pi}(v_2, \mathcal{P}v_1) \otimes' \bar{\Pi}(v_2, \mathcal{P}v_2)) \otimes' (\bar{\Pi}(v_2, \mathcal{P}v_1) \otimes' \bar{\Pi}(v_1, \mathcal{P}v_1))\| \\ &\geq \|\bar{\Pi}(v_2, \mathcal{P}v_1) \otimes' \bar{\Pi}(v_2, \mathcal{P}v_2)\| \otimes' \|\bar{\Pi}(v_2, \mathcal{P}v_1) \otimes' \bar{\Pi}(v_1, \mathcal{P}v_1)\| \\ &\geq h \|\mathcal{P}(v_1) \otimes' \mathcal{P}(v_2)\| \otimes' \|\xi v_1 \otimes' \xi v_2\|. \end{aligned}$$

This means that

$$\|\mathcal{P}v_1 \otimes' \mathcal{P}v_2\| \leq \frac{1}{h \otimes' 1} \|\xi v_1 \otimes' \xi v_2\| \quad h > 1.$$

Since  $\xi$  is continuous so  $\mathbb{P}$  is continuous function of  $\xi(\Omega)$  into  $\Omega$ . Also  $\mathbb{P} \circ \xi$  is a continuous function of  $\Omega$  into  $\Omega$  and  $\mathbb{P} \circ \xi(\Omega) \in \tilde{S}_{cp}$ . So by Schauder second theorem [119] there is  $\kappa \in \Omega$  such that  $\mathbb{P} \circ \xi(\kappa) = \kappa$  or  $\mathbb{P}(\xi\kappa) = \kappa$ . Since for  $\xi\kappa \in \xi(\Omega)$  there is unique  $\mathbb{P}(\xi\kappa) \in \Omega$  such that  $\overline{\mathbb{P}}(\xi\kappa, \mathbb{P}(\xi\kappa)) = \mathbb{P}(\xi\kappa)$  and also  $\mathbb{P}(\xi\kappa) = \kappa$ , therefore  $\overline{\mathbb{P}}(\xi\kappa, \kappa) = \kappa$ . ■

**Remark 2.4.** In the above theorem,  $\xi$  is a mapping of  $\Omega$  into  $\Omega$  while in Theorem 2.4  $\xi$  is a mapping of  $\Omega$  into  $\mathcal{D}$ .

**Corollary 2.5.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\mathbb{P}}_{cl,cv}(\mathcal{D})$ . Suppose  $\xi : \Omega \rightarrow \Omega$  and  $\zeta : \Omega \rightarrow \mathcal{D}$  with

- (1)  $\xi$  is continuous such that  $\xi(\Omega) \subset C \in \overline{\mathbb{P}}_{cp}(\mathcal{D})$ ;
- (2)  $\zeta$  is an expansive mapping;
- (3)  $v \in \xi(\Omega)$  implies  $\Omega \subseteq \xi v \otimes \zeta(\Omega)$ .

Then there exists  $\kappa \in \Omega$  with  $(\xi \circ \xi)\kappa \otimes \zeta\kappa = \kappa$ .

**Proof.** Take  $\overline{\mathbb{P}}(\hat{u}, v) = \xi u \otimes \zeta v$  with  $\hat{u} = \xi v \in \xi(\Omega)$  and use Theorem 2.6. ■

**Theorem 2.7.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\mathbb{P}}_{cl,cv}(\mathcal{D})$ . Suppose  $\xi : \Omega \rightarrow Y$  be a mapping with  $\xi(\Omega) \in \overline{\mathbb{P}}_{prc}(Y)$ , where  $\xi$  is continuous and  $Y$  is MS. Let  $\overline{\mathbb{P}} : \overline{\xi(\Omega)} \times \Omega \rightarrow \mathcal{D}$  be a mapping which is continuous with the family  $\{\overline{\mathbb{P}}(v, \cdot) : v \in \overline{\xi(\Omega)}\}$  equiexpansive and  $\Omega \subseteq \overline{\mathbb{P}}(v, \Omega)$  for  $v \in \overline{\xi(\Omega)}$ . Then there is  $\kappa \in \Omega$  such that  $\overline{\mathbb{P}}(\xi\kappa, \kappa) = \kappa$ .

**Proof.** For  $\hat{u} \in \overline{\xi(\Omega)}$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  by  $H(v) = \overline{\mathbb{P}}(\hat{u}, v)$ . The mapping  $H$  is expansive because

$$\|H(v_1) \otimes' H(v_2)\| = \|\overline{\mathbb{P}}(\hat{u}, v_1) \otimes' \overline{\mathbb{P}}(\hat{u}, v_2)\| \geq h \|v_1 \otimes' v_2\|.$$

Also  $\Omega \subseteq \overline{\mathbb{P}}(\hat{u}, \Omega) = H(\Omega)$ . Hence (Theorem 2.2) there is point say  $\mathbb{P}\hat{u} \in \Omega$  which is unique with  $\mathbb{P}\hat{u} = \overline{\mathbb{P}}(\hat{u}, \mathbb{P}\hat{u})$ . We show that the mapping  $\mathbb{P}$  of  $\overline{\xi(\Omega)}$  into  $\Omega$  is continuous. Consider  $\overline{\xi(\Omega)}$  in which there is a sequence  $v_n$  whose convergent point is  $v_0$ . Then

$$\begin{aligned} \|\mathbb{P}(v_n) \otimes' \mathbb{P}(v_0)\| &= \|\overline{\mathbb{P}}(v_n, \mathbb{P}(v_n)) \otimes' \overline{\mathbb{P}}(v_0, \mathbb{P}(v_0))\| \\ &= \|(\overline{\mathbb{P}}(v_n, \mathbb{P}(v_n)) \otimes' \overline{\mathbb{P}}(v_n, \mathbb{P}(v_0))) \otimes' (\overline{\mathbb{P}}(v_0, \mathbb{P}(v_0)) \otimes' \overline{\mathbb{P}}(v_n, \mathbb{P}(v_0)))\| \\ &\geq \|\overline{\mathbb{P}}(v_n, \mathbb{P}(v_n)) \otimes' \overline{\mathbb{P}}(v_n, \mathbb{P}(v_0))\| \otimes' \|\overline{\mathbb{P}}(v_0, \mathbb{P}(v_0)) \otimes' \overline{\mathbb{P}}(v_n, \mathbb{P}(v_0))\| \\ &\geq h \|\mathbb{P}(v_n) \otimes' \mathbb{P}(v_0)\| \otimes' \|\overline{\mathbb{P}}(v_0, \mathbb{P}(v_0)) \otimes' \overline{\mathbb{P}}(v_n, \mathbb{P}(v_0))\| \end{aligned}$$

and therefore

$$\|\mathbb{P}(v_n) \otimes' \mathbb{P}(v_0)\| \leq \frac{1}{h \otimes' 1} \|\overline{\mathbb{P}}(v_0, \mathbb{P}(v_0)) \otimes' \overline{\mathbb{P}}(v_n, \mathbb{P}(v_0))\|.$$

Since  $\overline{\mathbb{P}}$  is continuous therefore  $\mathbb{P}$  is continuous. The operator  $\mathbb{P}$  maps  $\overline{\xi(\Omega)}$  into  $\Omega$  and is continuous. Since  $\xi(\Omega)$  is relatively compact therefore by [[77], page 412]  $\mathbb{P}(\xi(\Omega))$  is relatively compact. Let  $co(\mathbb{P}(\xi(\Omega)))$  be the convex hull of  $\mathbb{P}(\xi(\Omega))$ . By [[118], page195]  $co(\mathbb{P}(\xi(\Omega)))$  is relatively compact. Since  $\Omega$  is closed and convex so  $\overline{co(\mathbb{P}(\xi(\Omega)))} \subseteq \Omega$  and by [[56], page 415]  $\overline{co(\mathbb{P}(\xi(\Omega)))} \subseteq \Omega$ . Let  $P = \overline{co(\mathbb{P}(\xi(\Omega)))}$  then  $P \subseteq \Omega$  and  $\mathbb{P} \circ \xi(P) \subseteq P$ . Since  $P = \overline{co(\mathbb{P}(\xi(\Omega)))}$  also, therefore  $P$  is compact and convex. So by first Schauder theorem [119] there is a point  $\kappa \in P$  such that  $\mathbb{P}(\xi\kappa) = \kappa$ . Thus  $\kappa = \overline{\mathbb{P}}(\xi\kappa, \kappa)$  by similar arguments in Theorem 2.4. ■

**Remark 2.5.** Above is the variant result of Theorem 2.2 in [72].

## 2.3 Applications

Suppose  $\mathcal{D}$  be a given BS. For  $c > 0$  denote  $J = [0, c]$  a closed interval in  $\mathbb{R}$ . Let  $C(J, \mathcal{D})$  be the BS in which there are continuous functions whose domain is  $J$ . Here  $\|\nu\| = \sup_{\varrho \in J} |\nu(\varrho)|$ , where  $\nu \in C(J, \mathcal{D})$ . Let  $U \in \overline{\mathbb{P}}_{cl, cv}(\mathcal{D})$ . Consider the following controlled system

$$\begin{cases} D^\alpha \nu(\varrho) = \otimes' \xi \nu(\varrho) \otimes \overline{\mathbb{P}}(\varrho, \nu(\varrho)) \otimes C(\varrho) \nu(\varrho), & \varrho \in J := [0, c], \nu \in U \text{ and } \alpha \in (0, 1) \\ \nu(0) = \nu_0. \end{cases} \quad (A)$$

The mild solution corresponding to  $\nu$  of (A) is given by [125],

$$\nu^\nu(\varrho) = \mathbb{P}(\varrho) \nu_0 \otimes \int_0^\varrho (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{I}(\varrho \otimes' \kappa) \overline{\mathbb{P}}(\kappa, \nu(\kappa)) d\kappa \otimes \int_0^\varrho (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{I}(\varrho \otimes' \kappa) C(\kappa) \nu(\kappa) d\kappa, \quad (B)$$

where

$$\mathbb{P}(\varrho) = \int_0^\infty \xi_\alpha(\theta) T(\varrho^\alpha \theta) d\theta,$$

$$\mathfrak{S}(\varrho) = \alpha \int_0^{\infty} \theta \xi_{\alpha}(\theta) T(\varrho^{\alpha} \theta) d\theta,$$

$\xi_{\alpha}$  is probability density function defined on  $(0, \infty)$ , for more details about the notations we refer the readers to [125]. Also

$$\|T(\varrho)\| \leq \Omega \text{ for some } \Omega > 1,$$

$$\begin{aligned} \|\mathfrak{P}(\varrho)\nu\| &\leq \Omega \|\nu\| \\ \|\mathfrak{S}(\varrho)\nu\| &\leq \frac{\alpha\Omega}{\Gamma(\alpha \oplus 1)} \|\nu\| \end{aligned}$$

and  $\mathfrak{P}(\varrho), \mathfrak{S}(\varrho)$  are strongly continuous for  $\varrho \geq 0$  and are compact for each  $\varrho > 0$ .

We can write (A) in the form

$$\bar{\Pi}(Qv, \nu)(\varrho) = \mathfrak{P}(\varrho)\nu_0 \otimes \int_0^{\varrho} (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{S}(\varrho \otimes' \kappa) \bar{\Pi}(\kappa, \nu(\kappa)) d\kappa \otimes Q(v),$$

where

$$Q(v)(\varrho) = \int_0^{\varrho} (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{S}(\varrho \otimes' \kappa) C(\kappa) v(\kappa) d\kappa,$$

Assume the following.

(H1).  $Q : \zeta_{\delta} \longrightarrow \mathcal{D}$  is continuous and expansive, i.e.,  $\|Q(v) \otimes' Q(v^*)\| \geq \ell \|v \otimes' v^*\|$  for all  $v, v^* \in \zeta$  and some  $\ell > 1$ .

(H2).  $\bar{\Pi} : \overline{Q(\zeta_{\delta})} \times \zeta_{\delta} \longrightarrow \mathcal{D}$  is continuous.

(H3).  $\zeta_{\delta} \subseteq \bar{\Pi}(v, \zeta_{\delta})$  for each  $v \in \overline{Q(\zeta_{\delta})}$ .

Where  $\zeta_{\delta}$  is a closed ball with radius  $\delta$  and centre at 0 in  $\mathcal{D}$ .

We show that  $\bar{\mathbb{I}}$  is equiexpansive. For a fixed  $\nu$ , and any  $v, \hat{u} \in \zeta_\delta$ , consider

$$\begin{aligned} \|\bar{\mathbb{I}}(\nu, v) \otimes' \bar{\mathbb{I}}(\nu, \hat{u})\| &= \left\| \begin{aligned} &\mathbb{P}(\varrho) \nu_0 \otimes \int_0^\varrho (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{S}(\varrho \otimes' \kappa) \bar{\mathbb{I}}(\kappa, \nu(\kappa)) d\kappa \otimes Q(v) \\ &\otimes' \mathbb{P}(\varrho) \nu_0 \otimes' \int_0^\varrho (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{S}(\varrho \otimes' \kappa) \bar{\mathbb{I}}(\kappa, \nu(\kappa)) d\kappa \otimes' Q(\hat{u}) \end{aligned} \right\| \\ &= \|Q(v) \otimes' Q(\hat{u})\| \geq \ell \|v \otimes' \hat{u}\|, \end{aligned}$$

which shows that  $\{\bar{\mathbb{I}}(v, \cdot) : v \in \overline{Q(\zeta_\delta)}\}$  is equiexpansive.

We prove compactness of  $Q$ . Continuity of  $\mathfrak{S}$  and  $C$  implies the continuity of  $Q$ , we show that  $Q$  is uniformly bounded on  $\zeta_\delta$ . For  $v \in \zeta_\delta$ , consider

$$\begin{aligned} \|Q(v)(\varrho)\| &= \left\| \int_0^\varrho (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{S}(\varrho \otimes' \kappa) C(\kappa) v(\kappa) d\kappa \right\| \\ &\leq \int_0^\varrho (\varrho \otimes' \kappa)^{\alpha \otimes' 1} \|\xi^q \mathfrak{S}(\varrho \otimes' \kappa)\| \|C(\kappa)\| \|v(\kappa)\| d\kappa \\ &\leq \frac{\|C\| \Omega_q \alpha \Gamma(2 \otimes' q)}{\Gamma(1 \otimes' \alpha(1 \otimes' q))} \delta \int_0^\varrho (\varrho \otimes' \kappa)^{\alpha \otimes' 1} d\kappa \\ &\leq \frac{\|C\| \Omega_q \alpha \Gamma(2 \otimes' q)}{\Gamma(1 \otimes' \alpha(1 \otimes' q))} \delta \frac{T^{\alpha(1 \otimes' q)}}{\alpha(1 \otimes' q)} := \varrho, \end{aligned}$$

which implies that  $Q$  is UB. Now for equicontinuity of  $Q$ , we consider for  $0 < \varrho_1 < \varrho_2 < T$ ,

$$\begin{aligned} \|Q(v)(\varrho_2) \otimes' Q(v)(\varrho_1)\| &= \left\| \begin{aligned} &\int_0^{\varrho_2} (\varrho_2 \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{S}(\varrho_2 \otimes' \kappa) C(\kappa) v(\kappa) d\kappa \\ &\otimes' \int_0^{\varrho_1} (\varrho_1 \otimes' \kappa)^{\alpha \otimes' 1} \mathfrak{S}(\varrho_1 \otimes' \kappa) C(\kappa) v(\kappa) d\kappa \end{aligned} \right\| \\ &\leq \frac{\|C\| \Omega_q \alpha \mathfrak{d}(2 \otimes' q)}{\mathfrak{d}(1 \otimes' \alpha(1 \otimes' q))} \delta \left| \int_0^{\varrho_2} (\varrho_2 \otimes' \kappa)^{\alpha \otimes' 1} d\kappa \otimes' \int_0^{\varrho_1} (\varrho_1 \otimes' \kappa)^{\alpha \otimes' 1} d\kappa \right| \\ &\leq \frac{\|C\| \Omega_q \mathfrak{d}(2 \otimes' q)}{\mathfrak{d}(1 \otimes' \alpha(1 \otimes' q))} \delta |\varrho_2^\alpha \otimes' \varrho_1^\alpha| \longrightarrow 0 \text{ as } \varrho_2 \longrightarrow \varrho_1, \end{aligned}$$

which is independent of  $v$ , and  $\|Q(v)(\varrho_2) \oplus' Q(v)(\varrho_1)\| \rightarrow 0$  as  $\varrho_2 \rightarrow \varrho_1$ . Applying Arzela-Ascoli theorem, we get  $Q$  as a compact operator. From (H3),  $\zeta_\delta \subseteq \overline{\Pi}(v, \zeta_\delta)$  for each  $v \in \overline{Q}(\zeta_\delta)$ . Assumptions of the Theorem 2.7 are satisfied to get a point  $z$  with  $\overline{\Pi}(Qz, z) = z$ , which is the solution of controlled problem (A).

Finally, we summarize the above discussion in the following way.

**Theorem 2.8.** Let  $\mathcal{D}$  be a BS and  $\zeta_\delta$  be closed ball at origin with radius  $\delta$ . Assume that  $Q : \zeta_\delta \rightarrow \mathcal{D}$  and  $\overline{\Pi} : \overline{Q}(\zeta_\delta) \times \zeta_\delta \rightarrow \mathcal{D}$  satisfy (H1)  $\oplus'$  (H3). Then a mild solution of controlled problem (A) exists.

## 2.4 Conclusion

Some variants of Krasnoselskii type FP theorems are presented using the notion of newly defined equiexpansive mappings. As an application, we present an ER for controlled problem with less conditions. This problem is also a generalized kind of Cauchy problem for evolution differential equations.

## Chapter 3

# Krasnoselskii-type Results for Equiexpansive and Equicontractive Non-compact Operators with Application in Radiative Transfer Equations

TH-26792

### 3.1 Introduction

Krasnoselskii FP theorem contains two operators, one is compact and the other is contraction. So the theorem applies to contraction and compact operators only. There are also expansive, and noncompact operators (KSC and condensing ) for which the FP investigation of Krasnoselskii type can be studied. For applications and physical significance of noncompact operators, see [132]. Also, for generalization, there is a need to study the FP results for the operator  $\bar{\mathbb{I}}$  of two variables the one special case of which is  $\bar{\mathbb{I}}(\nu, v) = \xi\nu \otimes \zeta v$ . This type of work can be seen in [31, 72, 89, 94, 129].

To generalize the concept of compact operators we need measure of non compactness which gives the deviation from relative compactness for a given set. Sadovskii [[132], p.500], using

the idea of MNC, generalized the Schauder FP theorem by introducing condensing operators a more general class than the compact operators.

Xiang and Yuan [[128], see also in [130]] proved the result as a variant of Banach contraction principle. They proved there is a unique FP for the expansive operators in a complete MS.

Suppose  $\Omega$  is the subset of a BS  $\mathcal{D}$  and consider the mapping  $\bar{\Pi} : \xi(\Omega) \times \Omega \longrightarrow \mathcal{D}$ . We study the ES of equation  $\nu = \bar{\Pi}(\xi\nu, \nu)$  by taking different assumptions on  $\xi$ ,  $\bar{\Pi}$  and  $\Omega$  and using MNC. We assume the following to prove our FP results;

- (i) the family  $\{\bar{\Pi}(\hat{u}, \cdot) : \hat{u} \in \xi(\Omega)\}$  is equiexpansive or equicontractive.
- (ii)  $\xi$  is  $k$ -set contractive operator of  $\Omega$  into  $\mathcal{D}$  or strictly  $k$ -set contractive operator.
- (iii)  $\Omega \subseteq \bar{\Pi}(\hat{u}, \Omega)$  for each fixed  $\hat{u} \in \xi(\Omega)$  with equiexpansive family while  $\bar{\Pi}(\xi(\Omega), \Omega) \subseteq \Omega$  with equicontractive family.

## 3.2 Main Results

This section is devoted to noncompact, equiexpansive and equicontractive mappings.

### 3.2.1 Results for equiexpansive mappings

Here are FP results for equiexpansive mappings.

**Theorem 3.1.** For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\Pi}_{b,d,cv}(\mathcal{D})$ . Consider a mapping  $\bar{\Pi}$  of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \bar{\Pi}(\hat{u}, \Omega)$  for  $\hat{u} \in \xi(\Omega)$  and  $\xi$  be a  $k$ -set contractive mapping of  $\Omega$  into  $\mathcal{D}$  with

- (1) the family  $\{\bar{\Pi}(\hat{u}, \cdot) : \hat{u} \in \xi(\Omega)\}$  is equiexpansive with  $h > k \otimes 1$ ;
- (2)  $\|\bar{\Pi}(\hat{u}, v) \otimes' \bar{\Pi}(\hat{u}', v)\| \leq \|\hat{u} \otimes' \hat{u}'\|$  for all  $\hat{u}, \hat{u}' \in \xi(\Omega)$  and  $v \in \Omega$ ;

Then there is  $\kappa \in \Omega$  such that  $\bar{\Pi}(\xi\kappa, \kappa) = \kappa$ .

**Proof.** Define  $H : \Omega \longrightarrow \mathcal{D}$  by  $H(v) = \bar{\Pi}(\hat{u}, v)$  for  $\hat{u} \in \xi(\Omega)$ . Then  $H$  is expansive because

$$\begin{aligned} & \|H(v_1) \otimes' H(v_2)\| \\ &= \|\bar{\Pi}(\hat{u}, v_1) \otimes' \bar{\Pi}(\hat{u}, v_2)\| \geq h \|v_1 \otimes' v_2\|, \end{aligned}$$

and  $\Omega \subseteq \bar{\Pi}(\hat{u}, \Omega) = H(\Omega)$ . From Theorem 2.2, there is a unique point in  $\Omega$  say  $\mathfrak{P}\hat{u}$  such that

$\mathbb{P}\hat{u} = \bar{\mathbb{P}}(\hat{u}, \mathbb{P}\hat{u})$ . Now

$$\begin{aligned}
\|\mathbb{P}u \otimes' \mathbb{P}u'\| &= \|\bar{\mathbb{P}}(u, \mathbb{P}u) \otimes' \bar{\mathbb{P}}(u', \mathbb{P}u')\| \\
&= \|(\bar{\mathbb{P}}(u', \mathbb{P}u) \otimes' \bar{\mathbb{P}}(u', \mathbb{P}u')) \otimes' (\bar{\mathbb{P}}(u', \mathbb{P}u) \otimes' \bar{\mathbb{P}}(u, \mathbb{P}u))\| \\
&\geq \|\bar{\mathbb{P}}(\hat{u}', \mathbb{P}u) \otimes' \bar{\mathbb{P}}(\hat{u}', \mathbb{P}u')\| \otimes' \|\bar{\mathbb{P}}(u', \mathbb{P}u) \otimes' \bar{\mathbb{P}}(u, \mathbb{P}u)\| \\
&\geq h \|\mathbb{P}u \otimes' \mathbb{P}u'\| \otimes' \|u \otimes u'\|.
\end{aligned}$$

This means that

$$\|\mathbb{P}u \otimes' \mathbb{P}u'\| \leq \frac{1}{h \otimes' 1} \|u \otimes u'\| \quad h > 1.$$

which indicates that  $\mathbb{P}$  is continuous from  $\xi(\Omega)$  into  $\Omega$ . Also  $\mathbb{P} \circ \xi$  is a continuous from  $\Omega$  into  $\Omega$ . From the above inequality, Lemma 1.1 and since  $\xi$  is  $k$ -set contractive mapping,

$$\wp(\mathbb{P} \circ \xi(\mathcal{U})) = \wp(\mathbb{P}(\xi(\mathcal{U}))) \leq \frac{1}{h \otimes' 1} \wp(\xi(\mathcal{U})) \leq \frac{k}{h \otimes' 1} \wp(\mathcal{U}) < \wp(\mathcal{U}),$$

for all  $\mathcal{U} \subseteq \Omega$ . Hence  $\mathbb{P} \circ \xi$  is condensing and by Sadovskii FP theorem, there is  $\kappa \in \Omega$  such that  $\mathbb{P} \circ \xi(\kappa) = \kappa$  or  $\mathbb{P}(\xi\kappa) = \kappa$ . Since for  $\xi\kappa \in \xi(\Omega)$  there is unique  $\mathbb{P}(\xi\kappa) \in \Omega$  such that  $\bar{\mathbb{P}}(\xi\kappa, \mathbb{P}(\xi\kappa)) = \mathbb{P}(\xi\kappa)$  and also  $\mathbb{P}(\xi\kappa) = \kappa$ , therefore  $\bar{\mathbb{P}}(\xi\kappa, \kappa) = \kappa$ . ■

**Corollary 3.1.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\mathbb{P}}_{b,cl,cv}(\mathcal{D})$ .  $\bar{\mathbb{P}}$  be the mapping of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \bar{\mathbb{P}}(\hat{u}, \Omega)$  for  $\hat{u} \in \xi(\Omega)$  and

- (1)  $\xi : \Omega \rightarrow \mathcal{D}$  is continuous such that  $\xi(\Omega) \subset C \in \bar{\mathbb{P}}_{cp}(\mathcal{D})$ ;
- (2) the family  $\{\bar{\mathbb{P}}(\hat{u}, \cdot) : \hat{u} \in \xi(\Omega)\}$  is equiexpansive with  $h > k \otimes 1$ ;
- (3)  $\|\bar{\mathbb{P}}(\hat{u}, v) \otimes' \bar{\mathbb{P}}(\hat{u}', v)\| \leq \|\hat{u} \otimes' \hat{u}'\|$  for all  $\hat{u}, \hat{u}' \in \xi(\Omega)$  and  $v \in \Omega$ ;

Then there is  $\kappa \in \Omega$  such that  $\bar{\mathbb{P}}(\xi\kappa, \kappa) = \kappa$ .

**Proof.** From (1) we have  $\xi(\Omega) \subseteq C$ , where  $C$  is the compact set. Therefore  $\wp(\xi(\mathcal{U})) \leq \wp(C) = 0$  for all  $\mathcal{U} \subseteq \Omega$ . This shows that  $\wp(\xi(\mathcal{U})) \leq k\wp(\mathcal{U})$  and  $\xi$  is KSC mapping. Hence the required result is obtained (Theorem 3.1). ■

**Remark 3.1.** Theorem 2.2 in [128] is the special case of the above corollary with assumption that  $\Omega$  is bounded set. This can be shown by defining  $\bar{\mathbb{P}}(\hat{u}, v) = \hat{u} \otimes \zeta v$  with  $\hat{u} = \xi v \in \xi(\Omega)$ , where  $\xi$  is continuous map from  $\Omega$  into  $\mathcal{D}$  and  $\zeta$  is expansive map from  $\Omega$  into  $\mathcal{D}$ .

**Corollary 3.2.** For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\mathbb{P}}_{b,cl,cv}(\mathcal{D})$ . Consider the mappings  $\xi$  and  $\zeta$  with

- (1)  $\xi : \Omega \longrightarrow \mathcal{D}$  is KSC mapping;
- (2)  $\zeta : \Omega \longrightarrow \mathcal{D}$  is an expansive mapping with  $h > k \otimes 1$ ;
- (3)  $\Omega \subseteq \hat{u} \otimes \zeta(\Omega)$  for  $\hat{u} \in \xi(\Omega)$ .

Then there is  $\kappa \in \Omega$  such that  $\xi\kappa \otimes \zeta\kappa = \kappa$ .

**Proof.** Define  $\bar{\Pi}(\hat{u}, \nu) = \hat{u} \otimes \zeta\nu$  with  $\hat{u} = \xi\nu \in \xi(\Omega)$ . Here  $\bar{\Pi} : \xi(\Omega) \times \Omega \longrightarrow \mathcal{D}$  is a mapping. Since  $\bar{\Pi}(\hat{u}, \Omega) = \hat{u} \otimes \zeta(\Omega)$  and from (3)  $\Omega \subseteq \hat{u} \otimes \zeta(\Omega)$ , therefore  $\Omega \subseteq \bar{\Pi}(\hat{u}, \Omega)$ . Now, from (2), since  $\zeta$  is an expansive mapping for  $h > k \otimes 1$  therefore

$$\|\bar{\Pi}(\hat{u}, \nu_1) \otimes' \bar{\Pi}(\hat{u}, \nu_2)\| = \|\zeta\nu_1 \otimes' \zeta\nu_2\| \geq h \|\nu_1 \otimes' \nu_2\|.$$

This shows that the family  $\{\bar{\Pi}(\hat{u}, \cdot) : \hat{u} \in \xi(\Omega)\}$  is equiexpansive with  $h > k \otimes 1$ . Also  $\xi : \Omega \longrightarrow \mathcal{D}$  is KSC mapping and

$$\|\bar{\Pi}(\hat{u}, \nu) \otimes' \bar{\Pi}(\hat{u}', \nu)\| = \|\hat{u} \otimes' \hat{u}'\| \leq \|\hat{u} \otimes' \hat{u}'\|.$$

Hence there is  $\kappa \in \Omega$  such that  $\kappa = \bar{\Pi}(\xi\kappa, \kappa) = \xi\kappa \otimes \zeta\kappa$ . ■

**Corollary 3.3.** For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\Pi}_{b,d,cv}(\mathcal{D})$ . Consider the mappings  $\xi$  and  $\zeta$  with

- (1)  $\xi : \Omega \longrightarrow \Omega \subseteq \mathcal{D}$  is nonexpansive and KSC mapping;
- (2)  $\zeta : \Omega \longrightarrow \mathcal{D}$  is an expansive mapping with  $h > k \otimes 1$ ;
- (3)  $\Omega \subseteq \xi\hat{u} \otimes \zeta(\Omega)$  for  $\xi\hat{u} \in \xi(\Omega)$ .

Then there is  $\kappa \in \Omega$  such that  $(\xi \circ \xi)\kappa \otimes \zeta\kappa = \kappa$ .

**Proof.** Define  $\bar{\Pi}(\hat{u}, \nu) = \xi\hat{u} \otimes \zeta\nu$  with  $\xi\hat{u} \in \xi(\Omega)$  (Theorem 3.1). Since  $\bar{\Pi}(\hat{u}, \Omega) = \xi\hat{u} \otimes \zeta(\Omega)$  and from (3)  $\Omega \subseteq \xi\hat{u} \otimes \zeta(\Omega)$  for  $\xi\hat{u} \in \xi(\Omega)$ , therefore  $\Omega \subseteq \bar{\Pi}(\hat{u}, \Omega)$ . Now since  $\zeta$  is expansive mapping therefore

$$\|\bar{\Pi}(\hat{u}, \nu_1) \otimes' \bar{\Pi}(\hat{u}, \nu_2)\| = \|\zeta\nu_1 \otimes' \zeta\nu_2\| \geq h \|\nu_1 \otimes' \nu_2\|.$$

Also from (1), we have

$$\|\bar{\Pi}(\hat{u}, \nu) \otimes' \bar{\Pi}(\hat{u}', \nu)\| = \|\mathbf{P}_1\hat{u} \otimes' \mathbf{P}_1\hat{u}'\| \leq \|\hat{u} \otimes' \hat{u}'\|.$$

Hence there is  $\kappa \in \Omega$  such that  $(\xi \circ \xi)\kappa \otimes \zeta\kappa = \kappa$ . ■

**Corollary 3.4.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\mathbb{I}}_{b,cl,cv}(\mathcal{D})$ . Consider the mappings  $\xi$  and  $\zeta$  with

- (1)  $\xi : \Omega \longrightarrow \Omega \subseteq \mathcal{D}$  is nonexpansive and  $\xi(\Omega) \subset C \in \overline{\mathbb{I}}_{cp}(\mathcal{D})$ ;
- (2)  $\zeta : \Omega \longrightarrow \mathcal{D}$  is an expansive mapping with  $h > k \otimes 1$ ;
- (3)  $\Omega \subseteq \xi u \otimes \zeta(\Omega)$  for  $\xi u \in \xi(\Omega)$ .

Then there is  $\kappa \in \Omega$  such that  $(\xi \circ \xi)\kappa \otimes \zeta\kappa = \kappa$ .

**Proof.** Using the similar method used in the proofs of above corollaries. ■

**Remark 3.2.** Theorem 3.1 is extended form of Theorems 2.2 and 2.6 in [128] and is variant of the Krasnoselskii FP theorem.

### 3.2.2 Results for equicontractive mappings

The following FP results are for equicontractive mappings.

**Theorem 3.2.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\mathbb{I}}_{b,cl,cv}(\mathcal{D})$ . Let  $\xi$  be a continuous operator of  $\Omega$  into  $\mathcal{D}$ . Let  $\overline{\mathbb{I}}$  be a mapping of  $\Omega \times \Omega$  into  $\Omega$  with

- (1)  $\wp(\overline{\mathbb{I}}(\mathcal{U}, \Omega)) < \wp(\mathcal{U})$  for all  $\mathcal{U} \subseteq \Omega$  with  $\wp(\mathcal{U}) \neq 0$ ;
- (2) the family  $\{\overline{\mathbb{I}}(\nu, \cdot) : \nu \in \Omega\}$  is equicontractive;
- (3)  $\|\overline{\mathbb{I}}(\nu, v) \otimes' \overline{\mathbb{I}}(\nu', v)\| \leq \|\xi\nu \otimes' \xi\nu'\|$  for all  $\nu, \nu' \in \Omega$  and  $v \in \Omega$ ;

Then there is  $\kappa \in \Omega$  such that  $\overline{\mathbb{I}}(\kappa, \kappa) = \kappa$ .

**Proof.** Define  $H : \Omega \longrightarrow \Omega$  by  $H(v) = \overline{\mathbb{I}}(\nu, v)$  for  $\nu \in \Omega$ . Then  $H$  is contraction and there exists only one point in  $\Omega$  say  $\mathbb{P}\nu$  such that  $\mathbb{P}\nu = \overline{\mathbb{I}}(\nu, \mathbb{P}\nu)$ . Now

$$\begin{aligned}
\|\mathbb{P}(\nu) \otimes' \mathbb{P}(\nu')\| &= \|\overline{\mathbb{I}}(\nu, \mathbb{P}\nu) \otimes' \overline{\mathbb{I}}(\nu', \mathbb{P}\nu')\| \\
&= \|(\overline{\mathbb{I}}(\nu', \mathbb{P}\nu) \otimes' \overline{\mathbb{I}}(\nu', \mathbb{P}\nu')) \otimes' (\overline{\mathbb{I}}(\nu', \mathbb{P}\nu) \otimes' \overline{\mathbb{I}}(\nu, \mathbb{P}\nu))\| \\
&\leq \|\overline{\mathbb{I}}(\nu', \mathbb{P}\nu) \otimes' \overline{\mathbb{I}}(\nu', \mathbb{P}\nu')\| \otimes \|\overline{\mathbb{I}}(\nu', \mathbb{P}\nu) \otimes' \overline{\mathbb{I}}(\nu, \mathbb{P}\nu)\| \\
&\leq \alpha \|\mathbb{P}\nu \otimes' \mathbb{P}\nu'\| \otimes \|\xi\nu \otimes' \xi\nu'\|.
\end{aligned}$$

This means that

$$\|\mathbb{P}\nu \otimes' \mathbb{P}\nu'\| \leq \frac{1}{1 \otimes' \alpha} \|\xi\nu \otimes' \xi\nu'\|.$$

Continuity of  $\xi$  implies that  $\mathbb{P} : \Omega \longrightarrow \Omega$  is continuous function. Since  $\mathcal{U} \subseteq \Omega$ , therefore  $\mathbb{P}(\mathcal{U}) \subseteq \mathbb{P}(\Omega) \subseteq \Omega$  and also  $\overline{\mathbb{I}}(\mathcal{U}, \mathbb{P}(\mathcal{U})) \subseteq \overline{\mathbb{I}}(\mathcal{U}, \Omega)$ . Hence  $\wp(\overline{\mathbb{I}}(\mathcal{U}, \mathbb{P}(\mathcal{U}))) \leq \wp(\overline{\mathbb{I}}(\mathcal{U}, \Omega))$ . Now

from (1) we have

$$\begin{aligned}
& \wp(\mathbb{P}(\mathcal{U})) \\
&= \wp(\overline{\mathbb{P}}(\mathcal{U}, \mathbb{P}(\mathcal{U}))) \\
&\leq \wp(\overline{\mathbb{P}}(\mathcal{U}, \Omega)) < \wp(\mathcal{U}).
\end{aligned}$$

Hence  $\mathbb{P}: \Omega \longrightarrow \Omega$  is condensing. By Sadovskii FP theorem, there exists  $\kappa \in \Omega$  such that  $\mathbb{P}\kappa = \kappa$ . As for  $\kappa \in \Omega$  there is unique  $\mathbb{P}\kappa \in \Omega$  such that  $\overline{\mathbb{P}}(\kappa, \mathbb{P}\kappa) = \mathbb{P}\kappa$  and also  $\mathbb{P}\kappa = \kappa$ , therefore  $\overline{\mathbb{P}}(\kappa, \kappa) = \kappa$ . ■

**Corollary 3.5.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\mathbb{P}}_{b,cl,cv}(\mathcal{D})$ . Consider the mappings  $\xi$  and  $\zeta$  of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi\nu \otimes \zeta\nu \in \Omega$  for all  $\nu, v$  in  $\Omega$ ;
- (2)  $\xi$  is condensing mapping;
- (3)  $\zeta$  is contraction and  $\xi(\Omega) \subset C \in \overline{\mathbb{P}}_{cp}(\mathcal{D})$ .

Then there is  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ .

**Proof.** Define  $\overline{\mathbb{P}}(\nu, v) = \xi\nu \otimes \zeta v$ . Obviously  $\overline{\mathbb{P}}: \Omega \times \Omega \longrightarrow \Omega$  and

$$\begin{aligned}
& \wp(\overline{\mathbb{P}}(\mathcal{U}, \Omega)) \\
&= \wp(\xi(\mathcal{U}) \otimes \zeta(\Omega)) \\
&\leq \wp(\xi(\mathcal{U})) \otimes \wp(\zeta(\Omega)) \\
&\leq \wp(\xi(\mathcal{U})) < \wp(\mathcal{U}).
\end{aligned}$$

Now using (3)

$$\|\overline{\mathbb{P}}(\nu, v_1) \otimes' \overline{\mathbb{P}}(\nu, v_2)\| = \|\zeta v_1 \otimes' \zeta v_2\| \leq \alpha \|v_1 \otimes' v_2\|$$

and, also

$$\|\overline{\mathbb{P}}(\nu, v) \otimes' \overline{\mathbb{P}}(\nu', v)\| = \|\xi\nu \otimes' \xi\nu'\| \leq \|\xi\nu \otimes' \xi\nu'\|.$$

Hence there exists  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ . ■

**Remark 3.3.** The above proved theorem 3.2 and the above corollary are variants of Krasnoselskii FP theorem.

**Corollary 3.6.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\mathbb{P}}_{b,cl,cv}(\mathcal{D})$ . Suppose  $\xi : \Omega \rightarrow \mathcal{D}$  is continuous mapping and  $\zeta : \Omega \rightarrow \mathcal{D}$  is contraction mapping such that

- (1)  $\xi\nu \otimes \zeta\nu \in \Omega$ ;
- (2)  $\wp(\xi(\mathcal{U}) \otimes \zeta(\Omega)) < \wp(\mathcal{U})$  for all  $\mathcal{U} \subseteq \Omega$  with  $\wp(\mathcal{U}) \neq 0$ .

Then there is  $\kappa \in \Omega$  such that  $\xi\kappa \otimes \zeta\kappa = \kappa$ .

**Proof.** Define  $\overline{\mathbb{P}}(\nu, v) = \xi\nu \otimes \zeta v$  (Theorem 3.2). Now since  $\zeta$  is contraction mapping therefore

$$\|\overline{\mathbb{P}}(\nu, v_1) \otimes' \overline{\mathbb{P}}(\nu, v_2)\| = \|\zeta v_1 \otimes' \zeta v_2\| \leq \alpha \|v_1 \otimes' v_2\|.$$

Also

$$\|\overline{\mathbb{P}}(\nu, v) \otimes' \overline{\mathbb{P}}(\nu', v)\| = \|\xi\nu \otimes' \xi\nu'\| \leq \|\xi\nu \otimes' \xi\nu'\|,$$

and using (2)

$$\wp(\overline{\mathbb{P}}(\mathcal{U}, \Omega)) = \wp(\xi(\mathcal{U}) \otimes \zeta(\Omega)) < \wp(\mathcal{U}).$$

Hence there is  $\kappa \in \Omega$  such that  $\kappa = \overline{\mathbb{P}}(\kappa, \kappa) = \xi\kappa \otimes \zeta\kappa$ . ■

**Theorem 3.3.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\mathbb{P}}_{b,cl,cv}(\mathcal{D})$  and  $\xi : \Omega \rightarrow \mathcal{D}$  is continuous. Let  $\overline{\mathbb{P}} : \xi(\Omega) \times \Omega \rightarrow \Omega$  be the mapping with

- (1)  $\wp(\overline{\mathbb{P}}(\xi(\mathcal{U}), \Omega)) < \wp(\mathcal{U})$  for all  $\mathcal{U} \subseteq \Omega$  with  $\wp(\mathcal{U}) \neq 0$ ;
- (2) the family  $\{\overline{\mathbb{P}}(\hat{u}, \cdot) : \hat{u} \in \xi(\Omega)\}$  is equicontractive;
- (3)  $\|\overline{\mathbb{P}}(\hat{u}, v) \otimes' \overline{\mathbb{P}}(\hat{u}', v)\| \leq \|\hat{u} \otimes' \hat{u}'\|$  for all  $\hat{u}, \hat{u}' \in \xi(\Omega)$  and  $v \in \Omega$ ;

Then there is  $\kappa \in \Omega$  with  $\overline{\mathbb{P}}(\xi\kappa, \kappa) = \kappa$ .

**Proof.** Let  $\hat{u} \in \xi(\Omega)$ , define a map  $H : \Omega \rightarrow \Omega$  by  $H(v) = \overline{\mathbb{P}}(\hat{u}, v)$ . Clearly  $H$  is contraction and there is unique point in  $\Omega$  say  $\mathbb{P}\hat{u}$  such that  $\mathbb{P}\hat{u} = \overline{\mathbb{P}}(\hat{u}, \mathbb{P}\hat{u})$ . Now

$$\begin{aligned} \|\mathbb{P}(\hat{u}) \otimes' \mathbb{P}(\hat{u}')\| &= \|\overline{\mathbb{P}}(\hat{u}, \mathbb{P}\hat{u}) \otimes' \overline{\mathbb{P}}(\hat{u}', \mathbb{P}\hat{u}')\| \\ &= \|(\overline{\mathbb{P}}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \overline{\mathbb{P}}(\hat{u}', \mathbb{P}\hat{u}')) \otimes' (\overline{\mathbb{P}}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \overline{\mathbb{P}}(\hat{u}, \mathbb{P}\hat{u}))\| \\ &\leq \|\overline{\mathbb{P}}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \overline{\mathbb{P}}(\hat{u}', \mathbb{P}\hat{u}')\| \otimes \|\overline{\mathbb{P}}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \overline{\mathbb{P}}(\hat{u}, \mathbb{P}\hat{u})\| \\ &\leq \alpha \|\mathbb{P}\hat{u} \otimes' \mathbb{P}\hat{u}'\| \otimes \|\hat{u} \otimes' \hat{u}'\|. \end{aligned}$$

This means that

$$\|\mathbb{P}\hat{u} \otimes' \mathbb{P}\hat{u}'\| \leq \frac{1}{1 \otimes' \alpha} \|\hat{u} \otimes' \hat{u}'\| \quad \alpha \in (0, 1),$$

which indicates that  $\mathbb{P}$  is continuous from  $\xi(\Omega)$  into  $\Omega$  and  $\mathbb{P} \circ \xi$  is a continuous from  $\Omega$  into  $\Omega$ .

Also for  $\mathcal{U} \subseteq \Omega$  and using (1) we get

$$\wp(\mathbb{P} \circ \xi(\mathcal{U})) = \wp(\mathbb{P}(\xi(\mathcal{U}))) = \wp(\overline{\mathbb{P}}(\xi(\mathcal{U}), \mathbb{P}(\xi(\mathcal{U}))) \leq \wp(\overline{\mathbb{P}}(\xi(\mathcal{U}), \Omega) < \wp(\mathcal{U}).$$

Thus  $\mathbb{P} \circ \xi$  is a condensing mapping of  $\Omega$  into  $\Omega$  and by Sadovskii FP theorem there is  $\kappa \in \Omega$  such that  $\mathbb{P}(\xi\kappa) = \kappa$ . Hence  $\kappa = \mathbb{P}(\xi\kappa) = \overline{\mathbb{P}}(\xi\kappa, \mathbb{P}(\xi\kappa)) = \overline{\mathbb{P}}(\xi\kappa, \kappa)$ . ■

**Corollary 3.7.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\mathbb{N}}_{b,cl,cv}(\mathcal{D})$ . Consider the mappings  $\xi$  and  $\zeta$  of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi\nu \otimes \zeta\nu \in \Omega$ ;
- (2)  $\zeta : \Omega \longrightarrow \mathcal{D}$  is contraction with  $k < 1$  and  $\zeta(\Omega) \subset C \in \overline{\mathbb{N}}_{cp}(\mathcal{D})$ ;
- (3)  $\xi : \Omega \longrightarrow \mathcal{D}$  is strictly  $(1 \otimes' k)$ -SC mapping;

Then  $\nu = \xi\nu \otimes \zeta\nu$  has a solution in  $\Omega$ .

**Proof.** Define  $\overline{\mathbb{P}}(\hat{u}, \nu) = \hat{u} \otimes \zeta\nu$  with  $\hat{u} = \xi\nu \in \xi(\Omega)$  (Theorem 3.3). Using (1)  $\overline{\mathbb{P}}(\hat{u}, \nu) \in \Omega$  for all  $\hat{u} = \xi\nu, \nu \in \Omega$  and  $\nu \in \Omega$ . Now since  $\zeta$  is contraction mapping therefore  $\|\overline{\mathbb{P}}(\hat{u}, \nu_1) \otimes' \overline{\mathbb{P}}(\hat{u}, \nu_2)\| = \|\zeta\nu_1 \otimes' \zeta\nu_2\| \leq k \|\nu_1 \otimes' \nu_2\|$ . Also  $\|\overline{\mathbb{P}}(\hat{u}, \nu) \otimes' \overline{\mathbb{P}}(\hat{u}', \nu)\| = \|\hat{u} \otimes' \hat{u}'\| \leq \|\hat{u} \otimes' \hat{u}'\|$ . Now for  $\mathcal{U} \subseteq \Omega$

$$\begin{aligned} & \wp(\overline{\mathbb{P}}(\xi(\mathcal{U}), \Omega)) \\ &= \wp(\xi(\mathcal{U}) \otimes \zeta(\Omega)) \\ &\leq \wp(\xi(\mathcal{U})) \otimes \wp(\zeta(\Omega)) \\ &< (1 \otimes' k)\wp(\mathcal{U}) < \wp(\mathcal{U}). \end{aligned}$$

Hence there is  $\kappa \in \Omega$  such that  $\kappa = \overline{\mathbb{P}}(\xi\kappa, \kappa) = \xi\kappa \otimes \zeta\kappa$ . ■

**Remark 3.4.** If we take  $\zeta = O$  with  $k = 0$  in the above corollary, we obtain Sadovskii FP theorem.

**Theorem 3.4.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\mathbb{N}}_{b,cl,cv}(\mathcal{D})$ . Let  $\overline{\mathbb{P}} : \xi(\Omega) \times \Omega \longrightarrow \Omega$  be a mapping such that

- (1)  $\|\bar{\Pi}(\hat{u}, v) \otimes' \bar{\Pi}(\hat{u}', v)\| \leq \|\hat{u} \otimes' \hat{u}'\|$ ;
- (2) the family  $\{\bar{\Pi}(\hat{u}, \cdot) : \hat{u} \in \xi(\Omega)\}$  is equicontractive with  $k < 1$ ;
- (3)  $\xi : \Omega \longrightarrow \mathcal{D}$  strictly  $(1 \otimes' k)$ -set contractive mapping.

Under these conditions  $\nu = \bar{\Pi}(\xi\nu, \nu)$  has a solution in  $\Omega$ .

**Proof.** Consider a mapping  $H : \Omega \longrightarrow \Omega$  by  $H(v) = \bar{\Pi}(\hat{u}, v)$ , for  $\hat{u} \in \xi(\Omega)$ . Clearly  $H$  is contraction. So there is point in  $\Omega$  say  $\mathbb{P}\hat{u}$  which is unique with  $\mathbb{P}\hat{u} = \bar{\Pi}(\hat{u}, \mathbb{P}\hat{u})$ . Now

$$\begin{aligned}
\|\mathbb{P}(\hat{u}) \otimes' \mathbb{P}(\hat{u}')\| &= \|\bar{\Pi}(\hat{u}, \mathbb{P}\hat{u}) \otimes' \bar{\Pi}(\hat{u}', \mathbb{P}\hat{u}')\| \\
&= \|(\bar{\Pi}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \bar{\Pi}(\hat{u}', \mathbb{P}\hat{u}')) \otimes' (\bar{\Pi}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \bar{\Pi}(\hat{u}, \mathbb{P}\hat{u}))\| \\
&\leq \|\bar{\Pi}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \bar{\Pi}(\hat{u}', \mathbb{P}\hat{u}')\| \otimes \|\bar{\Pi}(\hat{u}', \mathbb{P}\hat{u}) \otimes' \bar{\Pi}(\hat{u}, \mathbb{P}\hat{u})\| \\
&\leq k \|\mathbb{P}\hat{u} \otimes' \mathbb{P}\hat{u}'\| \otimes \|\hat{u} \otimes' \hat{u}'\|.
\end{aligned}$$

This means that

$$\|\mathbb{P}\hat{u} \otimes' \mathbb{P}\hat{u}'\| \leq \frac{1}{1 \otimes' k} \|\hat{u} \otimes' \hat{u}'\| \quad k \in (0, 1),$$

which indicates that  $\mathbb{P}$  is continuous from  $\xi(\Omega)$  into  $\Omega$  and  $\mathbb{P} \circ \xi$  is a continuous from  $\Omega$  into  $\Omega$ .

Also from (3), the above inequality and Lemma 1.1 we have

$$\begin{aligned}
\wp(\mathbb{P} \circ \xi(U)) &= \wp(\mathbb{P}(\xi(U))) \leq \frac{1}{1 \otimes' k} \wp(\xi(U)) \\
&< \frac{1 \otimes' k}{1 \otimes' k} \wp(U) = \wp(U).
\end{aligned}$$

for all  $U \subseteq \Omega$ . Hence  $\mathbb{P} \circ \xi$  is a condensing mapping of  $\Omega$  into  $\Omega$  and by Sadovskii FP theorem there is  $\kappa \in \Omega$  such that  $\mathbb{P}(\xi\kappa) = \kappa$ . Hence  $\kappa = \mathbb{P}(\xi\kappa) = \bar{\Pi}(\xi\kappa, \mathbb{P}(\xi\kappa)) = \bar{\Pi}(\xi\kappa, \kappa)$ . ■

**Corollary 3.8.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\Pi}_{b,cl,cv}(\mathcal{D})$ . Consider the mappings  $\xi$  and  $\zeta$  of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi\nu \otimes \zeta\nu \in \Omega$ ;
- (2)  $\zeta : \Omega \longrightarrow \mathcal{D}$  is contraction with  $k < 1$ ;
- (3)  $\xi$  is strictly  $(1 \otimes' k)$ -set contractive mapping;

Then  $\nu = \xi\nu \otimes \zeta\nu$  has a solution in  $\Omega$ .

**Proof.** Define  $\bar{\Pi}(\hat{u}, v) = \hat{u} \otimes \zeta v$  with  $\hat{u} = \xi\nu \in \xi(\Omega)$ . Using (1)  $\bar{\Pi}(\hat{u}, v) \in \Omega$  for all  $\hat{u} = \xi\nu, \nu \in \Omega$  and  $v \in \Omega$ . Now since  $\zeta$  is contraction mapping therefore  $\|\bar{\Pi}(\hat{u}, v_1) \otimes' \bar{\Pi}(\hat{u}, v_2)\| =$

$\|\zeta v_1 \otimes' \zeta v_2\| \leq k \|v_1 \otimes' v_2\|$ . Also  $\|\bar{\Pi}(\hat{u}, v) \otimes' \bar{\Pi}(\hat{u}', v)\| = \|\hat{u} \otimes' \hat{u}'\| \leq \|\hat{u} \otimes' \hat{u}'\|$ . Hence there is  $\kappa \in \Omega$  such that  $\kappa = \bar{\Pi}(\xi\kappa, \kappa) = \xi\kappa \otimes' \zeta\kappa$ . ■

**Remark 3.5.** If we take  $\zeta = O$  and  $k = 0$ , (in the above corollary) we obtain Sadovskii FP theorem. Taking  $\xi = O$ , we get Banach contraction principle.

**Corollary 3.9.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\Pi}_{b,cl,cv}(\mathcal{D})$ . Consider the mappings  $\xi$  and  $\zeta$  of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\xi\nu \otimes' \zeta\nu \in \Omega$ ;
- (2)  $\zeta : \Omega \longrightarrow \mathcal{D}$  is contraction with  $k < 1$ ;
- (3)  $\xi$  is compact and continuous;

Then  $\nu = \xi\nu \otimes' \zeta\nu$  has a solution in  $\Omega$ .

**Proof.** Since every compact operator is strictly  $(1 \otimes' k)$ -set contractive, therefore using the above corollary we get the required result. ■

**Remark 3.6.** Taking  $\zeta = O$ , we obtain Schauder FP theorem and  $\xi = O$  we obtain Banach contraction principle.

### 3.3 Applications

Consider the class of stationary radiative transfer equations in a channel:

$$v_3 \frac{\partial \xi}{\partial \nu}(\tau, \nu, v) \otimes' \sigma(\nu, v) \xi(\tau, \nu, v) \otimes' \lambda \xi(\tau, \nu, v) = \int_{U^2} r(\nu, v, \hat{u}, \xi(\tau, \nu, \hat{u})) d\hat{u} \quad (1)$$

in  $(0, 1) \times (0, 1) \times U^2$ ,  $\lambda \in \mathbb{C}$ , where  $U^2$  is a unit sphere in  $\mathbb{R}^3$  and  $\xi(\tau, \nu, v)$  is complex function (energy density function) with

$$\xi(\tau, \tau, v) = \xi^\tau(v) \quad (2)$$

Here

$$\begin{aligned} \xi^\tau(v) &= (1 \otimes' \tau)^{\frac{1}{\theta}} \xi(0, 0, v)|_{U^2} \otimes' \tau^{\frac{1}{\theta}} \xi(1, 1, v)|_{U^2} \\ &= (1 \otimes' \tau)^{\frac{1}{\theta}} H^1(\xi(1, 1, v)|_{U^2}) \otimes' \tau^{\frac{1}{\theta}} H^2(\xi(0, 0, v)|_{U^2}) \end{aligned}$$

with  $0 < \theta \leq 1$ . Clearly for  $\theta = 1$ , we get the problem (4.3)  $\otimes$  (4.4) in [131] and for  $\theta = 1$  and  $\tau = 0$  we get problem (4.1)  $\otimes$  (4.2) of [131] [and also see [23]]. In fact we generalize the problem and present an implicit way of finding solution of the problem (1)  $\otimes$  (2).

For the solutions of (1)  $\otimes$  (2), consider

(A1) :  $r \in C((0, 1) \times U^2 \times U^2 \times \mathbb{C})$ ,  $r(\nu, v, \hat{u}, \xi) = 0$  for every  $(\nu, v, \hat{u}, \xi)$  in  $(0, 1) \times U^2 \times U^2 \times \mathbb{C}$  and  $v_3 \leq \frac{1}{2}$ , there is some  $a \in L^1((0, 1) \times U^2)$  and  $q > 0$  such that

$$|r(\nu, v, \hat{u}, \xi) \otimes' r(\nu, v, \hat{u}, \zeta)| \leq a(\nu, \hat{u}) |\xi \otimes' \zeta|,$$

and  $|r(\nu, v, \hat{u}, \xi)| \leq qa(\nu, \hat{u})$  for every  $(\nu, v, \hat{u}, \zeta)$  in  $(0, 1) \times U^2 \times U^2 \times \mathbb{C}$ .

(A2) :  $H^i : C(U^2) \rightarrow C(U^2)$ , with  $H^i(0) = 0$  satisfies

$$|H^i(\zeta) \otimes' H^i(\xi)| \leq q |\zeta \otimes' \xi|$$

on  $U^2$  for every  $\xi, \zeta \in C(U^2)$ ,  $i = 1, 2$ .

(A3) :  $q \otimes \int_0^1 |\sigma(y, v)| dy \otimes |\lambda| \otimes \int_0^1 \int_{U^2} a(y, \hat{u}) d\hat{u} dy \leq \gamma < 1$  and  $2\gamma \leq q \otimes 1$ .

**Theorem 3.5.** Suppose (A1), (A2) and (A3) are satisfied, then there exists a solution  $\xi \in C([0, 1] \times [0, 1] \times U^2)$  of (1)  $\otimes$  (2).

**Proof.** Consider  $\mathfrak{Y} = \{\xi \in C([0, 1] \times [0, 1] \times U^2) : \xi(\tau, \nu, v) = 0 \text{ for } v_3 \leq \frac{1}{2}\}$ , be the BS with norm  $\|\xi\| = \sup\{|\xi(\tau, \nu, v)| : \xi(\tau, \nu, v) \in [0, 1] \times [0, 1] \times U^2\}$ . Let  $K = \{\zeta \in \mathfrak{Y} : \|\zeta\| \leq q_1\}$  be the convex and closed subset of  $\mathfrak{Y}$ . We define  $\bar{\Pi} : S(K) \times K \rightarrow K$  by

$$\begin{aligned} \bar{\Pi}(S\xi, \xi) &= \otimes' b v_3 \xi^\tau \otimes b \int_\tau^\nu \sigma(y, v) \xi(\tau, y, v) dy \otimes \lambda b \int_\tau^\nu \xi(\tau, y, v) dy \\ &\quad \otimes' b \int_\tau^\nu \int_{U^2} r(y, v, \hat{u}, \xi(\tau, y, \hat{u})) d\hat{u} dy \otimes (1 \otimes b v_3) \xi(\tau, \nu, v) \end{aligned}$$

where

$$\begin{aligned}
S(\xi) &= \otimes' b\nu_3 \xi^\tau \otimes b \int_{\tau}^{\nu} \sigma(y, v) \xi(\tau, y, v) dy \\
&\quad \otimes' \lambda b \int_{\tau}^{\nu} \xi(\tau, y, v) dy \otimes' b \int_{\tau}^{\nu} \int_{U^2} r(y, v, \hat{u}, \xi(\tau, y, \hat{u})) d\hat{u} dy.
\end{aligned}$$

Clearly every  $\zeta \in \mathbb{Y}$  satisfying  $\zeta = \bar{\Pi}(S\zeta, \zeta)$  is the solution of problem (1)  $\otimes$  (2). Now we show that  $\bar{\Pi}$  satisfies all conditions of Theorem 3.1. For this first consider

$$\begin{aligned}
|\bar{\Pi}(S\xi, \xi_1) \otimes' \bar{\Pi}(S\xi, \xi_2)| &= |(1 \otimes b\nu_3) \xi_1(\tau, \nu, v) \otimes' (1 \otimes b\nu_3) \xi_2(\tau, \nu, v)| \\
&= |(1 \otimes b\nu_3) (\xi_1(\tau, \nu, v) \otimes' \xi_2(\tau, \nu, v))| \\
&\geq \left(1 \otimes \frac{b}{2}\right) |\xi_1(\tau, \nu, v) \otimes' \xi_2(\tau, \nu, v)|
\end{aligned}$$

for  $\xi_1, \xi_2 \in K$ . Therefore we get

$$\|\bar{\Pi}(S\xi, \xi_1) \otimes' \bar{\Pi}(S\xi, \xi_2)\| \geq \left(1 \otimes \frac{b}{2}\right) \|\xi_1 \otimes' \xi_2\|$$

hence  $\bar{\Pi}(\zeta, \cdot)$  is equiexpansive. Now consider

$$\begin{aligned}
|\bar{\Pi}(S\xi_1, \zeta) \otimes' \bar{\Pi}(S\xi_2, \zeta)| &= \left| b\nu_3 (\xi_2^\tau \otimes' \xi_1^\tau) \otimes b \int_{\tau}^{\nu} \sigma(y, v) (\xi_1(\tau, y, v) \otimes' \xi_2(\tau, y, v)) dy \right. \\
&\quad \left. \otimes' \lambda b \int_{\tau}^{\nu} (\xi_1(\tau, y, v) \otimes' \xi_2(\tau, y, v)) dy \right. \\
&\quad \left. \otimes' b \int_{\tau}^{\nu} \int_{U^2} (r(y, v, \hat{u}, \xi_1(\tau, y, \hat{u})) \otimes' r(y, v, \hat{u}, \xi_2(\tau, y, \hat{u}))) d\hat{u} dy \right| \\
&= |S\xi_1 \otimes' S\xi_2|
\end{aligned}$$

this implies

$$\|\bar{\Pi}(S\xi_1, \zeta) \otimes' \bar{\Pi}(S\xi_2, \zeta)\| \leq \|S\xi_1 \otimes' S\xi_2\|$$

Which shows that  $\bar{\Pi}$  satisfies (2) of Theorem 3.1.

Now we will show that  $S$  is a KSC, for this consider,

$$\begin{aligned}
|S\xi_1 \otimes' S\xi_2| &\leq b \left( \begin{array}{l} q \|\xi_1 \otimes' \xi_2\| \otimes \int_0^1 |\sigma(y, v)| |\xi_1(\tau, y, v) \otimes' \xi_2(\tau, y, v)| dy \\ \otimes |\lambda| \int_0^1 |\xi_1(\tau, y, v) \otimes' \xi_2(\tau, y, v)| dy \\ \otimes \int_0^1 \int_{U^2} |r(y, v, \hat{u}, \xi_1(\tau, y, \hat{u})) \otimes' r(y, v, \hat{u}, \xi_2(\tau, y, \hat{u}))| d\hat{u} dy \end{array} \right) \\
&\leq b \left( \begin{array}{l} q \|\xi_1 \otimes' \xi_2\| \otimes \int_0^1 |\sigma(y, v)| |\xi_1(\tau, y, v) \otimes' \xi_2(\tau, y, v)| dy \\ \otimes |\lambda| \int_0^1 |\xi_1(\tau, y, v) \otimes' \xi_2(\tau, y, v)| dy \\ \otimes \int_0^1 \int_{U^2} a(y, \hat{u}) |\xi_1(\tau, y, \hat{u}) \otimes' \xi_2(\tau, y, \hat{u})| d\hat{u} dy \end{array} \right) \\
&\leq b \left( q \otimes \int_0^1 |\sigma(y, v)| dy \otimes |\lambda| \otimes \int_0^1 \int_{U^2} a(y, \hat{u}) d\hat{u} dy \right) \|\xi_1 \otimes' \xi_2\|
\end{aligned}$$

Clearly from above we have

$$\|S\xi_1 \otimes' S\xi_2\| \leq \gamma b \|\xi_1 \otimes' \xi_2\|,$$

which means  $S$  is  $\gamma b$ -set contraction (also for  $\gamma = \frac{1}{2}$ ). Hence all conditions of Theorem 3.1 are satisfied to obtain  $\zeta \in K$  such that  $\zeta = \bar{\Pi}(S\zeta, \zeta)$ , a solution of (1)  $\otimes$  (2). ■

### 3.4 Conclusion

In this chapter, some Krasnoselskii type FP results using the notions of equiexpansive and equicontractive mappings are proved. The condition of compactness in some results is removed using non-compact operators. Many well known FP results like Banach, Schauder, Krasnosilskii and Sadovski are generalized. An ER for radiative transfer equation is obtained.

## Chapter 4

# Krasnoselskii-type Fixed Point Results for Generalized Expansive and Equiexpansive Mappings with Applications

### 4.1 Introduction

Fractional evolution equations can provide a unique way to analyze the well-posedness of many complex systems. Most differential systems with FDs are providing useful tools for the representation of memory and inherited properties. The physical systems of fractional order are always more applicable than the classical integer order systems. Recently, existence of many fractional order controlled problems and fractional evolution differential equations are studied, see [32, 91, 90, 127, 125].

The celebrated results of FP theory are Banach FPT and Schauder's FPT. During the analysis of neutral and delayed DEs, Krasnoselskii observed that, in most of the cases, the solutions of these equations might be expressed as a sum of contractive and compact operators. Therefore Krasnoselskii [75] generalized both results of Banach and Schauder, for finding the FPs of sum of contractive and compact operators. Huge implementations for existence theory

of DEs and IEs can be seen in the following monographs [73, 105, 133].

Due to the involvement of contractive operators in Krasnoselskii's result, the ERs cover a small class of physical problems. To overcome this issue, some new results are required involving other less restrictive conditions like expansive operators. In this regard, we describe a new class of generalized equiexpansive mappings, assume  $F$  is an implicit function of two variables with some conditions and present many variants of Krasnoselskii's FP theorem.

## 4.2 Main Results

Let  $\mathcal{D}$  be a BS and  $\Omega \subseteq \mathcal{D}$ . Suppose  $\bar{\Pi} : \Omega \times \Omega \longrightarrow \mathcal{D}$  or  $\bar{\Pi} : \xi(\Omega) \times \Omega \longrightarrow \mathcal{D}$ . We study the FP results of the equations of types  $\nu = \bar{\Pi}(\xi\nu, \nu)$  or  $\nu = \bar{\Pi}(\nu, \nu)$  by using generalized expansive and equiexpansive conditions on the operator  $\bar{\Pi}$  such that  $\Omega \subseteq \bar{\Pi}(\xi\nu, \Omega)$  or  $\Omega \subseteq \bar{\Pi}(\nu, \Omega)$ .

**Definition 4.1** Let  $(\mathcal{D}, d_{\mathcal{D}})$  be a MS and  $\Omega$  be a subset of  $\mathcal{D}$ . A mapping  $\xi$  of  $\Omega$  into  $\mathcal{D}$  is generalized expansive if

$$cd_{\mathcal{D}}(\xi\nu, \xi\nu) \otimes ad_{\mathcal{D}}(\xi\nu, \nu) \otimes bd_{\mathcal{D}}(\xi\nu, \nu) \geq hd_{\mathcal{D}}(\nu, \nu) \quad a \geq 0, b \geq 0, c, h > 0, a \otimes b \otimes c < h.$$

**Theorem 4.1.** Assume that  $\Omega \neq \emptyset$  which is closed subset of a complete MS  $\mathcal{D}$  and  $\xi$  is one to one mapping of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\Omega \subseteq \xi(\Omega)$ ;
- (2)  $cd_{\mathcal{D}}(\xi\nu, \xi\nu) \otimes ad_{\mathcal{D}}(\xi\nu, \nu) \otimes bd_{\mathcal{D}}(\xi\nu, \nu) \geq hd_{\mathcal{D}}(\nu, \nu), \quad a \geq 0, b \geq 0, c, h > 0, a \otimes b \otimes c < h.$

Thus, there is unique  $\nu \in \Omega$  such that  $\xi\nu = \nu$ .

**Proof.** Since  $\xi$  is one to one, therefore the inverse of  $\xi : \Omega \longrightarrow \xi(\Omega)$  exists. For  $\nu, \vartheta \in \xi(\Omega)$  there are  $\nu, \vartheta \in \Omega$  such that  $\xi\nu = \nu$  and  $\xi\vartheta = \vartheta$ , then  $\nu = \xi^{*\prime 1}\nu$  and  $\vartheta = \xi^{*\prime 1}\vartheta$ . From (2)

$$hd_{\mathcal{D}}(\xi^{*\prime 1}\nu, \xi^{*\prime 1}\vartheta) \leq ad_{\mathcal{D}}(\nu, \xi^{*\prime 1}\nu) \otimes bd_{\mathcal{D}}(\vartheta, \xi^{*\prime 1}\vartheta) \otimes cd_{\mathcal{D}}(\nu, \vartheta)$$

for all  $\nu, \vartheta \in \xi(\Omega)$ . Thus

$$d_{\mathcal{D}}(\xi^{*\prime 1}\nu, \xi^{*\prime 1}\vartheta) \leq a'd_{\mathcal{D}}(\nu, \xi^{*\prime 1}\nu) \otimes b'd_{\mathcal{D}}(\vartheta, \xi^{*\prime 1}\vartheta) \otimes c'd_{\mathcal{D}}(\nu, \vartheta)$$

where  $a' \otimes b' \otimes c' = \frac{a}{h} \otimes \frac{b}{h} \otimes \frac{c}{h} < 1$ . Since  $\Omega \subseteq \xi(\Omega)$  therefore  $\xi^{*\prime 1}|_{\Omega} : \Omega \longrightarrow \Omega$  is Reich

contraction. Hence by [110] there is  $\nu \in \Omega$  such that  $\xi^{\otimes' 1} \nu = \nu$  and  $\nu = \xi \nu$ . For uniqueness, let  $\nu, v \in \Omega$  such that  $\xi \nu = \nu$  and  $\xi v = v$ . Thus, from (2),  $cd_{\mathcal{D}}(\nu, v) \geq hd_{\mathcal{D}}(\nu, v)$  which implies that  $\nu = v$ . ■

**Corollary 4.1.** Let  $\Omega$  be nonempty closed subset of a complete MS  $\mathcal{D}$  and  $\xi$  is a mapping of  $\Omega$  into  $\mathcal{D}$  with

- (1)  $\Omega \subseteq \xi(\Omega)$ ;
- (2)  $d_{\mathcal{D}}(\xi \nu, \xi v) \geq h' d_{\mathcal{D}}(\nu, v)$ ,  $h' > 1$ .

Thus, there is unique  $\nu \in \Omega$  such that  $\xi \nu = \nu$ .

**Proof.** From (2) we deduce that  $\xi$  is one to one. Using the above theorem, we can take  $a = b = 0$  and  $h > c > 0$  such that  $h' = \frac{h}{c}$ . Thus there is unique  $\nu \in \Omega$  such that  $\xi \nu = \nu$ . ■

**Remark 4.1.** The above corollary is the Theorem 2.1 in [128] (see also Theorem 2.1 in [130]).

**Definition 4.2.** Suppose  $\Omega$  be a nonempty subset of a BS  $\mathcal{D}$ .  $\bar{\Pi}$  be a mapping of  $\Omega \times \Omega$  into  $\mathcal{D}$ .  $\bar{\Pi}$  is called generalized equiexpansive if

$$c \|\bar{\Pi}(\nu, v) \otimes' \bar{\Pi}(\nu, v')\| \otimes a \|\bar{\Pi}(\nu, v) \otimes' v\| \otimes b \|\bar{\Pi}(\nu, v') \otimes' v'\| \geq h \|v \otimes' v'\|$$

for all  $(\nu, v), (\nu, v')$  in the domain of  $\bar{\Pi}$ ,  $a, b \geq 0$ ,  $c, h > 0$ ,  $a \otimes b \otimes c < h$ .

**Theorem 4.2.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\Pi}_d(\mathcal{D})$ . Let  $\bar{\Pi}$  be a generalized equiexpansive mapping of  $\Omega \times \Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \bar{\Pi}(\nu, \Omega)$  for all  $\nu \in \Omega$  and

- (1) For each  $\nu \in \Omega$ ,  $\bar{\Pi}(\nu, v) = \bar{\Pi}(\nu, v')$  yields  $v = v'$ ;
- (2)  $\|\bar{\Pi}(\nu, v) \otimes' \bar{\Pi}(\nu', v)\| \leq \|v \otimes' v'\|$  for all  $(\nu, v), (\nu', v') \in \Omega \times \Omega$ .

Then there is unique  $\nu \in \Omega$  such that  $\bar{\Pi}(\nu, \nu) = \nu$ .

**Proof.** For  $\nu \in \Omega$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  such that  $H(v) = \bar{\Pi}(\nu, v)$ . From (1)

$$c \|H(v) \otimes' H(v')\| \otimes a \|H(v) \otimes' v\| \otimes b \|H(v') \otimes' v'\| \geq h \|v \otimes' v'\|.$$

Also

$$\Omega \subseteq \bar{\Pi}(\nu, \Omega) = H(\Omega)$$

and, using (2),  $H$  is one to one function. Thus, by Theorem 4.1, there is unique  $\nu \in \Omega$  such

that  $\mathfrak{P}\nu = \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu)$ . Now

$$\begin{aligned}
& \|\mathfrak{P}(\nu) \otimes' \mathfrak{P}(\nu')\| \\
&= \|\bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu')\| \\
&= \|(\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu')) \otimes' (\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu))\| \\
&\geq \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu')\| \otimes' \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu)\| \\
&\geq \frac{h}{c} \|\mathfrak{P}\nu \otimes' \mathfrak{P}\nu'\| \otimes' \frac{a}{c} \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \mathfrak{P}\nu\| \otimes' \frac{b}{c} \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu') \otimes' \mathfrak{P}\nu'\| \\
&\quad \otimes' \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu)\| \\
&= \frac{h}{c} \|\mathfrak{P}\nu \otimes' \mathfrak{P}\nu'\| \otimes' \frac{a}{c} \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu)\| \otimes' \frac{b}{c} \|\mathfrak{P}\nu' \otimes' \mathfrak{P}\nu'\| \\
&\quad \otimes' \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu)\|.
\end{aligned}$$

simplifying

$$\|\mathfrak{P}(\nu) \otimes' \mathfrak{P}(\nu')\| \leq \frac{a \otimes c}{h \otimes c} \|\bar{\mathfrak{I}}(\nu', \mathfrak{P}\nu) \otimes' \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu)\| \leq \frac{a \otimes c}{h \otimes c} \|\nu \otimes' \nu'\|,$$

which yields that  $\mathfrak{P}$  is a contraction mapping of  $\Omega$  into  $\Omega$ . Thus there is unique  $\nu \in \Omega$  such that  $\mathfrak{P}\nu = \nu$ . Since for this  $\nu \in \Omega$  there is a unique  $\mathfrak{P}\nu \in \Omega$  such that  $\mathfrak{P}\nu = \bar{\mathfrak{I}}(\nu, \mathfrak{P}\nu)$ , therefore there is unique  $\nu$  satisfying  $\bar{\mathfrak{I}}(\nu, \nu) = \nu$ . ■

**Corollary 4.2.** Let  $\mathfrak{D}$  be a BS and  $\Omega \in \bar{\mathfrak{I}}_{\alpha}(\mathfrak{D})$ . Let  $\bar{\mathfrak{I}}$  be a mapping of  $\Omega \times \Omega$  into  $\mathfrak{D}$  with  $\Omega \subseteq \bar{\mathfrak{I}}(\nu, \Omega)$  for all  $\nu \in \Omega$  and

(1)  $c \|\bar{\mathfrak{I}}(\nu, v) \otimes' \bar{\mathfrak{I}}(\nu, v')\| \geq h \|v \otimes' v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\bar{\mathfrak{I}}$ ,  $c, h > 0$ ,  $2c < h$ ;

(2)  $\|\bar{\mathfrak{I}}(\nu, v) \otimes' \bar{\mathfrak{I}}(\nu', v)\| \leq \|\nu \otimes' \nu'\|$  for all  $(\nu, v), (\nu', v') \in \Omega \times \Omega$ .

Then there is unique  $\nu \in \Omega$  such that  $\bar{\mathfrak{I}}(\nu, \nu) = \nu$ .

**Proof.** From (1),  $\bar{\mathfrak{I}}(\nu, v) = \bar{\mathfrak{I}}(\nu, v')$  yields  $v = v'$ . Putting  $a = b = 0$  in generalized equiexpansive condition, (Theorem 4.2) the required result is obtained. ■

**Corollary 4.3.** For a given BS  $\mathfrak{D}$  and  $\Omega \in \bar{\mathfrak{I}}_{\alpha}(\mathfrak{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathfrak{D}$  such that  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for all  $\nu \in \Omega$  and

(1)  $c \|\zeta v \otimes' \zeta v'\| \otimes a \|\xi\nu \otimes' (I \otimes' \zeta)v\| \otimes b \|\xi\nu \otimes' (I \otimes' \zeta)v'\| \geq h \|v \otimes' v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\bar{\mathfrak{I}}$ ,  $a \geq 0, b \geq 0, c, h > 0, a \otimes b \otimes c < h, a \otimes c < h \otimes c$ ;

(2)  $\zeta$  is one to one mapping;

(3)  $\|\xi\nu \otimes' \xi\nu'\| \leq \|\nu \otimes' \nu'\|$  for all  $\nu, \nu' \in \Omega$ .

Thus, there is unique  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ .

**Proof.** Define  $\bar{\Pi}(\nu, v) = \xi\nu \otimes \zeta v$ . Clearly  $\bar{\Pi}$  is a mapping of  $\Omega \times \Omega$  into  $\mathcal{D}$ . Also  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega) = \bar{\Pi}(\nu, \Omega)$ . From (1)

$$\begin{aligned} h \|v \otimes' v'\| &\leq c \|\zeta v \otimes' \zeta v'\| \otimes a \|\xi\nu \otimes' (I \otimes' \zeta)v\| \otimes b \|\xi\nu \otimes' (I \otimes' \zeta)v'\| \\ &= c \|(\xi\nu \otimes \zeta v) \otimes' (\xi\nu \otimes \zeta v')\| \otimes a \|v \otimes' (\xi\nu \otimes \zeta v)\| \otimes b \|v' \otimes' (\xi\nu \otimes \zeta v')\| \\ &= c \|\bar{\Pi}(\nu, v) \otimes' \bar{\Pi}(\nu, v')\| \otimes a \|\bar{\Pi}(\nu, v) \otimes' v\| \otimes b \|\bar{\Pi}(\nu, v') \otimes' v'\|. \end{aligned}$$

Since  $\zeta$  is one to one mapping, so  $\zeta v = \zeta v'$  implies  $v = v'$  and hence  $\bar{\Pi}(\nu, v) = \bar{\Pi}(\nu, v')$  implies  $v = v'$ . Also

$$\|\bar{\Pi}(\nu, v) \otimes' \bar{\Pi}(\nu', v)\| = \|\xi\nu \otimes' \xi\nu'\| \leq \|\nu \otimes' \nu'\|.$$

Hence, by Theorem 4.2, there is unique  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ . ■

**Remark 4.2.** If  $\xi = O$  in the above corollary, then we acquire the result like Theorem 4.1.

**Corollary 4.4.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\Pi}_{cl}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for all  $\nu \in \Omega$  and

(1)  $c \|\zeta v \otimes' \zeta v'\| \geq h \|v \otimes' v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\bar{\Pi}$ ,  $c, h > 0$ ,  $2c < h$ ;

(2)  $\|\xi\nu \otimes' \xi\nu'\| \leq \|\nu \otimes' \nu'\|$  for all  $\nu, \nu' \in \Omega$ .

Then there is unique  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ .

**Proof.** From (1) we deduce that  $\zeta$  is one to one. Hence the result can be obtained by taking  $a = b = 0$  in Corollary 4.3. ■

**Theorem 4.3.** For a given BS  $\mathcal{D}$  and  $\Omega \in \bar{\Pi}_{cl}(\mathcal{D})$ . Let  $\bar{\Pi}$  be a mapping of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \bar{\Pi}(\xi\nu, \Omega)$  for all  $\xi\nu \in \xi(\Omega)$  such that

(1)  $c \|\bar{\Pi}(\xi\nu, v) \otimes' \bar{\Pi}(\xi\nu, v')\| \otimes a \|\bar{\Pi}(\xi\nu, v) \otimes' v\| \otimes b \|\bar{\Pi}(\xi\nu, v') \otimes' v'\| \geq h \|v \otimes' v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\bar{\Pi}$ ,  $a \geq 0, b \geq 0$ ,  $c, h > 0$ ,  $a \otimes b \otimes c < h$ ,  $a \otimes c < h \otimes' c$ ;

(2) For each  $\xi\nu \in \Omega$ ,  $\bar{\Pi}(\xi\nu, v) = \bar{\Pi}(\xi\nu, v')$  implies  $v = v'$  for all  $v, v' \in \Omega$ ;

(3)  $\|\bar{\Pi}(\xi\nu, v) \otimes' \bar{\Pi}(\xi\nu', v)\| \leq \|\nu \otimes' \nu'\|$  for all  $(\xi\nu, v), (\xi\nu', v) \in \xi(\Omega) \times \Omega$  and  $\xi$  is a mapping of  $\Omega$  into  $\mathcal{D}$ .

Then there is unique  $\nu \in \Omega$  such that  $\bar{\Pi}(\xi\nu, \nu) = \nu$ .

**Proof.** For  $\xi\nu \in \xi(\Omega)$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  such that  $H(v) = \bar{\Pi}(\xi\nu, v)$ . From (1)

$$c \|H(v) \otimes' H(v')\| \otimes a \|H(v) \otimes' v\| \otimes b \|H(v') \otimes' v'\| \geq h \|v \otimes' v'\|.$$

Also

$$\Omega \subseteq \bar{\Pi}(\xi\nu, \Omega) = H(\Omega)$$

and, using (2),  $H$  is one to one function. Thus, by Theorem 4.1, there is a unique point  $\mathbb{P}(\xi\nu) \in \Omega$  with  $\mathbb{P}(\xi\nu) = \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu))$ . Now

$$\begin{aligned} & \| \mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu') \| \\ &= \| \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu')) \| \\ &= \| (\bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu'))) \otimes' (\bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu))) \| \\ &\geq \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu')) \| \otimes' \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) \| \\ &\geq \left( \frac{h}{c} \| \mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu') \| \otimes' \frac{a}{c} \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \mathbb{P}(\xi\nu) \| \right. \\ &\quad \left. \otimes' \frac{b}{c} \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu')) \otimes' \mathbb{P}(\xi\nu') \| \otimes' \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) \| \right) \\ &= \left( \frac{h}{c} \| \mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu') \| \otimes' \frac{a}{c} \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) \| \right. \\ &\quad \left. \otimes' \frac{b}{c} \| \mathbb{P}(\xi\nu') \otimes' \mathbb{P}(\xi\nu') \| \otimes' \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) \| \right). \end{aligned}$$

simplifying

$$\begin{aligned} \| (\mathbb{P} \circ \xi)\nu \otimes' (\mathbb{P} \circ \xi)\nu' \| &\leq \frac{a \otimes c}{h \otimes' c} \| \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) \| \\ &\leq \frac{a \otimes c}{h \otimes' c} \| \nu \otimes' \nu' \|. \end{aligned}$$

which means that  $\mathbb{P} \circ \xi$  is a contraction mapping of  $\Omega$  into  $\Omega$  and there is unique  $\nu \in \Omega$  such that  $(\mathbb{P} \circ \xi)\nu = \nu$  or  $\mathbb{P}(\xi\nu) = \nu$ . Also for  $\xi\nu \in \xi(\Omega)$  there is unique  $\mathbb{P}(\xi\nu) \in \Omega$  such that  $\mathbb{P}(\xi\nu) = \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu))$ . Hence there is unique  $\nu$  such that  $\nu = \bar{\Pi}(\xi\nu, \nu)$ . ■

**Theorem 4.4.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\Pi}_{b,cv,d}(\mathcal{D})$ . Suppose  $\bar{\Pi}$  is a mapping of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \bar{\Pi}(\xi\nu, \Omega)$  for all  $\xi\nu \in \xi(\Omega)$  such that

$$(1) \ c \| \bar{\Pi}(\xi\nu, v) \otimes' \bar{\Pi}(\xi\nu, v') \| \otimes a \| \bar{\Pi}(\xi\nu, v) \otimes' v \| \otimes b \| \bar{\Pi}(\xi\nu, v') \otimes' v' \| \geq h \| v \otimes' v' \| \text{ for all } (v, v'),$$

$(\nu, \nu')$  in the domain of  $\bar{\mathbb{I}}$ ,  $a, b \geq 0$ ,  $c, h > 0$ ,  $a \otimes b \otimes c < h$ ;

(2) For each  $\nu \in \Omega$ ,  $\bar{\mathbb{I}}(\xi\nu, \nu) = \bar{\mathbb{I}}(\xi\nu, \nu')$  implies  $\nu = \nu'$  for all  $\nu, \nu' \in \Omega$ .

(3)  $\|\bar{\mathbb{I}}(\xi\nu, \nu) \otimes' \bar{\mathbb{I}}(\xi\nu', \nu)\| \leq \|\xi\nu \otimes' \xi\nu'\|$  for all  $(\xi\nu, \nu), (\xi\nu', \nu') \in \xi(\Omega) \times \Omega$  and  $\xi$  is a  $k$ -set contractive mapping of  $\Omega$  into  $\mathcal{D}$  for  $k < \frac{h \otimes' c}{a \otimes' c}$ .

Then there is  $\nu \in \Omega$  such that  $\bar{\mathbb{I}}(\xi\nu, \nu) = \nu$ .

**Proof.** For  $\xi\nu \in \Omega$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  such that  $H(\nu) = \bar{\mathbb{I}}(\xi\nu, \nu)$ . From (1)

$$c \|H(\nu) \otimes' H(\nu')\| \otimes a \|H(\nu) \otimes' \nu\| \otimes b \|H(\nu') \otimes' \nu'\| \geq h \|\nu \otimes' \nu'\|.$$

Also

$$\Omega \subseteq \bar{\mathbb{I}}(\xi\nu, \Omega) = H(\Omega)$$

and, using (2),  $H$  is one to one function. Thus, by Theorem 4.1, there is a unique  $\mathbb{P}(\xi\nu) \in \Omega$  such that  $\mathbb{P}(\xi\nu) = \bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu))$ . Now

$$\begin{aligned} & \|\mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu')\| \\ &= \|\bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu'))\| \\ &= \|(\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu')) \otimes' (\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu)))\| \\ &\geq \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu'))\| \otimes' \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu))\| \\ &\geq \left(\frac{h}{c} \|\mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu')\| \otimes' \frac{a}{c} \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \mathbb{P}(\xi\nu)\| \right. \\ &\quad \left. \otimes' \frac{b}{c} \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu')) \otimes' \mathbb{P}(\xi\nu')\| \otimes' \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu))\| \right) \\ &= \left(\frac{h}{c} \|\mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu')\| \otimes' \frac{a}{c} \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu))\| \right. \\ &\quad \left. \otimes' \frac{b}{c} \|\mathbb{P}(\xi\nu') \otimes' \mathbb{P}(\xi\nu')\| \otimes' \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu))\| \right). \end{aligned}$$

simplifying

$$\begin{aligned} \|(\mathbb{P} \circ \xi)\nu \otimes' (\mathbb{P} \circ \xi)\nu'\| &\leq \frac{a \otimes' c}{h \otimes' c} \|\bar{\mathbb{I}}(\xi\nu', \mathbb{P}(\xi\nu)) \otimes' \bar{\mathbb{I}}(\xi\nu, \mathbb{P}(\xi\nu))\| \\ &\leq \frac{a \otimes' c}{h \otimes' c} \|\xi\nu \otimes' \xi\nu'\|. \end{aligned}$$

which implies that  $\mathbb{P} \circ \xi$  is a continuous mapping of  $\Omega$  into  $\Omega$ . Now

$$\begin{aligned} \wp(\mathbb{P} \circ \xi(\mathcal{U})) &= \wp(\mathbb{P}(\xi(\mathcal{U}))) \\ &\leq \frac{a \otimes c}{h \otimes' c} \wp(\xi(\mathcal{U})) \\ &\leq k \frac{a \otimes c}{h \otimes' c} \wp(\mathcal{U}) < \wp(\mathcal{U}) \end{aligned}$$

for all  $\mathcal{U} \subseteq \Omega$ . By Sadovskii theorem FP theorem there is  $\nu \in \Omega$  such that  $(\mathbb{P} \circ \xi)\nu = \nu$  or  $\mathbb{P}(\xi\nu) = \nu$ . Now for  $\xi\nu \in \xi(\Omega)$  there is unique  $\mathbb{P}(\xi\nu) \in \Omega$  such that  $\mathbb{P}(\xi\nu) = \bar{\mathbb{P}}(\xi\nu, \mathbb{P}(\xi\nu))$ . Hence  $\nu = \mathbb{P}(\xi\nu) = \bar{\mathbb{P}}(\xi\nu, \mathbb{P}(\xi\nu)) = \bar{\mathbb{P}}(\xi\nu, \nu)$ . ■

**Corollary 4.5.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\mathbb{I}}_{b,cv,d}(\mathcal{D})$ . Suppose  $\bar{\mathbb{I}}$  is a mapping of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \bar{\mathbb{I}}(\xi\nu, \Omega)$  for all  $\xi\nu \in \xi(\Omega)$  such that

(1)  $c \|\bar{\mathbb{I}}(\xi\nu, \nu) \otimes' \bar{\mathbb{I}}(\xi\nu, \nu')\| \geq h \|v \otimes' v'\|$  for all  $(\nu, \nu), (\nu, \nu')$  in the domain of  $\bar{\mathbb{I}}$ ,  $c, h > 0$ ,  $c < h$ ;

(2)  $\|\bar{\mathbb{I}}(\xi\nu, \nu) \otimes' \bar{\mathbb{I}}(\xi\nu', \nu)\| \leq \|\xi\nu \otimes' \xi\nu'\|$  for all  $(\xi\nu, \nu), (\xi\nu', \nu) \in \xi(\Omega) \times \Omega$  and  $\xi$  is a  $k$ -set contractive mapping of  $\Omega$  into  $\mathcal{D}$  for  $k < \frac{h \otimes' c}{c}$ .

Then there is  $\nu \in \Omega$  such that  $\bar{\mathbb{I}}(\xi\nu, \nu) = \nu$ .

**Proof.** From (1),  $\bar{\mathbb{I}}(\xi\nu, \nu) = \bar{\mathbb{I}}(\xi\nu, \nu')$  implies  $\nu = \nu'$ . Putting  $a = b = 0$  in Theorem 4.4, we acquire the required result. ■

**Corollary 4.6.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\mathbb{I}}_{b,cv,d}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for all  $\nu \in \Omega$  and

(1)  $\zeta$  is one to one mapping;

(2)  $c \|\zeta v \otimes' \zeta v'\| \otimes a \|\xi\nu \otimes' (I \otimes' \zeta)v\| \otimes b \|\xi\nu \otimes' (I \otimes' \zeta)v'\| \geq h \|v \otimes' v'\|$  for all  $(\nu, \nu), (\nu, \nu')$  in the domain of  $\bar{\mathbb{I}}$ ,  $a, b \geq 0$ ,  $c, h > 0$ ,  $a \otimes b \otimes c < h$ ;

(3)  $\xi$  is a  $k$ -set contractive mapping of  $\Omega$  into  $\mathcal{D}$  for  $k < \frac{h \otimes' c}{a \otimes c}$ ;

Then there is  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$

**Proof.** Define  $\bar{\mathbb{I}}(\xi\nu, \nu) = \xi\nu \otimes \zeta\nu$ . Clearly  $\bar{\mathbb{I}}$  is a mapping of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$ . Also

$\Omega \subseteq \xi\nu \otimes \zeta(\Omega) = \overline{\Pi}(\xi\nu, \Omega)$ . From (2)

$$\begin{aligned} h \|v \otimes v'\| &\leq c \|\zeta v \otimes \zeta v'\| \otimes a \|\xi\nu \otimes (I \otimes \zeta)v\| \otimes b \|\xi\nu \otimes (I \otimes \zeta)v'\| \\ &= c \|(\xi\nu \otimes \zeta v) \otimes (\xi\nu \otimes \zeta v')\| \otimes a \|v \otimes (\xi\nu \otimes \zeta v)\| \otimes b \|v' \otimes (\xi\nu \otimes \zeta v')\| \\ &= c \|\overline{\Pi}(\xi\nu, v) \otimes \overline{\Pi}(\xi\nu, v')\| \otimes a \|\overline{\Pi}(\xi\nu, v) \otimes v\| \otimes b \|\overline{\Pi}(\xi\nu, v') \otimes v'\|. \end{aligned}$$

Since  $\zeta$  is one to one mapping, therefore  $\zeta v = \zeta v'$  implies  $v = v'$ , and hence  $\overline{\Pi}(\nu, v) = \overline{\Pi}(\nu', v)$  implies  $v = v'$ . Also

$$\|\overline{\Pi}(\nu, v) \otimes \overline{\Pi}(\nu', v)\| = \|\xi\nu \otimes \xi\nu'\| \leq \|\xi\nu \otimes \xi\nu'\|.$$

Hence, by Theorem 4.4, there is  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ . ■

**Corollary 4.7.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \overline{\Pi}_{b,cv,d}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for all  $\nu \in \Omega$ ;
- (2)  $c \|\zeta v \otimes \zeta v'\| \geq h \|v \otimes v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\overline{\Pi}$ ,  $c, h > 0$ ,  $c < h$ ;
- (3)  $\xi$  is a  $k$ -set contractive mapping of  $\Omega$  into  $\mathcal{D}$  for  $k < \frac{h \otimes c}{c}$ .

Then there is  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ .

**Proof.** (2) implies that  $\zeta$  is one to one mapping. Putting  $a = b = 0$  (in Corollary 4.6) the required result is obtained. ■

**Remark 4.3.** Corollary 4.7 is the Theorem 2.6 in [128].

**Corollary 4.8.** For a given BS  $\mathcal{D}$  and  $\Omega \in \overline{\Pi}_{b,cv,d}(\mathcal{D})$ . Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$  for all  $\nu \in \Omega$ ;
- (2)  $\xi$  is continuous and  $\xi(\Omega) \subset C \in \overline{\Pi}_{cp}(\mathcal{D})$ ;
- (3)  $c \|\zeta v \otimes \zeta v'\| \geq h \|v \otimes v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\overline{\Pi}$ ,  $c, h > 0$ ,  $c < h$ ;

Then there is  $\nu \in \Omega$  such that  $\xi\nu \otimes \zeta\nu = \nu$ .

**Proof.** From Corollary 4.5, Define  $\overline{\Pi} : \xi(\Omega) \times \Omega \longrightarrow \mathcal{D}$  by  $\overline{\Pi}(\xi\nu, v) = \xi\nu \otimes \zeta v$ . From (1), since  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega)$ , therefore  $\Omega \subseteq \xi\nu \otimes \zeta(\Omega) = \overline{\Pi}(\xi\nu, \Omega)$ . Now using (3)

$$c \|\overline{\Pi}(\xi\nu, v) \otimes \overline{\Pi}(\xi\nu, v')\| = c \|\zeta v \otimes \zeta v'\| \geq h \|v \otimes v'\|$$

and

$$\|\bar{\Pi}(\xi\nu, \nu) \otimes' \bar{\Pi}(\xi\nu', \nu)\| = \|\xi\nu \otimes' \xi\nu'\| \leq \|\xi\nu \otimes' \xi\nu'\|.$$

Since  $\xi(\Omega)$  lies in compact subset  $C$  of  $\mathcal{D}$ , therefore for  $U \subseteq \Omega$

$$\begin{aligned} & \wp(\xi(U)) \\ & \leq \wp(\xi(\Omega)) \\ & \leq \wp(C) = 0. \end{aligned}$$

So  $\xi$  is a  $k$ -set contractive mapping for  $k = 0$ . Hence there is  $\nu \in \Omega$  such that  $\nu = \bar{\Pi}(\xi\nu, \nu) = \xi\nu \otimes' \zeta\nu$ . ■

**Remark 4.4.** Corollary 4.8 is the Theorem 2.2 in [128] with the assumption that  $\Omega$  is bounded set.

**Theorem 4.5.** [[84]. p.392] If  $(\mathcal{D}, d_{\mathcal{D}})$  is a complete MS,  $0 \leq q < 1/2$  and  $\xi : \mathcal{D} \rightarrow \mathcal{D}$  is a map such that

$$d_{\mathcal{D}}(\xi\nu, \xi v) \leq q[d_{\mathcal{D}}(\nu, \xi\nu) \otimes d_{\mathcal{D}}(\nu, \xi v)]$$

for all  $\nu, v \in \mathcal{D}$ . Then  $\xi$  has unique FP.

**Theorem 4.6.** Let  $\Omega \neq \phi$  which is closed subset of a complete MS  $\mathcal{D}$  and  $\xi$  is a one to one mapping of  $\Omega$  into  $\mathcal{D}$  such that

- (1)  $\Omega \subseteq \xi(\Omega)$ ;
- (2)  $d_{\mathcal{D}}(\xi\nu, \nu) \otimes d_{\mathcal{D}}(\xi v, v) \geq h d_{\mathcal{D}}(\nu, v)$ ,  $h > 2$ .

Then there is unique  $\nu \in \Omega$  such that  $\xi\nu = \nu$ .

**Proof.** Use Theorem 4.1 Theorem 4.5. ■

**Theorem 4.7.** Suppose  $\Omega \neq \phi$  and closed subset of a BS  $\mathcal{D}$ .  $\bar{\Pi}$  is a mapping of  $\xi(\Omega) \times \Omega$  into  $\mathcal{D}$  such that  $\Omega \subseteq \bar{\Pi}(\xi\nu, \Omega)$  for all  $\xi\nu \in \xi(\Omega)$  and

- (1)  $\|\bar{\Pi}(\xi\nu, \nu) \otimes' v\| \otimes \|\bar{\Pi}(\xi\nu, v') \otimes' v'\| \geq h \|v \otimes' v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\bar{\Pi}$ ,  $h > 2$ ;
- (2) For each  $\nu \in \Omega$ ,  $\bar{\Pi}(\xi\nu, \nu) = \bar{\Pi}(\xi\nu, v')$  implies  $\nu = v'$  for all  $\nu, v' \in \Omega$ ;
- (3)  $\|\bar{\Pi}(\xi\nu, \nu) \otimes' \bar{\Pi}(\xi\nu', \nu)\| \leq \|\nu \otimes' \nu'\|$  for all  $(\xi\nu, \nu), (\xi\nu', \nu) \in \xi(\Omega) \times \Omega$  and  $\xi$  is a mapping of  $\Omega$  into  $\mathcal{D}$ .

Then there is unique  $\nu \in \Omega$  such that  $\bar{\Pi}(\xi\nu, \nu) = \nu$ .

**Proof.** For  $\xi\nu \in \Omega$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  such that  $H(v) = \bar{\Pi}(\xi\nu, v)$ . From (1)

$$\|H(v) \otimes' v\| \otimes \|H(v') \otimes' v'\| \geq h \|v \otimes' v'\|.$$

Also

$$\Omega \subseteq \bar{\Pi}(\xi\nu, \Omega) = H(\Omega)$$

and, using (2),  $H$  is one to one function. Therefore, there is a unique  $\mathbb{P}(\xi\nu) \in \Omega$  with  $\mathbb{P}(\xi\nu) = \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu))$ . Now putting  $v = \mathbb{P}(\xi\nu)$  and  $v' = \mathbb{P}(\xi\nu')$  in (1) we deduce

$$\begin{aligned} h \|\mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu')\| &\leq \|\bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) \otimes' \mathbb{P}(\xi\nu)\| \otimes \|\bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu')) \otimes' \mathbb{P}(\xi\nu')\| \\ &\leq \|\mathbb{P}(\xi\nu) \otimes' \mathbb{P}(\xi\nu)\| \otimes \|\bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu')) \otimes' \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu'))\| \\ &= \|\bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu')) \otimes' \bar{\Pi}(\xi\nu', \mathbb{P}(\xi\nu'))\| \\ &\leq \|v \otimes' v'\| \end{aligned}$$

This gives

$$\|(\mathbb{P} \circ \xi)\nu \otimes' (\mathbb{P} \circ \xi)\nu'\| \leq \frac{1}{h} \|v \otimes' v'\|.$$

which shows that  $\mathbb{P} \circ \xi$  is contraction mapping and there is unique  $\nu \in \Omega$  such that  $(\mathbb{P} \circ \xi)\nu = \nu$  or  $\mathbb{P}(\xi\nu) = \nu$ . Also for  $\xi\nu \in \xi(\Omega)$  there is unique  $\mathbb{P}(\xi\nu) \in \Omega$  such that  $\mathbb{P}(\xi\nu) = \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu))$ . Hence there is unique  $\nu$  such that  $\nu = \mathbb{P}(\xi\nu) = \bar{\Pi}(\xi\nu, \mathbb{P}(\xi\nu)) = \bar{\Pi}(\xi\nu, \nu)$ . ■

**Theorem 4.8.** Let  $\mathcal{D}$  be a BS and  $\Omega \in \bar{\Pi}_{cl}(\mathcal{D})$ . Suppose  $\bar{\Pi}$  is a mapping of  $\Omega \times \Omega$  into  $\mathcal{D}$  with  $\Omega \subseteq \bar{\Pi}(\nu, \Omega)$  for all  $\nu \in \Omega$  and

(1)  $\|\bar{\Pi}(\nu, v) \otimes' v\| \otimes \|\bar{\Pi}(\nu, v') \otimes' v'\| \geq h \|v \otimes' v'\|$  for all  $(\nu, v), (\nu, v')$  in the domain of  $\bar{\Pi}$ ,  $h > 2$ ;

(2) For each  $\nu \in \Omega$ ,  $\bar{\Pi}(\nu, v) = \bar{\Pi}(\nu, v')$  implies  $v = v'$ ;

(3)  $\|\bar{\Pi}(\nu, v) \otimes' \bar{\Pi}(\nu', v)\| \leq \|v \otimes' v'\|$  for all  $(\nu, v), (\nu', v) \in \Omega \times \Omega$ .

Thus, there is unique  $\nu \in \Omega$  such that  $\bar{\Pi}(\nu, \nu) = \nu$ .

**Proof.** For  $\nu \in \Omega$ , define a mapping  $H$  of  $\Omega$  into  $\mathcal{D}$  such that  $H(v) = \bar{\Pi}(\nu, v)$ . From (1)

$$\|H(v) \otimes' v\| \otimes \|H(v') \otimes' v'\| \geq h \|v \otimes' v'\|.$$

Also

$$\Omega \subseteq \bar{\mathbb{I}}(\nu, \Omega) = H(\Omega)$$

and, using (2),  $H$  is one to one function. Thus there is a unique  $\mathbb{P}\nu \in \Omega$  with  $\mathbb{P}\nu = \bar{\mathbb{I}}(\nu, \mathbb{P}\nu)$ . Now putting  $\nu = \mathbb{P}\nu$  and  $\nu' = \mathbb{P}\nu'$  in (1) we deduce

$$\begin{aligned} h \|\mathbb{P}\nu \otimes' \mathbb{P}\nu'\| &\leq \|\bar{\mathbb{I}}(\nu, \mathbb{P}\nu) \otimes' \mathbb{P}\nu\| \otimes \|\bar{\mathbb{I}}(\nu, \mathbb{P}\nu') \otimes' \mathbb{P}\nu'\| \\ &\leq \|\mathbb{P}\nu \otimes' \mathbb{P}\nu\| \otimes \|\bar{\mathbb{I}}(\nu, \mathbb{P}\nu') \otimes' \bar{\mathbb{I}}(\nu', \mathbb{P}\nu')\| \\ &= \|\bar{\mathbb{I}}(\nu, \mathbb{P}\nu') \otimes' \bar{\mathbb{I}}(\nu', \mathbb{P}\nu')\| \\ &\leq \|\nu \otimes' \nu'\| \end{aligned}$$

which shows that  $\mathbb{P}$  is contraction. Hence by similar arguments as in Theorem 4.7 there is unique  $\nu \in \Omega$  such that  $\bar{\mathbb{I}}(\nu, \nu) = \nu$ . ■

**Remark 4.5.** Theorems 4.1 and 4.2 are extensions of Theorem 2.1 in [128] (see also Theorem 2.1 in [130]).

**Remark 4.6.** Theorem 4.4 is generalization of Theorem 2.6 and Theorem 2.2 in [128].

### 4.3 Applications

Now we prove the ER for the solutions of Cauchy problem (A) given below, with suitable conditions on given functions. After that we propose a general class of integral equations and discuss the ESs. This integral equation presents many kinds of evolutions equations. We summarize the discussion in the form of theorems .

#### Problem 4.1

Consider the Cauchy problem

$$\begin{aligned} {}^{A\zeta C}D^\rho \nu(\sigma) &= A\nu(\sigma) \otimes \hbar(\sigma, \nu(\sigma)) \otimes \Phi(\sigma) \nu(\sigma) \\ \nu(0) &= \nu_0. \end{aligned} \tag{A}$$

The mild solution is given by;

$$\begin{aligned} \nu(\sigma) = & \mathbb{P}T_\rho(\sigma)\nu_0 \otimes \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho)\mathfrak{d}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \mathfrak{h}(\varrho, \nu(\varrho)) d\varrho \otimes \frac{\rho\mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \mathfrak{h}(\varrho, \nu(\varrho)) d\varrho \\ & \otimes \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho)\mathfrak{d}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \Phi(\varrho) \nu(\varrho) \otimes \frac{\rho\mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \Phi(\varrho) \nu(\varrho) d\varrho \end{aligned}$$

where

$$\mathbb{P} = \wp(\wp I \otimes' A)^{\otimes' 1}$$

and

$$K = \otimes' \wp A (\wp I \otimes' A)^{\otimes' 1}$$

with  $\wp = \frac{\zeta(\rho)}{1 \otimes' \rho}$  and

$$T_\rho(\sigma) = \mathbb{Y}_\rho(\otimes' K \sigma^\rho) = \frac{1}{2\pi i} \int_\Delta e^{\delta\sigma} \delta^{\rho \otimes' 1} (\delta^\rho I \otimes' K)^{\otimes' 1} d\delta,$$

$$S_\rho(\sigma) = \sigma^{\rho \otimes' 1} \mathbb{Y}_\rho(\otimes' K \sigma^\rho) = \frac{1}{2\pi i} \int_\Delta e^{\delta\sigma} (\delta^\rho I \otimes' K)^{\otimes' 1} d\delta,$$

here  $\Delta$  is a specific path lying on  $\mathbb{Y}_{(\beta, \varpi)}$  and  $\mathfrak{h} \in C(J \times \mathbb{Y}, \mathbb{Y})$ ,  $\Phi \in C(J, \mathbb{Y})$  where  $J = [0, T]$  for some  $T > 0$ , see [15]. If  $A \in A^\rho(\beta_0, \varpi_0)$ , thus  $\|T_\rho(\sigma)\| \leq \Omega e^{\varpi\sigma}$  and  $\|S_\rho(\sigma)\| \leq C_1 e^{\varpi\sigma} (1 \otimes \sigma^{\rho \otimes' 1})$  for every  $\sigma > 0$  and  $\varpi > \varpi_0$ . Set  $\Omega = \sup_{\sigma > 0} \|T_\rho(\sigma)\|$  and  $\Omega_1 = \sup_{\sigma > 0} C_1 e^{\varpi\sigma} (1 \otimes \sigma^{\rho \otimes' 1})$  then we get

$$\|T_\rho(\sigma)\| \leq \Omega \text{ and } \|S_\rho(\sigma)\| \leq \Omega_1.$$

Define  $\bar{\mathbb{J}}: \zeta_\delta \times \zeta_\delta \longrightarrow \mathcal{D}$  by

$$\begin{aligned}
\bar{\Pi}(\nu, v)(\sigma) &= \mathbb{P}T_\rho(\sigma) \nu_0 \otimes \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho) \mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \mathfrak{h}(\varrho, \nu(\varrho)) d\varrho \\
&\quad \otimes \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \mathfrak{h}(\varrho, \nu(\varrho)) d\varrho \otimes \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho) \mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \Phi(\varrho) v(\varrho) d\varrho \\
&\quad \otimes \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \Phi(\varrho) v(\varrho) d\varrho. \tag{B}
\end{aligned}$$

$$\text{set } \min \left\{ \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho) \mathfrak{b}(\rho)}, \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \right\} = K_{\mathbb{P}}.$$

Assume that the following conditions hold:

$$(H1). \left\| \int_0^\sigma \left\{ (\sigma \otimes' \varrho)^{\rho \otimes' 1} \otimes S_\rho(\sigma \otimes' \varrho) \right\} \Phi(\varrho) (v(\varrho) \otimes' \vartheta(\varrho)) d\varrho \right\| \geq K_S \|v \otimes' \vartheta\| \text{ for some } K_S > 0.$$

$$(H2). \|\mathfrak{h}(\varrho, \nu_1(\varrho)) \otimes' \mathfrak{h}(\varrho, \nu_2(\varrho))\| \leq L_h \|\nu_1 \otimes' \nu_2\| \text{ for some } L_h > 0.$$

$$(H3). \left( \frac{\|\mathbb{P}\|}{\zeta(\rho)} \left\| \frac{K(1 \otimes' \rho)}{\mathfrak{b}(\rho)} \right\| \frac{1}{\rho} \otimes \rho \|\mathbb{P}\| \Omega_1 \right) L_h \leq 1 \text{ and } 2 < K_{\mathbb{P}} K_S,$$

where  $\zeta_\delta$  is a closed ball with radius  $\delta$  and centre at 0 in  $\mathcal{D}$  such that  $\zeta_\delta \subseteq \bar{\Pi}(v, \zeta_\delta)$  for each  $v \in \zeta_\delta$ .

First we show that  $\bar{\Pi}$  satisfies (1) of Corollary 4.2,

$$\begin{aligned}
&c \|\bar{\Pi}(\nu, v) \otimes' \bar{\Pi}(\nu, \vartheta)\| \\
&= c \left\| \begin{aligned} &\frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho) \mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \Phi(\varrho) v(\varrho) d\varrho \otimes \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \Phi(\varrho) v(\varrho) d\varrho \\ &\otimes \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho) \mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \Phi(\varrho) \vartheta(\varrho) d\varrho \otimes \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \Phi(\varrho) \vartheta(\varrho) d\varrho \end{aligned} \right\| \\
&= c \left\| \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho) \mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \Phi(\varrho) (v(\varrho) \otimes' \vartheta(\varrho)) d\varrho \otimes \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \Phi(\varrho) (v(\varrho) \otimes' \vartheta(\varrho)) d\varrho \right\| \\
&\geq cK_{\mathbb{P}} \left\| \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \Phi(\varrho) (v(\varrho) \otimes' \vartheta(\varrho)) d\varrho \otimes \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \Phi(\varrho) (v(\varrho) \otimes' \vartheta(\varrho)) d\varrho \right\| \\
&= cK_{\mathbb{P}} \left\| \int_0^\sigma \left\{ (\sigma \otimes' \varrho)^{\rho \otimes' 1} \otimes S_\rho(\sigma \otimes' \varrho) \right\} \Phi(\varrho) (v(\varrho) \otimes' \vartheta(\varrho)) d\varrho \right\| \\
&\geq cK_{\mathbb{P}} K_S \|v \otimes' \vartheta\|
\end{aligned}$$

Now we see that  $\bar{\Pi}$  satisfies (2) of Corollary 4.2, consider

$$\begin{aligned}
& \left\| \bar{\Pi}(\nu_1, \nu) \otimes' \bar{\Pi}(\nu_2, \nu) \right\| \\
& \left\| \begin{aligned} & \mathbb{P}T_\rho(\sigma) \nu_0 \otimes \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho)\mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \hbar(\varrho, \nu_1(\varrho)) d\varrho \otimes \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \hbar(\varrho, \nu_1(\varrho)) d\varrho \\ & \otimes' \mathbb{P}T_\rho(\sigma) \nu_0 \otimes' \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho)\mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} \hbar(\varrho, \nu_2(\varrho)) d\varrho \otimes' \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) \hbar(\varrho, \nu_2(\varrho)) d\varrho \end{aligned} \right\| \\
& \leq \left\| \begin{aligned} & \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho)\mathfrak{b}(\rho)} \int_0^\sigma (\sigma \otimes' \varrho)^{\rho \otimes' 1} [\hbar(\varrho, \nu_1(\varrho)) \otimes' \hbar(\varrho, \nu_2(\varrho))] d\varrho \\ & \otimes' \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \int_0^\sigma S_\rho(\sigma \otimes' \varrho) [\hbar(\varrho, \nu_1(\varrho)) \otimes' \hbar(\varrho, \nu_2(\varrho))] d\varrho \end{aligned} \right\| \\
& \leq \left\| \frac{K\mathbb{P}(1 \otimes' \rho)}{\zeta(\rho)\mathfrak{b}(\rho)} \right\| \frac{1}{\rho} \|\hbar(\varrho, \nu_1(\varrho)) \otimes' \hbar(\varrho, \nu_2(\varrho))\| \\
& \quad \otimes \left\| \frac{\rho \mathbb{P}^2}{\zeta(\rho)} \right\| \Omega_1 \|\hbar(\varrho, \nu_1(\varrho)) \otimes' \hbar(\varrho, \nu_2(\varrho))\| \\
& \leq \left( \frac{\|\mathbb{P}\|}{\zeta(\rho)} \left\| \frac{K(1 \otimes' \rho)}{\mathfrak{b}(\rho)} \right\| \frac{1}{\rho} \otimes \rho \|\mathbb{P}\| \Omega_1 \right) \|\hbar(\varrho, \nu_1(\varrho)) \otimes' \hbar(\varrho, \nu_2(\varrho))\| \\
& \leq \left( \frac{\|\mathbb{P}\|}{\zeta(\rho)} \left\| \frac{K(1 \otimes' \rho)}{\mathfrak{b}(\rho)} \right\| \frac{1}{\rho} \otimes \rho \|\mathbb{P}\| \Omega_1 \right) L_h \|\nu_1 \otimes' \nu_2\|
\end{aligned}$$

since  $\left( \frac{\|\mathbb{P}\|}{\zeta(\rho)} \left\| \frac{K(1 \otimes' \rho)}{\mathfrak{b}(\rho)} \right\| \frac{1}{\rho} \otimes \rho \|\mathbb{P}\| \Omega_1 \right) L_h \leq 1$ , therefore  $\bar{\Pi}$  satisfies (2). From Corollary 4.2

there is a point  $\gamma \in \zeta_\delta$  such that  $\gamma = \bar{\Pi}(\gamma, \gamma)$ , which is the solution of our main problem (A).

### Problem 4.2

More generally, the above solution (A) can be written as

$$\nu(\sigma) = \zeta(\sigma) \nu_0 \otimes \lambda \int_0^\sigma h(\sigma) \hbar(\varrho, \nu(\varrho)) d\varrho \otimes \wp \int_0^\sigma k(\sigma) \hbar(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu(\sigma))$$

or

$$\nu(\sigma) = \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu(\sigma)) \quad (C)$$

Where  $h : C(J, \mathbb{R}) \rightarrow \mathbb{R}$ ,  $\zeta, k, \nu \in C(J, \mathbb{R})$  and  $\bar{h} \in C(J \times \mathbb{R}, \mathbb{R})$ . The integral equation (C) and the FP problem  $\nu = \bar{\Pi}(\nu, \nu)$  are equivalent, where

$$\bar{\Pi}(\nu, \nu) = \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu(\sigma)). \quad (D)$$

We assume the followings

(C1)  $h$  is injective and there exists  $\eta > 0$  such that

$$\|h(\nu) \otimes' h(\nu')\| \geq \eta \|\nu \otimes' \nu'\|, \text{ for all } \nu, \nu' \in \zeta_\delta.$$

(C2)

$$|h(\sigma, \nu(\sigma)) \otimes' h(\sigma, \nu'(\sigma))| \leq |\nu(\sigma) \otimes' \nu'(\sigma)| \text{ for all } \nu, \nu' \in \zeta_\delta \text{ and } \sigma \in J.$$

(C3)  $a \otimes b \otimes c < 2 \otimes \eta c$  and  $T(\wp \otimes \lambda) \Omega \leq 1$ , where  $\Omega = \max\{\|k\|, \|h\|\}$ .

We apply Theorem 4.2, to show that there exists  $\nu$  such that  $\nu = \bar{\Pi}(\nu, \nu)$ .

For this purpose, consider

$$\begin{aligned} |\bar{\Pi}(\nu, \nu) \otimes' \bar{\Pi}(\nu', \nu)| &= \left| \begin{array}{l} \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu(\sigma)) \\ \otimes' \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu'(\varrho)) d\varrho \otimes h(\nu'(\sigma)) \end{array} \right| \\ &\leq \left| \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes' \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu'(\varrho)) d\varrho \right| \\ &\leq (\wp \otimes \lambda) \Omega \int_0^\sigma |\bar{h}(\varrho, \nu(\varrho)) \otimes' \bar{h}(\varrho, \nu'(\varrho))| d\varrho, \end{aligned}$$

which implies using (C2)  $\otimes'$  (C3) that

$$\|\bar{\Pi}(\nu, \nu) \otimes' \bar{\Pi}(\nu', \nu)\| \leq T(\wp \otimes \lambda) \Omega \|\nu \otimes' \nu'\| \leq \|\nu \otimes' \nu'\|, \text{ since } T(\wp \otimes \lambda) \Omega \leq 1.$$

Hence the condition (2) of Theorem 4.2 is satisfied. Since  $h$  is injective,  $\bar{\Pi}(\nu, \nu) = \bar{\Pi}(\nu, \nu')$  implies  $\nu = \nu'$ , which is condition (1) of Theorem 4.2. Set

$$\min \left\{ \begin{array}{l} a \left\| \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu(\sigma)) \otimes' \nu(\sigma) \right\|, \\ b \left\| \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu'(\sigma)) \otimes' \nu'(\sigma) \right\| \end{array} \right\} \geq \|\nu \otimes' \nu'\|.$$

Also as

$$\begin{aligned} c \|\bar{\Pi}(\nu, \nu) \otimes' \bar{\Pi}(\nu, \nu')\| &= c \left\| \begin{array}{l} \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu(\sigma)) \\ \otimes' \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu'(\sigma)) \end{array} \right\| \\ &= c \|h(\nu(\sigma)) \otimes' h(\nu'(\sigma))\|, \end{aligned}$$

$$a \|\bar{\Pi}(\nu, \nu) \otimes' \nu\| = a \left\| \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu(\sigma)) \otimes' \nu(\sigma) \right\|,$$

and

$$b \|\bar{\Pi}(\nu, \nu') \otimes' \nu'\| = b \left\| \zeta(\sigma) \nu_0 \otimes \int_0^\sigma [\lambda h(\sigma) \otimes \wp k(\sigma)] \bar{h}(\varrho, \nu(\varrho)) d\varrho \otimes h(\nu'(\sigma)) \otimes' \nu'(\sigma) \right\|.$$

Then clearly

$$\begin{aligned} &c \|\bar{\Pi}(\nu, \nu) \otimes' \bar{\Pi}(\nu, \nu')\| \otimes a \|\bar{\Pi}(\nu, \nu) \otimes' \nu\| \otimes b \|\bar{\Pi}(\nu, \nu') \otimes' \nu'\| \\ &\geq 2 \|\nu \otimes' \nu'\| \otimes c \|h(\nu(\sigma)) \otimes' h(\nu'(\sigma))\| \\ &\geq (2 \otimes \eta c) \|\nu \otimes' \nu'\|. \end{aligned}$$

From Theorem 4.2 there is a point  $\nu \in \zeta_\delta$  with  $\nu = \bar{\Pi}(\nu, \nu)$ , which is solution of (C). We can get the following consequences.

**Theorem 4.9** Let  $\zeta, h, k, \hbar$  are as defined above, and  $\overline{\Pi}$  is defined in (D). If (C1), (C2) and (C3) are satisfied. Thus, there is a solution  $\nu \in C(J, \mathbb{R})$  of integral equation (C).

**Remark 4.7** Comparing equations ( $\zeta$ ) with (D), we can choose specific values of  $\zeta, h, k, \hbar$ , in this case, the above theorem also provide the ESs of the evolution equation (A).

In the following example, we will use above theorem and iterative scheme to find the approximate solution of given integral equation.

**Example 4.1** Consider

$$v(\iota) = \left( \iota \oplus e^{\otimes \iota} \right) \otimes' \iota \sin(v(\iota)) \otimes \int_0^\iota \frac{\sin(v(\iota)) v(\kappa)}{(\kappa \otimes e^{\otimes \kappa})} d\kappa,$$

a special kind of above integral equation (C) satisfying all conditions of above theorem for  $\iota \in [0, 1]$ . Here is the iterative sequence

$$v_{U \otimes 1}(\iota) = \left( \iota \otimes e^{\otimes \iota} \right) \otimes' \iota \sin(v_n(\iota)) \otimes \int_0^\iota \frac{\sin(v_n(\iota)) v_n(\kappa)}{(\kappa \otimes e^{\otimes \kappa})} d\kappa.$$

We take initial guess  $v_0(\iota) = 1 \otimes \frac{\iota^2}{2} \otimes' \frac{\iota^3}{6}$ , after two iterations we have the approximate solution  $v_2(\iota)$  and exact solution  $v(\iota) = \iota \otimes e^{\otimes \iota}$  given in Figure-1. The error is plotted in Figure-2.

## 4.4 Conclusion

We prove some new variants of Krasnoselskii's FP theorem for equiexpansive and generalized equiexpansive mappings. We consider the map  $\overline{\Pi}$  whose domain is either  $\xi(\Omega) \times \Omega$  or  $\Omega \times \Omega$  and range in a BS  $\mathcal{D}$ , where  $\xi$  is an operator from  $\Omega$  into  $\mathcal{D}$ . The operator equation  $\nu = \overline{\Pi}(\xi\nu, \nu)$  is solved with the assumption that  $\overline{\Pi}$  is either equiexpansive or generalized equiexpansive. We also apply the results to obtained the solutions of a general class of IEs which represents solutions of many evolution equations of fractional order.

## Chapter 5

# Fixed Points of Monotone Mappings Via Generalized-Measure of Non-compactness

### 5.1 Introduction

Condensing operators play key role in the existence theory of linear and nonlinear operators. For a given bounded  $\Omega \subseteq \mathcal{D}$ , Kuratowski in [79] defines the ball MNC  $\alpha$ , and Sadovskii [117] defines the Hausdorff MNC  $\beta$ , as follows.

$$\alpha(\Omega) = \inf \left\{ \lambda : \Omega \subset \bigcup_{i=0}^m \zeta_i, \delta(\zeta_i) \leq \lambda, m \in N \right\}.$$

Here  $\zeta_i$  are nonempty subsets of  $\mathcal{D}$ ,  $\lambda$  is positive and  $\delta$  denotes the diameter of a set, and

$$\beta(\Omega) = \inf \left\{ \lambda : \Omega \subset \bigcup_{i=0}^m B(\kappa_i, \lambda) \text{ for some } \kappa_i \in M, m \in N \right\},$$

where  $B(\kappa_i, \lambda)$  are balls which are open.

For a given BS  $\mathcal{D}$ , let us denote,  $\bar{\mathcal{I}}_b(\mathcal{D})$ ,  $\bar{\mathcal{I}}_{cv}(\mathcal{D})$ ,  $\bar{\mathcal{I}}_c(\mathcal{D})$ ,  $\bar{\mathcal{I}}_{b,cv}(\mathcal{D})$ , the set of all nonempty bounded, nonempty convex, nonempty closed and nonempty bounded-convex subsets of  $\mathcal{D}$  respectively. The above concepts are generalized in [16, 26, 27, 25, 60, 61, 112, 123]. Recently in

[51], the author introduced a generalized MNC for monotone operators in ordered BSs, while in [68], authors introduced cone MNC and proved some FP results.

Here, we introduce partial  $\mathbb{Y}$ -MNC in the settings of ordered  $\mathbb{Y}$ -MSs. We define partially continuous mappings, partially compact, partially bounded subsets and partially closed sets in ordered  $\mathbb{Y}$ -MSs. We prove FP results for partially continuous, partially bounded monotone mappings and sum of two monotone mappings and generalize the well known results of Darbo [43] and Krasnoselskii [76]. We prove the results for a class of IEs and provide some nontrivial examples.

## 5.2 Preliminaries

Here, we recall  $\mathbb{Y}$ -MSs [19], and also introduce some new notions in ordered  $\mathbb{Y}$ -MSs.

**Definition 5.1.** [19] Let  $\mathbb{Y}$  be a VS and  $\preceq$  be a partial order relation,  $\mathbb{Y}$  is ordered space if  
(O1)  $\forall \kappa, v$ , and  $\omega \in \mathbb{Y}$ ,  $\kappa \preceq v$  implies  $\kappa \oplus \omega \preceq v \oplus \omega$ ,  
(O2)  $\forall a \in \mathbb{R}^{\otimes}$  and  $\kappa \in \mathbb{Y}$  such that  $\kappa \succeq 0_{\mathbb{Y}}$ , implies that  $a\kappa \succeq 0_{\mathbb{Y}}$ .

If  $\mathbb{Y}$  is normed space such that

(O3) there is  $\lambda > 0$  and  $\forall \kappa, v \in \mathbb{Y}$ , with  $0_{\mathbb{Y}} \preceq \kappa \preceq v$  implies  $\|\kappa\|_{\mathbb{Y}} \leq \lambda \|v\|_{\mathbb{Y}}$ , in this case  $\mathbb{Y}$  is known as a normed ordered space.

Let  $\mathbb{Y}$  is ordered space. The element  $v$  in  $\mathbb{Y}$  is positive if  $v \succeq 0_{\mathbb{Y}}$ . The set of elements of  $\mathbb{Y}$  which are positive are denoted by  $\mathbb{Y}^{\otimes}$ , ( $\mathbb{Y}^{\otimes}$  is also known as a cone of  $\mathbb{Y}$ ). Here  $x \succ y$  mean  $x \succeq y$  but  $x \neq y$ .

According to [19], we define  $\mathbb{Y}$ -MS as follows.

**Definition 5.2.** Let  $\mathbb{Y}$  be a real ordered VS, and  $\mathcal{D} \neq \phi$  is a set. An  $\mathbb{Y}$ -valued function  $d^{\mathbb{Y}} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{Y}$  is  $\mathbb{Y}$ -metric on  $\mathcal{D}$ , if for all  $\kappa, v$  and  $\omega \in \mathcal{D}$ , the following hold:

- ( $\mathbb{Y}$ 1)  $d^{\mathbb{Y}}(\kappa, v) \succeq 0_{\mathbb{Y}}$  and  $d^{\mathbb{Y}}(\kappa, v) = 0_{\mathbb{Y}} \iff \kappa = v$ ,
- ( $\mathbb{Y}$ 2)  $d^{\mathbb{Y}}(\kappa, v) = d^{\mathbb{Y}}(v, \kappa)$ ,
- ( $\mathbb{Y}$ 3)  $d^{\mathbb{Y}}(\kappa, v) \preceq d^{\mathbb{Y}}(\kappa, \omega) \oplus d^{\mathbb{Y}}(\omega, v)$ .

Then  $(\mathcal{D}, d^{\mathbb{Y}})$  is an  $\mathbb{Y}$ -MS.

**Example 5.1.** Let  $\mathbb{Y} = C[0, 1]$ , then  $\text{int}(\mathbb{Y}^{\otimes}) = \{\hbar \in \mathbb{Y}^{\otimes} : \hbar(\kappa) \neq 0 \text{ for all } \kappa\}$  and  $\mathcal{D} = C[0, 1]$ , define  $d^{\mathbb{Y}} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbb{Y}$  by  $d^{\mathbb{Y}}(\hbar, h) = |\hbar \otimes' h|$ . Then  $d^{\mathbb{Y}}$  is an  $\mathbb{Y}$ -MS.

For more details and properties of  $\mathbb{Y}$ -MSs, we refer [19, 66, 86].

We assume that  $\text{int}(\mathbb{Y}^*)$  is nonempty unless otherwise stated. For  $x \in \text{int}(\mathbb{Y}^*)$  we denote it by  $x \gg 0_{\mathbb{Y}}$ . The following definitions will be crucial for our main results.

**Definition 5.3.** Let  $(\mathcal{D}, d_{\mathbb{Y}}, \leq)$  be an ordered  $\mathbb{Y}$ -MS, that is,  $(\mathcal{D}, d_{\mathbb{Y}})$  is an  $\mathbb{Y}$ -MS and  $\leq$  is a partial order on  $\mathcal{D}$ , let  $\xi : \mathcal{D} \rightarrow \mathcal{D}$  be a mapping then,  $\xi$  is called monotone increasing (monotone non-increasing) if for  $\varkappa \leq v$  implies  $\xi\varkappa \leq \xi v$  (resp,  $\xi x \geq \xi v$ ) for all  $\varkappa, v \in \mathcal{D}$ .

The following notions are more general than the similar notions defined in [51].

**Note:** There are two partial orders, the first one " $\preceq$ " on the BS  $\mathbb{Y}$  and the other one " $\leq$ " on the nonempty set  $\mathcal{D}$ .

**Definition 5.4.** Let  $(\mathcal{D}, d_{\mathbb{Y}}, \leq)$  be an ordered  $\mathbb{Y}$ -MS. A mapping  $\xi : \mathcal{D} \rightarrow \mathcal{D}$  is partially continuous (PC), at  $a \in \mathcal{D}$ , if for  $\epsilon \gg 0_{\mathbb{Y}}$  there is  $\delta \gg 0_{\mathbb{Y}}$  such that for  $x \in \mathcal{D}$  satisfying  $d(\varkappa, a) \prec \delta$ , with  $\varkappa \leq a$ , implies  $d(\xi\varkappa, \xi a) \prec \epsilon$ .

**Definition 5.5** For  $\varkappa \in \mathcal{D}$  and  $r \gg 0_{\mathbb{Y}}$ , we define partially ordered open ball (POB).

$$B(\varkappa, r, \leq) = \{v \in \mathcal{D} : v \leq \varkappa, \text{ and } d(\varkappa, v) \prec r\},$$

we denote the partially closed ball by  $B[\varkappa, r, \leq]$  and define in similar way.

**Definition 5.6.** A subset  $\Lambda$  of an  $\mathbb{Y}$ -MS  $\mathcal{D}$  is called partially bounded (PB) if there exists  $\lambda \gg 0_{\mathbb{Y}}$  such that  $d(\varkappa, v) \preceq \lambda$  for  $\varkappa, v \in \Lambda$  with  $\varkappa \leq v$ .  $\Lambda$  is said to be partially compact (PCP) if there exists finite number of POB that covers  $\Lambda^*$  where  $\Lambda^*$  is the set of all chains in  $\Lambda$ , and  $\Lambda$  is partially totally bounded if  $\Lambda^*$  can be covered by a finitely many POB with fixed radius  $r \gg 0_{\mathbb{Y}}$ .

**Definition 5.7.** A subset  $\Lambda$  of an  $\mathbb{Y}$ -MS  $\mathcal{D}$  is called partially precompact (PPRC), if  $\tilde{\Lambda} = \Lambda^* \cup \Lambda^\delta$  is partially compact, where  $\Lambda^\delta$  is the set of all limits of convergent chains in  $\Lambda$ , and  $\tilde{\Lambda}$  is known as the partial-closure (p-closure) of  $\Lambda$ .  $\Lambda$  is partially closed iff  $\tilde{\Lambda} \subseteq \Lambda^*$ .

**Remark 5.1** (a) Every continuous function is partially continuous.

(b) Every bounded set is partially bounded.

**Example 5.2** Let  $\mathcal{D} = \mathbb{R}$  with usual metric

$$d(\varkappa, v) = |\varkappa \otimes' v|$$

for all  $\varkappa \leq v$ , let  $\leq$  be usual order of reals and we say  $\varkappa \leq v$  iff  $\varkappa, v \in \mathcal{D}$  and  $\varkappa \leq v$ . Since  $\mathcal{D}$  is closed in  $\mathbb{R}$ , so  $\mathcal{D}$  is partially complete. The function  $\varpi : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\varpi(\varkappa) = \begin{cases} \varkappa \oplus 1 & \text{if } \varkappa \neq \frac{1}{2} \\ 0 & \text{if } \varkappa = \frac{1}{2} \end{cases}$$

is discontinuous but partially continuous on  $\mathbb{R}$  with partial order  $\leq$ .

### 5.3 Main results

Let  $\overline{\mathcal{D}}_{pb}(\mathcal{D})$  be the family in which there are nonempty PB subsets of an ordered  $\mathbb{Y}$ -MS  $(\mathcal{D}, d_{\mathbb{Y}}, \leq)$ . We define partial  $\mathbb{Y}$ -MNC as follows.

**Definition 5.8.** Let  $(\mathcal{D}, d_{\mathbb{Y}}, \leq)$  be ordered  $\mathbb{Y}$ -MS. The mapping  $\alpha_{\mathbb{Y}} : \overline{\mathcal{D}}_{pb}(\mathcal{D}) \rightarrow \mathbb{Y}^*$  is called partial  $\mathbb{Y}$ -MNC if for any chain  $\Lambda, \Omega \in \overline{\mathcal{D}}_{pb}(\mathcal{D})$ .

(M1)  $\alpha_{\mathbb{Y}}(\Lambda) = 0_{\mathbb{Y}}$  implies  $\Lambda$  is p-precompact, or  $\widetilde{\Lambda}$  is partially compact,

(M2)  $\Lambda \subseteq \Omega$  implies  $\alpha_{\mathbb{Y}}(\Lambda) \preceq \alpha_{\mathbb{Y}}(\Omega)$ ,

(M3) for any  $\Lambda \in \overline{\mathcal{D}}_{pb}(\mathcal{D})$ ,  $\alpha_{\mathbb{Y}}(\Lambda) = \alpha_{\mathbb{Y}}(\widetilde{\Lambda}) = \alpha_{\mathbb{Y}}(\{\varkappa\} \cup (\Lambda))$  for  $\varkappa \in \mathcal{D}$ ,

(M4) and for any decreasing sequence  $\{\Lambda_n\}$  in  $\overline{\mathcal{D}}_{pb}(\mathbb{Y})$  such that  $\lim_{n \rightarrow \infty} \alpha_{\mathbb{Y}}(\Lambda_n) = 0_{\mathbb{Y}}$ , implies that  $\Lambda_{\infty} = \bigcap_{n=1}^{\infty} \widetilde{\Lambda}_n$  is nonempty.

Our definition is more general than the definitions defined in [16, 25, 43, 51, 60, 68, 117], and observe that our structure is also more general. The next definition of  $k$ -chain contraction is crucial for the proof of our main results.

**Example 5.3.** Let  $(\mathcal{D}, \|\cdot\|, \leq)$  be an ordered BS. Let  $\alpha$  be a given Kuratowski MNC. For any convex and bounded subset  $\Omega$  of  $\mathcal{D}$ , define

$$\alpha_{\leq}(\Omega) = \begin{cases} \alpha(\Omega) & : \Omega \text{ is a chain in } \mathcal{D} \\ \lambda, & \text{otherwise} \end{cases},$$

where  $\lambda > 0$  be any given fixed real number. Then  $\alpha_{\leq}$  is a partial MNC which is not a Kuratowski MNC. If  $\Omega$  is compact subset of  $\mathcal{D}$  which not a chain (with respect to order  $\leq$  induced by a cone of  $\mathcal{D}$ ), we have  $\alpha_{\leq}(\Omega) = \lambda > 0$ , but if  $\Omega$  is a bounded chain itself which is compact, we get  $\alpha_{\leq}(\Omega) = 0$ , as desired.

**Definition 5.9.** Let  $(\mathcal{D}, d_Y, \leq)$  be ordered  $\mathbb{Y}$ -MS, a monotone mapping  $\xi : \mathcal{D} \longrightarrow \mathcal{D}$  is  $k$ -chain contraction if there exists  $k \in (0, 1)$  with partial  $\mathbb{Y}$ -MNC  $\alpha_Y$  such that

$$\alpha_Y(\xi(C)) \preceq k\alpha_Y(C)$$

for each bounded chain  $C \in \mathcal{D}$ .

**Theorem 5.1.** Let  $(\mathcal{D}, d_Y, \leq)$  be a complete ordered  $\mathbb{Y}$ -MS and  $\xi : \mathcal{D} \longrightarrow \mathcal{D}$  be a monotone PC and PB  $k$ -chain contraction. If there exists  $\varkappa_0 \in \mathbb{Y}$  such that  $\varkappa_0 \leq \xi\varkappa_0$ , then  $\xi$  has a FP.

**Proof.** Consider a sequence

$$\varkappa_{n \oplus 1} = \xi\varkappa_n, \text{ for } n \in \mathbb{N}.$$

Here  $\xi$  is monotone and  $\varkappa_0 \leq \xi\varkappa_0$ , so we have

$$\varkappa_0 \leq \xi\varkappa_0 \leq \xi^2\varkappa_0 \leq \dots \leq \xi^n\varkappa_0 \dots$$

or

$$\varkappa_0 \leq \varkappa_1 \leq \varkappa_2 \dots \varkappa_{n \oplus 1} \leq \varkappa_n \leq \dots$$

Now define

$$\Lambda_n = \{\varkappa_n, \varkappa_{n \oplus 1}, \varkappa_{n \oplus 2}, \dots\}$$

for  $n = 0, 1, 2, 3, \dots$ , clearly

$$\Lambda_{n \oplus 1} = \xi(\Lambda_n), \text{ for } n \in \mathbb{N}.$$

Consider,

$$\begin{aligned} \Lambda_0 &= \{\varkappa_0, \varkappa_1, \varkappa_2, \varkappa_3, \dots\} \\ &= \{\varkappa_0\} \cup \{\varkappa_1, \varkappa_2, \varkappa_3, \dots\} \\ &= \{\varkappa_0\} \cup \Lambda_1 \\ &= \{\varkappa_0\} \cup \xi(\Lambda_0) \end{aligned}$$

and

$$\begin{aligned}
\Lambda_1 &= \{\varkappa_1, \varkappa_2, \varkappa_3, \dots\} \\
&= \{\varkappa_1\} \cup \{\varkappa_2, \varkappa_3, \varkappa_4, \dots\} \\
&= \{\varkappa_1\} \cup \xi(\Lambda_1),
\end{aligned}$$

continuing,

$$\Lambda_n = \{\varkappa_n\} \cup \xi(\Lambda_n), \text{ for } n \in \mathbb{N}.$$

Also

$$\dots \Lambda_{n \oplus 1} \subset \Lambda_n \subset \dots \Lambda_1 \subset \Lambda_0,$$

therefore  $\Lambda_n$  is a decreasing. Consider

$$\begin{aligned}
\alpha_Y(\Lambda_n) &= \alpha_Y(\{\varkappa_n\} \cup \xi(\Lambda_n)) \\
&= \alpha_Y(\xi(\Lambda_n)) \\
&\preceq k\alpha_Y(\Lambda_{n \oplus 1}) \\
&\preceq k^2\alpha_Y(\Lambda_{n \oplus 2}) \dots \preceq k^n\alpha_Y(\Lambda_0).
\end{aligned}$$

As  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \alpha_Y(\Lambda_n) = 0_Y,$$

which implies that

$$\Lambda_\infty = \bigcap_{n=1}^{\infty} \widetilde{\Lambda}_n \neq \phi,$$

and for any  $n \in \mathbb{N}$ ,

$$\alpha_Y(\Lambda_\infty) \preceq \lim_{n \rightarrow \infty} \alpha_Y(\Lambda_n) = 0_Y,$$

therefore  $\Lambda_\infty$  is partially compact, also  $\xi(\Lambda_\infty) \subseteq \Lambda_\infty$ . As  $\lim_{n \rightarrow \infty} \varkappa_n \in \Lambda_\infty$ , there is  $k \in \mathbb{N}$  with  $\varkappa_n \in \Lambda_\infty$  for  $n \geq k$ . Since  $\Lambda_\infty$  is PCP so there is convergent sub-chain  $\{\varkappa_{n_m}\}_{m \geq k}$  of  $\{\varkappa_n\}_{n \geq k}$  in  $\Lambda_\infty$  such that

$$\lim_{m \rightarrow \infty} \varkappa_{n_m} = \varkappa,$$

and

$$\varkappa_{n,m} \preceq \varkappa \in \Lambda_\infty.$$

Since  $\xi$  is PC, it is continuous on every chain of  $\mathcal{D}$ , so we have

$$\varkappa = \xi\varkappa.$$

This proves the theorem. ■

If we assume in the above theorem an ordered BS  $\mathcal{D}$ , instead of ordered  $\mathbb{Y}$ -MS  $\mathcal{D}$ , then the following result is obtained.

**Theorem 5.2.** Let  $(\mathcal{D}, \|\cdot\|, \leq)$  be ordered BS and  $C \in \overline{\Pi}_{cl,b}(\mathcal{D})$ . Let  $\xi : C \rightarrow C$  be a monotone PC and PB  $k$ -chain contraction. If there exists  $\varkappa_0 \in C$  with  $\varkappa_0 \leq \xi\varkappa_0$ , then  $\xi$  has a FP in  $C$ .

**Proof.** Similar to the proof of above theorem. ■

**Corollary 5.1.** Let  $(\mathcal{D}, d_Y, \leq)$  be a complete ordered  $\mathbb{Y}$ -MS and  $\xi : \mathcal{D} \rightarrow \mathcal{D}$  be a monotone continuous and bounded  $k$ -chain contraction. If there exists  $\varkappa_0 \in \mathbb{Y}$  such that  $\varkappa_0 \leq \xi\varkappa_0$ , then  $\xi$  has a FP.

**Corollary 5.2.** Let  $(\mathcal{D}, d_Y, \leq)$  be a complete ordered MS and  $\xi : \mathcal{D} \rightarrow \mathcal{D}$  be a monotone continuous and bounded  $k$ -chain contraction. If there exists  $\varkappa_0 \in \mathbb{Y}$  such that  $\varkappa_0 \leq \xi\varkappa_0$ , then  $\xi$  has a FP.

**Remark 5.2.** Every Hausdorff MNC, ball MNC, cone MNC and MNC defined in [25, 60, 68, 117] is partial  $\mathbb{Y}$ -MNC.

By virtue of above Remark 5.2, and Theorem 5.1., we generalized the main results of [25, 40, 39, 60, 71, 117, 68, 111].

The next theorem generalizes the Krasnoselskii's FPT in the settings of monotone operators in ordered BSs.

**Theorem 5.3** Let  $(\mathcal{D}, \|\cdot\|, \leq)$  be ordered BS and  $\xi, \zeta : \mathcal{D} \rightarrow \mathcal{D}$  be two monotone PB operators such that,

- (a)  $\xi$  is PC and  $k$ -chain contraction;
- (b)  $\zeta$  is PC and PCP;

If there exists  $\varkappa_0 \in \mathcal{D}$  such that  $\varkappa_0 \leq \xi\varkappa_0 \oplus \zeta\varkappa_0$ , then there exists  $\omega \in \mathcal{D}$ , such that  $\omega = \xi\omega \oplus \zeta\omega$ .

**Proof.** Define a mapping  $\mathbb{P}: \mathcal{D} \rightarrow \mathcal{D}$  such that,

$$\mathbb{P}(\varkappa) = \xi \varkappa \otimes \zeta \varkappa \text{ for } \varkappa \in \mathcal{D},$$

since  $\mathbb{P}(\mathcal{D}) \subseteq \xi(\mathcal{D}) \otimes \zeta(\mathcal{D})$ , then  $\mathbb{P}$  is PB, also being sum of two partially continuous mappings,  $\mathbb{P}$  is partially continuous. Note that  $\varkappa_0 \leq \mathbb{P}(\varkappa_0)$ , and now consider for any bounded chain  $C$  of  $\mathcal{D}$ ,

$$\begin{aligned} \sigma_{\mathbb{Y}}(\mathbb{P}(C)) &\preceq \sigma_{\mathbb{Y}}(\zeta(C)) \otimes \sigma_{\mathbb{Y}}(\xi(C)) \\ &\preceq \sigma_{\mathbb{Y}}(\xi(C)), \text{ since } \zeta \text{ is PCP,} \\ &\preceq k\sigma_{\mathbb{Y}}(C), \end{aligned}$$

which implies that  $\mathbb{P}$  is a  $k$ -chain contraction. All conditions of Theorem 5.2., are satisfied to obtain the FP of  $\mathbb{P}$ . ■

**Corollary 5.3.** Let  $(\mathcal{D}, \|\cdot\|, \leq)$  be a BS with partial order  $\leq$ . Let  $\xi, \zeta : \mathcal{D} \rightarrow \mathcal{D}$  be two monotone bounded operators satisfying,

- (a)  $\xi$  is continuous and  $k$ -chain contraction;
- (b)  $\zeta$  is continuous and compact;

If there exists  $\varkappa_0 \in \mathcal{D}$  with  $\varkappa_0 \leq \xi \varkappa_0 \otimes \zeta \varkappa_0$ , then  $\zeta \varkappa \otimes \xi \varkappa = \varkappa$  has a solution.

**Corollary 5.4.** Let  $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}, \leq)$  be a BS with partial order  $\leq$ . Let  $\xi, \zeta : \mathcal{D} \rightarrow \mathcal{D}$  be two bounded operators with,

- (a)  $\xi$  is continuous and  $k$ -contraction;
- (b)  $\zeta$  is continuous and compact;

then  $\zeta \varkappa \otimes \xi \varkappa = \varkappa$  has a solution.

**Corollary 5.5.**  $(\mathcal{D}, \|\cdot\|_{\mathcal{D}}, \leq)$  be a BS with partial order  $\leq$ . Let  $\xi, \zeta : \mathcal{D} \rightarrow \mathcal{D}$  be two bounded operators satisfying,

- (a)  $\xi$  is continuous and KSC;
- (b)  $\zeta$  is continuous and compact;

then  $\zeta \varkappa \otimes \xi \varkappa = \varkappa$  has a solution.

Using Remark 5.2, and Theorem 5.3, the following result is obtained, which is a well known result of Krasnoselskii [76].

**Corollary 5.6.** [76] For a given  $\Omega$  which closed and convex subset of BS  $\mathcal{D}$ , assume that  $\zeta, \xi : \Omega \longrightarrow \mathcal{D}$  are the mappings satisfying

- (i)  $\zeta \varkappa \otimes \xi v \in \Omega$  for all  $\varkappa, v \in \Omega$ ,
- (ii)  $\zeta$  is continuous and  $\zeta(\Omega) \subset C \in \overline{\mathbb{D}}_{cp}(\mathcal{D})$ ,
- (iii)  $\xi$  is a Banach contraction, with  $\alpha < 1$ .

Then  $\zeta \varkappa \otimes \xi \varkappa = \varkappa$  has a solution.

## 5.4 Applications

The applications of presented theory have a broad spectrum, like existence theory of linear and nonlinear operators, optimization theory, game theory and Nash equilibria. We apply our results to find the ES of the following IE.

$$\varkappa(\nu) = \varpi(\nu, \varkappa(\theta(\nu))) \otimes \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) d\kappa \quad (4.1)$$

for  $\nu \in \mathbb{R}^{\otimes}$ . Where

- (i)  $\varpi : \mathbb{R}^{\otimes} \times \mathbb{R} \longrightarrow \mathbb{R}$  and  $\varrho : \mathbb{R}^{\otimes} \times \mathbb{R}^{\otimes} \times \mathbb{R} \longrightarrow \mathbb{R}$  are continuous, and
- (ii)  $\theta, \sigma, \gamma : \mathbb{R}^{\otimes} \longrightarrow \mathbb{R}^{\otimes}$ .

Consider  $B_C(\mathbb{R}^{\otimes}, \mathbb{R})$ , the normed linear space of all bounded real valued functions defined on the set of positive real numbers with supremum norm  $\|\cdot\|_{\mathbb{R}}$ . For any  $\varkappa, v \in B_C(\mathbb{R}^{\otimes}, \mathbb{R})$ , we define a partial order  $\leq$  by  $\varkappa \leq v$  iff  $\varkappa(\nu) \leq v(\nu)$  for all  $\nu > 0$ . The space  $(B_C(\mathbb{R}^{\otimes}, \mathbb{R}), \|\cdot\|_{\mathbb{R}}, \leq)$  is a regular ordered BS, for details see [51].

We define a partial  $\forall$ -MNC as follows.

For a bounded chain  $\Omega$  in  $B_C(\mathbb{R}^{\otimes}, \mathbb{R})$ , define

$$\Omega(\nu) = \{\varkappa(\nu) : v > 0, \varkappa \in \Omega\}$$

and

$$\sigma_{\forall}(\Omega) = \limsup_{\nu \rightarrow \infty} \delta(\Omega(\nu)),$$

where  $\delta(\Omega(\nu)) = \sup\{|\varkappa(\nu) \otimes' v(\nu)| : \varkappa, v \in \Omega\}$ , then clearly  $\sigma_{\forall}$  is a partial  $\forall$ -MNC. Assume

(A1)  $\alpha, \sigma, \gamma$  are continuous and  $\theta(\nu) \geq \nu$  for all  $\nu \in \mathbb{R}^{\otimes}$ .

(A2)  $0 \leq \varpi(\nu, \varkappa)$  for all  $(\nu, \varkappa) \in \mathbb{R}^{\otimes} \times \mathbb{R}$ , and there exists  $L > 0$  with

$$0 \leq \varpi(\nu, \varkappa) \otimes' \varpi(\nu, \nu) \leq L(\varkappa \otimes' \nu),$$

for all  $\varkappa \geq \nu$ .

(A3)  $\varpi(\nu, \varkappa)$  and  $\varrho(\nu, \kappa, \varkappa)$  are increasing in  $\varkappa$  for all  $\nu, \kappa \in \mathbb{R}^{\otimes}$ .

(A4) for some  $\hat{u} \in B_C(\mathbb{R}^{\otimes}, \mathbb{R})$ ,

$$\hat{u}(\nu) \leq \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) d\kappa$$

for all  $\nu \in \mathbb{R}^{\otimes}$ .

(A5) There exists functions  $a, b : \mathbb{R}^{\otimes} \rightarrow \mathbb{R}^{\otimes}$  such that  $\varrho$  is continuous with

$$|\varrho(\nu, \kappa, \varkappa)| \leq a(\nu) b(\kappa), \text{ for all } \kappa, \nu \in \mathbb{R}^{\otimes},$$

moreover the  $\lim_{\nu \rightarrow \infty} a(\nu) \int_0^{\sigma(\nu)} b(\kappa) d\kappa = 0$ .

(A6)  $|\varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) \otimes' \varrho(\nu, \kappa, \nu(\gamma(\kappa)))| \leq |\varkappa(\gamma(\kappa)) \otimes' \nu(\gamma(\kappa))|$  for all  $\kappa$ .

**Theorem 5.4.** Assume, (A1), (A3)  $\otimes'$  (A6) holds. Then IE

$$\varkappa(\nu) = \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) d\kappa$$

has a solution  $\hat{u}^*$  in  $B_C(\mathbb{R}^{\otimes}, \mathbb{R})$ .

The successive approximations for solution are defined by

$$\varkappa_n(\nu) = \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa_{n \otimes' 1}(\gamma(\kappa))) d\kappa$$

with  $\varkappa_0 = \hat{u}$  converges to  $\hat{u}^*$ .

**Proof.** We define

$$\xi \varkappa = \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) d\kappa,$$

then  $\xi$  is monotone, for this consider for  $\varkappa \leq v$ ,

$$\begin{aligned} \xi \varkappa(\nu) &= \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) d\kappa \\ &\leq \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, v(\gamma(\kappa))) d\kappa, \text{ using (A3),} \\ &= \xi v(\nu), \end{aligned}$$

for all  $\nu \in \mathbb{R}^{\otimes}$ . So  $\xi$  is monotone. Note that from (A4), there is  $\hat{u} \in B_C(\mathbb{R}^{\otimes}, \mathbb{R})$  with

$$\hat{u}(\nu) \leq \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) d\kappa$$

for all  $\nu \in \mathbb{R}^{\otimes}$ . Clearly using (A5),  $\xi$  is partially bounded. Since  $\varrho$  is continuous so  $\xi$  is also partially continuous. Now we show that  $\xi$  is  $k$ -chain contraction, let for  $\varkappa \geq v$ , in a bounded chain  $\Omega$ , consider

$$\begin{aligned} |\xi \varkappa(\nu) \otimes' \xi v(\nu)| &= \left| \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa(\gamma(\kappa))) d\kappa \otimes' \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, v(\gamma(\kappa))) d\kappa \right| \\ &\leq \int_0^{\sigma(\nu)} |\varkappa(\gamma(\kappa)) \otimes' v(\gamma(\kappa))| d\kappa \\ &\leq \sigma(\nu) \delta(\Omega(\nu)), \end{aligned}$$

which implies

$$\delta(\xi(\Omega(\nu))) \leq k \delta(\Omega(\nu)),$$

for all  $\nu \in \mathbb{R}^{\otimes}$ . Taking lim sup over  $\nu \rightarrow \infty$ , we have

$$\sigma_{\Upsilon}(\xi(\Omega)) \leq k \sigma_{\Upsilon}(\Omega),$$

with  $k = \sup \{ \sigma(\nu) : \nu > 0 \} < 1$ . Thus, from Theorem 5.2, FP of  $\xi$  is obtained. This proves the theorem. ■

Now we state the ER for the solution of our main integral equation (4.1).

**Theorem 5.5.** Assume that the hypothesis  $(A1) \otimes' (A6)$  holds. Then IE (4.1) has a solution  $\hat{u}^*$  in  $B_C(\mathbb{R}^{\otimes}, \mathbb{R})$ .

The successive approximations for solution are given

$$\varkappa_n(\nu) = \varpi(\nu, \varkappa_{n \otimes' 1}(\theta(\nu))) \otimes \int_0^{\sigma(\nu)} \varrho(\nu, \kappa, \varkappa_{n \otimes' 1}(\gamma(\kappa))) d\kappa$$

with  $\varkappa_0 = \hat{u}$  converges to  $\hat{u}^*$ .

**Proof.** Same as to the proof of the above theorem, using (A2) for function  $\varpi$ . ■

## 5.5 Conclusion

In this study a general criterion is established to deal with monotone operators and sum of two monotone operators to find their FPs. The notions of partially continuous mappings, partially bounded and partially compact sets are introduced, these notions are used to define partial  $\forall$ -MNC.

## Chapter 6

# Fuzzy Measure of Non-compactness with Applications

### 6.1 Introduction

In this chapter, unless otherwise stated, let  $\mathcal{D}$  be a BS over a field  $F$  and  $\bar{\mathcal{N}}_n(\mathcal{D})$ ,  $\bar{\mathcal{N}}_{cl}(\mathcal{D})$ ,  $\bar{\mathcal{N}}_{cv}(\mathcal{D})$ ,  $\bar{\mathcal{N}}_{prc}(\mathcal{D})$ ,  $\bar{\mathcal{N}}_{cp}(\mathcal{D})$ ,  $\bar{\mathcal{N}}_b(\mathcal{D})$  be the set of all nonempty, nonempty closed, nonempty convex, nonempty precompact, nonempty compact and nonempty bounded subsets of  $\mathcal{D}$  respectively. One can combine any of these sub-notations, for example  $\bar{\mathcal{N}}_{cl,b}(\mathcal{D})$  the set of all nonempty closed and bounded subsets of  $\mathcal{D}$ .

The lower and upper semi-continuity of multivalued operators is very important, and is used in many important results to ensure the existence of FPs. For example, see Kakutani-Fan [69] result to find the FPs of a upper semi-continuous multivalued operator. Also noncompact operators (condensing and KSCs) have gained importance in FP analysis.

Keeping in mind the importance of these concepts, we introduce fuzzy-MNC, fuzzy condensing and fuzzy KSCs and generalize Mönch [92], Darbo [43], Kakutani [58] and Bohnentblust-Karlin [30] FP theorems, indeed many results in the literature.

### 6.2 Preliminaries

The following definitions will play a crucial role in all coming results.

**Definition 6.1.** A fuzzy set  $\wp_F : \overline{\mathbb{I}}_b(\mathcal{D}) \times [0, \infty) \longrightarrow [0, 1]$  is called fuzzy MNC (FMNC) if for  $A, B \in \overline{\mathbb{I}}_b(\mathcal{D})$  and  $\iota > 0$ , the following conditions are satisfied;

- (1)  $\wp_F(A, 0) = 0$ ,
- (2) The family  $\{A \in \overline{\mathbb{I}}_b(\mathcal{D}) : \wp_F(A, \iota) = 1\}$  is nonempty and contained in  $\overline{\mathbb{I}}_{prc}(\mathcal{D})$ ,
- (3)  $\wp_F(A, \iota) \geq \wp_F(B, \iota)$  for  $A \subseteq B$ , and  $\iota > 0$ ,
- (4)  $\wp_F(A, \iota) = \wp_F(\overline{A}, \iota) = \wp_F(\text{conv}(A), \iota)$  for any  $A \in \overline{\mathbb{I}}_b(\mathcal{D})$ ,
- (5)  $\wp_F(A, \cdot) : [0, \infty) \longrightarrow [0, 1]$  is left continuous and  $\lim_{\iota \rightarrow \infty} \wp_F(A, \iota) = 1$ .
- (6) for any decreasing sequence  $\{A_n\}$  in  $\overline{\mathbb{I}}_b(\mathcal{D})$  such that  $\lim_{n \rightarrow \infty} \wp_F(A_n, \iota) = 1$ , implies  $A_\infty = \bigcap_{n=1}^{\infty} \overline{A_n}$  is nonempty.

$\wp_F$  is called fuzzy sublinear if

- (7)  $\wp_F(kA, \iota) = \wp_F\left(A, \frac{\iota}{|k|}\right)$  for  $k \in F$ ,
- (8)  $\wp_F(A \otimes B, \iota \otimes \kappa) \geq \wp_F(A, \iota) * \wp_F(B, \kappa)$ , where  $*$  is continuous  $\iota$ -norm.

**Remark 6.1.** For a given MNC  $\alpha$ , (resp,  $\beta$  or  $\wp$ ), and for  $\iota > 0$ , we define

$$\wp_F(\Omega, \iota) = \frac{\iota}{\iota \otimes \alpha(\Omega)},$$

then  $\wp_F$  is FMNC, called standard FMNC induced by  $\alpha$ , (resp,  $\beta$  or  $\wp$ ).

**Examples 6.1.** (1) Let  $\hbar$  of  $\mathbb{R}^{\otimes}$  into  $\mathbb{R}^{\otimes}$  be increasing and continuous, for  $\delta > 0$ , define a function

$$\wp_F(\Omega, \iota) = \frac{\hbar(\iota)}{\hbar(\iota) \otimes \delta \cdot \alpha(\Omega)},$$

then  $\wp_F$  is a FMNC with  $*$  as a usual multiplication  $\cdot$ . The standard FMNC is a special case.

(2) Define with the same function  $\hbar$

$$\wp_F(\Omega, \iota) = e^{\otimes' \frac{\alpha(\Omega)}{\hbar(\iota)}},$$

then  $\wp_F$  is a FMNC with  $*$  as a usual multiplication  $\cdot$ .

(3) Define with the same function  $\hbar$

$$\wp_F(\Omega, \iota) = 1 \otimes' \frac{\alpha(\Omega)}{\hbar(\iota)},$$

then  $\wp_F$  is a FMNC with  $*$  as Lukasiewicz-t-norm  $\mathcal{L}$ .

**Remark 6.2.** If  $(X, N, *)$  is a fuzzy BS [42], then  $\wp_F : \bar{\mathbb{I}}_{b,cv}(X) \times [0, \infty) \longrightarrow [0, 1]$  defined by any (1)  $\otimes'$  (3) in above Examples is a FMNC and in this whole fuzzy structure we can not use any crisp MNC. In this environment the existence theory of fuzzy DEs and IEs is not explored.

The concept of FMNC is different from the concept of fuzzy measure, we define fuzzy measure and fuzzy measure space then explain how both concept are different.

**Definition 6.2.** ([62] chapter 6) Let  $X$  be a set and  $\mathfrak{B}$  be Borel field of  $X$ . A set function  $\sigma : \mathfrak{B} \longrightarrow [0, 1]$  is a fuzzy measure (FM) if

$$(FM1) \quad \sigma(\emptyset) = 0 \text{ and } \sigma(X) = 1,$$

$$(FM2) \quad \text{for } A, B \in \mathfrak{B} \text{ and } A \subset B, \text{ then } \sigma(A) \leq \sigma(B),$$

$$(FM3) \quad \text{if } \{F_n\}_{n \geq 1} \text{ is a monotone sequence (in the sense of inclusion) in } \mathfrak{B}, \text{ then } \lim_{n \rightarrow \infty} \sigma(F_n) = \sigma\left(\lim_{n \rightarrow \infty} F_n\right).$$

**Remark 6.3.** Note that if we assume  $\wp_F(\Omega, \cdot)$  is independent of  $\iota$  (just to behave like fuzzy measure). Since  $\phi$  being a finite set is compact set then  $\wp_F(\phi, \cdot) = 1$  but clearly from (FM1)  $\sigma(\emptyset) = 0$ , so we cannot develop any useful relationship between FMNC and FM.

We define fuzzy  $\wp_F$ -condensing and fuzzy  $\wp_F$ - $k$ -set contractions by means of FMNC.

**Definition 6.3.** A mapping  $\xi : \Omega \longrightarrow \bar{\mathbb{I}}_{cl,cv,b}(\Omega)$  is fuzzy  $\wp_F$ -condensing if there is FMNC  $\wp_F$  with

$$\wp_F(\xi(\Omega), \iota) > \wp_F(\Omega, \iota)$$

for each  $\Omega \in \bar{\mathbb{I}}_{cl,cv,b}(\emptyset)$ .

**Definition 6.4.** A mapping  $\xi : \Omega \longrightarrow \bar{\mathbb{I}}_{cl,cv,b}(\Omega)$  is called a fuzzy  $\wp_F$ -KSC if there exists some  $k \in (0, 1)$  and a FMNC  $\wp_F$  such that

$$\wp_F(\xi(\Omega), k\iota) \geq \wp_F(\Omega, \iota)$$

for each  $\Omega \in \bar{\mathbb{I}}_{cl,cv,b}(\emptyset)$ .

**Remark 6.4.** In [100], it has been proved that every multivalued  $k$ -contraction  $\xi$  is a  $\alpha$ -KSC and hence a  $\wp_F$ -KSC.

### 6.3 Main Results

Now we present our first results regarding the FP of a continuous self operator on  $\Omega \in \overline{\mathbb{I}}_{b,cl,cv}(\mathcal{D})$ . This theorem generalizes the results of [4] and [121]. This result has a unique importance because it will be very useful in the existence theory.

**Theorem 6.1.** Let  $\Omega \in \overline{\mathbb{I}}_{b,cl,cv}(\mathcal{D})$  with  $0 \in \Omega$ .  $\xi$  is continuous and self mapping with domain  $\Omega$ . If

$$V = Conv(\xi(V)) \text{ or } V = \xi(V) \cup \{0\} \Rightarrow \wp_F(V, \iota) = 1, \text{ for } \iota > 0,$$

holds for each  $V \subseteq \Omega$ , then  $\xi$  has a FP.

**Proof.** Let  $w_0 = 0$  and  $w_{n+1} = \xi(w_n)$ , where  $n = 0, 1, 2, 3, \dots$ . Let  $W = \{w_n : n = 0, 1, 2, 3, \dots\}$ , then clearly  $W = \xi(W) \cup \{0\}$  and using hypothesis of theorem we have  $\wp_F(W, \iota) = 1$ , therefore  $W \in \overline{\mathbb{I}}_{pre}(\infty)$ . If  $K$  is a set in which there are limit points of  $W$ , then  $K = \xi(K)$  (see [119] page 34, 7(iii)). Let

$$\overline{\mathbb{I}} = \{Y \subseteq \Omega : K \subset Y \text{ and } Conv\xi(Y) \subset Y\}.$$

Clearly  $\Omega \in \overline{\mathbb{I}}$  so  $\overline{\mathbb{I}}$  is nonempty. Let  $U = \bigcap_{Y \in \overline{\mathbb{I}}} Y$ , as  $K \subset U$  so  $U$  is nonempty and  $K = \xi(K) \subset Conv\xi(K) \subset Conv\xi(U)$  which implies

$$K \subset Conv\xi(U). \quad (*)$$

Also since  $Conv\xi(U) \subset Conv\xi(Y) \subset Y$  for all  $Y \in \overline{\mathbb{I}}$ , therefore

$$Conv\xi(U) \subset \bigcap_{Y \in \overline{\mathbb{I}}} Y = U, \quad (**)$$

which implies  $U \in \overline{\mathbb{I}}$ . Now as  $Conv\xi(Conv\xi(U)) \subset Conv\xi(U)$ , and  $K \subset Conv\xi(U)$  by (\*), therefore  $Conv\xi(U) \in \overline{\mathbb{I}}$ . Also as  $Conv\xi(U) \subset U$  by (\*\*) and obviously  $U \subset Conv\xi(U) \in \overline{\mathbb{I}}$  ( $U$  being intersection of all elements of  $\overline{\mathbb{I}}$ ), this implies  $U = Conv(\xi(U))$ . Hence by assumption in statement of theorem  $\wp_F(U, \iota) = 1$ , so  $U \in \overline{\mathbb{I}}_{pre}(\Omega)$  and hence  $\overline{U} \in \overline{\mathbb{I}}_{cp}(\Omega)$ . Applying Schauder's FP theorem on mapping  $\xi|_{\overline{U}}$  (i.e,  $\xi : \overline{U} \rightarrow \overline{U}$ ),  $\xi$  has a FP. ■

Next result is regarding the FP of a given multivalued fuzzy  $\wp_F$ -KSC using FMNC. This

result generalizes the well known results of Darbo [43], and [16].

**Theorem 6.2.** Every continuous fuzzy  $\wp_F$ -KSC on  $\Omega \in \overline{\Pi}_{cl,cv,b}(\Omega)$  has a FP and  $Fix_\xi \in \overline{\Pi}_{prc}(\mathcal{D})$ .

**Proof.** For any  $\Omega_0 \in \overline{\Pi}_{cl,cv,b}(\Omega)$  define the sequence

$$\Omega = \Omega_0$$

and

$$\Omega_{n\oplus 1} = \overline{Conv(\xi(\Omega_n))}, \quad n = 0, 1, 2, 3, \dots$$

As  $\Omega_0$  is closed and convex, then  $\Omega_1 = \overline{Conv(\xi(\Omega_0))} \subset \Omega_0$  and by induction we have

$$\dots \Omega_{n\oplus 1} \subset \Omega_n \dots \Omega_2 \subset \Omega_1 \subset \Omega_0,$$

which implies that  $\Omega_n$  is a non-increasing sequence. Now consider for  $\iota > 0$ ,

$$\begin{aligned} \wp_F(\Omega_{n\oplus 1}, \iota) &= \wp_F(\overline{Conv(\xi(\Omega_n))}, \iota) \\ &= \wp_F(\xi(\Omega_n), \iota) \\ &\geq \wp_F\left(\Omega_n, \frac{\iota}{k}\right) \\ &= \wp_F\left(\overline{Conv(\xi(\Omega_{n\oplus' 1}))}, \frac{\iota}{k}\right) \\ &\geq \wp_F\left(\Omega_{n\oplus' 1}, \frac{\iota}{k^2}\right) \\ &\vdots \\ &\geq \wp_F\left(\Omega_0, \frac{\iota}{k^n}\right) \longrightarrow 1. \end{aligned}$$

We conclude that

$$\lim_{n \rightarrow \infty} \wp_F(\Omega_{n\oplus 1}, \iota) = 1,$$

which implies  $\Omega_\infty = \bigcap_{n=1}^{\infty} \overline{\Omega_n}$  is nonempty. Since  $\Omega_\infty \subseteq \Omega_n$  for all  $n \in \mathbb{N}$ , therefore (using (3) in Definition ),  $\wp_F(\Omega_\infty, \iota) \geq \lim_{n \rightarrow \infty} \wp_F(\Omega_{n\oplus 1}, \iota) = 1$  which implies  $\overline{\Omega_\infty} = \Omega_\infty$  (by construction  $\Omega_\infty$  is closed set), is compact and convex. Therefore by Lemma 1.2, the operator  $\xi : \Omega_\infty \rightarrow \overline{\Pi}_{cv,cp}(\Omega_\infty)$  is upper semi-continuous, and application of Kakutani theorem implies  $\xi$  has a FP.

Finally to prove  $Fix_\xi \in \overline{\mathbb{I}}_{pre}(\mathcal{D})$ . Consider for  $k \in (0, 1)$  for  $\iota > 0$ ,

$$\begin{aligned} \wp_F(Fix_\xi, \iota) &\geq \wp_F(\xi^n(Fix_\xi), \iota), \\ &\geq \wp_F\left(Fix_\xi, \frac{\iota}{k^n}\right) \longrightarrow 1, \text{ as } n \longrightarrow \infty. \end{aligned}$$

This implies

$$\wp_F(Fix_\xi, \iota) = 1.$$

It means that  $Fix_\xi$  is precompact. This proves the theorem. ■

**Remark 6.5.** From Remark 6.1, we conclude that every  $\alpha$ -condensing/ $k$ -set contraction (resp,  $\beta$  or  $\wp$ -condensing/ $k$ -set contraction ) operator is a fuzzy  $\wp_F$ -condensing/ $\wp_F$ - $k$ -set contraction operator.

From above definitions and results we have the following corollaries.

**Corollary 6.1.** Every continuous  $\wp$ - $k$ -set ( $\beta$ - $k$ -set,  $\alpha$ - $k$ -set ), contraction on  $\Omega \in \overline{\mathbb{I}}_{cl,cv,b}(\mathcal{D})$  has a FP.

Now we prove Krasnoselskii-type FP theorem where functions are multivalued.

**Theorem 6.3.** Let  $\Omega \in \overline{\mathbb{I}}_{b,cl,cv}(\mathcal{D})$  and  $\wp_F$  be a fuzzy sublinear, fuzzy MNC. Let  $\zeta, \xi : \Omega \longrightarrow \overline{\mathbb{I}}_{cl,cv}(\Omega)$  be multivalued functions with,

- (a)  $\zeta \in \check{S}_{cl}$  and  $\wp_F$ - $\frac{k}{2}$ -SC for some  $k$  between 0 and 1;
- (b)  $\xi \in \check{S}_{cl,cp}$ ;
- (c)  $\zeta u \otimes \xi u \subset \Omega$  for  $\hat{u} \in \Omega$ .

Then there is a solution of operator inclusion  $\hat{u} \in \zeta u \otimes \xi u$  and the solution set  $Fix_{\zeta \otimes \xi} \in \overline{\mathbb{I}}_{pre}(\mathcal{D})$ .

**Proof.** Define  $\psi : \Omega \longrightarrow \overline{\mathbb{I}}_{cl,cv}(\Omega)$  by

$$\psi(\hat{u}) = \zeta u \otimes \xi u$$

for each  $\hat{u} \in \Omega$ . As  $\zeta$  is closed and  $\xi$  is compact, therefore their sum  $\psi$  is also closed . Also the sum is Minkowski therefore  $\psi$  is convex. Therefore  $\psi : \Omega \longrightarrow \overline{\mathbb{I}}_{cl,cv}(\Omega)$  is well defined. Also for any nonempty  $B \subseteq \Omega$ ,

$$\psi(B) \subseteq \Omega$$

which implies  $\psi(B)$  is bounded as  $\Omega$  is bounded, and by definition

$$\psi(B) \subseteq \zeta(B) \otimes \xi(B).$$

Now using fuzzy sublinearity of  $\wp_F$ , for some  $k \in (0, 1)$ , and  $\iota > 0$ , consider

$$\begin{aligned} \wp_F(\psi(B), k\iota) &\geq \wp_F\left(\zeta(B) \otimes \xi(B), \frac{k}{2}\iota \otimes \frac{k}{2}\iota\right) \\ &\geq \wp_F\left(\zeta(B), \frac{k}{2}\iota\right) * \wp_F\left(\xi(B), \frac{k}{2}\iota\right), \end{aligned}$$

since  $\xi$  is compact therefore  $\wp_F(\xi(B), \frac{k}{2}\iota) = 1$ , so

$$\begin{aligned} \wp_F(\psi(B), k\iota) &\geq \wp_F\left(\zeta(B), \frac{k}{2}\iota\right) * 1 \\ &= \wp_F\left(\zeta(B), \frac{k}{2}\iota\right) \\ &\geq \wp_F(B, \iota), \text{ using the fact that } \zeta \text{ is } \wp_F\text{-}\frac{k}{2}\text{-set contraction.} \end{aligned}$$

Hence from Theorem 6.2, there is  $z \in \psi(z)$ , the required solution of the operator inclusion  $z \in \zeta z \otimes \xi z$ . Next we claim

$$\wp_F(\text{Fix}_{\zeta \otimes \xi}, \iota) = 1.$$

On contrary, suppose that for  $\iota > 0$ ,

$$\wp_F(\text{Fix}_{\zeta \otimes \xi}, \iota) < 1.$$

Then for some  $k \in (0, 1)$

$$1 > \wp_F(\text{Fix}_{\zeta \otimes \xi}, \iota) \geq \wp_F(\psi^n(\text{Fix}_{\zeta \otimes \xi}), \iota) \geq \wp_F\left(\text{Fix}_{\zeta \otimes \xi}, \frac{\iota}{k^n}\right) \longrightarrow 1, \text{ as } n \longrightarrow \infty,$$

a contradiction, therefore

$$\wp_F(\text{Fix}_{\zeta \otimes \xi}, \iota) = 1.$$

This proves the theorem. ■

The following corollaries are due to Remark 6.4 and 6.5.

**Corollary 6.2.** [49] Let  $\Omega \in \overline{\Pi}_{cl,cv,b}(\mathcal{D})$  and  $\wp$  be a sublinear MNC. Let  $\zeta, \xi : \Omega \longrightarrow \overline{\Pi}_{cl,cv}(\Omega)$  are functions with,

- (a)  $\zeta \in \check{S}_{cl}$  and  $\wp$ -KSC (resp,  $\alpha$  or  $\beta$ -KSC ),
- (b)  $\xi \in \check{S}_{cl,cp}$ ,
- (c)  $\zeta u \circledast \xi u \subset \Omega$  for  $\hat{u} \in \Omega$ .

Then there exist a solution of operator inclusion  $\hat{u} \in \zeta u \circledast \xi u$  and the solution set  $\zeta_{\zeta \circledast \xi} \in \overline{\Pi}_{cp}(\mathcal{D})$ .

In [100] it has been proved that every multivalued  $k$ -contraction  $\xi$  is  $\alpha$ -KSC and hence a  $\wp_F$ -KSC. So we can make the following statement.

**Corollary 6.3.** [100] Let  $\Omega \in \overline{\Pi}_{cl,cv,b}(\mathcal{D})$  and  $\zeta, \xi : \Omega \longrightarrow \overline{\Pi}_{cl,cv}(\Omega)$  are functions with,

- (a)  $\zeta \in \check{S}_{cl}$  and multivalued  $k$ -C, for some  $k \in (0, 1)$ ,
- (b)  $\xi \in \check{S}_{cl,cp}$ ,
- (c)  $\zeta u \circledast \xi u \subset \Omega$  for  $\hat{u} \in \Omega$ .

Then there is a solution of operator inclusion  $\hat{u} \in \zeta u \circledast \xi u$  and the solution set  $\zeta_{\zeta \circledast \xi} \in \overline{\Pi}_{cp}(\mathcal{D})$ .

## 6.4 Applications in fractional differential equations

In this section, some ERs for Caputo Fabrizio fractional anti-periodic BVP of order  $1 < \alpha \leq 2$  are proved. We use our Theorem 6.1, to find the sufficient conditions for ES of said Caputo Fabrizio fractional BVP. For more applications related to the FP theory and the existence theory of IEs and DEs, kindly see [63, 87, 88, 108].

Now we discuss the conditions for the ESs of the following CF-fractional anti-periodic boundary value problem (CFFBVP).

$$({}^{CF}D_b^\alpha y)(t) = f(t, y(t)), \quad 1 < \alpha \leq 2, \quad 0 < t < \xi, \quad (4.1)$$

with

$$y(0) = \circledast y(\xi) \quad \text{and} \quad \dot{y}(0) = \circledast \dot{y}(\xi), \quad (4.2)$$

where  $f$  of  $[0, \xi] \times \mathbb{R}$  into  $\mathbb{R}$  is continuous and  $y(t) \in H^1(0, \xi)$ .

**Lemma 6.1.** Let  $K : [0, \xi] \longrightarrow \mathbb{R}$  be continuous. A function  $y(t) \in H^1(0, \xi)$  is solution of

following anti-periodic BVP

$$({}^{CF}D_b^\alpha y)(\iota) = K(\iota), \quad 1 < \alpha \leq 2, \quad 0 < \iota < \xi, \quad (4.3)$$

with boundary conditions (4.2) if and only if  $y(\iota)$  is solution of following IE

$$y(\iota) = \hbar(\iota) \otimes \int_0^\xi G(\iota, \kappa) K(\kappa) d\kappa,$$

where,

$$G(\iota, \kappa) = \begin{cases} \frac{(\xi \otimes' 2\iota)(\alpha \otimes' 1) \otimes (2 \otimes' \alpha) 2}{4B(\alpha \otimes' 1)} \oplus \frac{(\alpha \otimes' 1)[2\iota \otimes' (\kappa \otimes' \xi)]}{2B(\alpha \otimes' 1)} & 0 \leq \kappa \leq \iota \\ \frac{(\xi \otimes' 2\iota)(\alpha \otimes' 1) \otimes' 2(2 \otimes' \alpha)}{4B(\alpha \otimes' 1)} \otimes' \frac{(\alpha \otimes' 1)}{2B(\alpha \otimes' 1)} (\xi \otimes' \kappa) & \iota \leq \kappa \leq \xi \end{cases},$$

and

$$\hbar(\iota) = \frac{(\xi \otimes' 2\iota)(2 \otimes' \alpha)}{4B(\alpha \otimes' 1)} K(\xi).$$

**Proof.** We have

$$({}^{CF}D_b^\alpha y)(\iota) = K(\iota), \quad 1 < \alpha \leq 2, \quad 0 < \iota < \xi.$$

Applying above Proposition 1.1, we have

$$y(\iota) = c_1 \otimes c_2 \iota \otimes \frac{(2 \otimes' \alpha)}{B(\alpha \otimes' 1)} \int_0^\iota K(\kappa) d\kappa \otimes \frac{(\alpha \otimes' 1)}{B(\alpha \otimes' 1)} \int_0^\iota (\iota \otimes' \kappa) K(\kappa) d\kappa.$$

Differentiating both sides, we have

$$\dot{y}(\iota) = c_2 \otimes \frac{(2 \otimes' \alpha)}{B(\alpha \otimes' 1)} K(\iota) \otimes \frac{(\alpha \otimes' 1)}{B(\alpha \otimes' 1)} \int_0^\iota K(\kappa) d\kappa$$

Using  $\dot{y}(0) = \otimes' \dot{y}(\xi)$ , with assumption given in [3], i.e.,  $K(0) = 0$ , we have

$$c_2 = \otimes' \left( c_2 \otimes \frac{(2 \otimes' \alpha)}{B(\alpha \otimes' 1)} K(\xi) \otimes \frac{(\alpha \otimes' 1)}{B(\alpha \otimes' 1)} \int_0^\xi K(\kappa) d\kappa \right),$$

which implies

$$c_2 = \otimes' \frac{1}{2} \left( \frac{(2 \otimes' \alpha)}{B(\alpha \otimes' 1)} K(\xi) \otimes \frac{(\alpha \otimes' 1)}{B(\alpha \otimes' 1)} \int_0^\xi K(\kappa) d\kappa \right).$$

Now by using  $y(0) = \otimes' y(\xi)$ , we have

$$c_1 = \otimes' \left( c_1 \otimes c_2 \xi \otimes \frac{(2 \otimes' \alpha)}{B(\alpha \otimes' 1)} \int_0^\xi K(\kappa) d\kappa \otimes \frac{(\alpha \otimes' 1)}{B(\alpha \otimes' 1)} \int_0^\xi (\xi \otimes' \kappa) K(\kappa) d\kappa \right),$$

which gives

$$c_1 = \otimes' c_2 \frac{\xi}{2} \otimes' \frac{(2 \otimes' \alpha)}{2B(\alpha \otimes' 1)} \int_0^\xi K(\kappa) d\kappa \otimes' \frac{(\alpha \otimes' 1)}{2B(\alpha \otimes' 1)} \int_0^\xi (\xi \otimes' \kappa) K(\kappa) d\kappa.$$

From  $c_2$  and above,

$$\begin{aligned} c_1 &= \frac{\xi(2 \otimes' \alpha)}{4B(\alpha \otimes' 1)} K(\xi) \otimes \frac{\xi(\alpha \otimes' 1)}{4B(\alpha \otimes' 1)} \int_0^\xi K(\kappa) d\kappa \otimes' \frac{(2 \otimes' \alpha)}{2B(\alpha \otimes' 1)} \int_0^\xi K(\kappa) d\kappa \\ &\quad \otimes' \frac{(\alpha \otimes' 1)}{2B(\alpha \otimes' 1)} \int_0^\xi (\xi \otimes' \kappa) K(\kappa) d\kappa. \end{aligned}$$

Thus the solution is

$$\begin{aligned} y(\iota) &= \frac{(\xi \otimes' 2\iota)(2 \otimes' \alpha)}{4B(\alpha \otimes' 1)} K(\xi) \otimes \int_0^\iota \frac{(\xi \otimes' 2\iota)(\alpha \otimes' 1) \otimes (2 \otimes' \alpha) 2}{4B(\alpha \otimes' 1)} K(\kappa) d\kappa \\ &\quad \otimes \int_0^\iota \frac{(\alpha \otimes' 1)[2\iota \otimes' (\kappa \otimes' \xi)]}{2B(\alpha \otimes' 1)} K(\kappa) d\kappa \otimes \int_\iota^\xi \frac{(\xi \otimes' 2\iota)(\alpha \otimes' 1) \otimes' 2(2 \otimes' \alpha)}{4B(\alpha \otimes' 1)} K(\kappa) d\kappa \\ &\quad \otimes' \int_\iota^\xi \frac{(\alpha \otimes' 1)}{2B(\alpha \otimes' 1)} (\xi \otimes' \kappa) K(\kappa) d\kappa. \end{aligned}$$

Let

$$G(\iota, \kappa) = \begin{cases} \frac{(\xi \otimes' 2\iota)(\alpha \otimes' 1) \otimes (2 \otimes' \alpha) 2}{4B(\alpha \otimes' 1)} \otimes \frac{(\alpha \otimes' 1)[2\iota \otimes' (\kappa \otimes' \xi)]}{2B(\alpha \otimes' 1)} & 0 \leq \kappa \leq \iota \\ \frac{(\xi \otimes' 2\iota)(\alpha \otimes' 1) \otimes' 2(2 \otimes' \alpha)}{4B(\alpha \otimes' 1)} \otimes' \frac{(\alpha \otimes' 1)}{2B(\alpha \otimes' 1)} (\xi \otimes' \kappa) & \iota \leq \kappa \leq \xi \end{cases},$$

hence, we have

$$y(\iota) = \hbar(\iota) \otimes \int_0^\xi G(\iota, \kappa) K(\kappa) d\kappa.$$

This proves the lemma. ■

Clearly  $G(\iota, \kappa)$  is continuous on  $J \times J$  and denote  $G^* = \sup_{(\iota, \kappa) \in J \times J} \|G(\iota, \kappa)\|$ . As  $\hbar$  is continuous on  $J$  therefore there exists  $\hbar^* = \sup_{\iota \in J} \|\hbar(\iota)\|$ .

To prove the ER of the CF fractional DE (4.1) with BC (4.2), assume;

(C1)  $f: J \times \mathcal{D} \rightarrow \mathcal{D}$  is a Caratheodory function.

(C2) There exists  $q \in L^1(J, \mathbb{R}^{\otimes})$ , with

$$\|f(\iota, \hat{u})\| \leq q(\iota) \|\hat{u}\|,$$

for  $\iota \in J$  and  $\hat{u} \in \mathcal{D}$ .

(C3) For each  $\iota \in J$  and each bounded subset  $\beta$  of  $\mathcal{D}$ , we have

$$\lim_{h \rightarrow 0^{\otimes}} (f(J_{\iota, h} \times \beta)) \leq q(\iota) \alpha(\beta),$$

where  $J_{\iota, h} = J \cap [\iota \otimes' h, \iota]$ .

**Theorem 6.4.** If the conditions (C1)  $\otimes'$  (C3) are satisfied and

$$G^* \int_0^\xi q(\kappa) d\kappa < 1$$

holds, then the CFFBVP (4.1)-(4.2) has a solution.

**Proof.** Define an operator  $Q: H \rightarrow H$  by

$$Q(\hat{u}(\iota)) = \hbar(\iota) \otimes \int_0^\xi G(\iota, \kappa) f(\kappa, \hat{u}(\kappa)) d\kappa.$$

Clearly the solution of CFFBVP (4.1)  $\otimes'$  (4.2) exists if and only if  $Q$  has a FP.

Set

$$r \geq \frac{\hbar^*}{1 \otimes' G^* \int_0^\xi q(\kappa) d\kappa},$$

and define the set

$$B_r = \{v \in H : \|v\|_\infty \leq r\}.$$

The set  $B_r \in \overline{\mathcal{D}}_{cl,cv,b}(\mathcal{D})$ . Now we explain that  $Q$  satisfies all conditions of Theorem 6.1.

First we show that  $Q \in \check{S}_{cl}$ . Consider  $v_n$  in  $H$  with  $v_n \rightarrow v \in H$ . Now for each  $\iota \in J$ ,

$$\begin{aligned} \|Q(v_n) \otimes' Q(v)\| &= \left\| \int_0^\xi G(\iota, \kappa) f(\kappa, v_n(\kappa)) d\kappa \otimes' \int_0^\xi G(\iota, \kappa) f(\kappa, v(\kappa)) d\kappa \right\| \\ &= \left\| \int_0^\xi G(\iota, \kappa) [f(\kappa, v_n(\kappa)) \otimes' f(\kappa, v(\kappa))] d\kappa \right\| \\ &\leq \int_0^\xi \|G(\iota, \kappa)\| \|f(\kappa, v_n(\kappa)) \otimes' f(\kappa, v(\kappa))\| d\kappa \\ &\leq G^* \int_0^\xi \|f(\kappa, v_n(\kappa)) \otimes' f(\kappa, v(\kappa))\| d\kappa. \end{aligned}$$

Since  $f$  is a Caratheodory function, then by "Lebesgue dominated convergence theorem" [5]

$$\|f(\kappa, v_n(\kappa)) \otimes' f(\kappa, v(\kappa))\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Therefore  $Q$  is continuous.

Now we show that  $Q$  maps  $B_r$  into  $B_r$ . For each  $v \in B_r$  and  $\iota \in J$ , we have

$$\begin{aligned} \|Q(v(\iota))\| &\leq \|\hbar(\iota)\| \otimes \int_0^\xi \|G(\iota, \kappa)\| \|f(\kappa, v(\kappa))\| d\kappa \\ &\leq \|\hbar(\iota)\| \otimes \int_0^\xi \|G(\iota, \kappa)\| q(\kappa) \|v(\kappa)\| d\kappa \end{aligned}$$

$$\leq \hbar^* \otimes G^* r \int_0^\xi q(\kappa) d\kappa \leq r$$

which was required.

Next to show that  $Q(B_r)$  is bounded and equicontinuous. From above step  $Q(B_r)$  is bounded. Now for  $\iota_1, \iota_2 \in J$  with  $\iota_1 < \iota_2$  and  $v \in B_r$ , consider

$$\begin{aligned} \|Q(v(\iota_2)) \otimes' Q(v(\iota_1))\| &\leq \|\hbar(\iota_1) \otimes' \hbar(\iota_2)\| \\ &\quad \otimes \left\| \int_0^\xi [G(\iota_2, \kappa) \otimes' G(\iota_1, \kappa)] f(\kappa, v(\kappa)) d\kappa \right\| \\ &\leq \|\hbar(\iota_1) \otimes' \hbar(\iota_2)\| \\ &\quad \otimes r \left\| \int_0^\xi q(\kappa) [G(\iota_2, \kappa) \otimes' G(\iota_1, \kappa)] d\kappa \right\|. \end{aligned}$$

As  $\iota_1 \rightarrow \iota_2$  the right to the inequality approaches to 0.

Let  $U \subset B_r$  with  $U \subset \overline{\text{Conv}}(Q(U) \cup \{0\})$ . As  $U$  is equicontinuous & bounded which implies the function  $\hat{u}(\iota) := \alpha(U(\iota))$  is continuous on  $J$ . Using Remark , we have

$$\wp_F(U, \tau) = \frac{\tau}{\tau \otimes \alpha(U)},$$

for some  $\tau > 0$ . Now consider

$$\begin{aligned} \hat{u}(\iota) &= \alpha(U(\iota)) \\ &\leq \alpha(Q(U)(\iota) \cup \{0\}) \end{aligned}$$

which implies

$$\|\hat{u}(\iota)\|_\infty \leq \|\hat{u}(\iota)\|_\infty G^* \int_0^\xi q(\kappa) d\kappa$$

since  $G^* \int_0^\xi q(\kappa) d\kappa < 1$ , so we have  $\hat{u}(\iota) = 0$  for  $\iota \in J$ , so  $\hat{u} = 0$ , therefore  $\alpha(U(\iota)) = 0$  for all  $\iota \in J$ , hence

$$\wp_F(U, \tau) = 1,$$

therefore by Theorem 6.2,  $\bar{U}$  is compact in  $B_r$  by virtue of Arzila Ascoli. Hence by Theorem 6.2,  $Q$  has a FP which was needed. ■

## 6.5 Conclusion

We define the notions of fuzzy MNC, fuzzy  $k$ -set contractions and fuzzy condensing mappings. Using these notions we prove some FP results which generalize the well known results of Schauder, Monch, Sadovski and Krasnoselskii. We provide some examples and applications to validate our definitions and results.

## Chapter 7

# Analysis of Schauder and Krasnoselskii Theorems by Weakening the Condition of Normed Space

### 7.1 Introduction

Topological FP theory is very important branch of nonlinear and convex analysis. It has a strong and useful history of more than a century. In 1912 Brouwer proved his well-known FP theorem. The most applicable generalization of Brouwer's theorem was presented in 1930 by Schauder using compact operators. Krasnoselskii, in 1958, combined Schauder theorem and Banach contraction principle to consider the FP problem:

$$\xi\nu \oplus \zeta\nu = \nu, \quad \nu \in \Omega,$$

where  $\Omega$  is a set in the BS  $\mathcal{D}$ . Keeping the importance of this type of problems, many researchers have solved the operator equation  $\xi\nu \oplus \zeta\nu = \nu$  under different assumptions on  $\xi, \zeta$  and  $\Omega$ . These types of results can be seen in [33, 41, 106, 103, 126, 128, 129, 130].

In this chapter we generalize Schauder, Krasnoselskii and some related FP theorems by

wakening the condition of normed space. We assume a linear space  $\mathcal{D}$  with metric  $\mathfrak{d}$  and the following two properties:

$$(a^*) \quad \mathfrak{d}(\omega \otimes \gamma, \nu \otimes \gamma) = \mathfrak{d}(\omega, \nu) \text{ for all } \omega, \nu, \gamma \in \mathcal{D};$$

$$(b^*) \quad \mathfrak{d}(\iota\omega, \iota\nu) \leq \iota\mathfrak{d}(\omega, \nu) \text{ for all } \omega, \nu \in \mathcal{D} \text{ and } 0 \leq \iota \leq 1.$$

If the metric  $\mathfrak{d}$  is induced from the normed space then  $\mathfrak{d}$  satisfy the following properties [44]

$$(a) \quad \mathfrak{d}(\omega \otimes \gamma, \nu \otimes \gamma) = \mathfrak{d}(\omega, \nu) \text{ for all } \omega, \nu, \gamma \in \mathcal{D}; \text{ (translation invariance)}$$

$$(b) \quad \mathfrak{d}(\iota\omega, \iota\nu) = |\iota| \mathfrak{d}(\omega, \nu) \text{ for all } \omega, \nu \in \mathcal{D} \text{ and } \iota \in \mathbb{R}. \text{ (Absolute homogeneity)}$$

**Remark 7.1.** Clearly the condition  $(b^*)$  is weaker than the condition  $(b)$ . If  $\mathcal{D}$  is linear space with norm then;

(1) the open ball is convex and ;

(2) both the binary operations are continuous.

However in a linear space  $\mathcal{D}$  with metric  $\mathfrak{d}$ , these conditions are not true in general.

## 7.2 Main Results

Norfolk [96] established the condition on MS under which the open ball is convex. This is represented below.

**Lemma 7.1.** Let  $\mathcal{D}$  be a linear space with metric  $\mathfrak{d}$  with  $(a^*)$  and  $(b^*)$ . Then the open ball  $B_{\mathfrak{d}}(\omega, \epsilon)$  is convex.

**Proof.** Let  $\nu, \gamma \in B_{\mathfrak{d}}(\omega, \epsilon)$  then

$$\begin{aligned} \mathfrak{d}(\omega, \iota\nu \otimes (1 \otimes' \iota)\gamma) &= \mathfrak{d}(0, (\iota\nu \otimes (1 \otimes' \iota)\gamma) \otimes' \omega) \\ &= \mathfrak{d}(0, \iota(\nu \otimes' \omega) \otimes (1 \otimes' \iota)(\gamma \otimes' \omega)) \\ &\leq \mathfrak{d}(0, \iota(\nu \otimes' \omega)) \otimes \mathfrak{d}(\iota(\nu \otimes' \omega), (\iota(\nu \otimes' \omega) \otimes (1 \otimes' \iota)(\gamma \otimes' \omega))) \\ &\leq \iota\mathfrak{d}(0, (\nu \otimes' \omega)) \otimes (1 \otimes' \iota)\mathfrak{d}(0, (\gamma \otimes' \omega)) \\ &= \iota\mathfrak{d}(\omega, \nu) \otimes (1 \otimes' \iota)\mathfrak{d}(\omega, \gamma) \\ &\leq \max \{\mathfrak{d}(\omega, \nu), \mathfrak{d}(\omega, \gamma)\} \\ &< \epsilon. \end{aligned}$$

which shows that  $B_{\mathfrak{d}}(\omega, \epsilon)$  is convex.

Now we show that if  $\mathcal{D}$  is linear space with metric  $\mathfrak{d}$  with  $(a^*)$  and  $(b^*)$ , then addition and scalar multiplication (in first variable) are continuous.

**Lemma. 7.2.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the condition  $(a^*)$ . If  $(\omega_n)$  and  $(\nu_n)$  be sequences converge to  $\omega$  and  $\nu$  respectively, then  $(\omega_n \otimes \nu_n)$  converges to  $\omega \otimes \nu$ .

**Proof:** Trivial.

**Lemma. 7.3.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  and if  $a_n$  in  $\mathbb{R}$  with  $a_n \rightarrow a$  then, for fixed  $\omega$ ,  $a_n \omega \rightarrow a\omega$ .

**Proof.** Since  $a_n$  converges to  $a$  therefore for  $\epsilon > 0$  there is  $N$  with

$$\mathfrak{d}_{\mathbb{R}}(a_n, a) < \epsilon \text{ for every } n > N.$$

Suppose for fixed  $\omega$ ,  $\mathfrak{d}(\omega, 0) = c$ , then

$$\begin{aligned} \mathfrak{d}(a_n \omega, a\omega) &= \mathfrak{d}(a_n \omega \otimes' a\omega, 0) \\ &= \mathfrak{d}((a_n \otimes' a)\omega, 0) \\ &= \mathfrak{d}((a_n \otimes' a)\omega, (a_n \otimes' a)0). \end{aligned} \tag{1}$$

Also

$$\begin{aligned} \mathfrak{d}(a_n \omega, a\omega) &= \mathfrak{d}(0, a\omega \otimes' a_n \omega) \\ &= \mathfrak{d}((a \otimes' a_n)\omega, (a \otimes' a_n)0). \end{aligned} \tag{2}$$

Since  $a_n$  converges to  $a$  there is  $\Omega_1 > N$  such that  $|a_n \otimes' a| < 1$  for all  $n > \Omega_1$ . So from (1) and (2) and using  $\mathfrak{d}(\iota\omega, \iota\nu) \leq \iota\mathfrak{d}(\omega, \nu)$  for all  $\omega, \nu \in \mathcal{D}$  and  $0 \leq \iota \leq 1$  (for every  $n > \Omega_1$ , if  $a_n \otimes' a > 0$  then by (1) and if  $a_n \otimes' a < 0$  then by (2)) we have

$$\begin{aligned} \mathfrak{d}(a_n \omega, a\omega) &\leq |a_n \otimes' a| \mathfrak{d}(\omega, 0) \\ &= c \mathfrak{d}_{\mathbb{R}}(a_n, a) \\ &\leq c\epsilon. \end{aligned}$$

**Lemma 7.4.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  and  $\Omega^* \in \check{S}_{cp}$ . Then ,for each  $\epsilon > 0$ , there is  $\Omega \subset \Omega^*$  which is finite and a continuous mapping  $G$  of  $\Omega^*$  into  $co(\Omega)$  such that  $\mathfrak{d}(G\omega, \omega) < \epsilon$ .

**Proof.** Since  $\Omega^* \in \check{S}_{cp}$  therefore there is  $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\} \subset \Omega^*$  so that open balls  $B_{\mathfrak{d}}(\omega_i, \epsilon)$  cover  $B$ . Define

$$x_i(\omega) = \begin{cases} \epsilon \otimes' \mathfrak{d}(\omega, \omega_i) & \mathfrak{d}(\omega, \omega_i) < \epsilon \\ 0 & \mathfrak{d}(\omega, \omega_i) \geq \epsilon \end{cases},$$

where  $x_i(\omega)$  is a continuous function. Moreover, if  $\omega \in B_{\mathfrak{d}}(\omega_i, \epsilon)$  then  $x_i(\omega)$  is nonzero. Therefore for each  $\omega \in \Omega^*$  some  $x_i(\omega)$  in nonzero. Define a mapping  $G$  of  $\Omega^*$  into  $co(\Omega)$  by

$$G(\omega) = \frac{\sum_i x_i(\omega)\omega_i}{\sum x_i(\omega)}$$

By Lemmas 7.2 and 7.3  $G \in \check{S}_{ct}$ . Since  $G(\omega)$  is a convex combination of the points  $\omega_i$  that are in the open ball  $B_{\mathfrak{d}}(\omega, \epsilon)$  and  $B_{\mathfrak{d}}(\omega, \epsilon)$  is convex open ball for each  $\omega \in \Omega^*$ , therefore we have

$$\mathfrak{d}(G\omega, \omega) < \epsilon.$$

**Remark 7.2.** Lemma 7.4 is the generalization of the projection theorem in [119].

**Lemma 7.5.**  $co(\Omega)$ , where  $\Omega$  is finite, is compact in a linear space  $\mathcal{D}$  with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$ .

**Proof.** Let  $\Omega \subseteq \mathcal{D}$  with  $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ . Define a mapping  $f$  of  $\mathbb{R}^n$  into  $\mathcal{D}$  by

$$f(a_1, a_2, \dots, a_n) = a_1\omega_1 \otimes a_2\omega_2 \otimes \dots \otimes a_n\omega_n$$

and

$$K = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{R}, a_1 \otimes a_2 \otimes \dots \otimes a_n = 1\}.$$

Clearly  $f$  is continuous on  $K$  ( by Lemmas 7 and 8). Since  $K$  is closed and bounded, therefore its continuous image  $f(K) = co(\Omega)$  is compact.

**Corollary 7.1.** [85]  $co(\Omega) \in \overline{\Pi}_{cp}(\mathcal{D})$  where  $\Omega \subseteq \mathcal{D}$  and finite.

**Lemma 7.6.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  and  $f$  is a linear functional of  $\mathcal{D}$  into  $\mathbb{R}$ . If  $f$  possesses continuity at 0, then  $f$  is continuous on whole space  $\mathcal{D}$ .

**Proof.** Suppose  $f$  is continuous at 0. Let  $\omega_n$  be a sequence in  $\mathcal{D}$  that is convergent to  $\omega$ . Define  $a_n = \omega_n \otimes (\otimes' \omega)$ . By lemma 7.1,  $a_n$  converges to 0. This means that  $f(a_n)$  converges to  $f(0) = 0$ . Since  $f$  is linear,  $f(\omega_n)$  converges to  $f(\omega)$ .

**Lemma 7.7.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  with scalar multiplication is continuous in the 2nd variable and, for  $0 < a < 1$ ,  $h$  be a function of  $\mathcal{D}$  into  $\mathcal{D}$  defined by  $h(\omega) = a\omega$ . Then  $h$  is homeomorphism.

**Proof.** Let  $\omega_n \rightarrow \omega$ , then for  $\epsilon > 0$  there is  $N$  with

$$\mathfrak{d}(\omega_n, \omega) < \epsilon \text{ for every } n > N.$$

Now

$$\mathfrak{d}(h(\omega_n), h(\omega)) = \mathfrak{d}(a\omega_n, a\omega) \leq a\mathfrak{d}(\omega_n, \omega) < a\epsilon.$$

This shows that  $h$  is continuous. Also  $h$  is linear and bijective. Its inverse is given by  $h(\kappa) = a'\kappa$ , where  $aa' = a'a = 1$ . Since scalar multiplication is continuous in the second variable therefore  $h$  is continuous. Hence  $h$  is homeomorphism.

**Corollary 7.2.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  such that scalar multiplication is continuous in 2nd variable and, for  $\otimes' 1 < a < 0$ ,  $h$  be a function of  $\mathcal{D}$  into  $\mathcal{D}$  defined by  $h(\omega) = a\omega$ . Then  $h$  is homeomorphism.

**Lemma 7.8.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  with scalar multiplication is continuous in 2nd variable and  $U$  is open set. Then, for  $0 < a < 1$ ,  $aU$  is open in  $\mathcal{D}$ .

**Proof.** Define  $h$  of  $\mathcal{D}$  into  $\mathcal{D}$  by  $h(\omega) = a\omega$  for  $0 < a < 1$ . By Lemma 7.7,  $h$  is homeomorphism. Since  $U \in \mathcal{D}$  is open, therefore  $h(U) = aU$  is open.

**Lemma 7.9.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  such that scalar multiplication is continuous in the second variable.  $f$  is a linear of  $\mathcal{D}$  into  $\mathbb{R}$  with  $f(\omega)$  is nonzero for some  $\omega$ . If  $f \in \check{S}_b$  in some neighborhood of 0, then  $f \in \check{S}_{ct}$ .

**Proof.** If  $f \in \check{S}_b$  in some neighborhood  $U$  of 0, then there is  $\Omega_1 > 0$  such that  $\mathfrak{d}_{\mathbb{R}}(f(\omega), 0) <$

$\Omega_1$  for all  $\omega \in U$ . Let  $\varepsilon > 0$  and consider  $S_\varepsilon(0) = \{\omega \in \mathbb{R} : \mathfrak{d}_{\mathbb{R}}(\omega, 0) < \varepsilon\}$ . set  $\Omega_2 = \frac{\varepsilon}{\Omega_1^*}U$  such that  $\frac{\varepsilon}{\Omega_1^*} < 1$  for  $\Omega_1^* > \Omega_1$ . Here  $\Omega_2$  is a neighborhood of 0 (Lemma 7.8). We have to show that  $f(\Omega_2) \subset S_\varepsilon(0)$ . Let  $\omega \in \Omega_2$  then  $\omega = \frac{\varepsilon}{\Omega_1^*}\hat{u}$  for some  $\hat{u}$  in  $U$  and  $\frac{\Omega_1^*}{\varepsilon}\omega = \frac{\Omega_1^*}{\varepsilon}(\frac{\varepsilon}{\Omega_1^*}\hat{u}) = (\frac{\Omega_1^*}{\varepsilon}\frac{\varepsilon}{\Omega_1^*})\hat{u} = \hat{u} \in U$  so that  $\mathfrak{d}_{\mathbb{R}}(f(\frac{\Omega_1^*}{\varepsilon}\omega), 0) < \Omega_1$ . Since  $f$  is linear and  $\mathfrak{d}_{\mathbb{R}}$  is homogeneous, therefore  $\mathfrak{d}_{\mathbb{R}}(f(\omega), 0) < \varepsilon$ . This means that  $f(\omega) \in S_\varepsilon(0)$ . Hence  $f(\Omega_2) \subset S_\varepsilon(0)$  and  $f$  is continuous at 0. By the Lemma 7.6  $f$  is continuous at  $\mathfrak{D}$ .

**Lemma 7.10.** Suppose  $\mathfrak{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  such that scalar multiplication is continuous in the second variable and  $U \subseteq \mathfrak{D}$  is neighborhood about 0. Then  $U$  contains a balanced neighborhood, say  $\Omega_2$ , of 0.

**Proof.** Let  $U \subseteq \mathfrak{D}$  is neighborhood about 0. Define a function  $h$  of  $\mathfrak{D}$  into  $\mathfrak{D}$  by  $h(\omega) = a\omega$  for fixed  $a \in (\mathfrak{D}'1, 1) \mathfrak{D}'\{0\}$ . Then  $h$  is a homeomorphism by (Lemma 7.7). So for  $U$  in  $\mathfrak{D}$  there is some  $\mathfrak{Y}$  (containing 0) in  $\mathfrak{D}$ , for every fixed  $a \in (\mathfrak{D}'1, 1) \mathfrak{D}'\{0\}$ , such that  $h(\mathfrak{Y}) = a\mathfrak{Y} \subset U$ . Let  $\Omega_2 = \cup a\mathfrak{Y}$ . Then  $0 \in \Omega_2 \subset U$  and  $\Omega_2$  is a neighborhood of 0. Let  $\omega \in \Omega_2$  and  $\alpha \in \mathbb{R}$  such that  $|\alpha| \leq 1$ . If  $\omega \in \Omega_2$ , then  $\omega \in a\mathfrak{Y}$  for some  $a\mathfrak{Y}$  and, hence,  $\alpha\omega \in \alpha(a\mathfrak{Y}) = (\alpha a)\mathfrak{Y}$ . Since  $|\alpha a| \leq |a| < 1$  therefore  $\alpha\omega \in \Omega_2$  and  $\Omega_2$  is balanced.

**Lemma 7.11.** Let  $\mathfrak{D}$  be  $n$ -dimensional linear space with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  such that scalar multiplication is continuous in the second variable. Then  $\mathbb{R}^n$  is (linearly) homeomorphic to  $\mathfrak{D}$ .

**Proof.** Let  $\beta = \{\omega_1, \omega_2, \dots, \omega_n\}$  and  $\Upsilon = \{w_1, w_2, \dots, w_n\}$  be bases sets of  $\mathbb{R}^n$  and  $\mathfrak{D}$  respectively. Define a mapping  $\bar{\mathfrak{I}}$  of  $\mathbb{R}^n$  into  $\mathfrak{D}$  by  $\bar{\mathfrak{I}}(\omega) = \bar{\mathfrak{I}}(a_1\omega_1 \otimes a_2\omega_2 \otimes \dots \otimes a_n\omega_n) = a_1\omega_1 \otimes a_2\omega_2 \otimes \dots \otimes a_n\omega_n$ . Then  $\bar{\mathfrak{I}}$  is linear and bijective. Also  $\bar{\mathfrak{I}}$  is continuous by (Lemmas 7.2 and 7.3). So  $\bar{\mathfrak{I}}^{\mathfrak{D}'1}$  of  $\mathfrak{D}$  into  $\mathbb{R}^n$  exists and is linear. We have to show that  $\bar{\mathfrak{I}}^{\mathfrak{D}'1}$  is continuous. Consider the set  $S = \{(a_1, a_2, \dots, a_n) \in \mathbb{R}^n : a_1^2 \otimes a_2^2 \otimes \dots \otimes a_n^2 = 1\}$ . The set  $S$  is compact in  $\mathbb{R}^n$ . Since  $\bar{\mathfrak{I}}$  is continuous of  $\mathbb{R}^n$  into  $\mathfrak{D}$  therefore  $\bar{\mathfrak{I}}(S)$  is compact in  $\mathfrak{D}$ . Also,  $\bar{\mathfrak{I}}(S)$  is closed in  $\mathfrak{D}$  because  $\mathfrak{D}$  is a MS. Set  $\Xi = \mathfrak{D} \mathfrak{D}'\bar{\mathfrak{I}}(S)$ . Since  $\bar{\mathfrak{I}}$  is linear and bijective and 0 is not in  $S$ , the element 0 of  $\mathfrak{D}$  does not belong to  $\bar{\mathfrak{I}}(S)$ . Hence the set  $\Xi$  is an open set containing 0. By Lemma 7.10 there is a balanced neighborhood  $\Omega_2$  of 0 such that  $\Omega_2$  is a subset of  $\Xi$  and hence the intersection of  $\Omega_2$  and  $\bar{\mathfrak{I}}(S)$  is empty. Since  $\bar{\mathfrak{I}}^{\mathfrak{D}'1}$  is bijective and linear, therefore intersection of  $\bar{\mathfrak{I}}^{\mathfrak{D}'1}(\Omega_2)$  and  $S$  is empty, and  $\bar{\mathfrak{I}}^{\mathfrak{D}'1}(\Omega_2)$  is a balanced neighborhood of 0. Now we show that if  $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \bar{\mathfrak{I}}^{\mathfrak{D}'1}(\Omega_2)$  then  $\omega_1^2 \otimes \omega_2^2 \otimes \dots \otimes \omega_n^2 < 1$ . Let  $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \bar{\mathfrak{I}}^{\mathfrak{D}'1}(\Omega_2)$ .

Suppose  $\omega_1^2 \otimes \omega_2^2 \otimes \dots \otimes \omega_n^2 \geq 1$ , then  $\frac{\omega}{\sqrt{\omega_1^2 \otimes \omega_2^2 \otimes \dots \otimes \omega_n^2}} \in S$ . Also  $\frac{\omega}{\sqrt{\omega_1^2 \otimes \omega_2^2 \otimes \dots \otimes \omega_n^2}} \in \bar{\mathcal{I}}^{\otimes'1}(\Omega_2)$  because  $\bar{\mathcal{I}}^{\otimes'1}(\Omega_2)$  is a balanced neighborhood of 0. Since intersection of  $\bar{\mathcal{I}}^{\otimes'1}(\Omega_2)$  and  $S$  is empty, so our supposition is wrong and  $\omega_1^2 \otimes \omega_2^2 \otimes \dots \otimes \omega_n^2 < 1$  for  $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \bar{\mathcal{I}}^{\otimes'1}(\Omega_2)$ . Since  $\bar{\mathcal{I}}^{\otimes'1}$  is a linear transformation of  $\mathcal{D}$  into  $\mathbb{R}^n$  by  $\bar{\mathcal{I}}^{\otimes'1}(v) = (f^1(v), f^2(v), \dots, f^n(v))$ , so  $f^i$ , for  $i$  from 1 to  $n$ , is a linear functional of  $\mathcal{D}$  into  $\mathbb{R}$ . Since  $\Omega_2$  is a open neighborhood of 0 and  $\bar{\mathcal{I}}^{\otimes'1}(v) \in \bar{\mathcal{I}}^{\otimes'1}(\Omega_2)$  for  $v \in \Omega_2$ , therefore  $(f^1(v))^2 \otimes (f^2(v))^2 \otimes \dots \otimes (f^n(v))^2 < 1$  and hence  $\mathfrak{d}_{\mathbb{R}^n}(\bar{\mathcal{I}}^{\otimes'1}(v), 0) < 1$ . It means that  $\bar{\mathcal{I}}^{\otimes'1}$  is bounded at  $\Omega_2$  and also  $f^i$  is bounded at  $\Omega_2$ . Since  $f^i$  is linear functional which is bounded in open neighborhood  $\Omega_2$  of 0, therefore by Lemma 7.9  $f^i$  is continuous at  $\mathcal{D}$  and hence  $\bar{\mathcal{I}}^{\otimes'1}$  is continuous at  $\mathcal{D}$ .

**Theorem 7.1.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  such that scalar multiplication is continuous in the second variable,  $\Omega_1 \in \bar{\mathcal{I}}_{cl,cv}(\mathcal{D})$  and  $S \in \check{S}_{ct}$  whose domain is  $\Omega_1$  and image lies in a compact set  $\Omega^* \subset \Omega_1$ . Then  $S$  has a FP.

**Proof.** The set  $\Omega^*$  is compact, therefore (by Lemma 7.4) for  $n = 1, 2, 3, \dots$ , there is a finite set  $\Omega \subset \Omega^*$  and  $G_n \in \check{S}_{ct}$  of  $\Omega^*$  into  $co(\Omega)$  such that  $\mathfrak{d}(G_n \omega, \omega) < \frac{1}{n}$ . Here  $co(\Omega) \subset \Omega_1$  because  $\Omega \subset \Omega^* \subset \Omega_1$ . Define a mapping  $\mathcal{H} = G_n \circ S$  of  $co(\Omega)$  into  $co(\Omega)$ .  $\mathcal{H}$  is continuous because  $G_n$  and  $S$  are continuous. Since  $\Omega$  is finite set,  $co(\Omega) \subset \Omega_1$  and span of  $\Omega$  is finite dimensional subspace that is linearly homeomorphic to euclidean space, therefore  $co(\Omega) \in \check{S}_{cp,cv}(\text{span of } \Omega)$  that is homeomorphic to compact convex subset of finite dimensional euclidean space. So  $\mathcal{H}$  has a FP (Brouwer theorem) and there is  $m_n$  with  $\mathcal{H}m_n = m_n$ . Also

$$\mathfrak{d}(Sm_n, \kappa_n) \leq \mathfrak{d}(Sm_n, \mathcal{H}m_n) \otimes \mathfrak{d}(\mathcal{H}m_n, m_n) = \mathfrak{d}(Sm_n, \mathcal{H}m_n) = \mathfrak{d}(Sm_n, G_n \circ Sm_n) \leq \frac{1}{n}.$$

By [[119], Theorem 1.1.3]  $S$  has FP.

**Remark 7.3.** If the set  $\Omega$  in the above theorem is linearly independent, Then FP can be obtained.

**Proof:** Suppose  $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\} \subseteq \mathcal{D}$  which is linearly independent and  $K = \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{R}, a_1 \otimes a_2 \otimes \dots \otimes a_n = 1\}$ . Define a function  $f$  from  $\mathbb{R}^n$  into  $\mathcal{D}$  such that  $f(a_1, a_2, \dots, a_n) = a_1 \omega_1 \otimes a_2 \omega_2 \otimes \dots \otimes a_n \omega_n$ . Here  $f$  is continuous on  $K$  by Lemma 7.2 and 7.3 and  $f(K) = co(\Omega)$ . Also  $f$  is bijective on  $K$  and  $K$  is compact. Therefore (munkres)  $f$  is homeomorphism. Since  $K$  has FP property and homeomorphic to  $co(\Omega)$ , so  $co(\Omega)$  has FP property [119].

**Corollary 7.3.** [119] (Schauder second theorem) Let  $\Omega_1 \in \overline{\Pi}_{cv}(\mathcal{D})$  and  $S \in \check{S}_{ct}$  of  $\Omega_1$  into  $\Omega^* \subset \Omega_1$ , where  $\Omega^* \in \check{S}_{cp}$ . Then  $S$  has a FP.

**Proof.** Since  $\mathcal{D}$  is normed space so scalar multiplication is continuous. Also the metric  $\mathfrak{d}$  induced by normed space satisfy the conditions  $(a^*)$  and  $(b^*)$ .

**Lemma 7.12.** Suppose  $\mathcal{D}$  be a space which is linear with metric  $\mathfrak{d}$  with  $(a^*)$  and  $\Omega_1 \subseteq \mathcal{D}$ . If  $\zeta$  is a contraction of  $\Omega_1$  into  $\mathcal{D}$  then  $I \otimes' \zeta$  is a homeomorphism of  $\Omega_1$  into  $(I \otimes' \zeta)\Omega_1$ .

**Proof.** Clearly  $I \otimes' \zeta$  is continuous. From second triangle inequality

$$|\mathfrak{d}(\omega, \gamma) \otimes' \mathfrak{d}(\nu, w)| \leq \mathfrak{d}(\omega, \nu) \otimes' \mathfrak{d}(\gamma, w)$$

we have

$$\mathfrak{d}(\omega, 0) \otimes' \mathfrak{d}(\nu, 0) \leq \mathfrak{d}(\omega, \nu).$$

Replacing  $\omega$  by  $\omega \otimes' \nu$  and  $\nu$  by  $\zeta\omega \otimes' \zeta\nu$  we obtain

$$\begin{aligned} & \mathfrak{d}(\omega \otimes' \nu, 0) \otimes' \mathfrak{d}(\zeta\omega \otimes' \zeta\nu, 0) \\ & \leq \mathfrak{d}(\omega \otimes' \nu, \zeta\omega \otimes' \zeta\nu) \end{aligned}$$

Now since  $\mathfrak{d}(\omega \otimes' \nu, 0) = \mathfrak{d}(\omega, \nu)$ ,  $\mathfrak{d}(\zeta\omega \otimes' \zeta\nu, 0) = \mathfrak{d}(\zeta\omega, \zeta\nu)$ , and

$$\begin{aligned} \mathfrak{d}(\omega \otimes' \nu, \zeta\omega \otimes' \zeta\nu) &= \mathfrak{d}(\omega \otimes' \zeta\omega, \nu \otimes' \zeta\nu) \\ &= \mathfrak{d}((1 \otimes' \zeta)\omega, (1 \otimes' \zeta)\nu), \end{aligned}$$

therefore

$$\mathfrak{d}(\omega, \nu) \otimes' \mathfrak{d}(\zeta\omega, \zeta\nu) \leq \mathfrak{d}((1 \otimes' \zeta)\omega, (1 \otimes' \zeta)\nu)$$

Since  $\zeta$  is contraction,

$$\begin{aligned} \mathfrak{d}((1 \otimes' \zeta)\omega, (1 \otimes' \zeta)\nu) &\geq \mathfrak{d}(\omega, \nu) \otimes' \mathfrak{d}(\zeta\omega, \zeta\nu) \\ &\geq \mathfrak{d}(\omega, \nu) \otimes' k\mathfrak{d}(\omega, \nu) \\ &= (1 \otimes' k)\mathfrak{d}(\omega, \nu), \quad k \in (0, 1) \end{aligned}$$

so that  $I \otimes' \zeta$  is bijective function and the inverse of  $(I \otimes' \zeta)$  exists. For  $\hat{u}$  and  $v$  in  $(I \otimes' \zeta)\Omega_1$  there are  $\omega$  and  $\nu$  in  $\Omega_1$   $\hat{u} = (I \otimes' \zeta)\omega$ ,  $v = (I \otimes' \zeta)\nu$  and  $\omega = (I \otimes' \zeta)^{\otimes'1}\hat{u}$ ,  $\nu = (I \otimes' \zeta)^{\otimes'1}v$ .

Putting in the above inequality

$$\mathfrak{d}(\hat{u}, v) \geq (1 \otimes' k)\mathfrak{d}((I \otimes' \zeta)^{\otimes'1}\hat{u}, (I \otimes' \zeta)v)$$

implies

$$\mathfrak{d}((I \otimes' \zeta)^{\otimes'1}\hat{u}, (I \otimes' \zeta)^{\otimes'1}v) \leq \frac{1}{(1 \otimes' k)}\mathfrak{d}(\hat{u}, v).$$

Thus  $(I \otimes' \zeta)^{\otimes'1}$  is continuous.

**Remark 7.4.** Lemma 7.12 is generalization of the Lemma 4.4.2 in [119].

**Theorem 7.2.** For a given complete linear space  $\mathcal{D}$  with metric  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  such that scalar multiplication is continuous in the second variable and  $\Omega_1 \in \overline{\mathbb{I}}_{cl,cv}(\mathcal{D})$ . Suppose that  $\xi$  and  $\zeta$  are mappings of  $\Omega_1$  into  $\mathcal{D}$  such that

1  $\xi\omega \otimes \zeta\nu \in \Omega_1$  for all  $\omega, \nu \in \Omega_1$ ;

2  $\xi \in \check{S}_{cp,ct}$ ;

3  $\zeta$  is contraction;

Then there is a  $\omega$  in  $\Omega_1$  satisfying  $\xi\omega \otimes \zeta\omega = \omega$ .

**Proof** For each fixed  $\hat{u}$  in  $\Omega_1$ , Consider a function  $\mathcal{H}$  of  $\Omega_1$  into  $\Omega_1$  by  $\mathcal{H}(\gamma) = \zeta\gamma \otimes \xi\hat{u}$ . The function  $\mathcal{H}$  is contraction mapping because

$$\mathfrak{d}(\mathcal{H}(\gamma), \mathcal{H}(\nu)) = \mathfrak{d}(\zeta\gamma \otimes \xi\hat{u}, \zeta\nu \otimes \xi\hat{u}) = \mathfrak{d}(\zeta\gamma, \zeta\nu) \leq \alpha\mathfrak{d}(\gamma, \nu).$$

Thus there is a unique  $\kappa$  in  $\Omega_1$  with  $\mathcal{H}(\kappa) = \kappa$  that is  $\zeta\kappa \otimes \xi\hat{u} = \kappa$ . Thus using Lemma 7.12  $\kappa = (I \otimes' \zeta)^{\otimes'1}\xi\hat{u} \in \Omega_1$ . Since  $(I \otimes' \zeta)^{\otimes'1}$  is continuous and  $\xi$  is compact and continuous, therefore  $(I \otimes' \zeta)^{\otimes'1}\xi$  is continuous and compact mapping of  $\Omega_1$  into  $\Omega_1$ . By Theorem 7.1.,  $(I \otimes' \zeta)^{\otimes'1}\xi$  has FP  $\omega$  in  $\Omega_1$  such that  $\xi\omega \otimes \zeta\omega = \omega$ .

**Corollary 7.4.** [119] (Krasnoselskii) Let  $\mathcal{D}$  be a BS and  $\Omega_1 \in \overline{\mathbb{I}}_{cl,cv}(\mathcal{D})$ . Suppose that  $\xi$  and  $\zeta$  are mappings of  $\Omega_1$  into  $\mathcal{D}$  with

1  $\xi\omega \otimes \zeta\nu \in \Omega_1$  for all  $\omega, \nu \in \Omega_1$ ;

2  $\xi \in \check{S}_{cp,ct}$ ;

3  $\zeta$  is contraction;

Then there is a  $\omega$  in  $\Omega_1$  with  $\xi\omega \otimes \zeta\omega = \omega$ .

**Proof.** If  $\mathcal{D}$  is BS then the scalar multiplication is continuous and  $\mathfrak{d}$  induced by BS satisfy the conditions  $(a^*)$  and  $(b^*)$ . Also  $\mathcal{D}$  is complete space.

**Lemma 7.13.** [130] Let  $\Omega_1 \in \overline{\mathfrak{N}}_{cl,cv}(\mathcal{D})$ , where  $\mathcal{D}$  is a complete MS and  $\zeta$  is an expansive mapping of  $\Omega_1$  into  $\mathcal{D}$  with  $\Omega_1 \subset \zeta(\Omega_1)$ . Then there exists a unique  $\kappa \in \Omega_1$  such that  $\zeta\kappa = \kappa$ .

**Theorem 7.3.** Let  $\mathcal{D}$  be a complete linear space with  $\mathfrak{d}$  satisfying the conditions  $(a^*)$  and  $(b^*)$  such that scalar multiplication is continuous in the second variable and  $\Omega_1 \neq \phi$  which is closed convex. Suppose  $\xi$  and  $\zeta$  are mappings of  $\Omega_1$  into  $\mathcal{D}$  such that

(1)  $\xi \in \check{S}_{cp,ct}$ ;

(2)  $\zeta$  is an expansive mapping for  $h > 1$ ;

(3) For  $\hat{u} \in \xi(\Omega_1)$  implies  $\Omega_1 \subset \{\hat{u}\} \otimes \zeta(\Omega_1)$ .

Then there exists  $\kappa \in \Omega_1$  such that  $\xi\kappa \otimes \zeta\kappa = \kappa$ .

**Proof:** Let  $\hat{u} \in \xi(\Omega_1)$ . Define a mapping  $\mathcal{H}$  of  $\Omega_1$  into  $\mathcal{D}$  by  $\mathcal{H}(\omega) = \hat{u} \otimes \zeta(\omega)$ . The mapping  $\mathcal{H}$  is expansive because

$$\mathfrak{d}(\mathcal{H}(\omega), \mathcal{H}(\nu)) = \mathfrak{d}(\zeta(\omega) \otimes \hat{u}, \zeta(\nu) \otimes \hat{u}) = \mathfrak{d}(\zeta(\omega), \zeta(\nu)) \geq h\mathfrak{d}(\omega, \nu).$$

Also, by (3),  $\Omega_1 \subset \{\hat{u}\} \otimes \zeta(\Omega_1) = \mathcal{H}(\Omega_1)$ . By Lemma 7.13, there is a unique  $G\hat{u} \in \Omega_1$  such that  $G\hat{u} = \hat{u} \otimes \zeta(G\hat{u})$ . Now

$$\begin{aligned} \mathfrak{d}(\zeta(G(\hat{u}_1)), \zeta(G(\hat{u}_2))) &= \mathfrak{d}(G(\hat{u}_1) \otimes' \hat{u}_1, G(\hat{u}_2) \otimes' \hat{u}_2) \\ &\leq \mathfrak{d}(G(\hat{u}_1), G(\hat{u}_2)) \otimes \mathfrak{d}(\hat{u}_1, \hat{u}_2). \end{aligned}$$

Since  $\zeta$  is expansive,

$$\mathfrak{d}(\zeta(G(\hat{u}_1)), \zeta(G(\hat{u}_2))) \geq h\mathfrak{d}(G(\hat{u}_1), G(\hat{u}_2)).$$

From the above expressions

$$h\mathfrak{d}(G(\hat{u}_1), G(\hat{u}_2)) \leq \mathfrak{d}(G(\hat{u}_1), G(\hat{u}_2)) \otimes \mathfrak{d}(\hat{u}_1, \hat{u}_2).$$

Simplifying, we obtain

$$\mathfrak{d}(G(\hat{u}_1), G(\hat{u}_2)) \leq \frac{1}{h \otimes' 1} \mathfrak{d}(\hat{u}_1, \hat{u}_2).$$

This shows that the mapping  $G$  of  $\xi(\Omega_1)$  into  $\Omega_1$  is continuous. Since  $\xi$  is continuous, therefore  $G \circ \xi$  is continuous mapping of  $\Omega_1$  into  $\Omega_1$ . Also  $\xi \in \check{S}_{cp}$  and  $G \in \check{S}_{ct}$ . Hence  $G \circ \xi$  is a mapping of  $\Omega_1$  into a compact subset of  $\Omega_1$ . Thus by Theorem 7.1 there is a point  $m \in \Omega_1$  such that  $(G \circ \xi)m = m$ . Since  $\xi m \in \xi(\Omega_1)$  therefore there is unique  $G(\xi m)$  such that  $G(\xi m) = \xi m \otimes \zeta(G(\xi m))$ . Hence  $m = G(\xi m) = \xi m \otimes \zeta(G(\xi m)) = \xi m \otimes \zeta m$ .

**Remark 7.5.** The above theorem is variant of Krasnoselskii FP theorem and generalization of the Theorem 2.2 in [128].

### 7.3 Conclusion

An interesting approach is used to prove Schauder's first and second theorems in a generalized topological structure. Using these results Krasnosilskii's FP theorem is also generalized. These results constitute a base for existence theory of DEs and fuzzy DEs in more general spaces.

## Chapter 8

# Existence Results for Fractional Order Boundary Value Problems

### 8.1 Introduction

Fractional calculus is a natural generalization of ordinary calculus when the order of the derivative is non-integer. Many fractional operators are defined like, Riemman-Liouville, Hadamard, Caputo and Grunwald-Letnikov [37, 99, 105]. The choice of the operator depends upon the considered system. Due to the variations in the physical systems, many researchers are defining a number of fractional operators, for details, we refer the readers to the article [122].

Many of the complex physical problems may be better understood in the framework of fractional differential equations. One can find applications of fractional calculus in diverse fields like biology, chemistry, physics, fluid mechanics, economics and social sciences, etc. [1, 73, 81, 105, 133].

For theoretical details of fractional differential equations (FDEs) see [73, 82] and the references cited therein. The ERs for non-integer order differential equations are discussed in many articles, for example, see [24, 35, 53, 59]. The theory of fractional equations involving different kinds of BCs has always been remained a field of interest in physical sciences. Integral and Non-local BCs are mostly used where the classical BCs are failed to examine many physical properties of the models. Many researchers have already been involved in the existence theory of BVPs involving integral and non-local boundary conditions, for instant see

[8, 9, 10, 13, 12, 11, 7, 97, 115]. Recently Alsaedi et al. [18], discussed the existence theory of the following second order BVP with non-separated, non-local, multi-point integral BCs on an generalized domain.

$$\begin{cases} \hat{u}''(\iota) = f(\iota, \hat{u}(\iota)), & a < \iota < \Sigma, \quad a, \Sigma \in \mathbb{R}, \\ \alpha_1 \hat{u}(a) \oplus \alpha_2 \hat{u}(\Sigma) = \alpha_3 \int_a^\zeta \hat{u}(\tau) d\tau \oplus \sum_{\iota=1}^{\hat{n}} \gamma_\iota \hat{u}(v_\iota), \\ \beta_1 \hat{u}'(a) \oplus \beta_2 \hat{u}'(\Sigma) = \beta_3 \int_a^\zeta \hat{u}'(\kappa) d\kappa \oplus \sum_{\iota=1}^{\hat{n}} \rho_\iota \hat{u}'(v_\iota), \end{cases} \quad (1.1)$$

where  $a \leq \zeta \leq v_i \leq \Sigma$ . Motivated by [18], we prove the ERs for the following Caputo FBVP

$${}^c D^{\hat{n}} \hat{u}(\iota) = \mathbb{P}(\iota, \hat{u}(\iota)) \quad \text{where } 1 < \hat{n} \leq 2, \quad (1.2)$$

with boundary conditions

$$\begin{aligned} \gamma_1 \hat{u}(a) \oplus \gamma_2 \hat{u}(\Sigma) &= \gamma_3 \int_a^\zeta \hat{u}(\tau) d\tau \oplus \gamma_4 \sum_{\iota=1}^e \vartheta_\iota \hat{u}(v_\iota) \\ \varrho_1 {}^c D^{\mathfrak{k}} \hat{u}(a) \oplus \varrho_2 {}^c D^{\mathfrak{k}} \hat{u}(\Sigma) &= \varrho_3 \int_a^\zeta {}^c D^{\mathfrak{k}} \hat{u}(\tau) d\tau \oplus \varrho_4 \sum_{\iota=1}^e \rho_\iota {}^c D^{\mathfrak{k}} \hat{u}(v_\iota), \quad 0 < \mathfrak{k} \leq 1. \end{aligned} \quad (1.3)$$

Clearly for  $\hat{n} = 2$ ,  $\mathfrak{k} = 1$  and  $\gamma_4 = \varrho_4 = 1$ , the above problem (1.2)  $\oplus$  (1.3) reduces to (1.1).

The remaining part of the chapter is arranged in this way. Section 8.2 contains the ERs for the FBVP (1.2)-(1.3) which are proved by applying Schaefer-and Krasnoselskii's FP theorems. In Section 8.3, existence of unique solution is discussed via Banach contraction result.

## 8.2 Existence results via Schaefer and Krasnoselskii's fixed point theorems

Let  $\mathcal{D} = C[a, \Sigma]$  is BS in which there are continuous functions whose domain is  $[a, \Sigma] \subseteq \mathbb{R}$  with

$$\|\hat{u}\| = \sup_{\iota \in [a, \Sigma]} |\hat{u}(\iota)| \quad (2.1)$$

for  $\hat{u} \in \mathcal{D}$ . Set

$$\alpha_1 = \gamma_1 \otimes \gamma_2 \otimes' \gamma_3 (\zeta \otimes' a) \otimes' \gamma_4 \sum_{\iota=1}^e \vartheta_\iota \quad (2.2)$$

and

$$\beta = \varrho_2 \frac{(\Sigma \otimes' a)^{1 \otimes' \mathfrak{k}}}{\mathfrak{d}(2 \otimes' \mathfrak{k})} \otimes' \varrho_3 \frac{(\zeta \otimes' a)^{2 \otimes' \mathfrak{k}}}{\mathfrak{d}(3 \otimes' \mathfrak{k})} \otimes' \varrho_4 \sum_{\iota=1}^e \rho_\iota \frac{(v_\iota \otimes' a)^{1 \otimes' \mathfrak{k}}}{\mathfrak{d}(2 \otimes' \mathfrak{k})}. \quad (2.3)$$

Here is very important result.

**Lemma 8.1.** Let  $\sigma \in \mathcal{D}$  and  $\alpha_1$  and  $\beta$  are nonzero. Then solution of linear problem (2.4) with BCs (1.3)

$${}^c D^{\hat{n}} \hat{u}(\iota) = \sigma(\iota), \text{ for } a < \iota < \Sigma \quad (2.4)$$

has an integral representation given as

$$\begin{aligned} \hat{u}(\iota) = & \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n})} \sigma(\varepsilon) d\varepsilon \otimes \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \varepsilon)^{\hat{n} \otimes' \mathfrak{k}} \sigma(\varepsilon) d\varepsilon \\ & \otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota) \right) (\Sigma \otimes' \varepsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \sigma(\varepsilon) d\varepsilon \\ & \otimes' \frac{1}{K} \sum_{\iota=1}^e \int_a^{v_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(v_\iota \otimes' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_4}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \rho_\iota \eta(\iota) \right) (v_\iota \otimes' \varepsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \sigma(\varepsilon) d\varepsilon \end{aligned} \quad (2.5)$$

where

$$\alpha_2 = \gamma_2 (\Sigma \otimes' a) \otimes' \gamma_3 \frac{(\zeta \otimes' a)^2}{2} \otimes' \gamma_4 \sum_{\iota=1}^e \vartheta_\iota (v_\iota \otimes' a), \quad (2.6)$$

$$\eta(\iota) = \alpha_1 (\iota \otimes' a) \otimes' \alpha_2 \quad (2.7)$$

and

$$K = \beta \alpha_1. \quad (2.8)$$

**Proof.** The general solution for linear problem

$${}^c D^{\hat{n}} \hat{u}(\iota) = \sigma(\iota), \text{ } a < \iota < \Sigma \text{ and } 1 < \hat{n} \leq 2$$

as given in [73], is

$$\hat{u}(\iota) = \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n})} \sigma(\varepsilon) d\varepsilon \otimes' d_1 \otimes' d_2 (\iota \otimes' a) \quad (2.9)$$

for some real constants  $d_1$  and  $d_2$ . Making use of (1.3), (2.2) and (2.6) in (2.9), we have

$$d_1 \alpha_1 \otimes d_2 \alpha_2 = \gamma_2 \int_a^\Sigma \frac{(\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon \otimes' \gamma_3 \int_a^\zeta \frac{(\zeta \otimes' \varepsilon)^{\dot{n}}}{\mathfrak{d}(\dot{n} \otimes 1)} \sigma(\varepsilon) d\varepsilon \\ \otimes' \gamma_4 \sum_{\iota=1}^e \vartheta_\iota \int_a^{v_\iota} \frac{(v_\iota \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon.$$

$$d_1 = \frac{1}{\alpha_1} \left( \otimes' d_2 \alpha_2 \otimes' \gamma_2 \int_a^\Sigma \frac{(\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon \right. \\ \left. \otimes' \gamma_3 \int_a^\zeta \frac{(\zeta \otimes' \varepsilon)^{\dot{n}}}{\mathfrak{d}(\dot{n} \otimes 1)} \sigma(\varepsilon) d\varepsilon \otimes' \gamma_4 \sum_{\iota=1}^e \vartheta_\iota \int_a^{v_\iota} \frac{(v_\iota \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon \right). \quad (2.10)$$

From (2.9)

$$\hat{u}(\iota) = \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon \otimes' d_1 \otimes' d_2 \iota.$$

Taking Caputo FD of order  $\mathfrak{k} \in (0, 1)$  on both sides and using its properties, we get

$${}^c D^{\mathfrak{k}} \hat{u}(\iota) = \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\dot{n} \otimes' \mathfrak{k} \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' \mathfrak{k})} \sigma(\varepsilon) d\varepsilon \otimes' d_2 \frac{(\iota \otimes' a)^{1 \otimes' \mathfrak{k}}}{\mathfrak{d}(2 \otimes' \mathfrak{k})}. \quad (2.11)$$

Now using the second boundary condition from (1.3) in (2.11) we get

$$\varrho_2 \int_a^\Sigma \frac{(\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' \mathfrak{k} \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' \mathfrak{k})} \sigma(\varepsilon) d\varepsilon \otimes' \varrho_2 d_2 \frac{(\Sigma \otimes' a)^{1 \otimes' \mathfrak{k}}}{\mathfrak{d}(2 \otimes' \mathfrak{k})} \\ = \varrho_3 \int_a^\zeta \int_a^\tau \frac{(\tau \otimes' \varepsilon)^{\dot{n} \otimes' \mathfrak{k} \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' \mathfrak{k})} \sigma(\varepsilon) d\varepsilon d\tau \otimes' \varrho_3 d_2 \frac{(\zeta \otimes' a)^{2 \otimes' \mathfrak{k}}}{(2 \otimes' \mathfrak{k}) \mathfrak{d}(2 \otimes' \mathfrak{k})} \\ \otimes' \varrho_4 \sum_{\iota=1}^e \rho_\iota \int_a^{v_\iota} \frac{(v_\iota \otimes' \varepsilon)^{\dot{n} \otimes' \mathfrak{k} \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' \mathfrak{k})} \sigma(\varepsilon) d\varepsilon \otimes' \varrho_4 \sum_{\iota=1}^e \rho_\iota d_2 \frac{(v_\iota \otimes' a)^{1 \otimes' \mathfrak{k}}}{\mathfrak{d}(2 \otimes' \mathfrak{k})},$$

simplifying, we obtain

$$\begin{aligned}
& d_2 \left( \varrho_2 \frac{(\Sigma \otimes' a)^{1 \otimes' k}}{\mathfrak{d}(2 \otimes' k)} \otimes' \varrho_3 \frac{(\zeta \otimes' a)^{2 \otimes' k}}{(2 \otimes' k) \mathfrak{d}(2 \otimes' k)} \otimes' \varrho_4 \sum_{i=1}^e \rho_i \frac{(v_i \otimes' a)^{1 \otimes' k}}{\mathfrak{d}(2 \otimes' k)} \right) \\
= & \varrho_2 \int_a^\Sigma \frac{(\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' k)} \sigma(\varepsilon) d\varepsilon \otimes' \varrho_4 \sum_{i=1}^e \rho_i \int_a^{v_i} \frac{(v_i \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' k)} \sigma(\varepsilon) d\varepsilon \\
& \otimes' \varrho_3 \int_a^\zeta \frac{(\zeta \otimes' \varepsilon)^{\dot{n} \otimes' k}}{(\dot{n} \otimes' k) \mathfrak{d}(\dot{n} \otimes' k)} \sigma(\varepsilon) d\varepsilon.
\end{aligned}$$

Using (2.3) in above equation, we have

$$\begin{aligned}
d_2 = & \frac{1}{\beta} \left( \varrho_2 \int_a^\Sigma \frac{(\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' k)} \sigma(\varepsilon) d\varepsilon \otimes' \varrho_3 \int_a^\zeta \frac{(\zeta \otimes' \varepsilon)^{\dot{n} \otimes' k}}{\mathfrak{d}(\dot{n} \otimes' k \otimes' 1)} \sigma(\varepsilon) d\varepsilon \right. \\
& \left. \otimes' \varrho_4 \sum_{i=1}^e \rho_i \int_a^{v_i} \frac{(v_i \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' k)} \sigma(\varepsilon) d\varepsilon \right), \tag{2.12}
\end{aligned}$$

substituting (2.12) in (2.10) gives

$$\begin{aligned}
d_1 = & \frac{\otimes' \alpha_2}{\beta \alpha_1} \left( \varrho_2 \int_a^\Sigma \frac{(\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' k)} \sigma(\varepsilon) d\varepsilon \otimes' \varrho_3 \int_a^\zeta \frac{(\zeta \otimes' \varepsilon)^{\dot{n} \otimes' k}}{\mathfrak{d}(\dot{n} \otimes' k \otimes' 1)} \sigma(\varepsilon) d\varepsilon \right. \\
& \left. \otimes' \varrho_4 \sum_{i=1}^e \rho_i \int_a^{v_i} \frac{(v_i \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1}}{\mathfrak{d}(\dot{n} \otimes' k)} \sigma(\varepsilon) d\varepsilon \right) \otimes' \frac{1}{\alpha_1} \left( \gamma_2 \int_a^\Sigma \frac{(\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon \right. \\
& \left. \otimes' \gamma_3 \int_a^\zeta \frac{(\zeta \otimes' \varepsilon)^{\dot{n}}}{\mathfrak{d}(\dot{n} \otimes' 1)} \sigma(\varepsilon) d\varepsilon \otimes' \gamma_4 \sum_{i=1}^e \vartheta_i \int_a^{v_i} \frac{(v_i \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon \right). \tag{2.13}
\end{aligned}$$

Making use of (2.7), (2.8), (2.12) and (2.13) in (2.9), we get the required solution

$$\begin{aligned}
\hat{u}(\iota) = & \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\dot{n} \otimes' 1}}{\mathfrak{d}(\dot{n})} \sigma(\varepsilon) d\varepsilon \\
& \otimes' \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \varepsilon)^k}{\mathfrak{d}(\dot{n} \otimes' 1)} \otimes' \frac{\varrho_3}{\mathfrak{d}(\dot{n} \otimes' k \otimes' 1)} \eta(\iota) \right) (\zeta \otimes' \varepsilon)^{\dot{n} \otimes' k} \sigma(\varepsilon) d\varepsilon \\
& \otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \varepsilon)^k}{\mathfrak{d}(\dot{n})} \otimes' \frac{\varrho_2}{\mathfrak{d}(\dot{n} \otimes' k)} \eta(\iota) \right) (\Sigma \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1} \sigma(\varepsilon) d\varepsilon \\
& \otimes' \frac{1}{K} \sum_{i=1}^e \int_a^{v_i} \left( \beta \gamma_4 \vartheta_i \frac{(v_i \otimes' \varepsilon)^k}{\mathfrak{d}(\dot{n})} \otimes' \frac{\varrho_4 \rho_i}{\mathfrak{d}(\dot{n} \otimes' k)} \eta(\iota) \right) (v_i \otimes' \varepsilon)^{\dot{n} \otimes' k \otimes' 1} \sigma(\varepsilon) d\varepsilon.
\end{aligned}$$

■

The solution of BVP (1.2)-(1.3) exists if and only if the following operator  $\Delta : \mathcal{D} \rightarrow \mathcal{D}$  defined by

$$\begin{aligned} \Delta(\hat{u}(\iota)) &= \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n})} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \\ &\otimes \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \varepsilon)^{\hat{n} \otimes' \mathfrak{k}} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \\ &\otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota) \right) (\Sigma \otimes' \varepsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \\ &\otimes \frac{1}{K} \sum_{\iota=1}^e \int_a^{v_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(v_\iota \otimes' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_4 \rho_\iota}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota) \right) (v_\iota \otimes' \varepsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon, \end{aligned} \quad (2.14)$$

has a FP.

To reduce computational complexities we set

$$\begin{aligned} c &= \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{1}{|\alpha_1|} \left( |\gamma_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n} \otimes 2)} \otimes |\gamma_2| \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \sum_{\iota=1}^e |\gamma_4| \frac{\vartheta_\iota (v_\iota \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)} \right) \\ &\otimes \frac{|\alpha_1 (\Sigma \otimes' a) \otimes' \alpha_2|}{K} \left( |\varrho_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 2)} \otimes |\varrho_2| \frac{(\Sigma \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \otimes \sum_{\iota=1}^e |\varrho_4| \frac{\rho_\iota (v_\iota \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \right). \end{aligned} \quad (2.15)$$

$$\begin{aligned} d &= \left| \frac{1}{\alpha_1} \right| \left( |\gamma_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n} \otimes 2)} \otimes |\gamma_2| \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes |\gamma_4| \sum_{\iota=1}^e \vartheta_\iota \frac{(v_\iota \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)} \right) \\ &\otimes \left| \frac{\alpha_1 (\Sigma \otimes' a) \otimes' \alpha_2}{K} \right| \left( |\varrho_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 2)} \otimes |\varrho_2| \frac{(\Sigma \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \otimes |\varrho_4| \sum_{\iota=1}^e \frac{\rho_\iota (v_\iota \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \right). \end{aligned} \quad (2.16)$$

The following is the ERs by applying Schaefer FP theorem (Theorem 1.5).

**Theorem 8.1.** Suppose that  $\mathbf{P} \in \check{S}_{ct}$  which is real valued on  $[a, \Sigma] \times \mathbb{R}$ . If  $\mathbf{P}$  satisfies for some  $L_1 > 0$ ,

$$|\mathbf{P}(\iota, \hat{u}(\iota))| \leq L_1$$

for all  $\iota \in [a, \Sigma]$ ,  $\hat{u} \in \mathbb{R}$ . Then the solution of BVP (1.2)-(1.3) exists.

**Proof.** First the completely continuity of the operator  $\Delta : \mathcal{D} \rightarrow \mathcal{D}$  will be proved. Since  $\mathbf{P} \in \check{S}_{ct}$ , this implies that  $\Delta \in \check{S}_{ct}$ . For a positive constant  $\epsilon$ , define  $B_\epsilon = \{\hat{u} \in \mathcal{D} : \|\hat{u}\| \leq \epsilon\} \subset \mathcal{D}$ ,

we prove that  $\Delta(B_\epsilon)$  is bounded. For  $\hat{u} \in B_\epsilon$ ,  $\iota \in [a, \Sigma]$ , consider

$$\begin{aligned}
\|(\Delta u)\| &= \sup_{\iota \in [a, \Sigma]} \left| \int_a^\iota \frac{(\iota \otimes' \epsilon)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n})} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right. \\
&\quad \otimes \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k}} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \\
&\quad \otimes \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota) \right) (\Sigma \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \\
&\quad \otimes \frac{1}{K} \sum_{i=1}^e \int_a^{v_i} \left( \beta \gamma_4 \vartheta_i \frac{(v_i \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_4 \rho_i}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota) \right) (v_i \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \left. \right| \\
&\leq |\mathbb{P}(\epsilon, \hat{u}(\epsilon))| \left[ \frac{(\iota \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{1}{|\alpha_1|} (|\gamma_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes 1}}{\mathfrak{d}(\hat{n} \otimes 2)} \otimes |\gamma_2| \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \sum_{i=1}^e |\gamma_4| \frac{\vartheta_i (v_i \otimes' a)^{\hat{n}}}{\mathfrak{d}(\hat{n} \otimes 1)}) \right. \\
&\quad \otimes \left. \frac{|\alpha_1 (\iota \otimes' a) \otimes' \alpha_2|}{k} \left( |\varrho_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' \mathfrak{k} \otimes 1}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 2)} \otimes |\varrho_2| \frac{(\Sigma \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \otimes |\varrho_4| \frac{\rho_i (v_i \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \right) \right] \\
&\leq L_1 c, \tag{2.17}
\end{aligned}$$

where  $c$  is defined by (2.15).

Next we prove equicontinuity of  $\Delta$ . For  $a \leq \iota_1 \leq \iota_2 \leq \Sigma$  and  $\hat{u} \in B_\epsilon$ , we consider

$$\begin{aligned}
|(\Delta u)(\iota_2) \otimes' (\Delta u)(\iota_1)| &\leq \left| \int_a^{\iota_2} \frac{(\iota_2 \otimes' \epsilon)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n})} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right| \\
&\quad \otimes \left| \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \eta(\iota_2) \right) (\zeta \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k}} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right| \\
&\quad \otimes \left| \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota_2) \right) (\Sigma \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right| \\
&\quad \otimes \left| \frac{1}{K} \sum_{i=1}^e \int_a^{v_i} \left( \beta \gamma_4 \vartheta_i \frac{(v_i \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_4 \rho_i}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota_2) \right) (v_i \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right| \\
&\quad \otimes \left| \int_a^{\iota_1} \frac{(\iota_1 \otimes' \epsilon)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n})} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right| \\
&\quad \otimes \left| \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \eta(\iota_1) \right) (\zeta \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k}} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right| \\
&\quad \otimes \left| \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \epsilon)^{\hat{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota_1) \right) (\Sigma \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbb{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right|
\end{aligned}$$

$$\begin{aligned}
& \otimes \left| \frac{1}{K} \sum_{\iota=1}^e \int_a^{v_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(v_\iota \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_4 \rho_\iota}{\mathfrak{b}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota_1) \right) (v_\iota \otimes' \varepsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathfrak{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \right| \\
& \leq L_1 \left[ \frac{(\iota_2 \otimes' \iota_1)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \frac{(\iota_1 \otimes' a)^{\hat{n} \otimes' (\iota_2 \otimes' a)^{\hat{n}}}}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \frac{(\iota_2 \otimes' \iota_1)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \right. \\
& \quad \left. \otimes \left| \frac{\alpha_1 (\iota_2 \otimes' \iota_1)}{K} \right| \left( |\varrho_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' \mathfrak{k} \otimes 1}}{\mathfrak{b}(\hat{n} \otimes' \mathfrak{k} \otimes 2)} \otimes |\varrho_2| \frac{(\Sigma \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{b}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \otimes \sum_{\iota=1}^e |\varrho_4| \frac{\rho_\iota (v_\iota \otimes' a)^{\hat{n} \otimes' \mathfrak{k}}}{\mathfrak{b}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \right). \right.
\end{aligned}$$

As  $\iota_1 \rightarrow \iota_2$ , the above expression approaches to zero independent of  $\hat{u} \in B_\varepsilon$ . Thanks to Arzelá Ascoli theorem, which implies here  $\Delta$  is completely continuous. At last, we show  $V = \{\hat{u} \in \mathcal{D} : \hat{u} = \lambda \Delta u, 0 \leq \lambda \leq 1\} \in \overline{\mathfrak{N}}_b(\mathcal{D})$ . For  $\hat{u} \in V$  and  $\iota \in [a, \Sigma]$ , and using inequalities (2.17) and (2.15), we have

$$\|\hat{u}\| = \sup_{\iota \in [a, \Sigma]} |\lambda(\Delta u)\iota| \leq L_1 c.$$

Therefore via Theorem 2.2,  $\Delta$  has a FP in  $\mathcal{D}$ . ■

We apply the Krasnoselskii theorem ( Theorem 1.4 ) to prove the following ER.

**Theorem 8.2.** Let  $\mathfrak{P}$  be a continuous real function on  $[a, \Sigma] \times \mathbb{R}$  with properties:

( $\mathcal{D}_1$ )  $|\mathfrak{P}(\iota, \hat{u}) \otimes' \mathfrak{P}(\iota, v)| \leq L |\hat{u} \otimes' v|$ , for all  $\iota \in [a, \Sigma]$ ,  $L > 0$ , and  $\hat{u}, v \in \mathbb{R}$ ,

( $\mathcal{D}_2$ ) for some  $\wp \in C([a, \Sigma], \mathbb{R}^\otimes)$  satisfying  $|\mathfrak{P}(\iota, \hat{u})| \leq \wp(\iota)$ ,  $\forall (\iota, \hat{u}) \in [a, \Sigma] \times \mathbb{R}$ .

If  $d$  in (2.16) satisfies

$$d < \frac{1}{L}, \tag{2.18}$$

then there a solution of problem (1.2)-(1.3) exists.

**Proof.** Consider a set  $B_\varepsilon = \{\hat{u} \in \mathcal{D} : \|\hat{u}\| \leq \varepsilon\}$  with  $\varepsilon \geq c \|\hat{u}\|$ , clearly  $B_\varepsilon \in \overline{\mathfrak{N}}_{cd}(\mathcal{D})$ , where  $c$  is given in (2.15). We decompose the operator  $\Delta$  defined in (2.14) into sum of two operators  $\Delta_1$  and  $\Delta_2$  on  $B_\varepsilon$  as follows:

$$(\Delta_1 \hat{u})(\iota) = \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{b}(\hat{n})} \mathfrak{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon$$

and

$$\begin{aligned}
(\Delta_2 \hat{u})(\iota) &= \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \varepsilon)^{\hat{n} \otimes' k} \mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \\
&\otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (\Sigma \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \\
&\otimes \frac{1}{K} \sum_{\iota=1}^e \int_a^{v_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(v_\iota \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_4 \rho_\iota}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (v_\iota \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon.
\end{aligned}$$

For  $\hat{u}, v \in B_\varepsilon$ , consider

$$\begin{aligned}
&\|\Delta_1 \hat{u} \otimes \Delta_2 v\| \\
&= \sup_{\iota \in [a, \Sigma]} \left| \int_a^\iota \frac{(\iota \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{b}(\hat{n})} \mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \right. \\
&\quad \otimes \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \varepsilon)^{\hat{n} \otimes' k} \mathbb{P}(\varepsilon, v(\varepsilon)) d\varepsilon \\
&\quad \otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (\Sigma \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbb{P}(\varepsilon, v(\varepsilon)) d\varepsilon \\
&\quad \left. \otimes \frac{1}{K} \sum_{\iota=1}^e \int_a^{v_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(v_\iota \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_4 \rho_\iota}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (v_\iota \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbb{P}(\varepsilon, v(\varepsilon)) d\varepsilon \right| \\
&\leq \|\wp\| \sup_{\iota \in [a, \Sigma]} \left[ \frac{(\iota \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \left| \frac{\beta \gamma_3}{k} \right| \frac{(\zeta \otimes' a)^{\hat{n} \otimes 1}}{\mathfrak{b}(\hat{n} \otimes 2)} \otimes \left| \frac{\beta \gamma_2}{k} \right| \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \sum_{\iota=1}^e \left| \frac{\beta \gamma_4}{k} \right| \frac{\vartheta_\iota (v_\iota \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \right. \\
&\quad \left. \otimes \left| \frac{\varrho_3 \eta(\iota)}{k} \right| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' k \otimes 1}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 2)} \otimes \left| \frac{\varrho_2 \eta(\iota)}{k} \right| \frac{(\Sigma \otimes' a)^{\hat{n} \otimes' k}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \otimes \left| \frac{\varrho_4 \eta(\iota)}{k} \right| \frac{\rho_\iota (v_\iota \otimes' a)^{\hat{n} \otimes' k}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \right] \\
&\leq \|\wp\| \left[ \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \frac{1}{|\alpha_1|} \left( \left| \gamma_3 \right| \frac{(\zeta \otimes' a)^{\hat{n} \otimes 1}}{\mathfrak{b}(\hat{n} \otimes 2)} \otimes \left| \gamma_2 \right| \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \sum_{\iota=1}^e \left| \gamma_4 \right| \frac{\vartheta_\iota (v_\iota \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \right) \right. \\
&\quad \left. \otimes \frac{|\alpha_1 (\Sigma \otimes' a) \otimes' \alpha_2|}{k} \left( \left| \varrho_3 \right| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' k \otimes 1}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 2)} \otimes \left| \varrho_2 \right| \frac{(\Sigma \otimes' a)^{\hat{n} \otimes' k}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \otimes \left| \varrho_4 \right| \frac{\rho_\iota (v_\iota \otimes' a)^{\hat{n} \otimes' k}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \right) \right] \\
&\leq \|\wp\| c \leq \varepsilon.
\end{aligned}$$

Thus  $\Delta_1 \hat{u} \otimes \Delta_2 v \subset B_\varepsilon$ , which verifies assumption (i) in Theorem 1.4. For  $\hat{u}, v \in B_\varepsilon$ , consider

$$\begin{aligned} & \|\Delta_2 \hat{u} \otimes \Delta_2 v\| \\ &= \sup_{\iota \in [a, \Sigma]} \left[ \left[ \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \varepsilon)^{\hat{n} \otimes' k} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \right. \right. \\ & \quad \otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (\Sigma \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \\ & \quad \left. \otimes \frac{1}{K} \sum_{\iota=1}^e \int_a^{\nu_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(\nu_\iota \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_4 \rho_\iota}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (\nu_\iota \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \right] \\ & \quad \otimes' \left[ \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \varepsilon)^{\hat{n} \otimes' k} \mathbf{P}(\varepsilon, v(\varepsilon)) d\varepsilon \right. \\ & \quad \otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (\Sigma \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbf{P}(\varepsilon, v(\varepsilon)) d\varepsilon \\ & \quad \left. \otimes \frac{1}{K} \sum_{\iota=1}^e \int_a^{\nu_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(\nu_\iota \otimes' \varepsilon)^k}{\mathfrak{b}(\hat{n})} \otimes \frac{\varrho_4 \rho_\iota}{\mathfrak{b}(\hat{n} \otimes' k)} \eta(\iota) \right) (\nu_\iota \otimes' \varepsilon)^{\hat{n} \otimes' k \otimes' 1} \mathbf{P}(\varepsilon, v(\varepsilon)) d\varepsilon \right] \Big\| \end{aligned}$$

using (D1)

$$\begin{aligned} & \leq L \left[ \frac{1}{\alpha_1} \left( |\gamma_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes 1}}{\mathfrak{b}(\hat{n} \otimes 2)} \otimes |\gamma_2| \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \otimes |\gamma_4| \sum_{\iota=1}^e \vartheta_\iota \frac{(\nu_\iota \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \right) \right. \\ & \quad \left. \otimes \left| \frac{\alpha_1 (\Sigma \otimes' a)^{\otimes' \alpha_2}}{K} \right| \left( |\varrho_3| \frac{(\zeta \otimes' a)^{\hat{n} \otimes' k \otimes 1}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 2)} \otimes |\varrho_2| \frac{(\Sigma \otimes' a)^{\hat{n} \otimes' k}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \otimes |\varrho_4| \sum_{\iota=1}^e \frac{\rho_\iota (\nu_\iota \otimes' a)^{\hat{n} \otimes' k}}{\mathfrak{b}(\hat{n} \otimes' k \otimes 1)} \right) \right] \|\hat{u} \otimes v\|. \\ & = Ld \|\hat{u} \otimes v\|. \end{aligned}$$

From (2.18), we have  $Ld < 1$ , so  $\Delta_2$  is contraction.

Next, we show  $\Delta_1 \in \check{\mathcal{S}}_{cp, ct}$ . The continuity of  $\mathbf{P}$  implies the continuity of  $\Delta_1$  and since  $\|\Delta_1 \hat{u}\| \leq \|\varphi\| \frac{(\Sigma \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)}$ , therefore  $\Delta_1$  is UB on  $B_\varepsilon$ .

Set  $\sup_{\iota \in [a, \Sigma] \times \beta_\varepsilon} |\mathbf{P}(\iota, \hat{u})| = \mathbf{P}$  and for  $a \leq \iota_1 \leq \iota_2 \leq \Sigma$ , consider

$$\begin{aligned} & \|\Delta_1 \hat{u}(\iota_2) \otimes \Delta_1 \hat{u}(\iota_1)\| \\ &= \left\| \int_a^{\iota_2} \frac{(\iota_2 \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{b}(\hat{n})} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \otimes' \int_a^{\iota_1} \frac{(\iota_1 \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{b}(\hat{n})} \mathbf{P}(\varepsilon, \hat{u}(\varepsilon)) d\varepsilon \right\| \\ &\leq \mathbf{P} \left\| \int_a^{\iota_1} \frac{(\iota_2 \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{b}(\hat{n})} d\varepsilon \otimes' \int_{\iota_1}^{\iota_2} \frac{(\iota_2 \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{b}(\hat{n})} d\varepsilon \otimes' \int_a^{\iota_1} \frac{(\iota_1 \otimes' \varepsilon)^{\hat{n} \otimes' 1}}{\mathfrak{b}(\hat{n})} d\varepsilon \right\| \\ &= \left[ \frac{(\iota_1 \otimes' a)^{\hat{n}} \otimes' (\iota_2 \otimes' a)^{\hat{n}}}{\mathfrak{b}(\hat{n} \otimes 1)} \right] \mathbf{P}. \end{aligned}$$

As  $\iota_1 \rightarrow \iota_2$ , on R.H.S, we get zero and is independent of  $\hat{u}$ . Therefore  $\Delta_1 \in \bar{\mathcal{I}}_{pre}(\varnothing)$  and using Arzelá Ascoli theorem  $\Delta_1 \in \bar{\mathcal{I}}_{cp}(\varnothing)$ . Therefore the problem (1.2)-(1.3) has at least one solution

due to Krasnoselskii's theorem. ■

### 8.3 Uniqueness of solution via Banach contraction

Here is the result of uniqueness.

**Theorem 8.3.** Suppose that  $\mathbf{P} \in \check{S}_{ct}$  which is real valued on  $[a, \Sigma] \times \mathbb{R}$  satisfying  $(\mathcal{D}1)$ . If  $L < c^{\otimes'1}$ , where  $c$  is defined in (2.15), then a unique solution of the problem (1.2)-(1.3) exists.

**Proof.** Set

$$\sup_{\iota \in [a, \Sigma]} \mathbf{P}(\iota, 0) = L_2 \quad (3.1)$$

and choose  $\epsilon \geq \frac{\epsilon L_2}{1 \otimes' L c} > 0$ . We show that the mapping  $\Delta$  defined in (2.14) satisfies  $\Delta(B_\epsilon) \subset B_\epsilon$ .

For  $\hat{u} \in B_\epsilon$  and  $\iota \in [a, \Sigma]$ , we consider

$$\begin{aligned} |\mathbf{P}(\iota, \hat{u}(\iota))| &= |[\mathbf{P}(\iota, \hat{u}(\iota)) \otimes' \mathbf{P}(\iota, 0)] \otimes \mathbf{P}(\iota, 0)| \\ &\leq |\mathbf{P}(\iota, \hat{u}(\iota)) \otimes' \mathbf{P}(\iota, 0)| \otimes |\mathbf{P}(\iota, 0)| \\ &\leq L \|\hat{u}\| \otimes L_2. \end{aligned}$$

Now for  $\hat{u} \in B_\epsilon$ , consider

$$\begin{aligned} \|\Delta(\hat{u})\| &= \sup_{\iota \in [a, \Sigma]} \left| \int_a^\iota \frac{(\iota \otimes' \epsilon)^{\hat{n} \otimes' 1}}{\mathfrak{d}(\hat{n})} \mathbf{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right. \\ &\quad \otimes \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \otimes' \epsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n} \otimes 1)} \otimes \frac{\varrho_3}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k} \otimes 1)} \eta(\iota) \right) (\zeta \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k}} \mathbf{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \\ &\quad \otimes' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \otimes' \epsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_2}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota) \right) (\Sigma \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbf{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \\ &\quad \left. \otimes \frac{1}{K} \sum_{\iota=1}^e \int_a^{v_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(v_\iota \otimes' \epsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \otimes \frac{\varrho_4 \rho_\iota}{\mathfrak{d}(\hat{n} \otimes' \mathfrak{k})} \eta(\iota) \right) (v_\iota \otimes' \epsilon)^{\hat{n} \otimes' \mathfrak{k} \otimes' 1} \mathbf{P}(\epsilon, \hat{u}(\epsilon)) d\epsilon \right| \\ &\leq (L\epsilon \otimes L_2) c \leq \epsilon. \end{aligned}$$

Thus  $\Delta(B_\epsilon) \subset B_\epsilon$ .

Now to prove  $\Delta$  contraction. For  $\hat{u}, v \in \mathcal{D}$ ,

$$\begin{aligned}
\|\Delta u \circledast' \Delta v\| &= \sup_{\iota \in [a, \Sigma]} \left| \int_a^\iota \frac{(\iota \circledast' \varepsilon)^{\hat{n} \circledast' 1}}{\mathfrak{d}(\hat{n})} (\mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) \circledast' \mathbb{P}(\varepsilon, v(\varepsilon))) d\varepsilon \right. \\
&\quad \left. \circledast' \frac{1}{K} \int_a^\zeta \left( \beta \gamma_3 \frac{(\zeta \circledast' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n} \circledast' 1)} \circledast' \frac{\varrho_3}{\mathfrak{d}(\hat{n} \circledast' \mathfrak{k} \circledast' 1)} \eta(\iota) \right) (\zeta \circledast' \varepsilon)^{\hat{n} \circledast' \mathfrak{k}} (\mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) \circledast' \mathbb{P}(\varepsilon, v(\varepsilon))) d\varepsilon \right. \\
&\quad \left. \circledast' \frac{1}{K} \int_a^\Sigma \left( \beta \gamma_2 \frac{(\Sigma \circledast' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \circledast' \frac{\varrho_2}{\mathfrak{d}(\hat{n} \circledast' \mathfrak{k})} \eta(\iota) \right) (\Sigma \circledast' \varepsilon)^{\hat{n} \circledast' \mathfrak{k} \circledast' 1} (\mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) \circledast' \mathbb{P}(\varepsilon, v(\varepsilon))) d\varepsilon \right. \\
&\quad \left. \circledast' \frac{1}{K} \sum_{\iota=1}^e \int_a^{v_\iota} \left( \beta \gamma_4 \vartheta_\iota \frac{(v_\iota \circledast' \varepsilon)^{\mathfrak{k}}}{\mathfrak{d}(\hat{n})} \circledast' \frac{\varrho_4 \rho_\iota}{\mathfrak{d}(\hat{n} \circledast' \mathfrak{k})} \eta(\iota) \right) (v_\iota \circledast' \varepsilon)^{\hat{n} \circledast' \mathfrak{k} \circledast' 1} (\mathbb{P}(\varepsilon, \hat{u}(\varepsilon)) \circledast' \mathbb{P}(\varepsilon, v(\varepsilon))) d\varepsilon \right| \\
&\leq Lc \|\hat{u} \circledast' v\|, \tag{3.2}
\end{aligned}$$

Since  $Lc < 1$ , therefore  $\Delta$  is contraction, so a unique solution of (1.2)-(1.3) exists. ■

The following is the example that illustrate the above theorem.

**Example 8.1** Consider non-separated multi-point fractional BVP

$$\begin{aligned}
{}^c D^{1.75} \hat{u}(\iota) &= \frac{1}{\sqrt{\iota \circledast' 14}} \tan^{\circledast' 1} \left( \hat{u}(\iota)^{0.5} \circledast' e^\iota \cos \iota \right) \circledast' \sin \iota, \quad 2 < \iota < 3, \\
\gamma_1 \hat{u}(2) \circledast' \gamma_2 \hat{u}(3) &= \gamma_3 \int_a^\zeta \hat{u}(\tau) d\tau \circledast' \gamma_4 \sum_{\iota=1}^4 \vartheta_\iota \hat{u}(v_\iota), \\
\varrho_1 {}^c D^{0.5} \hat{u}(2) \circledast' \varrho_2 {}^c D^{0.5} \hat{u}(3) &= \varrho_3 \int_a^\zeta {}^c D^{0.5} \hat{u}(\tau) d\tau \circledast' \varrho_4 \sum_{\iota=1}^e \rho_\iota {}^c D^{0.5} \hat{u}(v_\iota),
\end{aligned}$$

where  $\gamma_1 = 1, \gamma_2 = \frac{1}{2}, \gamma_3 = 1, \gamma_4 = \frac{1}{3}, \varrho_1 = 1 = \varrho_2 = \varrho_3, \varrho_4 = \frac{1}{3}, \vartheta_1 = \frac{1}{10}, \vartheta_2 = \frac{3}{20}, \vartheta_3 = \frac{1}{4}, \vartheta_4 = 1, v_1 = \frac{11}{5}, v_2 = \frac{12}{5}, v_3 = \frac{13}{5}, v_4 = \frac{14}{5}, \rho_1 = \rho_2 = \frac{17}{19}, \rho_3 = \rho_4 = \frac{9}{11}$ . Note that  $|\mathbb{P}(\iota, \hat{u}(\iota))| \leq \frac{\pi}{4} \circledast' 1 = L$  and  $|\mathbb{P}(\iota, \hat{u}) \circledast' \mathbb{P}(\iota, v)| \leq L |\hat{u} \circledast' v|$ , by assuming  $L = \frac{1}{4}$ . Since  $Lc < 1$ , so by Theorem 1.1, unique solution is obtained.

## 8.4 Conclusion

In this chapter, we apply Krasnosilskii and Shaefer fixed point theorem to find the criterion for existence of solution to the given fractional boundary value problem. For unique solution, we

use Banach contraction principle. Example is provided to validate our results.

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