

Fixed Point Results for Generalized Contractions in Metric Spaces with Applications

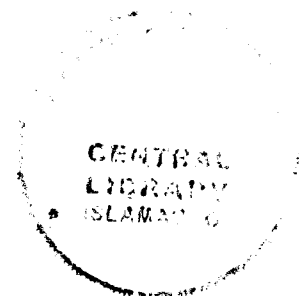


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2022**



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Fixed point theory

Metric spaces

Contraction mappings

Nonlinear functional analysis

Mathematics - Applications

Differential equations, nonlinear

Fixed Point Results for Generalized Contractions in Metric Spaces with Applications



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A Thesis Submitted in Partial Fulfilment

of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

at the Department of Mathematics and Statistics

Faculty of Basic and Applied Sciences

International Islamic University, Islamabad

Pakistan

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2022

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This is to certify that the research work presented in this thesis, entitled: **Fixed Point Results for Generalized Contractions in Metric Spaces with Applications** was conducted by **Mr. Eskandar Ameer Abdullah Ahmed**, Reg. No. **59-FBAS/PHDMA/F15** under the supervision of **Prof. Dr. M. Arshad Zia** no part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Department of Mathematics & Statistics, FBAS, IIU, Islamabad** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics, Department of Mathematics & Statistics, Faculty of Basic & Applied Science, International Islamic University, Sector H-10, Islamabad, Pakistan.**

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DEDICATED TO....

**My Parents, Wife, daughters and Teachers for
supporting and encouraging me.**

ACKNOWLEDGEMENTS

My sincere thanks are due to my benign supervisor Prof. Dr. Muhammad Arshad. It was all because of his unstinting support that my thesis could be completed successfully. I am extremely thankful to the Chairman Department of Mathematics International Islamic University, Islamabad, Pakistan, for his kind patronage throughout my Ph.D. coursework and research work. It would be an injustice not to mention the encouragement and support offered by the faculty of International Islamic University, and the Department of Mathematics. I thank them cordially for facilitating me throughout my stay at the University. I offer my sincere thanks to Taiz University, Taiz 6803, Yemen, for its financial support throughout my Ph.D. My special thanks go to Professor Dr. Hassen Aydi, Institut Supérieur d'informatique et des Techniques de Communication, H. Sousse, Université de Sousse, Tunisia, for his support and guidance throughout this thesis.

I am grateful to all my colleagues from International Islamic University Islamabad, especially Dr. Muhammad Nazam, Dr. Sami Ullah Khan and Dr. Aftab Hussien for their continued moral support during my research.

It is difficult to overstate my deepest gratitude to my parents for their immense love, support, encouragement, constant patience and prayers throughout my life, without them this effort would have been worth nothing.

Eskandar Ameer Abdullah Ahmed
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List of Publications

The list of the research articles, deduced from the work presented in this thesis, published in the international journals (ESCI & SCI) is given below.

1. **E. Ameer**, M. Arshad, W. Sintunavarat, On New Coincidence Points Results for a Mapping and a Relation Approach to the Integral Equations, **Differential Equations and Dynamical Systems**, (2019), doi.org/10.1007/s12591-01900506-7.
2. **E. Ameer**, M. Nazam, H. Aydi, M. Arshad, On $(\Lambda, \Upsilon, \mathcal{R})$ -contractions and applications to nonlinear matrix equations, **Mathematics** 2019,7,443; doi:10.3390/math7050443.
3. **E. Ameer**, H. Aydi, M. Arshad, H. Alsamir, M. Selmi Noorani, Hybrid multivalued type contraction mappings in α_K -complete partial b-metric spaces and applications, **Symmetry** 2019, 11, 86; doi:10.3390/sym11010086.
4. **E. Ameer**, H. Huang, M. Nazam, M. Arshad, Fixed point theorems for multivalued γ -FG-contractions with (α^*, β^*) -admissible mappings in partial b-metric spaces and application, **U. P. B. Sci. Bull., Series A**, Vol. 81, Iss. 2, 2019.
5. **E. Ameer**, H. Aydi, M. Arshad, M. D. I. Sen, Hybrid Ćirić type Graphic (Υ, Λ) Contraction Mappings with Applications to Electric Circuit and Fractional Differential Equations, **Symmetry** 2020, 12, 467; doi:10.3390/sym12030467.
6. **E. Ameer**, H. Aydi, H. Işık, M. Nazam, V. Parvaneh, M. Arshad, Some Existence Results for a System of Nonlinear Fractional Differential Equations, **Journal of Mathematics**, Vol. 2020, Article ID 4786053, 17 pages.
7. **E. Ameer**, M. Arshad, Two New Fixed Point Results for Generalized Wardowski Type Contractions, **Journal of Analysis & Number Theory**, 5 (1) (2017), 1-9.
8. **E. Ameer**, H. Aydi, M. Arshad, On Fuzzy Fixed Points and an Application to Ordinary Fuzzy Differential Equations, **Journal of Function Spaces**, Vol. 2020, Article ID 8835751, 12 page.

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Preface

The research work presented in this dissertation is based on "metric fixed point theory" in particular: on generalizations of Banach's fixed point theorem in three abstract metric spaces. We formulate a method for investigations of fixed points of generalized \mathbb{F} -contractions in partial b -metric and Branciari metric spaces (see Chapter 2 for details). We investigate criteria for existence of coincidence and common fixedpoints of improved θ -contractions in metric spaces (see Chapter 3). We investigate criteria for existence of common fixed points of $(\Lambda, \Upsilon, \acute{R}_{\mathfrak{S}})$ -contractions (where, $\acute{R}_{\mathfrak{S}}$ is an arbitrary binary relation) on metric spaces and prove some fuzzy common fixed point results for fuzzy mappings in metric spaces (see Chapter 4). We also compose three concepts namely α_s^* -admissibility, multivalued (Υ, Λ) -contraction and partial b -metric space and investigate common fixed points of two self-mappings (see Chapter 5). We investigate criteria for existence of common fixed points of generalized graphic (Υ, Λ) -contractions in partial b -metric spaces with graph (see Chapter 5). This thesis presents a comprehensive and comparative study of existence of fixed and common fixedpoints of generalized contractions in three spaces: (1) metric space (2) Branciari metric space (3) partial b -metric space.

This dissertation comprises five chapters.

Chapter 1 deals with the general introduction of fixed point theory. It defines fundamental concepts, notations and some of the well-known results.

Chapter 2 aims to introduce the notion of cyclic (α_*, β_*) -type- γ - $\mathbb{F}G$ -contraction for multivalued mapping and establish some new fixed point theorems in partial b -metric spaces. Moreover, the application of nonlinear integral equation is shown. These results have been published in **U. P. B. Sci. Bull., Series A, Vol. 81, Iss. 2, 2019.**

We present the concept of Branciari \mathbb{F} -rational contractions and discuss some related fixed point results in Branciari metric spaces. Some examples are given to illustrate the usability of the obtained results. These results have been published in **Journal of Analysis & Number Theory, 5 (1) (2017), 1-9.**

Chapter 3 deals with the generalized θ -contraction mappings. We discuss some coincidence point theorems for a mapping and a relation. Moreover, As the application of integral equation is given. These results have been published in **Differential Equations and Dynamical**

Systems, (2019), doi.org/10.1007/s12591-01900506-7.

We discuss generalized Suzuki type (θ, \hat{C}) -rational contractions, generalized Suzuki type Ćirić JS -contractions (θ -contractions) and corresponding common fixedpoint results for four self-mappings. We employ our result to study the existence of the solution of the system of fractional differential equations. These results have been published in **Journal of Mathematics, Vol. 2020, Article ID 4786053, 17 pages.**

Chapter 4 aims to study $(\Lambda, \Upsilon, \hat{R}_3)$ -contraction mappings and to establish related common fixed point theorems in complete metric spaces. An application and an example are given to support the results.

These results have been published in **Mathematics 2019, 7, 443; doi:10.3390/math7050443.**

Furthermore, we study generalized almost (Υ, Λ) -contractions and corresponding common fuzzy fixed point theorems for two fuzzy maps defined on complete metric spaces. These results have been published in "**Journal of Function Spaces, Vol. 2020, Article ID 8835751, 12 page**".

Chapter 5 deals with the notion of generalized multivalued $(\alpha_s^*, \Upsilon, \Lambda)$ -contractions and provide common fixed point theorems in α_s -complete partial b -metric spaces. We give a application to illustrate our results. These results have been published in "**Symmetry 2019, 11, 86; doi:10.3390/sym11010086**".

We discuss some common fixed point theorems for Ćirić type rational graphic (Υ, Λ) -contractions on partial b -metric spaces endowed with graph and present some applications. These results have been published in "**Symmetry2020,12,467;doi:10.3390/sym12030467**".

Chapter 1

Introduction and Preliminaries

In "metric fixed point theory (MFPT)" the Banach Contraction Principle (BCP) is one of the most important result of analysis and considered to be the main source of MFPT. It is the most widely applied fixed point (FP) result in many branches of mathematics because it requires the structure of complete metric space (CMS) with contractive condition on the map which is easy to test in this setting. The BCP has been generalized in many different directions. In fact, there is vast amount of literature dealing with generalizations of this remarkable result.

Nadler [93] extended BCP to multi valued mapping and introduced the concept of multi-valued contractions and gave FP results for multi-valued contractions in a CMS. Subsequently many authors extended Nadler's FP result in different way.

Samet et al. [108] initiated the notions of single-valued α -admissible maps and α - ψ -contraction self maps and proved FP theorems by using these notions. Further Mohammadi et al. [85] extended some results on FPs of α - ψ -Ćirić generalized multi-functions. Asl et al. [64] introduced the notion of α^* - ψ -Geraghty contractive multi-functions and established FP results for multifunctions. Popescu [100] introduced single-valued triangular α -orbital admissible maps. Karapinar [74] investigated FPs of generalized α - ψ -Geraghty contraction type mappings via triangular α -admissible maps (see also [35, 36, 37, 66, 85, 105, 111, 112, 63, 29]). Alizadeh *et al.* [3] initiated the concept of cyclic (α, β) -admissible mappings, (α, β) - (ψ, ϕ) -contractive and weak (α, β, ψ) -rational contraction mappings.

Wardowski [120] introduced a new family of mappings known as F-contraction. He gave an interesting generalization of BCP. Wardowski et. al [121] initiated the concept of F-weak

contraction. Secelean [109], established FP results for F-contractions by iterated function systems. Piri et al. [97] extended the result of Wardowski by applying some weaker conditions on the selfmap in a CMS. Cosentino and Vetro [50] presented some FP results of Hardy-Rogers-type for self-mappings on CMSs and complete ordered metric spaces. Hussain and Salimi [99] introduced an α -GF-contraction with respect to a general family of functions G and established Wardowski type FP results in metric and ordered metric spaces. Further Acar and Altun [1] extended multi-valued F-contractions with δ -Distance and established FP results in CMSs. Recently, numerous research papers on F-contractions have been published (see for instant, ([2, 54, 90, 21, 4, 5, 84, 110, 102, 103, 122])).

Jleli and Samet [66] introduced a new generalization of BCP called JS-contraction (also called θ -contraction). Later on, Arshad et al. [5], Hussain et al. [58], Liet al. [83], Ahmed et al. [19], Parvaneh et al. [99], Hussain et al. [59], Al-Sulami et al. [23], HanÇer et al. [60] (see also [119]) and Altun et al. [26] extended the result of Jleli and Samet [66] in different settings. Liu et al. [80] introduced (Υ, Λ) -type Suzuki contractions, and corresponding FP results in CMSs.

The idea of a b -metric space was introduced by Bakhtin [44]. Subsequently, Czerwik [52, 53] initiated the study of FP theorems in an b -metric space and proved an analogue of Banach's FP theorem. Afterwards, numerous research articles have been published on FP theorems for various classes of single-valued and multi-valued operators in b -metric spaces (see for example, ([61, 62, 69, 76, 101, 116])). Matthews [86, 87] initiated the concept of a partial metric space and extended BCP in the setting of partial metric spaces. Shukla [115] introduced the notion of a partial b -metric space. Branciari [45] announced the notion of a Branciari metric space.

In this chapter, we recollect some basic notions and explain the terminology used throughout this thesis. Some existing known FP theorems are given without proof.

Throughout this thesis, we denote the intervals $(0, +\infty)$, $[0, +\infty)$ and $(-\infty, +\infty)$ by \mathbb{R}^+ , \mathbb{R}_0^+ and \mathbb{R} respectively. Denote metric space, complete metric space and Cauchy sequence by MS, CMS and CS respectively. We use some more notations defined as

$$\begin{aligned} \partial(r, j) &= \partial_j^r; & \partial_\infty(r, j) &= \partial_\infty|_j^r; & \partial_b(r, j) &= \partial_b|_j^r; & P(r, j) &= P_j^r; \\ P_b(r, j) &= P_b|_j^r; & \partial_B(r, j) &= \partial_B|_j^r; & \mathbb{P}(r, j) &= \mathbb{P}_j^r; & D_{P_b}(r, A) &= D_{P_b}|_A^r; \\ \hat{H}_{P_b}(A, B) &= \hat{H}_{P_b}|_B^A; & \hat{H}(A, B) &= \hat{H}_B^A; & \alpha(r, y) &= \alpha_j^r; & \alpha_s(r, y) &= \alpha_s|_j^r. \end{aligned}$$

1.1 Some basic concepts

Definition 1.1.1 Let \mathfrak{S} be a nonempty set. $\hat{E}, \hat{S} : \mathfrak{S} \longrightarrow \mathfrak{S}$, then $r \in \mathfrak{S}$ is called

- (i) FP if image $\hat{S}(r)$ coincides with r (i.e., $\hat{S}(r) = r$).
- (ii) Common fixed point (CPF) of the pair (\hat{E}, \hat{S}) if $\hat{E}(r) = \hat{S}(r) = r$.
- (iii) Coincidence point (COP) of the pair (\hat{E}, \hat{S}) if $\hat{E}(r) = \hat{S}(r)$.

Theorem 1.1.2 [40] Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \longrightarrow \mathfrak{S}$ a contraction mapping (i.e., if there exists $h \in [0, 1)$, such that

$$\partial_{\hat{S}(j)}^{\hat{S}(r)} \leq h\partial_j^r, \text{ for each } r, j \in \mathfrak{S}.$$

Then \hat{S} has a unique FP in \mathfrak{S}

Suzuki [114] proved a generalized BCP in CMSs as follows:

Theorem 1.1.3 [114] Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a self-mapping. Assume that for all $r, j \in \mathfrak{S}$ with $r \neq j$,

$$\frac{1}{2}\partial_{\hat{S}(r)}^r < \partial_j^r \implies \partial_{\hat{S}(j)}^{\hat{S}(r)} < \partial_j^r.$$

Then \hat{S} has a unique FP in \mathfrak{S} .

Definition 1.1.4 [71] Self maps \hat{S} and \hat{E} of a MS (\mathfrak{S}, ∂) are compatible if and only if $\lim_{n \rightarrow \infty} \partial_{\hat{E}(\hat{S}(J_n))}^{\hat{S}(\hat{E}(J_n))} = 0$, whenever, $\{J_n\}$ is a sequence in \mathfrak{S} such that, $\lim_{n \rightarrow \infty} \hat{S}(J_n) = \lim_{n \rightarrow \infty} \hat{E}(J_n) = \check{t}$, for some point $\check{t} \in \mathfrak{S}$.

Lèmma 1.1.5 Let (\mathfrak{S}, ∂) be a MS. If there exist two sequences $\{r_n\}, \{v_n\}$ such that

$$\lim_{n \rightarrow \infty} \partial_{v_n}^{r_n} = 0, \lim_{n \rightarrow \infty} r_n = \check{t}, \text{ for some } \check{t} \in \mathfrak{S}.$$

Then,

$$\lim_{n \rightarrow \infty} v_n = \check{t}.$$

If $(r, j) \in \acute{R}_{\mathfrak{S}}$, then we express it by $r\acute{R}_{\mathfrak{S}}j$ and it is said that “ r is related to j ”. Here, $\acute{R}_{\mathfrak{S}}$ is a binary-relation on a nonempty set \mathfrak{S} and (\mathfrak{S}, ∂) is a MS endowed with a binary relation $\acute{R}_{\mathfrak{S}}$.

Definition 1.1.6.[22, 76] Let $\hat{S} : \mathfrak{S} \longrightarrow \mathfrak{S}$. Then, $\acute{R}_{\mathfrak{S}}$ is called \hat{S} -closed, if for any $r, j \in \mathfrak{S}$,

$$(r, j) \in \acute{R}_{\mathfrak{S}} \implies (\hat{S}(r), \hat{S}(j)) \in \acute{R}_{\mathfrak{S}}.$$

Definition 1.1.7. [76] For $r, j \in \mathfrak{S}$, a path of length $n \in \mathbb{N}$ in $\hat{R}_{\mathfrak{S}}$ from r to j is a finite sequence $\{r_0, r_1, r_2, \dots, r_n\} \subseteq \mathfrak{S}$ such that

- (i) $r_0 = r$ and $r_n = j$;
- (ii) $(r_i, r_{i+1}) \in \hat{R}_{\mathfrak{S}}$ for each $i \in \{0, 1, 2, \dots, n-1\}$.

Notice that a path of length n involves $n+1$ elements of \mathfrak{S} , although they are not necessarily distinct. "Denote by $\Gamma(r, j, \hat{R}_{\mathfrak{S}})$ the set of all paths in $\hat{R}_{\mathfrak{S}}$ from r to j ."

Definition 1.1.8. [104] A (\mathfrak{S}, ∂) is $\hat{R}_{\mathfrak{S}}$ -regular if for every $\{r_n\} \subseteq \mathfrak{S}$,

$$(r_n, r_{n+1}) \in \hat{R}_{\mathfrak{S}} \text{ and } r_n \rightarrow r \in \mathfrak{S} \text{ implies } (r_n, r) \in \hat{R}_{\mathfrak{S}}, \text{ for each } n \in \mathbb{N}.$$

Definition 1.1.9. [122] Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ be self-maps. Then, $\hat{R}_{\mathfrak{S}}$ is called (\hat{S}, \hat{E}) -closed, if for any $r, j \in \mathfrak{S}$,

$$(r, j) \in \hat{R}_{\mathfrak{S}} \Rightarrow (\hat{S}(r), \hat{E}(j)), (\hat{E}(r), \hat{S}(j)) \in \hat{R}_{\mathfrak{S}}.$$

Definition 1.1.10 Let \mathfrak{S} be a nonempty set and $2^{\mathfrak{S}}$ denote collection of all nonempty subset of \mathfrak{S} . Then $\hat{S} : \mathfrak{S} \rightarrow 2^{\mathfrak{S}}$ is called set-valued map. A point $r \in \mathfrak{S}$ is called:

- (i) Fixed point of \hat{S} if $r \in \hat{S}(r)$.
- (ii) Common fixed point of the pair (\hat{S}, \hat{E}) if $r \in \hat{S}(r) \cap \hat{E}(r)$.

Let (\mathfrak{S}, ∂) be a MS. we shall denote by, $CB(\mathfrak{S})$ the class of all bounded and closed subsets of \mathfrak{S} , $K(\mathfrak{S})$ the class of all compact subsets of \mathfrak{S} and, $CL(\mathfrak{S})$ the class of all closed subsets of \mathfrak{S} . We define

$$D(r, B) = \inf_{j \in B} \partial_j^r, \text{ for } r \in \mathfrak{S} \text{ and } A, B \in CB(\mathfrak{S}).$$

Let \hat{H} be the Pompeiu-Hausdorff metric induced by ∂ , i.e.,

$$\hat{H}_B^A = \max \left\{ \sup_{r \in A} \partial_B^r, \sup_{j \in B} \partial_A^j \right\}, \text{ for all } A, B \in CB(\mathfrak{S}).$$

Lemma 1.1.11 [93] If $\check{A}, \check{Z} \in CB(\mathfrak{S})$, then, $\partial_{\check{Z}}^\eta \leq \hat{H}_{\check{Z}}^{\check{A}}$, for every $\eta \in \check{A}$.

Lemma 1.1.12 [93] If $\check{A}, \check{Z} \in CB(\mathfrak{S})$, then there exists $\check{\epsilon} \in \check{Z}$ such that, $\partial_{\check{\epsilon}}^{\bar{a}} \leq \hat{H}_{\check{Z}}^{\check{A}} + \check{N}$, for $\check{N} > 0, \bar{a} \in \check{A}$.

Definition 1.1.13 [93] A mapping $\hat{S} : \mathfrak{S} \rightarrow CB(\mathfrak{S})$ is set-valued contraction if there exists $\lambda \in [0, 1)$, such that,

$$\hat{H}_{\hat{S}(j)}^{\hat{S}(r)} \leq \lambda \partial_j^r, \text{ for all } r, j \in \mathfrak{S}.$$

Theorem 1.1.14 [93] Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \rightarrow CB(\mathfrak{S})$ a set-valued contraction. Then \hat{S} has a fixed point $r^* \in \mathfrak{S}$ such that $r^* \in \hat{S}(r^*)$.

A fuzzy set (FS) in \mathfrak{S} is a function with domain \mathfrak{S} and range $[0, 1]$, $\tilde{A}(\omega)$ is called the grade of membership of $\omega \in \mathfrak{S}$ in \tilde{A} (\tilde{A} is a fuzzy set). The $\check{\epsilon}$ -level set of \tilde{A} is symbolized and given by,

$$[\tilde{A}]_{\check{\epsilon}} = \left\{ \mathfrak{S} : \tilde{A}(\omega) \geq \check{\epsilon} \right\} \text{ if } \check{\epsilon} \in (0, 1), \quad [\tilde{A}]_0 = \overline{\left\{ \mathfrak{S} : \tilde{A}(\omega) > 0 \right\}}.$$

For $A, B \in \mathcal{F}(\mathfrak{S})$, ($\mathcal{F}(\mathfrak{S})$ is the class of all FSs in a MS \mathfrak{S}) $A \subset B$ means $A(\omega) \leq B(\omega)$ for each $\omega \in \mathfrak{S}$. If $[A]_{\check{\epsilon}}, [B]_{\check{\epsilon}} \in CB(\mathfrak{S})$, $\forall \alpha \in [0, 1]$. Then,

$$\begin{aligned} \mathbb{P}_{\check{\epsilon}}(A, B) &= \inf_{\check{r} \in [A]_{\check{\epsilon}}, \check{J} \in [B]_{\check{\epsilon}}} \partial_{\check{J}}^{\check{r}}, & D_{\check{\epsilon}}(A, B) &= \hat{H}_{[B]_{\check{\epsilon}}}^{[A]_{\check{\epsilon}}}, \\ \mathbb{P}_B^A &= \sup_{\check{\epsilon}} \mathbb{P}_{\check{\epsilon}}(A, B), & \partial_{\infty|B}^A &= \sup_{\check{\epsilon}} D_{\check{\epsilon}}(A, B). \end{aligned}$$

A FS A in a metric linear space (MLS) V is called an approximate quantity (AQ) if and only if $[A]_{\check{\epsilon}}$ is compact and convex in V for each $\check{\epsilon} \in [0, 1]$ and $\sup_{\check{r} \in V} A(\check{r}) = 1$.

$A_E(V) = \left\{ \check{A} : \check{A} \text{ is an AQs in } V \right\}$. Let \mathfrak{S}_1 be any set, \mathfrak{S}_2 a MS. A map \hat{S} is called fuzzy map (in short FM) if \hat{S} is a mapping from \mathfrak{S}_1 into $\mathcal{F}(\mathfrak{S}_2)$. A FM $\hat{S} : \mathfrak{S}_1 \rightarrow \mathcal{F}(\mathfrak{S}_2)$ is a fuzzy subset on $\mathfrak{S}_1 \times \mathfrak{S}_2$ with a membership function $\hat{S}(\check{r})(\check{J})$. The function $\hat{S}(\check{r})(\check{J})$ is the grade of membership of \check{J} in $\hat{S}(\check{r})$.

Definition 1.1.15 Let $\hat{S}, \Gamma : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$. Then σ in \mathfrak{S} is called:

1. An $\check{\epsilon}$ -fuzzy fixed point of Γ if $\exists \check{\epsilon} \in [0, 1]$ such that $\sigma \in [\Gamma u]_{\check{\epsilon}}$.
2. A common α -fuzzy FP of \hat{S} and Γ if $\sigma \in \left[\hat{S}u \right]_{\check{\epsilon}} \cap [\Gamma u]_{\check{\epsilon}}$. When $\check{\epsilon} = 1$ is called a common FP of FMs.

Lemma 1.1.16 [38]. Let V be a MLS, $\hat{S} : V \rightarrow A_E(V)$ be a FM and $\check{\sigma}_0 \in V$. Then $\exists \check{\sigma}_1 \in V$ such that $\{\check{\sigma}_1\} \subset \hat{S}(\check{\sigma}_0)$.

Lemma 1.1.17 [39] Let (\mathfrak{S}, ∂) be a MS, $\check{r}^* \in \omega$, and $\hat{S}, \Gamma : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ be FM such that $\hat{S}(\check{r})$ is non empty compact set for each $\check{r} \in \mathfrak{S}$. Then $\check{r}^* \in \hat{S}(\check{r}^*) \Leftrightarrow \hat{S}(\check{r}^*)(\check{r}^*) \geq \hat{S}(\check{r}^*)(\check{r}), \forall \check{r} \in \mathfrak{S}$.

1.2 Some abstract spaces

In this section, we discuss some abstract spaces, that is, a partial b-metric space (abbreviated as PbMS) and a Branciari metric space (abbreviated as BMS).

1.2.1 Partial b-metric spaces

Czerwik [53] announced the notion of b-metric space (abbreviated as bMS).

Definition 1.2.1 [53] Let $\mathfrak{S} \neq \emptyset$ and $s \geq 1$. A function $\partial_b : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ is a b-metric (bM) if, for each $r, j, v \in \mathfrak{S}$,

- (i) $\partial_b|_j^r = 0$ if and only if $r = j$;
- (ii) $\partial_b|_j^r = \partial_b|_r^j$;
- (iii) $\partial_b|_j^r \leq s \left[\partial_b|_v^r + \partial_b|_j^v \right]$.

$(\mathfrak{S}, \partial_b)$ is called a bMS with coefficient s .

Matthews [89] introduced the concept of partial metric space (abbreviated as PMS).

Definition 1.2.2 [89] A partial metric (PM) on a set $\mathfrak{S} \neq \emptyset$ is a function $P : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ such that for each $r, j, v \in \mathfrak{S}$,

- (P1) $P_r^r = P_j^j = P_j^j$ if and only if $r = j$,
- (P2) $P_r^r \leq P_j^r$,
- (P3) $P_j^r = P_r^j$,
- (P4) $P_j^r \leq P_v^r + P_j^v - P_v^v$.

(\mathfrak{S}, P) is called a PMS.

Shukla [115] extended both the concepts of bMS and PMS by introducing the PbMS.

Definition 1.2.3 [115] A partial b-metric (PbM) on a set $(\mathfrak{S} \neq \emptyset)$ is a function $P_b : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ such that $\forall r, j, v \in \mathfrak{S}$,

$$(P_{b1}) \quad P_b|_r^r = P_b|_j^r = P_b|_j^j \text{ if and only if } r = j,$$

$$(P_{b2}) \quad P_b|_r^r \leq P_b|_j^r,$$

$$(P_{b3}) \quad P_b|_j^r = P_b|_r^j,$$

(P_{b4}) there exists a real number $s \geq 1$ such that

$$P_b|_j^r \leq s[P_b|_v^r + P_b|_j^v] - P_b|_v^v.$$

(\mathfrak{S}, P_b) is called a PbMS with coefficient s .

Remark 1.2.4 The self distance $P_b|_r^r$, referred to the size or weight of r , is a feature used to describe the amount of information contained in r .

Remark 1.2.5 Obviously, a PMS is also a PbMS with coefficient $s = 1$. A bMS is also a PbMS with the same coefficient and zero self-distance. While, the converse of each fact need not hold.

Example 1.2.6 [115] Define the mapping $P_b : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}_0^+$ by

$$P_b|_j^r = (\max\{r, j\})^k + |r - j|^k, \quad k > 1, \text{ for all } r, j \in \mathfrak{S}$$

is a PbM on \mathbb{R}^+ with $s = 2^k$. For any $r > 0$, we have $P_b|_r^r = r^k \neq 0$, so P_b is not a bM on \mathbb{R}^+ . Also, for $r = 5, j = 1$ and $v = 4$, we have, $P_b|_j^r = 5^k + 4^k$ and $P_b|_v^r + P_b|_j^v - P_b|_v^v = 5^k + 1 + 4^k + 3^k - 4^k = 5^k + 1 + 3^k$, so

$$P_b|_j^r > P_b|_v^r + P_b|_j^v - P_b|_v^v.$$

Then P_b is not a PM on \mathbb{R}^+ .

Proposition 1.2.7 [91] Every PbM P_b defines a bM ∂_{P_b} where,

$$\partial_{P_b}(r, j) = 2 P_b|_j^r - P_b|_r^r - P_b|_j^j,$$

for all $r, j \in \mathfrak{S}$.

Definition 1.2.8 [115] Let (\mathfrak{S}, P_b) be a PbMS with a coefficient $s \geq 1$.

- (i) A sequence $\{r_n\}$ in \mathfrak{S} converges to a point $r \in \mathfrak{S}$ if, $\lim_{n \rightarrow \infty} P_b|_{r_n}^{r_n} = P_b|_r^r$.
- (ii) $\{r_n\}$ is CS if $\lim_{n, m \rightarrow \infty} P_b|_{r_m}^{r_n}$ exists and is finite.
- (iii) (\mathfrak{S}, P_b) is complete if every CS in \mathfrak{S} converges to a point v in \mathfrak{S} such that

$$\lim_{n \rightarrow \infty} P_b|_v^{r_n} = P_b|_v^v = \lim_{n, m \rightarrow \infty} P_b|_{r_m}^{r_n}.$$

Lemma 1.2.9 [91] Let (\mathfrak{S}, P_b) be a PbMS.

- (1) Every CS in $(\mathfrak{S}, \partial_{P_b})$ is also CS in (\mathfrak{S}, P_b) and vice versa;
- (2) $(\mathfrak{S}, \partial_{P_b})$ is a complete \Leftrightarrow (\mathfrak{S}, P_b) is complete;
- (3) The sequence $\{r_n\}$ in $(\mathfrak{S}, \partial_{P_b})$ converges to some $v \in \mathfrak{S}$ iff

$$\lim_{n \rightarrow \infty} P_b|_v^{r_n} = P_b|_v^v = \lim_{n, m \rightarrow \infty} P_b|_{r_m}^{r_n}.$$

Moreover, let (\mathfrak{S}, P_b) be a PbMS. Then, we define

$$D_{P_b}|_A^r = \inf_{a \in A} P_b|_a^r, \quad \delta_{P_b}(A, B) = \sup_{a \in A} P_b|_B^a,$$

for $r \in \mathfrak{S}$ and $A, B \in CB_{P_b}(\mathfrak{S})$ (where, $CB_{P_b}(\mathfrak{S})$ is the class of all bounded and closed subsets of \mathfrak{S}). Define a mapping $\hat{H}_{P_b} : CB_{P_b}(\mathfrak{S}) \times CB_{P_b}(\mathfrak{S}) \rightarrow [0, \infty)$ by

$$\hat{H}_{P_b}|_B^A = \max \{ \delta_{P_b}(A, B), \delta_{P_b}(B, A) \},$$

Lemma 1.2.10 [55] Let (\mathfrak{S}, P_b) be a PbMS. For $A \in CB_{P_b}(\mathfrak{S})$ and $r \in \mathfrak{S}$, then $D_{P_b}|_A^r = P_b|_r^r$ if and only if $r \in \bar{A} = A$, where \bar{A} is the closure of A .

Lemma 1.2.11 [55] Let (\mathfrak{S}, P_b) be a PbMS. For any $A, B, C \in CB_{P_b}(\mathfrak{S})$,

$$(H_1) \quad \hat{H}_{P_b}|_A^A \leq \hat{H}_{P_b}|_B^A,$$

$$(H_2) \quad \hat{H}_{P_b} \Big|_B^A = \hat{H}_{P_b} \Big|_A^B,$$

$$(H_3) \quad \hat{H}_{P_b} \Big|_B^A \leq s[\hat{H}_{P_b} \Big|_C^A + \hat{H}_{P_b} \Big|_B^C] - \inf_{c \in C} P_b|_c^c.$$

Lemma 1.2.12 [55] Let (\mathfrak{S}, P_b) be a PbMS and $B \in CB_{P_b}(\mathfrak{S})$. If $r \in \mathfrak{S}$ and $P_b|_B^r < c$ where $c > 0$, then there exists $j \in B$ such that $P_b|_j^r < c$.

Lemma 1.2.13 [55] Let (\mathfrak{S}, P_b) be a PbMS, $A, B \in CB_{P_b}(\mathfrak{S})$ and $q > 1$. Then, for all $\zeta \in A$, there exists $\eta \in B$ such that $P_b|_\eta^\zeta \leq q \hat{H}_{P_b} \Big|_B^A$.

1.2.2 Branciari metric spaces

Branciari [45] refined the notion of metric to get a new distance function by substituting the triangle inequality with the quadrilateral inequality. This refined metric function was called Branciari metric.

Definition 1.2.14 [45] Let $\mathfrak{S} \neq \emptyset$ and $\partial_B : \mathfrak{S} \times \mathfrak{S} \longrightarrow \mathbb{R}_0^+$ be a function such that $\forall r, j \in \mathfrak{S}$ and all distinct points $u, v \in \mathfrak{S} \setminus \{r, j\}$,

$$(i) \quad \partial_B|_j^r = 0 \iff r = j,$$

$$(ii) \quad \partial_B|_j^r = \partial_B|_r^j,$$

$$(iii) \quad \partial_B|_j^r \leq \partial_B|_u^r + \partial_B|_v^u + \partial_B|_j^v.$$

Then $(\mathfrak{S}, \partial_B)$ is called a BMS.

As it is mentioned above, such spaces are called also generalized metric space (GMS), rectangular metric space (RMS) in the literature. We assert that BMS is more suitable regarding the fact that several extension of metric is called general metric.

Definition 1.2.15 Let $(\mathfrak{S}, \partial_B)$ be a BMS, $\{r_n\} \subset \mathfrak{S}$ and $r \in \mathfrak{S}$, then $\{r_n\}$ is convergent to $r \iff \partial_B|_r^{r_n} \longrightarrow 0$ as $n \longrightarrow \infty$ (or $r_n \longrightarrow r$).

Definition 1.2.16 Let $(\mathfrak{S}, \partial_B)$ be a BMS and $\{r_n\} \subset \mathfrak{S}$ be a sequence. $\{r_n\}$ is CS $\iff \partial_B|_{r_m}^{r_n} \longrightarrow 0$ as $n, m \longrightarrow \infty$.

Definition 1.2.17 Let $(\mathfrak{S}, \partial_B)$ be a BMS. $(\mathfrak{S}, \partial_B)$ is complete \iff if every CS in \mathfrak{S} converges to some element in \mathfrak{S} .

Remark 1.2.18 Obviously, every MS is a BMS. But the converse is not necessarily true.

Example 1.2.19 Let $\mathfrak{S} = [-2, -1] \cup \{0\} \cup [1, 2]$. Define $\partial_B : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ as follows

$$\begin{aligned}\partial_B|_r^r &= 0, \text{ for all } r \in \mathfrak{S}, \\ \partial_B|_2^1 &= \partial_B|_1^2 = 3, \\ \partial_B|_{-1}^1 &= \partial_B|_1^{-1} = \partial_B|_2^{-1} = \partial_B|_{-1}^2 = 1, \\ \partial_B|_j^r &= |r - j|, \quad \text{otherwise.}\end{aligned}$$

Then $(\mathfrak{S}, \partial_B)$ is a complete BMS, but it is not MS because ∂_B does not satisfy triangle inequality on \mathfrak{S} . Indeed,

$$3 = \partial_B|_2^1 > \partial_B|_{-1}^1 + \partial_B|_2^{-1} = 1 + 1 = 2.$$

Proposition 1.2.20 [70] Let $(\mathfrak{S}, \partial_B)$ be a BMS, $\{r_n\}$ be a CS in $(\mathfrak{S}, \partial_B)$ and $\partial_B|_r^{r_n} \rightarrow 0$ as $n \rightarrow \infty$ for some $r \in \mathfrak{S}$. Then $\partial_B|_j^{r_n} \rightarrow \partial_B|_j^r$ as $n \rightarrow \infty$ for all $j \in \mathfrak{S}$. In particular, $\{r_n\}$ does not converge to j if $j \neq r$.

1.3 α -Admissible and cyclic (α, β) -admissible mappings

Samet et al. [108] announced the notions of α -admissible mappings.

Definition 1.3.1. [108] Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ and $\alpha : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$. We say that \hat{S} is α -admissible if $r, j \in \mathfrak{S}$, $\alpha_j^r \geq 1 \Rightarrow \alpha_{\hat{S}(j)}^{\hat{S}(r)} \geq 1$.

Example 1.3.2. [72] Consider $\mathfrak{S} = \mathbb{R}_0^+$, and define $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ and $\alpha : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by $\hat{S}(r) = 2r$, for all $r, j \in \mathfrak{S}$ and

$$\alpha_j^r = \begin{cases} e^{\frac{j}{r}} & \text{if } r \geq j, r \neq 0 \\ 0 & \text{if } r < j. \end{cases}$$

Then \hat{S} is α -admissible.

Hussain et al. [63] (see also [62]) extended the notions of α -admissible mappings.

Definition 1.3.3. [63] Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a map and $\alpha, \eta : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$. The mapping \hat{S} is called (α, η) -admissible if $r, j \in \mathfrak{S}$,

$$\alpha_j^r \geq 1 \Rightarrow \alpha_{\hat{S}(j)}^{\hat{S}(r)} \geq 1 \text{ and } \eta_j^r \leq 1 \Rightarrow \eta_{\hat{S}(j)}^{\hat{S}(r)} \leq 1.$$

Definition 1.3.4 [63] Given $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ and $\alpha, \eta : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$. \hat{S} is said be triangular (α, η) -admissible if

$$(T1) \ , \ \alpha_j^r \geq 1 \implies \alpha_{\hat{S}(j)}^{\hat{S}(r)} \geq 1, \ r, j \in \mathfrak{S};$$

$$(T2) \ \eta_j^r \leq 1 \implies \eta_{\hat{S}(j)}^{\hat{S}(r)} \leq 1, \ r, j \in \mathfrak{S};$$

$$(T2) \ \alpha_u^r \geq 1 \text{ and } \alpha_j^u \geq 1 \implies \alpha_j^r \geq 1, \text{ for each } r, u, j \in \mathfrak{S};$$

$$(T4) \ \eta_u^r \leq 1 \text{ and } \eta_j^u \leq 1 \implies \eta_j^r \leq 1, \text{ for each } r, u, j \in \mathfrak{S}.$$

Definition 1.3.5 [88] Given $\hat{S} : \mathfrak{S} \rightarrow CB(\mathfrak{S})$ and $\alpha : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ be a function. Such \hat{S} is said be α^* -admissible if for $r, j \in \mathfrak{S}$ with $\alpha_j^r \geq 1$, we have $\alpha^*|_{\hat{S}(j)}^{\hat{S}(r)} \geq 1$, where $\alpha^*|_B^A = \inf \{ \alpha_j^r : r \in A, j \in B \}$.

Definition 1.3.6 [92] The PbMS (\mathfrak{S}, P_b) is called α_s -complete if every CS $\{r_n\}$ in (\mathfrak{S}, P_b) satisfying $\alpha_s|_{r_{n+1}}^{r_n} \geq s^2, \forall n \in \mathbb{N}$ converges in \mathfrak{S} .

On the other hand, Alizadeh *et al.* [3] introduced the concept of cyclic (α, β) -admissible mapping as follows:

Definition 1.3.7 [3] Let \mathfrak{S} be a nonempty set and $\alpha, \beta : \mathfrak{S} \rightarrow \mathbb{R}_0^+$ be mappings. A self-mapping \hat{S} on \mathfrak{S} is called a cyclic (α, β) -admissible if,

$$\alpha(\mathfrak{J}) \geq 1 \ (\mathfrak{J} \in \mathfrak{S}) \implies \beta(\hat{S}(\mathfrak{J})) \geq 1,$$

$$\beta(\mathfrak{J}) \geq 1 \ (\mathfrak{J} \in \mathfrak{S}) \implies \alpha(\hat{S}(\mathfrak{J})) \geq 1.$$

Example 1.3.8 [3] Let $\hat{S} : \mathbb{R} \rightarrow \mathbb{R}$ by defined by $\hat{S}(\mathfrak{J}) = -(\mathfrak{J} + \mathfrak{J}^3)$. Define $\alpha, \beta : \mathbb{R} \rightarrow \mathbb{R}_0^+$ by $\alpha(\mathfrak{J}) = e^{\mathfrak{J}}, \beta(\mathfrak{J}) = e^{-\mathfrak{J}}$, for each $\mathfrak{J} \in \mathbb{R}$. Then \hat{S} is a cyclic (α, β) -admissible mapping.

1.4 F-contractions

Wordowski [120] introduced F-contraction:

Definition 1.4.1 [120] Let (\mathfrak{S}, ∂) be a MS. $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ is said to be an F-contraction if there exist $F \in F$ and $\vartheta \in \mathbb{R}^+$ such that for all $r, j \in \mathfrak{S}$,

$$\partial_{\hat{S}(j)}^{\hat{S}(r)} > 0 \implies \vartheta + F \left(\partial_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq F \left(\partial_j^r \right), \quad (1.1)$$

where F is the set of functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying:

(F1) F is strictly increasing;

(F2) $\forall \{\tilde{\partial}_n\}_{n=1}^\infty \subset \mathbb{R}^+, \lim_{n \rightarrow \infty} \tilde{\partial}_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} F(\tilde{\partial}_n) = -\infty$;

(F3) $\exists 0 < \hat{h} < 1$ such that $\lim_{\tilde{\partial} \rightarrow 0^+} \tilde{\partial}^{\hat{h}} F(\tilde{\partial}) = 0$.

Example 1.4.3 [120] Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined by $F(t) = \ln t$. Then $F \in F$. Note that (1.1) reduces to the following

$$\partial_{\hat{S}(j)}^{\hat{S}(r)} \leq e^{-\vartheta} \partial_j^r, \text{ for all } r, j \in \mathfrak{S}, \hat{S}r \neq \hat{S}j.$$

It is clear that for $r, j \in \mathfrak{S}$ such that $\hat{S}(r) = \hat{S}(j)$ the inequality $\partial_{\hat{S}(j)}^{\hat{S}(r)} \leq e^{-\vartheta} \partial_j^r$, also true, i.e., \hat{S} is a Banach contraction.

Example 1.4.4 [120] If $F(t) = \ln t + t, t \in \mathbb{R}^+$ then $F \in F$ and (1.1) reduces to ξ

$$\frac{\partial_{\hat{S}(j)}^{\hat{S}(r)}}{\partial_j^r} e^{[\partial_{\hat{S}(j)}^{\hat{S}(r)} - \partial_j^r]} \leq e^{-\vartheta}, \text{ for all } r, j \in \mathfrak{S}, \hat{S}r \neq \hat{S}j.$$

Remark 1.4.5. From (F1) and (1.1) it is easy to conclude that every F -contraction is necessarily continuous.

Theorem 1.4.6 [120] Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ is F -contraction. Then \exists only one FP $r^* \in \mathfrak{S}$ of \hat{S} .

Piri and Kumam [97] modified the notion of F -contractions by changing (F3) by

(F'3) F is continuous.

Denote F^* the class of functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying (F1), (F2) and (F'3).

Example 1.4.7 The following are some examples of functions belonging to F^* :

$$(1) F_1(t) = \ln t, t > 0 \quad (2) F_2(t) = t - \frac{1}{t}, t > 0 \quad (3) F_3(t) = \frac{e^t}{1 - e^{2t}}, t > 0$$

Remark 1.4.8. Piri and Kumam [97] showed that $F \setminus F^* \neq \emptyset, F^* \setminus F \neq \emptyset$ and $F \cap F^* \neq \emptyset$.

More precisely, for $F(t) = -\frac{1}{t^q}; q \geq 1$, we have $F \in F^*, F \notin F$, and for $F(t) = -\frac{1}{(t+[t])^k}; k \in (0, \acute{a}), \acute{a} < 1$, we have $F \in F, F \notin F^*$. For $F(t) = \ln(t)$, we have $F \in F \cap F^*$.

Acar et al. [2] introduced the concept of generalized multivalued F -contraction mappings.

Definition 1.4.9. [2] Let (\mathfrak{S}, ∂) be a MS and $\hat{S} : \mathfrak{S} \rightarrow CB(\mathfrak{S})$ be a multivalued mapping.

Then \hat{S} is said to be a generalized multivalued F-contraction (GM F-C) if $\mathbf{F} \in F$ and there exists $\vartheta \in \mathbb{R}^+$ such that

$$r, j \in \mathfrak{S}, \hat{H}_{\hat{S}(j)}^{\hat{S}(r)} > 0 \implies \vartheta + \mathbf{F}(\hat{H}_{\hat{S}(j)}^{\hat{S}(r)}) \leq \mathbf{F}(\mathcal{M}_1(r, j)),$$

where, \bar{A}

$$\mathcal{M}_1(r, j) = \max\{\partial_j^r, \mathfrak{D}_{\hat{S}(r)}^r, \mathfrak{D}_{\hat{S}(j)}^j, \frac{1}{2}[\mathfrak{D}_{\hat{S}(j)}^r + \mathfrak{D}_{\hat{S}(r)}^j]\}.$$

Theorem 1.4.10 [2] Let (\mathfrak{S}, ∂) be a CMS, and $\hat{S} : \mathfrak{S} \rightarrow K(\mathfrak{S})$ be a GM F-C. If \hat{S} or F is continuous, then \hat{S} has a FP in \mathfrak{S} .

Theorem 1.4.11 [110] Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \rightarrow CL(\mathfrak{S})$ a multivalued mapping. Assume that there exist $\mathbf{F} \in F$ and $\vartheta \in \mathbb{R}^+$ such that

$$2\vartheta + \mathbf{F}(\hat{H}_{\hat{S}(j)}^{\hat{S}(r)}) \leq \mathbf{F}(\mu_1 \partial_j^r + \mu_2 \partial_{\hat{S}(r)}^r + \mu_3 \partial_{\hat{S}(j)}^j + b\mu_4 \partial_{\hat{S}(j)}^r + \mu_5 \partial_{\hat{S}(r)}^j)$$

for all $r, j \in \mathfrak{S}$, with $\hat{S}(r) \neq \hat{S}(j)$, where $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \geq 0$, $\mu_1 + \mu_2 + \mu_3 + 2\mu_4 = 1$ and $\mu_3 \neq 1$. Then \hat{S} has a FP.

In light with ([102],[51],[17]), we denote the set of all functions $\mathbf{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$ by Δ_F which satisfy following axioms;

(Δ_1) \mathbf{F} is strictly increasing,

(Δ_2) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} t_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \mathbf{F}(t_n) = -\infty$,

(Δ_3) $\exists h \in (0, 1)$ such that $\lim_{t \rightarrow 0^+} t^h \mathbf{F}(t) = 0$,

(Δ_4) $\mathbf{F}(st_n) \leq \mathbf{F}(t_{n-1}) + G(\gamma(t_{n-1})) \Rightarrow \mathbf{F}(s^n t_n) \leq \mathbf{F}(s^{n-1} t_{n-1}) + G(\gamma(t_{n-1}))$,
 $(G, \gamma) \in \Delta_{G, \gamma}$,

(Δ_5) $\mathbf{F}(\inf C) = \inf \mathbf{F}(C)$ for all $C \subseteq \mathbb{R}^+$ with $\inf C > 0$.

Let $\Delta_{G, \gamma}$ represents the family of pairs (G, γ) , where $G : \mathbb{R}^+ \rightarrow \mathbb{R}$ and $\gamma : \mathbb{R}_0^+ \rightarrow [0, 1)$ are mappings such that

(Δ_6) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \sup G(t_n) \geq 0 \Leftrightarrow \lim_{n \rightarrow \infty} \sup t_n \geq 1$,

(Δ_7) for each sequence $\{t_n\} \subseteq \mathbb{R}_0^+$, $\lim_{n \rightarrow \infty} \sup \gamma(t_n) = 1 \Rightarrow \lim_{n \rightarrow \infty} t_n = 0$,

(Δ_8) for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\sum_{n=1}^{\infty} G(\gamma(t_n)) = -\infty$.

Example 1.4.12. If $F(t) = G(t) = \ln(t)$ and $\gamma(t) = k \in (0, 1)$, then $F \in \Delta_F$ and $(G, \gamma) \in \Delta_{G, \gamma}$.

1.5 θ -contractions

Let $\Theta := \{\theta : \mathbb{R}^+ \rightarrow (1, \infty)\}$ such that θ satisfies $\Theta_1, \Theta_2, \Theta_3$. where,

(Θ_1) θ is non-decreasing;

(Θ_2) for every sequence $\{\dot{\lambda}_n\} \subset \mathbb{R}^+$,

$$\lim_{n \rightarrow \infty} \theta(\dot{\lambda}_n) = 1 \iff \lim_{n \rightarrow \infty} \dot{\lambda}_n = 0^+;$$

(Θ_3) there exist $0 < q < 1$ and $\ell \in (0, \infty]$ such that $\lim_{\dot{\lambda} \rightarrow 0^+} \frac{\theta(\dot{\lambda}) - 1}{\dot{\lambda}^q} = \ell$.

Jaleli et al.[66] introduced the following concept of θ -contraction (or JS-contraction):

Definition 1.5.1 [66] Let (\mathfrak{S}, ∂) be a MS. The map $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ is called θ -contraction, if $\exists k \in [0, 1)$ and $\theta \in \Theta$ such that,

$$r, j \in \mathfrak{S}, \partial_{\hat{S}(j)}^{\hat{S}(r)} > 0 \Rightarrow \theta \left(\partial_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq [\theta(\partial_j^r)]^k, \quad (1.2)$$

Example 1.5.2 Define $\theta_1, \theta_2 : \mathbb{R}^+ \rightarrow (1, \infty)$ by,

$$\theta_1(t) = e^{\sqrt{t}}, \quad \theta_2(t) = e^{\sqrt{te^t}},$$

for each $t \in \mathbb{R}^+$. Then θ_1 and $\theta_2 \in \Theta$.

Theorem 1.5.3 [66] Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be an JS-contraction. Then \hat{S} has a unique FP.

Consistent with [19], let $\Theta^* := \{\theta : \mathbb{R}^+ \rightarrow (1, \infty) : \theta \text{ satisfies } (\Theta_1), (\Theta_2), (\Theta'_3)\}$, where (Θ'_3): θ is continuous.

Note that $(\Theta 3)$ and $(\Theta' 3)$ are independent of each other, (see [19]).

Consistent with [80], Ξ is the class of functions $\theta : \mathbb{R}^+ \rightarrow (1, \infty)$ satisfying (Θ_1) , $(\Theta' 2)$ and $(\Theta' 3)$, where

$$(\Theta' 2): \inf_{t \in \mathbb{R}^+} \theta(t) = 1.$$

The set of functions Θ^* is not empty:

Example 1.5.4 [19] Define $\theta_3, \theta_4, \theta_5, \theta_6 : \mathbb{R}^+ \rightarrow (1, \infty)$ by,

$$\begin{aligned} \theta_3(\check{t}) &= e^{\check{t}}, & \theta_4(\check{t}) &= \cosh \check{t}; \\ \theta_5(\check{t}) &= e^{\sqrt{\check{t}}e^{\check{t}}}, & \theta_6(\check{t}) &= 1 + \ln(\check{t} + 1). \end{aligned}$$

Then $\theta_3, \theta_4, \theta_5, \theta_6 \in \Theta^*$.

Hussain et al. [58] modified the family Θ of mappings in the following way.

Let $\Psi = \{\theta : \mathbb{R}_0^+ \rightarrow [1, \infty) : \theta \text{ satisfies } \Psi 1 - \Psi 5\}$, where,

(Ψ_1) θ is non-decreasing;

(Ψ_2) $\theta(\check{\delta}) = 1 \Leftrightarrow \check{\delta} = 0$;

(Ψ_3) for every sequence $\{\check{\delta}_n\} \subset \mathbb{R}^+$,

$$\lim_{n \rightarrow \infty} \theta(\check{\delta}_n) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} \check{\delta}_n = 0^+;$$

(Ψ_4) $\exists \lambda \in (0, 1)$, $\ell \in (0, \infty]$ such that $\lim_{\check{\delta} \rightarrow 0^+} \frac{\theta(\check{\delta}) - 1}{\check{\delta}^\lambda} = \ell$;

(Ψ_5) $\theta(\check{\delta}_1 + \check{\delta}_2) \leq \theta(\check{\delta}_1)\theta(\check{\delta}_2)$.

The functions $\theta_7(\check{\delta}) = e^{\sqrt{\check{\delta}}}$, $\theta_8(\check{\delta}) = 5^{\sqrt{\check{\delta}}}$, for each $\check{\delta} \in [0, \infty)$, are members of Ψ .

Let $\Delta_{\hat{C}}$ denotes the class of functions $\hat{C} : \mathbb{R}^{+4} \rightarrow \mathbb{R}^+$ satisfying:

(\hat{C}^*) : for all $\check{t}_1, \check{t}_2, \check{t}_3, \check{t}_4 \in \mathbb{R}^+$ with $\check{t}_1\check{t}_2\check{t}_3\check{t}_4 = 0$ there exists $\mathbb{k} \in [0, 1)$ such that $\hat{C}(\check{t}_1, \check{t}_2, \check{t}_3, \check{t}_4) = \mathbb{k}$.

Example 1.5.5 [99]

1. If $\hat{C}(\check{t}_1, \check{t}_2, \check{t}_3, \check{t}_4) = \check{h}e^{v \min\{t_1, t_2, t_3, t_4\}}$ where $v \in \mathbb{R}^+$ and $\check{h} \in [0, 1)$, then $\hat{C} \in \Delta_{\hat{C}}$.
2. If $\hat{C}(\check{t}_1, \check{t}_2, \check{t}_3, \check{t}_4) = \check{n} \min\{\check{t}_1, \check{t}_2, \check{t}_3, \check{t}_4\} + \check{h}$, where $\check{n} \in \mathbb{R}^+$, $\check{h} \in [0, 1)$, then $\hat{C} \in \Delta_{\hat{C}}$.

We define the following class of functions, which was considered in [18].

$$\Phi = \left\{ \varphi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R} \text{ and, } \varphi(s, t) = \frac{1}{2}s - t \right\}.$$

[18]

HanÇer et al. [60] (see also [119]) extended the concept of θ -contraction to multivalued mappings as follows.

Definition 1.5.1 [60] Let (\mathfrak{S}, ∂) be a MS, $\hat{S} : \mathfrak{S} \rightarrow CB(\mathfrak{S})$ and $\theta \in \Theta$. Then \hat{S} is said to be a multivalued θ -contraction if there exists $k \in [0, 1)$ such that for each $r, j \in \mathfrak{S}$ with $\hat{H}_{\hat{S}(j)}^{\hat{S}(r)} > 0$,

$$\theta\left(\hat{H}_{\hat{S}(j)}^{\hat{S}(r)}\right) \leq [\theta(\partial_j^r)]^k.$$

1.6 (Υ, Λ) -contractions

Throughout this thesis, Λ_Δ is the family of functions $\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying [80]:

(Λ_1) Λ is nondecreasing;

(Λ_2) for every sequence $\{t_n\} \subset \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \Upsilon(t_n) = 0 \iff \lim_{n \rightarrow \infty} t_n = 0$;

(Λ_3) Λ is continuous.

The function ([41]) $\Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a comparison if:

(i) $a_1 < a_2 \Rightarrow \Upsilon(a_1) \leq \Upsilon(a_2)$;

(ii) $\lim_{n \rightarrow \infty} \Upsilon^n(\varsigma) = 0$ for each $\varsigma > 0$.

Denote by Ω the class of comparison functions. If $\Upsilon \in \Omega$, then $\Upsilon(\varsigma) < \varsigma$ for every $\varsigma > 0$.

Example 1.6.1 [41] The following functions $\Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \in \Omega$:

$$\begin{aligned} (1) \quad \Upsilon(\varsigma) &= b\varsigma, 0 < b < 1, & (3) \quad \Upsilon(\varsigma) &= \frac{\varsigma}{\varsigma+1}. \\ (2) \quad \Upsilon(t) &= \begin{cases} \frac{1}{2}, & 0 < \varsigma < 1, \\ \frac{\varsigma}{3}, & 1 \leq \varsigma, \end{cases} \end{aligned}$$

Example 1.6.2 [80] Define: (1) $\Lambda_1(\dot{u}) = \dot{u}$, (2) $\Lambda_2(\dot{u}) = \sqrt{\dot{u}\sqrt{\dot{u}}}$, (3) $\Lambda_3(\dot{u}) = \dot{u}e^{\dot{u}}$. Then $\Lambda_1, \Lambda_2, \Lambda_3 \in \Lambda_\Delta$.

Liu et al. [80] announced (Υ, Λ) -type Suzuki contractions ((Υ, Λ) -TS-contraction) and corresponding FP results.

Theorem 1.6.3 Let (\mathfrak{S}, ∂) be a MS and $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a (Υ, Λ) -TS-contraction that is, if $\exists \Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that $\forall r, j \in \mathfrak{S}$ with $\hat{S}(r) \neq \hat{S}(j)$

$$\frac{1}{2}\partial_{\hat{S}(r)}^r < \partial_j^r \implies \Lambda\left(\partial_{\hat{S}(j)}^{\hat{S}(r)}\right) \leq \Upsilon[\Lambda(\mathcal{M}_2(r, j))], \text{ where,} \quad (1.3)$$

$$\mathcal{M}_2(r, j) = \max\left\{\partial_j^r, \partial_{\hat{S}(r)}^r, \partial_{\hat{S}(j)}^j, \frac{1}{2}\partial_{\hat{S}(j)}^r, \partial_{\hat{S}(r)}^j\right\}.$$

If Υ is continuous. Then \hat{S} has a unique FP.

Lemma 1.6.4 [80] If $\Lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (Λ_1) , (Λ_3) and $\inf_{t \in \mathbb{R}^+} \Lambda(t) = 0$ and $\{t_k\}_k \subset \mathbb{R}^+$ is a sequence. Then, $\lim_{k \rightarrow \infty} \Lambda(t_k) = 0 \iff \lim_{k \rightarrow \infty} t_k = 0$.

Chapter 2

Fixed Point Theorems for F-Contractions in Abstract Spaces

Wardowski [120] presented F-contraction and established a Banach FP result concerning F-contraction. His findings were followed by Secelean [109], Piri and Kumam [97], Cosentino and Vetro [50], Acar et al. [2], Acar and Altun [1], Minak et al. [84], Sgroi et al.[110], Ahmad et al. [16, 28], Kutbi et al. [75], Parvaneh *et al.* [102], Padhan *et al.* [103] and many other continue these investigation on F-contraction and obtained FP theorems. This chapter consists of three sections.

In section 2.1, we evolve the notion of F-contraction by using the idea of Alizadeh *et al.* [3] (the concept of cyclic (α, β) -admissible mappings) and introduce the notion of cyclic (α_*, β_*) -type- γ -FG-contraction for multivalued valued mapping and obtain some new FP theorems in a complete PbMS. In section 2.2, we introduce the existence theorem for nonlinear integral equations. In section 2.3, we initiate the notion of Branciari F-rational contraction for single valued mappings and established FP theorems on a BMS.

Results given in this chapter have been published in ([10],[6]).

2.1 Fixed points of cyclic (α_*, β_*) -type- γ -FG-contractions in partial b -metric spaces

Here, $\mathfrak{S} \neq \emptyset$ endowed with a PbM P_b , that is, (\mathfrak{S}, P_b) is PbMS with coefficient $s \geq 1$. Further, in results it is considered as a complete PbMS. Unless otherwise stated. Also note that $\alpha, \beta : \mathfrak{S} \rightarrow \mathbb{R}_0^+$ is mappings and A, B are subsets of \mathfrak{S} . We begin with the following definitions:

Definition 2.1.1 A map $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ is called a cyclic (α_*, β_*) -admissible mapping if,

$$\alpha(r) \geq 1 \ (r \in \mathfrak{S}) \Rightarrow \beta_*(\hat{S}(r)) \geq 1, \text{ where } \beta_*(A) = \inf_{a \in A} \beta(a),$$

and

$$\beta(r) \geq 1 \ (r \in \mathfrak{S}) \Rightarrow \alpha_*(\hat{S}(r)) \geq 1, \text{ where } \alpha_*(B) = \inf_{b \in B} \alpha(b).$$

Definition 2.1.2 A mapping $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ is called multivalued P_b -continuous at point $r \in \mathfrak{S}$ if

$$\lim_{n \rightarrow \infty} P_b|_r^{r_n} = P_b|_r^r \Rightarrow \lim_{n \rightarrow \infty} \hat{H}_{P_b} \Big|_{\hat{S}(r_n)}^{\hat{S}(r_n)} = \hat{H}_{P_b} \Big|_{\hat{S}(r)}^{\hat{S}(r)}.$$

Definition 2.1.3. A map $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ is called cyclic (α_*, β_*) -type- γ -FG-contraction, if there exist $F \in \Delta_F$, $(G, \gamma) \in \Delta_{G, \gamma}$ such that for each $r, j \in \mathfrak{S}$, $\alpha(r)\beta(j) \geq 1$ and $\hat{H}_{P_b} \Big|_{\hat{S}(j)}^{\hat{S}(r)} > 0$ imply

$$F(\alpha(r)\beta(j)s \hat{H}_{P_b} \Big|_{\hat{S}(j)}^{\hat{S}(r)}) \leq F(\mathcal{M}_3(r, j)) + G(\gamma(\mathcal{M}_3(r, j))), \quad (2.1)$$

where

$$\mathcal{M}_3(r, j) = \max \left\{ P_b|_j^r, D_{P_b}|_{\hat{S}(j)}^j, D_{P_b}|_{\hat{S}(r)}^r, \frac{D_{P_b}|_{\hat{S}(j)}^r + D_{P_b}|_{\hat{S}(r)}^j}{2s} \right\}. \quad (2.2)$$

Theorem 2.1.4. Let $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ be a cyclic (α_*, β_*) -type- γ -FG-contraction mapping satisfying:

- (1) either $\exists r_0 \in \mathfrak{S}$ such that $\alpha(r_0) \geq 1$ or $\exists j_0 \in \mathfrak{S}$ such that $\beta(j_0) \geq 1$,
- (2) \hat{S} is multivalued P_b -continuous,
- (3) \hat{S} is cyclic (α_*, β_*) -admissible.

Then \hat{S} has a FP.

Proof. Let $r_0 \in \mathfrak{S}$ be such that $\alpha(r_0) \geq 1$. Since \hat{S} is cyclic (α_*, β_*) -admissible mapping, there exist $r_1 \in \hat{S}(r_0)$, $r_2 \in \hat{S}(r_1)$ such that

$$\alpha(r_0) \geq 1 \Rightarrow \beta(r_1) \geq \beta_*(\hat{S}(r_0)) \geq 1 \Rightarrow \alpha(r_2) \geq \alpha_*(\hat{S}(r_1)) \geq 1. \quad (2.3)$$

Because of $\alpha(r_0)\beta(r_1) \geq 1$, it is easy to see that

$${}_s D_{P_b}|_{\hat{S}(r_1)}^{r_1} \leq {}_s \hat{H}_{P_b}|_{\hat{S}(r_1)}^{\hat{S}(r_0)} \leq \alpha(r_0)\beta(r_1) {}_s \hat{H}_{P_b}|_{\hat{S}(r_1)}^{\hat{S}(r_0)}$$

by (Δ_1) , we have

$$\begin{aligned} \mathbf{F}({}_s D_{P_b}|_{\hat{S}(r_1)}^{r_1}) &\leq \mathbf{F}(\alpha(r_0)\beta(r_1) {}_s \hat{H}_{P_b}|_{\hat{S}(r_1)}^{\hat{S}(r_0)}) \\ &\leq \mathbf{F}(\mathcal{M}_3(r_0, r_1)) + G(\gamma(\mathcal{M}_3(r_0, r_1))). \end{aligned} \quad (2.4)$$

The axiom (Δ_5) implies that $\mathbf{F}({}_s D_{P_b}|_{\hat{S}(r_1)}^{r_1}) = \inf_{r \in \hat{S}(r_1)} \mathbf{F}({}_s P_b|_r^{r_1})$. Thus, there exists $r = r_2 \in \hat{S}(r_1)$ such that $\mathbf{F}({}_s D_{P_b}|_{\hat{S}(r_1)}^{r_1}) = \mathbf{F}({}_s P_b|_{r_2}^{r_1})$ and the inequality (2.4) implies

$$\mathbf{F}({}_s P_b|_{r_2}^{r_1}) \leq \mathbf{F}(\mathcal{M}_3(r_0, r_1)) + G(\gamma(\mathcal{M}_3(r_0, r_1))), \quad (2.5)$$

where

$$\begin{aligned} \mathcal{M}_3(r_0, r_1) &= \max \left\{ P_b|_{r_1}^{r_0}, D_{P_b}|_{\hat{S}(r_1)}^{r_1}, D_{P_b}|_{\hat{S}(r_0)}^{r_0}, \frac{D_{P_b}|_{\hat{S}(r_1)}^{r_0} + D_{P_b}|_{\hat{S}(r_0)}^{r_1}}{2s} \right\} \\ &\leq \max \left\{ P_b|_{r_1}^{r_0}, P_b|_{r_2}^{r_1}, \frac{P_b|_{r_2}^{r_0} + P_b|_{r_1}^{r_1}}{2s} \right\} \\ &\leq \max \left\{ P_b|_{r_1}^{r_0}, P_b|_{r_2}^{r_1}, \frac{s[P_b|_{r_1}^{r_0} + P_b|_{r_2}^{r_1}]}{2s} \right\} \\ &\leq \max \{ P_b|_{r_1}^{r_0}, P_b|_{r_2}^{r_1} \}. \end{aligned}$$

If $\mathcal{M}_3(r_0, r_1) \leq P_b|_{r_2}^{r_1}$, then (2.5) yields that

$$\mathbf{F}({}_s P_b|_{r_2}^{r_1}) \leq \mathbf{F}(P_b|_{r_2}^{r_1}) + G(\gamma(\mathcal{M}_3(r_0, r_1))),$$

which implies $G(\gamma(\mathcal{M}_3(r_0, r_1))) \geq 0$ and by (Δ_6) we get $\gamma(\mathcal{M}_3(r_0, r_1)) \geq 1$. This is a contradiction to definition of γ . Thus, $\mathcal{M}_3(r_0, r_1) \leq P_b|_{r_1}^{r_0}$. By (2.5), we get

$$\mathbf{F}(s P_b|_{r_2}^{r_1}) \leq \mathbf{F}(P_b|_{r_1}^{r_0}) + G(\gamma(\mathcal{M}_3(r_0, r_1))).$$

Similarly, there exists $r_3 \in \hat{S}(r_2)$ such that

$$\mathbf{F}(s P_b|_{r_3}^{r_2}) \leq \mathbf{F}(P_b|_{r_2}^{r_1}) + G(\gamma(\mathcal{M}_3(r_1, r_2))).$$

Continuing in this fashion we produce a sequence $\{r_n\} \subset \mathfrak{S}$ such that $r_{n+1} \in \hat{S}(r_n)$, for ever $n \in \mathbb{N}$, $\alpha(r_n)\beta(r_{n+1}) \geq 1$, and

$$\mathbf{F}(s P_b|_{r_{n+1}}^{r_n}) \leq \mathbf{F}(P_b|_{r_n}^{r_{n-1}}) + G(\gamma(\mathcal{M}_3(r_{n-1}, r_n))). \quad (2.6)$$

By (2.6) and axiom (Δ_4) ,

$$\mathbf{F}(s^n P_b|_{r_{n+1}}^{r_n}) \leq \mathbf{F}(s^{n-1} P_b|_{r_n}^{r_{n-1}}) + G(\gamma(\mathcal{M}_3(r_{n-1}, r_n))),$$

for each $n \in \mathbb{N}$. This implies,

$$\begin{aligned} \mathbf{F}(s^n P_b|_{r_{n+1}}^{r_n}) &\leq \mathbf{F}(s^{n-2} P_b|_{r_{n-1}}^{r_{n-2}}) + G(\gamma(\mathcal{M}_3(r_{n-2}, r_{n-1}))) \\ &\quad + G(\gamma(\mathcal{M}_3(r_{n-1}, r_n))). \end{aligned}$$

Thus,

$$\mathbf{F}(s^n P_b|_{r_{n+1}}^{r_n}) \leq \mathbf{F}(P_b|_{r_1}^{r_0}) + \sum_{i=1}^n G(\gamma(\mathcal{M}_3(r_{i-1}, r_i))). \quad (2.7)$$

Taking $n \rightarrow \infty$ in (2.7), we get

$$\lim_{n \rightarrow \infty} \mathbf{F}(s^n P_b|_{r_{n+1}}^{r_n}) = -\infty.$$

From (Δ_2) , we obtain,

$$\lim_{n \rightarrow \infty} (s^n P_b|_{r_{n+1}}^{r_n}) = 0.$$

By (Δ_3) , there exists, $0 < \mathfrak{h} < 1$ such that,

$$\lim_{n \rightarrow \infty} (s^n P_b|_{r_{n+1}}^{r_n})^{\mathfrak{h}} \mathbb{F}(s^n P_b|_{r_{n+1}}^{r_n}) = 0.$$

By (2.7), for each $n \in \mathbb{N}$,

$$\begin{aligned} & (s^n P_b|_{r_{n+1}}^{r_n})^{\mathfrak{h}} \mathbb{F}(s^n P_b|_{r_{n+1}}^{r_n}) - (s^n P_b|_{r_{n+1}}^{r_n})^{\mathfrak{h}} \mathbb{F}(P_b|_{r_1}^{r_0}) \\ & \leq (s^n P_b|_{r_{n+1}}^{r_n})^{\mathfrak{h}} \sum_{i=1}^n G(\gamma(\mathcal{M}_3(r_{i-1}, r_i))) \leq 0. \end{aligned} \quad (2.8)$$

Letting $n \rightarrow \infty$ in (2.8), we get,

$$\lim_{n \rightarrow \infty} \left(\sum_{i=1}^n G(\gamma(\mathcal{M}_3(r_{i-1}, r_i))) (s^n P_b|_{r_{n+1}}^{r_n})^{\mathfrak{h}} \right) = 0.$$

Thus, $\exists n_1 \in \mathbb{N}$ such that

$$\begin{aligned} & \sum_{i=1}^n G(\gamma(\mathcal{M}_3(r_{i-1}, r_i))) (s^n P_b|_{r_{n+1}}^{r_n})^{\mathfrak{h}} \leq 1, \forall n \geq n_1, \text{ or} \\ & s^n P_b|_{r_{n+1}}^{r_n} \leq \frac{1}{A_n^{1/\mathfrak{h}}}, \text{ for all } n \geq n_1, \text{ where, } A_n = \sum_{i=1}^n G(\gamma(\mathcal{M}_3(r_{i-1}, r_i))). \end{aligned} \quad (2.9)$$

Using (2.9), we get for $m \geq n \geq n_1$,

$$\begin{aligned} P_b|_{r_m}^{r_n} & \leq \sum_{i=n}^{m-1} s^i P_b|_{r_{i+1}}^{r_i} - \sum_{i=n+1}^{m-1} s^{i-(n+1)} P_b|_{r_i}^{r_i} \\ & \leq \sum_{i=n}^{m-1} s^i P_b|_{r_{i+1}}^{r_i} \leq \sum_{i=n}^{\infty} s^i P_b|_{r_{i+1}}^{r_i} \leq \sum_{i=n}^{\infty} \frac{1}{A_i^{1/\mathfrak{h}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{A_i^{1/\mathfrak{h}}}$ entails $\{r_n\}$ is a CS in (\mathfrak{S}, P_b) , As (\mathfrak{S}, P_b) is a complete PbMS, so $\exists r \in \mathfrak{S}$ such that

$$P_b|_r^r = \lim_{n \rightarrow \infty} P_b|_r^{r_n} = \lim_{n, m \rightarrow \infty} P_b|_{r_m}^{r_n} = 0.$$

By multivlued P_b -continuity of \hat{S} we get,

$$\lim_{n \rightarrow \infty} D_{P_b}|_{\hat{S}(r)}^{r_{n+1}} \leq \lim_{n \rightarrow \infty} \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r_n)} = \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}. \quad (2.10)$$

Using (P_{b4}) , we have

$$\begin{aligned} D_{P_b}|_{\hat{S}(r)}^r &\leq s[P_b|_{r_{n+1}}^r + D_{P_b}|_{\hat{S}(r)}^{r_{n+1}}] - P_b(r_{n+1}, r_{n+1}) \\ &\leq s[P_b|_{r_{n+1}}^r + D_{P_b}|_{\hat{S}(r)}^{r_{n+1}}] \end{aligned}$$

Letting $n \rightarrow \infty$ and using (2.10),

$$D_{P_b}|_{\hat{S}(r)}^r \leq \lim_{n \rightarrow \infty} s P_b|_{r_{n+1}}^r + \lim_{n \rightarrow \infty} s D_{P_b}|_{\hat{S}(r)}^{r_{n+1}} \leq s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}.$$

So we have $D_{P_b}(r, \hat{S}(r)) \leq s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}$. We will show that $r \in \hat{S}r$. Suppose that $r \notin \hat{S}(r)$. By Lemma 1.2.10, we obtain that $D_{P_b}(r, \hat{S}(r)) \neq 0$, which implies that

$$\begin{aligned} F(s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}) &\leq F(\alpha(r) \cdot \beta(r) s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}) \\ &\leq F(\mathcal{M}_3(r, r)) + G(\gamma(\mathcal{M}_3(r, r))), \text{ where,} \end{aligned}$$

$$\mathcal{M}_3(r, r) = \max \left\{ P_b|_r^r, D_{P_b}|_{\hat{S}(r)}^r, D_{P_b}|_{\hat{S}(r)}^r, \frac{D_{P_b}|_{\hat{S}(r)}^r + D_{P_b}|_{\hat{S}(r)}^r}{2s} \right\} = D_{P_b}|_{\hat{S}(r)}^r.$$

We get

$$\begin{aligned} F(s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}) &\leq F(D_{P_b}|_{\hat{S}(r)}^r) + G(\gamma(D_{P_b}|_{\hat{S}(r)}^r)) \\ &\leq F(s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}) + G(\gamma(s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)})). \end{aligned}$$

Since $G(\gamma(s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)})) \geq 0$, which yields that $\gamma(s \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r)}) \geq 1$, a contradiction. Therefore, $r \in \hat{S}(r)$ and hence \hat{S} has a FP in \mathfrak{S} . ■

Example 2.1.5 Consider $\mathfrak{S} = \{0, 1, 2, 3\}$ endowed with the PbM $P_b : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ defined by

$$P_b|_j^r = |r - j|^2 + \max\{r, j\}^2, \text{ for all } r, j \in \mathfrak{S}.$$

Clearly, (\mathfrak{S}, P_b) is a complete PbMS with coefficient $s = 4$, but

it is not a PMS since $P_b|_3^0 = 18 > 14 = P_b|_1^0 + P_b|_3^1 - P_b|_1^1$. Define mappings $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$

and $\gamma : \mathbb{R}_0^+ \rightarrow [0, 1)$, by

$$\hat{S}(0) = \hat{S}(1) = \{1, 2\}, \hat{S}(2) = \hat{S}(3) = \{1\} \text{ and } \gamma(t) = \frac{9}{10}.$$

Define mappings $\alpha, \beta : \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by

$$\alpha(r) = \begin{cases} \frac{r+4}{4} & \text{if } r \in \{2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta(r) = \begin{cases} \frac{r+5}{5} & \text{if } r \in \{2, 3\}, \\ 0 & \text{otherwise.} \end{cases}$$

For, $r \in \{2, 3\}$, we have

$$\alpha(r) \geq 1 \Rightarrow \beta_*(\hat{S}(r)) = \beta_*(\{1\}) = \frac{1+5}{5} \geq 1,$$

and

$$\beta(r) \geq 1 \Rightarrow \alpha_*(\hat{S}(r)) = \alpha_*(\{1\}) = \frac{1+4}{4} \geq 1.$$

Hence, \hat{S} is cyclic (α_*, β_*) -admissible mapping. Now, assume that $r, j \in \mathfrak{S}$ are such that $\alpha(r)\beta(j) \geq 1$, then we have $r, j \in \{2, 3\}$ and $\hat{H}_{P_b}|_{\hat{S}(j)}^{\hat{S}(r)} > 0$ imply

$$\begin{aligned} \mathbf{F}(\alpha(r)\beta(j)s \hat{H}_{P_b}|_{\hat{S}(j)}^{\hat{S}(r)}) &= \mathbf{F}(\alpha(r)\beta(j)s \hat{H}_{P_b}|_{\{1\}}^{\{1\}}) \\ &= \mathbf{F}(\alpha(r)\beta(j)s P_b|_1^1) \\ &\leq \mathbf{F}(P_b|_j^r) + G(\gamma(P_b|_j^r)) \\ &\leq \mathbf{F}(\mathcal{M}_3(r, j)) + G(\gamma(\mathcal{M}_3(r, j))). \end{aligned}$$

For $\mathbf{F}(t) = G(t) = \ln(t)$, $t > 0$, all hypotheses of Theorem 2.1.4 are obeyed. Hence, \hat{S} has a FP. Following Corollary provides a generalization of the results in [120], [103] in the set up of a PbMS.

Corollary 2.1.6. Let $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ be a multivalued mapping satisfying following conditions:

- (1) there exists $r_0 \in \mathfrak{S}$ such that $\alpha(r_0) \geq 1$ or there exists $j_0 \in \mathfrak{S}$ such that $\beta(j_0) \geq 1$;
- (2) \hat{S} is multivalued P_b -continuous;
- (3) there exist $\vartheta \in \mathbb{R}^+$, for all $r, j \in \mathfrak{S}$; $\alpha(r)\beta(j) \geq 1$ and

$$\hat{H}_{P_b} \Big|_{\hat{S}(j)}^{\hat{S}(r)} > 0 \Rightarrow \vartheta + \mathbf{F}(\alpha(r)\beta(j)s \hat{H}_{P_b} \Big|_{\hat{S}(j)}^{\hat{S}(r)}) \leq \mathbf{F}(\mathcal{M}_3(r, j)),$$

- (4) \hat{S} is cyclic (α_*, β_*) -admissible.

Then, \hat{S} has a FP.

Proof. Put $\gamma(t) = k, G(t) = \ln(t)$, where $k \in (0, 1)$ and $\vartheta = -\ln(k)$ in the Theorem 2.1.4 .

■

Corollary 2.1.7. Let $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ be a multivalued mapping satisfying following conditions:

- (1) there exists $r_0 \in \mathfrak{S}$ such that $\alpha(r_0) \geq 1$ or there exists $j_0 \in \mathfrak{S}$ such that $\beta(j_0) \geq 1$;
- (2) \hat{S} is multivalued P_b -continuous;
- (3) for each $r, j \in \mathfrak{S}$ with, $\alpha(r)\beta(j) \geq 1$,

$$\alpha(r)\beta(j)s \hat{H}_{P_b} \Big|_{\hat{S}(j)}^{\hat{S}(r)} \leq \gamma(\mathcal{M}_3(r, j))\mathcal{M}_3(r, j),$$

- (4) \hat{S} is cyclic (α_*, β_*) -admissible.

Then \hat{S} has a FP.

Proof. Set $\mathbf{F}(t) = G(t) = \ln(t)$ in Theorem 2.1.4. ■

Definition 2.1.8. Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a self-mapping. Then \hat{S} is γ -FG-contraction, if there exist $\mathbf{F} \in \Delta_F, (G, \gamma) \in \Delta_{G, \gamma}$ such that,

$$\mathbf{F}(s P_b \Big|_{\hat{S}(j)}^{\hat{S}(r)}) \leq \mathbf{F}(\mathcal{M}_4(r, j)) + G(\gamma(\mathcal{M}_4(r, j))),$$

for all $r, j \in \mathfrak{S}$, $P_b|_{\hat{S}(j)}^{\hat{S}(r)} > 0$, where

$$\mathcal{M}_4(r, j) = \max \left\{ P_b|_j^r, P_b|_{\hat{S}(j)}^j, P_b|_{\hat{S}(r)}^r, \frac{P_b|_{\hat{S}(j)}^r + P_b|_{\hat{S}(r)}^j}{2s} \right\}.$$

Corollary 2.1.9. Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a γ -FG-contraction. If \hat{S} is P_b -continuous, then \hat{S} has a FP.

Proof. Consider Picard iterative sequence $\{r_n : r_n = \hat{S}(r_{n-1})\}_{n \in \mathbb{N}}$ in the proof of Theorem 2.1.4. This proof contains similar steps as in the proof of Theorem 2.1.4, so, we omit details.

■

"Corollary" 2.1.10. Let $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ be a multivalued mapping, such that

$$s \hat{H}_{P_b}|_{\hat{S}(j)}^{\hat{S}(r)} \leq \lambda \mathcal{M}_3(r, j),$$

for some $\lambda \in [0, 1)$ and for all $r, j \in \mathfrak{S}$. Then \hat{S} has a FP.

Proof. Set $\mathbf{F}(t) = t$, $G(t) = (1 - k)t$, $\gamma(t) = k$, $k \in (0, 1)$ and $\alpha(r) = \beta(j) = 1$ in Theorem 2.1.4. ■

Corollary 2.1.11. Let $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ be a multivalued mapping, such that

$$\hat{H}_{P_b}|_{\hat{S}(j)}^{\hat{S}(r)} \leq \lambda \mathcal{M}_3(r, j),$$

for some $\lambda \in [0, 1)$ and for all $r, j \in \mathfrak{S}$. Then \hat{S} has a FP.

Proof. Set $\mathbf{F}(t) = t$, $G(t) = (\lambda - 1)t$, $\gamma(t) = \lambda$, $\lambda \in \mathbb{R}_0^+$, $k = 1$ and $\alpha(r) = \beta(j) = 1$ in Theorem 2.1.4. ■

2.2 Application to nonlinear integral equations

In this section, We investigate the existence of the solution of nonlinear integral equation (NIE). Consider the following NIE:

$$r(\tilde{u}) = g(t) + \lambda \int_0^1 \kappa(\tilde{u}, \tilde{a}) f(\tilde{a}, r(\tilde{a})) d\tilde{a}, \tilde{u}, \tilde{a} \in I = [0, 1], \lambda \geq 0. \quad (2.11)$$

Assume the following conditions:

(a) $g : I \rightarrow \mathbb{R}$ is a continuous mapping;

(b) $f : I \times X \rightarrow \mathbb{R}$ is a continuous mapping and there exists a constant $\delta \in [0, 1)$ such that for all $r, j \in \mathfrak{S}$;

$$|f(\tilde{u}, r(\tilde{u})) - f(\tilde{u}, j(\tilde{u}))| \leq \delta \sqrt{\ln(\mathcal{M}_5(r(\tilde{u}), j(\tilde{u})) e^{G(\gamma(\mathcal{M}_5(r(\tilde{u}), j(\tilde{u}))))})};$$

(c) $\kappa : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous at $\tilde{u} \in [0, 1]$ for each $\check{a} \in [0, 1]$ and measurable at $\check{a} \in [0, 1]$ for each $\tilde{u} \in [0, 1]$, such that,

$$\sup_{\tilde{u} \in I} \int_0^1 \kappa(\tilde{u}, \check{a}) d\check{a} \leq \mathbb{K};$$

(d) $\lambda \mathbb{K} \delta \leq 1$;

(e) $\ln(s P_b |_{\hat{S}(j)}^{\hat{S}(r)}) \leq \sup_{\tilde{u} \in I} |\hat{S}(r(\tilde{u})) - \hat{S}(j(\tilde{u}))|^2$.

Let $\mathfrak{S} = C([0, 1])$ be the space of all continuous real valued mappings defined on $[0, 1]$.

Define mapping $P_b : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by

$$P_b(r, j) = \left(\sup_{\tilde{u} \in I} |r(\tilde{u}) - j(\tilde{u})| + \eta \right)^2 \text{ for all } r, j \in \mathfrak{S} \text{ and } \eta > 0. \quad (2.12)$$

Then (\mathfrak{S}, P_b) is a complete PbMS with $s = 2 > 1$.

Lemma 2.2.1. Let $\mathfrak{S} = C(I)$. Define mapping $P_b : \mathfrak{S} \times \mathfrak{S} \rightarrow [0, \infty)$ as in (2.12). Then

$$\mathcal{M}_4(r, j) = \sup_{\tilde{u} \in I} \mathcal{M}_5(r(\tilde{u}), j(\tilde{u})).$$

$$\mathcal{M}_5(r(\tilde{u}), j(\tilde{u})) = \left\{ \begin{array}{l} \left(\sup_{\tilde{u} \in I} |r(\tilde{u}) - j(\tilde{u})| + \eta \right)^2, \left(\sup_{\tilde{u} \in I} |j(\tilde{u}) - \hat{S}(j(\tilde{u}))| + \eta \right)^2, \\ \left(\sup_{\tilde{u} \in I} |r(\tilde{u}) - \hat{S}(r(\tilde{u}))| + \eta \right)^2, \\ \frac{\left(\sup_{\tilde{u} \in I} |r(\tilde{u}) - \hat{S}(j(\tilde{u}))| + \eta \right)^2 + \left(\sup_{\tilde{u} \in I} |j(\tilde{u}) - \hat{S}(r(\tilde{u}))| + \eta \right)^2}{2s} \end{array} \right\}.$$

Proof. Since,

$$P_b(r, j) = \left(\sup_{\tilde{u} \in I} |r(\tilde{u}) - j(\tilde{u})| + \eta \right)^2 \text{ for all } r, j \in \mathfrak{S} \text{ and } \eta > 0.$$

Result follows. ■

Theorem 2.2.2. Let $\mathfrak{S} = C(I)$. Define the mapping $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\hat{S}r(t) = g(\tilde{u}) + \lambda \int_0^1 \kappa(\tilde{u}, \check{a}) f(\check{a}, r(\check{a})) d\check{a}, \tilde{u} \in I = [0, 1], \lambda \geq 0. \quad (2.13)$$

If the conditions (a)-(e) hold, then the NIE (2.11) has a solution in \mathfrak{S} .

Proof. We note that $r^*(\cdot) \in \mathfrak{S}$ is a solution of (2.11) iff $r^*(\cdot) \in \mathfrak{S}$ is a FP of the mapping \hat{S} defined in (2.13).

By assumption (b), one has

$$\begin{aligned} \left| \hat{S}(r(\tilde{u})) - \hat{S}(j(\tilde{u})) \right|^2 &= \left| \lambda \int_0^1 \kappa(\tilde{u}, \check{a}) f(\check{a}, r(\check{a})) d\check{a} - \lambda \int_0^1 \kappa(\tilde{u}, \check{a}) f(\check{a}, j(\check{a})) d\check{a} \right|^2 \\ &\leq \lambda^2 \left(\int_0^1 \kappa(\tilde{u}, \check{a}) |f(\check{a}, r(\check{a})) - f(\check{a}, j(\check{a}))| d\check{a} \right)^2 \\ &\leq \lambda^2 \left(\int_0^1 \kappa(\tilde{u}, \check{a}) \delta \sqrt{\ln(\mathcal{M}_5(r(\check{a}), j(\check{a})) e^{G(\gamma(\mathcal{M}_5(r(\check{a}), j(\check{a}))))})} d\check{a} \right)^2 \\ &\leq \lambda^2 \delta^2 \ln \left(\sup_{\check{a} \in I} \mathcal{M}_5(r(\check{a}), j(\check{a})) e^{G(\gamma(\sup_{\check{a} \in I} \mathcal{M}_5(r(\check{a}), j(\check{a}))))} \right) \\ &\quad \left(\int_0^1 \kappa(\tilde{u}, \check{a}) d\check{a} \right)^2. \end{aligned}$$

By assumption (c) and (d), we have

$$\begin{aligned} \sup_{\tilde{u} \in I} \left| \hat{S}(r(\tilde{u})) - \hat{S}(j(\tilde{u})) \right|^2 &\leq \lambda^2 \delta^2 \mathbb{K}^2 \ln \left(\sup_{\check{a} \in I} \mathcal{M}_5(r(\check{a}), j(\check{a})) e^{G(\gamma(\sup_{\check{a} \in I} \mathcal{M}_5(r(\check{a}), j(\check{a}))))} \right) \\ &\leq \left(\sup_{\check{a} \in I} \mathcal{M}_5(r(\check{a}), j(\check{a})) e^{G(\gamma(\sup_{\check{a} \in I} \mathcal{M}_5(r(\check{a}), j(\check{a}))))} \right). \end{aligned}$$

By assumption (e) and Lemma 2.2.1, we have

$$\begin{aligned} \ln(s P_b|_{\hat{S}(j)}^{\hat{S}(r)}) &\leq \ln \left(\mathcal{M}_4(r, j) e^{G(\gamma(\mathcal{M}_4(r, j)))} \right) \\ &\leq \ln(\mathcal{M}_4(r, j)) + G(\gamma(\mathcal{M}_4(r, j))). \end{aligned}$$

Now $\mathbf{F}(\zeta) = \ln(\zeta)$ satisfies all the conditions of Corollary 2.1.9, and so the integral equations given by (2.11) has a solution. ■

2.3 Fixed points of Branciari F-rational contractions in Branciari metric spaces

Here, we initiate Branciari F-rational contraction and provide related FP results on BMSs.

Definition 2.3.1 Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a self-mapping defined on a BMS $(\mathfrak{S}, \partial_B)$. Then \hat{S} is called Branciari F-rational contraction (BF-R-contraction), if there exist $\mathbf{F} \in F$, $\vartheta > 0$ such that $\forall r, j \in \mathfrak{S}$, $\partial_B|_{\hat{S}(j)}^{\hat{S}(r)} > 0$, implies,

$$\vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq \mathbf{F}(\mathcal{M}_6(r, j)), \quad (2.14)$$

where,

$$\mathcal{M}_6(r, j) = \max \left\{ \partial_B|_j^r, \partial_B|_{\hat{S}(r)}^r, \partial_B|_{\hat{S}(j)}^j, \frac{\partial_B|_{\hat{S}(r)}^r \partial_B|_{\hat{S}(j)}^j}{1 + \partial_B|_j^r}, \frac{\partial_B|_{\hat{S}(r)}^r \partial_B|_{\hat{S}(j)}^j}{1 + \partial_B|_{\hat{S}(j)}^{\hat{S}(r)}} \right\}. \quad (2.15)$$

Theorem 2.3.2 Let $(\mathfrak{S}, \partial_B)$ be a complete BMS and $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a BF-R-contraction. If \hat{S} or \mathbf{F} is continuous, then \hat{S} has a unique FP in \mathfrak{S} .

Proof. Let $r \in \mathfrak{S}$ be an arbitrary point. If $\hat{S}^n(r) = \hat{S}^{n+1}(r)$, $n \in \mathbb{N}$. Then, $\hat{S}^n r$ is a FP of \hat{S} . So, suppose that $\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} > 0$, for each $n \in \mathbb{N}$. Then, from (2.14), we have,

$$\vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right) \leq \mathbf{F} \left(\mathcal{M}_6(\hat{S}^{n-1}(r), \hat{S}^n(r)) \right), \text{ for each } n \in \mathbb{N}, \quad (2.16)$$

where

$$\begin{aligned}
\mathcal{M}_6(\hat{S}^{n-1}(r), \hat{S}^n(r)) &= \max \left\{ \begin{array}{l} \partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}, \partial_B|_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)}, \partial_B|_{\hat{S}(\hat{S}^n(r))}^{\hat{S}^n(r)}, \\ \frac{\partial_B|_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)} \partial_B|_{\hat{S}(\hat{S}^n(r))}^{\hat{S}^n(r)}}{1 + \partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}}, \frac{\partial_B|_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)} \partial_B|_{\hat{S}(\hat{S}^n(r))}^{\hat{S}^n(r)}}{1 + \partial_B|_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)}} \end{array} \right\} \\
&= \max \left\{ \begin{array}{l} \partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}, \partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}, \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}, \\ \frac{\partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}}{1 + \partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}}, \frac{\partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}}{1 + \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}} \end{array} \right\} \\
&\leq \max\{\partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}, \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}\}.
\end{aligned}$$

Now if, $\mathcal{M}_6(\hat{S}^{n-1}(r), \hat{S}^n(r)) \leq \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}$, then (2.16) turns into

$$\vartheta + \mathbf{F}\left(\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}\right) \leq \mathbf{F}\left(\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}\right),$$

a contradiction. Hence, $\mathcal{M}_6(\hat{S}^{n-1}(r), \hat{S}^n(r)) \leq \partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}$. Thus, (2.16) turns into

$$\mathbf{F}\left(\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}\right) \leq \mathbf{F}\left(\partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}\right) - \vartheta, \quad \forall n \in \mathbb{N}. \quad (2.17)$$

Iteratively, we find that

$$\begin{aligned}
\mathbf{F}\left(\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}\right) &\leq \mathbf{F}\left(\partial_B|_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)}\right) - \vartheta \\
&\leq \mathbf{F}\left(\partial_B|_{\hat{S}^{n-1}(r)}^{\hat{S}^{n-2}(r)}\right) - 2\vartheta \\
&\leq \mathbf{F}\left(\partial_B|_{\hat{S}^{n-2}(r)}^{\hat{S}^{n-3}(r)}\right) - 3\vartheta \\
&\vdots \\
&\leq \mathbf{F}\left(\partial_B|_{\hat{S}(r)}^r\right) - n\vartheta.
\end{aligned} \quad (2.18)$$

Taking limit as $n \rightarrow \infty$ in (2.18), one get

$$\lim_{n \rightarrow \infty} \mathbf{F}\left(\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}\right) = -\infty.$$

By (F2), we get,

$$\lim_{n \rightarrow \infty} \partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} = 0 \quad (2.19)$$

Now from (F3), there exists $\hat{h} \in (0, 1)$ such that,

$$\lim_{n \rightarrow \infty} \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right) = 0. \quad (2.20)$$

By (2.18), we have

$$\begin{aligned} & \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right) - \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}(r)}^r \right) \\ & \leq \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \left[\mathbf{F} \left(\partial_B |_{\hat{S}(r)}^r \right) - n\vartheta \right] - \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}(r)}^r \right) \\ & = -n\tau \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \leq 0. \end{aligned} \quad (2.21)$$

Setting $n \rightarrow \infty$ in (2.21) and from (2.19) and (2.20),

$$\lim_{n \rightarrow \infty} n \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} = 0, \quad (2.22)$$

and hence

$$\lim_{n \rightarrow \infty} n^{\frac{1}{\hat{h}}} \partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} = 0. \quad (2.23)$$

Then, $\exists n_1 \in \mathbb{N}$ such that, $n \left[\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \leq 1, \forall n \geq n_1$. Thus, one has,

$$\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \leq \frac{1}{n^{\frac{1}{\hat{h}}}}. \quad (2.24)$$

Now, we will prove that \hat{S} has a periodic point (PEP). Suppose, the contrary, that is, \hat{S} has no PEP, then $\hat{S}^n(r) \neq \hat{S}^m(r)$ for each $n, m \in \mathbb{N}$ such that $n \neq m$. By (2.14), we get

$$\begin{aligned} & \vartheta + \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) \\ & \leq \mathbf{F} \left(\max \left\{ \begin{array}{l} \partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^{n-1}(r)}, \partial_B |_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)}, \partial_B |_{\hat{S}(\hat{S}^{n+1}(r))}^{\hat{S}^{n+1}(r)}, \\ \frac{\partial_B |_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}(\hat{S}^{n+1}(r))}^{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^{n-1}(r)}}, \frac{\partial_B |_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}(\hat{S}^{n+1}(r))}^{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}(\hat{S}^{n-1}(r))}^{\hat{S}^{n-1}(r)}} \end{array} \right\} \right) \end{aligned}$$

$$= \mathbf{F} \left(\max \left\{ \begin{array}{l} \partial_B |_{\hat{S}^{n+1}(r)}, \partial_B |_{\hat{S}^n(r)}, \partial_B |_{\hat{S}^{n+2}(r)}, \\ \frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+1}(r)}}, \frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+2}(r)}} \end{array} \right\} \right). \quad (2.25)$$

As \mathbf{F} is increasing, we obtain from (2.25)

$$\vartheta + \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)} \right) \leq \max \left\{ \begin{array}{l} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)} \right), \mathbf{F} \left(\partial_B |_{\hat{S}^n(r)} \right), \\ \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+1}(r)}} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+2}(r)}} \right) \end{array} \right\}. \quad (2.26)$$

Consider I is the set of $n \in \mathbb{N}$ such that,

$$\begin{aligned} u_n &= \max \left\{ \begin{array}{l} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)} \right), \mathbf{F} \left(\partial_B |_{\hat{S}^n(r)} \right), \\ \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+1}(r)}} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+2}(r)}} \right) \end{array} \right\} \\ &= \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)} \right). \end{aligned}$$

If $|I| < \infty$ then $\exists N \in \mathbb{N}$ such that for each $n \geq N$,

$$\max \left\{ \begin{array}{l} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)} \right), \mathbf{F} \left(\partial_B |_{\hat{S}^n(r)} \right), \\ \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+1}(r)}} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+2}(r)}} \right) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \mathbf{F} \left(\partial_B |_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \right), \\ \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^{n+1}(r)} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^{n-1}(r)}} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)}} \right) \end{array} \right\}.$$

In this case, we obtain from (2.26)

$$\begin{aligned} & \vartheta + \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) \\ & \leq \max \left\{ \begin{array}{l} \mathbf{F} \left(\partial_B |_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \right), \\ \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^{n+1}(r)} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^{n-1}(r)}} \right), \\ \mathbf{F} \left(\frac{\partial_B |_{\hat{S}^n(r)}^{\hat{S}^{n-1}(r)} \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^{n+1}(r)}}{1 + \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)}} \right) \end{array} \right\}, \end{aligned}$$

for all $n \geq N$. Setting $n \rightarrow \infty$ and using (2.19), we get

$$\lim_{n \rightarrow \infty} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) = -\infty.$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, then we denote also by $\{u_n\}$, such that

$$u_n = \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^{n-1}(r)} \right), \text{ for } n \text{ large enough.}$$

In this case, we get from (2.26)

$$\vartheta + \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) \leq \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^{n-1}(r)} \right)$$

Iteratively, we find that

$$\begin{aligned}
\mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) &\leq \mathbf{F} \left(\partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^{n-1}(r)} \right) - \vartheta \\
&\leq \mathbf{F} \left(\partial_B |_{\hat{S}^n(r)}^{\hat{S}^{n-2}(r)} \right) - 2\vartheta \\
&\leq \mathbf{F} \left(\partial_B |_{\hat{S}^{n-1}(r)}^{\hat{S}^{n-3}(r)} \right) - 3\vartheta \\
&\quad \vdots \\
&\leq \mathbf{F} \left(\partial_B |_{\hat{S}^2(r)}^r \right) - n\vartheta, \forall n \in \mathbb{N}.
\end{aligned} \tag{2.27}$$

Letting $n \rightarrow \infty$ in (2.27), we get

$$\lim_{n \rightarrow \infty} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) = -\infty. \tag{2.28}$$

Then in all cases, (2.28) holds. From (2.28) and (F2), we obtain,

$$\lim_{n \rightarrow \infty} \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} = 0. \tag{2.29}$$

From (F3), there exists $0 < \hat{h} < 1$ such that,

$$\lim_{n \rightarrow \infty} \left[\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) = 0. \tag{2.30}$$

By (2.27), we have

$$\begin{aligned}
&\left[\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right) - \left[\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}^2(r)}^r \right) \\
&\leq \left[\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \left[\mathbf{F} \left(\partial_B |_{\hat{S}^2(r)}^r \right) - n\vartheta \right] - \left[\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \mathbf{F} \left(\partial_B |_{\hat{S}^2(r)}^r \right) \\
&= -n\tau \left[\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} \leq 0.
\end{aligned} \tag{2.31}$$

Setting $n \rightarrow \infty$ in (2.31) and using (2.29) and (2.30),

$$\lim_{n \rightarrow \infty} n \left[\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \right]^{\hat{h}} = 0, \tag{2.32}$$

and so

$$\lim_{n \rightarrow \infty} n^{\frac{1}{h}} \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} = 0. \quad (2.33)$$

Hence, there exists $n_2 \in \mathbb{N}$ such that

$$\partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} \leq \frac{1}{n^{\frac{1}{h}}}, \forall n \geq n_2. \quad (2.34)$$

Let $\xi = \max\{n_0, n_1\}$. we consider two cases.

Case 1: If $m > 2$ is odd, then writing $m = 2L + 1, L \geq 1$, using (2.24), for each $n \geq \xi$, we obtain

$$\begin{aligned} \partial_B |_{\hat{S}^{n+m}(r)}^{\hat{S}^n(r)} &\leq \partial_B |_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} + \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^{n+1}(r)} + \dots + \partial_B |_{\hat{S}^{n+2L+1}(r)}^{\hat{S}^{n+2L}(r)} \\ &\leq \frac{1}{n^{\frac{1}{h}}} + \frac{1}{(n+1)^{\frac{1}{h}}} + \dots + \frac{1}{(n+2L)^{\frac{1}{h}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}. \end{aligned}$$

Case 2: If $m > 2$ is even, then writing $m = 2L, L \geq 2$, using (2.10) and (2.20), for each $n \geq \xi$, we have

$$\begin{aligned} \partial_B |_{\hat{S}^{n+m}(r)}^{\hat{S}^n(r)} &\leq \partial_B |_{\hat{S}^{n+2}(r)}^{\hat{S}^n(r)} + \partial_B |_{\hat{S}^{n+3}(r)}^{\hat{S}^{n+2}(r)} + \dots + \partial_B |_{\hat{S}^{n+2L}(r)}^{\hat{S}^{n+2L-1}(r)} \\ &\leq \frac{1}{n^{\frac{1}{h}}} + \frac{1}{(n+2)^{\frac{1}{h}}} + \dots + \frac{1}{(n+2L-1)^{\frac{1}{h}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}. \end{aligned}$$

Thus, combining all cases, we have

$$\partial_B |_{\hat{S}^{n+m}(r)}^{\hat{S}^n(r)} \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}} \text{ for all } n \geq \xi, m \in \mathbb{N}.$$

Since $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}$ is convergent series, we deduce that $\{\hat{S}^n(r)\}$ is a CS. As $(\mathfrak{S}, \partial_B)$ is complete, there $z \in \mathfrak{S}$ such that $\hat{S}^n(r) \rightarrow z$ as $n \rightarrow \infty$. Now we Suppose that \hat{S} is continuous. Then,

$$z = \lim_{n \rightarrow \infty} \hat{S}^{n+1}(r) = \lim_{n \rightarrow \infty} \hat{S}(\hat{S}^n(r)) = \hat{S}\left(\lim_{n \rightarrow \infty} \hat{S}^n(r)\right) = \hat{S}(z).$$

Next, we Suppose that \mathbf{F} is continuous. Without restriction of the generality, we can assume that $\hat{S}^n(r) \neq z$ for each n . Assume that $\partial_B|_{\hat{S}(z)}^z > 0$,

$$\vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}(z)}^{\hat{S}^{n+1}(r)} \right) \leq \mathbf{F} \left(\max \left\{ \begin{array}{l} \partial_B|_z^{\hat{S}^n(r)}, \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}, \partial_B|_{\hat{S}(z)}^z, \\ \frac{\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \partial_B|_{\hat{S}(z)}^z}{1 + \partial_B|_z^{\hat{S}^n(r)}}, \frac{\partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)} \partial_B|_{\hat{S}(z)}^z}{1 + \partial_B|_{\hat{S}^{n+1}(r)}^{\hat{S}^n(r)}} \end{array} \right\} \right).$$

Letting $n \rightarrow \infty$ in the above inequality, using Proposition 1.2.20, we obtain

$$\vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}(z)}^z \right) \leq \mathbf{F} \left(\partial_B|_{\hat{S}(z)}^z \right),$$

a contradiction. Thus, $z = \hat{S}(z)$, which is also a contradiction with the assumption: \hat{S} does not have a PEP. Thus \hat{S} has a PEP, say z of period q . Assume that the set of FPs of \hat{S} is empty. Then,

$$q > 1 \text{ and } \partial_B|_{\hat{S}(z)}^z > 0.$$

From (2.1), we get

$$\vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}(z)}^z \right) = \vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}^{q+1}(z)}^{\hat{S}^q(z)} \right) \leq \mathbf{F} \left(\partial_B|_{\hat{S}^q(z)}^{\hat{S}^{q-1}(z)} \right).$$

This implies

$$\begin{aligned} \mathbf{F} \left(\partial_B|_{\hat{S}(z)}^z \right) &\leq \mathbf{F} \left(\partial_B|_{\hat{S}^q(z)}^{\hat{S}^{q-1}(z)} \right) - \vartheta \\ &\leq \dots \leq \mathbf{F} \left(\partial_B|_{\hat{S}(z)}^z \right) - q\vartheta < \mathbf{F} \left(\partial_B|_{\hat{S}(z)}^z \right), \end{aligned}$$

a contradiction. Thus, \hat{S} has at least one FP. Now we assume that $z, u \in \mathfrak{S}$ are two FPs of \hat{S} such that $\partial_B|_u^z = \partial_B|_{\hat{S}(u)}^{\hat{S}(z)} > 0$. By (2.14), we get

$$\vartheta + \mathbf{F} \left(\partial_B|_u^z \right) = \vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}(u)}^{\hat{S}(z)} \right) \leq \mathbf{F} \left(\partial_B|_u^z \right),$$

this is a contradiction. Therefore, \hat{S} has a unique FP. ■

As a MS is a BMS, we can define \mathbf{F} -rational contraction:

Definition 2.3.3 Let (\mathfrak{S}, ∂) be a MS. Then, $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ is called \mathbf{F} -rational contraction

(F-R-contraction), there exist $\mathbf{F} \in F$ and $\vartheta > 0$ such that, $\forall r, j \in \mathfrak{S}$, $\partial_{\hat{S}(j)}^{\hat{S}(r)} > 0$, implies,

$$\vartheta + \mathbf{F} \left(\partial_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq \mathbf{F} (\mathcal{M}_7(r, j)),$$

where,

$$\mathcal{M}_7(r, j) = \max \left\{ \partial_j^r, \partial_{\hat{S}(r)}^r, \partial_{\hat{S}(j)}^j, \frac{\partial_{\hat{S}(r)}^r \partial_{\hat{S}(j)}^j}{1 + \partial_j^r}, \frac{\partial_{\hat{S}(r)}^r \partial_{\hat{S}(j)}^j}{1 + \partial_{\hat{S}(j)}^{\hat{S}(r)}} \right\}.$$

Theorem 2.3.4 Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a F-R-contraction. If \hat{S} or \mathbf{F} is continuous, then \exists only one FP of \hat{S} in \mathfrak{S} .

Definition 2.3.5 Let $(\mathfrak{S}, \partial_B)$ be a BMS. $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ is a Branciari F-contraction (BF-contraction), if $\exists \mathbf{F} \in F$ and $\vartheta > 0$ such that

$$\forall r, j \in \mathfrak{S}, \partial_B|_{\hat{S}(j)}^{\hat{S}(r)} > 0 \Rightarrow \vartheta + \mathbf{F} \left(\partial_B|_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq \mathbf{F} \left(\partial_B|_j^r \right).$$

Theorem 2.3.6 Let $(\mathfrak{S}, \partial_B)$ be a complete BMS and $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a BF-contraction. Then \hat{S} has a unique FP in \mathfrak{S} .

Example 2.3.7 Let $\mathfrak{S} = \{1, 2, 3, 4\}$. Define a function $\partial_B : X \times X \rightarrow \mathbb{R}_0^+$ by,

$$\begin{aligned} \partial_B|_2^1 &= \partial_B|_1^2 = 3, \\ \partial_B|_3^2 &= \partial_B|_2^3 = \partial_B|_3^1 = \partial_B|_1^3 = 1, \\ \partial_B|_4^1 &= \partial_B|_1^4 = \partial_B|_4^2 = \partial_B|_2^4 = \partial_B|_4^3 = \partial_B|_3^4 = 4. \end{aligned}$$

Clearly, $(\mathfrak{S}, \partial_B)$ is a complete BMS. But it is not MS because ∂_B does not satisfy triangle inequality on \mathfrak{S} . Indeed,

$$3 = \partial_B|_2^1 > \partial_B|_3^1 + \partial_B|_2^3 = 1 + 1 = 2.$$

Define $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\hat{S}(r) = \begin{cases} 3 & \text{if } r \in \{1, 2, 3\}, \\ 1 & \text{if } r = 4, \end{cases}$$

and $\mathbf{F} : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $F(u) = \ln(u)$. Now, for $r \in \{1, 2, 3\}$, $j = 4$, where $\vartheta = 1$, one has

$$\vartheta + \mathbf{F} \left(\partial_B \Big|_{\hat{S}(4)}^{\hat{S}(r)} \right) = 1 + \mathbf{F} \left(\partial_B \Big|_1^3 \right) \leq \mathbf{F} \left(\partial_B \Big|_4^r \right).$$

Then, \hat{S} is BF-contraction, and \hat{S} has a unique FP.

Example 2.3.9 Let $\mathfrak{S} = \{0, \frac{8}{3}, 7\}$ endowed with the usual metric $\partial_j^r = |r - j|$ for each $r, j \in \mathfrak{S}$.

Define a mapping, $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ by,

$$\hat{S}(r) = \begin{cases} \frac{8}{3} & r \in \{0, \frac{8}{3}\}, \\ 0 & r = 7. \end{cases}$$

Then, (\mathfrak{S}, ∂) is a CMS (also a complete BMS). As \hat{S} is not continuous, so \hat{S} is not F-contraction (by Remark 1.4.5).

Now, For $r \in \{0, \frac{8}{3}\}$ and $j = 7$, we have

$$\partial_{\hat{S}(7)}^{\hat{S}(r)} = \partial_0^{\frac{8}{3}} = \left| \frac{8}{3} - 0 \right| = \frac{8}{3} > 0, \text{ and } \mathcal{M}_7(r, 7) = 7.$$

So, by choosing, $\mathbf{F}(u) = \ln u + u$ and $\vartheta \in (0, 4.965]$, we see that $\forall r, j \in \mathfrak{S}$, $\hat{S}(r) \neq \hat{S}(j)$

$$\vartheta + \mathbf{F} \left(\partial_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq \mathbf{F} \left(\mathcal{M}_7(r, j) \right).$$

Hence, \hat{S} is BF-R-contraction (or F-R-contraction) and so, \hat{S} has a unique FP.

Chapter 3

Fixed point Theorems for θ -Contractions in Metric Spaces

Jleli et al. [66, 65] announced θ -contraction (or JS-contraction). Li et al. [83] presented θ -quasi-contraction and obtained related FP results. Altun et al. [26] obtained some FP theorems for new class of θ -contraction. The work of Jleli et al. [66, 65] has been extended by many authors, see, ([19, 59, 58, 99, 5]).

In first two sections of This chapter, we study the existence of a COPs results for mappings and relations satisfying new θ -contractive condition on MS and provide application to Volterra integral equations, while in the remaining three sections, we discuss the concept of generalized Suzuki type θ -contractions for single valued mappings, existence of their common FPs and apply these results to prove the existence of the solution for a system of nonlinear fractional differential equations.

Results given in this chapter have been published in ([7],[12]).

3.1 Coincidence points theorems for single valued mappings and relations

Consistent with [30], let $\check{A}_1 \neq \emptyset$ and $\check{A}_2 \neq \emptyset$. A relation $\check{R}_e : \check{A}_1 \rightsquigarrow \check{A}_2$ is called left-total if for every $r \in \check{A}_1 \exists$ a $j \in \check{A}_2$ such that $r\check{R}_ej$ (i.e., r is \check{R}_e -related to j). A relation $\check{R}_e : \check{A}_1 \rightsquigarrow \check{A}_2$

is called right-total if for each $j \in \check{A}_2$ there exists an $r \in \check{A}_1$ such that $r\check{R}_ej$. A relation $\check{R}_e : \check{A}_1 \rightsquigarrow \check{A}_2$ is called functional if, $r \in \check{A}_1$ and $j, x \in \check{A}_2$ with $r\check{R}_ej$ and $r\check{R}_ex$ imply $j = x$.

A mapping $\hat{S} : \check{A}_1 \longrightarrow \check{A}_2$ is a relation from \check{A}_1 to \check{A}_2 which is both functional and left-total. For a relation $\check{R}_e : \check{A}_1 \rightsquigarrow \check{A}_2$, $E \subseteq \check{A}_1$ we define $\check{R}_e(E)$, domain of \check{R}_e ($Do(\check{R}_e)$) and range of \check{R}_e ($Ran(\check{R}_e)$) by

$$\check{R}_e(E) = \left\{ j \in \check{A}_2 : r\check{R}_ej \text{ for some } r \in E \right\}, \quad Do(\check{R}_e) = \left\{ r \in \check{A}_1 : \check{R}_e(\{r\}) \neq \emptyset \right\},$$

and

$$Ran(\check{R}_e) = \left\{ j \in \check{A}_2 : j \in \check{R}_e(\{r\}) \text{ for some } r \in Do(\check{R}_e) \right\}.$$

The set of relations from \check{A}_1 to \check{A}_2 is denoted by $\check{R}_e(\check{A}_1, \check{A}_2)$. Thus the collection $\mu(\check{A}_1, \check{A}_2)$ of all maps from \check{A}_1 to \check{A}_2 is a proper subcollection of $\check{R}_e(\check{A}_1, \check{A}_2)$. An element $w \in \check{A}_1$ is called a COP of a mapping $\hat{S} : \check{A}_1 \longrightarrow \check{A}_2$ and a relation $\check{R}_e : \check{A}_1 \rightsquigarrow \check{A}_2$ if $\hat{S}(w) \in \check{R}_e(\{w\})$. If \check{R}_e is a mapping, then $\hat{S}(w) = \check{R}_e(w)$. For a relation $\check{R}_e : \mathfrak{S} \rightsquigarrow \hat{Y}$ and $u, v \in Do(\check{R}_e)$, we define

$$D_{\check{R}_e(\{v\})}^{\check{R}_e(\{u\})} = \inf_{u\check{R}_er, v\check{R}_ej} \partial_j^r.$$

In this section, we investigate COPs for mappings and relations satisfying satisfying θ -contractive condition ($\theta \in \Theta^*$).

Theorem 3.1.1. Let $\mathfrak{S} \neq \emptyset$ and (\hat{Y}, ∂) be a MS. Assume that $\hat{S} : \mathfrak{S} \longrightarrow \hat{Y}$ is a map and $\check{R}_e : \mathfrak{S} \rightsquigarrow \hat{Y}$ is a left-total relation, $Ran(\hat{S}) \subseteq Ran(\check{R}_e)$ and there exist $\theta \in \Theta^*$ and $k \in (0, 1)$ such that,

$$\theta \left(\partial_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq \left[\theta \left(D_{\check{R}_e(\{j\})}^{\check{R}_e(\{r\})} \right) \right]^k, \quad \text{for each } r, j \in \mathfrak{S} \text{ with } \hat{S}(r) \neq \hat{S}(j). \quad (3.1)$$

If $Ran(\hat{S})$ or $Ran(\check{R}_e)$ is complete, then \check{R}_e and \hat{S} have a COP.

Proof. Let $r_0 \in \mathfrak{S}$ be an arbitrary point and let $j_1 = \hat{S}(r_0)$. Since $Ran(\hat{S}) \subseteq Ran(\check{R}_e)$, we can select $r_1 \in \mathfrak{S}$ such that $r_1\check{R}_ej_1$. Repeating the above argument, inductively, we can define two sequences $\{r_n\} \subseteq \mathfrak{S}$ and $\{j_n\} \subseteq Ran(\check{R}_e)$ such that $j_{n+1} = \hat{S}(r_n)$ and $r_{n+1}\check{R}_ej_{n+1}$ for each

$n \in \mathbb{N}$. If $\exists n^* \in \mathbb{N}$ such that $\hat{S}(r_{n^*-1}) = \hat{S}(r_{n^*})$, then,

$$r_{n^*} \check{R}_e j_{n^*} \implies r_{n^*} \check{R}_e \hat{S}(r_{n^*-1}) \implies r_{n^*} \check{R}_e \hat{S}(r_{n^*}).$$

Hence r_{n^*} is a COP of \hat{S} and \check{R}_e . Suppose that $\hat{S}(r_{n-1}) \neq \hat{S}(r_n)$ for each $n \in \mathbb{N}$. By (3.1), one gets

$$\theta \left(\partial_{j_{n+1}}^{j_n} \right) = \theta \left(\partial_{\hat{S}(r_n)}^{\hat{S}(r_{n-1})} \right) \leq \left[\theta \left(D_{\check{R}_e(\{r_n\})}^{\check{R}_e(\{r_{n-1}\})} \right) \right]^k \quad (3.2)$$

for each $n \in \mathbb{N}$. Since $r_n \check{R}_e j_n$ for each $n \in \mathbb{N}$, by the definition of D , we get

$$D_{\check{R}_e(\{r_n\})}^{\check{R}_e(\{r_{n-1}\})} \leq \partial_{j_n}^{j_{n-1}}. \quad (3.3)$$

From (3.2) and (3.3), we obtain

$$\theta \left(\partial_{j_{n+1}}^{j_n} \right) \leq \left[\theta \left(\partial_{j_n}^{j_{n-1}} \right) \right]^k. \quad (3.4)$$

It yields that

$$1 < \theta \left(\partial_{j_{n+1}}^{j_n} \right) \leq [\theta(\partial(j_0, j_1))]^{k^n}, \text{ for each } n \in \mathbb{N}. \quad (3.5)$$

Letting $n \rightarrow \infty$ in (3.5), we get

$$\lim_{n \rightarrow \infty} \theta \left(\partial_{j_{n+1}}^{j_n} \right) = 1. \quad (3.6)$$

By ($\Theta 2$), we have

$$\lim_{n \rightarrow \infty} \partial_{j_{n+1}}^{j_n} = 0. \quad (3.7)$$

Now, we shall show that $\{j_n\}$ is a CS. Assume that there are $\epsilon > 0$ and sequences $\{p(n)\}$ and $\{q(n)\}$ of natural numbers such that

$$p(n) > q(n) > n \text{ and } \partial_{j_{q(n)}}^{j_{p(n)}} \geq \epsilon, \partial_{j_{q(n)}}^{j_{p(n)-1}} < \epsilon. \quad (3.8)$$

By (3.8) and the triangle inequality, we have

$$\epsilon \leq \partial_{j_{q(n)}}^{j_{p(n)}} \leq \partial_{j_{p(n)-1}}^{j_{p(n)}} + \partial_{j_{q(n)}}^{j_{p(n)-1}} \leq \partial_{j_{p(n)}}^{j_{p(n)-1}} + \epsilon. \quad (3.9)$$

Taking $n \rightarrow \infty$ in (3.9) and from (3.7),

$$\lim_{n \rightarrow \infty} \partial_{j_q(n)}^{j_{p(n)}} = \epsilon. \quad (3.10)$$

From (3.7), there is $n_1 \in \mathbb{N}$ such that,

$$\partial_{j_{p(n)+1}}^{j_{p(n)}} < \frac{\epsilon}{4} \text{ and } \partial_{j_{q(n)+1}}^{j_{q(n)}} < \frac{\epsilon}{4}, \quad (3.11)$$

for each $n \geq n_1$. Next, we will claim that

$$\partial_{j_{q(n)+1}}^{j_{p(n)+1}} > 0, \quad (3.12)$$

$\forall n \geq n_1$. Suppose that $\exists n \geq n_1$ such that

$$\partial_{j_{q(n)+1}}^{j_{p(n)+1}} = 0. \quad (3.13)$$

It follows from (3.8), (3.11) and (3.13) that

$$\begin{aligned} \epsilon &\leq \partial_{j_q(n)}^{j_{p(n)}} \leq \partial_{j_{p(n)+1}}^{j_{p(n)}} + \partial_{j_{q(n)+1}}^{j_{p(n)+1}} + \partial_{j_q(n)}^{j_{q(n)+1}} \\ &< \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \end{aligned} \quad (3.14)$$

this is a contradiction. Then (3.12) holds. From (3.1), we obtain

$$\theta(\epsilon) \leq \theta \left(\partial_{j_{q(n)+1}}^{j_{p(n)+1}} \right) \leq \left[\theta \left(\partial_{j_q(n)}^{j_{p(n)}} \right) \right]^k. \quad (3.15)$$

Letting $n \rightarrow \infty$ in (3.15) and using (Θ'3) and (3.11), one has

$$\theta(\epsilon) \leq [\theta(\epsilon)]^k < \theta(\epsilon),$$

which contradicts our supposition. Hence, $\{j_n\}$ is a CS in $\text{Ran}(\check{R}_e)$. Here, we assume that $\text{Ran}(\check{R}_e)$ is complete. Then $\exists u \in \text{Ran}(\check{R}_e)$ such that $\lim_{n \rightarrow \infty} j_n = u$. As $u \in \text{Ran}(\check{R}_e)$, there is

$w \in \mathfrak{S}$ such that $w\check{R}_e u$. Let

$$\mathfrak{I} := \{n \in \mathbb{N} \cup \{0\} : \partial_{\hat{S}(w)}^{\hat{S}(r_n)} = 0\}.$$

- If \mathfrak{I} is infinite, we obtain $u = \hat{S}(w)$ and so $\hat{S}(w) \in \check{R}_e(\{w\})$, that is, w is a COP of \hat{S} and \check{R}_e .
- If \mathfrak{I} is finite, there is a subsequence $\{r_{n_\alpha}\}$ such that $\partial_{\hat{S}(w)}^{\hat{S}(r_{n_\alpha})} \neq 0$, that is, $\hat{S}(r_{n_\alpha}) \neq \hat{S}(w)$. From (3.1), we get

$$\theta\left(\partial_{\hat{S}(w)}^{j_{n_\alpha}}\right) = \theta\left(\partial_{\hat{S}(w)}^{\hat{S}(r_{n_\alpha-1})}\right) \leq \left[\theta\left(D_{\check{R}_e(\{w\})}^{\check{R}_e(r_{n_\alpha-1})}\right)\right]^k \leq \left[\theta\left(\partial_u^{j_{n_\alpha-1}}\right)\right]^k < \theta\left(\partial_u^{j_{n_\alpha-1}}\right).$$

Letting $\alpha \rightarrow \infty$ and using $(\Theta 2)$, we get $\lim_{\alpha \rightarrow \infty} \theta\left(\partial_u^{j_{n_\alpha-1}}\right) = 1$. This implies that $\lim_{\alpha \rightarrow \infty} \theta\left(\partial_{\hat{S}(w)}^{j_{n_\alpha-1}}\right) = 1$, which further implies $\lim_{\alpha \rightarrow \infty} \partial_{\hat{S}(w)}^{j_{n_\alpha-1}} = 0$. Therefore $\partial_{\hat{S}(w)}^u = 0$ and so $u = \hat{S}(w)$. Hence $\hat{S}(w) \in \check{R}_e(\{w\})$, that is, w is a COP of \hat{S} and \check{R}_e .

In the case of $Ran(\hat{S})$ is complete, as $Ran(\hat{S}) \subseteq Ran(\check{R}_e)$, so there is $u^* \in Ran(\check{R}_e)$ such that $\lim_{n \rightarrow \infty} j_n = u^*$. The remaining part of the proof is the same as in the previous case. ■

Theorem 3.1.2. Let $\mathfrak{S} \neq \emptyset$ and (\hat{Y}, ∂) be a MS. Assume that $\hat{S}, \check{R}_e : \mathfrak{S} \rightarrow \hat{Y}$ are two maps with $Ran(\hat{S}) \subseteq Ran(\check{R}_e)$ and there exist $\theta \in \Theta^*$ and $0 < k < 1$ such that

$$\theta\left(\partial_{\hat{S}(j)}^{\hat{S}(r)}\right) \leq \left[\theta\left(\partial_{\check{R}_e(j)}^{\check{R}_e(r)}\right)\right]^k, \text{ for each } r, j \in \mathfrak{S} \text{ with } \hat{S}(r) \neq \hat{S}(j). \quad (3.16)$$

If $Ran(\hat{S})$ or $Ran(\check{R}_e)$ is complete, then \hat{S} and \check{R}_e have a COP in \mathfrak{S} . Furthermore, if either \hat{S} or \check{R}_e is injective, then \hat{S} and \check{R}_e have a unique COP in \mathfrak{S} .

Proof. From Theorem 3.1.1, we get \hat{S} and \check{R}_e have a coincidence point in \mathfrak{S} . Now we suppose that $w_1, w_2 \in \mathfrak{S}$ such that $w_1 \neq w_2$, $\hat{S}(w_1) = \check{R}_e(w_1)$ and $\hat{S}(w_2) = \check{R}_e(w_2)$. If \mathfrak{S} or \check{R}_e is injective, then we get $\partial_{\check{R}_e(w_2)}^{\check{R}_e(w_1)} > 0$. From (3.16), we obtain

$$\theta\left(\partial_{\check{R}_e(w_2)}^{\check{R}_e(w_1)}\right) = \theta\left(\partial_{\hat{S}(w_2)}^{\hat{S}(w_1)}\right) \leq \left[\theta\left(\partial_{\check{R}_e(w_2)}^{\check{R}_e(w_1)}\right)\right]^k < \theta\left(\partial_{\check{R}_e(w_2)}^{\check{R}_e(w_1)}\right),$$

a contradiction. Therefore \hat{S} and \check{R}_e have a unique COP in \mathfrak{S} . ■

If we set $\mathfrak{S} = \hat{Y}$, $\check{R}_e = I$ (the identity mapping on \mathfrak{S}) in Theorem 3.1.2, then we get.

Corollary 3.1.3. Let (\mathfrak{S}, ∂) be a CMS and $\hat{S} : \mathfrak{S} \longrightarrow \mathfrak{S}$ be a map. If there exist $\theta \in \Theta^*$ and $0 < \mathbf{k} < 1$ such that

$$\theta \left(\partial_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq [\theta (\partial_j^r)]^{\mathbf{k}}, \text{ for each } r, j \in \mathfrak{S} \text{ with } \hat{S}(r) \neq \hat{S}(j).$$

Then, \exists only one FP of \hat{S} in \mathfrak{S} .

3.2 Application to Volterra integral equations

We investigate the existence of solutions of the Volterra integral equation by applying the obtained results in the section 3.1,

$$\varkappa(r(\iota)) = \omega(\iota) + \int_0^\iota \tilde{U}(\iota, s, \eta(r(s))) ds, \quad (3.17)$$

where $r : \hat{h} = [0, \hat{e}] \rightarrow \mathbb{R}$ is unknown, where, $\hat{e} > 0$, $\omega : \hat{h} \rightarrow \mathbb{R}$ and $\eta, \varkappa : \mathbb{R} \rightarrow \mathbb{R}$ are given functions and the kernel \tilde{U} of the integral equation is defined on $\hat{h} \times \hat{h} \times \mathbb{R}$

Let $C(\hat{h}, \mathbb{R})$ be the space of all continuous real-valued functions defined on \hat{h} .

Theorem 3.2.1. Consider the equation (3.17). Assume:

(I₁) \tilde{U} , \varkappa , ω and η are continuous and \varkappa is injective;

(I₂) there is $\mathbf{k} \in (0, 1)$ such that

$$1 + \int_0^\iota |\tilde{U}(\iota, s, \check{a}_1) - \tilde{U}(\iota, s, \check{a}_2)| ds \leq \left[1 + \frac{1}{\hat{e}} \int_0^\iota |\check{a}_1 - \check{a}_2| ds \right]^{\mathbf{k}} \quad (3.18)$$

for each $\iota \in \hat{h}$ and $\check{a}_1, \check{a}_2 \in \mathbb{R}$ with $\check{a}_1 \neq \check{a}_2$;

(I₃) for each $r \in C(\hat{h}, \mathbb{R})$, $\exists j \in C(\hat{h}, \mathbb{R})$ such that

$$\varkappa(j(\iota)) = \omega(\iota) + \int_0^\iota \tilde{U}(\iota, s, \eta(r(s))) ds, \text{ for each } \iota \in \hat{h}; \quad (3.19)$$

(I₄) for all $\check{a}_1, \check{a}_2 \in \mathbb{R}$, $|\eta(\check{a}_1) - \eta(\check{a}_2)| \leq |\varkappa(\check{a}_1) - \varkappa(\check{a}_2)|$.

(I_5) the set $\mathcal{U} := \left\{ \hat{h} \ni \iota \mapsto \varkappa(r(\iota)) : r \in C(\hat{h}, \mathbb{R}) \right\}$ is complete.

Then \exists only one solution of (3.17).

Proof. Let $\mathfrak{S} = \hat{Y} := C(\hat{h}, \mathbb{R})$ and $\partial : C(\hat{h}, \mathbb{R}) \times C(\hat{h}, \mathbb{R}) \rightarrow \mathbb{R}_0^+$ be given by

$$\partial_j^r = \sup_{\iota \in \hat{h}} \{|r(\iota) - j(\iota)|\}, \text{ for each } r, j \in C(\hat{h}, \mathbb{R}). \quad (3.20)$$

Obviously $(C(\hat{h}, \mathbb{R}), \partial)$ is a CMS. Define two maps $\hat{S}, \check{R} : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\left(\hat{S}(r) \right) (\iota) = \omega(\iota) + \int_0^\iota \tilde{U}(\iota, s, \eta(r(s))) ds \quad \text{and} \quad (\check{R}(r))(\iota) = \varkappa(r(\iota)),$$

for every $\iota \in \hat{h}$, $r \in \mathfrak{S}$. From (I_5), we get $\text{Ran}(\check{R}_e)$ is complete. Next, we will prove that $\text{Ran}(\hat{S}) \subseteq \text{Ran}(\check{R}_e)$. Let $r \in \text{Ran}(\hat{S})$, so there is $r^* \in \mathfrak{S}$ such that $r = \hat{S}(r^*) \in \mathfrak{S}$. By the condition (I_3), there is $j \in \mathfrak{S}$ such that

$$\varkappa(j(\iota)) = \omega(\iota) + \int_0^\iota \tilde{U}(\iota, s, \eta(r^*(s))) ds; \quad (3.21)$$

for every $t \in \hat{h}$. Thus, $\check{R}_e(j) = \hat{S}(r^*) = r$ and so $r \in \text{Ran}(\check{R}_e)$. Hence $\text{Ran}(\hat{S}) \subseteq \text{Ran}(\check{R}_e)$.

Next, we shall show that \hat{S} satisfies condition (3.16) with $\theta : \mathbb{R}^+ \rightarrow (1, \infty)$ defined by

$\theta(\tilde{\theta}) = 1 + \tilde{\theta}$ for $\forall \tilde{\theta} \in \mathbb{R}^+$. Let $r, j \in \mathfrak{S}$ with $\hat{S}(r) \neq \hat{S}(j)$ and $\iota \in \hat{h}$. Then,

$$\begin{aligned}
1 + \left| \left(\hat{S}(r) \right) (\iota) - \left(\hat{S}(j) \right) (\iota) \right| &= 1 + \left| \int_0^\iota \tilde{U}(\iota, s, \eta(r(s))) ds - \int_0^\iota \tilde{U}(\iota, s, \eta(j(s))) ds \right| \\
&\leq 1 + \int_0^\iota \left| \tilde{U}(\iota, s, \eta(r(s))) - \tilde{U}(\iota, s, \eta(j(s))) \right| ds \\
&\leq \left[1 + \frac{1}{\hat{e}} \int_0^\iota |\eta(r(s)) - \eta(j(s))| ds \right]^{\mathbf{k}} \\
&\leq \left[1 + \frac{1}{\hat{e}} \int_0^\iota |\varkappa(r(s)) - \varkappa(j(s))| ds \right]^{\mathbf{k}} \\
&= \left[1 + \frac{1}{\hat{e}} \int_0^\iota |(\check{R}_e(r))(s) - (\check{R}_e(j))(s)| ds \right]^{\mathbf{k}} \\
&\leq \left[1 + \frac{1}{\hat{e}} \int_0^\iota \partial_{\check{R}_e(j)}^{\check{R}_e(r)} ds \right]^{\mathbf{k}} \\
&\leq \left[1 + \frac{\iota}{\hat{e}} \partial_{\check{R}_e(j)}^{\check{R}_e(r)} \right]^{\mathbf{k}} \leq \left[1 + \partial_{\check{R}_e(j)}^{\check{R}_e(r)} \right]^{\mathbf{k}} = \left[\theta(\partial_{\check{R}_e(j)}^{\check{R}_e(r)}) \right]^{\mathbf{k}}.
\end{aligned}$$

This implies that

$$1 + \partial_{\hat{S}(j)}^{\hat{S}(r)} \leq \left[\theta(\partial_{\check{R}_e(j)}^{\check{R}_e(r)}) \right]^{\mathbf{k}} \quad (3.22)$$

and thus

$$\theta(\partial_{\hat{S}(j)}^{\hat{S}(r)}) \leq \left[\theta(\partial_{\check{R}_e(j)}^{\check{R}_e(r)}) \right]^{\mathbf{k}}. \quad (3.23)$$

Therefore, \hat{S} satisfies the (3.16) (with $\theta \in \Theta^*$). The injection of \varkappa implies the injection of \check{R}_e . Thus, from Theorem 3.1.2, \hat{S} and \check{R}_e have a unique coincidence point in \mathfrak{S} . Thus, the equation (3.17) has only one solution.

Example 3.2.2. Consider

$$r(\iota) - \iota^3 \sin(\iota^2 + 1) = \int_0^\iota \left[\frac{\iota^2 + s^3}{\sqrt{1 + \iota^2 + s^2}} + \left(\frac{\sqrt{2 + 2\sqrt{2}} - 1}{2 + 4\sqrt{2}} \right) r(s) \right] ds, \quad (3.24)$$

where $r : [0, 1] \rightarrow \mathbb{R}$ is unknown. We apply Theorem 3.2.1 for solving the Volterra integral

equation (3.24). Let $\hat{e} = 1$ and $\omega : [0, \hat{e}] \rightarrow \mathbb{R}$ be defined by

$$\omega(\iota) = \iota^3 \sin(\iota^2 + 1)$$

for each $\iota \in \hat{h}$. Define maps $\eta, \varkappa : \mathbb{R} \rightarrow \mathbb{R}$ and $\tilde{U} : \hat{h} \times \hat{h} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varkappa(s) = s, \quad \eta(s) = \frac{s}{2}, \quad \text{for every } s \in \mathbb{R} \text{ and}$$

$$\tilde{U}(\iota, s, \nu) = \frac{\iota^2 + s^3}{\sqrt{1 + \iota^2 + s^2}} + \left(\frac{\sqrt{2 + 2\sqrt{2}} - 1}{1 + 2\sqrt{2}} \right) \nu, \quad \text{for every } \iota \in \hat{h} \text{ and } \nu \in \mathbb{R}.$$

Then the equation (3.24) is equivalent to the Volterra integral equation

$$\varkappa(r(\iota)) = \omega(\iota) + \int_0^\iota \tilde{U}(\iota, s, \eta(r(s))) ds. \quad (3.25)$$

Next, we will show that conditions (I_1) - (I_5) hold. ■

(♣₁) It is easy to see that \tilde{U} , \varkappa , ω and η are continuous and \varkappa is injective. Then (I_1) holds.

(♣₂) We shall prove that there exists $\mathbf{k} = \frac{1}{2} \in (0, 1)$ such that

$$1 + \int_0^\iota |\tilde{U}(\iota, s, \check{a}_1) - \tilde{U}(\iota, s, \check{a}_2)| ds \leq \left[1 + \frac{1}{\hat{e}} \int_0^\iota |\check{a}_1 - \check{a}_2| ds \right]^{\mathbf{k}} \quad (3.26)$$

for all $\iota \in \hat{h}$ and $\check{a}_1, \check{a}_2 \in \mathbb{R}$ with $\check{a}_1 \neq \check{a}_2$.

First, we shall define $\check{A} : \mathbb{R}^+ \rightarrow \mathbb{R}$ by,

$$\check{A}(\alpha) = \frac{\sqrt{1 + \alpha} - 1}{\alpha}, \quad \text{for each } \alpha \in \mathbb{R}^+.$$

We know that the minimum of \check{A} is $\frac{\sqrt{2+2\sqrt{2}}-1}{1+2\sqrt{2}}$ and so

$$\frac{\sqrt{2 + 2\sqrt{2}} - 1}{1 + 2\sqrt{2}} \leq \check{A}(\alpha) = \frac{\sqrt{1 + \alpha} - 1}{\alpha}, \quad \text{for each } \alpha \in \mathbb{R}^+.$$

This implies that

$$1 + \frac{\sqrt{2+2\sqrt{2}}-1}{1+2\sqrt{2}}\alpha \leq \sqrt{1+\alpha}, \text{ for each } \alpha \in \mathbb{R}^+. \quad (3.27)$$

Next, we shall prove that (3.26) holds. Let $\iota \in \hat{h} = [0, 1]$ and $\check{a}_1, \check{a}_2 \in \mathbb{R}$ with $\check{a}_1 \neq \check{a}_2$. By (3.27),

$$\begin{aligned} 1 + \int_0^\iota |\tilde{U}(\iota, s, \check{a}_1) - \tilde{U}(\iota, s, \check{a}_2)| ds &= 1 + \left(\frac{\sqrt{2+2\sqrt{2}}-1}{1+2\sqrt{2}} \right) \int_0^\iota |\check{a}_1 - \check{a}_2| ds \\ &\leq \sqrt{1 + \int_0^\iota |\check{a}_1 - \check{a}_2| ds} \\ &= \sqrt{1 + \frac{1}{\hat{e}} \int_0^\iota |\check{a}_1 - \check{a}_2| ds} \\ &= \left[1 + \frac{1}{\hat{e}} \int_0^\iota |\check{a}_1 - \check{a}_2| ds \right]^k. \end{aligned}$$

Hence (I_2) holds.

(♣₃) For every $r \in C(\hat{h}, \mathbb{R})$, there is $j \in C(\hat{h}, \mathbb{R})$ defined by

$$j(\iota) = \iota^3 \sin(\iota^2 + 1) + \int_0^\iota \left[\frac{\iota^2 + s^3}{\sqrt{1 + \iota^2 + s^2}} + \left(\frac{\sqrt{2+2\sqrt{2}}-1}{2+4\sqrt{2}} \right) r(s) \right] ds$$

for all $\iota \in \hat{h}$ such that

$$\begin{aligned} \varkappa(j(\iota)) &= j(\iota) \\ &= \iota^3 \sin(\iota^2 + 1) \\ &\quad + \int_0^\iota \left[\frac{\iota^2 + s^3}{\sqrt{1 + \iota^2 + s^2}} + \left(\frac{\sqrt{2+2\sqrt{2}}-1}{2+4\sqrt{2}} \right) r(s) \right] ds \\ &= \omega(\iota) + \int_0^\iota \tilde{U}(\iota, s, \eta(r(s))) ds, \text{ for each } \iota \in \hat{h}. \end{aligned}$$

Hence (I_3) holds.

(♣₄) For every $\check{a}_1, \check{a}_2 \in \mathbb{R}$, we have

$$|\eta(\check{a}_1) - \eta(\check{a}_2)| = \left| \frac{\check{a}_1}{2} - \frac{\check{a}_2}{2} \right| \leq |\check{a}_1 - \check{a}_2| = |\varkappa(\check{a}_1) - \varkappa(\check{a}_2)|.$$

Then (I_4) holds.

(♣₅) Since $\mathcal{U} := C(\hat{h} = [0, 1], \mathbb{R})$, we have \mathcal{U} is complete. Thus, (I_5) holds.

Thus, all hypotheses of Theorem 3.2.1 are obeyed and so the equation (3.25) has a unique solution. Then, the equation (3.24) has a unique solution.

3.3 Generalized Suzuki type θ -rational contractions and related common fixed point theorems

In this section, we present generalized Suzuki type (θ, \hat{C}) -rational contractions (GST (θ, \hat{C}) -R-contractions) based on four self-mappings and obtain related CPF theorems.

Definition 3.3.1 Let $q, \mu, \xi_1, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$ be self-maps on a $\text{MS}(\mathfrak{S}, \partial)$. q, μ, ξ_1 and ξ_2 form a GST (θ, \hat{C}) -R-contraction, if, for each $r, j \in \mathfrak{S}$, with $\partial_{\mu(j)}^{q(r)} > 0$, for $\theta \in \Theta^*$, $\hat{C} \in \Delta_{\hat{C}}$ and $\varphi \in \Phi$,

$$\varphi \left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\xi_1(r)} \right) < 0 \Rightarrow \theta \left(\partial_{\mu(j)}^{q(r)} \right) \leq [\theta(\mathcal{M}_8(r, j))]^{\hat{C}(U(r, j))}, \quad (3.28)$$

where

$$\mathcal{M}_8(r, j) = \max \left\{ \partial_{\xi_2(j)}^{\xi_1(r)}, \partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\mu(j)}, \frac{\partial_{\xi_2(j)}^{q(r)} + \partial_{\xi_1(r)}^{\mu(j)}}{2}, \frac{(1 + \partial_{\xi_1(r)}^{q(r)}) \partial_{\xi_2(j)}^{\mu(j)}}{1 + \partial_{\xi_2(j)}^{\xi_1(r)}} \right\}, \quad (3.29)$$

and

$$U(r, j) = \left\{ \partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\mu(j)}, \partial_{\xi_2(j)}^{q(r)}, \partial_{\xi_1(r)}^{\mu(j)} \right\}. \quad (3.30)$$

Let us define the following conditions:

(\hat{C}_1): The pairs (q, ξ_1) , (μ, ξ_2) of self-mappings defined on \mathfrak{S} are compatible.

(\hat{C}_2): ξ_2 and ξ_1 are continuous on \mathfrak{S} .

Theorem 3.3.2 Let (\mathfrak{S}, ∂) be a CMS. Assume that the mappings $q, \mu, \xi_1, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$ form a

GST (θ, \hat{C}) -R-contraction with, $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$ and $\mu(\mathfrak{S}) \subseteq \xi_1(\mathfrak{S})$. If the conditions (\hat{C}_1) and (\hat{C}_2) hold, then q, μ, ξ_1 and ξ_2 have a unique CPF in \mathfrak{S} .

Proof. Let $r_0 \in \mathfrak{S}$. Since $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$, there exists $r_1 \in \mathfrak{S}$ such that $q(r_0) = \xi_2(r_1)$. As $\mu(r_1) \in \xi_1(\mathfrak{S})$, we can select $r_2 \in \mathfrak{S}$ such that $\mu(r_1) = \xi_1(r_2)$. Thus, r_{2n+1} and r_{2n+2} are selected in \mathfrak{S} such that $q(r_{2n}) = \xi_2(r_{2n+1})$ and $\mu(r_{2n+1}) = \xi_1(r_{2n+2})$. Define $\{\mathfrak{R}_n\} \subset \mathfrak{S}$ such that

$$\mathfrak{R}_{2n} = q(r_{2n}) = \xi_2(r_{2n+1})$$

and

$$\mathfrak{R}_{2n+1} = \mu(r_{2n+1}) = \xi_1(r_{2n+2}),$$

for $n = 0, 1, 2, \dots$. Assume that $\partial_{\Gamma(r_{2n+1})}^{q(r_{2n})} > 0$ and since

$$\frac{1}{2} \partial_{\xi_1(r_{2n})}^{q(r_{2n})} = \frac{1}{2} \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}} < \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} = \partial_{\xi_2(r_{2n+1})}^{\xi_1(r_{2n})},$$

we have

$$\varphi \left(\partial_{\xi_1(r_{2n})}^{q(r_{2n})}, \partial_{\xi_2(r_{2n+1})}^{\xi_1(r_{2n})} \right) < 0.$$

Hence, from the contractive condition (3.28),

$$1 < \theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) = \theta \left(\partial_{\Gamma(r_{2n+1})}^{q(r_{2n})} \right) \leq [\theta(\mathcal{M}_8(r_{2n}, r_{2n+1}))]^{\hat{C}(U(r_{2n}, r_{2n+1}))}, \quad (3.31)$$

for each $n \in \mathbb{N}$, where

$$\begin{aligned} U(r_{2n}, r_{2n+1}) &= \left\{ \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n}}, \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}} \right\} \\ &= \left\{ \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}}, 0, \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}} \right\}. \end{aligned}$$

As $\partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}} \cdot \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}} \cdot \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}} \cdot 0 = 0$, so from (\hat{C}^*) , there exists $z \in [0, 1)$ such that

$$\hat{C}(U(r_{2n}, r_{2n+1})) = \hat{C} \left(\partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}}, 0, \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}} \right) = z, \quad (3.32)$$

and

$$\begin{aligned}
\mathcal{M}_8(r_{2n}, r_{2n+1}) &= \max \left\{ \frac{\partial_{\xi_2}^{\xi_1(r_{2n})}(r_{2n+1}), \partial_{\xi_1}^q(r_{2n}), \partial_{\xi_2}^{\Gamma(r_{2n+1})}}{\frac{\partial^q(r_{2n})}{\xi_2(r_{2n+1})} + \partial_{\xi_1}^{\Gamma(r_{2n+1})}} \frac{(1 + \partial_{\xi_1}^q(r_{2n})) \partial_{\xi_2}^{\Gamma(r_{2n+1})}}{1 + \partial_{\xi_2}^{\xi_1(r_{2n})}} \right\} \\
&= \max \left\{ \frac{\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}}, \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}}}{\frac{\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n}} + \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}}}{2}} \frac{(1 + \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}}) \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}}}{1 + \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}}} \right\} \\
&= \max \left\{ \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}}, \frac{\partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}}}{2} \right\}.
\end{aligned}$$

Since

$$\frac{\partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}}}{2} \leq \frac{\partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}}}{2} + \frac{\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}}}{2} \leq \max \left\{ \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n-1}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}} \right\},$$

one writes

$$\mathcal{M}_8(r_{2n}, r_{2n+1}) = \max \left\{ \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}}, \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}} \right\}. \quad (3.33)$$

If for some n , $\mathcal{M}_8(r_{2n}, r_{2n+1}) = \partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}}$, then from (3.31), (3.32) and (3.33), we have

$$\theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \leq \left[\theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \right]^z < \theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right),$$

a contradiction. Therefore,

$$\theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \leq \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} \right) \right]^z.$$

Thus, we have

$$\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) \leq \left[\theta \left(\partial_{\mathfrak{R}_n}^{\mathfrak{R}_{n-1}} \right) \right]^z, \text{ for each } n \in \mathbb{N}.$$

This implies that

$$1 < \theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) \leq \left[\theta \left(\partial_{\mathfrak{R}_n}^{\mathfrak{R}_{n-1}} \right) \right]^{z^2} \leq \dots \leq \left[\theta \left(\partial_{\mathfrak{R}_1}^{\mathfrak{R}_0} \right) \right]^{z^n}, \quad (3.34)$$

for each $n \in \mathbb{N}$. Taking $n \rightarrow \infty$ in (3.34) and since $\theta \in \Theta^*$, we get

$$\lim_{n \rightarrow \infty} \theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) = 1.$$

By (Θ2), we get

$$\lim_{n \rightarrow \infty} \partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} = 0. \quad (3.35)$$

To prove $\{\mathfrak{R}_n\}$ is a CS, we assume that $\{\mathfrak{R}_n\}$ is not a CS, then $\exists \varepsilon > 0$ and subsequences $\{n(k)\}_{k=1}^\infty \subset \mathbb{N}$, and $\{m(k)\}_{k=1}^\infty \subset \mathbb{N}$ such that

$$m(k) > n(k) > k, \quad \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)}} \geq \varepsilon, \quad \partial_{\mathfrak{R}_{m(k)-1}}^{\mathfrak{R}_{n(k)}} < \varepsilon,$$

for each $k \in \mathbb{N}$. Therefore,

$$\varepsilon \leq \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)}} \leq \partial_{\mathfrak{R}_{m(k)-1}}^{\mathfrak{R}_{n(k)}} + \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{m(k)-1}} < \varepsilon + \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{m(k)-1}}. \quad (3.36)$$

By applying $\lim_{k \rightarrow \infty}$ in (3.36) and from (3.35),

$$\lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)}} = \varepsilon. \quad (3.37)$$

Using the triangular inequality, we have

$$\partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)}} \leq \partial_{\mathfrak{R}_{n(k)+1}}^{\mathfrak{R}_{n(k)}} + \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)+1}}, \quad (3.38)$$

and

$$\partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)+1}} \leq \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{n(k)+1}} + \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)}}. \quad (3.39)$$

By taking upper limit as $k \rightarrow \infty$ in (3.38), (3.39) and applying (3.35) and (3.37),

$$\varepsilon \leq \lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)+1}} \leq \varepsilon.$$

Thus,

$$\lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)+1}} = \varepsilon. \quad (3.40)$$

Similarly, we can obtain

$$\lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)+1}}^{\mathfrak{R}_{n(k)+1}} = \lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)+1}}^{\mathfrak{R}_{n(k)}} = \varepsilon. \quad (3.41)$$

Assume that $\partial_{\mu(r_{m(k)+1})}^q(r_{n(k)+1}) > 0$ and Since

$$\frac{1}{2} \partial_{\xi_1(r_{m(k)})}^q(r_{n(k)}) = \frac{1}{2} \partial_{\mathfrak{R}_{m(k)-1}}^{\mathfrak{R}_{n(k)}} < \partial_{\mathfrak{R}_{m(k)-1}}^{\mathfrak{R}_{n(k)}} = \partial_{\xi_2(r_{n(k)+1})}^{\xi_1(r_{m(k)})},$$

we get

$$\varphi \left(\partial_{\xi_1(r_{m(k)})}^q(r_{n(k)}), \partial_{\xi_2(r_{n(k)+1})}^{\xi_1(r_{m(k)})} \right) < 0.$$

Hence, from (3.28),

$$\theta \left(\partial_{\mathfrak{R}_{m(k)+1}}^{\mathfrak{R}_{n(k)+1}} \right) = \theta \left(\partial_{\mu(r_{m(k)+1})}^q(r_{n(k)+1}) \right) \leq [\theta(\mathcal{M}_8(r_{n(k)}, r_{m(k)}))] \hat{C}(U(r_{n(k)}, r_{m(k)})), \text{ where,} \quad (3.42)$$

$$U(r_{n(k)}, r_{m(k)}) = \left\{ \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{n(k)+1}}, \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{m(k)+1}}, \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)+1}}, \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{m(k)+1}} \right\}. \quad (3.43)$$

Setting $k \rightarrow \infty$ in (3.43),

$$\lim_{k \rightarrow \infty} U(r_{n(k)}, r_{m(k)}) = \left\{ \begin{array}{l} \lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{n(k)+1}}, \lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{m(k)+1}}, \\ \lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)+1}}, \lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{m(k)+1}} \end{array} \right\} = \{0, 0, \varepsilon, \varepsilon\}.$$

By \hat{C}^* , there exists $z \in [0, 1)$ such that $\hat{C}(0, 0, \varepsilon, \varepsilon) = z$. Using the continuity of \hat{C} and (3.43),

$$\hat{C} \left(\lim_{k \rightarrow \infty} U(r_{n(k)}, r_{m(k)}) \right) \leq z. \quad (3.44)$$

Also,

$$\begin{aligned} \mathcal{M}_8(r_{n(k)}, r_{m(k)}) &= \max \left\{ \begin{array}{l} \partial_{\xi_2(r_{m(k)+1})}^{\xi_1(r_{n(k)+1})}, \partial_{\xi_1(r_{n(k)+1})}^q(r_{n(k)+1}), \partial_{\xi_2(r_{m(k)+1})}^{\mu(r_{m(k)+1})}, \\ \frac{\partial_{\xi_2(r_{m(k)+1})}^q(r_{n(k)+1}) + \partial_{\xi_1(r_{n(k)+1})}^{\mu(r_{m(k)+1})}}{2}, \frac{\left(1 + \partial_{\xi_1(r_{n(k)+1})}^q(r_{n(k)+1})\right) \partial_{\xi_2(r_{m(k)+1})}^{\mu(r_{m(k)+1})}}{1 + \partial_{\xi_1(r_{n(k)+1})}^{\xi_2(r_{m(k)+1})}} \end{array} \right\} \\ &= \max \left\{ \begin{array}{l} \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)}}, \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{n(k)+1}}, \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{m(k)+1}}, \\ \frac{\partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)+1}} + \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{m(k)+1}}}{2}, \frac{\left(1 + \partial_{\mathfrak{R}_{n(k)}}^{\mathfrak{R}_{n(k)+1}}\right) \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{m(k)+1}}}{1 + \partial_{\mathfrak{R}_{m(k)}}^{\mathfrak{R}_{n(k)}}} \end{array} \right\}. \quad (3.45) \end{aligned}$$

Taking $k \rightarrow \infty$ (3.45) and using (3.35), (3.37), (3.40) and (3.41), we get

$$\lim_{k \rightarrow \infty} \mathcal{M}_8(r_{n(k)}, r_{m(k)}) \leq \varepsilon. \quad (3.46)$$

Thus, from $(\Theta'3)$, (3.41), (3.42), (3.44) and (3.46), we have

$$\begin{aligned} \theta(\varepsilon) &= \theta \left(\lim_{k \rightarrow \infty} \partial_{\mathfrak{R}_{m(k)+1}}^{\mathfrak{R}_{n(k)+1}} \right) \leq \left[\theta \left(\lim_{k \rightarrow \infty} \mathcal{M}_8(r_{n(k)}, r_{m(k)}) \right) \right]^{\hat{C} \left(\lim_{k \rightarrow \infty} U(r_{n(k)}, r_{m(k)}) \right)} \\ &\leq [\theta(\varepsilon)]^z < \theta(\varepsilon), \end{aligned}$$

a contradiction. Then, $\{\mathfrak{R}_n\}$ is a CS. As \mathfrak{S} is a CMS, so there exists $r^* \in \mathfrak{S}$ such that

$$\lim_{n \rightarrow \infty} \partial_{r^*}^{\mathfrak{R}_n} = 0,$$

$$\lim_{n \rightarrow \infty} q(r_{2n}) = \lim_{n \rightarrow \infty} \xi_2(r_{2n+1}) = \lim_{n \rightarrow \infty} \mu(r_{2n+1}) = \lim_{n \rightarrow \infty} \xi_1(r_{2n+2}) = r^*.$$

As ξ_1 is continuous, so

$$\lim_{n \rightarrow \infty} \xi_1(q(r_{2n})) = \xi_1(r^*) = \lim_{n \rightarrow \infty} \xi_1(\xi_1(r_{2n+2})),$$

Since the pair (q, ξ_1) is a compatible,

$$\lim_{n \rightarrow \infty} \partial_{\xi_1(q(r_{2n}))}^{q(\xi_1(r_{2n}))} = 0.$$

From Lemma 1.1.5, we have

$$\lim_{n \rightarrow \infty} q(\xi_1(r_{2n})) = \xi_1(r^*).$$

Put $r = \xi_1(r_{2n})$ and $j = r_{2n+1}$ in (3.28) and assume that $\partial_{r^*}^{\xi_1(r^*)} > 0$, and

$$\frac{1}{2} \partial_{\xi_1(\tilde{S}(r_{2n}))}^{q(\xi_1(r_{2n}))} < \partial_{\xi_2(r_{2n+1})}^{\xi_1(\xi_1(r_{2n}))}.$$

Hence,

$$\varphi \left(\partial_{\xi_1(\xi_1(r_{2n}))}^{q(\xi_1(r_{2n}))}, \partial_{\xi_2(r_{2n+1})}^{\xi_1(\xi_1(r_{2n}))} \right) < 0,$$

and by (3.28), we get

$$\theta \left(\partial_{\mu(r_{2n+1})}^q(\xi_1(r_{2n})) \right) \leq [\theta(\mathcal{M}_8(r_{2n}, r_{2n+1}))]^{\hat{C}(U(r_{2n}, r_{2n+1}))}, \quad (3.47)$$

where

$$\mathcal{M}_8(r_{2n}, r_{2n+1}) = \max \left\{ \begin{array}{l} \partial_{\xi_2(r_{2n+1})}^{\xi_1(r_{2n})}, \partial_{\xi_1(\xi_1(r_{2n}))}^q(\xi_1(r_{2n})), \partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})}, \\ \frac{\partial_{\xi_2(r_{2n+1})}^q(\xi_1(r_{2n})) + \partial_{\xi_1(\xi_1(r_{2n}))}^{\mu(r_{2n+1})}}{2}, \frac{(1 + \partial_{\xi_1(\xi_1(r_{2n}))}^q(\xi_1(r_{2n}))) \partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})}}{1 + \partial_{\xi_2(r_{2n+1})}^{\xi_1(\xi_1(r_{2n}))}} \end{array} \right\},$$

and

$$U(r_{2n}, r_{2n+1}) = \left\{ \partial_{\xi_1(\xi_1(r_{2n}))}^q(\xi_1(r_{2n})), \partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})}, \partial_{\xi_2(r_{2n+1})}^q(\xi_1(r_{2n})), \partial_{\xi_1(\xi_1(r_{2n}))}^{\mu(r_{2n+1})} \right\}.$$

Setting $n \rightarrow \infty$ in (3.47), one has

$$\theta \left(\partial_{r^*}^{\xi_1(r^*)} \right) \leq \left[\theta \left(\partial_{r^*}^{\xi_1(r^*)} \right) \right]^z < \theta \left(\partial_{r^*}^{\xi_1(r^*)} \right),$$

a contradiction. Thus, $\partial_{r^*}^{\xi_1(r^*)} = 0$ and so $\xi_1(r^*) = r^*$. As ξ_2 is continuous,

$$\lim_{n \rightarrow \infty} \xi_2(\mu(r_{2n+1})) = \lim_{n \rightarrow \infty} \xi_2(\xi_2(r_{2n+1})) = \xi_2(r^*).$$

Since the pair (μ, ξ_2) is a compatible,

$$\lim_{n \rightarrow \infty} \partial_{\xi_2(\mu(r_{2n+1}))}^{\mu(\xi_2(r_{2n+1}))} = 0.$$

From Lemma 1.1.5,

$$\lim_{n \rightarrow \infty} \mu(\xi_2(r_{2n+1})) = \xi_2(r^*).$$

Set $r = r_{2n}$ and $j = \xi_2(r_{2n+1})$ in (3.28) and, assume that $\partial_{\tilde{T}(r^*)}^{r^*} > 0$, and

$$\frac{1}{2} \partial_{\xi_1(r_{2n})}^q(r_{2n}) < \partial_{\xi_2(\tilde{T}(r_{2n+1}))}^{\xi_1(r_{2n})}.$$

Therefore,

$$\varphi \left(\partial_{\xi_1(r_{2n})}^q(r_{2n}), \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\xi_1(r_{2n})} \right) < 0.$$

By (3.28), one gets

$$\theta \left(\partial_{\mu(\xi_2(r_{2n+1}))}^{q(r_{2n})} \right) \leq [\theta(\mathcal{M}_8(r_{2n}, r_{2n+1}))]^{\hat{C}(U(r_{2n}, r_{2n+1}))}, \quad (3.48)$$

where

$$\mathcal{M}_8(r_{2n}, r_{2n+1}) = \max \left\{ \begin{array}{l} \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\xi_1(r_{2n})}, \partial_{\xi_1(r_{2n})}^{q(r_{2n})}, \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\mu(\xi_2(r_{2n+1}))}, \\ \frac{\partial_{\xi_2(\xi_2(r_{2n+1}))}^{q(r_{2n})} + \partial_{\xi_1(r_{2n})}^{\mu(\xi_2(r_{2n+1}))}}{2}, \frac{(1 + \partial_{\xi_1(r_{2n})}^{q(r_{2n})}) \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\mu(\xi_2(r_{2n+1}))}}{1 + \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\xi_1(r_{2n})}} \end{array} \right\},$$

$$U(r_{2n}, r_{2n+1}) = \left\{ \partial_{\xi_1(r_{2n})}^{q(r_{2n})}, \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\mu(\xi_2(r_{2n+1}))}, \partial_{\xi_2(\xi_2(r_{2n+1}))}^{q(r_{2n})}, \partial_{\xi_1(r_{2n})}^{\mu(\xi_2(r_{2n+1}))} \right\}.$$

Setting the upper limit in (3.48), we obtain

$$\theta \left(\partial_{\xi_2(r^*)}^{r^*} \right) \leq \left[\theta \left(\partial_{\xi_2(r^*)}^{r^*} \right) \right]^z < \theta \left(\partial_{\xi_2(r^*)}^{r^*} \right),$$

a contradiction. Thus, $\partial_{\xi_2(r^*)}^{r^*} = 0$ and $r^* = \xi_2(r^*)$. Suppose that $\partial_{r^*}^{q(r^*)} > 0$, and

$$\frac{1}{2} \partial_{\xi_1(r^*)}^{q(r^*)} < \partial_{\xi_2(r_{2n+1})}^{\xi_1(r^*)}.$$

Then

$$\varphi \left(\partial_{\xi_1(r^*)}^{q(r^*)}, \partial_{\xi_2(r_{2n+1})}^{\xi_1(r^*)} \right) < 0,$$

and by (3.28), we get

$$\theta \left(\partial_{\mu(r_{2n+1})}^{q(r^*)} \right) \leq [\theta(\mathcal{M}_8(r^*, r_{2n+1}))]^{\hat{C}(U(r^*, r_{2n+1}))}, \quad (3.49)$$

where

$$\mathcal{M}_8(r^*, r_{2n+1}) = \max \left\{ \begin{array}{l} \partial_{\xi_2(r_{2n+1})}^{\xi_1(r^*)}, \partial_{\xi_1(r^*)}^{q(r^*)}, \partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})}, \frac{\partial_{\xi_2(r_{2n+1})}^{q(r^*)} + \partial_{\xi_1(r^*)}^{\mu(r_{2n+1})}}{2}, \\ \frac{(1 + \partial_{\xi_1(r^*)}^{q(r^*)}) \partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})}}{1 + \partial_{\xi_2(r_{2n+1})}^{\xi_1(r^*)}} \end{array} \right\},$$

and

$$U(r^*, r_{2n+1}) = \left\{ \partial_{\xi_1(r^*)}^{q(r^*)}, \partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})}, \partial_{\xi_2(r_{2n+1})}^{q(r^*)}, \partial_{\xi_1(r^*)}^{\mu(r_{2n+1})} \right\}.$$

Setting the upper limit in (3.49), and since $\xi_1(r^*) = \xi_2(r^*) = r^*$, so

$$\theta\left(\partial_{r^*}^{q(r^*)}\right) \leq \left[\theta\left(\partial_{r^*}^{q(r^*)}\right)\right]^z < \theta\left(\partial_{r^*}^{q(r^*)}\right),$$

a contradiction. Thus, $\partial_{r^*}^{q(r^*)} = 0$ and $r^* = q(r^*)$. Finally, assume that $\partial_{\mu(r^*)}^{r^*} > 0$, and since $q(r^*) = \xi_2(r^*) = \xi_1(r^*) = r^*$, so,

$$\varphi\left(0, \partial_{\xi_2(r^*)}^{r^*}\right) < 0$$

and, from (3.28),

$$\theta\left(\partial_{\mu(r^*)}^{r^*}\right) = \theta\left(\partial_{\mu(r^*)}^{q(r^*)}\right) \leq \left[\theta\left(\partial_{\mu(r^*)}^{r^*}\right)\right]^z < \theta\left(\partial_{\mu(r^*)}^{r^*}\right),$$

a contradiction. Hence, $\partial_{\mu(r^*)}^{r^*} = 0$ and so $r^* = \mu(r^*)$. Therefore, r^* is a CFP of ξ_1, μ, q and ξ_2 . Assume that v^* is another CFP of ξ_1, μ, q and ξ_2 such that $r^* \neq v^*$, $\varphi\left(0, \partial_{\xi_2(v^*)}^{\xi_1(r^*)}\right) < 0$, then from (3.28),

$$\theta\left(\partial_{v^*}^{r^*}\right) = \theta\left(\partial_{\mu(v^*)}^{q(r^*)}\right) \leq \left[\theta\left(\mathcal{M}_8(r^*, v^*)\right)\right]^{\hat{C}(U(r^*, v^*))},$$

where,

$$\mathcal{M}_8(r^*, v^*) = \max \left\{ \partial_{\xi_2(v^*)}^{\xi_1(r^*)}, \partial_{\xi_1(r^*)}^{q(r^*)}, \partial_{\xi_2(v^*)}^{\mu(v^*)}, \frac{\partial_{\xi_2(v^*)}^{q(r^*)} + \partial_{\xi_1(r^*)}^{\mu(v^*)}}{2}, \frac{\left(1 + \partial_{\xi_1(r^*)}^{q(r^*)}\right) \partial_{\xi_2(v^*)}^{\mu(v^*)}}{1 + \partial_{\xi_2(v^*)}^{\xi_1(r^*)}} \right\},$$

$$U(r^*, v^*) = \left\{ \partial_{\xi_1(r^*)}^{q(r^*)}, \partial_{\xi_2(v^*)}^{\mu(v^*)}, \partial_{\xi_2(v^*)}^{q(r^*)}, \partial_{\xi_1(r^*)}^{\mu(v^*)} \right\}.$$

It implies that

$$\theta\left(\partial_{v^*}^{r^*}\right) \leq \left[\theta\left(\partial_{v^*}^{r^*}\right)\right]^z < \theta\left(\partial_{v^*}^{r^*}\right),$$

a contradiction. Hence $r^* = v^*$. Hence r^* is the unique CFP of ξ_1, Γ, q and ξ_2 . ■

Corollary 3.3.3. Let (\mathfrak{S}, ∂) be a CMS, and $q, \mu, \xi_1, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that if, $\forall r, j \in \mathfrak{S}$

with $\partial_{\mu(j)}^{q(r)} > 0$ for some $\theta \in \Theta^*$, $z \in [0, 1)$ and $\varphi \in \Phi$,

$$\varphi \left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\xi_1(r)} \right) < 0 \Rightarrow \theta \left(\partial_{\mu(j)}^{q(r)} \right) \leq [\theta(\mathcal{M}_8(r, j))]^z,$$

with $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$ and $\mu(\mathfrak{S}) \subseteq \xi_1(\mathfrak{S})$. If the conditions (\hat{C}_1) and (\hat{C}_2) hold, then q, μ, ξ_1 and ξ_2 have a unique CFP in \mathfrak{S} .

Example 3.3.4. Let $\mathfrak{S} = [0, 1]$ and define the function $\partial : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by $\partial_j^r = |r - j|$. Clearly, (\mathfrak{S}, ∂) is a CMS. Let $\theta(\varsigma) = e^\varsigma$, and $\varphi(s, \varsigma) = \frac{s}{2} - \varsigma$, then $\theta \in \Theta^*$, $\varphi \in \Phi$. Define $\xi_1, \mu, q, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\xi_1(r) = \left(\frac{r}{3}\right)^8, \quad \mu(r) = \left(\frac{r}{3}\right)^8, \quad q(r) = \left(\frac{r}{3}\right)^{16}, \quad \xi_2(r) = \left(\frac{r}{3}\right)^4.$$

Then, $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$ and $\mu(\mathfrak{S}) \subseteq \xi_1(\mathfrak{S})$. If $\{r_n\}$ is a sequence in \mathfrak{S} such that for some $\varsigma \in \mathfrak{S}$,

$$\lim_{n \rightarrow \infty} q(r_n) = \lim_{n \rightarrow \infty} \xi_1(r_n) = \varsigma,$$

Hence

$$\lim_{n \rightarrow \infty} |q(r_n) - \varsigma| = \lim_{n \rightarrow \infty} \left| \hat{S}(r_n) - \varsigma \right| = 0,$$

and,

$$\lim_{n \rightarrow \infty} \left| \left[\frac{r_n}{3} \right]^{16} - \varsigma \right| = \lim_{n \rightarrow \infty} \left| \left[\frac{r_n}{3} \right]^8 - \varsigma \right| = 0.$$

Then,

$$\lim_{n \rightarrow \infty} \left| [r_n]^{16} - 3^{16}\varsigma \right| = \lim_{n \rightarrow \infty} \left| [r_n]^8 - 3^8\varsigma \right| = 0.$$

We conclude $\varsigma^{\frac{1}{16}} = \varsigma^{\frac{1}{8}}$ (by uniqueness of limit), then, $\varsigma \in \{0, 1\}$. By continuity of q and ξ_1 , one gets

$$\begin{aligned} \lim_{n \rightarrow \infty} \partial_{\xi_1(q(r_n))}^{q(\xi_1(r_n))} &= \lim_{n \rightarrow \infty} |q(\xi_1(r_n)) - \xi_1(q(r_n))| \\ &= |q(\varsigma) - \xi_1(\varsigma)| = |0 - 0| = 0, \text{ for } \varsigma = 0 \in \mathfrak{S}. \end{aligned}$$

Thus, the pair (q, ξ_1) is compatible. Similarly, the pair (μ, ξ_2) is compatible. Define $\hat{C} : \mathbb{R}^{+4} \rightarrow$

\mathbb{R}^+ as $\hat{C}(s_1, s_2, s_3, s_4) = \frac{9}{10}$. Now for each $r, j \in \mathfrak{S}$ with $\partial_{\mu(j)}^{q(r)} > 0$,

$$\varphi\left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\xi_1(r)}\right) < 0 \Rightarrow \theta\left(\partial_{\Gamma(j)}^{q(r)}\right) \leq [\theta(\mathcal{M}_8(r, j))]^{\hat{C}(U(r, j))}.$$

Thus by Theorem 3.3.2, q, μ, ξ_1 and ξ_2 have a unique CFP.

Corollary 3.3.5. Let (\mathfrak{S}, ∂) be a CMS, and $q, \mu, \xi_1, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that if, $\forall r, j \in \mathfrak{S}$, with $\partial_{\mu(j)}^{q(r)} > 0$, for some $\theta \in \Theta^*$, $\varphi \in \Phi$ and $\hat{C} : \mathbb{R}^4 \rightarrow \mathbb{R}^+$,

$$\varphi\left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\xi_1(r)}\right) < 0 \Rightarrow \theta\left(\partial_{\mu(j)}^{q(r)}\right) \leq \left[\theta\left(\max\left\{\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\mu(j)}\right\}\right)\right]^{\hat{C}(U(r, j))},$$

with $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$ and $\mu(\mathfrak{S}) \subseteq \xi_1(\mathfrak{S})$. If the conditions (\hat{C}_1) and (\hat{C}_2) hold, then q, μ, ξ_1 and ξ_2 have a unique CFP in \mathfrak{S} .

Corollary 3.3.6. Let (\mathfrak{S}, ∂) be a CMS, and $q, \mu, \xi_1, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that if, $\forall r, j \in \mathfrak{S}$, with $\partial_{\mu(j)}^{q(r)} > 0$, for some $\theta \in \Theta^*$, $\varphi \in \Phi$ and $\hat{C} : \mathbb{R}^4 \rightarrow \mathbb{R}^+$,

$$\begin{aligned} \varphi\left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\xi_1(r)}\right) < 0 &\Rightarrow \theta\left(\partial_{V(j)}^{q(r)}\right) \\ &\leq \left[\theta\left(\max\left\{\partial_{\xi_2(j)}^{\xi_1(r)}, \partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\mu(j)}\right\}\right)\right]^{\hat{C}(U(r, j))}, \end{aligned}$$

with $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$ and $\mu(\mathfrak{S}) \subseteq \xi_1(\mathfrak{S})$. If the conditions (\hat{C}_1) and (\hat{C}_2) hold, then q, μ, ξ_1 and ξ_2 have a unique CFP in \mathfrak{S} .

3.4 Common fixed points of generalized Suzuki type Ćirić JS-contractions

In this section, we introduce generalized Suzuki type Ćirić JS-contractions (GSTĆ-JS-contractions) and prove related CFP results in CMS.

Definition 3.4.1. Let (\mathfrak{S}, ∂) be a MS, and $q, \mu, \xi_1, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$ be selfmaps. q, μ, ξ_1 and ξ_2 form a GSTĆ-JS-contraction, if, for each $r, j \in \mathfrak{S}$, for $\theta \in \Psi$, $\varphi \in \Phi$,

$$\begin{aligned} \varphi\left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\xi_1(r)}\right) < 0 &\Rightarrow \theta\left(\partial_{\Gamma(j)}^{q(r)}\right) \\ &\leq \left[\theta\left(\partial_{\xi_2(j)}^{\xi_1(r)}\right)\right]^{e_1} \left[\theta\left(\partial_{\xi_1(r)}^{q(r)}\right)\right]^{e_2} \left[\theta\left(\partial_{\xi_2(j)}^{\Gamma(j)}\right)\right]^{e_3} \left[\theta\left(\partial_{\xi_2(j)}^{q(r)} + \partial_{\xi_1(r)}^{\Gamma(j)}\right)\right]^{e_4}, \end{aligned} \quad (3.50)$$

where, $e_1, e_2, e_3, e_4 \geq 0$ with $e_1 + e_2 + e_3 + 2e_4 < 1$.

The result is as follows:

Theorem 3.4.2. Let (\mathfrak{S}, ∂) be a CMS. Suppose that $q, \mu, \xi_1, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$ form a GSTC-*JS*-contractions with $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$ and $\mu(\mathfrak{S}) \subseteq \xi_1(\mathfrak{S})$. If the conditions (\hat{C}_1) and (\hat{C}_2) hold, then q, μ, ξ_1 and ξ_2 have a unique CFP in \mathfrak{S} .

Proof. Let $r_0 \in \mathfrak{S}$. As $q(\mathfrak{S}) \subseteq \xi_2(\mathfrak{S})$, there exists $r_0 \in \mathfrak{S}$ such that $q(r_0) = \xi_2(r_1)$. As $\mu(r_1) \in \xi_1(\mathfrak{S})$, we can select $r_2 \in \mathfrak{S}$ such that $\mu(r_1) = \xi_1(r_2)$. Thus, r_{2n+1} and r_{2n+2} are selected in \mathfrak{S} such that $q(r_{2n}) = \xi_2(r_{2n+1})$ and $\mu(r_{2n+1}) = \xi_1(r_{2n+2})$. Define $\{\mathfrak{R}_n\} \subset \mathfrak{S}$ such that

$$\mathfrak{R}_{2n} = q(r_{2n}) = \xi_2(r_{2n+1}) \text{ and } \mathfrak{R}_{2n+1} = \mu(r_{2n+1}) = \xi_1(r_{2n+2}),$$

for $n = 0, 1, 2, \dots$. Since

$$\frac{1}{2} \partial_{\xi_1(r_{2n})}^{q(r_{2n})} = \frac{1}{2} \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}} < \partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} = \partial_{\xi_2(r_{2n+1})}^{\xi_1(r_{2n})},$$

we have

$$\varphi \left(\partial_{\xi_1(r_{2n})}^{q(r_{2n})}, \partial_{\xi_2(r_{2n+1})}^{\xi_1(r_{2n})} \right) < 0.$$

Hence, from (3.50),

$$\begin{aligned} \theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) &= \theta \left(\partial_{\mu(r_{2n+1})}^{q(r_{2n})} \right) & (3.51) \\ &\leq \left[\theta \left(\partial_{\xi_2(r_{2n+1})}^{\xi_1(r_{2n})} \right) \right]^{e_1} \left[\theta \left(\partial_{\xi_1(r_{2n})}^{q(r_{2n})} \right) \right]^{e_2} \left[\theta \left(\partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})} \right) \right]^{e_3} \left[\theta \left(\partial_{\xi_2(r_{2n+1})}^{q(r_{2n})} + \partial_{\xi_1(r_{2n})}^{\mu(r_{2n+1})} \right) \right]^{e_4} \\ &= \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} \right) \right]^{e_1} \left[\theta \left(\partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n}} \right) \right]^{e_2} \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}} \right) \right]^{e_3} \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n}} + \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}} \right) \right]^{e_4} \\ &\leq \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} \right) \right]^{e_1+e_2} \cdot \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n+1}} \right) \right]^{e_3} \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} + \partial_{\mathfrak{R}_{2n-1}}^{\mathfrak{R}_{2n+1}} \right) \right]^{e_4}. \end{aligned}$$

By (Ψ_5) , we have for every $n \in \mathbb{N}$,

$$\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} + \partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \leq \theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} \right) \cdot \theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right),$$

Hence, (3.51) becomes

$$1 < \theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \leq \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} \right) \right]^{e_1+e_2+e_4} \cdot \left[\theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \right]^{e_3+e_4}.$$

It implies that

$$\left[\theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \right]^{1-e_3-e_4} \leq \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} \right) \right]^{e_1+e_2+e_4}.$$

Therefore,

$$\theta \left(\partial_{\mathfrak{R}_{2n+1}}^{\mathfrak{R}_{2n}} \right) \leq \left[\theta \left(\partial_{\mathfrak{R}_{2n}}^{\mathfrak{R}_{2n-1}} \right) \right]^{\frac{e_1+e_2+e_4}{1-e_3-e_4}}.$$

Thus, we have,

$$\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) \leq \left[\theta \left(\partial_{\mathfrak{R}_n}^{\mathfrak{R}_{n-1}} \right) \right]^{\frac{e_1+e_2+e_4}{1-e_3-e_4}}, \text{ for every } n \in \mathbb{N}.$$

It implies that

$$\begin{aligned} 1 < \theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) &\leq \left[\theta \left(\partial_{\mathfrak{R}_n}^{\mathfrak{R}_{n-1}} \right) \right]^{\left(\frac{e_1+e_2+e_4}{1-e_3-e_4} \right)^1} \leq \left[\theta \left(\partial_{\mathfrak{R}_{n-1}}^{\mathfrak{R}_{n-2}} \right) \right]^{\left(\frac{e_1+e_2+e_4}{1-e_3-e_4} \right)^2} \\ &\leq \dots \leq \left[\theta \left(\partial_{\mathfrak{R}_1}^{\mathfrak{R}_0} \right) \right]^{\left(\frac{e_1+e_2+e_4}{1-e_3-e_4} \right)^n}, \text{ for every } n \in \mathbb{N}. \end{aligned} \quad (3.52)$$

Taking the limit as $n \rightarrow \infty$ in (3.52) and knowing that $\theta \in \Psi$, we have

$$\lim_{n \rightarrow \infty} \theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) = 1.$$

By (Ψ_3) , we get

$$\lim_{n \rightarrow \infty} \partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} = 0. \quad (3.53)$$

From condition (Ψ_4) , there exist $\lambda \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \frac{\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) - 1}{\left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda} = \ell.$$

Suppose that $\ell < \infty$. Let $\eta = \frac{\ell}{2} > 0$. From the definition of the limit, $\exists n_0 \in \mathbb{N}$ such that for each $n \geq n_0$,

$$\left| \frac{\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) - 1}{\left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda} - \ell \right| \leq \eta.$$

This implies

$$\frac{\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) - 1}{\left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda} \geq \ell - \eta = \eta.$$

Then

$$n \left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda \leq \xi n \left[\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) - 1 \right],$$

where $\xi = \frac{1}{\eta}$. Assume that $\ell = \infty$. Let $\eta > 0$ be an arbitrary number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$,

$$\frac{\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) - 1}{\left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda} \geq \eta.$$

Which implies

$$n \left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda \leq \xi n \left[\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) - 1 \right],$$

where, $\xi = \frac{1}{\eta}$. Then, in each cases, there exists $\xi \in \mathbb{R}^+$, $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$,

$$n \left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda \leq \xi n \left[\theta \left(\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right) - 1 \right].$$

By using (3.52), we get

$$n \left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda \leq \xi n \left(\left[\theta \left(\partial_{\mathfrak{R}_1}^{\mathfrak{R}_0} \right) \right]^{\left(\frac{c_1 + c_2 + c_4}{1 - c_3 - c_4} \right)^n} - 1 \right) \text{ for each } n \geq n_0. \quad (3.54)$$

Taking $n \rightarrow \infty$ in (3.54), we obtain

$$\lim_{n \rightarrow \infty} n \left[\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \right]^\lambda = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\partial_{\mathfrak{R}_{n+1}}^{\mathfrak{R}_n} \leq \frac{1}{n^{\frac{1}{\lambda}}} \text{ for each } n \geq n_1. \quad (3.55)$$

To prove $\{\mathfrak{R}_n\}$ is a CS, we use (3.55) and for $m > n \geq n_1$,

$$\text{for } m > n \geq n_1, \partial_{\mathfrak{R}_m}^{\mathfrak{R}_n} \leq \sum_{i=n}^{m-1} \partial_{\mathfrak{R}_{i+1}}^{\mathfrak{R}_i} \leq \sum_{i=n}^{\infty} \partial_{\mathfrak{R}_{i+1}}^{\mathfrak{R}_i} \leq \sum_{i=n}^{\infty} \frac{1}{i^\lambda}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^\lambda}$ is convergent, we find that $\lim_{n,m \rightarrow \infty} \partial_{\mathfrak{R}_m}^{\mathfrak{R}_n} = 0$. Thus $\{\mathfrak{R}_n\}$ is a CS. As \mathfrak{S} is a CMS, there exists $r^* \in \mathfrak{S}$ such that $\lim_{n \rightarrow \infty} \partial_{r^*}^{\mathfrak{R}_n} = 0$ and

$$\lim_{n \rightarrow \infty} q(r_{2n}) = \lim_{n \rightarrow \infty} \xi_2(r_{2n+1}) = \lim_{n \rightarrow \infty} \mu(r_{2n+1}) = \lim_{n \rightarrow \infty} \xi_1(r_{2n+2}) = r^*,$$

Since ξ_1 is continuous, so

$$\lim_{n \rightarrow \infty} \xi_1(q(r_{2n})) = \xi_1(r^*) = \lim_{n \rightarrow \infty} \xi_1(\xi_1(r_{2n+2})),$$

As (q, ξ_1) is a compatible pair,

$$\lim_{n \rightarrow \infty} \partial_{\xi_1(q(r_{2n}))}^{q(\xi_1(r_{2n}))} = 0.$$

From Lemma 1.1.5, we have

$$\lim_{n \rightarrow \infty} q(\xi_1(r_{2n})) = \xi_1(r^*).$$

Put $r = \xi_1(r_{2n})$ and $j = r_{2n+1}$ in (3.50) and if $\partial_{r^*}^{\xi_1(r^*)} > 0$, we get,

$$\frac{1}{2} \partial_{\xi_1(\xi_1(r_{2n}))}^{q(\xi_1(r_{2n}))} < \partial_{\xi_2(r_{2n+1})}^{\xi_1(\xi_1(r_{2n}))}.$$

Hence,

$$\varphi \left(\partial_{\xi_1(\xi_1(r_{2n}))}^{q(\xi_1(r_{2n}))}, \partial_{\xi_2(r_{2n+1})}^{\xi_1(\xi_1(r_{2n}))} \right) < 0,$$

and from (3.50), we get

$$\begin{aligned} & \theta \left(\partial_{\Gamma(r_{2n+1})}^{q(\xi_1(r_{2n}))} \right) \tag{3.56} \\ & \leq \left[\theta \left(\partial_{\xi_2(r_{2n+1})}^{\xi_1(\xi_1(r_{2n}))} \right) \right]^{e_1} \cdot \left[\theta \left(\partial_{\xi_1(\xi_1(r_{2n}))}^{q(\xi_1(r_{2n}))} \right) \right]^{e_2} \left[\theta \left(\partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})} \right) \right]^{e_3} \cdot \left[\theta \left(\partial_{\xi_2(r_{2n+1})}^{q(\xi_1(r_{2n}))} + \partial_{\xi_1(\xi_1(r_{2n}))}^{\mu(r_{2n+1})} \right) \right]^{e_4}. \end{aligned}$$

Setting $n \rightarrow \infty$ in (3.56), we obtain

$$\begin{aligned} \theta \left(\partial_{r^*}^{\xi_1(r^*)} \right) &\leq \left[\theta \left(\partial_{r^*}^{\xi_1(r^*)} \right) \right]^{e_1} \cdot \left[\theta \left(\partial_{r^*}^{\xi_1(r^*)} + \partial_{\xi_1(r^*)}^{r^*} \right) \right]^{e_4} \\ &\leq \left[\theta \left(\partial_{r^*}^{\xi_1(r^*)} \right) \right]^{e_1+2e_4} < \theta \left(\partial_{r^*}^{\xi_1(r^*)} \right), \end{aligned}$$

a contradiction. Thus, $\partial_{r^*}^{\xi_1(r^*)} = 0$ and so $\xi_1(r^*) = r^*$. As ξ_2 is continuous,

$$\lim_{n \rightarrow \infty} \xi_2(\mu(r_{2n+1})) = \xi_2(r^*) = \lim_{n \rightarrow \infty} \xi_2(\xi_2(r_{2n+1})).$$

As (μ, ξ_2) is a compatible pair, so

$$\lim_{n \rightarrow \infty} \partial_{\xi_2(\mu(r_{2n+1}))}^{\mu(\xi_2(r_{2n+1}))} = 0.$$

By Lemma 1.1.5,

$$\lim_{n \rightarrow \infty} \mu(\xi_2(r_{2n+1})) = \xi_2(r^*).$$

Set $r = r_{2n}$ and $j = \xi_2(r_{2n+1})$ in (3.50) and assume that $\partial_{\xi_2(r^*)}^{r^*} > 0$, and

$$\frac{1}{2} \partial_{\xi_1(r_{2n})}^{q(r_{2n})} < \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\xi_1(r_{2n})}.$$

Therefore,

$$\varphi \left(\partial_{\xi_1(r_{2n})}^{q(r_{2n})}, \partial_{\xi_2(\xi_2(r_{2n+1}))}^{\xi_1(r_{2n})} \right) < 0,$$

and from (3.50), one gets

$$\begin{aligned} &\theta \left(\partial_{\mu(\xi_2(r_{2n+1}))}^{q(r_{2n})} \right) \tag{3.57} \\ &\leq \left[\theta \left(\partial_{\xi_2(\xi_2(r_{2n+1}))}^{\xi_1(r_{2n})} \right) \right]^{e_1} \cdot \left[\theta \left(\partial_{\xi_1(r_{2n})}^{q(r_{2n})} \right) \right]^{e_2} \left[\theta \left(\partial_{\xi_2(\xi_2(r_{2n+1}))}^{\mu(\xi_2(r_{2n+1}))} \right) \right]^{e_3} \cdot \left[\theta \left(\partial_{\xi_2(\xi_2(r_{2n+1}))}^{q(r_{2n})} + \partial_{\xi_1(r_{2n})}^{\mu(\xi_2(r_{2n+1}))} \right) \right]^{e_4}. \end{aligned}$$

Setting $n \rightarrow \infty$ in (3.57), we get

$$\theta \left(\partial_{\xi_2(r^*)}^{r^*} \right) \leq \left[\theta \left(\partial_{\xi_2(r^*)}^{r^*} \right) \right]^{e_1+2e_4} < \theta \left(\partial_{\xi_2(r^*)}^{r^*} \right),$$

a contradiction. Thus, $\partial_{\xi_2(r^*)}^{r^*} = 0$ and $r^* = \xi_2(r^*)$. Suppose that $\partial_{r^*}^{q(r^*)} > 0$, and

$$\frac{1}{2}\partial_{\xi_1(r^*)}^{q(r^*)} < \partial_{\xi_2(r_{2n+1})}^{\xi_1(r^*)}.$$

Then,

$$\varphi\left(\partial_{\xi_1(r^*)}^{q(r^*)}, \partial_{\xi_2(r_{2n+1})}^{\xi_1(r^*)}\right) < 0,$$

and from (3.50), we get

$$\begin{aligned} & \theta\left(\partial_{\mu(r_{2n+1})}^{q(r^*)}\right) \tag{3.58} \\ \leq & \left[\theta\left(\partial_{\xi_2(r_{2n+1})}^{\xi_1(r^*)}\right)\right]^{e_1} \cdot \left[\theta\left(\partial_{\xi_1(r^*)}^{q(r^*)}\right)\right]^{e_2} \left[\theta\left(\partial_{\xi_2(r_{2n+1})}^{\mu(r_{2n+1})}\right)\right]^{e_3} \cdot \left[\theta\left(\partial_{\xi_2(r_{2n+1})}^{q(r^*)} + \partial_{\xi_1(r^*)}^{\mu(r_{2n+1})}\right)\right]^{e_4}. \end{aligned}$$

Setting $n \rightarrow \infty$ in (3.58), and Since $r^* = \xi_2(r^*) = \xi_1(r^*)$, so

$$\theta\left(\partial_{r^*}^{q(r^*)}\right) \leq \left[\theta\left(\partial_{r^*}^{q(r^*)}\right)\right]^{e_2+e_4} < \theta\left(\partial_{r^*}^{q(r^*)}\right),$$

a contradiction. Then, $\partial_{r^*}^{q(r^*)} = 0$ and so $r^* = q(r^*)$. Finally, assume that $\partial_{\mu(r^*)}^{r^*} > 0$, and since $r^* = \xi_2(r^*) = \xi_1(r^*) = q(r^*)$ so, one has,

$$\varphi\left(0, \partial_{\xi_2(r^*)}^{r^*}\right) < 0$$

and from (3.50),

$$\theta\left(\partial_{\mu(r^*)}^{r^*}\right) = \theta\left(\partial_{\mu(r^*)}^{q(r^*)}\right) < \theta\left(\partial_{\mu(r^*)}^{r^*}\right),$$

a contradiction. Then, $\partial_{\mu(r^*)}^{r^*} = 0$ and so $r^* = \mu(r^*)$. Hence, r^* is a CFP of ξ_1, μ, q and ξ_2 .

Next, assume that v^* is another CFP of ξ_1, μ, q and ξ_2 such that $r^* \neq v^*$, we get

$$\varphi\left(0, \partial_{\xi_2(v^*)}^{\xi_1(r^*)}\right) < 0,$$

then from (3.50),

$$\begin{aligned} \theta\left(\partial_{v^*}^{r^*}\right) &= \theta\left(\partial_{\mu(v^*)}^{q(r^*)}\right) \\ &\leq \left[\theta\left(\partial_{\xi_2(v^*)}^{\xi_1(r^*)}\right)\right]^{e_1} \cdot \left[\theta\left(\partial_{\xi_1(r^*)}^{q(r^*)}\right)\right]^{e_2} \left[\theta\left(\partial_{\xi_2(v^*)}^{\mu(v^*)}\right)\right]^{e_3} \cdot \left[\theta\left(\partial_{\xi_2(v^*)}^{q(r^*)} + \partial_{\xi_1(r^*)}^{\mu(v^*)}\right)\right]^{e_4}. \end{aligned}$$

It implies that

$$\theta\left(\partial_{v^*}^{r^*}\right) \leq \left[\theta\left(\partial_{v^*}^{r^*}\right)\right]^{e_1+2e_4} < \theta\left(\partial_{v^*}^{r^*}\right),$$

a contradiction. Then $r^* = v^*$. Thus r^* is the unique CFP of ξ_1, μ, q and ξ_2 . ■

Example 3.4.3. Let $\mathfrak{S} = [0, \infty)$ and define the function $\partial : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by $\partial(r, j) = |r - j|$. Clearly, (\mathfrak{S}, ∂) is a CMS. Let $\theta(\varsigma) = e^{\sqrt{\varsigma}}$, and $\varphi(s, \varsigma) = \frac{s}{2} - \varsigma$, then $\theta \in \Psi$, $\varphi \in \Phi$. Define $\xi_1, \mu, q, \xi_2 : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\begin{aligned} \xi_1(r) &= e^{7r} - 1, & \mu(r) &= \ln\left(1 + \frac{r}{6}\right), \\ q(r) &= \ln\left(1 + \frac{r}{6}\right), & \xi_2(r) &= e^{6r} - 1. \end{aligned}$$

Obviously, $q(\mathfrak{S}) = \xi_2(\mathfrak{S}) = \mu(\mathfrak{S}) = \xi_1(\mathfrak{S})$ if a sequence $\{r_n\} \subset \mathfrak{S}$ such that

$$\lim_{n \rightarrow \infty} q(r_n) = \lim_{n \rightarrow \infty} \xi_1(r_n) = \varsigma, \text{ for } \varsigma \in \mathfrak{S},$$

then,

$$\lim_{n \rightarrow \infty} |q(r_n) - \varsigma| = \lim_{n \rightarrow \infty} |\xi_1(r_n) - \varsigma| = 0,$$

and,

$$\lim_{n \rightarrow \infty} \left| \ln\left(1 + \frac{r_n}{6}\right) - \varsigma \right| = \lim_{n \rightarrow \infty} |e^{7r_n} - 1 - \varsigma| = 0.$$

Then,

$$\lim_{n \rightarrow \infty} \left| r_n - (6e^t - 6) \right| = \lim_{n \rightarrow \infty} \left| r_n - \frac{\ln(1 + \varsigma)}{7} \right| = 0.$$

It gives that $6e^\varsigma - 6 = \frac{\ln(1 + \varsigma)}{7}$, then $\varsigma = 0$. By continuity of q and ξ_1 , we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \partial_{\xi_1(q(r_n))}^{q(\xi_1(r_n))} &= \lim_{n \rightarrow \infty} |q(\xi_1(r_n)) - \xi_1(q(r_n))| \\ &= |q(\varsigma) - \xi_1(\varsigma)| = |0 - 0| = 0, \text{ for } \varsigma = 0 \in \mathfrak{S}. \end{aligned}$$

Therefore, the pair (q, ξ_1) is compatible. in a similar way, the pair (μ, ξ_2) is compatible. $\forall r, j \in \mathfrak{S}$ with

$$\varphi\left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\mu(j)}\right) < 0,$$

we have

$$\theta \left(\partial_{\mu(j)}^{q(r)} \right) \leq \left[\theta \left(\partial_{\xi_2(j)}^{\xi_1(r)} \right) \right]^{e_1} \left[\theta \left(\partial_{\xi_1(r)}^{q(r)} \right) \right]^{e_2} \left[\theta \left(\partial_{\xi_2(j)}^{\mu(j)} \right) \right]^{e_3} \left[\theta \left(\partial_{\xi_2(j)}^{q(r)} + \partial_{\xi_1(r)}^{\mu(j)} \right) \right]^{e_4},$$

where, $e_1 = e_2 = e_3 = 0, e_4 = 0.95$. Hence by Theorem 3.4.2, q, μ, ξ_1 and ξ_2 have a unique CFP.

3.5 Application to nonlinear fractional differential equations

Here, we apply the result given by the section 3.3 (Theorem 3.3.2) to show the existence of a solution of the system of nonlinear fractional differential equations (NFDE). Let $\mathfrak{S} = C(\hat{I}, \mathbb{R})$ be the space of all continuous functions on $\hat{I} = [0, 1]$. Define $\partial : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by

$$\partial_j^r = \|r - j\|_\infty = \max_{t \in \hat{I}} |r(t) - j(t)|, \quad r, j \in \mathfrak{S}.$$

Then, $\mathfrak{S} = C(\hat{I}, \mathbb{R})$ is CMS.

Consider the following system:

$$\begin{cases} {}^C D^\alpha r(t) = \bar{U}_1(t, \xi_1(r(t))) \\ {}^C D^\alpha j(t) = \bar{U}_2(t, \xi_2(j(t))) \end{cases}, \quad (3.59)$$

with boundary conditions

$$\begin{cases} r(0) = 0, \quad I r(1) = r'(0) \\ j(0) = 0, \quad Q j(1) = j'(0) \end{cases}$$

${}^C D^\alpha$ denotes the Caputo fractional derivative of order α , defined by

$$\begin{cases} {}^C D_1^\alpha \bar{U}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \bar{U}_2^n(s) ds, \\ {}^C D_2^\alpha \bar{U}(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\alpha-1} \bar{U}_2^n(s) ds, \end{cases}$$

(where we consider

$$n - 1 < \alpha, \quad \alpha < 1 \text{ and } n = [\alpha] + 1,$$

and $I^\alpha \bar{U}_1$ and $I^\alpha \bar{U}_2$ denote the Riemann-Liouville fractional integral of order α of continuous functions \bar{U}_1 and \bar{U}_2 , given by

$$\begin{cases} I^\alpha \bar{U}_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_1(s) ds, \text{ with } \alpha > 0, \\ Q^\alpha \bar{U}_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_2(s) ds, \text{ with } \alpha > 0. \end{cases}$$

The system (3.59) can be written in the following form.

$$\begin{cases} r(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_1(s, \xi_1(r(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_1(u, \xi_1(r(u))) dud s, \\ j(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_2(s, \xi_2(j(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_2(u, \xi_2(j(u))) dud s \end{cases}$$

Define the mappings $q, \Gamma : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\begin{cases} q(r(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_1(s, \xi_1(r(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_1(u, \xi_1(r(u))) dud s, \\ \mu(j(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_2(s, \xi_2(j(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_2(u, \xi_2(j(u))) dud s \end{cases}$$

Theorem 3.5.1. Suppose that the following conditions hold:

- (i) $\bar{U}_1, \bar{U}_2 : \hat{I} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions;
- (ii) $\bar{U}_1(s, \cdot), \bar{U}_2(s, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are increasing functions,
- (iii) for each $r, j \in \mathfrak{S}$ with $q(r) \leq \xi_2(j)$ and $|q(r(s)) - \mu(j(s))| > 0$, $\tau > 0$, we have

$$|\bar{U}_1(s, \xi_1(r(s))) - \bar{U}_2(s, \xi_2(j(s)))| \leq \frac{e^{-\tau} \Gamma(\alpha + 1)}{4} \mathcal{M}_8(r, j),$$

- (iv) $\exists r_0, j_0 \in C(\hat{I}, \mathbb{R})$ such that for each $t \in \hat{I}$,

$$\begin{cases} r_0(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_1(s, \xi_1(r_0(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_1(u, \xi_1(r_0(u))) dud s, \\ j_0(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_2(s, \xi_2(j_0(s))) ds + \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_2(u, \xi_2(j_0(u))) dud s \end{cases}$$

- (v) if there exists a sequence $\{r_n\}$ in \mathfrak{S} such that

$$\lim_{n \rightarrow \infty} \partial_{\xi_1(q(r_n))}^{\mu(\xi_1(r_n))} = 0 \text{ and } \lim_{n \rightarrow \infty} \partial_{\xi_2(\mu(r_n))}^{\mu(\xi_2(r_n))} = 0,$$

whenever,

$$\lim_{n \rightarrow \infty} q(r_n) = \lim_{n \rightarrow \infty} \xi_1(r_n) = t, \text{ and } \lim_{n \rightarrow \infty} \mu(r_n) = \lim_{n \rightarrow \infty} \xi_2(r_n) = t, \text{ for } t \in \mathfrak{S}.$$

Then the system (3.59) has a solution.

Proof. From (iii) and (iv), one has,

$$\begin{aligned} |q(r(t)) - \mu(j(t))| &= \left| \begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_1(s, \xi_1(r(s))) ds \\ &- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \bar{U}_2(s, \xi_2(j(s))) ds \\ &+ \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_1(u, \xi_1(r(u))) du ds \\ &- \frac{2t}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} \bar{U}_2(u, \xi_2(j(u))) du ds \end{aligned} \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |\bar{U}_1(s, \xi_1(r(s))) - \bar{U}_2(s, \xi_2(j(s)))| ds \\ &\quad + \frac{2}{\Gamma(\alpha)} \int_0^1 \int_0^s (s-u)^{\alpha-1} |\bar{U}_1(s, \xi_1(r(s))) - \bar{U}_2(u, \xi_2(j(u)))| du ds \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \cdot \int_0^t (t-s)^{\alpha-1} \mathcal{M}_8(r, j) ds \\ &\quad + \frac{2}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \cdot \int_0^1 \int_0^s (s-u)^{\alpha-1} \mathcal{M}_8(r, j) du ds \\ &\leq \frac{1}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \mathcal{M}_8(r, j) \int_0^t (t-s)^{\alpha-1} ds \\ &\quad + \frac{2}{\Gamma(\alpha)} \frac{e^{-\tau} \Gamma(\alpha+1)}{4} \mathcal{M}_8(r, j) \int_0^1 \int_0^s (s-u)^{\alpha-1} du ds \\ &\leq \frac{e^{-\tau} \Gamma(\alpha) \cdot \Gamma(\alpha+1)}{4 \Gamma(\alpha) \cdot \Gamma(\alpha+1)} \mathcal{M}_8(r, j) + 2e^{-\tau} \mathfrak{B}(\alpha+1, 1) \frac{\Gamma(\alpha) \cdot \Gamma(\alpha+1)}{4 \Gamma(\alpha) \cdot \Gamma(\alpha+1)} \mathcal{M}_8(r, j) \\ &\leq \frac{e^{-\tau}}{4} \mathcal{M}_8(r, j) + \frac{e^{-\tau}}{2} \mathcal{M}_8(r, j) \end{aligned}$$

where, β is the beta function. Which implies,

$$\partial_{\mu(j)}^{q(r)} \leq \frac{3}{4} e^{-\tau} \mathcal{M}_8(r, j).$$

It follows that

$$\begin{aligned} e^{\sqrt{\partial_{\mu(j)}^{q(r)}}} &\leq e^{\sqrt{\frac{3}{4} e^{-\tau} \mathcal{M}_8(r, j)}} = e^{\sqrt{\frac{3}{4} e^{-\tau}} \sqrt{\mathcal{M}_8(r, j)}} \\ &= \left[e^{\sqrt{\mathcal{M}_8(r, j)}} \right]^{\sqrt{\frac{3}{4} e^{-\tau}}} = \left[e^{\sqrt{\mathcal{M}_8(r, j)}} \right]^{\hat{C}(U(r, j))}, \end{aligned}$$

where, $\hat{C}(t_1, t_2, t_3, t_4) = \sqrt{\frac{3}{4e^\tau}} < 1$ and $\theta(t) = e^{\sqrt{t}}$. As this inequality holds for each $r, j \in \mathfrak{S}$ with $q(r) \leq \xi_2(j)$, so it is true for any $\varphi \in \Phi$,

$$\varphi \left(\partial_{\xi_1(r)}^{q(r)}, \partial_{\xi_2(j)}^{\xi_1(r)} \right) < 0 \Rightarrow \theta \left(\partial_{\mu(j)}^{q(r)} \right) \leq [\theta(\mathcal{M}_8(r, j))]^{\hat{C}(U(r, j))}.$$

Then, q, μ, ξ_1 and ξ_2 form a GST (θ, \hat{C}) -R-contraction. Hence, from Theorem 3.3.2, q, Γ, ξ_1 and ξ_2 have a CFP, that is, the system (3.59) has a solution. ■

Chapter 4

Fixed point Theorems for (Υ, Λ) -Contractions in Metric Spaces

Liu et al. [80] announced the notion of (Υ, Λ) -contractions, which encompasses both θ -contractions ($\theta \in \Xi$) and \mathbf{F} -contractions ($\mathbf{F} \in F^*$) and they established new related FP theorems in CMSs. Recently, Ameer et al. [14, 15] established some FP theorems for multivalued (Υ, Λ) -contractions in MSs and bMSs and applied these results to show the existence of solution of functional equations. The intention of this chapter is to study the notion of $(\Lambda, \Upsilon, \acute{R}_{\mathfrak{S}})$ -contraction self mappings, (where $\acute{R}_{\mathfrak{S}}$ is an arbitrary binary relation) and discuss related CFP theorems on CMSs. As application, we apply our main theorems to show the existence of a solution of the nonlinear matrix equations. We further discuss some common \check{e} -fuzzy FP theorems for a pair of fuzzy mappings which are generalized almost (Υ, Λ) -contractions in CMSs. **Results given in this chapter have been published in ([8],[13]).**

4.1 Common fixed points of $(\Lambda, \Upsilon, \acute{R}_{\mathfrak{S}})$ -contractions

Here, \mathfrak{S} is a nonempty set and $(\mathfrak{S}, \vartheta)$ is MS. Further, in results it is considered as a CMS. Also note that $\acute{R}_{\mathfrak{S}}$ is a binary relation on \mathfrak{S} . Unless otherwise stated.

Here, we announce $(\Lambda, \Upsilon, \acute{R}_{\mathfrak{S}})$ -contraction under an arbitrary binary relation $\acute{R}_{\mathfrak{S}}$ and investigate the existence of CFPs for those two self-maps satisfying $(\Lambda, \Upsilon, \acute{R}_{\mathfrak{S}})$ -contraction condition. We start with the following definition.

Definition 4.1.1 Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. Consider $\chi = \left\{ (r, j) \in \hat{R}_{\mathfrak{S}} : \partial_{\hat{E}(j)}^{\hat{S}(r)} > 0 \right\}$. The pair (\hat{S}, \hat{E}) is called $(\Lambda, \Upsilon, \hat{R}_{\mathfrak{S}})$ -contraction if \exists a continuous $\Upsilon \in \Omega$ and $\Lambda \in \Lambda_{\Delta}$ such that, for each $(r, j) \in \chi$

$$\Lambda \left(\partial_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \partial_j^r, \frac{\partial_{\hat{E}(j)}^r \partial_j^{\hat{S}(r)}}{1 + \partial_j^r} \right\} \right) \right). \quad (4.1)$$

$M \left((\hat{S}, \hat{E}); \hat{R}_{\mathfrak{S}} \right)$ is the class of all order pairs $(r, j) \in \mathfrak{S} \times \mathfrak{S}$ such that $(\hat{S}(r), \hat{E}(j)) \in \hat{R}_{\mathfrak{S}}$.

Let us define the following conditions:

- (\star_1): $\hat{R}_{\mathfrak{S}}$ is (\hat{S}, \hat{E}) -closed, (\star_2): $\hat{R}_{\mathfrak{S}}$ is \hat{S} -closed,
- (\star_3): (\mathfrak{S}, ∂) is $\hat{R}_{\mathfrak{S}}$ -regular, (\star_4): \hat{S} and \hat{E} are continuous,
- (\star_5): \hat{S} is continuous.

Theorem 4.1.2. If $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ satisfy:

- (a) $M \left((\hat{S}, \hat{E}); \hat{R}_{\mathfrak{S}} \right) \neq \emptyset$,
- (b) condition (\star_1) holds,
- (c) (\hat{S}, \hat{E}) is $(\Lambda, \Upsilon, \hat{R}_{\mathfrak{S}})$ -contraction,
- (d) condition (\star_4) holds.

Then \exists a CFP of \hat{S} and \hat{E} in \mathfrak{S} .

Proof. Let $(r_0, r_1) \in M \left((\hat{S}, \hat{E}); \hat{R}_{\mathfrak{S}} \right)$, then $(\hat{S}(r_0), \hat{E}(r_1)) \in \hat{R}_{\mathfrak{S}}$. Define the sequence $\{r_n\}$ in \mathfrak{S} by

$$\begin{cases} r_{2n+1} = \hat{S}(r_{2n}), \\ r_{2n+2} = \hat{E}(r_{2n+1}) \end{cases}, \quad n \in \mathbb{N}.$$

If $r_{2n^*+1} = r_{2n^*}$ for $n^* \in \mathbb{N}$. Hence, r_{2n^*} is a CFP of \hat{S} and \hat{E} . If $r_{2n+1} \neq r_{2n}, \forall n \in \mathbb{N} \cup \{0\}$.

Then, $\partial_{\hat{E}(r_{2n+1})}^{\hat{S}(r_{2n})} > 0$ for each $n \in \mathbb{N} \cup \{0\}$. Since $\hat{R}_{\mathfrak{S}}$ is (\hat{S}, \hat{E}) -closed, we get

$$\begin{aligned} (r_1, r_2) &= (\hat{S}(r_0), \hat{E}(r_1)) \in \hat{R}_{\mathfrak{S}}, & (r_2, r_3) &= (\hat{E}(r_1), \hat{S}(r_2)) \in \hat{R}_{\mathfrak{S}}, \\ (r_3, r_4) &= (\hat{S}(r_2), \hat{E}(r_3)) \in \hat{R}_{\mathfrak{S}}, & (r_4, r_5) &= (\hat{E}(r_3), \hat{S}(r_4)) \in \hat{R}_{\mathfrak{S}}, \\ \dots & & (r_{2n}, r_{2n+1}) &= (\hat{E}(r_{2n-1}), \hat{S}(r_{2n})) \in \hat{R}_{\mathfrak{S}}. \end{aligned}$$

Thus, $(r_{2n}, r_{2n+1}) \in \chi$, for all $n \in \mathbb{N} \cup \{0\}$. By setting $r = r_{2n}$ and $j = r_{2n-1}$ in (4.1), and by using (c), we have

$$\begin{aligned}
\Lambda \left(\partial_{r_{2n+1}}^{r_{2n}} \right) &= \Lambda \left(\partial_{r_{2n}}^{r_{2n+1}} \right) = \Lambda \left(\partial_{\hat{E}(r_{2n-1})}^{\hat{S}(r_{2n})} \right) \\
&\leq \Upsilon \left(\Lambda \left(\max \left\{ \partial_{r_{2n-1}}^{r_{2n}}, \frac{\partial_{\hat{E}(r_{2n-1})}^{r_{2n}} \partial_{r_{2n-1}}^{r_{2n-1}}}{1 + \partial_{r_{2n-1}}^{r_{2n}}} \right\} \right) \right) \\
&= \Upsilon \left(\Lambda \left(\max \left\{ \partial_{r_{2n-1}}^{r_{2n}}, \frac{\partial_{r_{2n}}^{r_{2n}} \partial_{r_{2n-1}}^{r_{2n-1}}}{1 + \partial_{r_{2n-1}}^{r_{2n}}} \right\} \right) \right) \\
&= \Upsilon \left(\Lambda \left(\partial_{r_{2n-1}}^{r_{2n}} \right) \right), \text{ for all } n \in \mathbb{N} \cup \{0\}.
\end{aligned}$$

Similarly, setting $r = r_{2n}$ and $j = r_{2n+1}$ in (4.1), and again from (c), we get

$$\begin{aligned}
\Lambda \left(\partial_{r_{2n+2}}^{r_{2n+1}} \right) &= \Lambda \left(\partial_{\hat{E}(r_{2n+1})}^{\hat{S}(r_{2n})} \right) \\
&\leq \Upsilon \left(\Lambda \left(\max \left\{ \partial_{r_{2n+1}}^{r_{2n}}, \frac{\partial_{\hat{E}(r_{2n+1})}^{r_{2n}} \partial_{r_{2n+1}}^{r_{2n+1}}}{1 + \partial_{r_{2n+1}}^{r_{2n}}} \right\} \right) \right) \\
&= \Upsilon \left(\Lambda \left(\max \left\{ \partial_{r_{2n+1}}^{r_{2n}}, \frac{\partial_{r_{2n+2}}^{r_{2n}} \partial_{r_{2n+1}}^{r_{2n+1}}}{1 + \partial_{r_{2n+1}}^{r_{2n}}} \right\} \right) \right) \\
&= \Upsilon \left(\Lambda \left(\partial_{r_{2n+1}}^{r_{2n}} \right) \right).
\end{aligned}$$

In general, we have

$$\Lambda \left(\partial_{r_{n+1}}^{r_n} \right) \leq \Upsilon \left(\Lambda \left(\partial_{r_n}^{r_{n-1}} \right) \right), \text{ for each } n \in \mathbb{N}.$$

This implies

$$\Lambda \left(\partial_{r_{n+1}}^{r_n} \right) \leq \Upsilon \left(\Lambda \left(\partial_{r_n}^{r_{n-1}} \right) \right) \leq \Upsilon^2 \left(\Lambda \left(\partial_{r_{n-1}}^{r_{n-2}} \right) \right) \leq \dots \leq \Upsilon^n \left(\Lambda \left(\partial_{r_1}^{r_0} \right) \right).$$

Taking $n \rightarrow \infty$, we obtain

$$0 \leq \lim_{n \rightarrow \infty} \Lambda \left(\partial_{r_{n+1}}^{r_n} \right) \leq \lim_{n \rightarrow \infty} \Upsilon^n \left(\Lambda \left(\partial_{r_1}^{r_0} \right) \right) = 0.$$

This implies

$$\lim_{n \rightarrow \infty} \Lambda \left(\partial_{r_{n+1}}^{r_n} \right) = 0.$$

From Lemma 1.6.4 and (Λ_2) ,

$$\lim_{n \rightarrow \infty} \partial_{r_{n+1}}^{r_n} = 0. \quad (4.2)$$

To prove $\{r_n\}$ is a CS, suppose that there is $\varepsilon > 0$ and $\{\hat{h}_n\}_{n=1}^{\infty}$ and $\{\tilde{I}_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N} \cup \{0\}$,

$$\hat{h}_n > \tilde{I}_n > n, \partial_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \geq \varepsilon \text{ and } \partial_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}} < \varepsilon.$$

Thus,

$$\varepsilon \leq \partial_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \leq \partial_{r_{\hat{h}(n)-1}}^{r_{\hat{h}(n)}} + \partial_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}} < \varepsilon + \partial_{r_{\hat{h}(n)-1}}^{r_{\hat{h}(n)}}. \quad (4.3)$$

Taking $n \rightarrow \infty$ in (4.3), we get

$$\lim_{n \rightarrow \infty} \partial_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} = \varepsilon. \quad (4.4)$$

Again,

$$\partial_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \leq \partial_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}} + \partial_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)+1}} + \partial_{r_{\tilde{I}(n)}}^{r_{\tilde{I}(n)+1}}, \quad (4.5)$$

and

$$\partial_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)+1}} \leq \partial_{r_{\hat{h}(n)}}^{r_{\hat{h}(n)+1}} + \partial_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} + \partial_{r_{\tilde{I}(n)+1}}^{r_{\tilde{I}(n)}}. \quad (4.6)$$

Taking $n \rightarrow \infty$ in (4.5) and (4.6), we get

$$\lim_{n \rightarrow \infty} \partial_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)+1}} = \varepsilon. \quad (4.7)$$

Consider $(r_{\hat{h}(n)}, r_{\tilde{I}(n)}) \in \acute{R}_{\mathfrak{S}}$. Since $\acute{R}_{\mathfrak{S}}$ is (\hat{S}, \hat{E}) -closed,

$$\left(\hat{S} \left(r_{\hat{h}(n)} \right), \hat{E} \left(r_{\tilde{I}(n)} \right) \right) = \left(r_{\hat{h}(n)+1}, r_{\tilde{I}(n)+1} \right) \in \acute{R}_{\mathfrak{S}},$$

and from (4.7), we have $\partial_{\hat{E}\left(\begin{smallmatrix} r_{\hat{h}(n)} \\ r_{\hat{l}(n)} \end{smallmatrix}\right)}^{\hat{S}\left(\begin{smallmatrix} r_{\hat{h}(n)} \\ r_{\hat{l}(n)} \end{smallmatrix}\right)} > 0$. Thus, $(r_{\hat{h}(n)+1}, r_{\hat{l}(n)+1}) \in \mathcal{X}$. From (4.1), we have

$$\begin{aligned}
0 &< \Lambda\left(\partial_{r_{\hat{l}(n)+1}}^{r_{\hat{h}(n)+1}}\right) = \Lambda\left(\partial_{\hat{E}\left(\begin{smallmatrix} r_{\hat{h}(n)} \\ r_{\hat{l}(n)} \end{smallmatrix}\right)}^{\hat{S}\left(\begin{smallmatrix} r_{\hat{h}(n)} \\ r_{\hat{l}(n)} \end{smallmatrix}\right)}\right) \\
&\leq \Upsilon\left(\Lambda\left(\max\left\{\partial_{r_{\hat{l}(n)}}^{r_{\hat{h}(n)}}, \frac{\partial_{\hat{E}\left(\begin{smallmatrix} r_{\hat{h}(n)} \\ r_{\hat{l}(n)} \end{smallmatrix}\right)}^{\hat{S}\left(\begin{smallmatrix} r_{\hat{h}(n)} \\ r_{\hat{l}(n)} \end{smallmatrix}\right)}}{1 + \partial_{r_{\hat{l}(n)}}^{r_{\hat{h}(n)}}}\right\}\right)\right) \\
&= \Upsilon\left(\Lambda\left(\max\left\{\partial_{r_{\hat{l}(n)}}^{r_{\hat{h}(n)}}, \frac{\partial_{r_{\hat{l}(n)+1}}^{r_{\hat{h}(n)}} \partial_{r_{\hat{h}(n)+1}}^{r_{\hat{l}(n)}}}{1 + \partial_{r_{\hat{l}(n)}}^{r_{\hat{h}(n)}}}\right\}\right)\right) \\
&\leq \Upsilon\left(\Lambda\left(\max\left\{\partial_{r_{\hat{l}(n)}}^{r_{\hat{h}(n)}}, \frac{[\partial_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}} + \partial_{r_{\hat{l}(n)+1}}^{r_{\hat{h}(n)+1}]} \partial_{r_{\hat{h}(n)+1}}^{r_{\hat{l}(n)}}}{1 + \partial_{r_{\hat{l}(n)}}^{r_{\hat{h}(n)}}}\right\}\right)\right).
\end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (4.2), (4.4) and (4.7), we get

$$\begin{aligned}
\Lambda(\varepsilon) &\leq \Lambda\left(\lim_{n \rightarrow \infty} \partial_{r_{\hat{l}(n)+1}}^{r_{\hat{h}(n)+1}}\right) \leq \lim_{n \rightarrow \infty} \Upsilon\left(\Lambda\left(\max\left\{\frac{\partial_{r_{\hat{l}(n)}}^{r_{\hat{h}(n)}}}{[\partial_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}} + \partial_{r_{\hat{l}(n)+1}}^{r_{\hat{h}(n)+1}]} \partial_{r_{\hat{h}(n)+1}}^{r_{\hat{l}(n)}}}\right\}\right)\right) \\
&= \Upsilon\left(\Lambda\left(\varepsilon, \frac{\varepsilon \cdot \varepsilon}{1 + \varepsilon}\right)\right) \leq \Upsilon\left(\Lambda\left(\varepsilon, \frac{\varepsilon^2}{\varepsilon}\right)\right) = \Upsilon(\Lambda(\varepsilon)) < \Lambda(\varepsilon),
\end{aligned}$$

a contradiction. Then, $\{r_n\}$ is CS in \mathfrak{S} . Since (\mathfrak{S}, ∂) , so we can find $r^* \in \mathfrak{S}$ such that

$$\lim_{n \rightarrow \infty} \partial_{r^*}^{r_n} = 0. \quad (4.8)$$

Next, since \hat{E} and \hat{S} are continuous and

$$\lim_{n \rightarrow \infty} \partial_{r^*}^{r_{2n}} = 0 = \lim_{n \rightarrow \infty} \partial_{r^*}^{r_{2n-1}},$$

we find that,

$$\lim_{n \rightarrow \infty} \partial_{\hat{S}(r^*)}^{r_{2n+1}} = \lim_{n \rightarrow \infty} \partial_{\hat{S}(r^*)}^{\hat{S}(r_{2n})} = 0, \quad (4.9)$$

and

$$\lim_{n \rightarrow \infty} \partial_{\hat{E}(r^*)}^{r_{2n}} = \lim_{n \rightarrow \infty} \partial_{\hat{E}(r^*)}^{\hat{E}(r_{2n-1})} = 0.$$

This leads us to conclude that $\hat{S}(r^*) = r^*$ and $\hat{E}(r^*) = r^*$. Thus $\hat{S}(r^*) = \hat{E}(r^*) = r^*$. Hence, \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$. ■

Next theorem shows the uniqueness of the CFP.

Theorem 4.1.3. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ and $\hat{R}_{\mathfrak{S}}$ be a transitive relation (TR) on \mathfrak{S} . Assume that:

- (a) $M\left(\left(\hat{S}, \hat{E}\right); \hat{R}_{\mathfrak{S}}\right) \neq \emptyset$ and $\Gamma((r, j), \hat{R}_{\mathfrak{S}}) \neq \emptyset$,
- (b) \exists a continuous $\Upsilon \in \Omega$ and a function $\Lambda \in \Lambda_{\Delta}$ so that for all $(r, j) \in \chi$,

$$\Lambda\left(\partial_{\hat{E}(j)}^{\hat{S}(r)}\right) \leq \Upsilon\left(\Lambda\left(\max\left\{\frac{1}{2}\partial_j^r, \frac{\partial_{\hat{E}(j)}^r \partial_{\hat{S}(r)}^j}{2[1 + \partial_j^r]}\right\}\right)\right), \quad (4.10)$$

(c) condition (\star_4) holds,

(d) condition (\star_1) holds.

Then \exists only one CFP of \hat{S} and \hat{E} in \mathfrak{S} .

Proof. By Theorem 4.1.2, \hat{S} and \hat{E} have a CFP. If v and v^* are two CFPs of \hat{S} and \hat{E} such that $v \neq v^*$. As $\Gamma((v, v^*), \hat{R}_{\mathfrak{S}})$ is the set of paths in $\hat{R}_{\mathfrak{S}}$ from v to v^* , there exists a sequence $\{w_0, w_1, w_2, \dots, w_L\}$ in $\hat{R}_{\mathfrak{S}}$ from v to v^* (i.e., \exists a path of finite length L) with,

$$w_0 = v, w_L = v^*, (w_j, w_{j+1}) \in \hat{R}_{\mathfrak{S}} \text{ for } j = 0, 1, 2, \dots, (L-1).$$

As \hat{R} is a TR, so

$$(v, w_1) \in \hat{R}_{\mathfrak{S}}, (w_1, w_2) \in \hat{R}_{\mathfrak{S}}, \dots, (w_{k-1}, v^*) \in \hat{R}_{\mathfrak{S}} \Rightarrow (v, v^*) \in \hat{R}_{\mathfrak{S}}.$$

By (4.10) with $r = v, j = v^*$,

$$\Lambda\left(\partial_{\hat{E}(v^*)}^{\hat{S}(v)}\right) \leq \Upsilon\left(\Lambda\left(\max\left\{\frac{1}{2}\partial_{v^*}^v, \frac{\partial_{\hat{E}(v^*)}^v \partial_{\hat{S}(v)}^{v^*}(v^*, \hat{S}(v))}{2[1 + \partial_{v^*}^v]}\right\}\right)\right).$$

This implies,

$$\begin{aligned}
\Lambda(\partial_{v^*}^v) &\leq \Upsilon \left(\Lambda \left(\max \left\{ \frac{1}{2} \partial_{v^*}^v, \frac{\partial_{v^*}^v \partial_{v^*}^v}{2[1 + \partial_{v^*}^v]} \right\} \right) \right) \\
&< \Upsilon \left(\Lambda \left(\max \left\{ \frac{1}{2} \partial_{v^*}^v, \frac{\partial_{v^*}^v}{2} \right\} \right) \right) \\
&= \Upsilon \left(\Lambda \left(\frac{1}{2} \partial_{v^*}^v \right) \right) < \Upsilon(\Lambda(\partial_{v^*}^v)) < \Lambda(\partial_{v^*}^v),
\end{aligned}$$

a contradiction. Then, $v = v^*$ and so \hat{S} and \hat{E} have a unique CFP. ■

In the following result we omit the continuity condition of \hat{S}, \hat{E} .

Theorem 4.1.4. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. Assum that:

- (a) $M \left((\hat{S}, \hat{E}); \hat{R}_{\mathfrak{S}} \right) \neq \emptyset$,
- (b) If \exists a continuous $\Upsilon \in \Omega$ and $\Lambda \in \Lambda_{\Delta}$ such that, for all $(r, j) \in \chi$, the inequality (4.1) holds,
- (c) condition (\star_3) holds,
- (d) condition (\star_1) holds.

Then \exists CFP of \hat{S} and \hat{E} in \mathfrak{S} .

Proof. By the analogous proof as in Theorem 4.1.2, we have obtained that $(r_n, r_{n+1}) \in \hat{R}_{\mathfrak{S}}$ and $r_n \rightarrow \hat{u}$ as $n \rightarrow \infty$, $\forall n \in \mathbb{N} \cup \{0\}$. As (\mathfrak{S}, ∂) is $\hat{R}_{\mathfrak{S}}$ -regular, so $(r_n, \hat{u}) \in \hat{R}_{\mathfrak{S}}$, $\forall n \in \mathbb{N}$. Let

$$\mathcal{A} = \left\{ n \in \mathbb{N} : \hat{S}(r_{2n}) = \hat{E}(\hat{u}) \text{ and } \hat{E}(r_{2n+1}) = \hat{S}(\hat{u}) \right\}.$$

Then we have two cases:

Case (1): If \mathcal{A} is finite, so $\exists n_0 \in \mathbb{N} \cup \{0\}$, $\hat{S}(r_{2n}) \neq \hat{E}(\hat{u})$ and $\hat{E}(r_{2n+1}) \neq \hat{S}(\hat{u})$, for each $n \geq n_0$. Since $r_{2n} \neq \hat{u}$ and $r_{2n+1} \neq \hat{u}$, so $\partial_{\hat{u}}^{r_{2n}} > 0$, $\partial_{\hat{u}}^{r_{2n+1}} > 0$ and $\partial_{\hat{E}(\hat{u})}^{\hat{S}(r_{2n})} > 0$, $\partial_{\hat{S}(\hat{u})}^{\hat{E}(r_{2n+1})} > 0$, for each $n \geq n_0$. Set $r = \hat{u}$ and $j = r_{2n+1}$ in (4.1), we obtain,

$$\Lambda \left(\partial_{\hat{E}(r_{2n+1})}^{\hat{S}(\hat{u})} \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \partial_{r_{2n+1}}^{\hat{u}}, \frac{\partial_{\hat{E}(r_{2n+1})}^{\hat{u}} \partial_{\hat{S}(\hat{u})}^{r_{2n+1}}}{1 + \partial_{r_{2n+1}}^{\hat{u}}} \right\} \right) \right).$$

This implies that

$$\begin{aligned}\Lambda\left(\partial_{r_{2n+2}}^{\hat{S}(\hat{u})}\right) &\leq \Upsilon\left(\Lambda\left(\max\left\{\partial_{r_{2n+1}}^{\hat{u}}, \frac{\partial_{r_{2n+2}}^{\hat{u}}\partial_{\hat{S}(\hat{u})}^{r_{2n+1}}}{1+\partial_{r_{2n+1}}^{\hat{u}}}\right\}\right)\right) \\ &< \Lambda\left(\max\left\{\partial_{r_{2n+1}}^{\hat{u}}, \frac{\partial_{r_{2n+2}}^{\hat{u}}\partial_{\hat{S}(\hat{u})}^{r_{2n+1}}}{1+\partial_{r_{2n+1}}^{\hat{u}}}\right\}\right).\end{aligned}$$

But a positive sequence $\{r_n\} = \left\{\max\left\{\partial_{r_{2n+1}}^{\hat{u}}, \frac{\partial_{r_{2n+2}}^{\hat{u}}\partial_{\hat{S}(\hat{u})}^{r_{2n+1}}}{1+\partial_{r_{2n+1}}^{\hat{u}}}\right\}\right\}$ and $\lim_{n \rightarrow \infty} r_n = 0$, From Lemma 1.6.4 and (Λ_2) , we obtain, $\lim_{n \rightarrow \infty} \Lambda(r_n) = 0$ and thus, $\lim_{n \rightarrow \infty} \Lambda\left(\partial_{r_{2n+2}}^{\hat{S}(\hat{u})}\right) = 0$. Again by (Λ_2) and Lemma 1.6.4, we obtain, $\lim_{n \rightarrow \infty} \partial_{r_{2n+2}}^{\hat{S}(\hat{u})} = 0$, $\lim_{n \rightarrow \infty} \partial_{r_{2n+2}}^{\hat{u}} = 0$. Hence,

$$\hat{S}(\hat{u}) = \hat{u}, \quad (4.11)$$

hence δ is the FP of \hat{S} . Similarly, Set $r = r_{2n}$ and $j = \hat{u}$ in (4.1), we obtain, $\lim_{n \rightarrow \infty} \Lambda\left(\partial_{\hat{E}(\hat{u})}^{r_{2n+1}}\right) = 0$. By (Λ_2) and Lemma 1.6.4, we get, $\lim_{n \rightarrow \infty} \partial_{\hat{E}(\hat{u})}^{r_{2n+1}} = 0$. Also, $\lim_{n \rightarrow \infty} \Lambda\left(\partial_{\hat{u}}^{r_{2n+1}}\right) = 0$, so,

$$\hat{E}(\hat{u}) = \hat{u}. \quad (4.12)$$

By (4.11) and (4.12), we get that

$$\hat{S}(\hat{u}) = \hat{E}(\hat{u}) = \hat{u}.$$

Case (2): If \mathcal{A} is infinite, so $\exists \{r_{2n(k)}\}$ of $\{r_n\}$ with $r_{2n(k)+1} = \hat{S}(r_{2n(k)}) = \hat{E}(\hat{u})$ such that $r_{2n(k)+2} = \hat{E}(r_{2n(k)+1}) = \hat{S}(\hat{u})$ for each $k \in \mathbb{N}$. But, $\lim_{n \rightarrow \infty} \partial_{\hat{u}}^{r_{2n(k)+1}} = 0$ and $\lim_{n \rightarrow \infty} \partial_{\hat{u}}^{r_{2n(k)+2}} = 0$. Then

$$\hat{S}(\hat{u}) = \hat{E}(\hat{u}) = \hat{u}.$$

Therefore, in cases (1) and (2), \hat{u} is a CFP of \hat{S} and \hat{E} .

Theorem 4.1.5. Let $\hat{R}_{\mathfrak{S}}$ be a TR on \mathfrak{S} and $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that ■

- (a) $M\left(\left(\hat{S}, \hat{E}\right); \hat{R}_{\mathfrak{S}}\right) \neq \emptyset$ and $\Gamma(r, j, \hat{R}_{\mathfrak{S}}) \neq \emptyset$,

(b) if \exists a continuous $\Upsilon \in \Omega$ and a function $\Lambda \in \Lambda_\Delta$ such that $\forall (r, j) \in \chi$,

$$\Lambda(\partial(\hat{S}(r), \hat{E}(j))) \leq \Upsilon \left(\Lambda \left(\max \left\{ \frac{1}{2} \partial(r, j), \frac{\partial(r, \hat{E}(j)) \partial(j, \hat{S}(r))}{2[1 + \partial(r, j)]} \right\} \right) \right),$$

(c) condition (\star_3) holds,

(d) condition (\star_1) holds.

Then \exists only one CFP of \hat{S} and \hat{E} .

Proof. The proof is the same as the proof of Theorems 4.1.4 and 4.1.3.

■

Corollary 4.1.6. Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume:

(a) $M(\hat{S}; \hat{R}_{\mathfrak{S}}) \neq \emptyset$,

(b) condition (\star_2) holds,

(c) either condition (\star_5) holds, or condition (\star_3) holds,

(d) \exists a continuous $\Upsilon \in \Omega$ and a function $\Lambda \in \Lambda_\Delta$ such that $\forall (r, j) \in \chi$,

$$\Lambda(\partial_{\hat{S}(j)}^{\hat{S}(r)}) \leq \Upsilon \left(\Lambda \left(\max \left\{ \partial_j^r, \frac{\partial_{\hat{S}(j)}^r \partial_{\hat{S}(r)}^j}{1 + \partial_j^r} \right\} \right) \right).$$

Then \hat{S} has a FP.

Proof. Set $\hat{S} = \hat{E}$ in Theorem 4.1.2 and Theorem 4.1.4. ■

Corollary 4.1.7. Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ and $\hat{R}_{\mathfrak{S}}$ be a transitive relation on \mathfrak{S} . Assume that:

(a) $M(\hat{S}; \hat{R}_{\mathfrak{S}}) \neq \emptyset$ and $\Gamma((r, j), \hat{R}_{\mathfrak{S}}) \neq \emptyset$,

(b) condition (\star_2) holds,

(c) either condition (\star_5) holds, or condition (\star_3) holds,

if \exists a continuous $\Upsilon \in \Omega$ and a function $\Lambda \in \Lambda_\Delta$ such that $\forall (r, j) \in \chi$, we have

$$\Lambda \left(\partial_{\hat{S}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \frac{1}{2} \partial_j^r, \frac{\partial_{\hat{S}(j)}^r \partial_{\hat{S}(r)}^j}{2 [1 + \partial_j^r]} \right\} \right) \right).$$

Then \hat{S} has a unique FP.

Proof. Set $\hat{S} = \hat{E}$ in Theorems 4.1.3 and 4.1.5. ■

Example 4.1.8. Consider $\mathfrak{S} = [0, 2]$. Let $\partial : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ be defined by $\partial(r, j) = |r - j|$, for each $r, j \in \mathfrak{S}$. Hence (\mathfrak{S}, ∂) is a CMS. Consider $\Lambda, \Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as $\Lambda(\check{a}) = \check{a}e^{\check{a}}$, $\Upsilon(\check{a}) = \frac{\check{a}}{2}$, $\check{a} > 0$. then, $\Lambda \in \Lambda_\Delta$ and $\Upsilon \in \Omega$ is continuous. Define,

$$\hat{R}_\mathfrak{S} = \left\{ \begin{array}{l} (0, 0), (0, \frac{1}{5}), (\frac{1}{5}, 0), (0, 1), \\ (1, 0), (1, 1), (2, 0) \end{array} \right\},$$

on \mathfrak{S} . Define $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ by,

$$\hat{S}(t) = \begin{cases} t & 0 \leq t \leq \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} < t \leq 2 \end{cases}, \hat{E}(t) = 0, \text{ for each } t \in \mathfrak{S}.$$

Obviously, $\hat{R}_\mathfrak{S}$ is (\hat{S}, \hat{E}) -closed and, \hat{S}, \hat{E} are continuous. Let

$$\chi = \left\{ (r, j) \in \hat{R}_\mathfrak{S} : \left| \hat{S}(r) - \hat{E}(j) \right| > 0 \right\},$$

then

$$\chi = \{(1, 0), (1, 1), (2, 0)\}.$$

Now, for each $(r, j) \in \chi$,

$$\Lambda \left(\partial_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \partial_j^r, \frac{\partial_{\hat{E}(j)}^r \partial_{\hat{S}(r)}^j}{1 + \partial_j^r} \right\} \right) \right).$$

Thus from Theorem 4.1.2, \hat{S} and \hat{E} have a CFP in \mathfrak{S} .

4.2 Some consequences

Next results generalize some recent FP theorems in the literature.

Corollary 4.2.1. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that

- (a) $M\left(\left(\hat{S}, \hat{E}\right); \hat{R}_{\mathfrak{S}}\right) \neq \emptyset$,
- (b) condition (\star_1) holds,
- (c) condition (\star_4) holds,
- (d) If $\exists \theta \in \Xi$ and $k \in (0, 1)$ such that, $\forall (r, j) \in \chi$,

$$\theta\left(\partial_{\hat{E}(j)}^{\hat{S}(r)}\right) \leq \left[\theta \left(\max \left\{ \partial_j^r, \frac{\partial_{\hat{E}(j)}^r \partial_{\hat{S}(r)}^j}{1 + \partial_j^r} \right\} \right) \right]^k.$$

Then \hat{S} and \hat{E} have a CFP.

Proof. Set $\Upsilon(\check{a}) := k\check{a}$ and $\Lambda(\check{a}) = \ln(\theta(\check{a}))$, in Theorem 4.1.2.

Corollary 4.2.2. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that ■

- (a) $M\left(\left(\hat{S}, \hat{E}\right); \hat{R}_{\mathfrak{S}}\right) \neq \emptyset$,
- (b) condition (\star_1) holds,
- (c) condition (\star_4) holds,
- (d) If $\exists F \in F^*$ and $\vartheta \in \mathbb{R}^+$ such that, $\forall (r, j) \in \chi$,

$$\vartheta + F\left(\partial_{\hat{E}(j)}^{\hat{S}(r)}\right) \leq F\left(\max \left\{ \partial_j^r, \frac{\partial_{\hat{E}(j)}^r \partial_{\hat{S}(r)}^j}{1 + \partial_j^r} \right\}\right).$$

Then \hat{S} and \hat{E} have a CFP.

Proof. Set $\Upsilon(\check{a}) = e^{-\vartheta}\check{a}$ and $\Lambda(\check{a}) = e^{F(\check{a})}$ in Theorem 4.1.2. ■

Let $\bar{\vartheta} : \mathbb{R}_0^+ \rightarrow [0, 1)$ be such that $\lim_{r \rightarrow t^+} \bar{\vartheta}(r) < 1, \forall t \in \mathbb{R}_0^+$.

Corollary 4.2.3. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that

- (a) $M\left(\left(\hat{S}, \hat{E}\right); \hat{R}_{\mathfrak{S}}\right) \neq \emptyset$,

- (b) condition (\star_1) holds,
- (c) condition (\star_4) holds,
- (d) If \exists a function $\bar{\delta}$ such that, $\forall (r, j) \in \chi$,

$$\partial_{\hat{E}(j)}^{\hat{S}(r)} \leq \bar{\delta} \left(\max \left\{ \partial_j^r, \frac{\partial_{\hat{E}(j)}^r \partial_j^{\hat{S}(r)}}{1 + \partial_j^r} \right\} \right) \cdot \max \left\{ \partial_j^r, \frac{\partial_{\hat{E}(j)}^r \partial_j^{\hat{S}(r)}}{1 + \partial_j^r} \right\}.$$

Then \hat{S} and \hat{E} have a CFP.

Proof. Set $\Upsilon(\check{\alpha}) := \bar{\delta}(\check{\alpha})\check{\alpha}$ and $\Lambda(\check{\alpha}) = \check{\alpha}$ in Theorem 4.1.2. ■

4.3 Applications to nonlinear matrix equations

In this section, we shall apply the obtained results in the section 4.1 to ensure the existence of a solution of nonlinear matrix equations (NMEs). Let us denote,

$$\begin{aligned} C_m(\tilde{n}) &:= \text{the class of } \tilde{n} \times \tilde{n} \text{ complex matrices } (M_s); \\ H_m(\tilde{n}) &:= \text{the class of } \tilde{n} \times \tilde{n} \text{ Hermitian } M_s; \\ P_m(\tilde{n}) &:= \text{the class of } \tilde{n} \times \tilde{n} \text{ positive definite } M_s; \\ H_m^+(\tilde{n}) &:= \text{the class of } \tilde{n} \times \tilde{n} \text{ positive semi-definite } M_s. \end{aligned}$$

Here, $P_m(\tilde{n}) \subseteq H_m(\tilde{n}) \subseteq C_m(\tilde{n})$, $H_m^+(\tilde{n}) \subseteq H_m(\tilde{n})$, $\Omega_1 \succ 0$ and $\Omega_1 \succeq 0$ means that $\Omega_1 \in P(\tilde{n})$ and $\Omega_1 \in H_m^+(\tilde{n})$, respectively. For $\Omega_1 - \Omega_2 \succeq 0$ and $\Omega_1 - \Omega_2 \succ 0$, we shall use $\Omega_1 \succeq \Omega_2$ and $\Omega_1 \succ \Omega_2$, resp.

We consider the following NME:

$$\hat{W} = D + \sum_{i=1}^n V_i^* \hat{W} V_i - \sum_{i=1}^n Z_i^* \hat{W} Z_i, \quad (4.13)$$

where, $D \in P_m(\tilde{n})$, V_i, Z_i are arbitrary $\tilde{n} \times \tilde{n}$ matrices.

Define the trace norm $\|\cdot\|_{tr}$ by $\|O\|_{tr} = \sum_{i=1}^n \theta_i(O)$, where $\theta_i(E)$, $i = 1, 2, \dots, n$, are the singularvalues of $O \in C_m(\tilde{n})$. $(H_m(\tilde{n}), \|\cdot\|_{tr})$ is a CMS (see [42],[106],[43]). $H_m(\tilde{n})$ is a poset with partial order \preceq , where $O_1 \preceq O_2 \Leftrightarrow O_2 \preceq O_1$.

Lemma 4.3.1. [106] If $O_1, O_2 \succeq 0$ are $\tilde{n} \times \tilde{n}$ matrices, then

$$0 \leq \text{tr}(O_1 O_2) = \|O_2\| \text{tr}(O_1).$$

Lemma 4.3.2. [101] If $O \in C_m(\tilde{n})$ with $O \prec I_n$, then $\|O\| < 1$.

Define the operator $\hat{E} : H_m(\tilde{n}) \rightarrow H_m(\tilde{n})$ by

$$\hat{E}(\aleph) = \frac{1}{2} \left(\hat{E}_1(\aleph) + \hat{E}_2(\aleph) \right),$$

where, $\hat{E}_1, \hat{E}_2 : H_m(\tilde{n}) \rightarrow H_m(\tilde{n})$ are defined as

$$\hat{E}_1(\aleph) = D + 2 \sum_{i=1}^n V_i^* \aleph V_i \text{ and } \hat{E}_2(\aleph) = D - 2 \sum_{i=1}^n Z_i^* \aleph Z_i. \quad (4.14)$$

Then the solutions of the equation (4.13) are the FPs of \hat{E} , which are the CFPs of \hat{E}_1 and \hat{E}_2 .

Theorem 4.3.3. Assume that:

(h1) there exist positive reals δ_1 and δ_2 such that $\sum_{i=1}^n V_i V_i^* \prec \delta_1 I_n$ and $\sum_{i=1}^n Z_i Z_i^* \prec \delta_2 I_n$;

(h2) for each $O_1, O_2 \in H_m(p)$ such that $(O_1, O_2) \in \preceq$,

$$\|O_1\|_{tr} + \|O_2\|_{tr} \leq \frac{1}{2\delta} \frac{U_1(O_1, O_2)}{U_1(O_1, O_2) + 1},$$

where

$$\delta = \max \{ \delta_1, \delta_2 \}, \quad U_1(O_1, O_2) = \max \left\{ \begin{array}{l} \|O_1 - O_2\|_{tr}, \\ \frac{\|O_1 - \hat{E}_2(O_2)\|_{tr} \|O_2 - \hat{E}_1(O_1)\|_{tr}}{1 + \|O_1 - O_2\|_{tr}} \end{array} \right\}.$$

Then the NME (4.13) has a solution in $H_m(\tilde{n})$.

Proof. As \hat{E}_1 and \hat{E}_2 are well defined and

$$(O_1, O_2) \in \preceq \Rightarrow \left(\hat{E}_1(O_1), \hat{E}_2(O_2) \right), \left(\hat{E}_2(O_1), \hat{E}_1(O_2) \right) \in \preceq,$$

so, \preceq on $H_m(\tilde{n})$ is (\hat{E}_1, \hat{E}_2) -closed. Now, we prove that \hat{E}_1 and \hat{E}_2 satisfy (4.1). Consider

$$\begin{aligned}
\left\| \hat{E}_1(O_1) - \hat{E}_2(O_2) \right\|_{tr} &= tr \left(\hat{E}_1(O_1) - \hat{E}_2(O_2) \right) = \\
&= 2tr \left(\sum_{i=1}^n (V_i^* O_1 V_i + Z_i^* O_2 Z_i) \right) \\
&= 2 \sum_{i=1}^n tr (V_i^* O_1 V_i + Z_i^* O_2 Z_i) \\
&= 2 \left(\sum_{i=1}^n tr (V_i V_i^* O_1) + \sum_{i=1}^n tr (Z_i Z_i^* O_2) \right) \\
&= 2 \left(tr \left(\sum_{i=1}^n V_i V_i^* O_1 \right) + tr \left(\sum_{i=1}^n Z_i Z_i^* O_2 \right) \right) \\
&\leq 2 \left(\left\| \sum_{i=1}^n V_i V_i^* \right\| \|O_1\|_{tr} + \left\| \sum_{i=1}^n Z_i Z_i^* \right\| \|O_2\|_{tr} \right) \\
&\leq 2 (\delta_1 \|O_1\|_{tr} + \delta_2 \|O_2\|_{tr}) \\
&\leq 2\delta \|O_1\|_{tr} + \|O_2\|_{tr}.
\end{aligned}$$

From conditions(h1) and (h2),

$$\left\| \hat{E}_1(O_1) - \hat{E}_2(O_2) \right\|_{tr} \leq \frac{U_1(O_1, O_2)}{U_1(O_1, O_2) + 1}.$$

Let $\Lambda, \Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined by $\Lambda(\check{a}) = \check{a}$, $\Upsilon(\check{a}) = \frac{\check{a}}{\check{a}+1}$. Hence,

$$\Lambda \left(\left\| \hat{E}_1(O_1) - \hat{E}_2(O_2) \right\|_{tr} \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \frac{\|O_1 - O_2\|_{tr},}{\frac{\|O_1 - \hat{E}_2(O_2)\|_{tr} \|O_2 - \hat{E}_1(O_1)\|_{tr}}{1 + \|O_1 - O_2\|_{tr}}} \right\} \right) \right).$$

Consequently,

$$\Lambda \left(\partial_{\hat{E}_2(O_2)}^{\hat{E}_1(O_1)} \right) \leq \Upsilon \left(\Lambda \left(\max \left\{ \partial_{O_2}^{O_1}, \frac{\partial_{\hat{E}_2(O_2)}^{O_1} \partial_{\hat{E}_1(O_1)}^{O_2}}{1 + \partial_{O_2}^{O_1}} \right\} \right) \right).$$

Thus, all the hypotheses of Theorem 4.1.2 hold. Then \hat{E}_1 and \hat{E}_2 have a CFP of \hat{E}_1 and \hat{E}_2 , (say O^*). Thus, \hat{E} has a FP and hence the equations (4.13) has a solution. ■

Theorem 4.3.4. Under the condition (h1) of Theorem 4.3.3, the equation (4.13) has only

one solution if $\hat{R}_{\mathfrak{S}}$ is TR and for each $O_1, O_2 \in H_m(p)$ such that $(O_1, O_2) \in \preceq$,

$$\|O_1\|_{tr} + \|O_2\|_{tr} \leq \frac{1}{2\delta} \frac{U_2(O_1, O_2)}{U_2(O_1, O_2) + 1},$$

where

$$\delta = \max\{\delta_1, \delta_2\}, \quad U_2(O_1, O_2) = \max \left\{ \begin{array}{l} \frac{1}{2} \|O_1 - O_2\|_{tr}, \\ \frac{\|O_1 - \hat{E}_2(O_2)\|_{tr} \|O_2 - \hat{E}_1(O_1)\|_{tr}}{2[1 + \|O_1 - O_2\|_{tr}]} \end{array} \right\}.$$

Proof. Using Theorem 4.1.3. and the proof is similar to the proof of Theorem 4.3.3. ■

4.4 Fuzzy fixed points of fuzzy mappings via generalized almost (Υ, Λ) -contractions

Here, we study the existence of common \check{e} -fuzzy FPs of fuzzy maps satisfying generalized almost (Υ, Λ) -contractions in CMS. (\mathfrak{S}, ∂) is assumed to be a CMS.

Theorem 4.4.1. Let $\hat{g}, Q : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ be FM and, $\forall \check{r} \in \mathfrak{S}$, $\exists \check{e}_{\hat{g}}(\check{r}), \check{e}_Q(\check{r}) \in (0, 1]$ such that $[\hat{g}(\check{r})]_{\check{e}_{\hat{g}}(\check{r})}$ and $[Q(\hat{J})]_{\check{e}_Q(\hat{J})}$ are non-empty, bounded and closed subsets (NBCS) of \mathfrak{S} . Assume that there exist $\Upsilon \in \Omega$, $\Lambda \in \Lambda_{\Delta}$ and $\varrho \geq 0$ such that, for every $\check{r}, \hat{J} \in \mathfrak{S}$, $\hat{H}_{[Q(\hat{J})]_{\check{e}_Q(\hat{J})}}^{[\hat{g}(\check{r})]_{\check{e}_{\hat{g}}(\check{r})}} > 0$, implies,

$$\Lambda \left(\hat{H}_{[Q(\hat{J})]_{\check{e}_Q(\hat{J})}}^{[\hat{g}(\check{r})]_{\check{e}_{\hat{g}}(\check{r})}} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_9(\check{r}, \hat{J}) \right) \right) + \varrho \aleph_1(\check{r}, \hat{J}), \quad (4.15)$$

where

$$\mathcal{M}_9(\check{r}, \hat{J}) = \max \left\{ \partial_{\hat{J}}^{\check{r}}, \partial_{[\hat{g}(\check{r})]_{\check{e}_{\hat{g}}(\check{r})}}^{\check{r}}, \partial_{[Q(\hat{J})]_{\check{e}_Q(\hat{J})}}^{\hat{J}}, \frac{\partial_{[Q(\hat{J})]_{\check{e}_Q(\hat{J})}}^{\check{r}} + \partial_{[\hat{g}(\check{r})]_{\check{e}_{\hat{g}}(\check{r})}}^{\hat{J}}}{2} \right\}, \quad (4.16)$$

and

$$\aleph_1(\check{r}, \hat{J}) = \min \left\{ \partial_{[\hat{g}(\check{r})]_{\check{e}_{\hat{g}}(\check{r})}}^{\check{r}}, \partial_{[Q(\hat{J})]_{\check{e}_Q(\hat{J})}}^{\hat{J}}, \partial_{[Q(\hat{J})]_{\check{e}_Q(\hat{J})}}^{\check{r}}, \partial_{[\hat{g}(\check{r})]_{\check{e}_{\hat{g}}(\check{r})}}^{\hat{J}} \right\}. \quad (4.17)$$

If Υ is continuous, then \exists some $\check{z} \in \mathfrak{S}$ such that $\check{z} \in [\hat{g}(\check{z})]_{\check{e}_{\hat{g}}(\check{z})} \cap [Q(\check{z})]_{\check{e}_Q(\check{z})}$.

Proof. Let $\check{r}_0 \in \mathfrak{S}$, then there exists $\check{e}_{\hat{g}}(\check{r}_0) \in (0, 1]$ such that $[\hat{g}(\check{r}_0)]_{\check{e}_{\hat{g}}(\check{r}_0)}$ is a NBCS of \mathfrak{S} . We denote $\check{e}_{\hat{g}}(\check{r}_0)$ by \check{e}_1 . Let $\check{r}_1 \in [\hat{g}(\check{r}_0)]_{\check{e}_1}$, then $\exists \check{e}_Q(\check{r}_1) \in (0, 1]$ such that $[Q(\check{r}_1)]_{\check{e}_Q(\check{r}_1)}$ is

a NBCS of \mathfrak{S} . By (Λ_1) , (4.15) and Lemma 1.1.11,

$$\Lambda \left(\partial_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{\check{r}_1} \right) \leq \Lambda \left(\hat{H}_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{[\hat{g}(\check{r}_0)]_{\varepsilon_{\hat{g}}(\check{r}_0)}} \right) \leq \Upsilon \left(\Lambda (\mathcal{M}_9(\check{r}_0, \check{r}_1)) \right) + \varrho \aleph_1(\check{r}_0, \check{r}_1), \quad (4.18)$$

where

$$\mathcal{M}_9(\check{r}_0, \check{r}_1) = \max \left\{ \partial_{\check{r}_1}^{\check{r}_0}, \partial_{[\hat{g}(\check{r}_0)]_{\varepsilon_{\hat{g}}(\check{r}_0)}}^{\check{r}_0}, \partial_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{\check{r}_1}, \frac{\partial_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{\check{r}_0} + \partial_{[\hat{g}(\check{r}_0)]_{\varepsilon_{\hat{g}}(\check{r}_0)}}^{\check{r}_1}}{2} \right\},$$

and

$$\aleph_1(\check{r}_0, \check{r}_1) = \min \left\{ \partial_{[\hat{g}(\check{r}_0)]_{\varepsilon_{\hat{g}}(\check{r}_0)}}^{\check{r}_0}, \partial_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{\check{r}_1}, \partial_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{\check{r}_0}, \partial_{[\hat{g}(\check{r}_0)]_{\varepsilon_{\hat{g}}(\check{r}_0)}}^{\check{r}_1} \right\}.$$

By (Λ_3) , we have,

$$\Lambda \left(\partial_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{\check{r}_1} \right) = \inf_{j \in [Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}} \Lambda \left(\partial_j^{\check{r}_1} \right).$$

Thus, there exists $\check{r}_2 \in [Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}$ such that

$$\Lambda \left(\partial_{[Q(\check{r}_1)]_{\varepsilon_Q(\check{r}_1)}}^{\check{r}_1} \right) = \Lambda \left(\partial_{\check{r}_2}^{\check{r}_1} \right)$$

Then from (4.18), we have

$$\Lambda \left(\partial_{\check{r}_2}^{\check{r}_1} \right) \leq \Upsilon \left(\Lambda (\mathcal{M}_9(\check{r}_0, \check{r}_1)) \right) + \varrho \aleph_1(\check{r}_0, \check{r}_1), \quad (4.19)$$

where

$$\mathcal{M}_9(\check{r}_0, \check{r}_1) = \max \left\{ \partial_{\check{r}_1}^{\check{r}_0}, \partial_{\check{r}_1}^{\check{r}_0}, \partial_{\check{r}_2}^{\check{r}_1}, \frac{\partial_{\check{r}_2}^{\check{r}_0} + \partial_{\check{r}_1}^{\check{r}_1}}{2} \right\} = \max \left\{ \partial_{\check{r}_1}^{\check{r}_0}, \partial_{\check{r}_2}^{\check{r}_1} \right\},$$

and

$$\aleph_1(\check{r}_0, \check{r}_1) = \min \left\{ \partial_{\check{r}_1}^{\check{r}_0}, \partial_{\check{r}_2}^{\check{r}_1}, \partial_{\check{r}_2}^{\check{r}_0}, \partial_{\check{r}_1}^{\check{r}_1} \right\} = 0.$$

If $\max \left\{ \partial_{\check{r}_1}^{\check{r}_0}, \partial_{\check{r}_2}^{\check{r}_1} \right\} = \partial_{\check{r}_2}^{\check{r}_1}$, then from (4.19), we have

$$\Lambda \left(\partial_{\check{r}_2}^{\check{r}_1} \right) \leq \Upsilon \left(\Lambda \left(\partial_{\check{r}_2}^{\check{r}_1} \right) \right) < \Lambda \left(\partial_{\check{r}_2}^{\check{r}_1} \right),$$

which is a contradiction. Thus, $\max \{ \partial_{\check{r}_1}^{\check{r}_0}, \partial_{\check{r}_2}^{\check{r}_1} \} = \partial_{\check{r}_1}^{\check{r}_0}$. By (4.19), we get that

$$\Lambda (\partial_{\check{r}_2}^{\check{r}_1}) \leq \Upsilon (\Lambda (\partial_{\check{r}_1}^{\check{r}_0})).$$

Next, $\exists \check{e}_{\hat{g}}(\check{r}_2) \in (0, 1]$ such that $[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}$ is a NBCS of \mathfrak{S} . From (Λ_1) , (4.15) and Lemma 1.1.11, we have

$$\Lambda \left(\partial_{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}}^{\check{r}_2} \right) \leq \Lambda \left(\hat{H}_{[Q(\check{r}_1)]_{\check{e}_Q(\check{r}_1)}}^{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}} \right) \leq \Upsilon (\Lambda (\mathcal{M}_9(\check{r}_1, \check{r}_2))) + \varrho \aleph_1(\check{r}_1, \check{r}_2), \quad (4.20)$$

where

$$\mathcal{M}_9(\check{r}_1, \check{r}_2) = \max \left\{ \partial_{\check{r}_2}^{\check{r}_1}, \partial_{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}}^{\check{r}_2}, \partial_{[Q(\check{r}_1)]_{\check{e}_Q(\check{r}_1)}}^{\check{r}_1}, \frac{\partial_{[Q(\check{r}_1)]_{\check{e}_Q(\check{r}_1)}}^{\check{r}_2} + \partial_{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}}^{\check{r}_1}}{2} \right\},$$

and

$$\aleph_1(\check{r}_1, \check{r}_2) = \min \left\{ \partial_{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}}^{\check{r}_2}, \partial_{[Q(\check{r}_1)]_{\check{e}_Q(\check{r}_1)}}^{\check{r}_1}, \partial_{[Q(\check{r}_1)]_{\check{e}_Q(\check{r}_1)}}^{\check{r}_2}, \partial_{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}}^{\check{r}_1} \right\}.$$

From (Λ_3) , we have,

$$\Lambda \left(\partial_{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}}^{\check{r}_2} \right) = \inf_{J_1 \in [\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}} \Lambda \left(\partial_{J_1}^{\check{r}_2} \right).$$

Then, there exists $\check{r}_3 \in [\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}$ such that

$$\Lambda \left(\partial_{[\hat{g}(\check{r}_2)]_{\check{e}_{\hat{g}}(\check{r}_2)}}^{\check{r}_2} \right) = \Lambda \left(\partial_{\check{r}_3}^{\check{r}_2} \right)$$

From (4.20), we get

$$\Lambda \left(\partial_{\check{r}_3}^{\check{r}_2} \right) \leq \Upsilon (\Lambda (\mathcal{M}_9(\check{r}_1, \check{r}_2))) + \varrho \aleph_1(\check{r}_1, \check{r}_2), \quad (4.21)$$

where

$$\mathcal{M}_9(\check{r}_1, \check{r}_2) = \max \left\{ \partial_{\check{r}_2}^{\check{r}_1}, \partial_{\check{r}_3}^{\check{r}_2}, \partial_{\check{r}_2}^{\check{r}_1}, \frac{\partial_{\check{r}_3}^{\check{r}_1} + \partial_{\check{r}_2}^{\check{r}_2}}{2} \right\} = \max \{ \partial_{\check{r}_2}^{\check{r}_1}, \partial_{\check{r}_3}^{\check{r}_2} \},$$

and

$$\aleph_1(\check{r}_1, \check{r}_2) = \min \{ \partial_{\check{r}_3}^{\check{r}_2}, \partial_{\check{r}_2}^{\check{r}_1}, \partial_{\check{r}_2}^{\check{r}_2}, \partial_{\check{r}_3}^{\check{r}_1} \} = 0.$$

If $\max \{ \partial_{\check{r}_2}^{\check{r}_1}, \partial_{\check{r}_3}^{\check{r}_2} \} = \partial_{\check{r}_3}^{\check{r}_2}$, then from (4.21), we have

$$\Lambda \left(\partial_{\check{r}_3}^{\check{r}_2} \right) \leq \Upsilon \left(\Lambda \left(\partial_{\check{r}_3}^{\check{r}_2} \right) \right) < \Lambda \left(\partial_{\check{r}_3}^{\check{r}_2} \right),$$

which is a contradiction. Thus, $\max \{ \partial_{\check{r}_2}^{\check{r}_1}, \partial_{\check{r}_3}^{\check{r}_2} \} = \partial_{\check{r}_2}^{\check{r}_1}$. By (4.21), we get that

$$\Lambda \left(\partial_{\check{r}_3}^{\check{r}_2} \right) \leq \Upsilon \left(\Lambda \left(\partial_{\check{r}_2}^{\check{r}_1} \right) \right). \quad (4.22)$$

Continuing in the same way, we get $\{ \check{r}_n \} \subset \mathfrak{S}$ such that $\check{r}_{2n+1} \in [\hat{g}(\check{r}_{2n})]_{\check{g}(\check{r}_{2n})}$, $\check{r}_{2n+2} \in [Q(\check{r}_{2n+1})]_{\check{g}_Q(\check{r}_{2n+1})}$, $\forall n \in \mathbb{N}$,

$$\Lambda \left(\partial_{\check{r}_{2n+2}}^{\check{r}_{2n+1}} \right) \leq \Upsilon \left(\Lambda \left(\partial_{\check{r}_{2n+1}}^{\check{r}_{2n}} \right) \right) \text{ and } \Lambda \left(\partial_{\check{r}_{2n+3}}^{\check{r}_{2n+2}} \right) \leq \Upsilon \left(\Lambda \left(\partial_{\check{r}_{2n+2}}^{\check{r}_{2n+1}} \right) \right). \quad (4.23)$$

Thus, from (4.23), we have

$$\Lambda \left(\partial_{\check{r}_{n+1}}^{\check{r}_n} \right) \leq \Upsilon \left(\Lambda \left(\partial_{\check{r}_n}^{\check{r}_{n-1}} \right) \right), \text{ for each } n \in \mathbb{N}. \quad (4.24)$$

which implies that

$$\Lambda \left(\partial_{\check{r}_{n+1}}^{\check{r}_n} \right) \leq \Upsilon \left(\Lambda \left(\partial_{\check{r}_n}^{\check{r}_{n-1}} \right) \right) \leq \Upsilon^2 \left(\Lambda \left(\partial_{\check{r}_{n-1}}^{\check{r}_{n-2}} \right) \right) \leq \dots \leq \Upsilon^n \left(\Lambda \left(\partial_{\check{r}_1}^{\check{r}_0} \right) \right).$$

Letting $n \rightarrow \infty$,

$$0 \leq \lim_{n \rightarrow \infty} \Lambda \left(\partial_{\check{r}_{n+1}}^{\check{r}_n} \right) \leq \lim_{n \rightarrow \infty} \Upsilon^n \left(\Lambda \left(\partial_{\check{r}_1}^{\check{r}_0} \right) \right) = 0,$$

Then,

$$\lim_{n \rightarrow \infty} \Lambda \left(\partial_{\check{r}_{n+1}}^{\check{r}_n} \right) = 0.$$

From (Λ_2) and Lemma 1.6.4, we obtain,

$$\lim_{n \rightarrow \infty} \partial_{\check{r}_{n+1}}^{\check{r}_n} = 0. \quad (4.25)$$

To prove $\{ \check{r}_n \}$ is a CS, we suppose that $\exists \varepsilon > 0$, sequences $\{ p_n \}_{n=1}^{\infty}$ and $\{ q_n \}_{n=1}^{\infty}$ of integers

suchthat $\forall n \in \mathbb{N}$,

$$p_n > q_n > n, \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)}} \geq \varepsilon, \text{ and}$$

$$\partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)-1}} < \varepsilon. \quad (4.26)$$

Thus,

$$\varepsilon \leq \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)}} \leq \partial_{\tilde{r}_{p(n)-1}}^{\tilde{r}_{p(n)}} + \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)-1}} \leq \partial_{\tilde{r}_{p(n)-1}}^{\tilde{r}_{p(n)}} + \varepsilon \quad (4.27)$$

Letting $n \rightarrow \infty$ in (4.27), we obtain

$$\lim_{n \rightarrow \infty} \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)}} = \varepsilon. \quad (4.28)$$

Again,

$$\partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)}} \leq \partial_{\tilde{r}_{p(n)+1}}^{\tilde{r}_{p(n)}} + \partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{p(n)+1}} + \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{q(n)+1}} \quad (4.29)$$

and

$$\partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{p(n)+1}} \leq \partial_{\tilde{r}_{p(n)}}^{\tilde{r}_{p(n)+1}} + \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)}} + \partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{q(n)}}. \quad (4.30)$$

Taking $n \rightarrow \infty$ in (4.29) and (4.30), we get

$$\lim_{n \rightarrow \infty} \partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{p(n)+1}} = \varepsilon. \quad (4.31)$$

From (4.15), we get

$$\Lambda \left(\partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{p(n)+1}} \right) \leq \Lambda \left(\hat{H} \begin{matrix} [\hat{g}(\tilde{r})]_{p(n)} \\ [Q(\tilde{r})]_{q(n)} \end{matrix} \Big|_{\tilde{r}_{\hat{g}(\tilde{r}_{p(n)})}} \Big|_{\tilde{r}_{Q(\tilde{r}_{q(n)})}} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_9 \left(\tilde{r}_{p(n)}, \tilde{r}_{q(n)} \right) \right) \right) + \varrho \mathfrak{N}_1 \left(\tilde{r}_{p(n)}, \tilde{r}_{q(n)} \right), \quad (4.32)$$

where

$$\begin{aligned} \mathcal{M}_9 \left(\tilde{r}_{p(n)}, \tilde{r}_{q(n)} \right) &= \max \left\{ \begin{array}{l} \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)}}, \partial_{[\hat{g}(\tilde{r})]_{p(n)} \Big|_{\tilde{r}_{\hat{g}(\tilde{r}_{p(n)})}}^{\tilde{r}_{p(n)}}}, \partial_{[Q(\tilde{r})]_{q(n)} \Big|_{\tilde{r}_{Q(\tilde{r}_{q(n)})}}^{\tilde{r}_{q(n)}}}, \\ \frac{\partial_{[Q(\tilde{r})]_{q(n)} \Big|_{\tilde{r}_{Q(\tilde{r}_{q(n)})}}^{\tilde{r}_{p(n)}} + \partial_{[\hat{g}(\tilde{r})]_{p(n)} \Big|_{\tilde{r}_{\hat{g}(\tilde{r}_{p(n)})}}^{\tilde{r}_{q(n)}}}}{2} \end{array} \right\} \\ &\leq \max \left\{ \partial_{\tilde{r}_{q(n)}}^{\tilde{r}_{p(n)}}, \partial_{\tilde{r}_{p(n)+1}}^{\tilde{r}_{p(n)}}, \partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{q(n)}}, \frac{\partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{p(n)}} + \partial_{\tilde{r}_{p(n)+1}}^{\tilde{r}_{q(n)}}}{2} \right\}, \end{aligned}$$

and

$$\begin{aligned} \aleph_1(\tilde{r}_{p(n)}, \tilde{r}_{q(n)}) &= \min \left\{ \begin{array}{l} \partial_{[\hat{g}(\tilde{r})]_{p(n)}]_{\tilde{e}_{\hat{g}}(\tilde{r}_{p(n)})}}^{\tilde{r}_{p(n)}}, \partial_{[Q(\tilde{r})]_{q(n)}]_{\tilde{e}_Q(\tilde{r}_{q(n)})}}^{\tilde{r}_{q(n)}} \\ \partial_{[Q(\tilde{r})]_{q(n)}]_{\tilde{e}_Q(\tilde{r}_{q(n)})}}^{\tilde{r}_{p(n)}}, \partial_{[\hat{g}(\tilde{r})]_{p(n)}]_{\tilde{e}_{\hat{g}}(\tilde{r}_{p(n)})}}^{\tilde{r}_{q(n)}} \end{array} \right\} \\ &\leq \min \left\{ \partial_{\tilde{r}_{p(n)+1}}^{\tilde{r}_{p(n)}}, \partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{q(n)}}, \partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{p(n)}}, \partial_{\tilde{r}_{p(n)+1}}^{\tilde{r}_{q(n)}} \right\} = 0. \end{aligned}$$

Setting $n \rightarrow \infty$ in (4.32), by using continuity of Υ , (A₃), (4.24), (4.26), (4.28) and (4.31), we get

$$\Lambda(\varepsilon) = \lim_{n \rightarrow \infty} \Lambda \left(\partial_{\tilde{r}_{q(n)+1}}^{\tilde{r}_{p(n)+1}} \right) \leq \lim_{n \rightarrow \infty} \Upsilon \left(\Lambda \left(\mathcal{M}_9(\tilde{r}_{p(n)}, \tilde{r}_{q(n)}) \right) \right) + \varrho \cdot 0 = \Upsilon(\Lambda(\varepsilon)) < \Lambda(\varepsilon),$$

a contradiction. Thus, $\{\tilde{r}_n\}$ is CS. As \mathfrak{S} is complete, so there is $\tilde{z} \in \mathfrak{S}$ such that $\lim_{n \rightarrow \infty} \partial_{\tilde{z}}^{\tilde{r}_n} = 0$. To prove $\tilde{z} \in [Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}$, we assume that $\tilde{z} \notin [Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}$ (i.e., $\partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{z}} > 0$), then $\exists n_0 \in \mathbb{N}$ and a subsequence $\{\tilde{r}_{n_k}\}$ of $\{\tilde{r}_n\}$ such that $\partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{r}_{2n_k+1}} > 0$, for each $n_k \geq n_0$. As $\partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{r}_{2n_k+1}} > 0$, for each $n_k \geq n_0$, so from (4.15), Lemma 1.1.11 and (A₁), we have

$$\Lambda \left(\frac{\tilde{r}_{2n_k+1}}{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}} \right) \leq \Lambda \left(\hat{H}_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\hat{g}(\tilde{r}_{2n_k})} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_9(\tilde{r}_{2n_k}, \tilde{z}) \right) \right) + \varrho \aleph_1(\tilde{r}_{2n_k}, \tilde{z}), \quad (4.33)$$

where

$$\begin{aligned} \mathcal{M}_9(\tilde{r}_{2n_k}, \tilde{z}) &= \max \left\{ \begin{array}{l} \partial_{\tilde{z}}^{\tilde{r}_{2n_k}}, \partial_{[\hat{g}(\tilde{r}_{2n_k})]_{\tilde{e}_{\hat{g}}(\tilde{r}_{2n_k})}}^{\tilde{r}_{2n_k}}, \partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{z}} \\ \frac{\partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{r}_{2n_k}} + \partial_{[\hat{g}(\tilde{r}_{2n_k})]_{\tilde{e}_{\hat{g}}(\tilde{r}_{2n_k})}}^{\tilde{z}}}{2} \end{array} \right\} \\ &\leq \max \left\{ \partial_{\tilde{z}}^{\tilde{r}_{2n_k}}, \partial_{\tilde{r}_{2n_k+1}}^{\tilde{r}_{2n_k}}, \partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{z}}, \frac{\partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{r}_{2n_k}} + \partial_{\tilde{r}_{2n_k+1}}^{\tilde{z}}}{2} \right\}, \end{aligned}$$

and

$$\begin{aligned} \aleph_1(\tilde{r}_{2n_k}, \tilde{z}) &= \min \left\{ \partial_{[\hat{g}(\tilde{r}_{2n_k})]_{\alpha_{\hat{g}}(\tilde{r}_{2n_k})}}^{\tilde{r}_{2n_k}}, \partial_{[Q(\tilde{z})]_{\alpha_Q(\tilde{z})}}^{\tilde{z}}, \partial_{[Q(\tilde{z})]_{\alpha_Q(\tilde{z})}}^{\tilde{r}_{2n_k}}, \partial_{[\hat{g}(\tilde{r}_{2n_k})]_{\alpha_{\hat{g}}(\tilde{r}_{2n_k})}}^{\tilde{z}} \right\} \\ &\leq \min \left\{ \partial_{\tilde{r}_{2n_k+1}}^{\tilde{r}_{2n_k}}, \partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{z}}, \partial_{[Q(\tilde{z})]_{\tilde{e}_Q(\tilde{z})}}^{\tilde{r}_{2n_k}}, \partial_{\tilde{r}_{2n_k+1}}^{\tilde{z}} \right\}. \end{aligned}$$

Letting $k \rightarrow \infty$ and using the continuity of Λ and Υ , we have

$$\Lambda \left(\partial_{[Q(\tilde{z})]_{\tilde{\epsilon}_Q(\tilde{z})}}^{\tilde{z}} \right) \leq \Upsilon \left(\Lambda \left(\partial_{[Q(\tilde{z})]_{\tilde{\epsilon}_Q(\tilde{z})}}^{\tilde{z}} \right) \right) + 0 < \Lambda \left(\partial_{[Q(\tilde{z})]_{\tilde{\epsilon}_Q(\tilde{z})}}^{\tilde{z}} \right),$$

a contradiction. Then, $\partial_{[Q(\tilde{z})]_{\tilde{\epsilon}_Q(\tilde{z})}}^{\tilde{z}} = 0$, and so $\tilde{z} \in [Q(\tilde{z})]_{\tilde{\epsilon}_Q(\tilde{z})}$. Similarly, we obtain $\tilde{z} \in [\hat{g}(\tilde{z})]_{\tilde{\epsilon}_{\hat{g}}(\tilde{z})}$. Hence, $\tilde{z} \in [\hat{S}(\tilde{z})]_{\tilde{\epsilon}_{\hat{S}}(\tilde{z})} \cap [\Gamma(\tilde{z})]_{\tilde{\epsilon}_{\Gamma}(\tilde{z})}$. ■

Example 4.4.2. Let $\mathfrak{S} = [0, 1]$ and $\partial : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}^+$ be defined by $\partial_{\hat{J}}^{\tilde{r}} = |\tilde{r} - \hat{J}|$, $\tilde{r}, \hat{J} \in \mathfrak{S}$. Consider $\Lambda, \Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as $\Lambda(t) = t$ and $\Upsilon(t) = \frac{99t}{100}$. Here, $\Lambda \in \Lambda_{\Delta}$ and $\Upsilon \in \Omega$ is continuous. For $\tilde{\epsilon} \in (0, 1]$, given $\hat{g}, Q : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ by

$$\hat{g}(\tilde{r})(t) = \begin{cases} \tilde{\epsilon}, & 0 \leq t \leq \frac{\tilde{r}}{60} \\ \frac{\tilde{\epsilon}}{2}, & \frac{\tilde{r}}{60} \leq t \leq \frac{\tilde{r}}{40} \\ \frac{\tilde{\epsilon}}{3}, & \frac{\tilde{r}}{40} \leq t \leq \frac{\tilde{r}}{20} \\ \frac{\tilde{\epsilon}}{5}, & \frac{\tilde{r}}{20} \leq t \leq 1 \end{cases},$$

and

$$Q(\tilde{r})(t) = \begin{cases} \tilde{\epsilon}, & 0 \leq t \leq \frac{\tilde{r}}{15} \\ \frac{\tilde{\epsilon}}{3}, & \frac{\tilde{r}}{15} \leq t \leq \frac{\tilde{r}}{10} \\ \frac{\tilde{\epsilon}}{4}, & \frac{\tilde{r}}{10} \leq t \leq \frac{\tilde{r}}{5} \\ \frac{\tilde{\epsilon}}{7}, & \frac{\tilde{r}}{5} \leq t \leq 1 \end{cases},$$

such that

$$[\hat{g}(\tilde{r})]_{\tilde{\epsilon}} = \left[0, \frac{\tilde{r}}{60} \right] \text{ and } [Q(\tilde{r})]_{\tilde{\epsilon}\alpha} = \left[0, \frac{\tilde{r}}{15} \right].$$

For $\tilde{r}, \hat{J} \in \mathfrak{S}$ with $\hat{H}_{[Q(\hat{J})]_{\tilde{\epsilon}_Q(\hat{J})}}^{[\hat{g}(\tilde{r})]_{\tilde{\epsilon}_{\hat{g}}(\tilde{r})}} > 0$ (i.e., $\tilde{r} \neq \hat{J} \neq 0$), we have,

$$\Lambda \left(\hat{H}_{[Q(\hat{J})]_{\tilde{\epsilon}_Q(\hat{J})}}^{[\hat{g}(\tilde{r})]_{\tilde{\epsilon}_{\hat{g}}(\tilde{r})}} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_9(\tilde{r}, \hat{J}) \right) \right) + \varrho \mathfrak{N}_1(\tilde{r}, \hat{J}).$$

Therefore, the hypotheses of Theorem 4.4.1 (with $\varrho = 1$) are satisfied and so $0 \in [\hat{g}(0)]_{\tilde{\epsilon}} \cap [Q(0)]_{\tilde{\epsilon}}$.

Corollary 4.4.3 Let $\hat{g}, Q : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ be FM and for all $\tilde{r} \in \mathfrak{S}$, there exist $\tilde{\epsilon}_{\hat{g}}(\tilde{r}), \tilde{\epsilon}_Q(\tilde{r}) \in (0, 1]$ such that $[\hat{g}(\tilde{r})]_{\tilde{\epsilon}_{\hat{g}}(\tilde{r})}, [Q(\hat{J})]_{\tilde{\epsilon}_Q(\hat{J})}$ are NBCS of \mathfrak{S} . Assume that there exist $\Upsilon \in \Omega, \Lambda \in \Lambda_{\Delta}$

such that

$$\hat{H}_{[Q(J)]_{\varepsilon_Q(J)}}^{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}} > 0 \implies \Lambda \left(\hat{H}_{[Q(J)]_{\varepsilon_Q(J)}}^{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_9 \left(\check{r}, \hat{J} \right) \right) \right).$$

for each $\check{r}, \hat{J} \in \mathfrak{S}$. If Υ is continuous, then $\exists \check{z} \in \mathfrak{S}$ such that $\check{z} \in [\hat{g}(\check{z})]_{\varepsilon_{\hat{g}(\check{z})}} \cap [Q(\check{z})]_{\varepsilon_Q(\check{z})}$.

Proof. Take $\varrho = 0$ in Theorem 4.4.1. ■

Corollary 4.4.4 Let $\hat{g} : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ be a FM and for all $\check{r} \in \mathfrak{S}$, $\exists \varepsilon_{\hat{g}(\check{r})} \in (0, 1]$ such that $[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}$ is a NBCS of \mathfrak{S} . Suppose that there exist $\Upsilon \in \Omega$, $\Lambda \in \Lambda_\Delta$ and $\varrho \geq 0$ such that

$$\hat{H}_{[\hat{g}(J)]_{\varepsilon_{\hat{g}(J)}}}^{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}} > 0 \implies \Lambda \left(\hat{H}_{[\hat{g}(J)]_{\varepsilon_{\hat{g}(J)}}}^{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{10} \left(\check{r}, \hat{J} \right) \right) \right) + \varrho \mathfrak{N}_2 \left(\check{r}, \hat{J} \right),$$

where

$$\mathcal{M}_{10} \left(\check{r}, \hat{J} \right) = \max \left\{ \partial_{\hat{J}}^{\check{r}}, \partial_{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}}^{\check{r}}, \partial_{[\hat{g}(\hat{J})]_{\varepsilon_{\hat{g}(\hat{J})}}}^{\hat{J}}, \frac{\partial_{[\hat{g}(\hat{J})]_{\varepsilon_{\hat{g}(\hat{J})}}}^{\check{r}} + \partial_{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}}^{\hat{J}}}{2} \right\},$$

and

$$\mathfrak{N}_2 \left(\check{r}, \hat{J} \right) = \min \left\{ \partial_{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}}^{\check{r}}, \partial_{[\hat{g}(\hat{J})]_{\varepsilon_{\hat{g}(\hat{J})}}}^{\hat{J}}, \partial_{[\hat{g}(\hat{J})]_{\varepsilon_{\hat{g}(\hat{J})}}}^{\check{r}}, \partial_{[\hat{g}(\check{r})]_{\varepsilon_{\hat{g}(\check{r})}}}^{\hat{J}} \right\},$$

for each $\check{r}, \hat{J} \in \mathfrak{S}$. If Υ is continuous, then $\exists \check{z} \in \mathfrak{S}$ such that $\check{z} \in [\hat{g}(\check{z})]_{\varepsilon_{\hat{g}(\check{z})}}$.

Proof. Setting $\hat{g} = Q$ in Theorem 4.4.1. ■

Now, we discuss some relation of FM and set-valued maps.

Theorem 4.4.5 Let $\overline{F}, \overline{G} : \mathfrak{S} \rightarrow CB(\mathfrak{S})$ be multivalued mappings. Suppose that $\exists \Upsilon \in \Omega$, $\Lambda \in \Lambda_\Delta$ and $\varrho \geq 0$ such that, for each $\check{r}, \hat{J} \in \mathfrak{S}$,

$$\hat{H}_{\overline{G}(\hat{J})}^{\overline{F}(\check{r})} > 0 \implies \Lambda \left(\hat{H}_{\overline{G}(\hat{J})}^{\overline{F}(\check{r})} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{11} \left(\check{r}, \hat{J} \right) \right) \right) + \varrho \mathfrak{N}_3 \left(\check{r}, \hat{J} \right), \text{ where,}$$

$$\mathcal{M}_{11} \left(\check{r}, \hat{J} \right) = \max \left\{ \partial_{\hat{J}}^{\check{r}}, \partial_{\overline{F}(\check{r})}^{\check{r}}, \partial_{\overline{G}(\hat{J})}^{\hat{J}}, \frac{\partial_{\overline{G}(\hat{J})}^{\check{r}} + \partial_{\overline{F}(\check{r})}^{\hat{J}}}{2} \right\},$$

and

$$\mathfrak{N}_3 \left(\check{r}, \hat{J} \right) = \min \left\{ \partial_{\overline{F}(\check{r})}^{\check{r}}, \partial_{\overline{G}(\hat{J})}^{\hat{J}}, \partial_{\overline{G}(\hat{J})}^{\check{r}}, \partial_{\overline{F}(\check{r})}^{\hat{J}} \right\}.$$

If Υ is continuous, then there exists $\check{z} \in \mathfrak{S}$ such that $\check{z} \in \overline{F}(\check{z}) \cap \overline{G}(\check{z})$.

Proof. Let $\check{\varepsilon} : \mathfrak{S} \rightarrow (0, 1]$ and $\hat{g}, Q : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ be defined by,

$$\hat{g}(\check{r})(\check{a}) = \begin{cases} \check{\varepsilon}(\check{r}), & \text{if } \check{a} \in \overline{F}(\check{r}) \\ 0, & \text{if } \check{a} \notin \overline{F}(\check{r}), \end{cases} \quad \text{and } Q(\check{r})(\check{a}) = \begin{cases} \check{\varepsilon}(\check{r}), & \text{if } \check{a} \in \overline{G}(\check{r}) \\ 0, & \text{if } \check{a} \notin \overline{G}(\check{r}). \end{cases}$$

Then,

$$[\hat{g}(\check{r})]_{\check{\varepsilon}(\check{r})} = \{\check{a} : \hat{g}(\check{r})(\check{a}) \geq \check{\varepsilon}(\check{r})\} = \overline{F}(\check{r}),$$

and

$$[Q(\check{r})]_{\check{\varepsilon}(\check{r})} = \{\check{a} : Q(\check{r})(\check{a}) \geq \check{\varepsilon}(\check{r})\} = \overline{G}(\check{r}).$$

Hence, from Theorem 4.4.1, $\exists \check{z} \in \mathfrak{S}$ such that $\check{z} \in [\hat{g}(\check{z})]_{\check{\varepsilon}(\check{z})} \cap [Q(\check{z})]_{\check{\varepsilon}(\check{z})} = \overline{F}(\check{z}) \cap \overline{G}(\check{z})$. ■

Theorem" 4.4.6 Let (\mathfrak{S}, ∂) be a complete MLS and $\hat{g}, Q : \mathfrak{S} \rightarrow A_E(\mathfrak{S})$ be FM. Suppose that $\exists \Upsilon \in \Omega$, $\Lambda \in \Lambda_\Delta$ and $\varrho \geq 0$ such that, $\forall \check{r}, \hat{J} \in \mathfrak{S}$,

$$\partial_\infty \left| \frac{\hat{g}(\check{r})}{Q(\hat{J})} \right| > 0 \implies \Lambda \left(\partial_\infty \left| \frac{\hat{g}(\check{r})}{Q(\hat{J})} \right| \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{12}(\check{r}, \hat{J}) \right) \right) + \varrho \mathfrak{N}_4(\check{r}, \hat{J}),$$

where

$$\mathcal{M}_{12}(\check{r}, \hat{J}) = \max \left\{ \mathbb{P}_{\check{r}}^{\check{r}}, \mathbb{P}_{\hat{g}(\check{r})}^{\check{r}}, \mathbb{P}_{Q(\hat{J})}^{\hat{J}}, \frac{\mathbb{P}_{Q(\hat{J})}^{\check{r}} + \mathbb{P}_{\hat{g}(\check{r})}^{\hat{J}}}{2} \right\}, \quad (4.34)$$

and

$$\mathfrak{N}_4(\check{r}, \hat{J}) = \min \left\{ \mathbb{P}_{\hat{g}(\check{r})}^{\check{r}}, \mathbb{P}_{Q(\hat{J})}^{\hat{J}}, \mathbb{P}_{Q(\hat{J})}^{\check{r}}, \mathbb{P}_{\hat{g}(\check{r})}^{\hat{J}} \right\}. \quad (4.35)$$

If Υ is continuous, then $\exists \check{z} \in \mathfrak{S}$ such that $\{\check{z}\} \subset \hat{g}(\check{z})$ and $\{\check{z}\} \subset Q(\check{z})$.

Proof. Let $\check{r} \in \mathfrak{S}$, then from Lemma 1.1.16, there is $\hat{J} \in \mathfrak{S}$ such that $\hat{J} \in [\hat{g}(\check{r})]_1$. Similarly, we can find $\hat{a} \in \mathfrak{S}$ such that $\hat{a} \in [Q(\check{r})]_1$. This implies that for every $\check{r} \in \mathfrak{S}$, $[\hat{g}(\check{r})]_{\check{\varepsilon}(\check{r})}$, $[Q(\check{r})]_{\check{\varepsilon}(\check{r})}$ are NBCS of \mathfrak{S} . As $\check{\varepsilon}(\check{r}) = \check{\varepsilon}(\hat{J}) = 1$,

$$\hat{H} \left| \frac{[\hat{g}(\check{r})]_{\check{\varepsilon}(\check{r})}}{[Q(\hat{J})]_{\check{\varepsilon}(\hat{J})}} \right| \leq \partial_\infty \left| \frac{\hat{g}(\check{r})}{Q(\hat{J})} \right|, \quad \text{for each } \check{r}, \hat{J} \in \mathfrak{S}.$$

From (Λ_1) , we get

$$\Lambda \left(\hat{H} \left| \frac{[\hat{g}(\check{r})]_{\check{\varepsilon}(\check{r})}}{[Q(\hat{J})]_{\check{\varepsilon}(\hat{J})}} \right| \right) \leq \Lambda \left(\partial_\infty \left| \frac{\hat{g}(\check{r})}{Q(\hat{J})} \right| \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{12}(\check{r}, \hat{J}) \right) \right) + \varrho \mathfrak{N}_4(\check{r}, \hat{J}).$$

Since $[\hat{g}(\tilde{r})]_1 \subseteq [\hat{g}(\tilde{r})]_{\tilde{z}(\tilde{r})}$ for every $\tilde{z} \in (0, 1]$,

$$\partial_\infty|_{[\hat{g}(\tilde{r})]_{\tilde{z}(\tilde{r})}} \leq \partial_\infty|_{[\hat{g}(\tilde{r})]_1}, \text{ for each } \alpha \in (0, 1].$$

Thus

$$\mathbb{P}_{\hat{g}(\tilde{r})}^{\tilde{r}} \leq \partial_\infty|_{[\hat{g}(\tilde{r})]_1}^{\tilde{r}}. \text{ Similarly, } \mathbb{P}_{Q(\tilde{r})}^{\tilde{r}} \leq \partial_\infty|_{[Q(\tilde{r})]_1}^{\tilde{r}}.$$

this implies that for each $\tilde{r}, \hat{J} \in \mathfrak{S}$,

$$\Lambda \left(\hat{H}_{[Q(\tilde{r})]_1}^{[\hat{g}(\tilde{r})]_1} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_9 \left(\tilde{r}, \hat{J} \right) \right) \right) + \varrho \aleph_1 \left(\tilde{r}, \hat{J} \right),$$

where

$$\mathcal{M}_9 \left(\tilde{r}, \hat{J} \right) = \max \left\{ \partial_{\hat{J}}^{\tilde{r}}, \partial_{[\hat{g}(\tilde{r})]_1}^{\tilde{r}}, \partial_{[Q(\hat{J})]_1}^{\hat{J}}, \frac{\partial_{[Q(\hat{J})]_1}^{\tilde{r}} + \partial_{[\hat{g}(\tilde{r})]_1}^{\hat{J}}}{2} \right\},$$

and

$$\aleph_1 \left(\tilde{r}, \hat{J} \right) = \min \left\{ \partial_{[\hat{g}(\tilde{r})]_1}^{\tilde{r}}, \partial_{[Q(\hat{J})]_1}^{\hat{J}}, \partial_{[Q(\hat{J})]_1}^{\tilde{r}}, \partial_{[\hat{g}(\tilde{r})]_1}^{\hat{J}} \right\}.$$

By Theorem 4.4.1, we get $\tilde{z} \in \mathfrak{S}$ such that $\tilde{z} \in [\hat{g}(\tilde{z})]_1 \cap [Q(\tilde{z})]_1$, that is, $\{\tilde{z}\} \subset \hat{g}(\tilde{z})$ and $\{\tilde{z}\} \subset Q(\tilde{z})$. ■

Corollary 4.4.7 Let (\mathfrak{S}, ∂) be a complete MLS and $\tilde{z}, Q : \mathfrak{S} \rightarrow A_E(\mathfrak{S})$ be FM. Suppose that $\exists \Upsilon \in \Omega$, $\Lambda \in \Lambda_\Delta$ such that, $\forall \tilde{r}, \hat{J} \in \mathfrak{S}$,

$$\partial_\infty|_{Q(\hat{J})}^{\hat{g}(\tilde{r})} > 0 \implies \Lambda \left(\partial_\infty|_{Q(\hat{J})}^{\hat{g}(\tilde{r})} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{12} \left(\tilde{r}, \hat{J} \right) \right) \right).$$

If Υ is continuous, then there exists $\tilde{z} \in \mathfrak{S}$ such that $\{\tilde{z}\} \subset \hat{g}(\tilde{z})$ and $\{\tilde{z}\} \subset Q(\tilde{z})$.

Proof. Setting $\varrho = 0$ in Theorem 4.4.6. ■

We denote by \tilde{g} (see [[117], [124)]) the multivalued map induced by $\hat{g} : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$, i.e.,

$$\tilde{g}(\tilde{r}) = \left\{ \hat{J} : \hat{g}(\hat{J})(\tilde{r}) = \max_{\tilde{a} \in \mathfrak{S}} \hat{g}(\tilde{r})(\tilde{a}) \right\}.$$

Corollary 4.4.8 Let $\hat{g}, Q : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ be FM such that $\forall \tilde{r} \in \mathfrak{S}$, $\tilde{g}(\tilde{r}), \tilde{Q}(\tilde{r})$ are NBCS of

\mathfrak{S} . Suppose that there exist $\Upsilon \in \Omega$, $\Lambda \in \Lambda_\Delta$ and $\varrho \geq 0$ such that, for each $\check{r}, \hat{J} \in \mathfrak{S}$,

$$\hat{H}_{\tilde{Q}(\hat{J})}^{\tilde{g}(\check{r})} > 0 \implies \Lambda \left(\hat{H}_{\tilde{Q}(\hat{J})}^{\tilde{g}(\check{r})} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{13} \left(\check{r}, \hat{J} \right) \right) \right) + \varrho \aleph_5 \left(\check{r}, \hat{J} \right), \text{ where}$$

$$\mathcal{M}_{13} \left(\check{r}, \hat{J} \right) = \max \left\{ \partial_{\check{J}}^{\check{r}}, \partial_{\tilde{g}(\check{r})}^{\check{r}}, \partial_{\tilde{Q}(\hat{J})}^{\hat{J}}, \frac{\partial_{\tilde{Q}(\hat{J})}^{\check{r}} + \partial_{\tilde{g}(\check{r})}^{\hat{J}}}{2} \right\}, \text{ and} \quad (4.36)$$

$$\aleph_5 \left(\check{r}, \hat{J} \right) = \min \left\{ \partial_{\tilde{g}(\check{r})}^{\check{r}}, \partial_{\tilde{Q}(\hat{J})}^{\hat{J}}, \partial_{\tilde{Q}(\hat{J})}^{\check{r}}, \partial_{\tilde{g}(\check{r})}^{\hat{J}} \right\}. \quad (4.37)$$

If Υ is continuous, then there exists $\check{z} \in \mathfrak{S}$ such that $\hat{g}(\check{z})(\check{z}) \geq \hat{g}(\check{z})(\check{r})$ and $Q(\check{z})(\check{z}) \geq Q(\check{z})(\check{r})$, for each $\check{r} \in \mathfrak{S}$.

Proof. From Theorem 4.4.5, $\exists \check{z} \in \mathfrak{S}$ such that $\check{z} \in \tilde{g}(\check{z}) \cap \tilde{Q}(\check{z})$. From Lemma 1.1.17, we get, for each $\check{r} \in \mathfrak{S}$, $\hat{g}(\check{z})(\check{z}) \geq \hat{g}(\check{z})(\check{r})$ and $Q(\check{z})(\check{z}) \geq Q(\check{z})(\check{r})$. ■

Corollary 4.4.9 Let $\hat{g}, Q : \mathfrak{S} \rightarrow \mathcal{F}(\mathfrak{S})$ be FM such that $\forall \check{r} \in \mathfrak{S}$, $\tilde{g}(\check{r}), \tilde{Q}(\check{r})$ are NBCS of \mathfrak{S} . Suppose that there exist $\Upsilon \in \Omega$, $\Lambda \in \Lambda_\Delta$ such that, for each $\check{r}, \hat{J} \in \mathfrak{S}$,

$$\hat{H}_{\tilde{Q}(\hat{J})}^{\tilde{g}(\check{r})} > 0 \implies \Lambda \left(\hat{H}_{\tilde{Q}(\hat{J})}^{\tilde{g}(\check{r})} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{13} \left(\check{r}, \hat{J} \right) \right) \right),$$

If Υ is continuous, then there exists $\check{z} \in \mathfrak{S}$ such that $\hat{g}(\check{z})(\check{z}) \geq \hat{g}(\check{z})(\check{r})$ and $Q(\check{z})(\check{z}) \geq Q(\check{z})(\check{r})$, for each $\check{r} \in \mathfrak{S}$.

Proof. Setting $\varrho = 0$ in Corollary 4.4.8. ■

Chapter 5

Fixed Point Theorems in Partial b -Metric Spaces

In this chapter, we extend the concept of (Υ, Λ) -contraction by using the idea of Popescu [100] (triangular α -orbital admissibility) and initiate the notion of generalized multivalued $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction pair of maps. Related common fixed point theorems are established in α_s -complete PbMSs. Furthermore, we introduce Ćirić type rational graphic (Υ, Λ) -contraction pair maps and discuss corresponding CFP theorems on PbMSs endowed with a directed graph (DG). We present examples to elaborate these results and give some applications.

Results given in this chapter have been published in ([9, 11]).

5.1 Common fixed points of $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction multivalued of mappings

Throughout this section, $\mathfrak{S} \neq \emptyset$ endowed with a PbM P_b , that is, (\mathfrak{S}, P_b) is PbMS. Further, in results it is considered as a complete PbMS. Unless otherwise stated. Also note that $\alpha_s : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ is a function and $\alpha_s^*|_B^A = \inf \{ \alpha_s|_j^r : r \in A, j \in B \}$. We start with the following definitions.

Definition 5.1.1. Given $s \geq 1$ and $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. The pair (\hat{S}, \hat{E}) is called triangular α_s^* -admissible (T- α_s^* -admissible) if:

- (i) (\hat{S}, \hat{E}) is α_s^* -admissible (i.e., $\alpha_s|_j^r \geq s^2$ implies $\alpha_s^*|_{\hat{E}(j)}^{\hat{S}(r)} \geq s^2$ and $\alpha_s^*|_{\hat{S}(j)}^{\hat{E}(r)} \geq s^2$);
(ii) $\alpha_s|_u^r \geq s^2$ and $\alpha_s|_j^u \geq s^2$ imply $\alpha_s|_j^r \geq s^2$.

Definition 5.1.2. Given $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. The pair (\hat{S}, \hat{E}) is called α_s^* -orbital admissible (α_s^* -O admissible) if:

$$\alpha_s^*|_{\hat{S}(r)}^r \geq s^2 \text{ and } \alpha_s^*|_{\hat{E}(r)}^r \geq s^2 \text{ imply } \alpha_s^*|_{\hat{E}(\hat{S}(r))}^{\hat{S}(r)} \geq s^2 \text{ and } \alpha_s^*|_{\hat{S}(\hat{E}(r))}^{\hat{E}(r)} \geq s^2.$$

Definition 5.1.3. Given $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. The pair (\hat{S}, \hat{E}) is called triangular α_s^* -orbital admissible (T- α_s^* -O admissible), if:

- (i) (\hat{S}, \hat{E}) is α_s^* -O admissible;
(ii) $\alpha_s|_j^r \geq s^2$, $\alpha_s^*|_{\hat{S}(j)}^j \geq s^2$ and $\alpha_s^*|_{\hat{E}(j)}^j \geq s^2$ imply $\alpha_s^*|_{\hat{S}(j)}^r \geq s^2$ and $\alpha_s^*|_{\hat{E}(j)}^r \geq s^2$.

Lemma 5.1.4. Given $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. Assume that (\hat{S}, \hat{E}) is T- α_s^* -O admissible and there is $r_0 \in \mathfrak{S}$ such that $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2$. Define a sequence $\{r_n\}$ in \mathfrak{S} by $r_{2i+1} \in \hat{S}(r_{2i})$ and $r_{2i+2} \in \hat{E}(r_{2i+1})$, where $i \in \mathbb{N} \cup \{0\}$. Then, $\alpha_s|_{r_m}^{r_n} \geq s^2 \forall n, m \in \mathbb{N}$ such that $m > n$.

Proof. Since $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} = \inf \{ \alpha_s|_{r_1}^{r_0} : r_1 \in \hat{S}(r_0) \} \leq \alpha_s|_{r_1}^{r_0} \geq s^2$, using the T- α_s^* -O admissibility of (\hat{S}, \hat{E}) , we have

$$\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2 \text{ implies } \alpha_s^*|_{\hat{E}(\hat{S}(r_0))}^{\hat{S}(r_0)} \leq \alpha_s^*|_{\hat{E}(r_1)}^{r_1} \leq \alpha_s|_{r_2}^{r_1} \geq s^2$$

and

$$\alpha_s^*|_{\hat{E}(r_1)}^{r_1} \geq s^2 \text{ implies } \alpha_s^*|_{\hat{S}(\hat{E}(r_1))}^{\hat{E}(r_1)} \leq \alpha_s^*|_{\hat{S}(r_2)}^{r_2} \leq \alpha_s|_{r_3}^{r_2} \geq s^2.$$

Thus, $\alpha_s|_{r_m}^{r_n} \geq s^2$, for each $n, m \in \mathbb{N}$ with, $m = n + 1$. By the T- α_s^* -O admissibility of (\hat{S}, \hat{E}) , we have $\alpha_s|_{r_m}^{r_n} \geq s^2$, for each $n, m \in \mathbb{N}$ with, $m > n$. ■

Definition 5.1.5. Let $\hat{S} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. Such \hat{S} is α_s^* - P_b -continuous on $(CB_{P_b}(\mathfrak{S}), \hat{H}_{P_b})$, if $\{r_n\} \subset \mathfrak{S}$ such that $\alpha_s|_{r_{n+1}}^{r_n} \geq s^2$ for each $n \in \mathbb{N}$ and $r \in \mathfrak{S}$ with $\lim_{n \rightarrow \infty} P_b|_r^{r_n} = 0$, then

$$\lim_{n \rightarrow \infty} \hat{H}_{P_b}|_{\hat{S}(r)}^{\hat{S}(r_n)} = 0.$$

Now, we initiate the notion of generalized $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction multivalued pair of mappings (generalized $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction).

Definition 5.1.6. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. The pair (\hat{S}, \hat{E}) is called a generalized

$(\alpha_s^*, \Upsilon, \Lambda)$ -contraction if $\exists \Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that for $r, j \in \mathfrak{S}$, $\alpha_s|_j^r \geq s^2$,

$$\hat{H}_{P_b} \Big|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \implies \Lambda \left(\alpha_s|_j^r \hat{H}_{P_b} \Big|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{14}(r, j) \right) \right), \quad (5.1)$$

where

$$\mathcal{M}_{14}(r, j) = \max \left\{ P_b|_j^r, D_{P_b}|_{\hat{S}(r)}^r, D_{P_b}|_{\hat{E}(j)}^j, \frac{D_{P_b}|_{\hat{E}(j)}^r + D_{P_b}|_{\hat{S}(r)}^j}{2s^2} \right\}. \quad (5.2)$$

Let us define the following conditions:

(\check{O}_1) \hat{S} and \hat{E} are α_s^* - P_b -continuous.

(\check{O}_2) If $\{r_n\} \subset \mathfrak{S}$ such that $\alpha_s|_{r_{n+1}}^{r_n} \geq s^2$ for each $n \in \mathbb{N}$ and $r_n \rightarrow r^* \in \mathfrak{S}$ as $n \rightarrow \infty$, then \exists a subsequence $\{r_{n(k)}\}$ of $\{r_n\}$ such that $\alpha_s|_{r^*}^{r_{n(k)}} \geq s^2$, $\forall k \in \mathbb{N}$.

Theorem 5.1.7. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. Suppose that

- (i) (\mathfrak{S}, d) is an α_s -complete PbMS;
- (ii) (\hat{S}, \hat{E}) is a generalized $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction;
- (iii) (\hat{S}, \hat{E}) is T- α_s^* -O admissible;
- (iv) there exists $r_0 \in \mathfrak{S}$ such that $\alpha_s|_{\hat{S}(r_0)}^{r_0} \geq s^2$;
- (v) (a) condition (\check{O}_1) holds;
- (b) condition (\check{O}_2) holds.

If Υ is continuous, then \hat{S} and \hat{E} have a CFP, say $r^* \in \mathfrak{S}$.

Proof. (a) Let $r_0 \in \mathfrak{S}$ be such that $\alpha_s|_{\hat{S}(r_0)}^{r_0} \geq s^2$. Choose $r_1 \in \hat{S}(r_0)$ such that $\alpha_s|_{r_1}^{r_0} \geq s^2$ and $r_1 \neq r_0$. Since,

$$D_{P_b}|_{\hat{E}(r_1)}^{r_1} \leq \hat{H}_{P_b} \Big|_{\hat{E}(r_1)}^{\hat{S}(r_0)}.$$

Since Λ is nondecreasing, we have,

$$0 < \Lambda \left(D_{P_b}|_{\hat{E}(r_1)}^{r_1} \right) \leq \Lambda \left(\hat{H}_{P_b} \Big|_{\hat{E}(r_1)}^{\hat{S}(r_0)} \right) \leq \Lambda \left(\alpha_s|_{r_1}^{r_0} \hat{H}_{P_b} \Big|_{\hat{E}(r_1)}^{\hat{S}(r_0)} \right). \quad (5.3)$$

By ($\Lambda 3$), $\Lambda \left(D_{P_b}|_{\hat{E}(r_1)}^{r_1} \right) = \inf_{j \in \hat{E}(r_1)} \Lambda \left(P_b|_j^{r_1} \right)$. Thus, there exists $r_2 \in \hat{E}(r_1)$ such that $\Lambda \left(D_{P_b}|_{\hat{E}(r_1)}^{r_1} \right) = \Lambda \left(P_b|_{r_2}^{r_1} \right)$. Then from (5.3), we have,

$$\Lambda \left(P_b|_{r_2}^{r_1} \right) \leq \Lambda \left(\hat{H}_{P_b} \Big|_{\hat{E}(r_1)}^{\hat{S}(r_0)} \right) \leq \Lambda \left(\alpha_s|_{r_1}^{r_0} \hat{H}_{P_b} \Big|_{\hat{E}(r_1)}^{\hat{S}(r_0)} \right). \quad (5.4)$$

Thus (5.4),

$$0 \leq \Lambda (P_b|_{r_2}^{r_1}) \leq \Lambda \left(\alpha_s|_{r_1}^{r_0} \hat{H}_{P_b} \Big|_{\hat{E}(r_1)}^{\hat{S}(r_0)} \right) \leq \Upsilon (\Lambda (\mathcal{M}_{14} (r_0, r_1))), \quad (5.5)$$

where

$$\begin{aligned} \mathcal{M}_{14} (r_0, r_1) &= \max \left\{ P_b|_{r_1}^{r_0}, D_{P_b}|_{\hat{S}(r_0)}^{r_0}, D_{P_b}|_{\hat{E}(r_1)}^{r_1}, \frac{D_{P_b}|_{\hat{E}(r_1)}^{r_0} + D_{P_b}|_{\hat{S}(r_0)}^{r_1}}{2s^2} \right\} \\ &\leq \max \left\{ P_b|_{r_1}^{r_0}, D_{P_b}|_{\hat{E}(r_1)}^{r_1}, \frac{D_{P_b}|_{\hat{E}(r_1)}^{r_0} + P_b|_{r_1}^{r_1}}{2s^2} \right\} \\ &\leq \max \left\{ P_b|_{r_1}^{r_0}, D_{P_b}|_{\hat{E}(r_1)}^{r_1} \right\}. \end{aligned}$$

If $\max \left\{ P_b|_{r_1}^{r_0}, D_{P_b}|_{\hat{E}(r_1)}^{r_1} \right\} = D_{P_b}|_{\hat{E}(r_1)}^{r_1}$, then from (5.5), we have

$$\Lambda (P_b|_{r_2}^{r_1}) \leq \Upsilon (\Lambda (P_b|_{r_2}^{r_1})) < \Lambda (P_b|_{r_2}^{r_1}),$$

which is a contradiction. Thus, $\max \left\{ P_b|_{r_1}^{r_0}, D_{P_b}|_{\hat{E}(r_1)}^{r_1} \right\} = P_b|_{r_1}^{r_0}$. By (5.5), we get that

$$\Lambda (P_b|_{r_2}^{r_1}) \leq \Upsilon (\Lambda (P_b|_{r_1}^{r_0})).$$

Similarly, for $r_2 \in \hat{E}(r_1)$ and $r_3 \in \hat{S}(r_2)$. We have

$$\begin{aligned} \Lambda (P_b|_{r_3}^{r_2}) &= \Lambda (D_{P_b}|_{\hat{S}(r_2)}^{r_2}) \leq \Lambda \left(\hat{H}_{P_b} \Big|_{\hat{S}(r_2)}^{\hat{E}(r_1)} \right) \leq \Lambda \left(\alpha_s|_{r_2}^{r_1} \hat{H}_{P_b} \Big|_{\hat{S}(r_2)}^{\hat{E}(r_1)} \right) \\ &\leq \Upsilon (\Lambda (\mathcal{M}_{14} (r_1, r_2))) \leq \Upsilon (\Lambda (P_b|_{r_2}^{r_1})). \end{aligned}$$

This implies that

$$\Lambda (P_b|_{r_3}^{r_2}) \leq \Upsilon (\Lambda (P_b|_{r_2}^{r_1})). \quad (5.6)$$

Thus we are able to build a sequence $\{r_n\}$ in \mathfrak{S} such that $r_{2i+1} \in \hat{S}(r_{2i})$ and $r_{2i+2} \in \hat{E}(r_{2i+1})$, $i = 0, 1, 2, \dots$. $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2$ and (\hat{S}, \hat{E}) is $T\text{-}\alpha_s^*\text{-O}$ admissible. By Lemma 5.1.4, we have $\alpha_s|_{r_{n+1}}^{r_n} \geq s^2$, for each $n \in \mathbb{N}$. For $i \in \mathbb{N}$, we have,

$$0 < \Lambda (P_b|_{r_{2i+2}}^{r_{2i+1}}) \leq \Lambda \left(\alpha_s|_{r_{2i+1}}^{r_{2i}} H_{P_b} \Big|_{\hat{E}(r_{2i+1})}^{\hat{S}(r_{2i})} \right) \leq \Upsilon (\Lambda (\mathcal{M}_{14} (r_{2i}, r_{2i+1}))), \quad (5.7)$$

where

$$\begin{aligned}
\mathcal{M}_{14}(r_{2i}, r_{2i+1}) &= \max \left\{ P_b|_{r_{2i+1}}^{r_{2i}}, D_{P_b}|_{\hat{S}(r_{2i})}^{r_{2i}}, D_{P_b}|_{\hat{E}(r_{2i+1})}^{r_{2i+1}}, \frac{D_{P_b}|_{\hat{E}(r_{2i+1})}^{r_{2i}} + D_{P_b}|_{\hat{S}(r_{2i})}^{r_{2i+1}}}{2s^2} \right\} \\
&\leq \max \left\{ P_b|_{r_{2i+1}}^{r_{2i}}, P_b|_{r_{2i+2}}^{r_{2i+1}}, \frac{P_b|_{r_{2i+2}}^{r_{2i}} + P_b|_{r_{2i+1}}^{r_{2i+1}}}{2s^2} \right\} \\
&\leq \max \left\{ P_b|_{r_{2i+1}}^{r_{2i}}, P_b|_{r_{2i+2}}^{r_{2i+1}} \right\}.
\end{aligned}$$

If $\max \left\{ P_b|_{r_{2i+1}}^{r_{2i}}, P_b|_{r_{2i+2}}^{r_{2i+1}} \right\} = P_b|_{r_{2i+2}}^{r_{2i+1}}$, then from (5.7) we have

$$\Lambda \left(P_b|_{r_{2i+2}}^{r_{2i+1}} \right) \leq \Upsilon \left(\Lambda \left(P_b|_{r_{2i+2}}^{r_{2i+1}} \right) \right) < \Lambda \left(P_b|_{r_{2i+2}}^{r_{2i+1}} \right),$$

which is a contradiction. Thus, $\max \left\{ P_b|_{r_{2i+1}}^{r_{2i}}, P_b|_{r_{2i+2}}^{r_{2i+1}} \right\} = P_b|_{r_{2i+1}}^{r_{2i}}$. By (5.7), we get that

$$\Lambda \left(P_b|_{r_{2i+2}}^{r_{2i+1}} \right) < \Upsilon \left(\Lambda \left(P_b|_{r_{2i+1}}^{r_{2i}} \right) \right).$$

This implies that

$$\Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) < \Upsilon \left(\Lambda \left(P_b|_{r_{2n+1}}^{r_{2n}} \right) \right), \quad \forall n \in \mathbb{N} \cup \{0\},$$

Thus

$$\Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) \leq \Upsilon \left(\Lambda \left(P_b|_{r_{2n+1}}^{r_{2n}} \right) \right) \leq \Upsilon^2 \left(\Lambda \left(P_b|_{r_{2n}}^{r_{2n-1}} \right) \right) \leq \dots \leq \Upsilon^n \left(\Lambda \left(P_b|_{r_1}^{r_0} \right) \right).$$

Letting $n \rightarrow \infty$ in the above inequality, we obtain

$$0 \leq \lim_{n \rightarrow \infty} \Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) \leq \lim_{n \rightarrow \infty} \Upsilon^n \left(\Lambda \left(P_b|_{r_1}^{r_0} \right) \right) = 0,$$

implies

$$\lim_{n \rightarrow \infty} \Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) = 0.$$

From $(\Lambda 2)$ and Lemma 1.6.4, we get

$$\lim_{n \rightarrow \infty} P_b|_{r_{2n+2}}^{r_{2n+1}} = 0. \quad (5.8)$$

We will show that $\{r_n\}$ is CS. We assume that $\{r_n\}$ is not CS, then there exist $\varepsilon > 0$ and a sequence $\{\hat{h}_n\}_{n=1}^{\infty}$ and $\{\tilde{I}_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, $\hat{h}_n > \tilde{I}_n > n$ with $P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \geq \varepsilon$, $P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}} < \varepsilon$. Therefore,

$$\begin{aligned} \varepsilon &\leq P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \leq s \left[P_b|_{r_{\hat{h}(n)-1}}^{r_{\hat{h}(n)}} + P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}} \right] - P_b|_{r_{\hat{h}(n)-1}}^{r_{\hat{h}(n)-1}} \\ &\leq s \left[P_b|_{r_{\hat{h}(n)-1}}^{r_{\hat{h}(n)}} + P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}} \right] < s\varepsilon + s P_b|_{r_{\hat{h}(n)-1}}^{r_{\hat{h}(n)}}. \end{aligned} \quad (5.9)$$

Taking $n \rightarrow \infty$ in (5.9) we get

$$\varepsilon < \lim_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} < s\varepsilon. \quad (5.10)$$

From triangular inequality, we have

$$\begin{aligned} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} &\leq s \left[P_b|_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}} + P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)+1}} \right] - P_b|_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)+1}} \\ &\leq s \left[P_b|_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}} + P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)+1}} \right], \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)+1}} &\leq s \left[P_b|_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}} + P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \right] - P_b|_{r_{\hat{h}(n)}}^{r_{\hat{h}(n)}} \\ &\leq s \left[P_b|_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}} + P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \right]. \end{aligned} \quad (5.12)$$

Setting $n \rightarrow \infty$ in (5.11) and using (5.8), (5.10), we get

$$\varepsilon \leq \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \leq s \left(\limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)+1}} \right).$$

Again, Setting $n \rightarrow \infty$ in (5.12) yields that

$$\varepsilon < \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)+1}} \leq s \left(\limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} \right) \leq s.s\varepsilon = s^2\varepsilon.$$

Thus

$$\frac{\varepsilon}{s} \leq \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)+1}} \leq s^2\varepsilon. \quad (5.13)$$

Similarly,

$$\frac{\varepsilon}{s} \leq \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)}} \leq s^2 \varepsilon. \quad (5.14)$$

By triangular inequality, we have

$$\begin{aligned} P_b|_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)-1}} &\leq s[P_b|_{r_{\hat{h}(n)}}^{r_{\hat{h}(n)-1}} + P_b|_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)}}] - P_b|_{r_{\hat{h}(n)}}^{r_{\hat{h}(n)}} \\ &\leq s[P_b|_{r_{\hat{h}(n)}}^{r_{\hat{h}(n)-1}} + P_b|_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)}}]. \end{aligned} \quad (5.15)$$

On letting $n \rightarrow \infty$ in (5.15) and using (5.8) and (5.14), we get

$$\limsup_{k \rightarrow \infty} P_b|_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)-1}} \leq s^3 \varepsilon. \quad (5.16)$$

From (5.1), we get,

$$\Lambda \left(\alpha_s|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}} P_b|_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)}} \right) \leq \Lambda \left(\alpha_s|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}} \hat{H}_{P_b} \left| \begin{array}{c} \hat{S}(r_{\tilde{I}(n)}) \\ \hat{E}(r_{\hat{h}(n)-1}) \end{array} \right. \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{14} \left(r_{\hat{h}(n)-1}, r_{\tilde{I}(n)} \right) \right) \right), \quad (5.17)$$

where,

$$\begin{aligned} \mathcal{M}_{14} \left(r_{\hat{h}(n)-1}, r_{\tilde{I}(n)} \right) &= \max \left\{ \begin{array}{c} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}}, D_{P_b}|_{r_{\tilde{I}(n)}}^{r_{\tilde{I}(n)}}, D_{P_b}|_{\hat{E}(r_{\hat{h}(n)-1})}^{r_{\hat{h}(n)-1}}, \\ \frac{D_{P_b}|_{\hat{E}(r_{\hat{h}(n)-1})}^{r_{\tilde{I}(n)}} + D_{P_b}|_{\hat{S}(r_{\tilde{I}(n)})}^{r_{\hat{h}(n)-1}}}{2s^2} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{c} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)-1}}, D_{P_b}|_{r_{\tilde{I}(n)+1}}^{r_{\tilde{I}(n)}}, D_{P_b}|_{r_{\hat{h}(n)}}^{r_{\hat{h}(n)-1}}, \\ \frac{D_{P_b}|_{r_{\hat{h}(n)}}^{r_{\tilde{I}(n)}} + D_{P_b}|_{r_{\tilde{I}(n)+1}}^{r_{\hat{h}(n)-1}}}{2s^2} \end{array} \right\}. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (5.8), (5.10) and (5.16), we get

$$\limsup_{n \rightarrow \infty} \mathcal{M}_{14} \left(r_{\hat{h}(n)-1}, r_{\tilde{I}(n)} \right) \leq \max \left\{ \varepsilon, \frac{s\varepsilon + s^3\varepsilon}{2s^2} \right\} \leq \max \left\{ \varepsilon, \frac{s^3\varepsilon + s^3\varepsilon}{2s^2} \right\} = s\varepsilon.$$

From (5.14), (A2), and by Lemma 1.6.4 since $\alpha_s|_{r_{\hat{I}(n)}^{r_{\hat{h}(n)}-1}} \geq s^2$, we get

$$\begin{aligned} \Lambda(s\varepsilon) &= \Lambda\left(s^2\frac{\varepsilon}{s}\right) \leq \Lambda\left(\alpha_s|_{r_{\hat{I}(n)}^{r_{\hat{h}(n)}-1}} \sup P_b|_{r_{\hat{I}(n)+1}}^{r_{\hat{h}(n)}}\right) \leq \lim_{n \rightarrow \infty} \Upsilon\left(\Lambda\left(\mathcal{M}_{14}\left(r_{\hat{h}(n)-1}, r_{\hat{I}(n)}\right)\right)\right) \\ &\leq \Upsilon(\Lambda(s\varepsilon)) < \Lambda(s\varepsilon). \end{aligned} \quad (5.18)$$

a contradiction. Hence, $\{r_n\}$ is CS in (\mathfrak{I}, P_b) and from Lemma 1.2.9, in the bMS $(\mathfrak{I}, \partial_{P_b})$. By Lemma 1.2.9, the α_s -completeness of the PbMS (\mathfrak{I}, P_b) implies the α_s -completeness of the bMS $(\mathfrak{I}, \partial_{P_b})$. So there exists $r^* \in \mathfrak{I}$ such that

$$\lim_{n \rightarrow \infty} \partial_{P_b}|_{r^*}^{r_n} = 0. \quad (5.19)$$

Because of Lemma 1.2.9, we have

$$\lim_{n \rightarrow \infty} P_b|_{r^*}^{r_n} = P_b|_{r^*}^{r^*} = \lim_{n \rightarrow \infty} P_b|_{r_m}^{r_n} = 0. \quad (5.20)$$

Hence,

$$\lim_{n \rightarrow \infty} P_b|_{r^*}^{r_n} = 0, \quad (5.21)$$

which implies,

$$\lim_{n \rightarrow \infty} P_b|_{r^*}^{r_{2i+1}} = \lim_{n \rightarrow \infty} P_b|_{r^*}^{r_{2i+2}} = 0. \quad (5.22)$$

Since \hat{S} is an α_s^* - P_b -continuous, $\lim_{n \rightarrow \infty} \hat{H}_{P_b}|_{\hat{S}(r^*)}^{\hat{S}(r_{2i+1})} = 0$. Thus

$$D_{P_b}|_{\hat{S}(r^*)}^{r^*} = \lim_{i \rightarrow \infty} D_{P_b}|_{\hat{S}(r^*)}^{r_{2i+2}} \leq \lim_{i \rightarrow \infty} \hat{H}_{P_b}|_{\hat{S}(r^*)}^{\hat{S}(r_{2i+1})} = 0,$$

and so, $r^* \in \hat{S}(r^*)$ and similarly, $r^* \in \hat{E}(r^*)$. Then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{I}$.

(b) From (a), we constructed a sequence $\{r_n\}$ in \mathfrak{I} defined by $r_{2i+1} \in \hat{S}(r_{2i})$ and $r_{2i+2} \in \hat{E}(r_{2i+1})$ with $\alpha_s|_{r_{n+1}}^{r_n} \geq s^2$, for each $n \in \mathbb{N} \cup \{0\}$. Also, $\{r_n\}$ converges to $r^* \in \mathfrak{I}$, and there is a subsequence $\{r_{n(k)}\}$ of $\{r_n\}$ such that $\alpha_s|_{r^*}^{r_{n(k)}} \geq s^2$ for every k . Thus,

$$\begin{aligned} \Lambda\left(D_{P_b}|_{\hat{E}(r^*)}^{r_{2n(k)+1}}\right) &\leq \Lambda\left(\hat{H}_{P_b}|_{\hat{E}(r^*)}^{\hat{S}(r_{2n(k)})}\right) \leq \Lambda\left(\alpha_s|_{r^*}^{r_{2n(k)}} \hat{H}_{P_b}|_{\hat{E}(r^*)}^{\hat{S}(r_{2n(k)})}\right) \\ &\leq \Upsilon\left(\Lambda\left(\mathcal{M}_{14}\left(r_{2n(k)}, r^*\right)\right)\right), \end{aligned} \quad (5.23)$$

where,

$$\begin{aligned} \mathcal{M}_{14}(r_{2n(k)}, r^*) &= \max \left\{ \begin{array}{l} P_b|_{r^*}^{r_{2n(k)}}, D_{P_b}|_{\hat{S}(r_{2n(k)})}^{r_{2n(k)}}, D_{P_b}|_{\hat{E}(r^*)}^{r^*}, \\ \frac{D_{P_b}|_{\hat{E}(r^*)}^{r_{2n(k)}} + D_{P_b}|_{\hat{S}(r_{2n(k)})}^{r^*}}{2s^2} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} P_b|_{r^*}^{r_{2n(k)}}, P_b|_{r_{2n(k)+1}}^{r_{2n(k)}}, D_{P_b}|_{\hat{E}(r^*)}^{r^*}, \\ \frac{D_{P_b}|_{\hat{E}(r^*)}^{r_{2n(k)}} + D_{P_b}|_{\hat{S}(r_{2n(k)})}^{r^*}}{2s^2} \end{array} \right\}. \end{aligned}$$

Since

$$\limsup_{k \rightarrow \infty} \frac{D_{P_b}|_{\hat{E}(r^*)}^{r_{2n(k)}} + D_{P_b}|_{\hat{S}(r_{2n(k)})}^{r^*}}{2s^2} \leq \frac{D_{P_b}|_{\hat{E}(r^*)}^{r^*} + P_b|_{r^*}^{r^*}}{2s^2},$$

by letting $k \rightarrow \infty$, we have $\lim_{k \rightarrow \infty} \mathcal{M}_{14}(r_{2n(k)}, r^*) = D_{P_b}|_{\hat{E}(r^*)}^{r^*}$. Suppose that $D_{P_b}|_{\hat{E}(r^*)}^{r^*} > 0$. From (5.23),

$$\Lambda \left(D_{P_b}|_{\hat{E}(r^*)}^{r_{2n(k)+1}} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{14}(r_{2n(k)}, r^*) \right) \right). \quad (5.24)$$

Letting $k \rightarrow \infty$ in (5.24) and by (Λ_3) and continuity of Υ ,

$$\Lambda \left(D_{P_b}|_{\hat{E}(r^*)}^{r^*} \right) \leq \Upsilon \left(\Lambda \left(D_{P_b}|_{\hat{E}(r^*)}^{r^*} \right) \right) < \Lambda \left(D_{P_b}|_{\hat{E}(r^*)}^{r^*} \right),$$

a contradiction. Thus, we obtain, $r^* \in \hat{E}(r^*)$. Similarly we can show that $r^* \in \hat{S}(r^*)$. Therefore \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$. ■

Corollary 5.1.8. Let $\hat{E}: \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. Suppose that:

- (i) (\mathfrak{S}, P_b) is an α_s -complete PbMS;
- (ii) \hat{E} is a generalized $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction, that is, if there exist $\Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that, for $r, j \in \mathfrak{S}$, $\alpha_s|_j^r \geq s^2$,

$$\hat{H}_{P_b}(\hat{E}(r), \hat{E}(j)) > 0 \implies \Lambda \left(\alpha_s|_j^r \hat{H}_{P_b}|_{\hat{E}(j)}^{\hat{E}(r)} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{15}(r, j) \right) \right), \text{ wh\u0117r\u0117,}$$

$$\mathcal{M}_{15}(r, j) = \max \left\{ P_b|_j^r, D_{P_b}|_{\hat{E}(r)}^r, D_{P_b}|_{\hat{E}(j)}^j, \frac{D_{P_b}|_{\hat{E}(j)}^r + D_{P_b}|_{\hat{E}(r)}^j}{2s^2} \right\}.$$

- iii) \hat{E} is T- α_s^* -O admissible;

(iv) there exists $r_0 \in \mathfrak{S}$ such that $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2$;

(v) (a) \hat{E} is an α_s^* - P_b -continuous;

(b) condition (\check{O}_2) holds.

If Υ is continuous, then \hat{E} has a FP $r^* \in \mathfrak{S}$.

Proof. Take $\hat{S} = \hat{E}$ in Theorem 5.1.7. ■

Example 5.1.9. Let $\mathfrak{S} = [0, 1]$. Take $P_b : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by $P_b|_j^r = |r - j|^2 + (\max\{r, j\})^2$, for each $r, j \in \mathfrak{S}$. Clearly, (\mathfrak{S}, P_b) is a complete PbMS with $s = 4$. Define $\Lambda, \Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\Lambda(t) = te^t$, $\Upsilon(t) = \frac{190t}{200}$. Then $\Lambda \in \Lambda_\Delta$ and $\Upsilon \in \Omega$ is continuous. Define $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$ by

$$\hat{S}(r) = \begin{cases} \left\{ \frac{8r}{1000} \right\}, & \text{if } 0 \leq r \leq \frac{1}{2} \\ \{1\}, & \text{if } \frac{1}{2} < r \leq 1. \end{cases} \quad \text{and } \hat{E}(r) = \{0\}, \text{ for each } r \in \mathfrak{S}.$$

Also, we define the function $\alpha_s : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by

$$\alpha_s|_j^r = \begin{cases} s^2, & \text{if } 0 \leq r, j \leq \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

If the sequence $\{r_n\}$ is CS with $\alpha_s|_{r_{n+1}}^{r_n} \geq s^2$ for each integer, then $\{r_n\} \subseteq [0, \frac{1}{2}]$. Since $([0, \frac{1}{2}], P_b)$ is a complete PbMS, $\{r_n\}$ converges in $[0, \frac{1}{2}] \subseteq \mathfrak{S}$. Thus (\mathfrak{S}, P_b) is an α_s -complete PbMS. Let $\alpha_s^*|_{\hat{S}(r)}^r \geq s^2$ and $\alpha_s^*|_{\hat{E}(r)}^r \geq s^2$, thus $r \in [0, \frac{1}{2}]$ and $\hat{S}(r), \hat{E}(r) \in [0, \frac{1}{2}]$ and so $\hat{S}^2(r) = \hat{S}(\hat{S}(r)), \hat{E}^2(r) = \hat{E}(\hat{E}(r)) \in [0, \frac{1}{2}]$, then $\alpha_s^*|_{\hat{E}^2(r)}^{\hat{S}(r)} \geq s^2$ and $\alpha_s^*|_{\hat{S}^2(r)}^{\hat{E}(r)} \geq s^2$. Thus, (\hat{S}, \hat{E}) is α_s^* -O admissible. Let $r, j \in \mathfrak{S}$ be such that $\alpha_s|_j^r \geq s^2$, $\alpha_s^*|_{\hat{S}(j)}^j \geq s^2$ and $\alpha_s^*|_{\hat{E}(j)}^j \geq s^2$. Clearly, $\alpha_s^*|_{\hat{S}(j)}^r \geq s^2$ and $\alpha_s^*|_{\hat{E}(j)}^r \geq s^2$. Then, (\hat{S}, \hat{E}) is T- α_s^* -O admissible. Let $\{r_n\}$ be a CS so that $\lim_{n \rightarrow \infty} P_b|_{r_n}^{r_n} = 0$ and $\alpha_s|_{r_{n+1}}^{r_n} \geq s^2$ for each $n \in \mathbb{N}$. Hence $\{r_n\} \subseteq [0, \frac{1}{2}]$ for all $n \in \mathbb{N}$.

Then $\lim_{n \rightarrow \infty} \hat{H}_{P_b}|_{\hat{E}(r)}^{\hat{E}(r_n)} = \lim_{n \rightarrow \infty} \hat{H}_{P_b}|_{\hat{E}(r)}^{\left\{ \frac{8r_n}{1000} \right\}} = \hat{H}_{P_b}|_{\hat{E}(r)}^{\left\{ \frac{8r}{1000} \right\}} = 0$. Hence \hat{E} is an α_s^* - P_b -continuous.

Similarly, \hat{S} is an α_s^* - P_b -continuous. Let $r_0 = \frac{1}{4}$. Then

$$\alpha_s^*|_{\hat{S}(\frac{1}{4})}^{\frac{1}{4}} = \alpha_s|_{\frac{8}{4000}}^{\frac{1}{4}} \geq s^2.$$

Let $r, j \in \mathfrak{S}$ be such that $\alpha_s|_j^r \geq s^2$. Then $r, j \in [0, \frac{1}{2}]$. $\forall r, j$ are nonzero and $r < j$,

$$\begin{aligned}
\Lambda \left(\alpha_s|_j^r \hat{H}_{P_b} \Big|_{\hat{E}(j)}^{\hat{S}(r)} \right) &= \Lambda \left(s^2 \hat{H}_{P_b} \Big|_{\{0\}}^{\left\{ \frac{8r}{1000} \right\}} \right) = \Lambda \left(16 \left[\left| \frac{8r}{1000} \right|^2 + \left(\frac{8r}{1000} \right)^2 \right] \right) \\
&= \Lambda \left(16 \left[\left| \frac{8r}{1000} \right|^2 + \left(\frac{8r}{1000} \right)^2 \right] \right) \\
&= \Lambda \left(\left(\frac{32}{1000} \right)^2 \left[|r|^2 + (r)^2 \right] \right) \\
&= \left(\frac{32}{1000} \right)^2 \left[|r|^2 + (r)^2 \right] e^{\left(\left(\frac{32}{1000} \right)^2 \left[|r|^2 + (r)^2 \right] \right)} \\
&\leq \frac{190}{200} \left[|r - j|^2 + (\max \{r, j\})^2 \right] e^{\left(\frac{190}{300} \left[|r - j|^2 + (\max \{r, j\})^2 \right] \right)} \\
&= \frac{190}{200} \left[P_b|_j^r \right] e^{\left(\frac{190}{300} \left[P_b|_j^r \right] \right)} \leq \frac{190}{200} \mathcal{M}_{14}(r, j) e^{\mathcal{M}_{14}(r, j)} \\
&= \frac{190}{200} \Lambda(\mathcal{M}_{14}(r, j)) = \Upsilon(\Lambda(\mathcal{M}_{14}(r, j))).
\end{aligned}$$

Thus all the conditions of Theorem 5.1.7 hold. Then, \hat{S} and \hat{E} have a CFP.

Definition 5.1.10. Let (\mathfrak{S}, P_b) be a PbMS. Given $\alpha_s : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ and $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. (\hat{S}, \hat{E}) is called an $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction if there exist $\Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that for $r, j \in \mathfrak{S}$, $\alpha_s|_j^r \geq s^2$,

$$\hat{H}_{P_b} \Big|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \implies \Lambda \left(\alpha_s|_j^r \hat{H}_{P_b} \Big|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(P_b|_j^r \right) \right).$$

Theorem 5.1.11. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow CB_{P_b}(\mathfrak{S})$. Suppose that:

- (i) (\mathfrak{S}, P_b, s) is an α_s -complete PbMS;
- (ii) (\hat{S}, \hat{E}) is an $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction;
- (iii) (\hat{S}, \hat{E}) is T- α_s^* -O admissible;
- (iv) there exists $r_0 \in \mathfrak{S}$ such that $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2$;
- (v) (a) condition (\check{O}_1) holds.
- (b) condition (\check{O}_2) holds.

If Υ is continuous, then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$.

Corollary 5.1.12. Let (\hat{Y}, d_{P_b}) be a bMS. Given $\hat{S}, \hat{E} : \hat{Y} \longrightarrow CB_b(\hat{Y})$. Suppose that:

- (i) (\hat{Y}, d_{P_b}) is an α_s -complete bMS;
- (ii) (\hat{S}, \hat{E}) is an $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction with respect to \hat{Y} ;
- (iii) (\hat{S}, \hat{E}) is T- α_s^* -O admissible;
- (iv) there exists $\hat{y}_0 \in \hat{Y}$ such that $\alpha_s^*|_{\hat{S}(\hat{y}_0)}^{\hat{y}_0} \geq s^2$;
- (v) (a) \hat{S} and \hat{E} are α_s^* - d_{P_b} -continuous.
 (b) If $\{\hat{y}_n\} \subset \hat{Y}$ such that $\alpha_s|_{\hat{y}_{n+1}}^{\hat{y}_n} \geq s^2, \forall n \in \mathbb{N}$ and $\hat{y}_n \longrightarrow \hat{y}^* \in \hat{Y}$ as $n \longrightarrow \infty$, then $\exists \{\hat{y}_{n(k)}\}$ of $\{\hat{y}_n\}$ such that $\alpha_s|_{\hat{y}^*}^{\hat{y}_{n(k)}} \geq s^2 \forall k \in \mathbb{N}$.

If Υ is continuous, then \exists a CFP $\hat{y}^* \in \hat{Y}$ of \hat{S} and \hat{E} .

Proof. Take $P_b|_r^r = 0$, for each $r \in \mathfrak{S}$ in Theorem 5.1.7. ■

Theorem 5.1.13 Let $\hat{S}, \hat{E} : \mathfrak{S} \longrightarrow CB_{P_b}(\mathfrak{S})$. Suppose that:

- (i) (\mathfrak{S}, P_b) is an α_s -complete PbMS;
- (ii) If there exist $\theta \in \Xi$ and $k \in (0, 1)$ such that, $\forall r, j \in \mathfrak{S}, \alpha_s|_j^r \geq s^2$,

$$\hat{H}_{P_b}|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \implies \theta \left(\alpha_s|_j^r \hat{H}_{P_b}|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq [\theta(\mathcal{M}_{14}(r, j))]^k;$$

- (iii) (\hat{S}, \hat{E}) is T- α_s^* -O admissible;
- (iv) there exists $r_0 \in \mathfrak{S}$ such that $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2$;
- (v) (a) condition (\check{O}_1) holds;
- (b) condition (\check{O}_2) holds.

Then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$.

Proof. Set $\Upsilon(\varsigma) = k\varsigma$ and $\Lambda(\varsigma) = \ln(\theta(\varsigma))$ in Theorem 5.1.7. ■

Theorem 5.1.14 Let $\hat{S}, \hat{E} : \mathfrak{S} \longrightarrow CB_{P_b}(\mathfrak{S})$. Assume that:

- (i) (\mathfrak{S}, P_b) is an α_s -complete PbMS;
- (ii) there exist $F \in F^*$ and $\vartheta > 0$ such that, for each $r, j \in \mathfrak{S}, \alpha_s|_j^r \geq s^2$,

$$\hat{H}_{P_b}|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \implies \vartheta + F \left(\alpha_s|_j^r \hat{H}_{P_b}|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq F(\mathcal{M}_{14}(r, j));$$

- (iii) (\hat{S}, \hat{E}) is T- α_s^* -O admissible;

(iv) there exists $r_0 \in \mathfrak{S}$ such that $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2$;

(v) (a) condition (\check{O}_1) holds;

(b) condition (\check{O}_2) holds.

Then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$.

Proof. Set $\Upsilon(\varsigma) = e^{-\vartheta}\varsigma$ and $\Lambda(\varsigma) = e^{\mathbf{F}(\varsigma)}$ in Theorem 5.1.7. ■

Theorem 5.1.15 Let $\hat{S}, \hat{E} : \mathfrak{S} \longrightarrow CB_{P_b}(\mathfrak{S})$. Assume that:

(i) (\mathfrak{S}, P_b) is an α_s -complete PbMS;

(ii) If for all $r, j \in \mathfrak{S}$, $\alpha_s|_j^r \geq s^2$,

$$\alpha_s|_j^r \hat{H}_{P_b} \Big|_{\hat{E}(j)}^{\hat{S}(r)} \leq \mathfrak{B}(\mathcal{M}_{14}(r, j)) \mathcal{M}_{14}(r, j) \text{ and}$$

$\mathfrak{B} : \mathbb{R}_0^+ \rightarrow [0, 1)$ is such that $\lim_{r \rightarrow t^+} \mathfrak{B}(r) < 1$ for every $t \in \mathbb{R}_0^+$;

(iii) (\hat{S}, \hat{E}) is T - α_s^* -O admissible;

(iv) there exists $r_0 \in \mathfrak{S}$ such that $\alpha_s^*|_{\hat{S}(r_0)}^{r_0} \geq s^2$;

(v) (a) condition (\check{O}_1) holds;

(b) condition (\check{O}_2) holds.

Then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$.

Proof. Set $\Upsilon(\varsigma) = \mathfrak{B}(\varsigma)\varsigma$ and $\Lambda(\varsigma) = \varsigma$ in Theorem 5.1.7. ■

5.2 Some common fixed point theorems for singlevalued mappings

Here, we introduce some FP theorems for self-mappings.

Definition 5.2.1. Let $\hat{S}, \hat{E} : \mathfrak{S} \longrightarrow \mathfrak{S}$. The pair (\hat{S}, \hat{E}) is called a generalized $(\alpha_s, \Upsilon, \Lambda)$ -contraction if $\exists \Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that $\forall r, j \in \mathfrak{S}$, with $\alpha_s|_j^r \geq s^2$,

$$P_b|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \Rightarrow \Lambda \left(\alpha_s|_j^r P_b|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{16}(r, j) \right) \right),$$

whère

$$\mathcal{M}_{16}(r, j) = \max \left\{ P_b|_j^r, P_b|_{\hat{S}(r)}^r, P_b|_{\hat{E}(j)}^j, \frac{P_b|_{\hat{E}(j)}^r + P_b|_{\hat{S}(r)}^j}{2s^2} \right\}.$$

Theorem 5.2.2. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. Assume that:

- (i) (\mathfrak{S}, P_b) is an α_s -complete PbMS;
- (ii) (\hat{S}, \hat{E}) is a generalized $(\alpha_s, \Upsilon, \Lambda)$ -contraction;
- (iii) (\hat{S}, \hat{E}) is T- α_s -O admissible;
- (iv) there exists $r_0 \in \mathfrak{S}$ such that $\alpha_s|_{\hat{S}(r_0)}^{r_0} \geq s^2$;
- (v)
 - (a) \hat{S} and \hat{E} are α_s - P_b -continuous.
 - (b) condition (\check{O}_2) holds.

If Υ is continuous, then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$.

Corollary 5.2.3. Let $(\mathfrak{S}, \preceq, P_b)$ be an ordered complete PbMS. Assume that $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ are weakly increasing maps [i.e., $\hat{S}(r) \preceq \hat{E}(\hat{S}(r))$ and $\hat{E}(j) \preceq \hat{S}(\hat{E}(j))$ hold for all $r, j \in \mathfrak{S}$] and satisfy:

- (i) if $\exists \Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that \forall comparable $r, j \in \mathfrak{S}$, (i.e., $r \preceq j$ or $j \preceq r$),

$$P_b|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \Rightarrow \Lambda \left(P_b|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda (\mathcal{M}_{16}(r, j)) \right);$$

- (ii) there exists $r_0 \in \mathfrak{S}$ such that $r_0 \preceq \hat{S}r_0$;
- (iii) (a) either, \hat{E} or \hat{S} is continuous.
- (b) If $\{r_n\} \subset \mathfrak{S}$ is a non-decreasing such that $r_n \rightarrow r^* \in \mathfrak{S}$ as $n \rightarrow \infty$, then $\exists \{r_{n(k)}\}$ of $\{r_n\}$ such that $r_{n(k)} \preceq r^* \forall k \in \mathbb{N}$.

If Υ is continuous, then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$.

Proof. Taking

$$\alpha_s|_j^r = \begin{cases} s^2, & r \preceq j \text{ or } j \preceq r, \\ 0, & \text{otherwise.} \end{cases}$$

in Theorem 5.1.7. ■

Jachymski [67] introduced \hat{G} -contraction on MSs.

Definition 5.2.4. [67] $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ is said a \hat{G} -contraction if

$$r, j \in \mathfrak{S}, (r, j) \in \bar{E}(\hat{G}) \Rightarrow (\hat{S}(r), \hat{S}(j)) \in \bar{E}(\hat{G})$$

and there exists $\lambda \in (0, 1)$ suchthat

$$r, j \in \mathfrak{S}, (r, j) \in \bar{E}(\hat{G}) \Rightarrow \partial_{\hat{S}(j)}^{\hat{S}(r)} \leq \lambda \partial_j^r.$$

Definition 5.2.5.[67] A mapping $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ is said a \hat{G} -continuous, if given $r \in \mathfrak{S}$ and sequence $\{r_n\}$ suchthat $r_n \rightarrow r$, as $n \rightarrow \infty$ and $(r_n, r_{n+1}) \in \bar{E}(\hat{G})$ for all $n \in \mathbb{N}$, implies $\hat{S}(r_n) \rightarrow \hat{S}(r)$.

Corollary 5.2.6. Let $(\mathfrak{S}, \hat{G}, P_b)$ be a complete PbMS endowed with a graph \hat{G} . Suppose that $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ satisfy:

(i) if there exist $\Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that, for all $r, j \in \mathfrak{S}$, with $(r, j) \in \bar{E}(\hat{G})$,

$$P_b|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \Rightarrow \Lambda \left(P_b|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{16}(r, j) \right) \right);$$

(ii) for $r, j \in \mathfrak{S}$, $(r, j) \in \bar{E}(\hat{G})$ implies $(\hat{S}(r), \hat{E}(\hat{S}(r))) \in \bar{E}(\hat{G})$ and $(\hat{E}(j), \hat{S}(\hat{E}(j))) \in \bar{E}(\hat{G})$;

(iii) there exists $r_0 \in \mathfrak{S}$ such that $(r_0, \hat{S}(r_0)) \in \bar{E}(\hat{G})$;

(iv) (a) either \hat{E} or \hat{S} is \hat{G} -continuous.

(b) If $\{r_n\} \subset \mathfrak{S}$ suchthat $(r_n, r_{n+1}) \in \bar{E}(\hat{G})$ and $r_n \rightarrow r^* \in \mathfrak{S}$ as $n \rightarrow \infty$, then $\exists \{r_{n(k)}\}$ of $\{r_n\}$ suchthat $(r_{n(k)}, r^*) \in \bar{E}(\hat{G}) \forall k \in \mathbb{N}$.

If Υ is continuous. Then \hat{S} and \hat{E} have a CFP $r^* \in \mathfrak{S}$.

Proof. Taking,

$$\alpha_s|_j^r = \begin{cases} s^2, & (r, j) \in \bar{E}(\hat{G}), \\ 0, & \text{otherwise.} \end{cases}$$

in Theorem 5.1.7. ■

Corollary 5.2.7. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ be suchthat:

(i) there exist $\Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that for $r, j \in \mathfrak{S}$,

$$P_b|_{\hat{E}(j)}^{\hat{S}(r)} > 0 \Rightarrow \Lambda \left(P_b|_{\hat{E}(j)}^{\hat{S}(r)} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{16}(r, j) \right) \right);$$

(ii) \hat{E} and \hat{S} are P_b -continuous.

If Υ is continuous, then \hat{S} and \hat{E} have a CFP, say $r^* \in \mathfrak{S}$.

Proof. The proof follows immediately from the proof of Theorem 5.1.7. ■

5.3 Application to functional equations

Here, we investigate the existence of the common solution of functional equations (FEs) arising in dynamic programming (DP) by applying obtained results in the section 5.1. Consider \mathbb{Y} and \mathbb{E} two Banach spaces, $\check{Z} \subseteq \mathbb{Y}$, $\mathcal{F} \subseteq \mathbb{E}$ and

$$\begin{aligned}\varpi & : \check{Z} \times \mathcal{F} \longrightarrow \check{Z} \\ \xi, \eta & : \check{Z} \times \mathcal{F} \longrightarrow \mathbb{R} \\ \Gamma, \zeta & : \check{Z} \times \mathcal{F} \times \mathbb{R} \longrightarrow \mathbb{R}\end{aligned}$$

For more details, see ([47, 48, 49, 96]), where \check{Z} is the state space and \mathcal{F} is the decision space. The problem of related DP is reduced to solve the FEs

$$p(r) = \sup_{j \in \mathcal{F}} \{\xi(r, j) + \Gamma(r, j, p(\varpi(r, j)))\}, \text{ for } r \in \check{Z} \quad (5.25)$$

$$q(r) = \sup_{j \in \mathcal{F}} \{\eta(r, j) + \zeta(r, j, q(\varpi(r, j)))\}, \text{ for } r \in \check{Z}. \quad (5.26)$$

Here, we show that the Equations (5.25) and (5.26) have at most one common and bounded solution. Define $P_b : B(\check{Z}) \times B(\check{Z}) \rightarrow \mathbb{R}_0^+$ by

$$P_b|_k^h = \sup_{r \in \check{Z}} |h(r) - k(r)|^2 + L, \quad L > 0, \quad (5.27)$$

for every $h, k \in B(\check{Z})$, (where, $B(\check{Z})$ is the class of all bounded real valued functions on \check{Z}).

Suppose that:

(B1) : Γ, ζ, ξ and η are continuous and bounded;

(B2) : For $r \in \check{Z}$, $h \in B(\check{Z})$, define $\bar{E}, A : B(\check{Z}) \longrightarrow B(\check{Z})$ by

$$\bar{E}(h(r)) = \sup_{j \in \mathcal{F}} \{\xi(r, j) + \Gamma(r, j, h(\varpi(r, j)))\}, \quad (5.28)$$

$$A(h(r)) = \sup_{j \in \mathcal{F}} \{\eta(r, j) + \zeta(r, j, h(\varpi(r, j)))\}. \quad (5.29)$$

Moreover, for every $(r, j) \in \check{Z} \times \mathcal{F}$, $h, k \in B(\check{Z})$, $t \in \check{Z}$ and $\sigma > 0$ implies

$$|\Gamma(r, j, h(t)) - \mathcal{C}(r, j, k(t))|^2 \leq \frac{9\mathcal{M}_{16}(h(t), k(t))}{10} - \sigma \quad (5.30)$$

whère

$$\mathcal{M}_{16}((h(t), k(t))) = \max\{P_b|_{k(t)}^{h(t)}, P_b|_{\bar{E}(h(t))}^{h(t)}, P_b|_{A(k(t))}^{k(t)}, \frac{P_b|_{A(k(t))}^{h(t)} + P_b|_{\bar{E}(h(t))}^{k(t)}}{2s^2}\}.$$

Theorem 5.3.1 Assume that the conditions (B1) and (B2) hold. Then Equations (5.25) èand (5.26) have a common and bounded solution in $B(\check{Z})$.

Proof. Obviously, $(B(\check{Z}), P_b)$ is a complete PbMS ($s = 4$). From (B1), (B2), $\bar{E}, A : B(\check{Z}) \rightarrow B(\check{Z})$. Given $\lambda > 0$ and $h_1, h_2 \in B(\check{Z})$. Select $r \in \check{Z}$ and $j_1, j_2 \in \mathcal{F}$ such that

$$\bar{E}(h_1) < \xi(r, j_1) + \Gamma(r, j_1, h_1(\varpi(r, j_1))) + \lambda \quad (5.31)$$

$$A(h_2) < \xi(r, j_2) + \mathcal{C}(r, j_2, h_2(\varpi(r, j_2))) + \lambda. \quad (5.32)$$

Further from (5.31) and (5.32), we have

$$\bar{E}(h_1) \geq \xi(r, j_2) + \Gamma(r, j_2, h_1(\varpi(r, j_2))) \quad (5.33)$$

$$A(h_2) \geq \xi(r, j_1) + \mathcal{C}(r, j_1, h_2(\varpi(r, j_1))). \quad (5.34)$$

Then (5.31) and (5.34) together with (5.30) imply,

$$\begin{aligned} \bar{E}(h_1(r)) - A(h_2(r)) &< \Gamma(r, j_1, h_1(\varpi(r, j_1))) - \mathcal{C}(r, j_1, h_2(\varpi(r, j_1))) + \lambda \quad (5.35) \\ &\leq |\Gamma(r, j_1, h_1(\varpi(r, j_1))) - \mathcal{C}(r, j_1, h_2(\varpi(r, j_1)))| + \lambda \\ &\leq \sqrt{\frac{9\mathcal{M}_{16}(h_1(r), h_2(r))}{10}} - \sigma + \lambda. \end{aligned}$$

Then (5.32) and (5.33) together with (5.30) imply

$$\begin{aligned}
A(h_2(r)) - \bar{E}(h_1(r)) &\leq \Gamma(r, j_2, h_1(\varpi(r, j_2))) - \mathcal{C}(r, j_2, h_2(\varpi(r, j_2))) + \lambda & (5.36) \\
&\leq |\Gamma(r, j_2, h_1(\varpi(r, j_2))) - \mathcal{C}(r, j_2, h_2(\varpi(r, j_2)))| + \lambda \\
&\leq \sqrt{\frac{9\mathcal{M}_{16}(h_1(r), h_2(r))}{10}} - \sigma + \lambda,
\end{aligned}$$

where,

$$\mathcal{M}_{16}(h_1(r), h_2(r)) = \max\{P_b|_{h_2(r)}^{h_1(r)}, P_b|_{\bar{E}(h_1(r))}^{h_1(r)}, P_b|_{A(h_2(r))}^{h_2(r)}, \frac{P_b|_{A(h_2(r))}^{h_1(r)} + P_b|_{\bar{E}(h_1(r))}^{h_2(r)}}{2s^2}\}.$$

From (5.35) and (5.36) and since $\lambda > 0$ is arbitrary, we obtain

$$|\bar{E}(h_1(r)) - A(h_2(r))| \leq \sqrt{\frac{9\mathcal{M}_{16}(h_1(r), h_2(r))}{10}} - \sigma.$$

Thus,

$$|\bar{E}(h_1(r)) - A(h_2(r))|^2 + \sigma \leq \frac{9\mathcal{M}_{16}(h_1(r), h_2(r))}{10}. \quad (5.37)$$

The inequality Equation (5.37) implies

$$P_b|_{A(h_2(r))}^{\bar{E}(h_1(r))} \leq \frac{9\mathcal{M}_{16}(h_1(r), h_2(r))}{10}. \quad (5.38)$$

Taking $\Lambda(t) = t$ and $\Upsilon(t) = \frac{9t}{10}$ for $t > 0$, we get

$$\Lambda\left(P_b|_{A(h_2(r))}^{\bar{E}(h_1(r))}\right) \leq \Upsilon\left(\Lambda\left(\mathcal{M}_{16}(h_1(r), h_2(r))\right)\right).$$

Thus, by Corollary 5.2.7, \bar{E} and A have a CFP, say $h^* \in B(\check{Z})$, and so, Equations (5.25) and (5.26) have a common solution, say $h^*(r)$. ■

5.4 Common fixed points of Ćirić type graphic (Υ, Λ) -contractions

Throughout this section, we denote by Δ the diagonal of $\mathfrak{S} \times \mathfrak{S}$ and by $F_P(\hat{S})$ the set of all FP of \hat{S} . Consider a DG \hat{G} which has no parallel edges such that the set $V(\hat{G})$ of its vertices coincides

with a PbMS (\mathfrak{S}, P_b) (i.e., $V(\hat{G}) := \mathfrak{S}$) and the set of its edges $(\bar{E}(\hat{G}) = \{(r, j) : (r, j) \in \mathfrak{S} \times \mathfrak{S}\})$ is such that $\Delta \subseteq \bar{E}(\hat{G})$. Then, \hat{G} is identify by the pair $(V(\hat{G}), \bar{E}(\hat{G}))$.

Property (C*): If for any $\{r_n\} \subset V(\hat{G})$ with $r_n \rightarrow r$ as $n \rightarrow \infty$, $(r_n, r_{n+1}) \in \bar{E}(\hat{G})$ for $n \in \mathbb{N} \Rightarrow \exists$ a subsequence $\{r_{n(k)}\}$ of $\{r_n\}$ with an edge between $r_{n(k)}$ and r for $k \in \mathbb{N}$.

Here, \hat{G} is a weighted graph such that the weight of each vertex r is $P_b(r, r)$, and the weight of every edge (r, j) is $P_b(r, j)$. Since (\mathfrak{S}, P_b) is a PbMS, the weight assigned to each vertex r need not to be zero, and whenever a zero weight is assigned to some edge (r, j) , it reduces to a loop (r, r) .

In this section, we introduce Ćirić type rational graphic (Υ, Λ) -contraction pair of maps (\hat{C} TRG (Υ, Λ) -contraction pair of maps) and provide corresponding CFP theorems. Here, $W(r)$ is the weight assigned to the vertex r , $W((r, j))$ is the weight assigned to the edge (r, j) and (\mathfrak{S}, P_b) is a PbMS endowed with a DG \hat{G} , ($s > 1$). Further, in results it is considered as a complete PbMS. Unless otherwise stated.

Definition 5.5.1 Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$. The pair (\hat{S}, \hat{E}) is a \hat{C} TRG (Υ, Λ) -contraction pair, if:

- (1) for each vertex $\nu \in \hat{G}$, $(\nu, \hat{S}(\nu)), (\nu, \hat{E}(\nu)) \in \bar{E}(\hat{G})$;
- (2) there exists $\Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that for each $r_1, r_2 \in \mathfrak{S}$, with $(r_1, r_2) \in \bar{E}(\hat{G})$ and $\hat{S}(r_1) \neq \hat{E}(r_2)$, we have

$$\Lambda \left(s P_b \Big|_{\hat{E}(r_2)}^{\hat{S}(r_1)} \right) \leq \Upsilon [\Lambda (\mathcal{M}_{17}(r_1, r_2))], \quad (5.39)$$

where,

$$\mathcal{M}_{17}(r_1, r_2) = \max \left\{ \begin{array}{l} P_b \Big|_{r_2}^{r_1}, P_b \Big|_{\hat{S}(r_1)}^{r_1}, P_b \Big|_{\hat{E}(r_2)}^{r_2}, \frac{P_b \Big|_{\hat{E}(r_2)}^{r_1} + P_b \Big|_{r_2}^{\hat{S}(r_1)}}{2s^2}, \\ \frac{P_b \Big|_{r_2}^{\hat{E}(r_2)} \left(1 + P_b \Big|_{\hat{S}(r_1)}^{r_1} \right)}{1 + P_b \Big|_{r_2}^{r_1}}. \end{array} \right\}. \quad (5.40)$$

Remark 5.5.2 If $\hat{S} = \hat{E}$, then \hat{S} is a \hat{C} TRG (Υ, Λ) -contraction.

Theorem 5.5.3 Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ be maps such that (\hat{S}, \hat{E}) is a \hat{C} TRG (Υ, Λ) -contraction pair. If Υ is continuous, then:

- (a) $F_P(\hat{S}) \neq \emptyset$ or $F_P(\hat{E}) \neq \emptyset$ if and only if $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$;
- (b) if $r^* \in F_P(\hat{S}) \cap F_P(\hat{E})$, then $W(r^*) = 0$;
- (c) $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$, provided that \hat{G} satisfies property (C*);

(d) $F_P(\hat{S}) \cap F_P(\hat{E})$ is a complete set if and only if $F_P(\hat{S}) \cap F_P(\hat{E})$ is a unit set

Proof. (a) Let $F_P(\hat{S}) \neq \emptyset$, so $\exists r^* \in F_P(\hat{S})$. Then \exists an edge between r^* and $\hat{S}(r^*)$, so $(r^*, \hat{S}(r^*)) \in \bar{E}(\hat{G})$. We shall show that $r^* \in F_P(\hat{E})$, that is, $W((r^*, \hat{E}(r^*))) = 0$. Assume on the contrary that $W((r^*, \hat{E}(r^*))) \neq 0$. Since $(r^*, \hat{E}(r^*)) \in \bar{E}(\hat{G})$ and (\hat{S}, \hat{E}) is a \acute{C} TRG (Υ, Λ) -contraction pair, so from (5.39),

$$\Lambda \left(P_b|_{\hat{E}(r^*)}^{r^*} \right) \leq \Lambda \left(s P_b|_{\hat{E}(r^*)}^{\hat{S}(r^*)} \right) \leq \Upsilon [\Lambda (\mathcal{M}_{17}(r^*, r^*))],$$

where

$$\mathcal{M}_{17}(r^*, r^*) = \max \left\{ P_b|_{r^*}^{r^*}, P_b|_{\hat{S}(r^*)}^{r^*}, P_b|_{\hat{E}(r^*)}^{r^*}, \frac{P_b|_{\hat{E}(r^*)}^{r^*} + P_b|_{\hat{S}(r^*)}^{r^*}}{2s^2}, \frac{P_b|_{\hat{E}(r^*)}^{r^*} (1 + P_b|_{\hat{S}(r^*)}^{r^*})}{1 + P_b|_{r^*}^{r^*}} \right\} = P_b|_{\hat{E}(r^*)}^{r^*}.$$

Thus,

$$\Lambda \left(P_b|_{\hat{E}(r^*)}^{r^*} \cdot \right) \leq \Upsilon \left[\Lambda \left(P_b|_{\hat{E}(r^*)}^{r^*} \cdot \right) \right] < \Lambda \left(P_b|_{\hat{E}(r^*)}^{r^*} \cdot \right).$$

a contradiction. Then, $W((r^*, \hat{E}(r^*))) = 0$, (i.e., $r^* = \hat{E}(r^*)$). Then,

$$r^* \in F_P(\hat{S}) \cap F_P(\hat{E}),$$

Hence,

$$F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset.$$

Conversely, let $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$. So there exists $r^* \in \mathfrak{S}$ such that $r^* \in F_P(\hat{S}) \cap F_P(\hat{E})$, then $r^* \in F_P(\hat{S})$ and $r^* \in F_P(\hat{E})$.

(b) Let $r^* \in F_P(\hat{S}) \cap F_P(\hat{E})$. Assume on the contrary that $W(r^*) \neq 0$. Since $(r^*, r^*) \in \bar{E}(\hat{G})$ and (\hat{S}, \hat{E}) is a \acute{C} TRG (Υ, Λ) -contraction pair, we obtain

$$\Lambda \left(P_b|_{r^*}^{r^*} \right) \leq \Lambda \left(s P_b|_{\hat{E}(r^*)}^{\hat{S}(r^*)} \right) \leq \Upsilon [\Lambda (\mathcal{M}_{17}(r^*, r^*))],$$

where

$$\begin{aligned} \mathcal{M}_{17}(r^*, r^*) &= \max \left\{ P_b|_{r^*}^{r^*}, P_b|_{\hat{S}(r^*)}^{r^*}, P_b|_{\hat{E}(r^*)}^{r^*}, \frac{P_b|_{\hat{E}(r^*)}^{r^*} + P_b|_{\hat{S}(r^*)}^{r^*}}{2s^2}, \right. \\ &\quad \left. \frac{P_b|_{\hat{E}(r^*)}^{r^*} (1 + P_b|_{\hat{S}(r^*)}^{r^*})}{1 + P_b|_{r^*}^{r^*}} \right\} \\ &= \max \left\{ P_b|_{r^*}^{r^*}, P_b|_{r^*}^{r^*}, P_b|_{r^*}^{r^*}, \frac{P_b|_{r^*}^{r^*} + P_b|_{r^*}^{r^*}}{2s^2}, \right. \\ &\quad \left. \frac{P_b|_{r^*}^{r^*} (1 + P_b|_{r^*}^{r^*})}{1 + P_b|_{r^*}^{r^*}} \right\} = P_b|_{r^*}^{r^*}. \end{aligned}$$

It implies that

$$\Lambda \left(P_b|_{r^*}^{r^*} \right) \leq \Upsilon \left[\Lambda \left(P_b|_{r^*}^{r^*} \right) \right] < \Lambda \left(P_b|_{r^*}^{r^*} \right),$$

a contradiction. Then, $W((r^*, r^*)) = 0$.

(c) Let $r_0 \in \mathfrak{S}$. If $r_0 \in F_P(\hat{S})$ or $r_0 \in F_P(\hat{E})$, then from (a) the proof is finished. Assume that $r_0 \notin F_P(\hat{S})$, then $\hat{S}(r_0) \neq r_0$. Since there is an edge between $\hat{S}(r_0)$ and r_0 that is, $(r_0, \hat{S}(r_0)) \in \bar{E}(\hat{G})$, this implies that there is $\hat{S}(r_0) = r_1 \in \mathfrak{S}$ such that $(r_0, r_1) \in E(G)$. Similarly, $(r_1, \hat{E}(r_1)) \in \bar{E}(\hat{G})$ implies $(r_1, r_2) \in \bar{E}(\hat{G})$. Thus, we obtain a sequence $\{r_n\} \subset \mathfrak{S}$ such that $(r_n, r_{n+1}) \in \bar{E}(\hat{G})$,

$$\hat{S}(r_{2n}) = r_{2n+1} \text{ and } r_{2n+2} = \hat{E}(r_{2n+1}) \text{ for each } n \in \mathbb{N}.$$

If $W((r_{2m}, r_{2m+1})) = 0$ for $m \in \mathbb{N}$, then $r_{2m} = r_{2m+1} = \hat{S}(r_{2m})$, thus $r_{2m} \in F_P(\hat{S})$, and from (a), $r_{2m} \in F_P(\hat{S}) \cap F_P(\hat{E})$. Hence there is nothing to prove. Suppose, that $W((r_{2n}, r_{2n+1})) \neq 0$ for every $n \in \mathbb{N}$, that is, $r_{2n} \neq r_{2n+1}$ for every $n \in \mathbb{N}$. From (5.39),

$$\Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) \leq \Lambda \left({}_s P_b|_{r_{2n+2}}^{r_{2n+1}} \right) = \Lambda \left({}_s P_b|_{\hat{E}(r_{2n+1})}^{\hat{S}(r_{2n})} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{17}(r_{2n}, r_{2n+1}) \right) \right), \quad (5.41)$$

where,

$$\begin{aligned}
\mathcal{M}_{17}(r_{2n}, r_{2n+1}) &= \max \left\{ P_b|_{r_{2n+1}}^{r_{2n}}, P_b|_{\tilde{S}(r_{2n})}^{r_{2n}}, P_b|_{\tilde{E}(r_{2n+1})}^{r_{2n+1}}, \frac{P_b|_{\tilde{E}(r_{2n+1})}^{r_{2n}} + P_b|_{\tilde{S}(r_{2n})}^{r_{2n+1}}}{2s^2}, \right. \\
&\quad \left. \frac{P_b|_{\tilde{E}(r_{2n+1})}^{r_{2n+1}} (1 + P_b|_{\tilde{S}(r_{2n})}^{r_{2n}})}{1 + P_b|_{r_{2n+1}}^{r_{2n}}} \right\} \\
&= \max \left\{ P_b|_{r_{2n+1}}^{r_{2n}}, P_b|_{r_{2n+1}}^{r_{2n}}, P_b|_{r_{2n+2}}^{r_{2n+1}}, \frac{P_b|_{r_{2n+2}}^{r_{2n}} + P_b|_{r_{2n+1}}^{r_{2n+1}}}{2s^2}, \right. \\
&\quad \left. \frac{P_b|_{r_{2n+2}}^{r_{2n+1}} (1 + P_b|_{r_{2n+1}}^{r_{2n}})}{1 + P_b|_{r_{2n+1}}^{r_{2n}}} \right\} \\
&= \max \left\{ P_b|_{r_{2n+1}}^{r_{2n}}, P_b|_{r_{2n+2}}^{r_{2n+1}} \right\}.
\end{aligned}$$

If $\max \left\{ P_b|_{r_{2n+1}}^{r_{2n}}, P_b|_{r_{2n+2}}^{r_{2n+1}} \right\} = P_b|_{r_{2n+2}}^{r_{2n+1}}$. Then from (5.41) we have,

$$\Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) \leq \Upsilon \left(\Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) \right) < \Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right),$$

a contradiction. Then, $\max \left\{ P_b|_{r_{2n+1}}^{r_{2n}}, P_b|_{r_{2n+2}}^{r_{2n+1}} \right\} = P_b|_{r_{2n+1}}^{r_{2n}}$ and

$$\Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) \leq \Upsilon \left(\Lambda \left(P_b|_{r_{2n+1}}^{r_{2n}} \right) \right) < \Lambda \left(P_b|_{r_{2n+1}}^{r_{2n}} \right), \text{ for each } n \in \mathbb{N}. \quad (5.42)$$

By (A1), we have

$$P_b|_{r_{2n+2}}^{r_{2n+1}} < P_b|_{r_{2n+1}}^{r_{2n}}, \text{ for each } n \in \mathbb{N}. \quad (5.43)$$

Analogously,

$$P_b|_{r_{2n+1}}^{r_{2n}} < P_b|_{r_{2n}}^{r_{2n-1}}, \text{ for each } n \in \mathbb{N}. \quad (5.44)$$

From (5.43) and (5.44), we conclude that $\{P_b|_{r_{2n}}^{r_{2n+1}}\}$ is a decreasing sequence and

$$\Lambda \left(P_b|_{r_{2n+1}}^{r_{2n}} \right) \leq \Upsilon \left(\Lambda \left(P_b|_{r_{2n}}^{r_{2n-1}} \right) \right) \leq \Upsilon^2 \left(\Lambda \left(P_b|_{r_{2n-1}}^{r_{2n-2}} \right) \right) \leq \dots \leq \Upsilon^{2n} \left(\Lambda \left(P_b|_{r_1}^{r_0} \right) \right). \quad (5.45)$$

Similarly, one gets

$$\Lambda \left(P_b|_{r_{2n+2}}^{r_{2n+1}} \right) \leq \Upsilon \left(\Lambda \left(P_b|_{r_{2n+1}}^{r_{2n}} \right) \right) \leq \Upsilon^2 \left(\Lambda \left(P_b|_{r_{2n}}^{r_{2n-1}} \right) \right) \leq \dots \leq \Upsilon^{2n+1} \left(\Lambda \left(P_b|_{r_1}^{r_0} \right) \right). \quad (5.46)$$

Letting $n \rightarrow \infty$ in (5.45) and (5.46), we get

$$0 \leq \lim_{n \rightarrow \infty} \Lambda \left(P_b|_{r_{n+1}}^{r_n} \right) \leq \lim_{n \rightarrow \infty} \Upsilon^n \left(\Lambda \left(P_b|_{r_1}^{r_0} \right) \right) = 0,$$

thus,

$$\lim_{n \rightarrow \infty} \Lambda \left(P_b|_{r_{n+1}}^{r_n} \right) = 0.$$

From (A2) and Lemma 1.6.4, we get

$$\lim_{n \rightarrow \infty} P_b|_{r_{n+1}}^{r_n} = 0. \quad (5.47)$$

Further, from (P_{b2}) we have

$$\lim_{n \rightarrow \infty} P_b|_{r_n}^{r_n} = 0. \quad (5.48)$$

To prove that $\{r_n\}$ is CS in (\mathfrak{S}, P_b) , we assume the contrary, that is, there exist $\varepsilon > 0$ and $\{\tilde{I}_n\}_{n=1}^{\infty} \subset \mathbb{N}$ and $\{\hat{h}_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that

$$\text{for every } n \in \mathbb{N}, \hat{h}_n > \tilde{I}_n > n, P_b|_{r_{\tilde{I}_n}}^{r_{\hat{h}_n}} \geq \varepsilon, P_b|_{r_{\tilde{I}_n-1}}^{r_{\hat{h}_n}} < \varepsilon.$$

. Therefore,

$$\begin{aligned} \varepsilon &\leq P_b|_{r_{\tilde{I}_n}}^{r_{\hat{h}_n}} \leq s \left[P_b|_{r_{\tilde{I}_n-1}}^{r_{\hat{h}_n}} + P_b|_{r_{\tilde{I}_n}}^{r_{\tilde{I}_n-1}} \right] - P_b|_{r_{\tilde{I}_n-1}}^{r_{\tilde{I}_n-1}} \\ &\leq s \left[P_b|_{r_{\tilde{I}_n-1}}^{r_{\hat{h}_n}} + P_b|_{r_{\tilde{I}_n}}^{r_{\tilde{I}_n-1}} \right] < s\varepsilon + s P_b|_{r_{\tilde{I}_n}}^{r_{\tilde{I}_n-1}}. \end{aligned} \quad (5.49)$$

Letting $n \rightarrow \infty$ in (5.49), we get

$$\varepsilon \leq \lim_{n \rightarrow \infty} P_b|_{r_{\tilde{I}_n}}^{r_{\hat{h}_n}} < s\varepsilon. \quad (5.50)$$

By (P_{b4}) , we have

$$P_b|_{r_{\tilde{I}_n}}^{r_{\hat{h}_n}} \leq s \left[P_b|_{r_{\hat{h}_n+1}}^{r_{\hat{h}_n}} + P_b|_{r_{\tilde{I}_n}}^{r_{\hat{h}_n+1}} \right] - P_b|_{r_{\hat{h}_n+1}}^{r_{\hat{h}_n+1}} \leq s \left[P_b|_{r_{\hat{h}_n+1}}^{r_{\hat{h}_n}} + P_b|_{r_{\tilde{I}_n}}^{r_{\hat{h}_n+1}} \right], \quad (5.51)$$

and

$$P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}+1}} \leq s[P_b|_{r_{\tilde{h}(n)+1}^{r_{\tilde{h}(n)}}} + P_b|_{r_{\tilde{j}(n)}^{r_{\tilde{h}(n)}}}] - P_b|_{r_{\tilde{h}(n)}^{r_{\tilde{h}(n)}}} \leq s[P_b|_{r_{\tilde{h}(n)+1}^{r_{\tilde{h}(n)}}} + P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}}}]. \quad (5.52)$$

Taking $n \rightarrow \infty$ in (5.51) and using (5.47), (5.50), we obtain

$$\varepsilon \leq \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}}} \leq s \left(\limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}+1}} \right).$$

Again, Taking $n \rightarrow \infty$ in (5.52), we get

$$\frac{\varepsilon}{s} < \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}+1}} \leq s \left(\limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}}} \right) \leq s^2 \varepsilon.$$

Thus

$$\frac{\varepsilon}{s} \leq \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}+1}} \leq s^2 \varepsilon. \quad (5.53)$$

Similarly,

$$\frac{\varepsilon}{s} \leq \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{i}(n)+1}^{r_{\tilde{h}(n)}}} \leq s^2 \varepsilon. \quad (5.1)$$

By (P_{b4}) , we have

$$\begin{aligned} P_b|_{r_{\tilde{i}(n)-1}^{r_{\tilde{h}(n)}+1}} &\leq s[P_b|_{r_{\tilde{h}(n)}^{r_{\tilde{h}(n)}+1}} + P_b|_{r_{\tilde{i}(n)-1}^{r_{\tilde{h}(n)}}}] - P_b|_{r_{\tilde{h}(n)}^{r_{\tilde{h}(n)}}} \leq s[P_b|_{r_{\tilde{h}(n)}^{r_{\tilde{h}(n)}+1}} + P_b|_{r_{\tilde{i}(n)-1}^{r_{\tilde{h}(n)}}}]. \\ &\leq s[P_b|_{r_{\tilde{h}(n)}^{r_{\tilde{h}(n)}+1}} + \varepsilon]. \end{aligned} \quad (5.55)$$

Letting $n \rightarrow \infty$ in (5.55) and using (5.47), we have

$$\limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{i}(n)-1}^{r_{\tilde{h}(n)}+1}} \leq s\varepsilon. \quad (5.56)$$

From (5.39), we get

$$0 < \Lambda \left(s P_b|_{r_{\tilde{i}(n)}^{r_{\tilde{h}(n)}+1}} \right) \leq \Lambda \left(s P_b|_{\hat{E}(r_{\tilde{i}(n)-1})}^{\hat{S}(r_{\tilde{h}(n)})} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{17} \left(r_{\tilde{h}(n)}, r_{\tilde{i}(n)-1} \right) \right) \right), \quad (5.57)$$

where

$$\mathcal{M}_{17}(r_{\hat{h}(n)}, r_{\tilde{I}(n)-1}) = \max \left\{ \begin{array}{l} P_b|_{r_{\tilde{I}(n)-1}}^{r_{\hat{h}(n)}}, P_b|_{\hat{S}(r_{\hat{h}(n)})}^{r_{\hat{h}(n)}}, P_b|_{\hat{E}(r_{\tilde{I}(n)-1})}^{r_{\tilde{I}(n)-1}}, \frac{P_b|_{\hat{E}(r_{\tilde{I}(n)-1})}^{r_{\hat{h}(n)}} + P_b|_{\hat{S}(r_{\hat{h}(n)})}^{r_{\tilde{I}(n)-1}}}{2s^2}, \\ \frac{P_b|_{r_{\tilde{I}(n)-1}}^{\hat{E}(r_{\tilde{I}(n)-1})} \left(1 + P_b|_{\hat{S}(r_{\hat{h}(n)})}^{r_{\hat{h}(n)}}\right)}{1 + P_b|_{r_{\tilde{I}(n)-1}}^{r_{\hat{h}(n)}}} \end{array} \right\}$$

$$\leq \max \left\{ \begin{array}{l} P_b|_{r_{\tilde{I}(n)-1}}^{r_{\hat{h}(n)}}, P_b|_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}}, P_b|_{r_{\tilde{I}(n)-1}}^{r_{\tilde{I}(n)-1}}, \\ \frac{P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)}} + P_b|_{r_{\hat{h}(n)+1}}^{r_{\tilde{I}(n)-1}}}{2s^2}, \frac{P_b|_{r_{\tilde{I}(n)-1}}^{r_{\tilde{I}(n)}} \left(1 + P_b|_{r_{\hat{h}(n)+1}}^{r_{\hat{h}(n)}}\right)}{1 + P_b|_{r_{\tilde{I}(n)-1}}^{r_{\hat{h}(n)}}} \end{array} \right\}.$$

Taking $n \rightarrow \infty$ and using (5.47), (5.50), (5.53) and (5.56), we get

$$\begin{aligned} \frac{\varepsilon}{s} &\leq \limsup_{n \rightarrow \infty} \mathcal{M}_{17}(r_{\hat{h}(n)}, r_{\tilde{I}(n)-1}) \leq \max \left\{ \varepsilon, \frac{s\varepsilon + s\varepsilon}{2s^2} \right\} \\ &= \max \left\{ \varepsilon, \frac{2s\varepsilon}{2s^2} \right\} = \max \left\{ \varepsilon, \frac{\varepsilon}{s} \right\} = \varepsilon. \end{aligned}$$

Thus,

$$\begin{aligned} \Lambda(\varepsilon) &= \Lambda\left(s \frac{\varepsilon}{s}\right) \leq \Lambda\left(s \limsup_{n \rightarrow \infty} P_b|_{r_{\tilde{I}(n)}}^{r_{\hat{h}(n)+1}}\right) \leq \lim_{n \rightarrow \infty} \Upsilon\left(\Lambda\left(\mathcal{M}_{17}(r_{\hat{h}(n)}, r_{\tilde{I}(n)-1})\right)\right) \quad (5.58) \\ &\leq \Upsilon(\Lambda(\varepsilon)) < \Lambda(\varepsilon), \end{aligned}$$

a contradiction. Hence, $\{r_n\}$ is CS in (\mathfrak{S}, P_b) and from Lemma 1.2.9, in the bMS $(\mathfrak{S}, \partial_{P_b})$. Since (\mathfrak{S}, P_b) is a complete PbMS, by Lemma 1.2.9, $(\mathfrak{S}, \partial_{P_b})$ is a complete bMS. Therefore, $\exists r^* \in \mathfrak{S}$ such that

$$\lim_{n \rightarrow \infty} \partial_{P_b}|_{r^*}^{r_n} = 0. \quad (5.59)$$

Again, by Lemma 1.2.9,

$$\lim_{n \rightarrow \infty} P_b|_{r^*}^{r_n} = P_b|_{r^*}^{r^*} = \lim_{n \rightarrow \infty} P_b|_{r_m}^{r_n} = 0. \quad (5.60)$$

Now, we show that $r^* \in F_P(\hat{S})$, that is, $W(r^*, \hat{S}(r^*)) = 0$. Assume that $P_b|_{\hat{S}(r^*)}^{r^*} > 0$. If $r_{2n+1} \in V(\hat{G})$, $n \in \mathbb{N}$, Hence $(r_{2n+1}, r_{2n+2}) = (r_{2n+1}, \hat{E}(\zeta_{2n+1})) \in \bar{E}(\hat{G})$. From property (C^*) , there exists a subsequence $\{r_{2n(k)+1}\}$ of $\{r_{2n+1}\}$ with an edge between $r_{2n(k)+1}$ and r^*

for $k \in \mathbb{N}$. Using (5.39), one gets

$$\begin{aligned}
\Lambda \left(P_b|_{r_{2n(k)+2}}^{\hat{S}(r^*)} \right) &\leq \Lambda \left(s P_b|_{\hat{E}(r_{2n(k)+1})}^{\hat{S}(r^*)} \right) \tag{5.61} \\
&\leq \Upsilon \left(\Lambda \left(\max \left\{ \frac{P_b|_{r_{2n(k)+1}}^{r^*}, P_b|_{\hat{S}(r^*)}^{r^*}, P_b|_{\hat{E}(r_{2n(k)+1})}^{r_{2n(k)+1}}}{P_b|_{\hat{E}(r_{2n(k)+1})}^{r^*} + P_b|_{\hat{S}(r^*)}^{r_{2n(k)+1}}, \frac{P_b|_{r_{2n(k)+1}}^{r_{2n(k)+1}} (1 + P_b|_{\hat{S}(r^*)}^{r^*})}{1 + P_b|_{r_{2n(k)+1}}^{r^*}} \right\} \right) \right) \\
&= \Upsilon \left(\Lambda \left(\max \left\{ \frac{P_b|_{r_{2n(k)+1}}^{r^*}, P_b|_{\hat{S}(r^*)}^{r^*}, P_b|_{r_{2n(k)+2}}^{r_{2n(k)+1}}}{P_b|_{r_{2n(k)+2}}^{r^*} + P_b|_{\hat{S}(r^*)}^{r_{2n(k)+1}}, \frac{P_b|_{r_{2n(k)+1}}^{r_{2n(k)+1}} (1 + P_b|_{\hat{S}(r^*)}^{r^*})}{1 + P_b|_{r_{2n(k)+1}}^{r^*}} \right\} \right) \right).
\end{aligned}$$

Setting $k \rightarrow \infty$ in (5.61), using (Λ_3) and since Υ is continuous, one gets

$$\Lambda \left(P_b|_{\hat{S}(r^*)}^{r^*} \right) < \Upsilon \left(\Lambda \left(P_b|_{\hat{S}(r^*)}^{r^*} \right) \right) < \Lambda \left(P_b|_{\hat{S}(r^*)}^{r^*} \right),$$

a contradiction. Then, $W \left(r^*, \hat{S}(r^*) \right) = 0$, that is, $r^* \in F_P \left(\hat{S} \right)$. Similarly, $r^* \in F_P \left(\hat{E} \right)$. Hence, $r^* \in F_P \left(\hat{S} \right) \cap F_P \left(\hat{E} \right)$. The proof of (c) is ended.

(d) Frist, we Suppose thar $F_P \left(\hat{S} \right) \cap F_P \left(\hat{E} \right)$ is complete. We will prove that $F_P \left(\hat{S} \right) \cap F_P \left(\hat{E} \right)$ is a unit set. Suppose that, there are $r^*, j^* \in F_P \left(\hat{S} \right) \cap F_P \left(\hat{E} \right)$ such that $r^* \neq j^*$. As $(r^*, j^*) \in \bar{E}(\hat{G})$, so from (5.39), we have

$$\begin{aligned}
\Lambda \left(P_b|_{j^*}^{r^*} \right) &\leq \Lambda \left(s P_b|_{\hat{E}(j^*)}^{\hat{S}(r^*)} \right) \\
&\leq \Upsilon \left(\Lambda \left(\max \left\{ \frac{P_b|_{j^*}^{r^*}, P_b|_{\hat{S}(r^*)}^{r^*}, P_b|_{\hat{E}(j^*)}^{j^*}}{P_b|_{\hat{E}(j^*)}^{r^*} + P_b|_{\hat{S}(r^*)}^{j^*}}, \frac{P_b|_{\hat{E}(j^*)}^{j^*} (1 + P_b|_{\hat{S}(r^*)}^{r^*})}{1 + P_b|_{j^*}^{r^*}} \right\} \right) \right) \\
&= \Upsilon \left(\Lambda \left(P_b|_{j^*}^{r^*} \right) \right) < \Lambda \left(P_b|_{j^*}^{r^*} \right),
\end{aligned}$$

a contradiction. Thus, $r^* = j^*$. Conversely, suppose that $F_P \left(\hat{S} \right) \cap F_P \left(\hat{E} \right)$ is a unit set, then $F_P \left(\hat{S} \right) \cap F_P \left(\hat{E} \right)$ is complete. ■

Example 5.5.4. Let $\mathfrak{S} = V \left(\hat{G} \right) = \{1, 2, 3, 4, 5\}$. Define $P_b : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ by $P_b|_{r_1}^{r_1} =$

$[\max \{r_1, r_2\}]^2, \forall r_1, r_2 \in \mathfrak{S}$. Then (P_b, \mathfrak{S}) is a complete PbMS ($s = 2$). Consider, (see Figure 1 represents the graph \hat{G} with all the possible cases)

$$\bar{E}(\hat{G}) = \left\{ \begin{array}{l} (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (2, 1), (4, 1), \\ (5, 1), (3, 2), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4) \end{array} \right\}.$$

Define $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\hat{S}(r_1) = \begin{cases} 1, & r_1 \in \{1, 2, 5\}, \\ 2, & r_1 \in \{3, 4\}. \end{cases} \quad \text{and} \quad \hat{E}(r_1) = \begin{cases} 1, & r_1 \in \{1, 5\}, \\ 2, & r_1 \in \{2, 3, 4\}. \end{cases},$$

and $\Lambda, \Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, by $\Lambda(t) = te^t, \Upsilon(t) = \frac{4t}{5}$. Clearly,

$$\text{for every vertex } \nu \in \hat{G}, (\nu, \hat{S}(\nu)), (\nu, \hat{E}(\nu)) \in \bar{E}(\hat{G}).$$

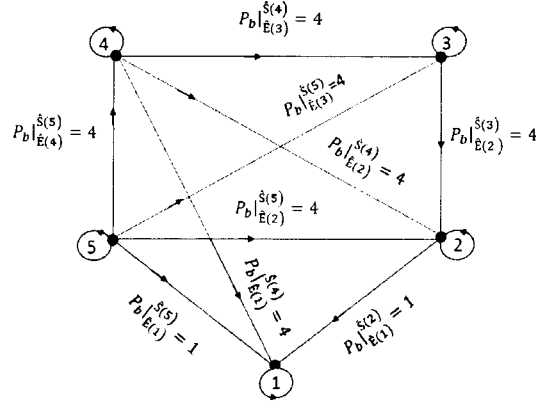
Now, for each $r_1, r_2 \in \mathfrak{S}$ with $(r_1, r_2) \in \bar{E}(\hat{G})$ and $r_1 \neq r_2$,

(r_1, r_2)	$\Lambda(sP_b _{\hat{E}(r_2)}^{\hat{S}(r_1)})$	$\Upsilon(\Lambda(M_{17}(r_1, r_2)))$
(2,1)	14.77	174.71
(4,1)	23,847.66	113,742,214.66
(5,1)	14.77	1,440,097,986,747.71
(3,2)	23,847.66	58,342.20
(4,2)	23,847.66	113,742,214.66
(5,2)	23,847.66	1,440,097,986,747.71
(4,3)	23,847.66	113,742,214.66
(5,3)	23,847.66	1,440,097,986,747.71
(5,4)	23,847.66	1,440,097,986,747.71

Hence, (\hat{S}, \hat{E}) is a \hat{C} TRG (Υ, Λ) -contraction pair. Hence from Theorem 5.5.3, \hat{S} and \hat{E} have

a CFP.

Figure 1



If $\hat{S} = \hat{E}$ in Theorem 5.5.3, we get the following corollary.

Corollary 5.5.5. Let $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a map such that \hat{S} is a \hat{C} TRG (Υ, Λ) -contraction. If Υ is continuous, then:

- (a) if $r^* \in F_P(\hat{S})$, then $W(r^*) = 0$;
- (b) $F_P(\hat{S}) \neq \emptyset$, provided that \hat{G} satisfies property (C^*) ;
- (c) $F_P(\hat{S})$ is a complete set $\Leftrightarrow F_P(\hat{S})$ is a unit set.

Theorem 5.5.6. Let (\mathfrak{S}, P_b) be a PMS with a DG \hat{G} , ($s > 1$). Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ be maps such that:

- (1) \forall vertex $\nu \in \hat{G}$, $(\nu, \hat{S}(\nu)), (\nu, \hat{E}(\nu)) \in \bar{E}(\hat{G})$;
- (2) $\exists \Upsilon \in \Omega$ and $\Lambda \in \Lambda_\Delta$ such that for each $r_1, r_2 \in \mathfrak{S}$ with $(r_1, r_2) \in \bar{E}(\hat{G})$ and $\hat{S}(r_1) \neq \hat{E}(r_2)$, we have

$$\Lambda \left(P_{\hat{E}(r_2)}^{\hat{S}(r_1)} \right) \leq \Upsilon \left[\Lambda \left(\mathcal{M}_{18}(r_1, r_2) \right) \right],$$

where

$$\mathcal{M}_{18}(r_1, r_2) = \max \left\{ P_{r_2}^{r_1}, P_{\hat{S}(r_1)}^{r_1}, P_{\hat{E}(r_2)}^{r_2}, \frac{P_{\hat{E}(r_2)}^{r_1} + P_{r_2}^{\hat{S}(r_1)}}{2s^2}, \frac{P_{r_2}^{\hat{E}(r_2)} \left(1 + P_{\hat{S}(r_1)}^{r_1} \right)}{1 + P_{r_2}^{r_1}} \right\}. \quad (5.62)$$

If Υ is continuous, then:

- (a) $F_P(\hat{S}) \neq \emptyset$ or $F_P(\hat{E}) \neq \emptyset$ if and only if $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$;

- (b) if $r^* \in F_P(\hat{S}) \cap F_P(\hat{E})$, then $W(r^*) = 0$;
- (c) $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$, provided that \hat{G} satisfies the property (\mathbb{C}^*) ;
- (d) $F_P(\hat{S}) \cap F_P(\hat{E})$ is complete set $\Leftrightarrow F_P(\hat{S}) \cap F_P(\hat{E})$ is a unit set.

Proof. Take $s = 1$ in Theorem 5.5.3. ■

Example 5.5.7. Let $\mathfrak{S} = V(\hat{G}) = [0, 1]$ and $P : \mathfrak{S} \times \mathfrak{S} \rightarrow \mathbb{R}_0^+$ be defined by $P_{r_2}^{r_1} = \max\{r_1, r_2\}$, for each $r_1, r_2 \in \mathfrak{S}$. Then (P, \mathfrak{S}) is a complete PMS. Consider,

$$\bar{E}(\hat{G}) = \{(r_1, r_2) : r_1, r_2 \in [0, 1]\}.$$

Define $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\hat{S}(r_1) = \frac{r_1}{4} \text{ and } \hat{E}(r_1) = \frac{r_1}{5},$$

and $\Lambda, \Upsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, by $\Lambda(t) = te^t$, $\Upsilon(t) = \frac{9t}{10}$. Clearly, for every vertex $\nu \in \hat{G}$, we have $(\nu, \hat{S}(\nu)), (\nu, \hat{E}(\nu)) \in \bar{E}(\hat{G})$. Now, for each $(r_1, r_2) \in \mathfrak{S}$, with $r_1 \neq r_2 \neq 0$,

$$\Lambda\left(P_{\hat{E}(r_2)}^{\hat{S}(r_1)}\right) \leq \Upsilon[\Lambda(\mathcal{M}_{18}(r_1, r_2))].$$

Therefore, (\hat{S}, \hat{E}) is a \acute{C} TRG (Υ, Λ) -contraction pair. Therefore, the hypotheses of Theorem 5.5.6 are satisfied, and \hat{S}, \hat{E} have a CFP.

5.5 Some consequences

In this section, we introduce some consequences for Theorem 5.5.3.

Corollary 5.6.1. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ be maps suchthat:

- (1) for each vertex $\nu \in \hat{G}$, $(\nu, \hat{S}(\nu)), (\nu, \hat{E}(\nu)) \in \bar{E}(\hat{G})$;
- (2) there exist $\theta \in \Xi$ and $0 < k < 1$ such that for each $r_1, r_2 \in \mathfrak{S}$, with $(r_1, r_2) \in \bar{E}(\hat{G})$ and $\hat{S}(r_1) \neq \hat{E}(r_2)$, we have

$$\theta\left(s P_b\left(\frac{\hat{S}(r_1)}{\hat{E}(r_2)}\right)\right) \leq [\theta(\mathcal{M}_{17}(r_1, r_2))]^k.$$

Then the following conditions hold:

- (a) $F_P(\hat{S}) \neq \emptyset$ or $F_P(\hat{E}) \neq \emptyset$ if and only if $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$;

- (b) if $r^* \in F_P(\hat{S}) \cap F_P(\hat{E})$, then $W(r^*) = 0$;
- (c) $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$, provided that \hat{G} satisfies property (\mathbb{C}^*) ;
- (d) $F_P(\hat{S}) \cap F_P(\hat{E})$ is a complete set if and only if $F_P(\hat{S}) \cap F_P(\hat{E})$ is a unit set.

Proof. Take $\Upsilon(t) = kt$ and $\Lambda(t) = \ln(\theta(t))$ in Theorem 5.5.3. ■

Corollary 5.6.2. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ be maps such that:

- (1) for each vertex $\nu \in \hat{G}$, $(\nu, \hat{S}(\nu)), (\nu, \hat{E}(\nu)) \in \bar{E}(\hat{G})$;
- (2) there exist $\mathbf{F} \in F^*$ and $\vartheta \in \mathbb{R}^+$ such that for each $(r_1, r_2) \in \mathfrak{S}$ with $(r_1, r_2) \in \bar{E}(\hat{G})$ and $P_b|_{\hat{E}(r_2)}^{\hat{S}(r_1)} > 0$, we have

$$\vartheta + \mathbf{F} \left(s P_b|_{\hat{E}(r_2)}^{\hat{S}(r_1)} \right) \leq \mathbf{F}(\mathcal{M}_{17}(r_1, r_2)).$$

Then the following conditions hold:

- (a) $F_P(\hat{S}) \neq \emptyset$ or $F_P(\hat{E}) \neq \emptyset$ if and only if $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$;
- (b) if $r^* \in F_P(\hat{S}) \cap F_P(\hat{E})$, then $W(r^*) = 0$;
- (c) $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$, provided that \hat{G} satisfies property (\mathbb{C}^*) ;
- (d) $F_P(\hat{S}) \cap F_P(\hat{E})$ is a complete set if and only if $F_P(\hat{S}) \cap F_P(\hat{E})$ is a unit set.

Proof. Set $\Upsilon(t) = e^{-\vartheta}t$ and $\Lambda(t) = e^{\mathbf{F}(t)}$ in Theorem 5.5.3. ■

Corollary 5.6.3. Let $\hat{S}, \hat{E} : \mathfrak{S} \rightarrow \mathfrak{S}$ be maps such that:

- (1) for every vertex $\nu \in \hat{G}$, $(\nu, \hat{S}(\nu)), (\nu, \hat{E}(\nu)) \in \bar{E}(\hat{G})$;
- (2) if for each $(r_1, r_2) \in \mathfrak{S}$, with $(r_1, r_2) \in \bar{E}(\hat{G})$,

$$P_{\hat{E}(r_2)}^{\hat{S}(r_1)} \leq \mathfrak{B}(\mathcal{M}_{17}(r_1, r_2)) \cdot \mathcal{M}_{17}(r_1, r_2),$$

and $\mathfrak{B} : \mathbb{R}_0^+ \rightarrow [0, 1)$ is such that $\lim_{r \rightarrow t^+} \mathfrak{B}(r) < 1, \forall t \in \mathbb{R}_0^+$.

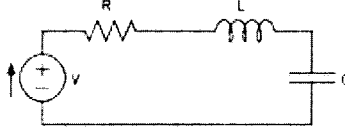
Then:

- (a) $F_P(\hat{S}) \neq \emptyset$ or $F_P(\hat{E}) \neq \emptyset$ if and only if $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$;
- (b) if $r^* \in F_P(\hat{S}) \cap F_P(\hat{E})$, then $W(r^*) = 0$;
- (c) $F_P(\hat{S}) \cap F_P(\hat{E}) \neq \emptyset$, provided that \hat{G} satisfies property (\mathbb{C}^*) ;
- (d) $F_P(\hat{S}) \cap F_P(\hat{E})$ is a complete set if and only if $F_P(\hat{S}) \cap F_P(\hat{E})$ is a unit set.

Proof. Take $\Upsilon(t) = \mathfrak{B}(t)t$ and $\Lambda(t) = t$ in Theorem 5.5.3. ■

5.6 Application to electric circuit equations

Here, we shall apply obtained results in the section 5.4 to study the existence of a solution of electric circuit equation (ECE) (see [21]). It is well known that electric circuit



contains a voltage V_e in series, a resistor R_{ec} , an inductor L_e , an electromotive force E_e and a capacitor C_e . If the current I_e is the rate of change of charge Q_e with respect to time t , we have,

$$I_e = \frac{dQ_e}{dt} \text{ and } (1) V_e = I_e R_{ec}, (2) V_e = \frac{Q_e}{C_e}, (3) V_e = L \frac{dI_e}{dt}.$$

From Kirchoff's voltage law,

$$I_e R_{ec} + \frac{Q_e}{C_e} + L \frac{dI_e}{dt} = V_e(t),$$

or

$$I_e R_{ec} + \frac{Q_e}{C_e} + L \frac{dI_e}{dt} = V_e(t), \quad Q_e(0) = 0, \quad Q'_e(0) = 0. \quad (5.63)$$

The Green function associated to (5.63) is given by

$$G(t, s) = \begin{cases} -se^{\tau(s-t)}, & \text{if } 0 \leq s \leq t \leq 1, \\ -te^{\tau(s-t)}, & \text{if } 0 \leq t \leq s \leq 1 \end{cases}$$

where $\tau \geq 1$ is calculated in terms of L and R .

Let $\mathfrak{S} = C([0, 1])$ be the set of all continuous functions defined on $\hbar = [0, 1]$. The PbM P_b on \mathfrak{S} is defined by

$$P_b|_{r_2}^{r_1} = \max_{0 \leq t \leq 1} |r_1(t) - r_2(t)|^2 e^{-4\tau t}.$$

Moreover, we define the graph \hat{G} with the partial ordered relation:

$$r_1, r_2 \in C(\hbar), r_1 \leq r_2 \Leftrightarrow r_1(t) \leq r_2(t), \forall t \in \hbar.$$

Let $\bar{E}(\hat{G}) = \{(r_1, r_2) \in \mathfrak{S} \times \mathfrak{S} : r_1 \leq r_2\}$. Then (P_b, \mathfrak{S}) is a complete PbMS ($s = 2$), endowed with a DG \hat{G} . Obviously, $\Delta = (\mathfrak{S} \times \mathfrak{S}) \in \bar{E}(\hat{G})$, and $(P_b, \mathfrak{S}, \hat{G})$ has property(C*).

Theorem 5.7.1. Let $\hat{S} : C(\hbar) \rightarrow C(\hbar)$ of a PbMS $(C(\hbar), P_b)$. Suppose that:

(1) \exists a non decreasing and continuous function $\tilde{U} : \hbar \times \mathbb{R} \rightarrow \mathbb{R}$ such that for each $r_1, r_2 \in C(\hbar)$, with $r_1 \leq r_2$,

$$\left| \tilde{U}(t, r_1) - \tilde{U}(t, r_2) \right| \leq \tau^2 e^{-\tau} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)},$$

where

$$\mathcal{M}_{\hat{S}}(r_1, r_2) = \max \left\{ P_b|_{r_2}^{r_1}, P_b|_{\hat{S}(r_1)}^{r_1}, P_b|_{\hat{S}(r_2)}^{r_2}, \frac{P_b|_{\hat{S}(r_2)}^{r_1} + P_b|_{r_2}^{\hat{S}(r_1)}}{2s^2}, \frac{P_b|_{r_2}^{\hat{S}(r_2)} \left(1 + P_b|_{\hat{S}(r_1)}^{r_1} \right)}{1 + P_b|_{r_2}^{r_1}} \right\}.$$

where $t \in \hbar$, and $\tau \geq 1$,

(2) $\forall r_1 \in C(\hbar)$, $r_1 \leq \int_0^1 G(t, s) \tilde{U}(t, r_1(s)) ds$, $\forall t \in \hbar$.

Then the equation (5.63) has only one solution.

Proof. The equation 5.63 is equivalent to the following equation,

$$r_1(t) = \int_0^1 G(t, s) \tilde{U}(t, r_1(s)) ds, t \in \hbar. \quad (5.64)$$

Define $\hat{S} : \mathfrak{S} \rightarrow \mathfrak{S}$ by

$$\hat{S}(r_1(t)) = \int_0^t G(t, s) \tilde{U}(t, r_1(s)) ds, t \in \hbar. \quad (5.65)$$

Then, r^* is a solution of (5.64) if and only if r^* is a FP of \hat{S} . By condition (2), clearly, for every $r_1 \in \mathfrak{S}$, we have $r_1 \leq \hat{S}(r_1)$, i.e., $(r_1, \hat{S}(r_1)) \in \bar{E}(\hat{G})$. Thus, from condition (2), $\mathfrak{S}_\phi = \left\{ r_1 \in \mathfrak{S} : r_1 \leq \hat{S}(r_1), \text{ i.e., } (r_1, \hat{S}(r_1)) \in \bar{E}(\hat{G}) \right\} \neq \emptyset$. Let $r_1, r_2 \in \mathfrak{S}$, then from condition

(1), we have

$$\begin{aligned}
\left| \hat{S}(r_1(t)) - \hat{S}(r_2(t)) \right| &\leq \int_0^t G(t,s) \left| \tilde{U}(t, r_1(s)) - \tilde{U}(t, r_2(s)) \right| ds \\
&\leq \int_0^t G(t,s) \tau^2 e^{-\tau} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)} ds \\
&\leq \int_0^t \tau^2 e^{-\tau} e^{-2\tau s} e^{2\tau s} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)} G(t,s) ds \\
&\leq \tau^2 e^{-\tau} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)} \int_0^t e^{2\tau s} G(t,s) ds \\
&\leq e^{-\tau} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)} \left[e^{2\tau t} (1 - 2t\tau + t\tau e^{-\tau t} - e^{-\tau t}) \right].
\end{aligned}$$

Thus,

$$\left| \hat{S}(r_1(t)) - \hat{S}(r_2(t)) \right| e^{-2\tau t} \leq e^{-\tau} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)} \left[1 - 2t\tau + t\tau e^{-\tau t} - e^{-\tau t} \right].$$

This implies that

$$\left| \hat{S}(r_1(t)) - \hat{S}(r_2(t)) \right| e^{-2\tau t} \leq e^{-\tau} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)} \left[1 - 2t\tau + t\tau e^{-\tau t} - e^{-\tau t} \right].$$

Since $1 - 2t\tau + t\tau e^{-\tau t} - e^{-\tau t} \leq 1$, so

$$\left| \hat{S}(r_1(t)) - \hat{S}(r_2(t)) \right| e^{-2\tau t} \leq e^{-\tau} \sqrt{\mathcal{M}_{\hat{S}}(r_1, r_2)}.$$

Hence,

$$P_b |_{\hat{S}(r_2(t))}^{\hat{S}(r_1(t))} \leq e^{-2\tau} \mathcal{M}_{\hat{S}}(r_1, r_2).$$

Taking $\Lambda(t) = t$ and $\Upsilon(t) = \frac{2}{e^{2\tau}} t$, one gets

$$\Lambda \left({}_s P_b |_{\hat{S}(r_2(t))}^{\hat{S}(r_1(t))} \right) \leq \frac{2}{e^{2\tau}} \Lambda \left(\mathcal{M}_{\hat{S}}(r_1, r_2) \right) = \Upsilon \left(\Lambda \left(\mathcal{M}_{\hat{S}}(r_1, r_2) \right) \right).$$

Thus,

$$\Lambda \left({}_s P_b |_{\hat{S}(r_2(t))}^{\hat{S}(r_1(t))} \right) \leq \Upsilon \left(\Lambda \left(\mathcal{M}_{\hat{S}}(r_1, r_2) \right) \right).$$

Therefore, from Corollary 5.5.5, \hat{S} has a FP. Hence, the equation (5.63) has a solution. ■

Conclusion and Future Work

In this thesis, we initiated the notion of cyclic (α_*, β_*) -type- γ -FG-contraction type for multivalued maps and established related FP results in PbMSs. These results are popularizations of recent FP theorems of Wardowski [120], Padhan *et al.* [103] and some other results in the literature. Then we applied these FP theorems to examine the existence of a solution of NIEs. We introduced the notion of Branciari F-contraction and established FP theorems for such contractions in BMSs.

Next, we introduced the new type of contractions for a mapping and a relation and proved certain COF theorems in CMSs. An application to the existence of a unique solution for the integral equation was also provided. We structured two CFP theorems comprising four self-mappings involving GST- (θ, \hat{C}) -R contractions and GST \acute{C} -JS-contractions in CMSs. The existence of the solution of NFDEs was shown through a CFP result.

The $(\Lambda, \Upsilon, \acute{R}_{\mathfrak{S}})$ -contraction considered in the section 4.1 of chapter 4 which encompasses both θ -contractions ($\theta \in \Xi$) and F-contractions ($F \in F^*$). The theorems obtained are applicable to solve NMEs. Furthermore, we investigated the existence of common $\check{\epsilon}$ -fuzzy FPs for two FMs under generalized almost (Υ, Λ) -contraction in CMSs and we also gave illustrative examples to elaborate these result. We further showed some relation of multi-valued mappings and FMs, which can be utilized to derive CFP for multivalued mappings. These FP results are generalizations of recent FP results of Liu et al. [80], Jleli et al. [66, 65], Wardowski [120] and Piri and Kumam [97] and some other results in the literature

Afterwards, we investigated CFP theorems for generalized $(\alpha_s^*, \Upsilon, \Lambda)$ -contraction multivalued pair of mappings in α_s -complete PbMSs. We applied these results to solve systems of FEs. Then, we introduced \acute{C} TRG (Υ, Λ) -contraction and investigated related CFP theorems for a pair of mappings in PbMSs endowed with a DG. Furthermore, we gave examples to clarify our results. Lastly, we applied these results to study the existence of a solution of ECEs. The new concepts lead to further investigations and applications.

The following open problems are suggested for future work.

- The results offered in chapter 2 can be extended by defining F-contractions for a pair of set valued mappings.

- what occurs if functions Υ and Λ are not continuous in chapters 4 and 5.
- A lot of research work can be done by defining a distance function on the subsets of a MS (or PbMS) instead of using Hausdorff metric.
- We intend to do some work on \mathbb{F} -contraction, θ -contraction and (Υ, Λ) -contraction in \mathbb{F} -metric spaces and (ϕ, ψ) -metric spaces.
- Instead of taking CMS, complete BMS and complete PbMS for the convergence of CS one may use some weaker constraint.
- We intend to do some work on \mathbb{F} -contraction, θ -contraction and (Υ, Λ) -contraction in M -metric spaces and Mb -metric spaces.
- Can we define a new type of contraction which encompasses both \mathbb{F} -contraction, θ -contraction and (Υ, Λ) -contraction.
- We feel that our results in the section 2.1 of chapter 2 and the section 5.1 of chapter 5 can be used for the existence of solution set of differential and integral inclusions.

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