Comparative study of solutions for non-Linear problems

By

Ahmed Zeeshan

A Thesis
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
MATHEMATICS

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Pakistan

2011

1. Linear programming

2. Programming mathematics

بسم الله الرحمن الرحيم

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Certificate

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE ***DOCTOR OF PHILOSOPHY in MATHEMATICS***

We accept this thesis as conforming to the required standard.

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Declaration

The work described in this thesis was carried out under the supervision and direction of Dr. Rahmat Ellahi, Department of Mathematics and Statistics, International Islamic University. No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning. The thesis is my original work except where due reference and credit is given.

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Ahmed Zeeshan

PhD. (Mathematics)

Dedication

This thesis is dedicated to my father,

my beloved mother,

my family and my uncle Dr. Saleem Asghar.

Acknowledgements

First and foremost, I am thankful to **Almighty Allah**, who created us, taught us everything we know, provided us with the balance, health, knowledge and Intelligence to explore his world. No doubt hesitation while saying فَبِأَيِّ آلَاءِ رَبِّكُمَا تُكَذِّبَانِ . I thank lord Almighty whose help and will I, today able to achieve yet another milestone of life. Salaam upon last Messenger of Allah, **Hazrat Muhammad** (SALLAH O ALI WALI WASALAM) who is forever a torch of gaudiness, a source of knowledge and blessings for entire creation. His teaching shows us a way to live with dignity, stand with honor and learn to be humble.

I express my profound gratitude to my respectable devoted supervisors, **Dr. Rahmat Ellahi**, who helped me through my PhD. with his kind and cooperative guidance. His many valuable comments and suggestion greatly improve my knowledge in the field. I am placing my earnest thanks to **Dr. Rahmat Ellahi**. I express my gratitude to all my teachers whose teaching has brought me to this stage of academic zenith. I couldn't thank my teachers specially Dr. sohail Nadeem, Dr. Malik M. Yousaf and Mr. Waseem siddique without whose continues help I couldn't stand here today.

I express my gratitude to **Dr. Mumtaz** president International Islamic University whose vital support enable me to complete my thesis well on time. I thank **Dr. Irshad Ahmed Arshad** chairman Department of Mathematics and statistics for providing support and helpful environment in the department.

I would like to thank all my **friends** specially Aftab, Waqar, Nouman, Arshad, Kashif, Arshad Raiz, Shafiq and Sana. My **collegues** Dr. Arshad Zia, Dr. Zaheer Abbas, Dr. Nasir Ali, Dr. Ahmer mehmood, Mr Naiz Ahmed and everyone directly or indirectly helped me and pray for me.

I am also grateful to Higher Education Commission of Pakistan (HEC) to provide funding during my research work under NRPU.

Finally, I pay my Gratitude to **Dr. F. M. Mahomed** who helped me during his visit to Pakistan in Lie Analysis, **Dr. S. Abbasbandy** and **Dr. Tariq Javed** for their time and support regarding HAM in chapter 2 and 9 of this work. **Dr. Tasweer Hayat** for his continues support and well wishing.

I am indebted to my Uncle and fatherly figure **Prof. Dr. Saleem Asghar** without whose continuous persuasion, encouragement, moral and academic support his kind, eminent, affectionate nature and encouraging behavior enlightened me to work in this field and achieve what is a distant dream for most and I express my gratitude to my beloved **family**, who are always there for encouragement and support, even in the gloomiest time of life they were always encouraging me and showered their everlasting love, care and support throughout my life. Their humble prayers have always been a source of success for me and whose sustained hope led myself to where, I am today.

Ahmad Zeeshan

PREFACE

This thesis deals with the comparative study of finding solutions of non-linear problems arising in fluid mechanics. The choice of such problem is challenging mathematically and desirable from applications point of view. The governing Naiver Stokes equations for the fluid flow are non-linear partial differential equations and it is overwhelmingly difficult to find exact analytic solution of these equations. Various approximate analytic methods and numerical techniques are available to address such problem to certain level of accuracy. In this thesis the applicabihty and consequences of these methods are investigated to make a comparison between these and to find the best solution for a variety of problems arising in the domain of non-Newtonian fluids.

In chapter one, we present basic ideas of nonlinear differential equations arising in fluid mechanics and discuss various analytic techniques used to solve these equations

Chapter two contains modeling and analytic solution for the flow of non-Newtonian fluid lubricating linear and parabolic slider bearings. Such fluids are important when polymers are added to lubricating oils for improving their viscosity index and make them less temperature dependent,Some interesting investigations explaining the phenomenon of non-Newtonian lubrication in bearings have been presented. Some numerical treatment of such problems was available in the literature and the main objective is to present analytical solution using HAM. The error analysis and the convergence questions are also addressed.

The results of this chapter are published in the journal of **Numer Methods for Partial Differential Eq 27; n/a. doi:10.10021num. 20578, 2010.**

In chapter four some fundamental problems such as Couette, Poiseuille and Generalized Couette flows in the presence of a slip condition have been studied. The slip condition gives rise to nonlinear boundary conditions in contrast to the usual no slip condition. The slip condition in non-Newtonian fluids are less attended which are otherwise important in such flows .Exact solutions are developed while solving the nonlinear governing equations with nonlinear boundary conditions. These observations are published in the journal of **Zeitschrift f¨ur angewandte Mathematik und Physik (ZAMP) 61; 877–888, 2010**.

The aim of chapter five is to examine the steady flow of an incompressible third grade fluid through a porous horizontal pipe with variable viscosity. A new dimension of variable viscosity has been added instead of most frequented constant viscosity assumptions. Only a limited work has been accomplished taking viscosity as temperature and space dependent. The solution for MHD third grade fluid flow in the horizontal porous pipe with viscous dissipation is addressed taking into consideration the slip effects. These results has been submitted for publication in the journal of **Transport in porous media (2011)**.

Chapter six deals with recent phenomenon of nanofluids (NF).A brief description of nano fluids goes like: The nanoparticles are ultra fine particles in the size of nanometer order. A base fluid with suspended nano size particles are called nanofluids. These suspended nanoparticles can change the transport and thermal properties of the base fluid. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis. The pipe flow of third grade nano fluid with variable viscosity is considered and solved by HAM. Effects of porosity and MHD are considered on emerging velocity field. These findings are submitted in the journal of **Applied mathematical modelling (2011)**.

In chapter seven, we extend the problem of chapter six by introducing linear partial slip effects and using Optimal Homotopic Asymptotic Method (OHAM) to solve differential equation of third grade nanofluid between coaxial porous cylinders with variable viscosity. The effects of heat transfer analysis and concentration of nanoparticles are considered in the presence of magnetohydrodynamic. The contents of this chapter are submitted in the journal of **Porous media (2011)**.

The aim of chapter eight is to examine the MHD steady flow of a third grade fluid in the annular region when both cylinders rotate with different but constant angular velocity. Approximate analytical solutions to the resulting nonlinear problem is derived when the magnetic and third grade material parameters are small. The contents of this chapter are published in **Commun Nonlinear Sci Numer Simulat, 15; 1224–1227, 2010**.

The main objective in chapter nine is to venture further in the regime of non-Newtonian fluids (Oldroyd constant) with nonlinear conditions while investigating the closed form solutions of Couette, Poiseuille and Generalized Couette flows. These findings are published in Commun Nonlinear Sci Numer Simulat, 15; 322–330, 2010.

In chapter three, we draw comparison of HAM with the other analytical techniques in finding the solution of second painleve equation. The results obtained with HAM are found to be better in accuracy than ADM, HPM and Legendre Tau. These results are published in the journal of **Numer Methods for Partial Differential Eq, 26; 1070–1078, 2010**.

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NOMENCLATURE

English Symbols

u, v .... Components of velocity,

p .... Non-dimensional pressure,

b............................. Width of slider bearings,

Re........................... Reynold number,

L ....Length of slider bearing,

q............................. Embedding parameter,

s.............................. Clearance ratio,

$R_{1m} - R_{12m}$..............mth order deformation equations,

B_0...........................Magnetic field,

P ....Non-dimensional porosity,

$\frac{d}{dt}$...........................Material derivative,

b ....Body forces,

T ....Stress tensor,

I ....Identity tensor,

$\mathbf{A}_n$ ....Rivilin Erickson tensor,

p ....Pressure,

c..............................Constant pressure gradient,

$\mathcal{L}$............................Linear operator,

$\mathcal{N}_i$...........................Nonlinear operator in HAM,

N ....Nonlinear operator in OHAM,

k_1 ....Permeability,

$\mathbf{V}$ ....Velocity vector,

R ....Radius,

r ....Dimensionless radius,

B_0 ....Magnetic field strength,

M ....Dimensionless Hartmann number,

P ....Dimensionless porosity,

$\mathbf{R}$ ....Darcy's resistance,

k ....Thermal conductivity,

N_b ....Brownian motion parameter,

N_t ....Thermophoresis parameter,

B_r ....Brownian diffusion coefficient,

G_r ....Thermophoresis diffusion coefficient,

Q .... heat flux,

e .... Internal energy,

c_p .... Specific heat,

c_f .... Dimensionless form of drag constant,

$\mathbf{E}$ .... Electric field intensity,

$\mathbf{B}$ .... Magnetic field,

$\mathbf{J}$ .... Induced current density,

$\hbar$ .... convergence parameter,

$\mathcal{B}$ .... Boundary conditions,

$C_1 - C_{15}$ ....Constant involved.

Greek Symbols

λ_1, λ_2.................... Non-dimensional second grade parameter,

σ .... Electric conductivity,

γ .... Slip length,

θ .... Dimensionless temperature profile,

ϕ .... Dimensionless concentration profile,

$\beta_1, \beta_2, \beta_3$ .... Third grade parameter,

$\lambda_3 - \lambda_9$ .... Oldroyd constant,

μ .... Kinematic viscosity,

α_1, α_2 .... Second grade parameter,

β..............................Non-dimensional third grade parameter,

α .... Small parameter,

Ω .... Angular velocity,

ψ............................ Porosity,

φ .... Unknown function

ρ .... Density,

ρ_e .... Charge density,

∇ .... Del operator,

Γ .... Non-dimensional heat transfer parameter.

Abbreviations

HAM...................... Homotopic Analysis Method,

OHAM .... Optimal Homotopic Asymptotic Method,

ADM .... Adomian Decomposition Method,

HPM .... Homotopic Perturbation Method,

PDE .... Partial Differential Equations

NF .... NanoFluid.

CHAPTER 1

PRELIMINARIES

Fundamental laws of nature can be generally formulated in terms of differential equations and their solutions and analysis is widely used in applied mathematics, physics, engineering and other fields of natural and social sciences. These differential equations may be linear or nonlinear. The solution of non linear equations is generally very complex and there are no well established methods to solve these equations. There are therefore some methods which provide approximate analytical and numerical solutions. We will be particularly interested in non linear partial differential equations occurring in non Newtonian fluid mechanics. In this chapter, we will discuss some recent analytical techniques which will be used in our subsequent analyses. Before that we present the formulation of the basic equations of fluid mechanics.

Newtonian and Non-Newtonian Fluids:

The fluids in which the shear stress is lineally proportional to the rate of shear strain are called Newtonian fluid. Water and air are considered as Newtonian fluid.

The fluids in which there is nonlinear proportionality between shear stress and rate of shear strain are termed as non-Newtonian fluids. A number of industrially important fluids such as molten plastics, polymers, pulps, foods and slurries display non-Newtonian fluid behavior.

Non-Newtonian fluid flow is a topic of great interest and several investigations have been made in this direction; for example the work appearing in the references [1] to [17]. The interest in these flows is generated due to their extensive applications in industry and engineering. An assessment of technological applications requires knowledge of the rheological characteristics of the non-Newtonian fluids. We now know that the features of non-Newtonian fluids, being different from the viscous fluids, cannot be predicted by employing a single constitutive equation. Therefore, several models of non-Newtonian fluids have been suggested. A second grade model is the simplest subclass of differential type fluids. The constitutive relations of non-Newtonian fluids further add complexities to the mathematical expressions. A novel feature of the problem governing the flow of non-Newtonian fluids, in general, is the presence of viscoelasticity of the fluid which increases the order of the differential equation.

Nanofluids:

The nanoparticles are ultra fine particles of the size of nanometer order. A base fluid with suspended nano size particles is called nanofluids (NF). Materials with sizes of nanometers possess unique physical and chemical properties These particles help in increasing the thermal conductivity of fluids. Although, the enhancement of thermal conductivity of conventional fluids by the suspension of solid particles, such as millimeter- or micrometer-sized particles, has been known for more than 100 years. However, they have not been of interest for practical applications due to problems of sedimentation, erosion, fouling and increased pressure drop of the flow channel. The nano size particles now have come to overcome these issues. Nanotechnology has been now widely used in industry. These suspended nanoparticles can change the transport and thermal properties of the base fluid. The model used for the nanofluid incorporates the effects of Brownian motion and thermophoresis. These are firstly introduced by Choi [18]. Choi et al. [19] showed that the addition of a small amount (less than 1 %)of volume of nanoparticles to conventional heat transfer liquids increased the thermal conductivity of the fluid two times (approximately). Khanafer et al. [20] seem to be the first who have examined heat transfer performance of nanofluids inside an enclosure taking into account the solid particle dispersion. After these studies, nanotechnology is considered by many to be one of the significant forces that drive the next major industrial revolution of this century. It aims at manipulating the structure of the matter at the molecular level with the goal for innovation in virtually every industry and its applications include biological sciences, physical sciences, electronics cooling, transportation, environment and national security etc. Some numerical and experimental studies on nanofluids can be found in [21] to [23].

Porous Medium:

A porous medium is a material consisting of a solid matrix with interconnected void (Pores). Examples of natural porous media are beach sand, sandstone, limestone, rye bread, wood, and the human lung. The porosity ψ of a porous medium is defined as the ratio of the void space to the total volume of the medium . Neild and Bejan explained different model and constitutive equations along with their merits and demerits in their book [24].

Henry Darcy after experimental investigations on steady-state unidirectional water flow in a uniform medium showed a relationship between flow rate and the applied pressure difference. In mathematical form this is expressed by

$$u = -\frac{k_1}{\mu}\frac{\partial p}{\partial x}. \tag{1.1}$$

Here μ is the dynamic viscosity of the fluid. The coefficient k_1 is independent of the nature of the fluid but it depends on the geometry of the medium. It has dimensions $(\text{length})^2$ and is called the specific permeability or intrinsic permeability of the medium. In three dimensions, Eq. (1.1) generalizes to

$$\mathbf{V} = -\frac{\mathbf{k_1}}{\mu}\nabla p, \tag{1.2}$$

where the permeability $\mathbf{k}_1$ is in general a second-order tensor. Rearranging the Eq. (1.2), we obtain

$$\nabla p = -\frac{\mu}{\mathbf{k}_1}\mathbf{V}. \tag{1.3}$$

Magnetohydrodynamics (MHD):

Magnetic fields are known to influence many natural and man-made flows. They are routinely used in industry to heat pump, stir and levitate liquid metals. There is the terrestrial magnetic field which is maintained by fluid motion in the earth's core, the solar magnetic field which generates sunspots and solar flares, and the galactic field which influences the formation of stars.

In MHD, the charge density ρ_e plays no significant part. The electric force, $q\mathbf{E}$, is minute by comparison with the Lorentz force, and that the contribution of $\partial\rho_e/\partial t$ to the charge conservation equation is also negligible (see [25]-[26]). Maxwell's equations can now be written as:

Solenoidal nature of **B**

$$\nabla.\mathbf{B} = 0. \tag{1.4}$$

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t}. \tag{1.5}$$

Ampere equation

$$\nabla \times \mathbf{B} = \mu\mathbf{J}. \tag{1.6}$$

Charge conservation

$$\nabla.\mathbf{J} = 0. \tag{1.7}$$

Lorentz Force

$$\mathbf{F} = \mathbf{J} \times \mathbf{B}. \tag{1.8}$$

Ohm's law

$$\mathbf{J} = \sigma\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right). \tag{1.9}$$

Slip Conditions:

With the advent of viscous fluid theory,it is widely accepted that the flow satisfies no slip boundary conditions. These conditions have been widely used in the fluid flow problems arising in Newtonian and non- Newtonian fluids [27] to [29]. However, there could be situations where no slip conditions are not adequate and it is more reasonable to assume slip conditions [30] to [33] instead. Such situations arise for fluid flow past permeable walls, slotted plates, rough and coated surfaces, gas and liquid flows in micro devices.

It is now established that slip effects may appear for two types of fluids i.e., rare field gases and fluids having much more elastic character. In these fluids, slippage appear subject to large tangential traction. Naiver [34] suggested slip condition at a rigid boundary which states that the fluid velocity at the plate is linearly proportional to the shear stress at the plate, later proposed independently by Maxwell [35] The constant of proportionality is called a slip length and may be regarded as the distance at which the velocity of the fluid is equal to that of the boundary [36]. Qian and Wang [37] showed that the amount of slip for Newtonian fluid is proportional to the shear rate. This is the simplest known boundary condition used to improve the no-slip condition.

The Naiver boundary conditions at the solid wall may be written as

$$\mathbf{V} = 2\gamma\mathbf{A}_1, \tag{1.10}$$

where γ is the slip length with the same sign as $\mathbf{A}_1$ is first rivilin erickson tensor.

1.1 Governing Equations of Fluid Mechanics

The basic equations of fluid dynamics are conservation of mass, momentum and energy equations. These equations show the time rate of change of mass, momentum or energy at a point in a fluid representing various physical mechanisms.

Continuity Equation

The partial differential equation expressing conservation of mass is called the continuity equation. The continuity equation involves only the fluid density and the fluid velocity. It is applicable to all fluids, compressible and incompressible, Newtonian and non-Newtonian. For the whole range of flow speeds it is given by

$$\frac{d\rho}{dt} + \mathbf{\nabla}.(\rho\mathbf{V}) = 0. \tag{1.11}$$

For an incompressible fluid, the density is constant and the continuity equation reduces to

$$\mathbf{\nabla}.\mathbf{V} = 0. \tag{1.12}$$

For nanofluid the additional equation for the continuity of nanoparticles in nanofluids.

$$\frac{\partial\varphi}{\partial t} + (\mathbf{V}.\mathbf{\nabla})\varphi = \mathbf{\nabla}.\left(D_B\mathbf{\nabla}\varphi + \frac{D_T}{\theta}\mathbf{\nabla}\theta\right), \tag{1.13}$$

where $\boldsymbol{\theta}$ is temperture and φ is volume fraction of nanoparticle.

Momentum Equation

The partial differential equations expressing conservation of linear momentum are called momentum equation. The equations representing Newtonian fluids are famously known as Naiver–Stokes equation. General momentum equation is given by

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}.\boldsymbol{\nabla})\mathbf{V}\right)=\rho\mathbf{b}+\boldsymbol{\nabla}.\mathbf{T}, \tag{1.14}$$

where $\mathbf{T}$ is Cauchy stress tensor, $\mathbf{b}$ is body forces.

Different fluids are defined by different Cauchy stress tensor, i.e.,

i. For inviscid fluid

$$\mathbf{T}=-p\mathbf{I}. \tag{1.15}$$

ii. For Newtonian fluid the tensor become

$$\mathbf{T}=-p\mathbf{I}+\mu\mathbf{A}_1, \tag{1.16}$$

where

$$A_1=(\text{grad}\,\mathbf{V})+(\text{grad}\,\mathbf{V})^T. \tag{1.17}$$

iii. Non- Newtonian second grade fluid

$$\mathbf{T}=-p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\left(\mathbf{A}_1\right)^2, \tag{1.18}$$

where $\alpha_{1,}$ and α_2 are the normal stress moduli. The second grade fluid model is compatible with thermodynamics when the Helmholtz free energy of the fluid is a minimum for the fluid in equilibrium. The Clausius–Duhem inequality and the assumption that the Helmholtz free energy is a minimum in equilibrium provide the following restrictions [38]

$$\mu\geqslant 0,\quad \alpha_1\geqslant 0,\quad \alpha_1+\alpha_2=0. \tag{1.19}$$

The fluid model of a second grade exhibits the normal stress effects only and cannot explain the shear thinning/shear thickening phenomena. The fluid model possessing shear thinning properties is known as a third grade fluid.

iv. Non- Newtonian third grade fluid

$$\mathbf{T} = -p\mathbf{I}+\mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{1.20}$$

here $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are Rivilin Erickson tensors. For $n > 1$, it is generally defined by

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}(\text{grad}\,\mathbf{V}) + (\text{grad}\,\mathbf{V})^T\mathbf{A}_{n-1}. \tag{1.21}$$

The material constants satisfy following thermodynamical constraints as defined [39]

$$\mu \geq 0,\ \alpha_1 \geq 0,\ \ |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3},\ \beta_1 = \beta_2 = 0,\ \beta_3 \geq 0 \tag{1.22}$$

v. Oldroyd model

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \tag{1.23}$$

$$\left.\begin{array}{c} \mathbf{S} + \lambda_1\frac{D\mathbf{S}}{Dt} + \frac{\lambda_3}{2}\left(\mathbf{S}\mathbf{A}_1 + \mathbf{A}_1\mathbf{S}\right) + \frac{\lambda_5}{2}\left(tr\mathbf{S}\right)\mathbf{A}_1 + \frac{\lambda_6}{2}\left[tr\left(\mathbf{S}\mathbf{A}_1\right)\right]\mathbf{I} \\ = \mu\left[\mathbf{A}_1 + \lambda_2\frac{D\mathbf{A}_1}{Dt} + \lambda_4\mathbf{A}_1^2 + \frac{\lambda_7}{2}\left[tr\left(\mathbf{A}_1^2\right)\right]\mathbf{I}\right] \end{array}\right\}, \tag{1.24}$$

where $\mathbf{S}$ is defined as

$$\mathbf{S} = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix}. \tag{1.25}$$

Energy Equation

The partial differential equation expressing conservation of energy for fluids is simply referred to as the energy equation. The derivation of this scalar equation is based on the thermodynamic principle that the time rate of change of internal energy plus kinetic energy for a volume of fluid is equal to the rate at which work is done on the fluid plus the rate at which heat is added to the fluid.

The energy equation is derived using the principle of conservation of heat energy. This is given by

$$\rho\frac{d\mathbf{e}}{dt} = \mathbf{T}.\,\mathrm{grad}\,\mathbf{V} - \mathrm{div}\,\mathbf{Q}. \tag{1.26}$$

where **e** is internal energy.

For nanofluids the equation modified as

$$\rho\frac{d\mathbf{e}}{dt} = \mathrm{div}\,\mathbf{Q} - (\rho c)_p\left(D_B\boldsymbol{\nabla}\varphi.\boldsymbol{\nabla}\theta + \frac{D_T}{\theta}\boldsymbol{\nabla}\theta.\boldsymbol{\nabla}\theta\right).$$

1.2 Methods of Solutions for Nonlinear Differential Equations

The nonlinear differential equations, nonlinear or complex boundary conditions, variable coefficients differential equations and coupled differential equations have little chances of getting exact or closed form solutions. This results in the need of approximate solutions using numerical or some approximate analytical techniques. With the advent of modern computers many numerical techniques have been evolved and exact numerical solutions can be obtained, However, analytical solutions are still important as they provide a standard for checking the accuracies of many approximate solutions which can be numerical or empirical. They can also be used as tests for verifying numerical schemes that are developed for studying more complex flow problems.

There are various approximate analytical methods to find the solutions of non linear governing equations in almost all branches of science and engineering. To name a few these are: lie symmetry methods [40] to [50], perturbation methods[51], artificial parameter method [52], Tanh method [53], Jacobi elliptic function method [54], Adomian decomposition method [55], homotopy perturbation method [56], modified homotopy perturbation method [57], variational method [58], iteration perturbation method [59] Homotopy Analysis Method [60] and Optimal Homotopic Asymptotic

Method [61]. Since we shall be using HAM and OHAM in our thesis, which are relatively recent methods,therefore we would like to give some mathematical details for the convenience of reading,

1.2.1 Homotopy Analysis Method

In 1992 Liao applied the concept of homotopy [62], a basic concept in topology [63], to get analytic approximations of nonlinear equations. The methodology is explained as below.

In order to describe the basic idea of HAM, presented by LIAO, we consider the following nonlinear differential equation

$$\mathcal{N}[\varphi(x)] = 0, \ \mathcal{B}(\varphi(x)) = 0, \tag{1.27}$$

where $\mathcal{N}$ is the nonlinear operator, φ is an unknown dependent function, $\mathcal{B}$ is the boundary conditions and x denotes the independent variable. The $zero - order$ deformation equation is written as

$$(1-q)\,\mathcal{L}[\widehat{\varphi}(x;q) - \varphi_0(x)] = q\hbar\mathcal{N}[\widehat{\varphi}(x;q)], \tag{1.28}$$

where $q \in [0,1]$ is called the embedding parameter, $\hbar$ is the non-zero-auxiliary parameter, $\mathcal{L}$ is the auxiliary linear operator, $\varphi_0(x)$ is the initial approximation which satisfy all the boundary conditions. It is vital that one has freedom to choose the initial approximation and auxiliary linear operator. In the above equation $q = 0$ and $q = 1$

$$\widehat{\varphi}(x;0) = \varphi_0(x) \ \text{ and } \widehat{\varphi}(x;1) = \varphi(x) \tag{1.29}$$

respectively. Thus as q increases from 0 to 1, the solution varies from initial approximation $\varphi_0(x)$ to the desired solution $\varphi(x)$. Expanding $\widehat{\varphi}(x;q)$ in the Taylor series with respect to q, one has

$$\widehat{\varphi}(x;q) = \varphi_0(x) + \sum_{m=1}^{\infty} \varphi_m(x) q^m, \qquad \varphi_m(x) = \frac{1}{m!} \frac{\partial^m \widehat{\varphi}(x,q)}{\partial q^m}\bigg|_{q=0}. \tag{1.30}$$

The mth order deformation equation is

$$\mathcal{L}\left[\varphi_m(\eta) - \chi_m \varphi_{m-1}(\eta)\right] = \hbar \mathcal{R}_m\left(\varphi_{m-1}\right), \tag{1.31}$$

where

$$\mathcal{R}_m\left(\varphi_{m-1}\right) = \frac{1}{(m-1)!} \frac{\partial^{m-1} \widehat{\varphi}(x,q)}{\partial q^{m-1}}\bigg|_{q=0} \tag{1.32}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{1.33}$$

Eq. (1.31) can be easily solved by using a symbolic computation software such as MAPLE or MATHEMATICA. If the auxiliary linear operator, the initial approximation and the auxiliary parameter $\hbar$ is properly chosen, the series (1.30) converges at $q = 1$ and one has

$$\varphi(x) = \varphi_0(x) + \sum_{m=1}^{\infty} \varphi_m(x) \tag{1.34}$$

which is the solutions of the original nonlinear Eq. (1.27). The higher order deformation equations can be found using [64]. HAM contains an auxiliary parameters $\hbar$, which provides us with a simple way to control and adjust the convergence of the series solution emerging in (1.34). A few recent investigations in the literature that contain Homotopy Analysis Method (HAM) solutions is mentioned in the references [65] to [77].

1.2.2 Optimal Homotopic Asymptotic Method

Recently, Marinca et al. [61] developed a very interesting method Optimal Homotopic Asymptotic Method (OHAM) to approximate the solution of nonlinear problems in the frame work of the homotopy analysis method. This method is not only valid for small (or large) values of physical parameter but also minimizes the residual error which shows its validity and great potential to solve the nonlinear problems. The application of this method in fluid mechanics, heat and mass transfer analysis has been successfully studied by Marinca et al [78] to [80].

In order to understand the method, we give a brief procedure as described by marinca in solving the following differential equation:

$$\mathcal{L}(v) + g + \mathbf{N}(v) = 0, \;\; B(v) = 0. \tag{1.35}$$

where g is a known function.

A homotopy is first construct as $\varphi(r,q) : R \times [0,1] \rightarrow R$ which satisfies

$$\left.\begin{array}{c} (1-q)\left[\mathcal{L}(\varphi(r,q)) + g(r)\right] = H(q)\left[\begin{array}{c} \mathcal{L}(\varphi(r,q)) \\ +\mathbf{N}(\varphi(r,q)) \end{array}\right] \\ B(\varphi(r,q)) = 0 \end{array}\right\}, \tag{1.36}$$

where $r \in R$ and $0 \leq q \leq 1$ is an embedding parameter, $H(q)$ is a non-zero auxiliary function for $q \neq 0$ and $H(0) = 0$, $\varphi_i(r,q)$ are unknown functions. the values $q = 0$ and $q = 1$ gives

$$\varphi(r,0) = v_0(r), \quad \varphi(r,1) = v(r).$$

let us choose the auxiliary function $H(q)$ in the form

$$H(q) = \sum_{j=1}^{k} q^j D_j, \tag{1.37}$$

where D_j are constants.

By Taylor's theorem one can write Eq. (1.36) to get $\varphi(r,q,D_j)$ in the following form

$$\varphi(r,q,D_j) = u(r) + \sum_{k\geq 1} u_k(r,D_j)\, q^k, \quad j = 1,2,...... \tag{1.38}$$

Using Eq. (1.36) and Eqs. (1.37) to (1.38) the *zeroth* and first order problems as follows

$$\mathcal{L}(u_0(r)) + g(r) = 0, \qquad B(u_0) = 0, \tag{1.39}$$

$$\mathcal{L}\left(u_1\left(r\right)\right)=K_1\mathbf{N}_0\left(u_0\left(r\right)\right),\qquad B\left(u_1\right)=0 \tag{1.40}$$

and the corresponding $k-th$ order equation will be defined by

$$\left.\begin{array}{c}\mathcal{L}\left(u_k\left(r\right)-u_{k-1}\left(r\right)\right)=D_k\mathbf{N}_0\left(u_0\left(r\right)\right)\\ +\sum\limits_{j=1}^{k-1}D_j\left[\begin{array}{c}\mathcal{L}_i\left(u_{k-j}\left(r\right)\right)\\ +\mathbf{N}_{i(k-j)}\left(u_0\left(r\right),u_1\left(r\right),\quad u_{k-j}\left(r\right)\right)\end{array}\right]\\ B\left(u_k,\frac{du_k}{dr}\right)=0;\qquad k=2,3,4...,\end{array}\right\}, \tag{1.41}$$

where

$$\mathbf{N}\left(u\left(r\right)\right)=\left.\begin{array}{c}\mathbf{N}_0\left(u_0\left(r\right)\right)+q\mathbf{N}_1\left(u_0\left(r\right),u_1\left(r\right)\right)\\ +q^2\mathbf{N}_2\left(u_0\left(r\right),u_1\left(r\right),u_2\left(r\right)\right)+...\end{array}\right\}. \tag{1.42}$$

The approximate solution of Eq. (1.36) can be determined in the form

$$u_i^{(m)}\left(r,q,D_j\right)=u_{i0}\left(r\right)+\sum_{k=1}^{m}u_{ik}\left(r,D_j\right). \tag{1.43}$$

By substitute Eq. (1.38) into Eq. (1.36) and as a result we get the following residual

$$Er\left(r,D_j\right)=\mathcal{L}\left(u^{(m)}\right)+g+\mathbf{N}\left(u^{(m)}\right). \tag{1.44}$$

If $Er\left(r,D_j\right)=0$ then $u^{(m)}\left(r,D_j\right)$ happens to be the exact solution. Generally such case will not arise for nonlinear problems, but we can minimize the functional by

$$Er_1\left(K_j\right)=\int_a^b Er^2\left(r,D_j\right)dr, \tag{1.45}$$

where a and b are two values, depending on the given problem for locating the desired D_j and finally the unknown constants D_j $(j=1,2,3,......m)$ can be optimally identified from the conditions

$$\frac{\partial Er_1}{\partial D_j}=0, \tag{1.46}$$

or simply we can use $Er\left(r,D_j\right)=0$ generates a set linear equations at different points in domain to gain the values of constants D_j, with these constants known, the approximate solution of order m is well determined now.

Chapter 2

A Study of Pressure Distribution for a Slider Bearing Lubricated with a Second Grade Fluid

In this chapter, the pressure distribution of slider bearing lubricated with second grade fluid for inclined and parabolic slider bearings is studied.Some numerical treatment for such problem for different fluids are available in the literature [81] to [84]. Interest here is to develop a series solution. A numerical solution is also computed. The solutions are valid not only for small but also for large values of all the emerging parameters. Convergence values and residual error are examined. Comparison between inclined and parabolic bearings is also presented.

2.1 Mathematical Formulation of the Problem

Assume the velocity field to be

$$\mathbf{V} = [u(x,y), v(x,y), 0],$$

Setting non dimensional parameters as,

$$x = \frac{\widehat{x}}{L},\ y = \frac{\widehat{y}}{\alpha L},\ u = \frac{\widehat{u}}{U},\ v = \frac{\widehat{v}}{\alpha U},\ p = \frac{\widehat{p}}{\mu \frac{U}{\alpha^2 L}},\ b = \frac{\widehat{b}(\widehat{x})}{\alpha L}. \tag{2.1}$$

Using Eqs. (1.12), (1.14), (1.18), (1.19) and (2.1)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\alpha^2 \mathrm{Re}}\frac{\partial p^*}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial \mathbf{T}_{xx}}{\partial x} + \frac{1}{\alpha \mathrm{Re}}\frac{\partial \mathbf{T}_{xy}}{\partial y}, \tag{2.3}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\alpha^4\,\mathrm{Re}}\frac{\partial p^*}{\partial y}+\frac{1}{\alpha\,\mathrm{Re}}\frac{\partial \mathbf{T}_{xy}}{\partial x}+\frac{1}{\alpha^2\,\mathrm{Re}}\frac{\partial \mathbf{T}_{yy}}{\partial y}, \tag{2.4}$$

where $\mathbf{T}_{xx}$, $\mathbf{T}_{xy}$ and $\mathbf{T}_{yy}$ are components of stress tensor $\mathbf{T}$, $\mathrm{Re}=\rho UL/\mu$ is the Reynolds number and α is the small parameter of same order as that of channel slope. Furthermore, in deriving these equations it is assumed that, in addition to the usual boundary layer approximations, the contribution due to the shear stresses is of the same order of magnitude as that due to the normal stresses and . Thus, both v and α_1/ρ are of $O(\delta^2)$, where δ is the boundary layer thickness.

It is intended to develop the lubrication equations for the second grade fluid for the geometries, where $\alpha \ll 1$ and for the Reynolds number such that $\alpha\,\mathrm{Re} \ll 1$. Under these assumptions, Eqs. (2.2) to (2.4) reduce to

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.5}$$

$$\begin{aligned} 0 &= -\frac{\partial p^*}{\partial x}+(2\lambda_1+\lambda_2)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)^2+\frac{\partial^2 u}{\partial y^2} \\ &\quad +\lambda_1\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^3 u}{\partial y^3}\right], \end{aligned} \tag{2.6}$$

$$0=-\frac{\partial p^*}{\partial y}+(2\lambda_1+\lambda_2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2. \tag{2.7}$$

Defining

$$p_1=p^*-(2\lambda_1+\lambda_2)(\frac{\partial u}{\partial y})^2, \tag{2.8}$$

Eqs. (2.5) -(2.7) become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.9}$$

$$\frac{\partial p_1}{\partial x}=\frac{\partial^2 u}{\partial y^2}+\lambda_1\left(\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^3 u}{\partial y^3}\right), \tag{2.10}$$

$$\frac{\partial p_1}{\partial y}=0, \tag{2.11}$$

in which $\lambda_1 = U\alpha_1/L\mu = \lambda$ and $\lambda_2 = U\alpha_2/L\mu$ are the dimensionless material constants. The boundary conditions are

$$u(x,0) = 1, \; u(x,b) = 0, \tag{2.12}$$

$$v(x,0) = 0, \; v(x,b) = 0. \tag{2.13}$$

2.2 Solutions of the Problem

The slider bearing for inclined and parabolic slider bearings are shown in the figures 2.1 and 2.2 respectively

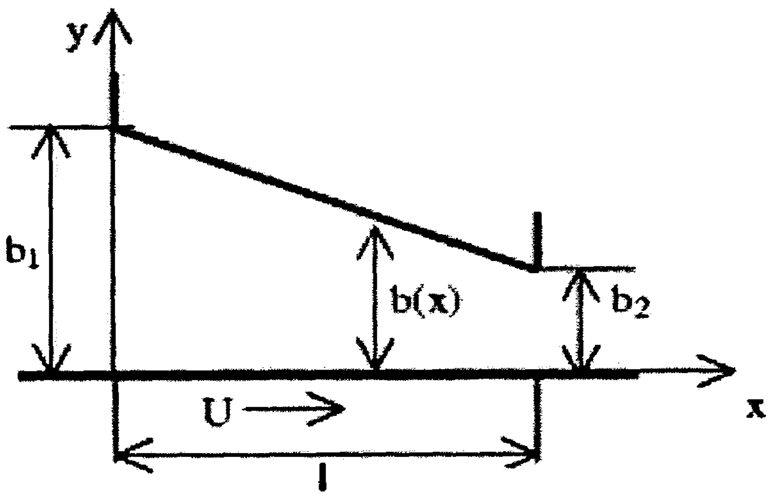

Fig. 2.1 : Inclined slider bearing.

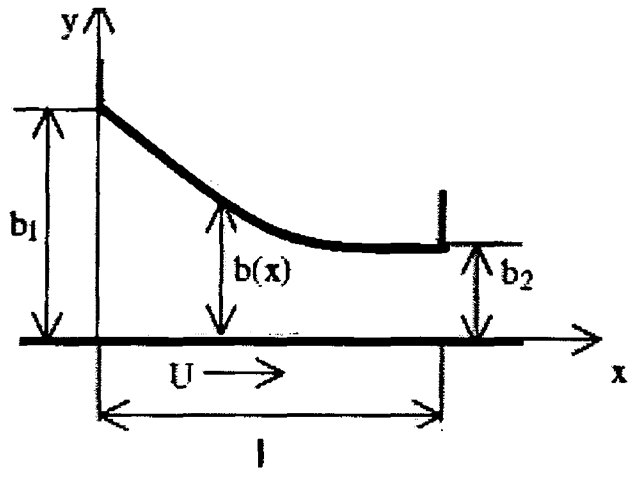

Fig. 2.2 : Parabolic slider bearing.

For the series solution of $u(x,y)$, we choose

$$u_o(x,y) = \frac{dp_0}{dx}\left(\frac{y^2}{2}-\frac{yb}{2}\right)+\left(1-\frac{y}{b}\right), \tag{2.14}$$

$$v_0(x,y) = \frac{6(1-s)y^2}{2b^2}, \tag{2.15}$$

$$p_0(x) = \frac{6(s-1)(x-1)x}{(1+s)b^2}, \tag{2.16}$$

as the initial approximations of u, v and p respectively which satisfy the corresponding boundary conditions. We use the method of higher order differential mapping in order to choose the linear operator $\mathcal{L}$

$$\mathcal{L}_1(f) = f'', \tag{2.17}$$

whereas the nonlinear operator $\mathcal{N}$ is

$$\mathcal{N}_1[\widehat{u}(y,q)] = -\frac{\partial p_1}{\partial x}+\frac{\partial^2\widehat{u}(y,q)}{\partial y^2}+\lambda\left(\begin{array}{c}\frac{\partial}{\partial x}\left(\widehat{u}(y,q)\frac{\partial^2\widehat{u}(y,q)}{\partial y^2}\right)-\frac{\partial\widehat{u}(y,q)}{\partial y}\frac{\partial^2\widehat{u}(y,q)}{\partial x\partial y}+\\ \widehat{v}(y,q)\frac{\partial^3\widehat{u}(y,q)}{\partial y^3}\end{array}\right). \tag{2.18}$$

If the convergence parameter is $\hbar$ and $0 \leq q \leq 1$ is an embedding parameter then the *zeroth* order deformation problem is

$$(1-q)\mathcal{L}_1[\widehat{u}(y,q)-u_0(y)] = q\hbar\mathcal{N}_1[\widehat{u}(y,q)], \tag{2.19}$$

$$\widehat{u}(0,q)=1,\ \widehat{u}(b,q)=0. \tag{2.20}$$

mth-order Deformation Equation

Taking derivative of Eq. (2.19) m times with respect to q and then setting $q=0$, one can write

$$\mathcal{L}_1[u_m(y)-\chi_m u_{m-1}(y)] = \hbar\mathcal{R}_{1m}(y), \tag{2.21}$$

$$u(0)=u(b)=0, \tag{2.22}$$

$$\mathcal{R}_{1m}(y) = -\frac{\partial p_{1m-1}}{\partial x} + \frac{\partial^2 u_{m-1}}{\partial y^2} + \lambda\left(\begin{array}{c}\sum_{n=0}^{m-1} v_{m-1-n}\frac{\partial^3 u_n}{\partial y^3} + \sum_{n=0}^{m-1} u_{m-1-n}\frac{\partial^3 u_n}{\partial x\partial y^2} + \\ \sum_{n=0}^{m-1} \frac{\partial u_{m-1-n}}{\partial x}\frac{\partial^2 u_n}{\partial y^2} - \sum_{n=0}^{m-1} \frac{\partial u_{m-1-n}}{\partial y}\frac{\partial^2 u_n}{\partial y\partial x}\end{array}\right). \tag{2.23}$$

Now Eq. (2.20) is solved by using MATHEMATICA.

From Eqs. (2.5) to (2.6) one can easily calculate v and p by using the following relations

$$v = -\int \frac{\partial u}{\partial x}dy, \tag{2.24}$$

$$p_1(0) = p_1(1) = 0. \tag{2.25}$$

To interpret the solution for the variation of pressure distribution with second grade fluid of lubricants $b(x)$ we consider the following two cases:

Case 1: Inclined slider bearing

$$b = 1-(1-s)x. \tag{2.26}$$

Case 2: Parabolic slider bearing

$$b = 1-(1-s)\left(x^2-2x\right), \tag{2.27}$$

where $s = b_2/b_1$.

2.3 Convergence of the Solution

Here, the convergence of the solutions containing the convergence parameter $\hbar$ is discussed. Figs. 2.3 and 2.4 are plotted for the $\hbar$−curves of inclined and parabolic slider bearings respectively. In Fig. 2.3 the $\hbar$−curve is plotted for 16th order approximation for dimensionless pressure and it is found that the range for admissible values of $\hbar$ is $-0.3 \leq \hbar \leq -0.1$. In Fig. 4, the $\hbar$−curve is plotted for 13th order approximation for dimensionless pressure and it is found that the range for admissible value of $\hbar$ is $-0.2 \leq \hbar \leq -0.1$. In Figs. 2.5 and 2.6, the graphs of residual errors for inclined and parabolic slider bearings are plotted respectively. For different values of convergence parameter $\hbar$, it is seen that the error is minimum at $\hbar = -0.24$ for inclined and at $\hbar = -0.15$ for parabolic slider bearings.

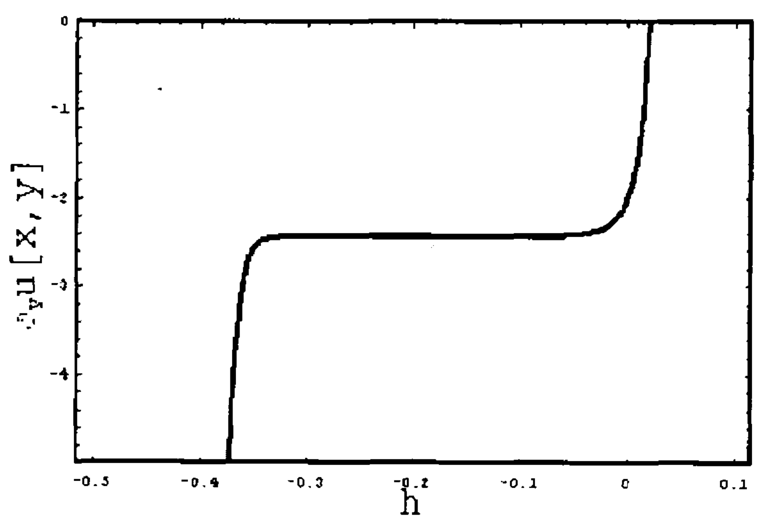

Fig. 2.3 : $\hbar$−curve for inclined slider bearing.

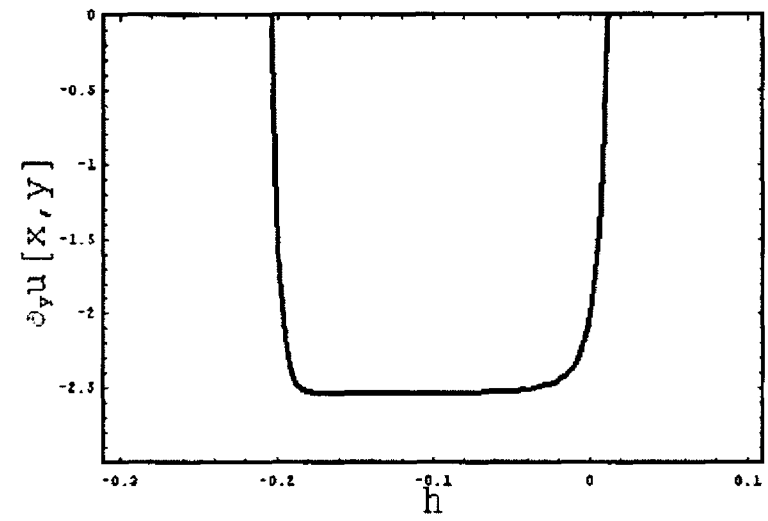

Fig. 2.4 : $\hbar$−curve for parabolic slider bearing.

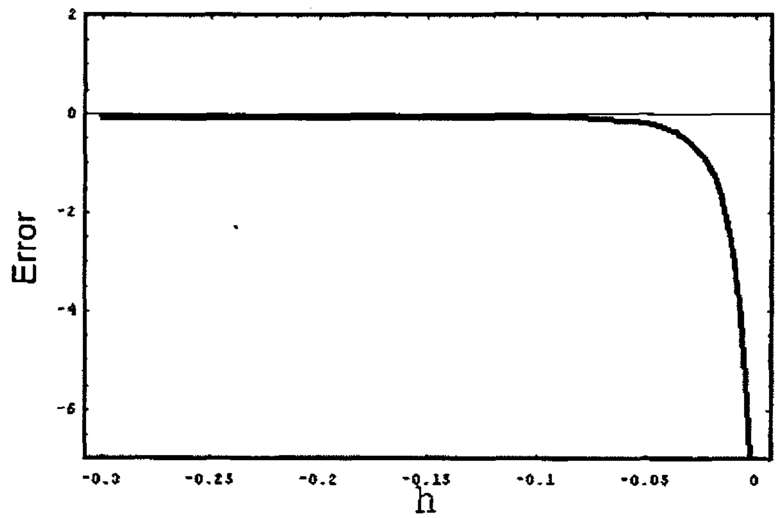

Fig. 2.5 : Residual for inclined slider bearing for different values of $\hbar$.

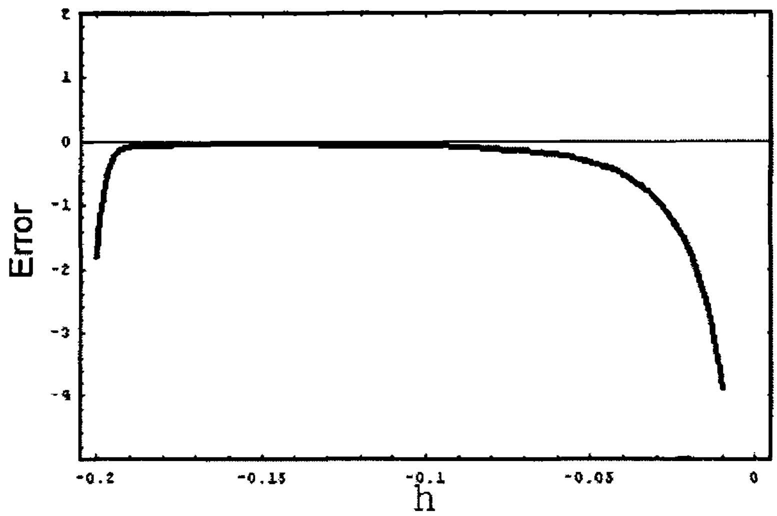

Fig. 2.6 : Residual for parabolic slider bearing for different values of $\hbar$.

2.4 Graphical Results

In this section, the pressure distribution is discussed for the various values of non-Newtonian parameter λ and clearance ratio s in the slider bearings. In Figs. 2.7 and 2.8, we observe that the effects of variation of non-Newtonian parameter λ in slider bearings when $s = 0.5$. For larger values of λ, it is found that an increase in λ increases the pressure. In Figs. 2.9 and 2.10, we reverse the order and now plot dimensionless pressure versus dimensionless length for constant $\lambda = 1$ and different values of clearance ratio s. Comparison of pressure distributions in the two slider bearings namely parabolic and inclined slider bearing at $\lambda = 1$ and $s = 0.5$ is given in Fig. 2.11.

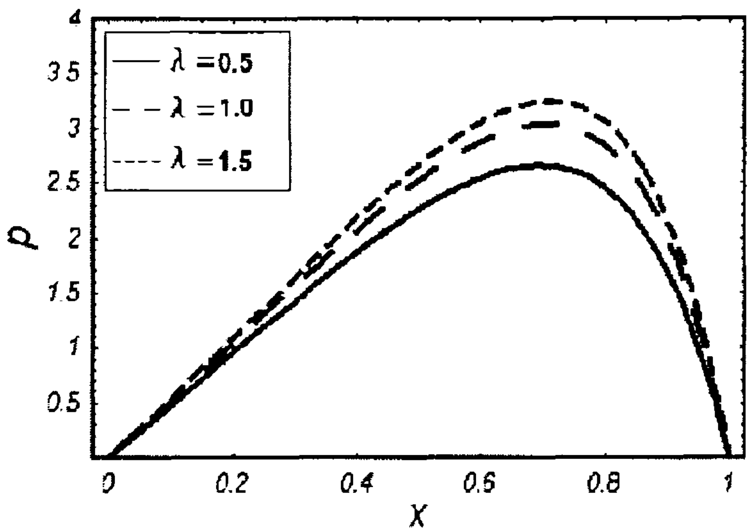

Fig. 2.7: Pressure distribution for various values of non-Newtonian parameter λ in inclined slider bearing for clearance ratio $s = 0.5$.

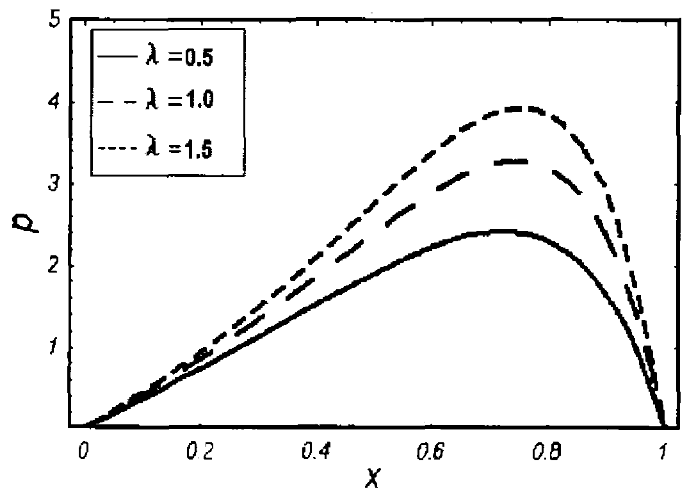

Fig. 2.8: Pressure distribution for slider bearings for various values of non-Newtonian parameter λ in parabolic slider bearing for clearance ratio $s = 0.5$.

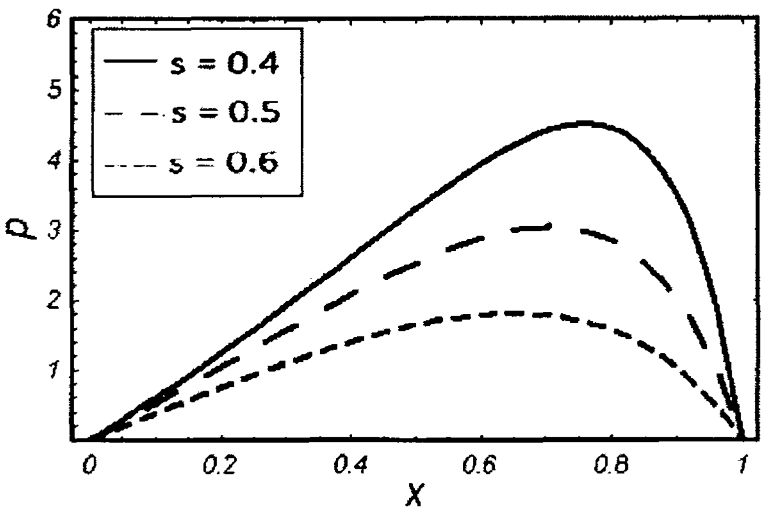

Fig. 2.9: Pressure distribution for various values of clearance ratio s in an inclined slider bearing for $\lambda = 1$.

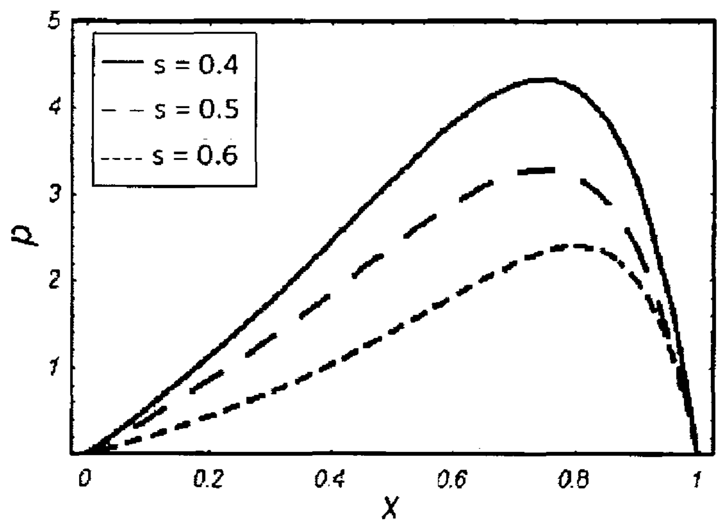

Fig. 2.10: Pressure distribution for various values of clearance ratio s in parabolic slider bearing.

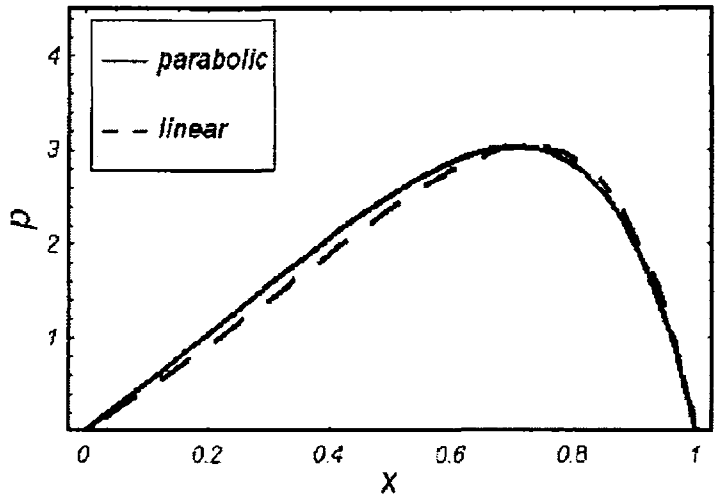

Fig. 2.11: Comparison of pressure distribution in parabolic and inclined slider bearings.

2.5 Numerical Results

Numerical results and Analytic solution obtained for pressure distribution by HAM are compared for inclined (Table 2.1) and parabolic (Table 2.2) slider bearings at $\lambda = 1$ and $s = 0.5$. Also comparison of pressure distribution between both type of bearings are shown in Table 2.3.

Table 2.1

x	Analytic	Numerical
0.1	0.471019	0.471028
0.2	0.970697	0.970700
0.3	1.49408	1.49418
0.4	2.02851	2.02856
0.5	2.54649	2.54652
0.6	2.99863	2.99864
0.7	3.26005	3.26007
0.8	3.14836	3.14839
0.9	2.28815	2.28818

Table 2.2

x	Analytic	Numerical
0.1	0.548634	0.548668
0.2	1.11304	1.11312
0.3	1.68339	1.68345
0.4	2.24195	2.24200
0.5	2.75340	2.75338
0.6	2.17395	2.17395
0.7	3.39191	3.39189
0.8	3.12742	3.12748
0.9	2.23005	2.23000

Table 2.3

x	Inclined	Parabolic
0.1	0.471019	0.548634
0.2	0.970697	1.11304
0.3	1.49408	1.68339
0.4	2.02851	2.24195
0.5	2.54649	2.7534
0.6	2.99863	2.17395
0.7	3.26005	3.39191
0.8	3.14836	3.12742
0.9	2.28815	2.23005

2.6 Conclusion

The main results are listed below.

- An increase in non-Newtonian parameter λ leads to an increase in pressure.
- An increase in r decreases the pressure.
- Comparison of slider bearings show that the pressure in parabolic slider bearing attains higher value.
- The pressure distribution in an inclined slider bearings is slightly greater than the parabolic slider bearing in the later part of flow i.e., $0.7 \leq x \leq 1$.
- Homotopy perturbation method (HPM) is the special case of HAM. For $\hbar = -1$ in Eq. (2.19) we get HPM solution.
- The residual is almost negligible (see Figs. 2.5 and 2.6).

- The analytic solutions for inclined and parabolic slider bearings with viscous fluid can be obtained by choosing $\lambda = 0$.

Chapter 3

Some Fundamental Flows Third Grade Fluid with Nonlinear Slip Conditions

The objective of this chapter is to study the Couette, Poiseuille and generalized Couette flows in the presence of a slip condition. Exact solutions have been constructed for the problems consisting of nonlinear equations with non-linear boundary conditions. The slip conditions in terms of shear stress are defined. Graphic and numerical results are presented.

3.1 Mathematical Formulation of the Problem

Considering the steady unidirectional flow described by the velocity field

$$\mathbf{V} = [u(y), 0, 0] \tag{3.1}$$

Setting the following dimensionless variables

$$u^* = \frac{u}{U_0},\ y^* = \frac{y}{h},\ \hat{p} = \frac{\hat{p}^* \mu \nu}{h^2},\ \beta = \frac{\beta_3}{\mu}\left(\frac{U_0}{h}\right)^2,$$

and by using Eqs. (1.2), (1.3) and (1.9), we get the dimensionless governing equation of the form

$$\frac{d}{dy}\left[\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3\right] = c, \tag{3.2}$$

where asterisks have been suppressed for simplicity, ν the kinematic viscosity, U_0 the characteristic velocity, $\hat{p}$ the modified pressure and $c = d\hat{p}/dx$.

3.2 Solution of the Problem

We use first integral approach to find the solution of three cases of flows, namely

1. Couette flow.

2. Poiseuille flow.

3. Generalized Couette flow.

3.2.1 Case 1: Couette Flow

Here the third grade fluid is bounded between two rigid plates a distant h apart. No pressure gradient is applied. The lower plate is suddenly jerked while the upper plate is fixed. The resulting dimensionless problem is

$$\frac{d}{dy}\left[\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3\right] = 0, \tag{3.3}$$

$$u(0) - \gamma\left[\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3\right]_{y=0} = 1, \tag{3.4}$$

$$u(1) + \gamma\left[\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3\right]_{y=1} = 0, \tag{3.5}$$

where γ^* $(= \gamma/h)$ is the slip parameter and asterisk is suppressed here.

Eq. (3.3) has first integral

$$\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3 = C_3. \tag{3.6}$$

After the use of Eqs. (3.4) and (3.5), the equivalent boundary conditions are

$$u(0) = \gamma C_3 + 1, \;\; u(1) = -\gamma C_3, \tag{3.7}$$

where C_3 is the constant to be selected.

The exact solution is given as follows

$$\frac{du}{dy} = \sqrt[3]{\frac{C_3}{4\beta} + \frac{1}{4\beta}\sqrt{C_3^2 + \frac{2}{27\beta}}} + \sqrt[3]{\frac{C_3}{4\beta} - \frac{1}{4\beta}\sqrt{C_3^2 + \frac{2}{27\beta}}} \cong \Delta. \tag{3.8}$$

Therefore,

$$u = \Delta y + C_4, \tag{3.9}$$

where C_4 is a further constant.

From Eqs. (3.7) and (3.9), we deduce

$$C_4 = \gamma C_3 + 1. \tag{3.10}$$

In view of Eqs. (3.6), (3.9) and (3.10) we obtain

$$\Delta = -2\gamma C_3 - 1. \tag{3.11}$$

Eqs. (3.8) and (3.11) imply that

$$\sqrt[3]{\frac{1}{4\beta}\sqrt{C_3^2 + \frac{2}{27\beta}} + \frac{C_3}{4\beta}} - \sqrt[3]{\frac{1}{4\beta}\sqrt{C_3^2 + \frac{2}{27\beta}} - \frac{C_3}{4\beta}} = -2\gamma C_3 - 1. \tag{3.12}$$

Thus the exact solution is

$$u = \Delta y + \gamma C_3 + 1, \tag{3.13}$$

with the condition that Eq. (3.12) be satisfied. The relation between β and γ for fixed C_3 is given in Table 3.1.

3.2.2 Case 2: Poiseuille Flow

Here an incompressible third grade fluid is bounded between the two stationary plates distant h apart and the flow is caused by an applied constant pressure gradient. The resulting dimensionless problem is of the form

$$\frac{d}{dy}\left[\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3\right] = c, \tag{3.14}$$

$$u(0)-\gamma\left[\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3\right]_{y=0}=0, \tag{3.15}$$

$$u(1)+\gamma\left[\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3\right]_{y=1}=0. \tag{3.16}$$

The first integral of Eq. (3.14) is

$$\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3=cy+C_5, \tag{3.17}$$

where C_5 is a constant to be selected. After the invocation of Eqs. (3.15) and (3.16), the boundary conditions reduce to

$$u(0) = \gamma C_5, \tag{3.18}$$

$$u(1) = -\gamma C_5-\gamma c. \tag{3.19}$$

The exact solution of Eq. (3.17) is

$$\begin{aligned} u \;=\; & \frac{9\left(y+\frac{C_5}{c}\right)}{8\,(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81(cy+C_5)^2+\frac{6}{\beta}}+9(cy+C_5)}{\beta}}-\sqrt[3]{\frac{\sqrt{81(cy+C_5)^2+\frac{6}{\beta}}-9(cy+C_5)}{\beta}}\right] \\ & -\frac{1}{24\,(6)^{\frac{2}{3}}\,c}\left[\sqrt[3]{\frac{\sqrt{81(cy+C_5)^2+\frac{6}{\beta}}-9(cy+C_5)}{\beta}}+\sqrt[3]{\frac{\sqrt{81(cy+C_5)^2+\frac{6}{\beta}}+9(cy+C_5)}{\beta}}\right] \\ & \times\sqrt{81\,(cy+C_5)^2+\frac{6}{\beta}}+C_6, \end{aligned} \tag{3.20}$$

where C_6 is a constant. In view of Eqs. (3.18) and (3.19), Eq. (3.20) yields the following two relations

$$\begin{aligned} \gamma C_5 \;=\; & \frac{9\left(\frac{C_5}{c}\right)}{8\,(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}+9C_5}{\beta}}-\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}-9C_5}{\beta}}\right] \\ & -\frac{1}{24\,(6)^{\frac{2}{3}}\,c}\left[\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}-9C_5}{\beta}}+\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}+9C_5}{\beta}}\right] \\ & \times\sqrt{81C_5^2+\frac{6}{\beta}}+C_6 \end{aligned} \tag{3.21}$$

and

$$
\begin{aligned}
-\gamma C_5-\gamma c &= \frac{9\left(1+\frac{C_5}{c}\right)}{8\,(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81(c+C_5)^2+\frac{6}{\beta}}+9(c+C_5)}{\beta}}-\sqrt[3]{\frac{\sqrt{81(c+C_5)^2+\frac{6}{\beta}}-9(c+C_5)}{\beta}}\right] \\
&\quad -\frac{1}{24\,(6)^{\frac{2}{3}}\,c}\left[\sqrt[3]{\frac{\sqrt{81(c+C_5)^2+\frac{6}{\beta}}-9(c+C_5)}{\beta}}+\sqrt[3]{\frac{\sqrt{81(c+C_5)^2+\frac{6}{\beta}}+9(c+C_5)}{\beta}}\right] \\
&\quad \times\sqrt{81\,(c+C_5)^2+\frac{6}{\beta}}+C_6. \qquad (3.22)
\end{aligned}
$$

The relation between β and γ for fixed c when $C_5 = 0$ and $C_5 \neq 0$ are given in Tables 3.2 and 3.3 respectively. The relation among β, γ, c and C_5 is obtained after eliminating C_6 between Eqs. (3.21) and (3.22).

3.2.3 Case 3: Generalized Couette Flow

In this case the geometrical description of the flow is similar to that of the previous one. Only the flow here is generated by a constant pressure gradient and sudden motion of the lower plate. The underlying dimensionless problem is

$$\frac{d}{dy}\left[\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3\right]=c, \qquad (3.23)$$

$$u(0)-\gamma\left[\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3\right]_{y=0}=1, \qquad (3.24)$$

$$u(1)+\gamma\left[\frac{du}{dy}+2\beta\left(\frac{du}{dy}\right)^3\right]_{y=1}=0. \qquad (3.25)$$

The first integral of Eq. (3.21) is the same as Eq. (3.17). The solution of Eq. (3.21) is Eq. (3.20). However, the boundary conditions reduce to

$$u(0) = \gamma C_5 + 1, \qquad (3.26)$$

$$u(1) = -\gamma C_5 - \gamma c. \qquad (3.27)$$

Again in view of Eqs. (3.26) and (3.27), we arrive at the following relation

$$
\begin{aligned}
\gamma C_5 + 1 &= \frac{9\left(\frac{C_5}{c}\right)}{8\,(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}+9C_5}{\beta}}-\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}-9C_5}{\beta}}\right] \\
&\quad -\frac{1}{24\,(6)^{\frac{2}{3}}\,c}\left[\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}-9C_5}{\beta}}+\sqrt[3]{\frac{\sqrt{81C_5^2+\frac{6}{\beta}}+9C_5}{\beta}}\right] \\
&\quad \times\sqrt{81C_5^2+\frac{6}{\beta}}+C_6 \qquad (3.28)
\end{aligned}
$$

and Eq. (3.22). The relation between β and γ for fixed c when $C_5 = 0$ and $C_5 \neq 0$ are given in Tables 3.4 and 3.5 respectively.

3.3 Graphical Results

In order to illustrate the influences of third grade parameter β, and slip parameter γ on the velocity u we have plotted figures. Figs. 3.1 and 3.2 for the Couette flow, Figs. 3.3 and 3.4 for the Poiseuille flow and Figs. 3.5 and 3.6 for the Generalized Couette flow.

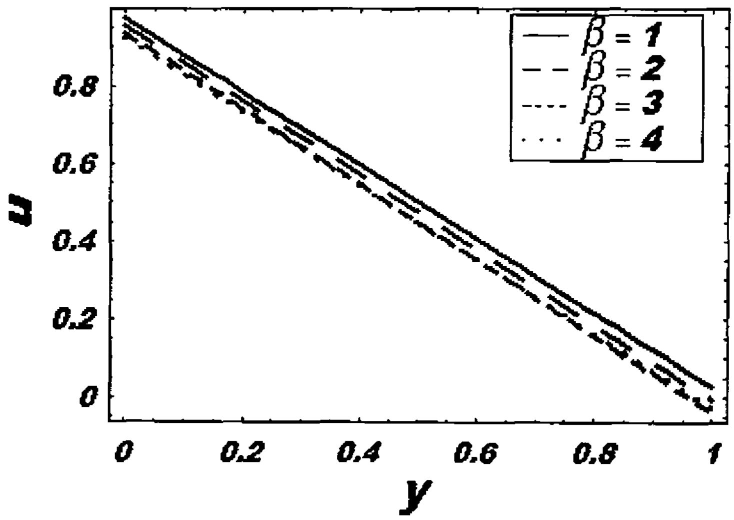

Fig. 3.1 : Velocity profile $u(y)$ for Couette flow for various values of β.

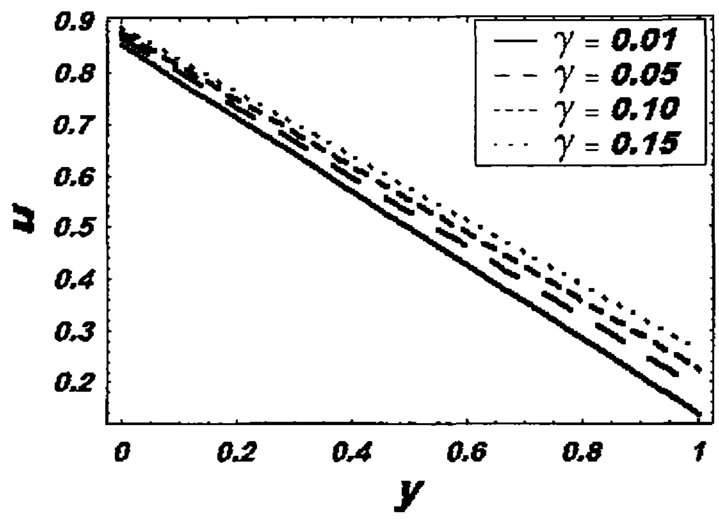

Fig. 3.2 : Velocity profile $u(y)$ for Couette flow for various values of γ.

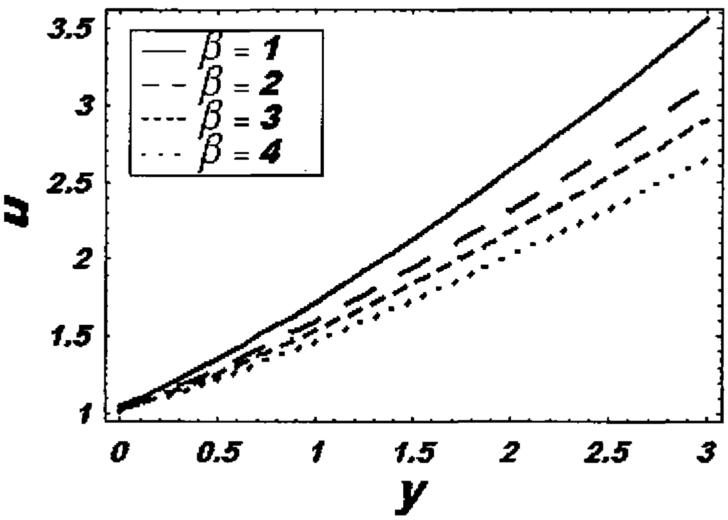

Fig. 3.3 : Velocity profile $u(y)$ for Poiseuille Flow for various values of β.

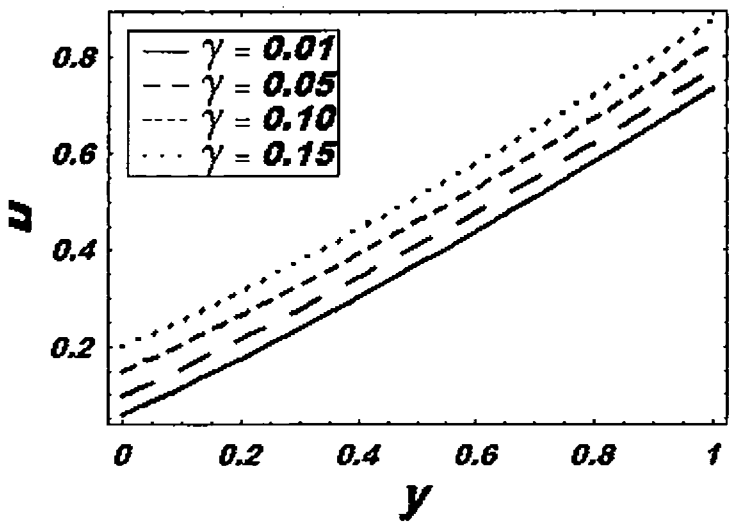

Fig. 3.4 : Velocity profile $u(y)$ for Poiseuille Flow for various values of γ.

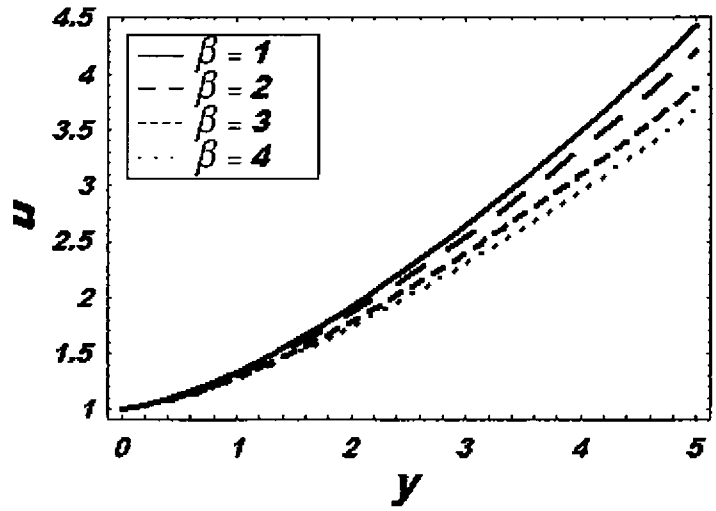

Fig. 3.5 : Velocity profile $u(y)$ for Generalized Couette flow for various values of β.

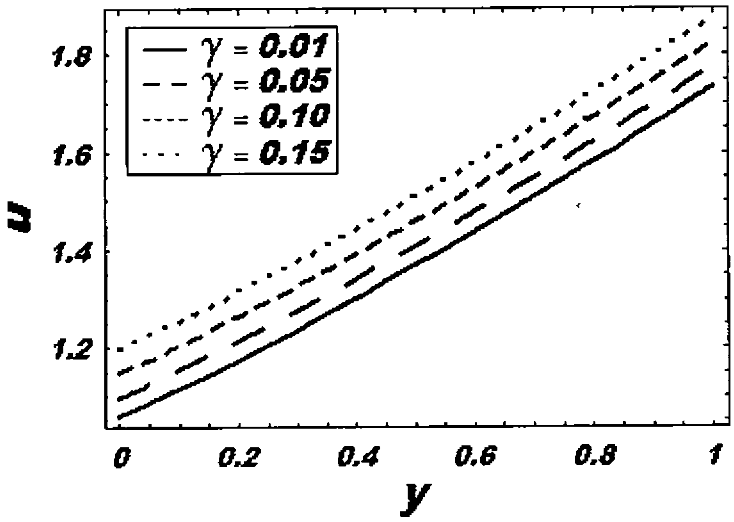

Fig. 3.6 : Velocity profile $u(y)$ for Generalized Couette flow for various values of γ.

3.4 Numerical Results

We now compute the numerical values of β, γ on c, C_5 and C_3. Table 3.1 corresponds to the first problem. Tables 3.2 is prepared for the second problem whereas Tables 3.3 is computed for the third problem.

Table 3.1

C_3	β	γ
1	1	0.7948
	2	0.7500
	3	0.7253
	4	0.7087

Table 3.2

C_5	c	β	γ
0	1	1	1.4905
		2	1.1268
		3	0.9616
		4	0.8609
1	1	1	1.3778
		2	1.0771
		3	0.9343
		4	0.8453

Table 3.3

C_5	c	β	γ
0	1	1	0.4905
		2	0.1268
		3	−0.0384
		4	−0.1391
1	1	1	1.0444
		2	0.7930
		3	0.6010
		4	0.5118

3.5 Conclusion

In this chapter the velocity profiles in a third grade fluid are found analytically. Three interesting cases of Couette flow, Poiseuille flow and generalized Couette flow are discussed. The first integral approach is used to find the velocity profiles. The variation of the third grade parameter β, pressure gradient c and slip parameter γ on the velocity profiles are illustrated. As a result, the following observations are made.

- In all three cases, we see that the flows show similar behavior of velocity profiles for different values of β, γ and c. However, as expected from the boundary conditions, the boundary layer for nonzero C_5 is translated when compared to the case $C_5 = 0$.

- It is noted that the solution of the first problem is linear whereas in the other two cases the solutions are nonlinear. Since all solutions are independent, one cannot obtain the solution of the Couette flow from the Generalized Couette flow by setting $c = 0$.

- Furthermore, a comparison between the third grade parameter β and the slip parameter γ are also presented in Tables 3.1 to 3.3. It is noted that the increase of β reduces γ in all cases and this fact is basically reflected from the tables as well.

- It is also worth mentioning that our exact analytical solutions are not only valid for small but also for large values of β.

Chapter 4

MHD Flow of a Third Grade Fluid with Variable Viscosity and Slip Effects in Porous Space Through a Cylinder

In this chapter, the motivation comes from a desire to understand the effects of magnetic field on the pipe flow of a third grade fluid. The fluid is electrically conducting under the application of a constant magnetic field. The flow is induced by a constant pressure gradient. The viscosity here depends upon the space coordinates. The relevant equations for flow and temperature have been solved analytically by using homotopy analysis method. Convergence of the obtained solutions is explicitly shown. The effects of the various parameters of interest on the velocity and temperature are pointed out. The heat transfer analysis is also examined. The solutions of arising problems have been presented for two cases of viscosity. Convergence of solution is obtained. Graphical results are also shown.

4.1 Mathematical Formulation of the Problem

Consider the steady unidirectional flow and heat transfer through a porous pipe. The velocity field is of the form

$$\mathbf{V} = [0, 0, v(r)]\,. \tag{4.1}$$

The governing momentum and energy equations can be expressed as

$$\frac{1}{r}\frac{d}{dr}(r\mu\frac{dv}{dr}) + \frac{2\beta_3}{r}\frac{d}{dr}\left[r(\frac{dv}{dr})^3\right] - \frac{\psi}{k_1}v = -\frac{\partial \widehat{p}}{\partial z} + \sigma B_0^2\, v, \tag{4.2}$$

$$\mu\left(\frac{dv}{dr}\right)^2+2\beta_3\left(\frac{dv}{dr}\right)^4+k\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)\right]=0, \tag{4.3}$$

where ψ is porosity, k_1 is the permeability, σ is the electrical conductivity, B_0 is the magnetic field strength and k is the thermal conductivity.

The corresponding slip conditions are

$$v(1)-\gamma^*\left[\frac{dv}{dr}(1)+\frac{\beta}{\mu}\left(\frac{dv}{dr}(1)\right)^3\right]=0;\ \theta(1)=0, \qquad \frac{dv}{dr}(0)=\frac{d\theta}{dr}(0)=0. \tag{4.4}$$

Putting

$$\begin{aligned}\beta &= \frac{2\beta_3 v_0^2}{\mu_0 R^2},\ c=\frac{\frac{\partial p_1}{\partial z}R^2}{\mu_0 v_0},\ r=\frac{\overline{r}}{R},\ v=\frac{\overline{v}}{v_0},\ \mu=\frac{\overline{\mu}}{\mu_0},\ \theta=\frac{\overline{\theta}-\theta_0}{\theta_1-\theta_0},\\ \Gamma &= \frac{\mu_0^2 v_0}{k(\theta_1-\theta_0)},\ P=\frac{\psi R^2}{k_1}, \gamma=\frac{\gamma^*}{R},\ M^2=\frac{\sigma B_0^2\, R^2}{\mu_0}.\end{aligned} \tag{4.5}$$

The non dimensional problem are

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\frac{dv}{dr}+\mu\frac{d^2v}{dr^2}+\frac{\beta}{r}\left(\frac{dv}{dr}\right)^3+3\beta\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}-P\left[\mu+\beta\left(\frac{dv}{dr}\right)^2\right]v-M^2v=c, \tag{4.6}$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dv}{dr}\right)^2\left(\mu(r)+\beta\left(\frac{dv}{dr}\right)^2\right)=0, \tag{4.7}$$

$$v(1)-\gamma\left[\frac{dv}{dr}(1)+\frac{\beta}{\mu}\left(\frac{dv}{dr}(1)\right)^3\right]=0;\ \theta(1)=0,\ \frac{dv}{dr}(0)=\frac{d\theta}{dr}(0)=0. \tag{4.8}$$

where R, P, v_0, μ_0, θ_0, $\overline{\theta}$ and θ_1 denote the radius, nondimensional form of porosity, reference velocity, reference viscosity, reference temperature, pipe temperature and fluid temperature respectively and bars have been dropped for simplicity.

4.2 Solution of the Problem

In this case, we find the solutions of the arising equations for the two values of dynamic viscosity.

4.2.1 Case 1: For constant viscosity

For constant viscosity model take

$$\mu = 1. \tag{4.9}$$

For HAM solution we select

$$v_0(r) = \frac{c\gamma}{4}(r^2 - 1 + 2\gamma), \qquad \theta_0 = \frac{c^2\Gamma(1 - r^4)}{64} \tag{4.10}$$

as initial approximations of v and θ respectively, which satisfy the corresponding boundary conditions. We use the method of higher order differential mapping, to choose the linear operator $\mathcal{L}_2$ which is defined by

$$\mathcal{L}_2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}, \tag{4.11}$$

such that

$$\mathcal{L}_2(C_7 + C_8 \ln r) = 0, \tag{4.12}$$

where C_7 and C_8 are the arbitrary constants.

The *zeroth*−order deformation problems

$$(1 - q)\mathcal{L}_2[v^*(r, q) - v_0(r)] = q\hbar\mathcal{N}_3[v^*(r, q), \theta^*(r, q)], \tag{4.13}$$

$$(1 - q)\mathcal{L}_2[\theta^*(r, q) - \theta_0(r)] = q\hbar\mathcal{N}_4[v^*(r, q), \theta^*(r, q)], \tag{4.14}$$

$$\begin{aligned} v^*(r,q) - \gamma\left[\frac{\partial v^*(r,q)}{\partial r} + \beta\left(\frac{\partial v^*(r,q)}{\partial r}\right)^2\right]\Bigg|_{r=1} &= 0, \quad \frac{\partial v^*(r,q)}{\partial r}\bigg|_{r=0} = 0, \\ \theta^*(r,q)\big|_{r=1} &= 0, \quad \frac{\partial \theta^*(r,q)}{\partial r}\bigg|_{r=0} = 0, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \mathcal{N}_3[v^*(r,q), \theta^*(r,q)] &= \frac{1}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\beta}{r}\left(\frac{dv^*}{dr}\right)^3 + 3\beta\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} \\ &\quad - Pv^* - M^2 v - c, \end{aligned} \tag{4.16}$$

$$\mathcal{N}_4[v^*(r,q), \theta^*(r,q)] = \frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2} + \Gamma\left(\frac{dv^*}{dr}\right)^2 + \Gamma\beta\left(\frac{dv^*}{dr}\right)^4. \tag{4.17}$$

For $q = 0$ and $q = 1$, we have

$$v^*(r,0) = v_0(r),\ \theta^*(r,0) = \theta_0(r)\ and\ v^*(r,1) = v(r),\ \theta^*(r,1) = \theta(r). \tag{4.18}$$

By Taylor's theorem and Eq. (4.18) we have

$$\begin{aligned} v^*(r,q) &= v_0(r) + \sum_{m=1}^{\infty}\left(\frac{1}{m!}\left.\frac{\partial^m v^*(r,q)}{\partial q^m}\right|_{q=0}\right)q^m, \\ \theta^*(r,q) &= \theta_0(r) + \sum_{m=1}^{\infty}\left(\frac{1}{m!}\left.\frac{\partial^m \theta^*(r,q)}{\partial q^m}\right|_{q=0}\right)q^m, \end{aligned} \tag{4.19}$$

Where the convergence of the series (4.19) depends upon the choice of $\hbar$, such that series converges at $q = 1$, then Eq. (4.19) becomes

$$\left.\begin{aligned} v(r) = v_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m v^*(r,q)}{\partial q^m}\right|_{q=0}, \\ \theta(r) = \theta_0(r) + \sum_{m=1}^{\infty}\frac{1}{m!}\left.\frac{\partial^m \theta^*(r,q)}{\partial q^m}\right|_{q=0} \end{aligned}\right\}. \tag{4.20}$$

mth order Deformation Equations

$$\mathcal{L}_2[v_m(r) - \chi_m v_{m-1}(r)] = \hbar\Re_{3m}(r), \tag{4.21}$$

$$\mathcal{L}_2[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar\Re_{4m}(r), \tag{4.22}$$

$$v_m(1) - \gamma\left[\frac{\partial v_m(1)}{\partial r} + \beta\left(\frac{\partial v_m(1)}{\partial r}\right)^3\right] = \theta_m(1) = 0, \qquad v_m'(0) = \theta_m'(0) = 0, \tag{4.23}$$

$$\Re_{2m}(r) = \left.\begin{array}{c} \frac{1}{r}\frac{dv_{m-1}}{dr} + \frac{d^2 v_{m-1}}{dr^2} + \frac{\beta}{r}\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr} \\ +3\beta\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2 v_i}{dr^2} - (1-\chi_m)c - qv_{m-1} \\ -P\beta v_i - M^2 v_{m-1} \end{array}\right\}, \tag{4.24}$$

$$\Re_{3m}(r) = \left.\begin{array}{c} \frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2} + \Gamma\sum_{k=0}^{m-1}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr} + \\ \beta\Gamma\sum_{k=0}^{m-1}\sum_{j=0}^{k}\sum_{i=0}^{j}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr} \end{array}\right\}. \tag{4.25}$$

4.2.2 Case 2: Space dependent viscosity

In this case we choose

$$\mu = r, \tag{4.26}$$

$$v_0(r) = \frac{c\gamma}{6}(r^2 - 1 + 2\gamma), \qquad \theta_0 = \frac{c^4\Gamma(1-r^2)}{64}, \tag{4.27}$$

and the linear operator

$$\mathcal{L}_3 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}, \tag{4.28}$$

such that

$$\mathcal{L}_3(C_9 + \frac{C_{10}}{r}) = 0, \tag{4.29}$$

where C_9 and C_{10} are the arbitrary constants. The *zeroth*− and *mth*−order order deformation problems in this case are developed as

$$(1-q)\mathcal{L}_3[v^*(r,q) - v_0(r)] = q\hbar\mathcal{N}_5[v^*(r,q), \theta^*(r,q)], \tag{4.30}$$

$$(1-q)\mathcal{L}_3[\theta^*(r,q) - \theta_0(r)] = q\hbar\mathcal{N}_6[v^*(r,q), \theta^*(r,q)], \tag{4.31}$$

$$\left.\begin{array}{c} v^*(r,q) - \gamma\left[\frac{\partial v^*(r,q)}{\partial r} + \frac{\beta}{r}\left(\frac{\partial v^*(r,q)}{\partial r}\right)^3\right]\Bigg|_{r=1} = 0, \quad \left.\frac{\partial v^*(r,q)}{\partial r}\right|_{r=0} = 0, \\ \theta^*(r,q)|_{r=1} = 0, \quad \left.\frac{\partial \theta^*(r,q)}{\partial r}\right|_{r=0} = 0 \end{array}\right\}, \tag{4.32}$$

$$\mathcal{L}_3[v_m(r) - \chi_m v_{m-1}(r)] = \hbar\Re_{5m}(r), \tag{4.33}$$

$$\mathcal{L}_3[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar\Re_{6m}(r), \tag{4.34}$$

$$v_m(1) - \gamma\left[\frac{\partial v_m(1)}{\partial r} + \frac{\beta}{r}\left(\frac{\partial v_m(1)}{\partial r}\right)^2\right] = 0, \theta_m(1) = 0, \; v_m'(0) = \theta_m'(0) = 0, \tag{4.35}$$

$$\begin{aligned} \mathcal{N}_5[v^*(r,q), \theta^*(r,q)] &= \frac{2}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\beta}{r^2}\left(\frac{dv^*}{dr}\right)^3 + \frac{3\beta}{r}\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} \\ &\quad - Pv^* - \frac{M^2v^*}{r} - \frac{c}{r}, \end{aligned} \tag{4.36}$$

$$\mathcal{N}_6[v^*(r,q), \theta^*(r,q)] = \frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2} + \Gamma\left(\frac{dv^*}{dr}\right)^2 + \Gamma\beta\left(\frac{dv^*}{dr}\right)^4 + \Gamma r\left(\frac{dv^*}{dr}\right)^2, \tag{4.37}$$

$$\Re_{5m}(r) = \left.\begin{array}{c} 2r\dfrac{dv_{m-1}}{dr} + r^2\dfrac{d^2v_{m-1}}{dr^2} + \beta\displaystyle\sum_{k=0}^{m-1}\sum_{i=0}^{k}(\frac{dv_{m-1-k}}{dr})\frac{dv_{k-1}}{dr}\frac{dv_i}{dr} \\ +3\beta r\displaystyle\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2v_i}{dr^2} - (1-\chi_m)cr \\ -M^2rv_{m-1} - qr^2v_{m-1} - P\beta rv_i, \end{array}\right\}, \tag{4.38}$$

$$\Re_{6m}(r) = \left.\begin{array}{c} \dfrac{1}{r}\dfrac{d\theta_{m-1}}{dr} + \dfrac{d^2\theta_{m-1}}{dr^2} + \Gamma r\displaystyle\sum_{k=0}^{m-1}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_k}{dr} \\ +\beta\Gamma\displaystyle\sum_{k=0}^{m-1}\sum_{j=0}^{k}\sum_{i=0}^{j}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-j}}{dr}\frac{dv_{j-i}}{dr}\frac{dv_i}{dr} \end{array}\right\}. \tag{4.39}$$

4.3 Convergence of the Solution

The most important aspect of series solution is to discuss the convergence of solution. In homotopy analysis method the convergence of series is ensured by using a auxiliary parameter $\hbar$ Figs. 4.1 and 4.2 provide the $\hbar-$curves in constant viscosity case for different sets of parameters M, P and γ. The admissible values of velocity are $-1.8 \leq \hbar \leq 0$ and for temperature are $-1.8 \leq \hbar \leq -0.2$. Figs. 4.3 and 4.4 represent the $\hbar-$curves for variable viscosity when $\mu = r$. The admissible ranges for both velocity and temperature profiles are $-1.8 \leq \hbar \leq -0.6$ and $-1.4 \leq \hbar \leq -0.5$, respectively. In Figs. 4.3 and 4.6, the graphs of residual errors for constant and variable viscosity are plotted, respectively.

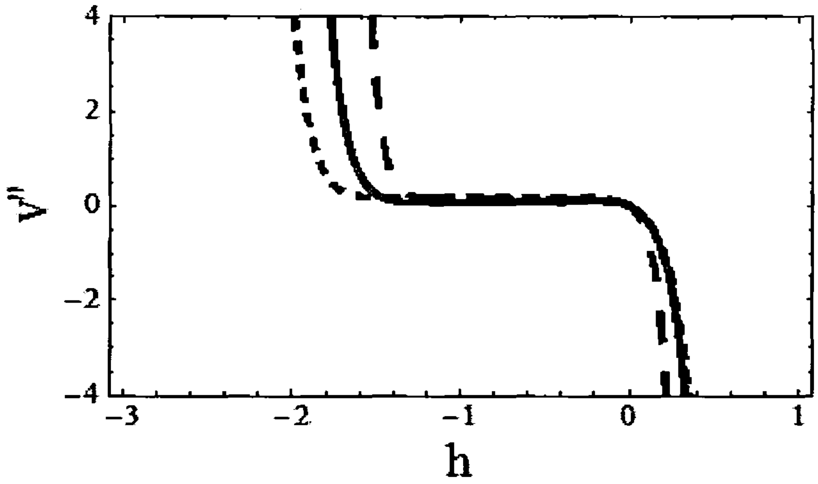

Fig. 4.1 : $\hbar$–curve for velocity profile in case of constant viscosity.

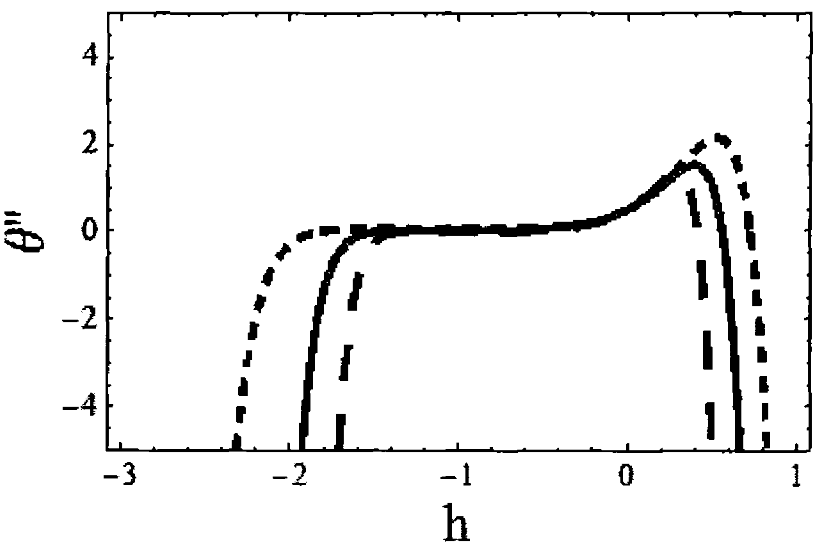

Fig. 4.2 : $\hbar$–curve for temperature in case of constant viscosity.

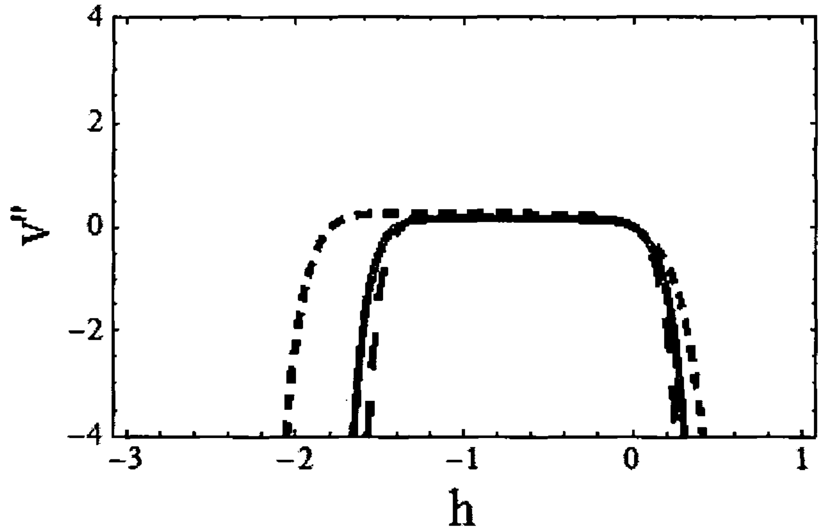

Fig. 4.3 : $\hbar$–curve for velocity profile in case of variable viscosity.

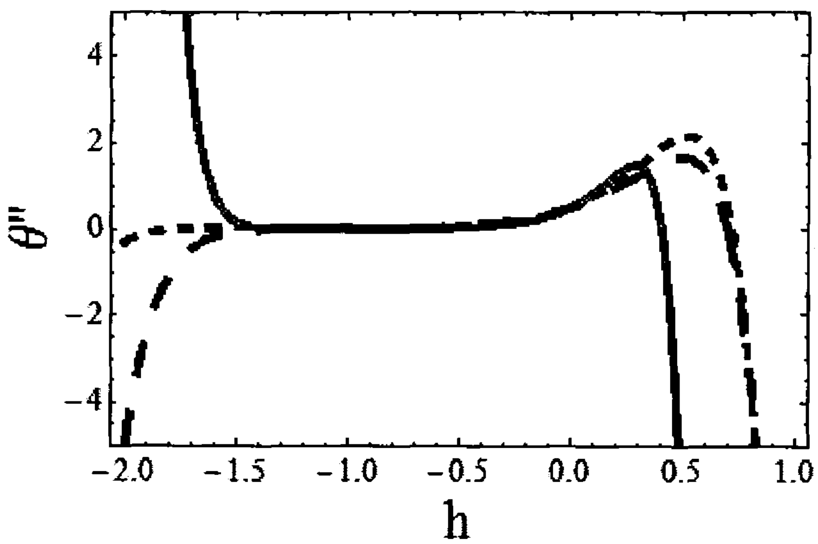

Fig. 4.4 : $\hbar$–curve for temperature in case of constant viscosity.

4.4 Graphical Results

In this we present the results by plotting both velocity v and temperature θ against the pipe radius r. Figs. (4.5 – 4.7) provide the variation of velocity with respect to slip parameter γ, magnetic parameter M and porosity P for suction and injection. Figs. (4.10 – 4.14) show the variation of v and θ when viscosity is space dependent the effect of the different parameter is shown. Note that the other parameters like third grade parameter β, pressure c and Γ are kept fixed.

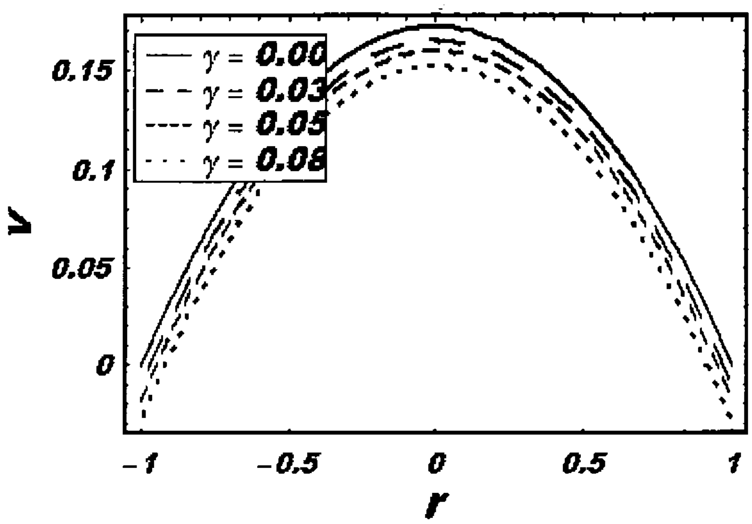

Fig. 4.5 : Variation of velocity with the change in slip parameter γ for $M = 1$ and $P = 1$.

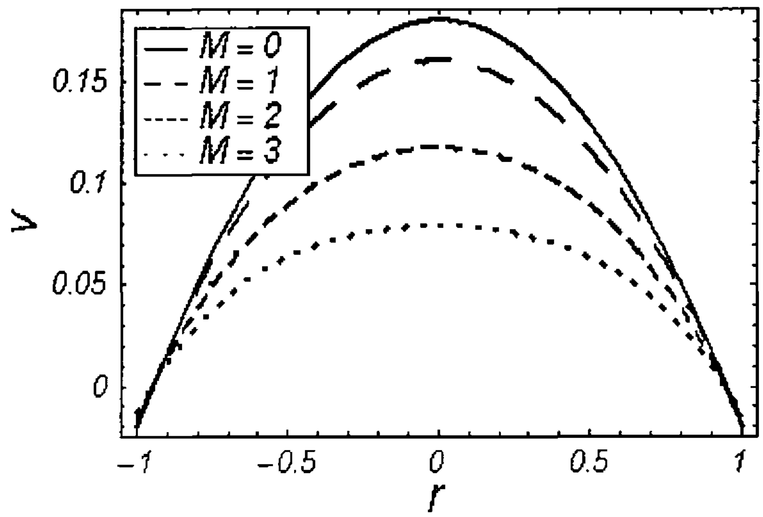

Fig. 4.6 : Variation of velocity with the change in M for $\gamma = 0.05$ and $P = 1$.

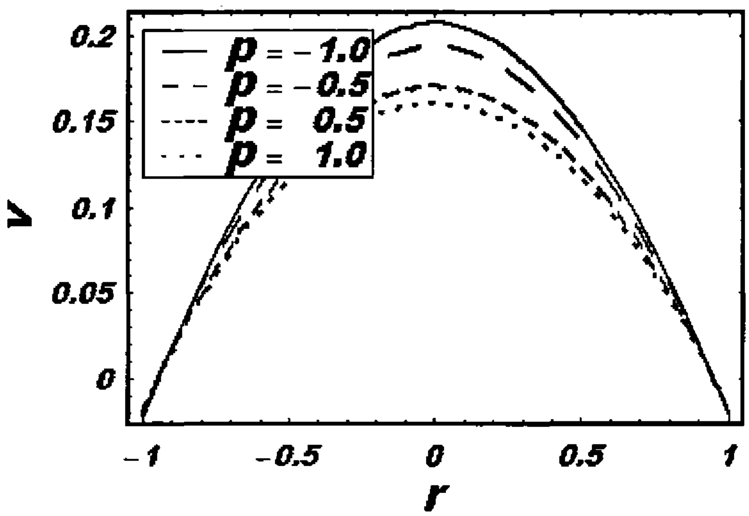

Fig. 4.7 : Variation of velocity with the change porosity P for $\gamma = 0.05$ and $M = 1$.

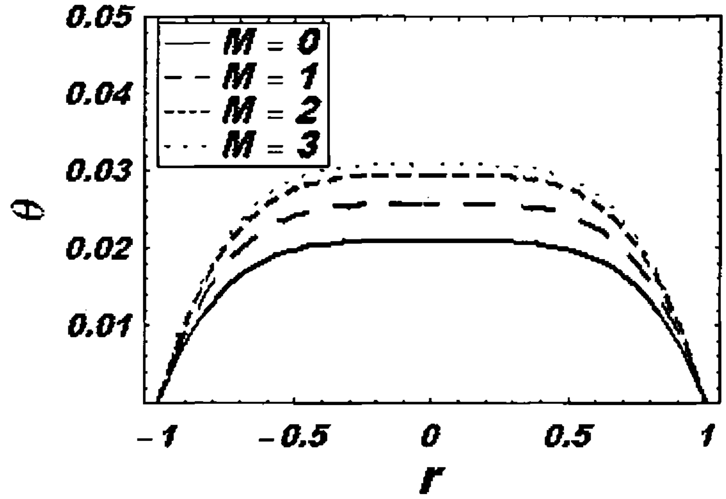

Fig. 4.8 : Variation of θ by changing values of M, with $P = 1$ and $\gamma = 1$.

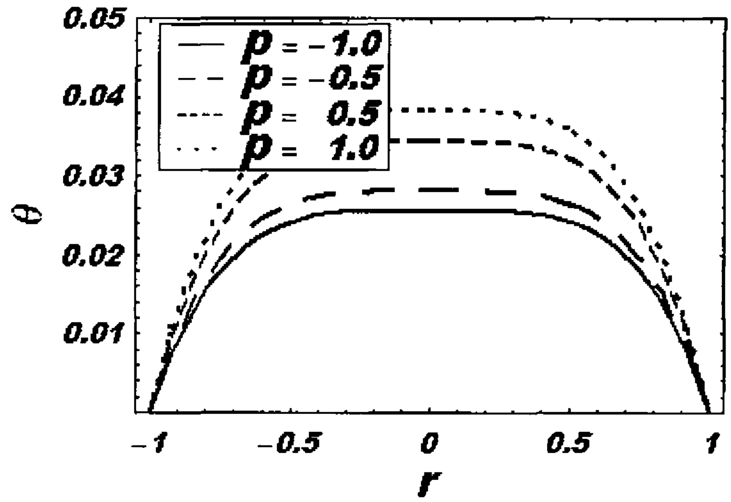

Fig. 4.9 : Variation of θ by changing values of P, with $M = 1$ and $\gamma = 0.05$.

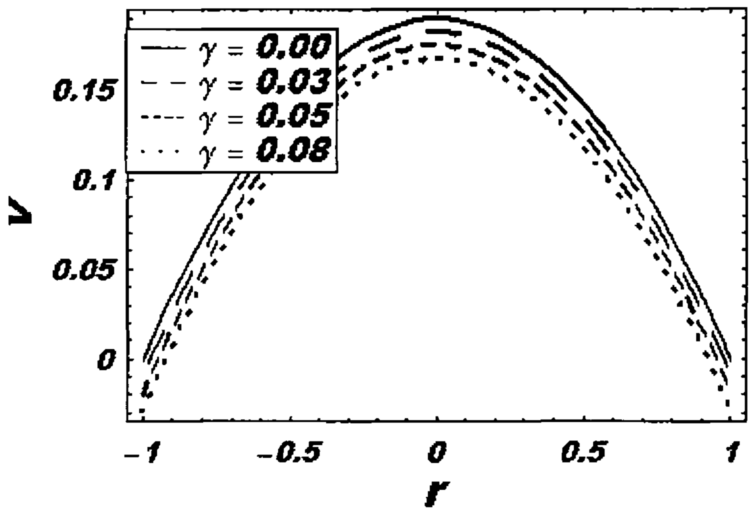

Fig. 4.10 : Variation of velocity with the change of slip parameter γ, for $P = 1$ and $M = 1$.

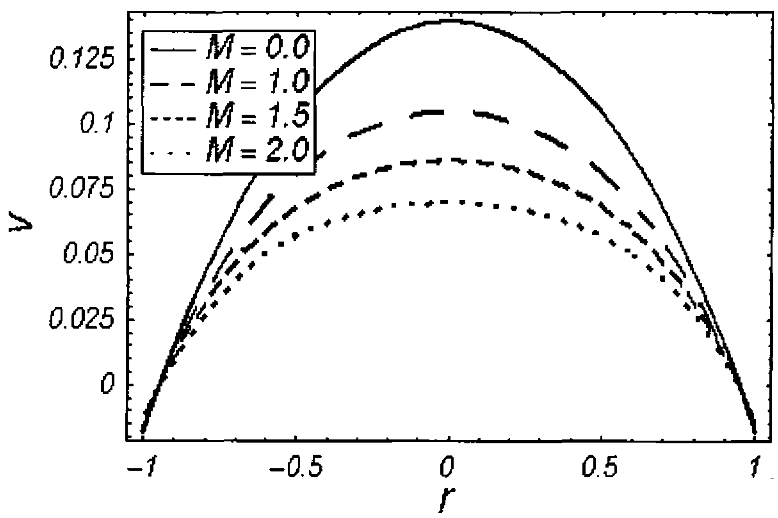

Fig. 4.11 : Variation of velocity with the change of M for $\gamma = 0.05$ and $P = 1$.

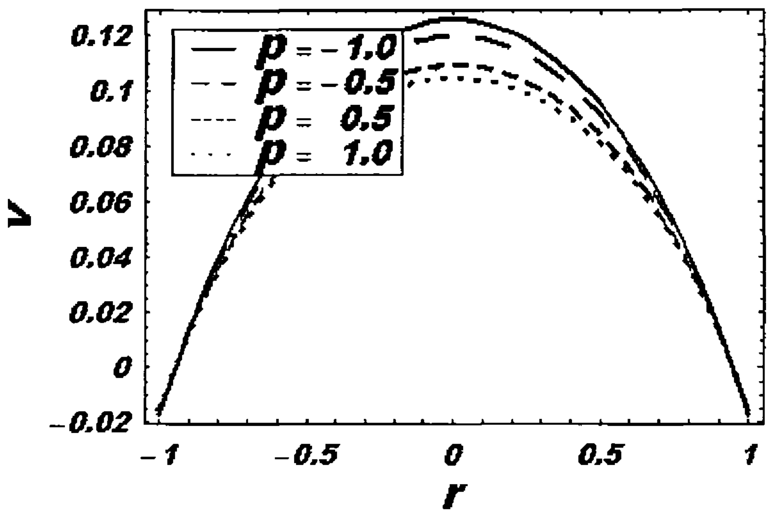

Fig. 4.12 : Variation of velocity with the change of porosity P for $\gamma = 0.05$ and $M = 1$.

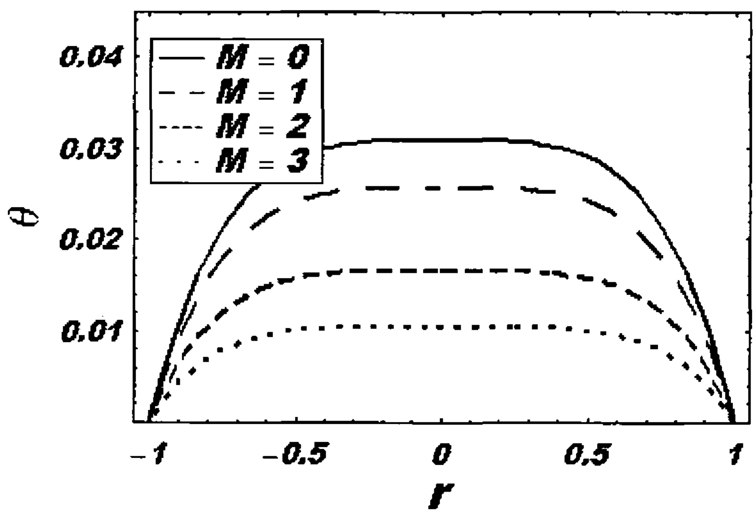

Fig. 4.13 : Variation of θ by changing values of M with $P = 1$ and $\gamma = 0.05$.

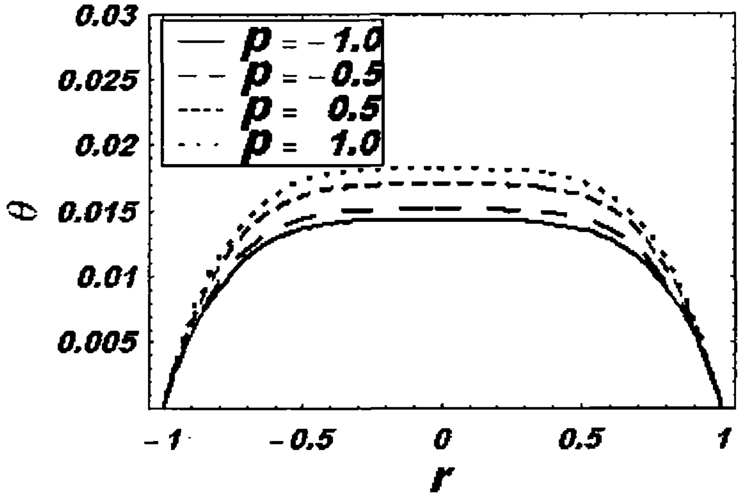

Fig. 4.14 : Variation of θ by changing values of P, with $\gamma = 0.05$ and $M = 1$.

4.5 Conclusion

The main emphasis in this study is given to the effects of MHD, porosity and slip parameter on a constant and variable viscosity for steady flow of a third grade fluid in a pipe. Series solutions have been developed and their convergence is carefully analyzed.

- The velocity and temperature are increasing function of M.
- Increase in suction suppression in the velocity and temperature.
- Increase in suction parameter the resistance in the flow increase developing a suppression in velocity v and temperature θ.
- There is decreases in v and θ when slip parameter γ is increased.
- The effects of γ, p and M on v and θ for variable viscosity are shown in Figs. 4.5 to 4.14.

Chapter 5

Effect of MHD on a Third Grade Nanofluid in a Coaxial Porous Cylinders

The objective of the present study is to analyze the effect of MHD on a third grade nanofluids (NF) in a coaxial porous cylinders. Assuming, a unidirectional, electrically conducting, an incompressible and thermodynamic third grade NF between two infinite coaxial cylinders. The outer cylinder is porous under the influence of perpendicular magnetic field. The flow is induced by a constant pressure gradient and motion of an inner cylinder with no-slip conditions is taken in account. The heat transfer analysis and nanoparticle concentration equations are also analyzed. The nonlinear governing equations have been solved by the homotopy analysis method which does not require any small or large parameters appearing in the problem. This method has already been successfully used to solve highly non-linear problems. Convergence of the obtained solutions are properly discussed. Two cases for variable viscosity and viscous dissipation are discussed.

5.1 Mathematical Formulation of Problem

Considering the velocity, temperature and nano particle concentration field as

$$\left.\begin{aligned} \mathbf{V} &= [0,0,v(r)] \\ \theta &= [0,0,\theta(r)] \\ \phi &= [0,0,\phi(r)] \end{aligned}\right\}. \tag{5.1}$$

The dimensionless problems can be written as follows:

$$\left.\begin{array}{c}\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\frac{dv}{dr}+\mu\frac{d^2v}{dr^2}+\frac{\beta}{r}(\frac{dv}{dr})^3+3\beta(\frac{dv}{dr})^2\frac{d^2v}{dr^2}= \\ c+P\mu v+M^2v-G_r\theta-B_r\phi\end{array}\right\}, \tag{5.2}$$

$$\alpha\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+N_b\frac{d\theta}{dr}\frac{d\phi}{dr}+\alpha_1 N_t\left(\frac{d\theta}{dr}\right)^2=0, \tag{5.3}$$

$$N_b\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}\right)+N_t\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=0. \tag{5.4}$$

The corresponding boundary conditions are

$$v(1)=1,\ v(2)=0;\ \theta(1)=1,\ \theta(2)=0;\ \phi(1)=1,\ \phi(2)=0. \tag{5.5}$$

The non-dimensional quantities are defined as

$$\begin{aligned}\phi &= \frac{\overline{\phi}-\phi_w}{\phi_m-\phi_w},\ B_r=\frac{(\rho_p-\rho_w)\,R^2\,(\phi_m-\phi_w)\,g}{\mu_0 v_0}, \\ G_r &= \frac{(\theta_m-\theta_w)\,\rho_{fw}R^2\,(1-\phi_w)\,g}{\mu_0 v_0},\end{aligned} \tag{5.6}$$

where ϕ_m is mass concentration, G_r is thermophoresis diffusion constant and B_r is Brownian diffusion constant.

5.2 Solution of the Problem

In this section, we find the series solutions of the nonlinear governing equations using homotopy analysis method for two cases namely; constant and variable viscosity.

5.2.1 Case 1: For Constant Viscosity

For constant viscosity model we select

$$v_0(r)=\frac{(8-r^3)}{7},\quad \theta_0=\frac{(8-r^3)}{7},\quad \phi_0=\frac{(8-r^3)}{7} \tag{5.7}$$

as initial approximations of v, θ and ϕ respectively, which satisfy the corresponding boundary conditions. We choose the linear operator $\mathcal{L}_1$ same as given in Eq. (9.1).

The *zeroth*−order deformation problems

$$(1-q)\mathcal{L}_1[v^*(r,q)-v_0(r)] = q\hbar\mathcal{N}_7[v^*(r,q),\theta^*(r,q),\phi^*(r,q)], \tag{5.8}$$

$$(1-q)\mathcal{L}_1[\theta^*(r,q)-\theta_0(r)] = q\hbar\mathcal{N}_8[v^*(r,q),\theta^*(r,q),\phi^*(r,q)], \tag{5.9}$$

$$(1-q)\mathcal{L}_1[\phi^*(r,q)-\phi_0(r)] = q\hbar\mathcal{N}_9[v^*(r,q),\theta^*(r,q),\phi^*(r,q)], \tag{5.10}$$

$$v^*(r,q)|_{r=1} = 1, \quad v^*(r,q)|_{r=2} = 0, \tag{5.11}$$

$$\theta^*(r,q)|_{r=1} = 1, \theta^*(r,q)|_{r=2} = 0, \tag{5.12}$$

$$\phi^*(r,q)|_{r=1} = 1, \phi^*(r,q)|_{r=2} = 0, \tag{5.13}$$

$$\mathcal{N}_7[v^*(r,q),\theta^*(r,q),\phi^*(r,q)] = \left.\begin{array}{c} \frac{1}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\beta}{r}\left(\frac{dv^*}{dr}\right)^3 - c \\ +3\beta\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} + G_r\theta^* + B_r\phi^* \\ -Pv^* - M^2v \end{array}\right\}, \tag{5.14}$$

$$\mathcal{N}_8[v^*(r,q),\theta^*(r,q),\phi^*(r,q)] = \frac{\alpha}{r}\frac{d\theta^*}{dr} + \alpha\frac{d^2\theta^*}{dr^2} + N_b\frac{d\theta^*}{dr}\frac{d\phi^*}{dr} + \alpha_1 N_t\left(\frac{d\theta^*}{dr}\right)^2, \tag{5.15}$$

$$\mathcal{N}_9[v^*(r,q),\theta^*(r,q),\phi^*(r,q)] = \frac{1}{r}\frac{d\phi^*}{dr} + \frac{d^2\phi^*}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2}\right). \tag{5.16}$$

For $q=0$ and $q=1$, we get

$$\left.\begin{array}{c} v^*(r,0) = v_0(r), \ \theta^*(r,0) = \theta_0(r), \ \phi^*(r,0) = \phi_0(r) \\ v^*(r,1) = v(r), \theta^*(r,1) = \theta(r), \ \phi^*(r,1) = \phi(r) \end{array}\right\}, \tag{5.17}$$

where q increases from 0 to 1, $v^*(r,q)$, $\theta^*(r,q)$, $\phi^*(r,q)$ varies from $v_0(r)$, $\theta_0(r)$, $\phi_0(r)$ to $v(r)$, $\theta(r)$, $\phi(r)$ respectively. By Taylor's theorem we have

$$\left.\begin{array}{l} v^*(r,q) = v_0(r) + \sum_{m=1}^{\infty}\left(\frac{1}{m!}\left.\frac{\partial^m v^*(r,q)}{\partial q^m}\right|_{q=0}\right)q^m \\ \theta^*(r,q) = \theta_0(r) + \sum_{m=1}^{\infty}\left(\frac{1}{m!}\left.\frac{\partial^m \theta^*(r,q)}{\partial q^m}\right|_{q=0}\right)q^m \\ \phi^*(r,q) = \phi_0(r) + \sum_{m=1}^{\infty}\left(\frac{1}{m!}\left.\frac{\partial^m \phi^*(r,q)}{\partial q^m}\right|_{q=0}\right)q^m \end{array}\right\}, \tag{5.18}$$

where the convergence of the series given in Eqs. (5.11) – (5.13) depends upon the choice of $\hbar$, such that series converges at $q = 1$, then Eq. (5.18) becomes

$$\left.\begin{aligned} v(r) &= v_0(r) + \sum_{m=1}^{\infty} \frac{1}{m!} \left.\frac{\partial^m v^*(r,q)}{\partial q^m}\right|_{q=0} \\ \theta(r) &= \theta_0(r) + \sum_{m=1}^{\infty} \frac{1}{m!} \left.\frac{\partial^m \theta^*(r,q)}{\partial q^m}\right|_{q=0} \\ \phi(r) &= \phi_0(r) + \sum_{m=1}^{\infty} \frac{1}{m!} \left.\frac{\partial^m \phi^*(r,q)}{\partial q^m}\right|_{q=0} \end{aligned}\right\}. \tag{5.19}$$

The mth order deformation problems are given by

$$\mathcal{L}_1[v_m(r) - \chi_m v_{m-1}(r)] = \hbar \Re_{7m}(r), \tag{5.20}$$

$$\mathcal{L}_1[\theta_m(r) - \chi_m \theta_{m-1}(r)] = \hbar \Re_{8m}(r), \tag{5.21}$$

$$\mathcal{L}_1[\phi_m(r) - \chi_m \phi_{m-1}(r)] = \hbar \Re_{9m}(r), \tag{5.22}$$

$$v_m(2) = 0, \theta_m(2) = 0, \phi_m(2) = 0,\ v_m(1) = 1, \theta_m(1) = 1, \phi_m(1) = 1, \tag{5.23}$$

where

$$\begin{aligned} \Re_{7m}(r) &= \frac{1}{r}\frac{dv_{m-1}}{dr} + \frac{d^2 v_{m-1}}{dr^2} + G_r \theta_m + B_r \phi_m - (1-\chi_m)c + \\ &\quad \frac{\beta}{r} \sum_{k=0}^{m-1} \sum_{i=0}^{k} \left(\frac{dv_{m-1-k}}{dr}\right) \frac{dv_{k-1}}{dr} \frac{dv_i}{dr} + 3\beta \sum_{k=0}^{m-1} \sum_{i=0}^{k} \left(\frac{dv_{m-1-k}}{dr}\right) \frac{dv_{k-1}}{dr} \frac{d^2 v_i}{dr^2} \\ &\quad - P v_{m-1} - M^2 v_{m-1}, \end{aligned} \tag{5.24}$$

$$\Re_{8m}(r) = \frac{\alpha}{r}\frac{d\theta_{m-1}}{dr} + \alpha \frac{d^2\theta_{m-1}}{dr^2} + N_b \sum_{k=0}^{m-1} \left(\frac{d\phi_{m-1-k}}{dr}\right) \frac{d\theta_k}{dr} + \alpha_1 N_t \sum_{k=0}^{m-1} \left(\frac{d\phi_{m-1-k}}{dr}\right) \frac{d\theta_k}{dr}, \tag{5.25}$$

$$\Re_{9m}(r) = \frac{1}{r}\frac{d\phi_{m-1}}{dr} + \frac{d^2\phi_{m-1}}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2}\right). \tag{5.26}$$

5.2.2 Case 2: For Space Dependent Viscosity

For space dependent viscosity, the *zeroth*− and *mth*−order order deformation problems with the same linear operator and initial guess are developed as

$$(1-q)\mathcal{L}_1[v^*(r,q) - v_0(r)] = q\hbar \mathcal{N}_{10}[v^*(r,q), \theta^*(r,q), \phi^*(r,q)], \tag{5.27}$$

$$(1-q)\mathcal{L}_1[\theta^*(r,q)-\theta_0(r)] = q\hbar\mathcal{N}_{11}[v^*(r,q),\theta^*(r,q),\phi^*(r,q)], \tag{5.28}$$

$$(1-q)\mathcal{L}_1[\phi^*(r,q)-\phi_0(r)] = q\hbar\mathcal{N}_{12}[v^*(r,q),\theta^*(r,q),\phi^*(r,q)], \tag{5.29}$$

$$\left.\begin{array}{l} v^*(r,q)|_{r=1} = 1, \;\; v^*(r,q)|_{r=2} = 0 \\ \theta^*(r,q)|_{r=1} = 1, \theta^*(r,q)|_{r=2} = 0 \\ \phi^*(r,q)|_{r=1} = 1, \phi^*(r,q)|_{r=2} = 0 \end{array}\right\}, \tag{5.30}$$

$$\mathcal{L}_1[v_m(r)-\chi_m v_{m-1}(r)] = \hbar\Re_{9m}(r), \tag{5.31}$$

$$\mathcal{L}_1[\theta_m(r)-\chi_m \theta_{m-1}(r)] = \hbar\Re_{10m}(r), \tag{5.32}$$

$$\mathcal{L}_1[\phi_m(r)-\chi_m \phi_{m-1}(r)] = \hbar\Re_{11m}(r), \tag{5.33}$$

$$\left.\begin{array}{l} v_m(2) = 0, \theta_m(2) = 0, \phi_m(2) = 0 \\ v_m(1) = 1, \theta_m(1) = 1, \phi_m(1) = 1 \end{array}\right\}, \tag{5.34}$$

$$\begin{aligned}\mathcal{N}_{10}[v^*(r,q),\theta^*(r,q)] &= \frac{2}{r}\frac{dv^*}{dr} + \frac{d^2v^*}{dr^2} + \frac{\beta}{r^2}\left(\frac{dv^*}{dr}\right)^3 + \frac{3\beta}{r}\left(\frac{dv^*}{dr}\right)^2\frac{d^2v^*}{dr^2} + G_r\theta^* \\ &\quad + B_r\phi^* - Pv^* - \frac{M^2v^*}{r} - \frac{c}{r},\end{aligned} \tag{5.35}$$

$$\mathcal{N}_{11}[v^*(r,q),\theta^*(r,q),\phi^*(r,q)] = \frac{\alpha}{r}\frac{d\theta^*}{dr} + \alpha\frac{d^2\theta^*}{dr^2} + N_b\frac{d\theta^*}{dr}\frac{d\phi^*}{dr} + \alpha_1 N_t\left(\frac{d\theta^*}{dr}\right)^2, \tag{5.36}$$

$$\mathcal{N}_{12}[v^*(r,q),\theta^*(r,q),\phi^*(r,q)] = \frac{1}{r}\frac{d\phi^*}{dr} + \frac{d^2\phi^*}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta^*}{dr} + \frac{d^2\theta^*}{dr^2}\right), \tag{5.37}$$

where

$$\begin{aligned}\Re_{9m}(r) &= 2r\frac{dv_{m-1}}{dr} + r^2\frac{d^2v_{m-1}}{dr^2} + \beta\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{dv_i}{dr} + \\ &\quad 3\beta r\sum_{k=0}^{m-1}\sum_{i=0}^{k}\left(\frac{dv_{m-1-k}}{dr}\right)\frac{dv_{k-1}}{dr}\frac{d^2v_i}{dr^2} - (1-\chi_m)cr \\ &\quad -M^2rv_{m-1} + G_r r\theta_m - qr^2v_{m-1} - Prv_i + B_r r\phi_m,\end{aligned} \tag{5.38}$$

$$\Re_{10m}(r) = \frac{\alpha}{r}\frac{d\theta_{m-1}}{dr} + \alpha\frac{d^2\theta_{m-1}}{dr^2} + N_b\sum_{k=0}^{m-1}\left(\frac{d\phi_{m-1-k}}{dr}\right)\frac{d\theta_k}{dr} + \alpha_1 N_t\sum_{k=0}^{m-1}\left(\frac{d\phi_{m-1-k}}{dr}\right)\frac{d\theta_k}{dr}, \tag{5.39}$$

$$\Re_{11m}(r) = \frac{1}{r}\frac{d\phi_{m-1}}{dr} + \frac{d^2\phi_{m-1}}{dr^2} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta_{m-1}}{dr} + \frac{d^2\theta_{m-1}}{dr^2}\right). \tag{5.40}$$

5.3 Graphical Results

Here we discuss the convergence of solutions. The convergence of the solution is discussed by drawing the $\hbar$−curve in Figs 5.1 to 5.4. These figures depict the convergence region and rate of approximation for the homotopy analysis method. It is noticed that the admissible values of $\hbar$ in all cases is approximately $-1.5 \leq \hbar \leq 0$. To see the effects of emerging parameters Figs. 5.5 to 5.12 have been displayed. The effects of MHD and porosity on velocity profile are shown in Figs. 5.5 to 5.8. Figs. 5.9 and 5.10 have been prepared to explain the variation of thermophoresis and Brownian parameters on the temperature distribution. Finally Figs. 5.11 to 5.12 bring out the influence on nanoparticle concentration.

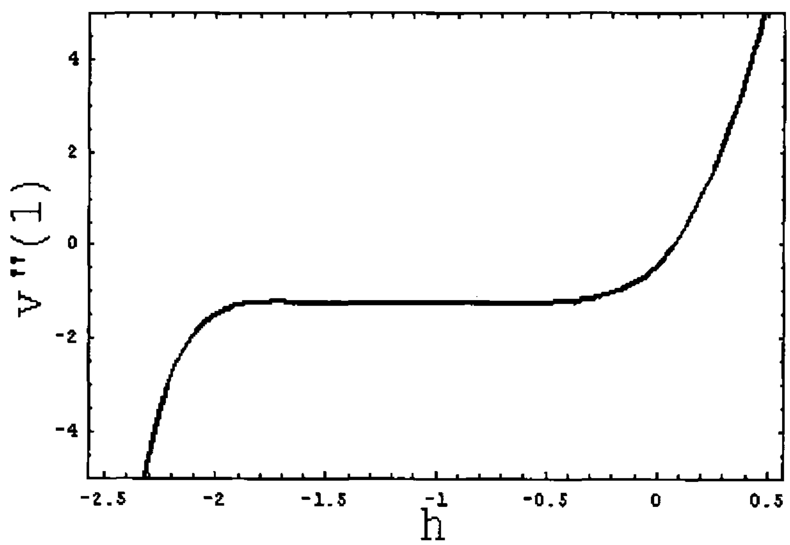

Fig. 5.1 : $\hbar$−curve for velocity profile for constant viscosity.

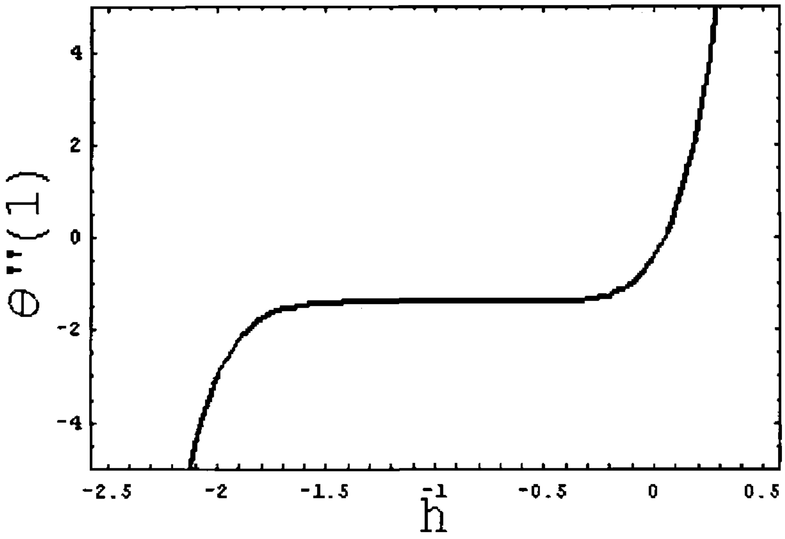

Fig. 5.2: $\hbar$−curve for temperature profile.

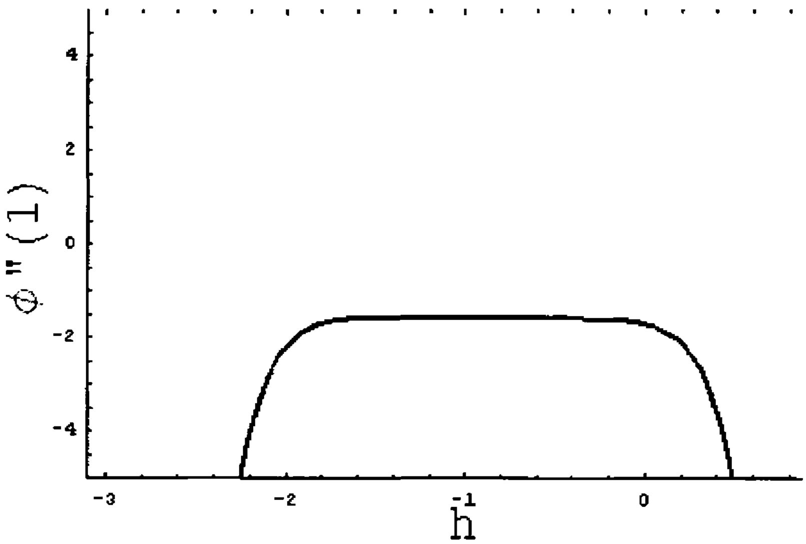

Fig. 5.3 : $\hbar$−curve for nano particle concentration profile.

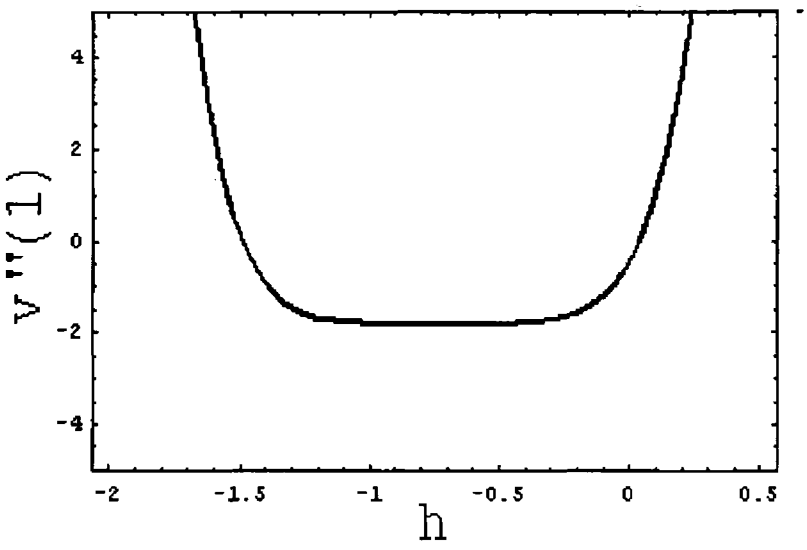

Fig. 5.4 : $\hbar$—curve for velocity profile for variable viscosity.

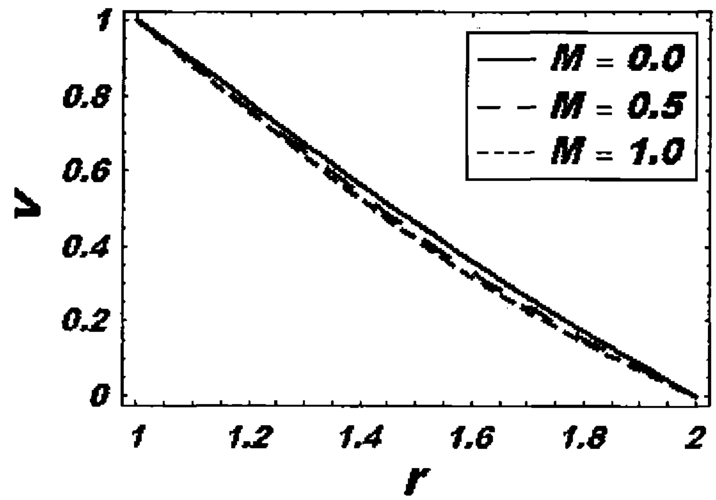

Fig. 5.5 : Effect of M on velocity profile when $N_b = 1, N_t = 1$ and $P = 0.5$ for constant viscosity.

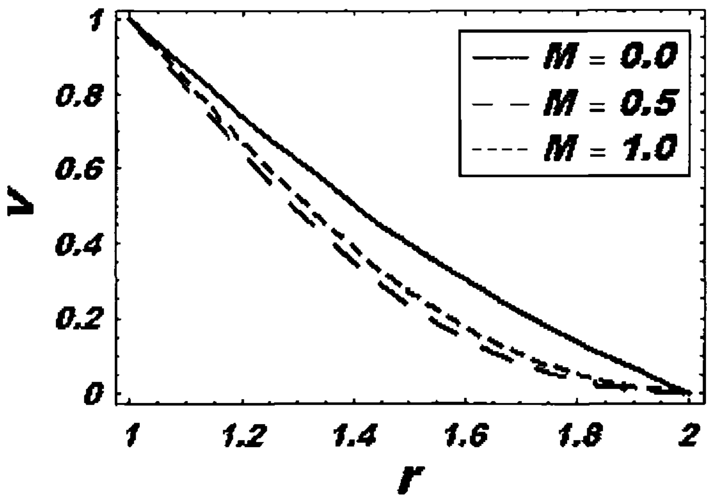

Fig. 5.6 : Effect of M on velocity profile when $N_b = 1, N_t = 1$ and $P = 0.5$ for variable viscosity.

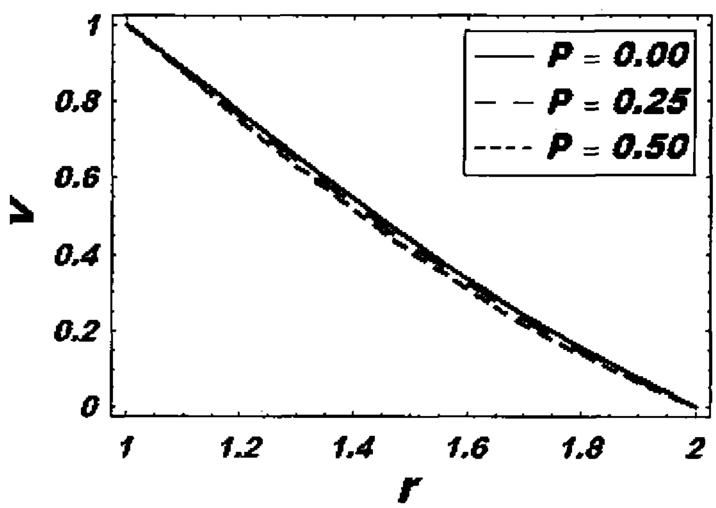

Fig. 5.7 : Effect of P on velocity profile when $N_b = 1, N_t = 1$ and $M = 0.5$ for constant viscosity.

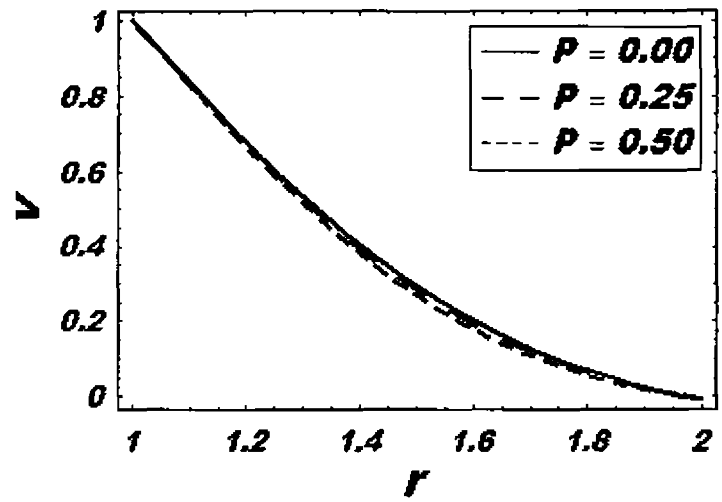

Fig. 5.8 : Effect of P on velocity profile when $N_b = 1, N_t = 1$ and $M = 0.5$ for variable viscosity.

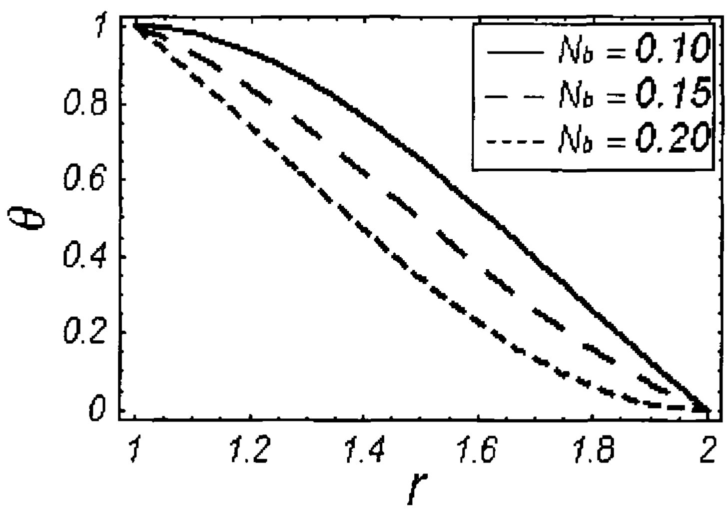

Fig. 5.9 : Effect of N_b on temperature distribution when $N_t = 0.1$.

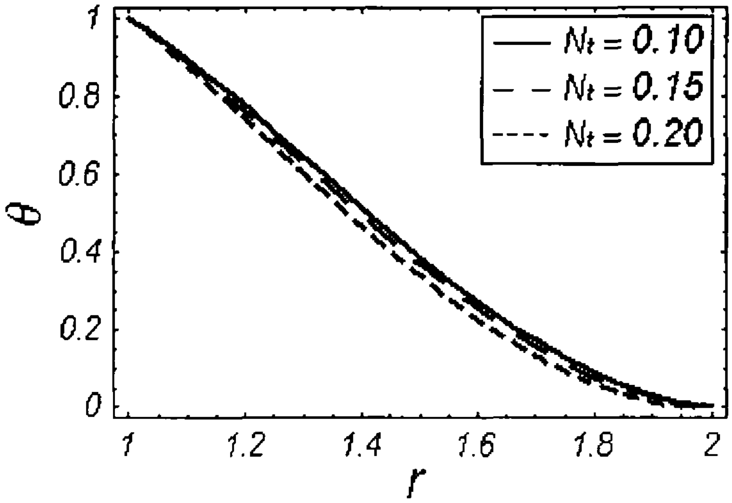

Fig. 5.10 : Effect of N_t on temperature distribution when $N_b = 0.1$.

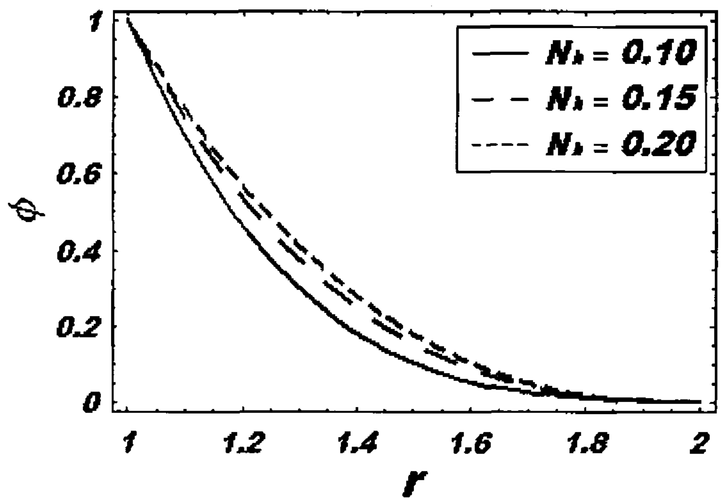

Fig. 5.11 : Effect of N_b on nanoparticle concentration when $N_t = 0.1$.

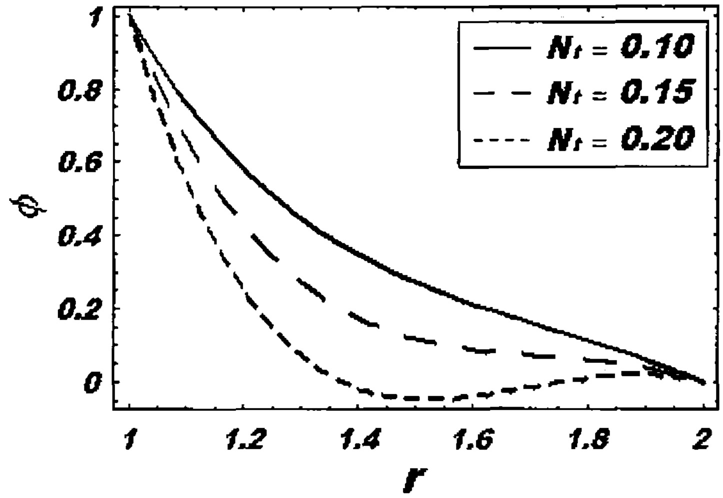

Fig. 5.12 : Effect of N_t on nanoparticle concentration when $N_b = 0.1$.

5.4 Conclusion

The effect of MHD on a third grade nanofluid in coaxial porous cylinders has been examined. In order to point out the salient features of the analysis of MHD and heat transfer for nanofluid the following discussions are set out. The graphs showing the behavior of the velocity temperature and nanoparticle concentration are plotted against r. Separate figures have been drawn in order to see the variation of each of the sundry parameter. To see the effects of emerging parameters for constant and variable viscosity Figs. 5.5 to 5.8 have been displayed. It is found that the velocity decreases with an increase in the values of M and P. The effects of N_b and N_t on nanoparticle concentration and temperature distribution are shown in Figs. 5.9 to 5.12. Figs. 5.9 and 5.10 have been prepared to explian the varition of N_b and N_t on the temperature distribution. Here, it is revealed that the thermal boundary layer thickness increases when large values of N_b have been taken into account and the thermal boundary layer decreases with increasing N_t. Figs. 5.11 and 5.12 bring out the influence of nanoparticle concentration for constant and variable viscosity. It is observed that the nanoparticles concentration increase with the decrease in N_b and decreases by increasing N_t.

Chapter 6

An Optimal Homotopy Asymptotic Method for Strongly Nonlinear Differential Equations of Nanofluid

In this chapter, an optimal homotopic asymptotic method is used to solve nonlinear differential equation of third grade NF between coaxial porous cylinders with variable viscosity. A unidirectional, electrically conducting and incompressible flow between two infinite porous coaxial cylinders under the influence of perpendicular magnetic field is considered. The flow is driven by constant pressure gradient and the motion of inner cylinder with partial slip conditions are considered on outer cylinder. The heat transfer and nanoparticles concentration equation are also taken into account. The effects of heat transfer analysis and concentration of nanoparticles are considered in the presence of magnetohydrodynamic and partial slip are also examined. The solutions are compared for slip length equals to zero with solutions of chapter six.

6.1 Mathematical Formulation of Problem

Considering the velocity, temperature and nano particle concentration field as

$$\left.\begin{aligned} \mathbf{V} &= [0,0,v(r)] \\ \theta &= [0,0,\theta(r)] \\ \phi &= [0,0,\phi(r)] \end{aligned}\right\}. \tag{6.1}$$

The problem in non-dimensional form is

$$\frac{d\mu}{dr}\frac{dv}{dr}+\frac{\mu}{r}\frac{dv}{dr}+\mu\frac{d^2v}{dr^2}+\frac{\beta}{r}\left(\frac{dv}{dr}\right)^3+3\beta\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} \tag{6.2}$$
$$=c+P\mu v+M^2v-G_r\theta-B_r\phi,$$

$$\alpha\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+N_b\frac{d\theta}{dr}\frac{d\phi}{dr}+\alpha_1N_t\left(\frac{d\theta}{dr}\right)^2=0, \tag{6.3}$$

$$N_b\left(\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}\right)+N_t\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=0, \tag{6.4}$$

subject to boundary conditions

$$v(1)=1,\ v(2)=\gamma\left[\frac{dv}{dr}(2)\right];\ \theta(1)=1,\ \theta(2)=0;\ \phi(1)=1,\ \phi(2)=0. \tag{6.5}$$

6.2 Solution of the Problem

In this section, we find the series solutions of the nonlinear governing equations using Optimal Homotopic Asymptotic Method for two cases of viscosity namely; constant and variable viscosity.

6.2.1 Case 1: For Constant Viscosity

For constant viscosity model, we choose the linear operator $\mathcal{L}_i$ which is defined by

$$\mathcal{L}_i\left(\varphi_i\left(r,p\right)\right)=\frac{\partial^2\varphi_i\left(r,p\right)}{\partial r^2}, \tag{6.6}$$

such that non linear operators are

$$\mathcal{N}_{14}\left(\varphi_i\left(r,p\right)\right) = \frac{1}{r}\frac{dv}{dr}+\frac{\beta}{r}\left(\frac{dv}{dr}\right)^3+3\beta\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2}+G_r\theta+B_r\phi$$
$$-Pv-M^2v, \tag{6.7}$$

$$\mathcal{N}_{15}[\varphi_i\left(r,p\right)]=\frac{1}{r}\frac{d\theta}{dr}+\frac{N_b}{\alpha}\frac{d\theta}{dr}\frac{d\phi}{dr}+\frac{\alpha_1N_t}{\alpha}\left(\frac{d\theta}{dr}\right)^2, \tag{6.8}$$

$$\mathcal{N}_{16}[\varphi_i\left(r,p\right)]=\frac{1}{r}\frac{d\phi}{dr}+\frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta}{dr}+\frac{d^2\theta}{dr^2}\right). \tag{6.9}$$

The corresponding conditions are

$$\varphi_1(1) = 1, \varphi_1(2) - \gamma\left[\frac{d\varphi_1}{dr}\right]_{r=2} = 0; \varphi_2(1) = 1, \varphi_2(2) = 0; \varphi_3(1) = 1, \varphi_3(2) = 0. \tag{6.10}$$

The *zeroth*−order deformation problems are of the form

$$\frac{d^2 v_0}{dr^2} = c, \qquad v_0(1) = 1, v_0(2) - \gamma\left[\frac{dv_0}{dr}\right]_{r=2} = 0, \tag{6.11}$$

$$\frac{d^2\theta_0}{dr^2} = 0, \qquad \theta_0(1) = 1, \theta_0(2) = 0, \tag{6.12}$$

$$\frac{d^2\phi_0}{dr^2} = 0, \qquad \phi_0(1) = 1, \phi_0(2) = 0. \tag{6.13}$$

First and second order problems are defined as

$$\frac{d^2 v_1}{dr^2} = C_{13}\left\{\begin{array}{c} \frac{1}{r}\frac{dv_0}{dr} + \frac{\beta}{r}\left(\frac{dv_0}{dr}\right)^3 + 3\beta\left(\frac{dv_0}{dr}\right)^2\frac{d^2v_0}{dr^2} + \\ G_r\theta_0 + B_r\phi_0 - Pv_0 - M^2 v_0 \end{array}\right\}, \tag{6.14}$$

$$\frac{d^2\theta_1}{dr^2} = C_{13}\left\{\frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2\right\}, \tag{6.15}$$

$$\frac{d^2\phi_1}{dr^2} = C_{13}\left\{\frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2}\right\}, \tag{6.16}$$

$$\begin{aligned}\frac{d^2 v_2}{dr^2} &= \frac{d^2 v_1}{dr^2} + C_{14}\left\{\begin{array}{c} \frac{1}{r}\frac{dv_0}{dr} + \frac{\beta}{r}\left(\frac{dv_0}{dr}\right)^3 + 3\beta\left(\frac{dv_0}{dr}\right)^2\frac{d^2v_0}{dr^2} \\ +G_r\theta_0 + B_r\phi_0 - Pv_0 - M^2 v_0 \end{array}\right\} \\ &+ C_{13}\left\{\begin{array}{c} \frac{d^2v_1}{dr^2} + \frac{1}{r}\frac{dv_1}{dr} + \frac{3\beta}{r}\left(\frac{dv_0}{dr}\right)^2\frac{dv_1}{dr} + +G_r\theta_1 + B_r\phi_1 \\ 3\beta\left[\left(\frac{dv_0}{dr}\right)^2\frac{d^2v_1}{dr^2} + 2\frac{dv_0}{dr}\frac{dv_1}{dr}\frac{d^2v_0}{dr^2}\right] - \\ Pv_1 - M^2 v_0 \end{array}\right\},\end{aligned} \tag{6.17}$$

$$\begin{aligned}\frac{d^2\theta_2}{dr^2} &= \frac{d^2\theta_1}{dr^2} + C_{14}\left\{\frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2\right\} + \\ &+C_{13}\left\{\frac{d^2\theta_1}{dr^2} + \frac{\alpha}{(\alpha+\alpha_1 N_t)\, r}\frac{d\theta_1}{dr} + \frac{N_b}{(\alpha+\alpha_1 N_t)}\left(\frac{d\theta_1}{dr}\frac{d\phi_0}{dr} + \frac{d\theta_0}{dr}\frac{d\phi_1}{dr}\right)\right\},\end{aligned} \tag{6.18}$$

$$\frac{d^2\phi_2}{dr^2} = \frac{d^2\phi_1}{dr^2} + C_{14}\left\{\frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2}\right\} + C_{13}\left\{\frac{d^2\phi_1}{dr^2} + \frac{1}{r}\frac{d\phi_1}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_1}{dr} + \frac{d^2\theta_1}{dr^2}\right\} \quad (6.19)$$

and so forth.

The solutions of the above deformation problems up to second order are

$$\begin{aligned}\widetilde{v} &= v_0 + v_1 + v_2\\ \widetilde{\theta} &= \theta_0 + \theta_1 + \theta_2 \quad ,\\ \widetilde{\phi} &= \phi_0 + \phi_1 + \phi_2\end{aligned} \quad (6.20)$$

where

$$v_0 = \frac{-4 - 2c + 2r + 3cr - cr^2 + 2\gamma + 3c\gamma - 4cr\gamma + cr^2\gamma}{2(1+\gamma)}, \quad (6.21)$$

$$\theta_0 = 2 - r, \quad (6.22)$$

$$\phi_0 = 2 - r, \quad (6.23)$$

$$v_1 = \frac{1}{24(1+\gamma)}\left[A_{11} + rA_{12} + r^2A_{13} + r^3A_{14} + r^4A_{15}\right], \quad (6.24)$$

$$\theta_1 = \frac{1}{2\alpha}\left[\begin{array}{c} 2C_{13}N_b - 3C_{13}N_br + C_{13}N_br^2 \\ -4C_{13}\alpha\ln 2 + 4C_{13}\alpha r\ln 2 - 2C_{13}\alpha\ln r\end{array}\right], \quad (6.25)$$

$$\phi_1 = \frac{2\left[-2C_{13}N_t\ln 2 + 2C_{13}rN_t\ln 2 - C_{13}N_tr\ln r\right]}{N_b}, \quad (6.26)$$

$$v_2 = \left[A_{21} + rA_{22} + r^2A_{23} + r^3A_{24} + r^4A_{25} + r^5A_{26} + r^6A_{27}\right], \quad (6.27)$$

$$\begin{aligned}\theta_2 = \; & \tfrac{(-1+r)}{6}\left[-6C_{13}\alpha\left\{N_b(r-2) + \alpha\ln 16\right\}\right] - 6C_{14}\alpha\\ & [N_b(r-2) + \alpha\ln 16] + C_{13}^2[N_b^2(6 - 7r + 2r^2)\\ & +3N_b(-2N_t(r-2)\alpha_1 + \alpha(6 - 12\ln 2 + r(\ln 16 - 3))]\\ & +6\alpha[N_t(r - 2 + r\ln 16 - \alpha_1\ln 16) - \alpha(2(\ln 2)^2\\ & +\ln 16 - \ln 4\ln 16)] + \tfrac{6r}{12\alpha}[-2C_{13}\alpha -\\ & 2C_{14}\alpha + C_{13}^2(N_b(r-3) + 2N_t(r - \alpha_1)\\ & +\alpha(\ln 16 - 2))\ln r + 6C_{13}^2r\alpha^2(\ln r)^2],\end{aligned} \quad (6.28)$$

$$
\phi_2 = \begin{array}{c} \frac{N_t}{2N_b^2}[8C_{13}N_b\,(r-1)\,\alpha\ln 2 + 8C_{14}N_b\,(r-1)\,\alpha\ln 2 + \\ C_{13}^2(-4N_t\,(r-1)\,\alpha\ln 2\ln 8 + N_b^2\,(4+2r^2-6\ln 2) - r \\ (-6+\ln 16) - N_b\,(r+1)\,\alpha\left(2\,(\ln r)^2 - \ln 4\ln 16 + \ln 4096\right)] \\ +\frac{r}{2N_b^2}[-4C_{13}N_b\alpha - 4C_{14}N_b\alpha + C_{13}^2(-3N_b^2 + 4N_t\alpha\ln 4 + \\ N_b\alpha\,(\ln 16 - 6))]\ln r - C_{13}^2\,(N_b+2N_t)\,r\alpha\,(\ln r)^2. \end{array} \tag{6.29}
$$

6.2.2 Case 2: For Space Dependent Viscosity

For space dependent viscosity model, defining non-linear operator as

$$
\left.\begin{array}{c} \mathcal{N}_{17}\left(\varphi_i\left(r,p\right)\right) = \frac{2}{r^2}\frac{dv}{dr} + \frac{\beta}{r^2}\left(\frac{dv}{dr}\right)^3 + \frac{3\beta}{r}\left(\frac{dv}{dr}\right)^2\frac{d^2v}{dr^2} \\ +\frac{G_r}{r}\theta + \frac{B_r}{r}\phi - \frac{P}{r}\left[1+\beta\left(\frac{dv}{dr}\right)^2\right]v - \frac{M^2}{r}v \end{array}\right\}, \tag{6.30}
$$

$$
\mathcal{N}_{18}[\varphi_i\,(r,p)] = \frac{1}{r}\frac{d\theta}{dr} + \frac{N_b}{\alpha}\frac{d\theta}{dr}\frac{d\phi}{dr} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{d\theta}{dr}\right)^2, \tag{6.31}
$$

$$
\mathcal{N}_{19}[\varphi_i\,(r,p)] = \frac{1}{r}\frac{d\phi}{dr} + \frac{N_t}{N_b}\left(\frac{1}{r}\frac{d\theta}{dr} + \frac{d^2\theta}{dr^2}\right) \tag{6.32}
$$

along with boundary conditions

$$
\begin{array}{c} \varphi_1(1) = 1, \varphi_1(2) - \gamma\left[\frac{d\varphi_1}{dr}\right]_{r=2} = 0 \\ \varphi_2(1) = 1, \varphi_2(2) = 0 \\ \varphi_3(1) = 1, \varphi_3(2) = 0 \end{array} \tag{6.33}
$$

The *zeroth* -order problem is given by

$$
\frac{d^2v_0}{dr^2} = c, \qquad v_0(1) = 1, v_0(2) = \gamma\left[\frac{dv_0}{dr}\right]_{r=2}, \tag{6.34}
$$

$$
\frac{d^2\theta_0}{dr^2} = 0, \qquad \theta_0(1) = 1, \theta_0(2) = 0, \tag{6.35}
$$

$$
\frac{d^2\phi_0}{dr^2} = 0, \qquad \phi_0(1) = 1, \phi_0(2) = 0. \tag{6.36}
$$

First order and second order problems are given by

$$
\frac{d^2v_1}{dr^2} = C_{15}\left\{\begin{array}{c} \frac{2}{r^2}\frac{dv_0}{dr} + \frac{\beta}{r^2}\left(\frac{dv_0}{dr}\right)^3 + \frac{3\beta}{r}\left(\frac{dv_0}{dr}\right)^2\frac{d^2v_0}{dr^2} \\ +\frac{G_r}{r}\theta_0 + \frac{B_r}{r}\phi_0 - \frac{P}{r}v_0 - \frac{M^2}{r}v_0 \end{array}\right\}, \tag{6.37}
$$

$$\frac{d^2\theta_1}{dr^2} = C_{15}\left\{\frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2\right\}, \tag{6.38}$$

$$\frac{d^2\phi_1}{dr^2} = C_{15}\left\{\frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2}\right\}, \tag{6.39}$$

$$\begin{aligned}\frac{d^2v_2}{dr^2} = \frac{d^2v_1}{dr^2} &+ \frac{C_{15}}{r}\left\{\begin{array}{c}\frac{2}{r}\frac{dv_0}{dr} + \frac{\beta}{r}\left(\frac{dv_0}{dr}\right)^3 + 3\beta\left(\frac{dv_0}{dr}\right)^2\frac{d^2v_0}{dr^2} \\ +G_{rr}\theta_0 + B_r\phi_0 - Pv_0 - M^2v_0\end{array}\right\} \\ &+\frac{C_{15}}{r}\left\{\begin{array}{c} r\frac{d^2v_1}{dr^2} + \frac{2}{r}\frac{dv_1}{dr} + \frac{3\beta}{r}\left(\frac{dv_0}{dr}\right)^2\frac{dv_1}{dr} + G_r\theta_1 + B_r\phi_1 \\ 3\beta\left(\left(\frac{dv_0}{dr}\right)^2\frac{d^2v_1}{dr^2} + 2\frac{dv_0}{dr}\frac{dv_1}{dr}\frac{d^2v_0}{dr^2}\right) - \\ Pv_1 - M^2v_0\end{array}\right\},\end{aligned} \tag{6.40}$$

$$\begin{aligned}\frac{d^2\theta_2}{dr^2} = \frac{d^2\theta_1}{dr^2} &+ C_{15}\left\{\frac{1}{r}\frac{d\theta_0}{dr} + \frac{N_b}{\alpha}\frac{d\theta_0}{dr}\frac{d\phi_0}{dr} + \frac{\alpha_1 N_t}{\alpha}\left(\frac{d\theta_0}{dr}\right)^2\right\} + \\ +C_{15}&\left\{\frac{d^2\theta_1}{dr^2} + \frac{\alpha}{(\alpha+\alpha_1 N_t)\, r}\frac{d\theta_1}{dr} + \frac{N_b}{(\alpha+\alpha_1 N_t)}\left(\frac{d\theta_1}{dr}\frac{d\phi_0}{dr} + \frac{d\theta_0}{dr}\frac{d\phi_1}{dr}\right)\right\},\end{aligned} \tag{6.41}$$

$$\begin{aligned}\frac{d^2\phi_2}{dr^2} = \frac{d^2\phi_1}{dr^2} &+ C_{15}\left\{\frac{1}{r}\frac{d\phi_0}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_0}{dr} + \frac{d^2\theta_0}{dr^2}\right\} \\ &+C_{15}\left\{\frac{d^2\phi_1}{dr^2} + \frac{1}{r}\frac{d\phi_1}{dr} + \frac{N_t}{N_b}\frac{1}{r}\frac{d\theta_1}{dr} + \frac{d^2\theta_1}{dr^2}\right\}.\end{aligned} \tag{6.42}$$

The solutions of the above deformation problems up to second order are

$$\begin{aligned}\widetilde{v} &= v_0 + v_1 + v_2, \\ \widetilde{\theta} &= \theta_0 + \theta_1 + \theta_2 \quad , \\ \widetilde{\phi} &= \phi_0 + \phi_1 + \phi_2\end{aligned} \tag{6.43}$$

where

$$v_0 = \frac{-4 - 2c + 2r + 3cr - cr^2 + 2\gamma + 3c\gamma - 4cr\gamma + cr^2\gamma}{2(1+\gamma)}, \tag{6.44}$$

$$\theta_0 = 2 - r, \tag{6.45}$$

$$\phi_0 = 2 - r, \tag{6.46}$$

$$v_1 = \frac{1}{48(1+\gamma)^4}\left[B_{11} + rB_{12} + r^2B_{13} + r^3B_{14}\right], \tag{6.47}$$

$$\theta_1 = \frac{1}{2\alpha}[2C_{15}N_b - 3C_{15}N_b r + C_{15}N_b r - 4C_{15}\alpha\ln 2 + 4C_{15}r\alpha\ln 2 - 2C_{15}r\alpha\ln r], \tag{6.48}$$

$$\phi_1 = \frac{2[-2C_{15}N_t \ln 2 + 2C_{15}N_t r \ln 2 - C_{15}N_t r \ln r]}{N_b}, \tag{6.49}$$

$$v_2 = \left[\frac{B_{21}}{r} + B_{22} + rB_{23} + r^2 B_{24} + r^3 B_{25} + r^4 B_{26}\right], \tag{6.50}$$

$$\theta_2 = \begin{array}{c} \frac{(-1+r)}{12\alpha^2}((-6C_{15}\alpha\,(N_b\,(-2+r) + \alpha \ln 16) - 6c2\alpha \\ (N_b(-2+r) + \alpha \ln 16) + C_{15}^2[N_b^2\,(6 - 7r + 2r^2) + 3N_b \\ (-2N_t\,(r-2)\,\alpha_1 + \alpha\,(6 - 12\ln 2 + r(\ln 16 - 3))] + 6\alpha \\ [N_t\,(r - 2 + r\ln 16 - \alpha_1 \ln 16) - \alpha\,(\ln 16 - \ln 4 \ln 16)] \\ \frac{6r}{12\alpha}[-2C_{15}\alpha - 2C_{15}\alpha + C_{15}^2(N_b\,(r-3) + 2N_t\,(r - \alpha_1) \\ +\alpha\,(\ln 16 - 2))C_{15}^2\alpha \ln r], \end{array} \tag{6.51}$$

$$\phi_2 = \begin{array}{c} \frac{N_t}{2N_b^2\alpha}[8C_{15}N_b\,(r-1)\,\alpha \ln 2 + 8C_{15}N_b\,(r-1)\,\alpha \ln 2 \\ +C_{15}^2(-12N_b\alpha \ln 2 - 6 - 12N_b\alpha \ln 2^2 + 2N_b^2 \\ (r - 2 + \ln 8) - 12N_t\alpha \ln 2^2) - N_b\,(r+1)\,\alpha \\ \left(2\,(\ln r)^2 - \ln 4 \ln 16 + \ln 4096\right)] + \frac{r \ln r}{2N_b^2} \\ [-4C_{15}N_b\alpha - 4C_{15}N_b\alpha + C_{15}^2(3N_b^2 + 8N_t\alpha \ln 2 \\ +N_b\alpha\,(4\ln 2 + 6))] + C_{15}^2\,(N_b + 2N_t)\,\alpha \ln r. \end{array} \tag{6.52}$$

The coefficients $A_{11} - A_{15}$, $A_{21} - A_{27}$, $B_{11} - B_{15}$ and $B_{21} - B_{27}$ are calculated using MATHEMATICA.

6.3 Graphical Results

The solution is obtained by optimal Homotopic Asymptotic Method. The investigation of the effect of magnetohydrodynamic parameter M, porosity P and slip parameter γ on velocity for both constant and variable viscosity are shown in Figs. 6.1 to 6.6. In Figs. 6.7 to 6.10 the effect of N_b and N_t are shown on nanoparticles concentration and temperature distribution.

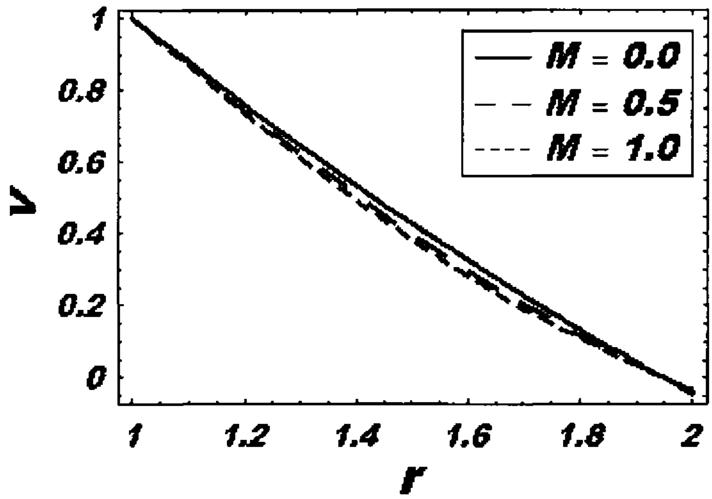

Fig. 6.1 : Effect of M on velocity profile when $N_b = 1$, $N_t = 1$, $p = 0.25$ and $\gamma = 0.05$ for constant viscosity.

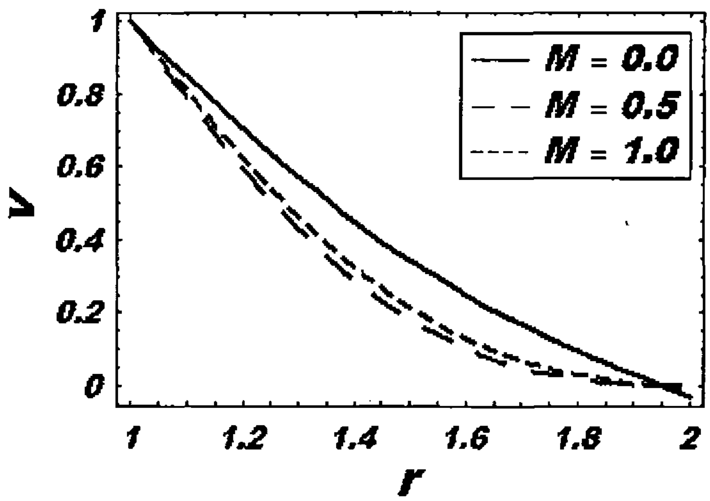

Fig. 6.2 : Effect of M on velocity profile when $N_b = 1, N_t = 1, p = 0.25$ and $\gamma = 0.05$ for variable viscosity.

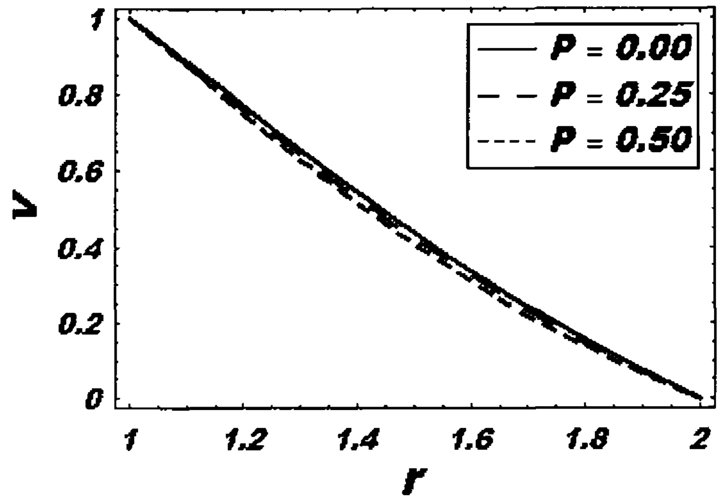

Fig. 6.3 : Effect of P on velocity profile when $N_b = 1$, $N_t = 1$, $M = 0.5$ and $\gamma = 0.05$ for constant viscosity.

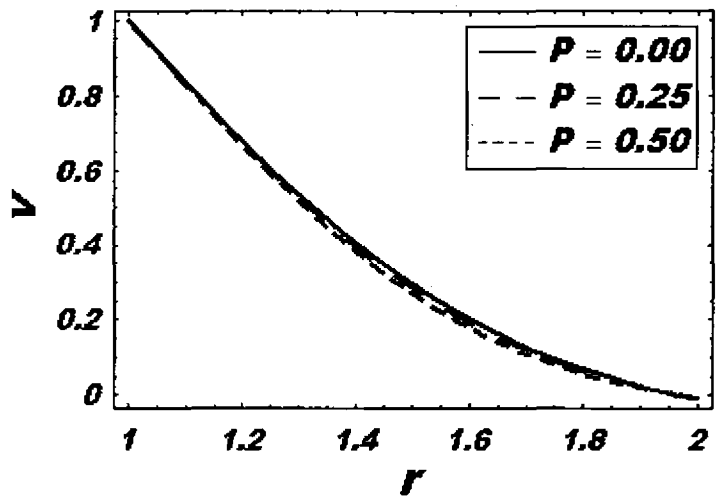

Fig. 6.4 : Effect of P on velocity profile when $N_b = 1$, $N_t = 1$, $M = 0.5$ and $\gamma = 0.05$ for variable viscosity.

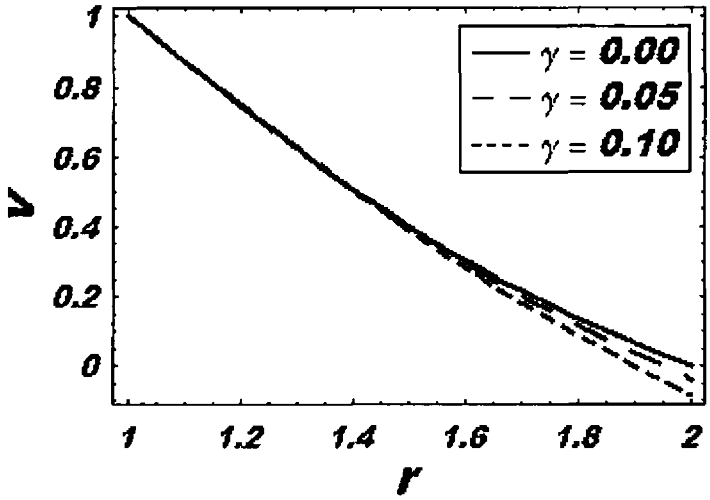

Fig. 6.5 : Effect of γ on velocity profile when $N_b = 1$, $M = 0.5$ and $P = 0.25$ for constant viscosity.

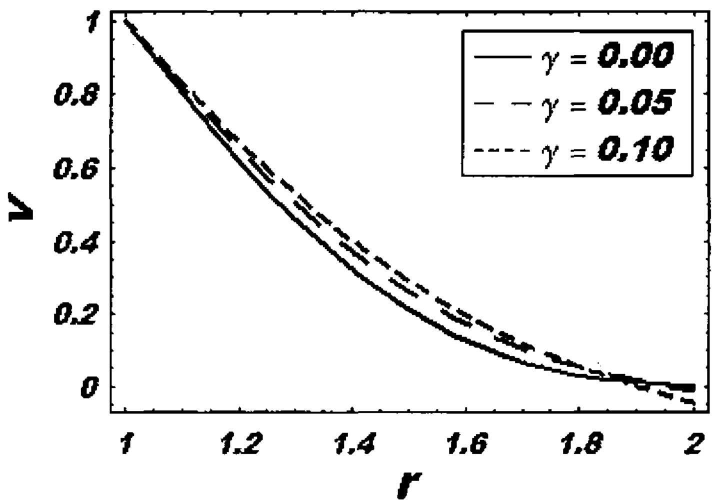

Fig. 6.6 : Effect of γ on velocity profile when $N_b = 1$, $N_t = 1$, $M = 0.5$ and $P = 0.25$ for variable viscosity.

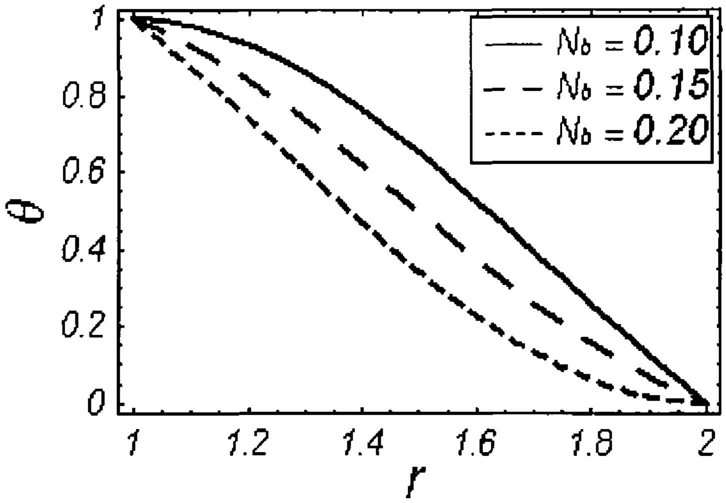

Fig. 6.7 : Effect of N_b on temperature distribution when $N_t = 0.1$.

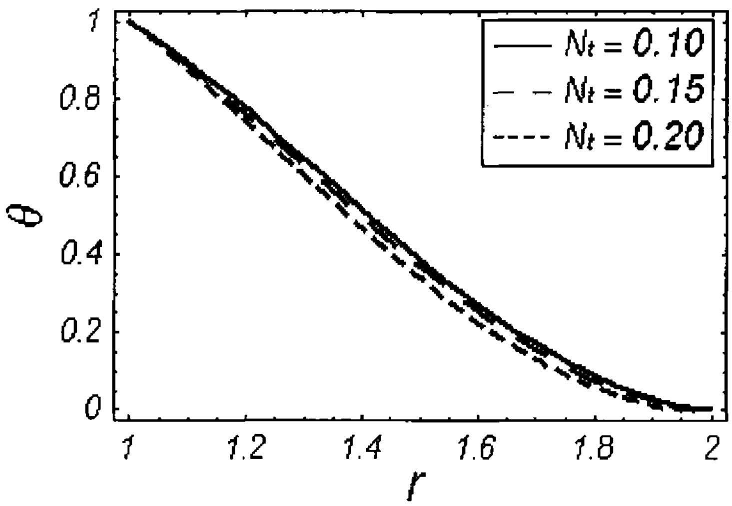

Fig. 6.8 : Effect of N_t on temperature distribution when $N_b = 0.1$.

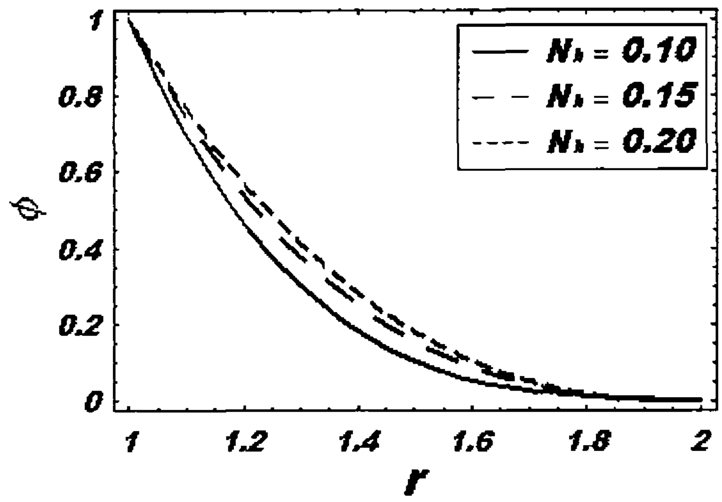

Fig. 6.9 : Effect of N_b on nanoparticles concentration when $N_t = 0.1$.

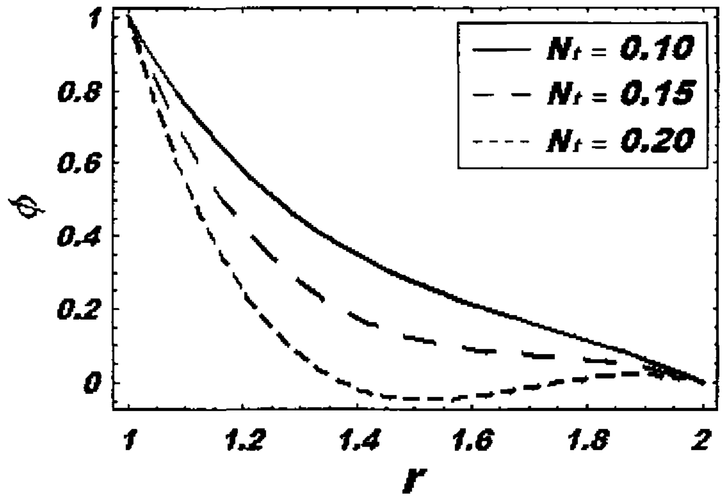

Fig. 6.10 : Effect of N_t on nanoparticles concentration when $N_b = 0.1$.

6.4 Numerical Results

The comparison with the HAM solution from previous chapter with obtained OHAM for $\gamma = 0$ with different values of the emerging parameters is given for different values. Tables 6.1 and 6.2 are constructed for velocity profile for constant and variable viscosity respectively. Tables 6.3 and 6.4 are for heat and nanoparticle concentration.

Table 6.1

M	P	r	$OHAM$	HAM	$Differnce$
1	0.25	1.2	0.74741378	0.74741463	8.5×10^{-7}
		1.4	0.52535753	0.52537756	2.003×10^{-5}
		1.6	0.33577505	0.33563555	1.395×10^{-4}
		1.8	0.16704901	0.16703548	1.353×10^{-5}
0.5	0.5	1.2	0.76462881	0.76462887	6.0×10^{-8}
		1.4	0.55632853	0.55628164	4.689×10^{-5}
		1.6	0.35626568	0.35692155	6.5587×10^{-4}
		1.8	0.16605189	0.16641218	3.6029×10^{-4}

Table 6.2

M	P	r	$OHAM$	HAM	$Differnce$
1	0.25	1.2	0.70250818	0.70251454	6.36×10^{-6}
		1.4	0.44967697	0.44969555	1.858×10^{-5}
		1.6	0.24829228	0.24828963	2.65×10^{-6}
		1.8	0.09394581	0.09394601	2.0×10^{-7}
0.5	0.5	1.2	0.64312867	0.6431581	2.943×10^{-5}
		1.4	0.35587631	0.35585551	2.08×10^{-5}
		1.6	0.15833565	0.15833286	2.79×10^{-6}
		1.8	0.04580312	0.04580024	2.88×10^{-6}

Table 6.3

N_b	N_t	r	$OHAM$	HAM	$Differnce$
0.1	0.1	1.2	0.73932345	0.7393236	1.5×10^{-7}
		1.4	0.46673214	0.46673212	2.0×10^{-8}
		1.6	0.22712271	0.22712301	3.0×10^{-7}
		1.8	0.06013782	0.06013742	4.0×10^{-7}
0.2	0.15	1.2	0.91716062	0.91716091	2.9×10^{-7}
		1.4	0.73405661	0.73405662	1.0×10^{-8}
		1.6	0.49341726	0.49341726	0.0
		1.8	0.23581424	0.23581433	9.0×10^{-8}

Table 6.4

N_b	N_t	r	$OHAM$	HAM	$Differnce$
0.1	0.1	1.2	0.46482512	0.46482537	2.03×10^{-6}
		1.4	0.18078829	0.18078827	1.91×10^{-7}
		1.6	0.05426173	0.05426168	5.07×10^{-6}
		1.8	0.01342030	0.01342030	0.0
0.2	0.15	1.2	0.54874301	0.54870510	5.359×10^{-5}
		1.4	0.26159829	0.26159836	7.0×10^{-7}
		1.6	0.09571721	0.09571718	3.6×10^{-7}
		1.8	0.01769187	0.01769188	1.7×10^{-7}

6.5 Conclusion

The effect of partial slip on MHD flow of a third grade nanofluid in a coaxial porous cylinders has been examined. In order to point out the salient features of the analysis of MHD and heat transfer for nanofluid the following discussions is presented. The graphs showing the behavior of the velocity temperature and nanoparticles concentration are plotted against r. Separate figures have been drawn in order to see the variation of each of the sundry parameter. To see the effects of emerging parameters for constant and variable viscosity Figs. 6.1 to 6.14 have been displayed. In Figs. 6.1 to 6.6, it is found that the velocity decreases with an increase in the values of M, γ and P. The effects of N_b and N_t on nanoparticles concentration and temperature distribution are shown in Figs. 6.7 to 6.14. Figs. 6.7 and 6.8 explain the variation of N_b and N_t on the temperature distribution. Here, it is revealed that the thermal boundary layer thickness increases when large values of N_b have been taken into account and he thermal boundary layer decreases with increasing N_t. Figs. 6.9 and 6.10 bring out the influence of nanoparticles concentration for constant and variable viscosity. It is observed that the nanoparticles concentration increases with the decrease in N_b and decreases by increasing N_t. Tables 6.1 to 6.4 shows that the results obtained HAM without slip effects in chapter six and the results obtained this chapter if we take $\gamma = 0$ are identical.

Chapter 7

Analytic Solutions for MHD Flow in an Annulus

In this chapter, flow of third grade fluid is discussed in a rotating frame though coaxial cylinder. An incompressible and homogeneous MHD third grade fluid between two cylinders rotating with constant but different angular velocities. The inner and outer cylinders have radii r_1 and r_2, respectively. These cylinders rotate with constant angular velocities Ω_1 and Ω_2. The exact solutions are calculated for zero and non-zero MHD.

7.1 Mathematical Formulation of the Problem

Considering the velocity, temperature and nano particle concentration field as

$$\mathbf{V} = [0, 0, v(r)]. \tag{7.1}$$

The mathematical statement of the problem is

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} + \epsilon\left(\frac{dv}{dr} - \frac{v}{r}\right)^2 \left[6\frac{d^2v}{dr^2} - \frac{2}{r}\frac{dv}{dr} + \frac{2v}{r^2}\right] = M^2 v \tag{7.2}$$

subject to the boundary conditions

$$\begin{aligned} v(r) &= 1 \text{ at } r = 1, \\ v(r) &= b \text{ at } r = R, \end{aligned} \tag{7.3}$$

where

$$\overline{v} = \frac{v}{r_1\Omega_1},\ \epsilon = \frac{\beta\Omega_1^2}{\mu},\ b = \frac{\Omega_2 r_2}{\Omega_1 r_1},\ R = \frac{r_2}{r_1}$$

and bars have been suppressed throughout.

7.2 Solution of the Problem.

We obtain analytic solutions with and without magnetic field.

7.2.1 Case-1: Exact Solution with Zero Magnetic Field

We first obtain exact solution when there is no magnetic field, i.e., $M = 0$. In this case we let

$$W = \frac{dv}{dr} - \frac{v}{r}. \tag{7.4}$$

Then Eq. (7.2) with $M = 0$ becomes

$$\frac{dW}{dr}\left(1 + 6\epsilon W^2\right) + \frac{2}{r}W\left(1 + 2\epsilon W^2\right) = 0 \tag{7.5}$$

which has first integral

$$W\left(1 + 2\epsilon W^2\right) = r^{-2}C_{17}, \tag{7.6}$$

where C_{17} is a constant. Thus

$$W = r^{-\frac{2}{3}}\left\{\begin{array}{c}\sqrt[3]{\frac{C_{17}}{4\epsilon} + \frac{1}{4\epsilon}\sqrt{C_{17}^2 + \frac{2}{27\epsilon}r^4}} + \\ \sqrt[3]{\frac{C_{17}}{4\epsilon} - \frac{1}{4\epsilon}\sqrt{C_{17}^2 + \frac{2}{27\epsilon}r^4}}\end{array}\right\} = r^{-\frac{2}{3}}\Gamma\left(C_{17}, \epsilon, r\right). \tag{7.7}$$

Therefore, we have the solution for v

$$v = r\int_1^r s^{-\frac{5}{3}}\Gamma\left(C_{17}, \epsilon, s\right)ds + r, \ v\left(R\right) = b. \tag{7.8}$$

7.2.2 Case-2: Solution with Magnetic Field

We now solve Eq. (7.2) for ϵ and M^2 small, i.e., we let (q_1 is the small parameter)

$$\begin{aligned} \epsilon &= q_1 v, \\ M^2 &= q_1 M^{*2} \end{aligned} \tag{7.9}$$

and assume a perturbation solution of the form

$$v = v_0 + q_1 v_1. \tag{7.10}$$

The substitution of Eq. (7.10) into Eq. (7.2) results in

$$\frac{d^2v_0}{dr^2}+\frac{1}{r}\frac{dv_0}{dr}-\frac{v_0}{r^2}=0, \tag{7.11}$$

$$\frac{d^2v_1}{dr^2}+\frac{1}{r}\frac{dv_1}{dr}-\frac{v_1}{r^2}+v\left(\frac{dv_0}{dr}-\frac{v_0}{r}\right)^2\left[6\frac{d^2v_0}{dr^2}-\frac{2}{r}\frac{dv_0}{dr}+\frac{2v_0}{r^2}\right]=M^{*2}v_0. \tag{7.12}$$

The solution of Eq. (7.11) subject to the boundary condition (7.3) is

$$v_0=\frac{Rb-1}{R^2-1}r+\frac{R(R-b)}{R^2-1}r^{-1}, \tag{7.13}$$

where $R\neq\pm1$. The insertion of Eq. (7.13) into Eq. (7.12) yields

$$\frac{d^2v_1}{dr^2}+\frac{1}{r}\frac{dv_1}{dr}-\frac{v_1}{r^2}=M^{*2}\left(\frac{Rb-1}{R^2-1}r+\frac{R(R-b)}{R^2-1}r^{-1}\right)-64v\frac{R^3(R-b)^3}{(R^2-1)^3}r^{-7}. \tag{7.14}$$

After some calculations the solution of (7.14) subject to the boundary conditions (7.3) is

$$v_1=\left.\begin{array}{l}\left[\frac{C_{20}\left(R^4-R^6+1\right)}{R^4}-C_{18}\left(1+R^2\right)-C_{19}\frac{R^2\ln R}{R^2-1}\right]r\\ +\left(\frac{C_{20}\left(R^6-1\right)}{R^4}+C_{18}R^2+C_{19}\frac{R^2\ln R}{R^2-1}\right)r^{-1}\\ +C_{18}r^3+C_{19}r\ln r-C_{20}r^{-5}\end{array}\right\}, \tag{7.15}$$

where C_{18}, C_{19} and C_{20} are given by

$$\begin{aligned}C_{18}&=\frac{1}{8}\frac{M^{*2}(Rb-1)}{R^2-1},\\ C_{19}&=\frac{1}{2}\frac{M^{*2}R(R-b)}{R^2-1},\\ C_{20}&=\frac{8}{3}v\frac{R^3(R-b)^3}{(R^2-1)^3}.\end{aligned} \tag{7.16}$$

Thus the first order approximate solution of Eq. (7.2) is (7.10) with v_0 and v_1 given by the Eq. (7.13), (7.15) and (7.16).

7.3 Graphical Results

In order to illustrate the influences of M^* and ϵ on the velocity u; we have plotted three figures. Fig. 7.1 for no magnetic field and Fig. 7.2 for magnetic field.

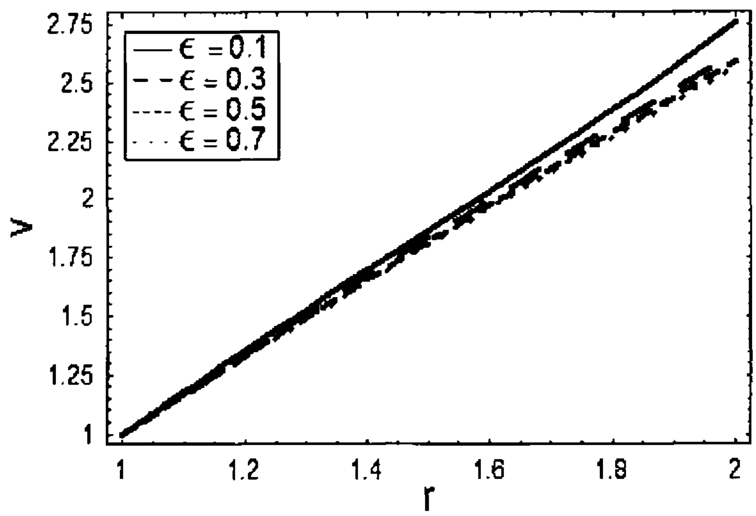

Fig. 7.1 : Profiles of dimensionless velocity u in the absence of magnetic field and with various values of ϵ

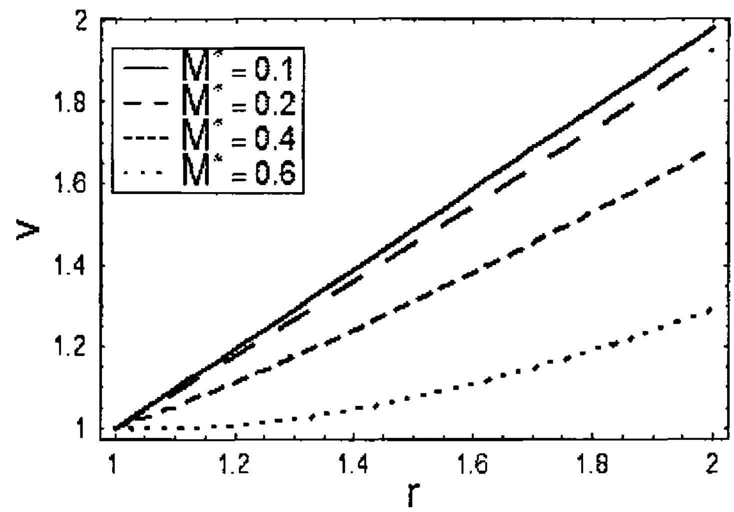

Fig. 7.2 : Velocity profile in the presence of magnetic field with various values of M^*.

7.4 Conclusion

As a result, the following observations are made.

- Our exact solutions are more general with variable boundary conditions. Therefore, one can easily obtain other exact solutions for different parameters with different boundaries.

- It is also worth mentioning that the exact solution in the absence of magnetic field is given for the first time here.

- the authors investigated this problem in the presence of magnetic field and obtained the numerical solutions.

Chapter 8

Exact Solutions for Flows of an Oldroyd 8-constant Fluid with Nonlinear Slip Conditions

The objective of the present chapter is to investigate the three nonlinear flow cases of an Oldroyd 8-constant fluid with slip conditions. Flow is considered between the concentric cylinders. In the first problem, the inner cylinder moves and the outer cylinder remains stationary. Second problem deals with Poiśeuille Flow. Third problem is for Generalized Couette Flow. All the differential systems are subjected to nonlinear differential equations and nonlinear boundary conditions. Exact solutions are developed and computations have been made for the salient features of the involved pertinent parameters.

8.1 Mathematical Formulation of Problem

The Cauchy stress tensor in an Oldroyd 8-constant is given in Eqs. (1.25) – (1.24). For unidirectional steady flow,

$$\mathbf{V} = [u(y), 0, 0]. \tag{8.1}$$

The continuity equation is identically satisfied and the equation of motion and the dimensionless variables, viz.

$$a_1 = \frac{a_1 U_0^2}{h^2}, \quad a_2^* = \frac{a_2 U_0^4}{h^2}.$$

yield

$$\frac{d}{dy}\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]=c, \tag{8.2}$$

the asterisks are omitted for brevity.

$$a_1=\lambda_3\left(\lambda_6+\lambda_9\right)-\left(\lambda_5+\lambda_7\right)\left(\lambda_6+\lambda_9-\lambda_4\right)-\frac{\lambda_7\lambda_9}{2}, \tag{8.3}$$

$$a_2=\lambda_3\left(\lambda_5+\lambda_8\right)-\left(\lambda_5+\lambda_7\right)\left(\lambda_5+\lambda_8-\lambda_5\right)-\frac{\lambda_7\lambda_8}{2}, \tag{8.4}$$

$$\hat{p}=p_1-\frac{1}{N_1}\left\{\begin{array}{c}\mu\left[\left(\lambda_4+\lambda_7\right)-\left(\lambda_3+\lambda_6\right)\right]\left(\frac{du}{dy}\right)^2\\+\mu\left[\left(\lambda_4+\lambda_7\right)a_2-a_1\left(\lambda_3+\lambda_6\right)\right]\left(\frac{du}{dy}\right)^4\end{array}\right\}, \tag{8.5}$$

$$N_1=1+a_2\left(\frac{du}{dy}\right)^2. \tag{8.6}$$

In the next section we find analytic solutions for three flows.

8.2 Solutions of the Problem

The exact solution of the problem is computed using first integral method for the cases of flow problem.

8.2.1 Case 1: Couette Flow

We investigate the steady flow of an Oldroyd 8-constant fluid between two rigid plates h apart. The lower plate at $y=0$ is suddenly moved while the upper plate (at $y=h$) is fixed. No pressure gradient is applied. The resulting mathematical problem is of the form

$$\frac{d}{dy}\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]=0, \tag{8.7}$$

$$u(0)-\gamma\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]_{y=0}=1, \tag{8.8}$$

$$u(1)+\gamma\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]_{y=1}=0, \tag{8.9}$$

where γ^* $(=\gamma/h)$ is the slip parameter, asterisk is suppressed here and dimensionless slip conditions (8.8) and (8.9) are defined in terms of the shear stress. Note that Eqs. (8.8) and (8.9) can be reduced into no-slip conditions iff $\gamma=0$.

A first integral of Eq. (8.7) is

$$\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)=C_{16}, \tag{8.10}$$

where C_{16} is an arbitrary constant. As a consequence of Eq. (8.10), the boundary conditions (8.8) and (8.9) become

$$u(0)=1+\gamma C_{16},\ u(1)=-\gamma C_{16}. \tag{8.11}$$

Since the arbitrary constant C_{16} appears in the boundary conditions (8.11), we can select any value for it.

The general solution of (8.10) is

$$u=\wp y+d, \tag{8.12}$$

where d is another constant and $\wp$ is

$$\wp = \sqrt[3]{\frac{\frac{1}{2}a_1C_{16}\left(a_1-\frac{1}{3}a_2\right)+\frac{1}{3^3}C_{16}a_2^3+\frac{1}{2}}{\sqrt{\frac{4}{3}a_1a_2^3C_{16}+\left(a_1^4-\frac{2}{3}a_1^3a_2-\frac{1}{3^3}a_1^2a_2^2\right)C_{16}+\frac{4}{3}a_1^3}}} + \sqrt[3]{\frac{\frac{1}{2}a_1C_{16}\left(a_1-\frac{1}{3}a_2\right)+\frac{1}{3^3}C_{16}^3a_2^3-\frac{1}{2}}{\sqrt{\frac{4}{3}a_1^2a_2^3C_{16}^4+\left(a_1^4-\frac{2}{3}a_1^3a_2-\frac{1}{3^3}a_1^2a_2^2\right)C_{16}^2+\frac{4}{3}a_1^3}}}.$$

The substitution of boundary conditions (8.11) into Eq. (8.12) yields the exact solution

$$u=\wp y+\gamma C_{16}+1, \tag{8.13}$$

together with the condition or relation

$$-2\gamma C_{16} - 1 = \not{0}. \tag{8.14}$$

8.2.2 Case 2: Poiseuille Flow

In this section an Oldroyd 8-constant fluid is bounded between two fixed plates. The flow is governed by an applied pressure gradient. The resulting mathematical problem is expressed as follows

$$\frac{d}{dy}\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right] = c, \tag{8.15}$$

$$u(0) - \gamma\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]_{y=0} = 0, \tag{8.16}$$

$$u(1) + \gamma\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]_{y=1} = 0. \tag{8.17}$$

A first integral of Eq. (8.15) is

$$\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right) = cy + C_{16}, \tag{8.18}$$

where C_{16} is an arbitrary constant. Hence the boundary conditions (8.16) and (8.17) become

$$u(0) = \gamma C_{16},\ u(1) = -\gamma C_{16} - \gamma c. \tag{8.19}$$

Since C_{16} is in the boundary condition (8.19), we can choose any value for C_{16}.

By means of the transformation

$$\begin{aligned} \bar{y} &= cy + C_{16}, \qquad c \neq 0 \\ \bar{u} &= cu, \end{aligned} \tag{8.20}$$

Eq. (8.18) becomes

$$\frac{1+a_1\left(\frac{d\overline{u}}{d\overline{y}}\right)^2}{1+a_2\left(\frac{d\overline{u}}{d\overline{y}}\right)^2}\left(\frac{d\overline{u}}{d\overline{y}}\right)=\overline{y} \tag{8.21}$$

and the boundary conditions (8.21) transform to

$$\begin{aligned}\overline{u}(d) &= c\gamma C_{16}, \\ \overline{u}(c+C_{16}) &= -c\gamma C_{16}-\gamma c^2.\end{aligned} \tag{8.22}$$

One can solve for $d\overline{u}/d\overline{y}$ from (8.21) to obtain

$$\begin{aligned}\frac{d\overline{u}}{d\overline{y}} &= \sqrt[3]{\begin{array}{c}\frac{a_1}{2}\overline{y}\left(a_1-\frac{1}{3}a_2\right)+\frac{1}{3^3}a_2^3\overline{y}^3+ \\ \frac{1}{2}\sqrt{\frac{4}{3}a_1^2a_2^3\overline{y}^4+\left(a_1^4-\frac{2}{3}a_1^3a_2-\frac{1}{3^3}a_1^2a_2^2\right)\overline{y}^2+\frac{4}{3}a_1^3}\end{array}} \\ &\quad + \sqrt[3]{\begin{array}{c}\frac{a_1}{2}\overline{y}\left(a_1-\frac{1}{3}a_2\right)+\frac{1}{3^3}a_2^3\overline{y}^3- \\ \frac{1}{2}\sqrt{\frac{4}{3}a_1^2a_2^3\overline{y}^4+\left(a_1^4-\frac{2}{3}a_1^3a_2-\frac{1}{3^3}a_1^2a_2^2\right)\overline{y}^2+\frac{4}{3}a_1^3}\end{array}} \\ &\cong \Lambda_1\left(a_1,a_2,\overline{y}\right).\end{aligned} \tag{8.23}$$

If the relation

$$\frac{4^3}{3^6}\alpha_1^5a_2^3=\left(a_1^4-\frac{2}{3}a_1^3a_2-\frac{1}{3^3}a_1^2a_2^2\right)^2 \tag{8.24}$$

holds, then one can obtain a simplification for $\Lambda_1\left(a_1,a_2,\overline{y}\right)$ given by

$$\begin{aligned}\Lambda_1\left(a_1,a_2,\overline{y}\right) &= \sqrt[3]{\frac{a_1}{2}\overline{y}\left(a_1-\frac{1}{3}a_2\right)+\frac{1}{3^3}a_2^3\overline{y}^3+3^{-\frac{3}{2}}\left(a_1a_2^{\frac{3}{2}}\overline{y}+a_1^{\frac{3}{2}}\right)} \\ &\quad +\sqrt[3]{\frac{a_1}{2}\overline{y}\left(a_1-\frac{1}{3}\alpha_2\right)+\frac{1}{3^3}a_2^3\overline{y}^3-3^{-\frac{3}{2}}\left(a_1a_2^{\frac{3}{2}}\overline{y}+a_1^{\frac{3}{2}}\right)}.\end{aligned}$$

The exact solution of Eq. (8.23) subject to the boundary conditions (8.22) is given by

$$\overline{u}\left(a_1,\alpha_2,\overline{y}\right)=\int_{C_{16}}^{\overline{y}}\Lambda_1\left(a_1,a_2,z\right)dz+cC_{16}\gamma \tag{8.25}$$

provided that the following relation holds

$$\int_{C_{16}}^{c+C_{16}}\Lambda_1\left(a_1,a_2,\overline{y}\right)d\overline{y}=-2c\gamma C_{16}-\gamma c^2. \tag{8.26}$$

8.2.3 Case 3: Generalized Couette Flow

Here the physical model is similar to that of the Couette flow. Additionally a constant pressure gradient is applied. The problem statement is

$$\frac{d}{dy}\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]=c, \tag{8.27}$$

$$u(0)-\gamma\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]_{y=0}=1, \tag{8.28}$$

$$u(1)+\gamma\left[\frac{1+a_1\left(\frac{du}{dy}\right)^2}{1+a_2\left(\frac{du}{dy}\right)^2}\left(\frac{du}{dy}\right)\right]_{y=1}=0. \tag{8.29}$$

As Eq. (8.27) is the same as Eq. (8.15), a first integral of Eq. (8.27) is (8.18) and the boundary conditions as a consequence become

$$u(0)=\gamma C_{16}+1,\ u(1)=-\gamma C_{16}-\gamma c. \tag{8.30}$$

The boundary conditions (8.30) transform to

$$\begin{aligned}\overline{u}(C_{16}) &= c\gamma C_{16}+c, \\ \overline{u}(c+C_{16}) &= -c\gamma C_{16}-\gamma c^2.\end{aligned} \tag{8.31}$$

Here too C_{16} is in the boundary conditions (8.30) or (8.31) and we can choose it to be any value. The transformation (8.20) reduces the Eq. (8.27) to Eq. (8.23). Eq. (8.23) now needs to be solved subject to the conditions (8.31) with the relation given below

$$\int_{C_{16}}^{c+C_{16}}\Lambda_1\left(a_1,a_2,\overline{y}\right)d\overline{y}=-2c\gamma C_{16}-\gamma c^2-c. \tag{8.32}$$

We obtain the exact solution

$$\overline{u}\left(a_1,\alpha_2,\overline{y}\right)=\int_{C_{16}}^{\overline{y}}\Lambda_1\left(a_1,a_2,z\right)dz+c\gamma C_{16}+c. \tag{8.33}$$

Here too Λ can be simplified as before if the relation (8.24) holds. Note that we cannot set $c = 0$ in (8.32) to obtain Eq. (8.7) which is the Couette flow as the transformation (8.20) breaks down when $c = 0$. Thus the solutions for the Couette and Generalized Couette flows are distinct.

8.3 Graphical Results

In order to illustrate the influences of non-Newtonian parameters a_1, a_2, and slip parameter γ on the velocity u, we have plotted eleven figures. Figs. 8.1–8.3 for the Couette flow, Figs. 8.4 to 8.6 for the Poiseuille Flow and Figs. 8.7 to 8.10 for the Generalized Couette Flow.

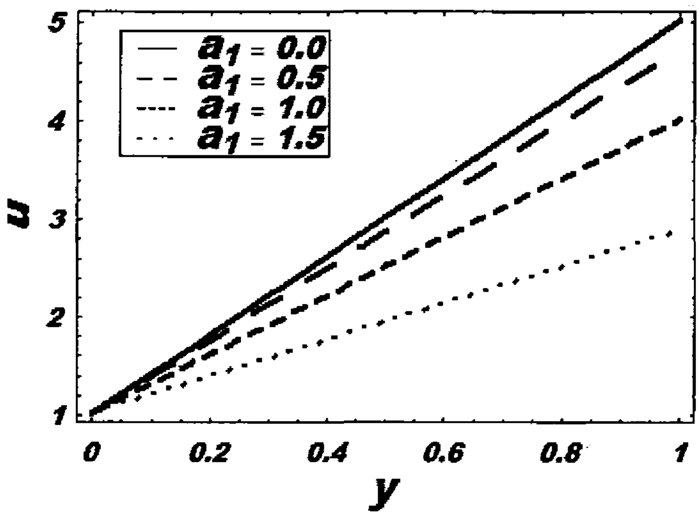

Fig. 8.1 : Velocity profile $u(y)$ for Couette flow with various values of the non-Newtonian parameter a_1 when a_2 and γ are fixed.

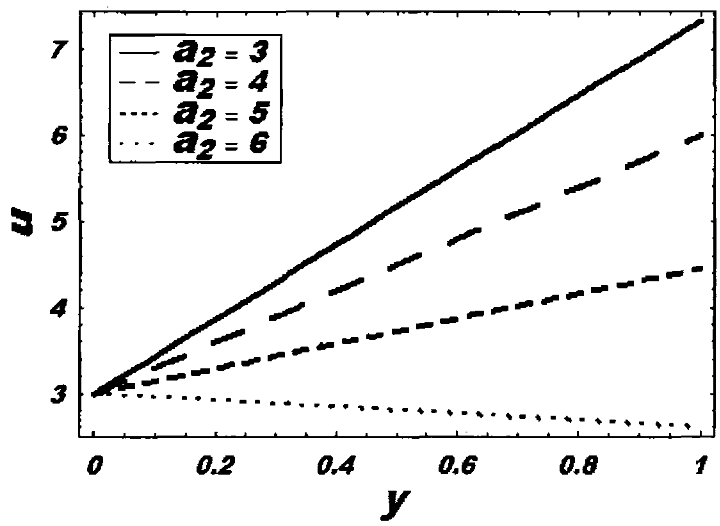

Fig. 8.2 : Velocity profile $u(y)$ for Couette flow with various values of the non-Newtonian parameter a_2 when a_1 and γ are fixed.

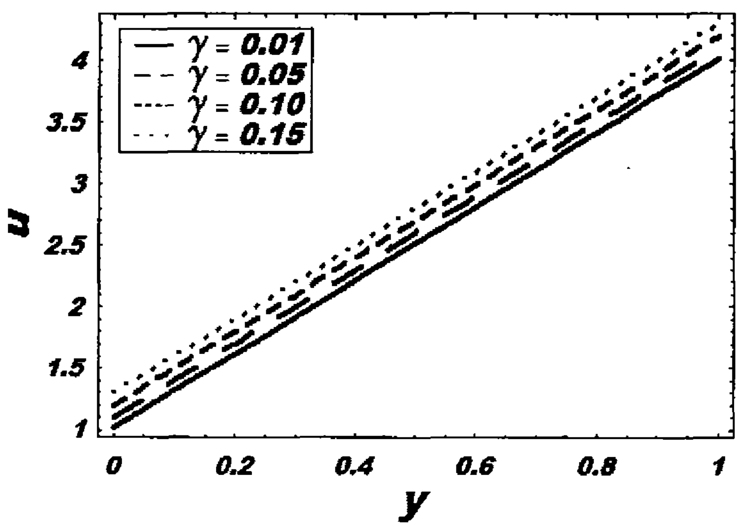

Fig. 8.3 : Velocity profile $u(y)$ for Couette flow with various values of the non-Newtonian parameter γ when a_1 and a_2 are fixed.

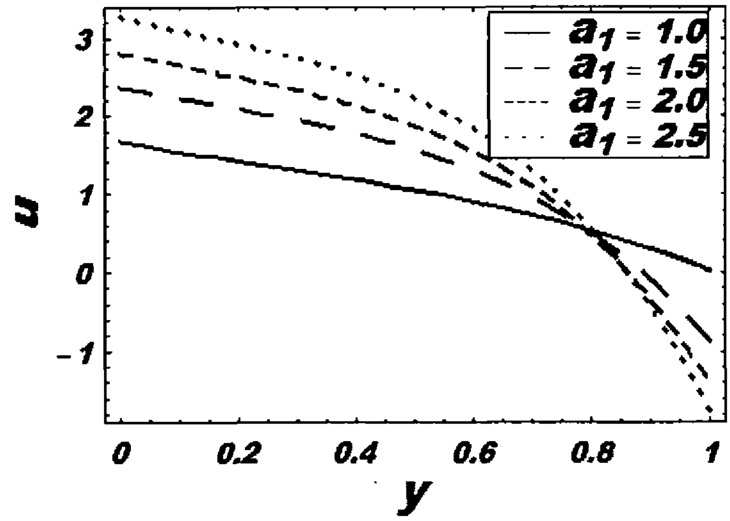

Fig. 8.4 : Velocity profile $u(y)$ for Poiseuille flow with various values of the non-Newtonian parameter a_1 when γ and a_2 are fixed.

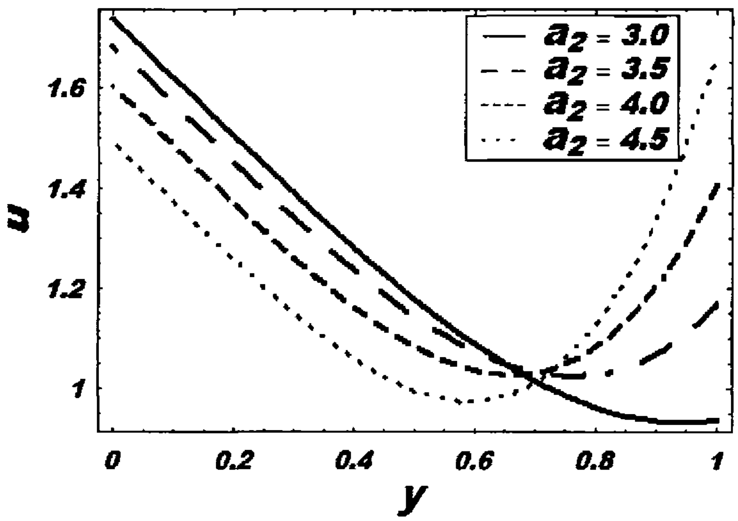

Fig. 8.5 : Velocity profile $u(y)$ for Poiseuille flow with various values of the non-Newtonian parameter a_2 when γ and a_1 are fixed.

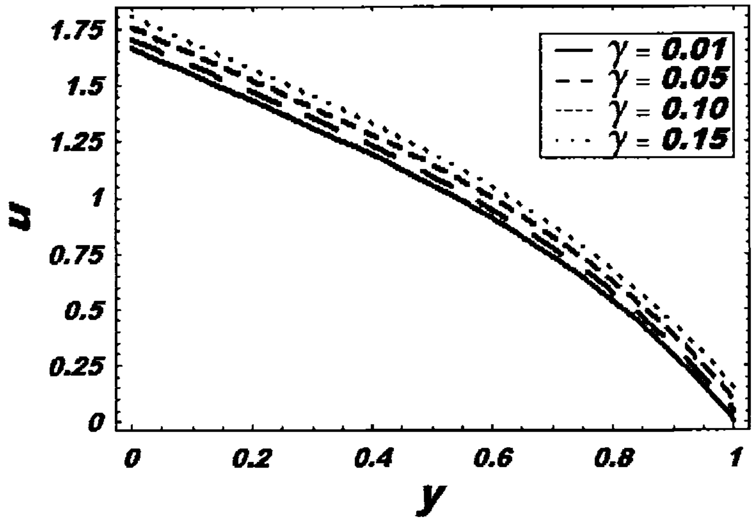

Fig. 8.6 : Velocity profile $u(y)$ for Poiseuille flow with various values of the non-Newtonian parameter γ when a_1 and a_2 are fixed.

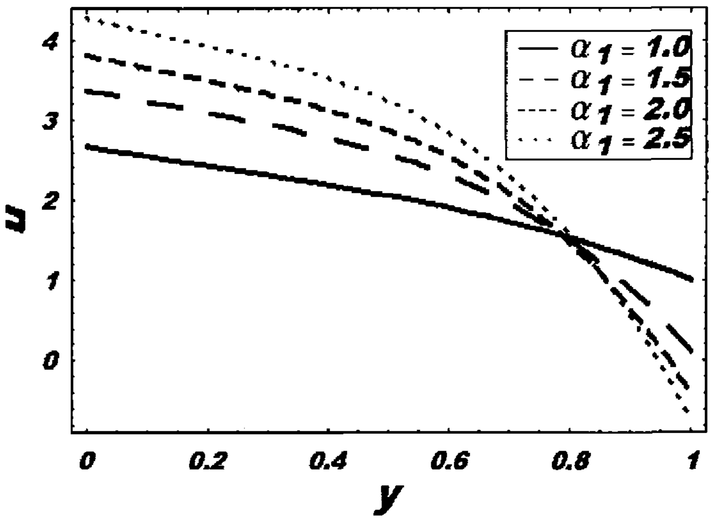

Fig. 8.7 : Velocity profile $u(y)$ for Generalized Couette Flow with various values of the non-Newtonian parameter a_1 when γ and a_2 are fixed.

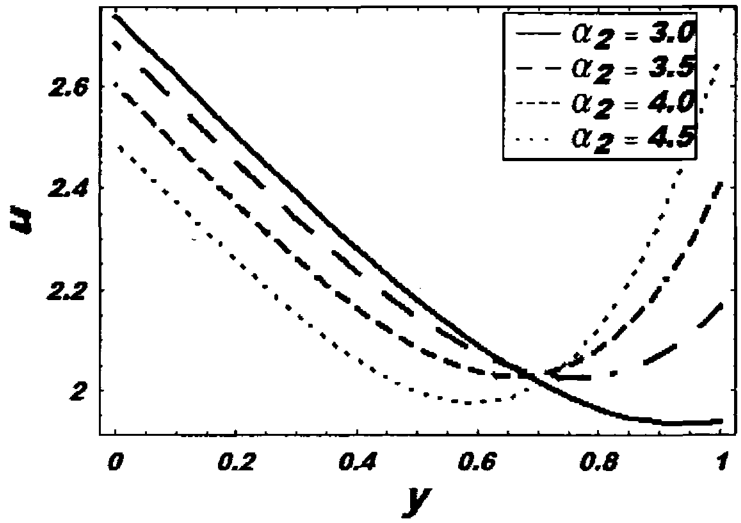

Fig. 8.8 : Velocity profile $u(y)$ for Generalized Couette Flow with various values of the non-Newtonian parameter a_2 when γ and a_1 are fixed.

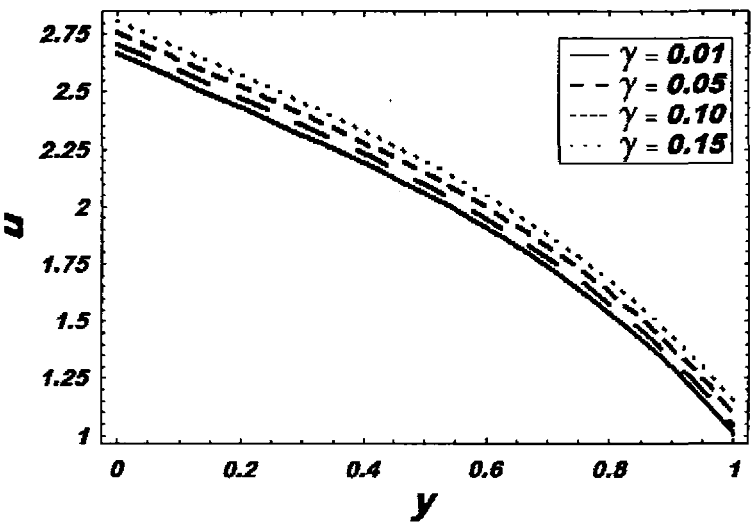

Fig. 8.9 : Velocity profile $u(y)$ for Generalized Couette Flow with various values of the non-Newtonian parameter γ when a_1 and a_2 are fixed.

8.4 Conclusion

In this work the velocity profiles in an Oldroyd 8-constant fluid are found analytically. Three interesting cases of Couette flow, Poiseuille flow and Generalized Couette flow subject to nonlinear slip boundaries conditions are discussed. The first integral approach is used to find the velocity profiles.

- It is noted that the solution of the first problem is linear whereas in the other two cases the solutions are nonlinear. Since all solutions are independent, one cannot obtain the solution of the Poiseuille flow from the Generalized Couette flow by setting $c = 0$ as the transformation that reduce the problem breaks down.

- As expected from the boundary conditions for $c > 0$, the velocity profile u for the generalized Couette flow are greater than to Couette flow. The curvature of the velocity profile will depend on the amplitude of the parameters

- It is also worth mentioning that our exact solutions are more general with variable boundary conditions. Therefore, one can easily obtain another exact solutions for different parameters with different boundaries. Moreover, our exact analytical solutions are not only valid for small but also for large values of all emerging parameters.

Chapter 9

Solution of Second Painlevé Equation by Homotopy Analysis Method

In this chapter, we presents the series solution of second type of Painlevé equation by using the HAM. Comparison of the present solution is given with the existing solutions by other methods for instant ADM, HPM and Legendre Tau Method. Numerical and Graphical results are presented.

9.1 Mathematical Formulation of the Problem

There are fifty canonical forms of Painlevé equations. Six of them define the Painlevé transcendent [87]. These equations require the introduction of new functions to solve them. Taking in account Dehan and Shakeri [86] solved the following problem with the help of HAM

$$u'' = 2u^3 + xu + \varrho, \tag{9.1}$$

with the initial conditions

$$u(0) = 1, \quad u'(0) = 1. \tag{9.2}$$

in which $\varrho = (-8k_1)^{1/2} - 1$ and k_1 is a constant of integration.

9.2 Solution of the Problem

Here we tend to solve it with Homotopic Analysis Method (HAM) and compare them with exiting solution of Dehan and Shakeri [86].

Zeroth-order Deformation Equation

The function $u(x)$ can be expressed by the set of base functions

$$\{x^i | i \geq 0\} \tag{9.3}$$

in the form

$$u(x) = \sum_{k=0}^{\infty} a_{m,i} x^i, \tag{9.4}$$

where $a_{m,i}$ are the coefficients. By considering the rule of solution expressions for $u(x)$ and Eqs. (9.1) and (9.2) one can choose

$$u_0(x) = \varrho \frac{x^2}{2} + 1 \tag{9.5}$$

as the initial approximation of $u(x)$. Using auxiliary linear operator as in Eq. (2.17)

$$\mathcal{L}_1 [C_1 x + C_2] = 0, \tag{9.6}$$

where C_1 and C_2 are arbitrary constants.

Eq. (9.1) suggests that the nonlinear operator is of the form

$$\mathcal{N}_2[\widehat{v}(x,q)] = \frac{\partial^2 \widehat{u}(x,q)}{\partial^2 x} - 2\left(\frac{\partial \widehat{u}(x,q)}{\partial x}\right)^3 - x\widehat{u}(x,q) - \varrho \tag{9.7}$$

The zeroth order deformation problem can be constructed by taking a non-zero auxiliary parameter $\hbar$

$$(1-q)\mathcal{L}_1[\widehat{u}(x,q) - u_0(x)] = q\hbar\mathcal{N}_2[\widehat{u}(x,q)], \tag{9.8}$$

$$\widehat{u}(0,q) = 1, \qquad \widehat{u}'(1,q) = 0 \tag{9.9}$$

where $q \in [0,1]$ is the embedding parameter. For $q = 0$ and $q = 1$, one respectively has

$$\widehat{u}(x,0) = u_0(x), \qquad \widehat{u}(x,1) = u(x). \tag{9.10}$$

When q increases from 0 to 1, $\widehat{u}(x,q)$ varies continuously from initial guess $u_0(x)$ to the final solution $u(x)$. By Taylor's theorem and Eq. (9.10) one can write

$$\widehat{u}(x,q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)\ q^m, \tag{9.11}$$

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \widehat{u}(x,q)}{\partial q^m}\bigg|_{q=0} \tag{9.12}$$

and convergence of series (9.12) depends upon $\hbar$. Assume that $\hbar$ is selected such that the series (9.12) is convergent at $q = 1$, then due to Eq. (9.11) one get

$$u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x). \tag{9.13}$$

mth-order Deformation Equation

Differentiating m times the zeroth order deformation Eq. (9.8) with respect to q. Dividing by $m!$ and finally setting $q = 0$ the following mth-order deformation problem can be obtained

$$\mathcal{L}_1[u_m(x) - \chi_m u_{m-1}(x)] = \hbar \mathcal{R}_{2m}(x), \tag{9.14}$$

$$u_m(0) = u'_m(1) = 0, \tag{9.15}$$

$$\mathcal{R}_{1m}(x) = u''_{m-1}(x) + 2\sum_{k=0}^{m-1} v'_{m-1-k}(x) \sum_{l=0}^{k} v'_{k-l}(x) v'_l(x) - x v_{m-1}(x) - \varrho(1-\chi_m). \tag{9.16}$$

This is an easy way to solve linear Eqs. (9.14) subject to conditions (9.15) in the order $m = 1, 2, 3, ...$ with the help of symbolic computation software MATHEMATICA.

9.3 Convergence of the Solution.

In this section we will discuss the convergence of our solution. The explicit, analytic expression given in Eq. 9.4 contains the auxiliary parameter $\hbar$. In Fig. 9.1 and 9.2 the $\hbar-$curves have been shown for the parameter $\varrho = 1$ and $\varrho = 2$. These figures depict the convergence region and rate of approximation for the homotopy analysis method. For this purpose $\hbar-$curves are sketched for 20th order of approximation to the corresponding solution of the problem for different values of ϱ. It is apparent from Fig. 9.1 that the range for the admissible values for $\hbar$ is $-1.8 \leq \hbar \leq -0.1$. Also the error of norm 2 with HAM by 20th-order approximation is calculated

$$\sqrt{\frac{1}{21}\sum_{i=0}^{20}(u(x_i))^2}, \tag{9.17}$$

where $u(x_i)$ are plotted in Fig. 9.3 for $\varrho = 1$ and in Fig. 9.5 for $\varrho = 2$. From Fig. 9.3 and 9.4 it can be seen that for $\varrho = 1$ error is minimum at $\hbar = -1.49$ and for $\varrho = 2$ error is minimum at $\hbar = -1.54$, also these values of $\hbar$ lies in admissible range of $\hbar$.

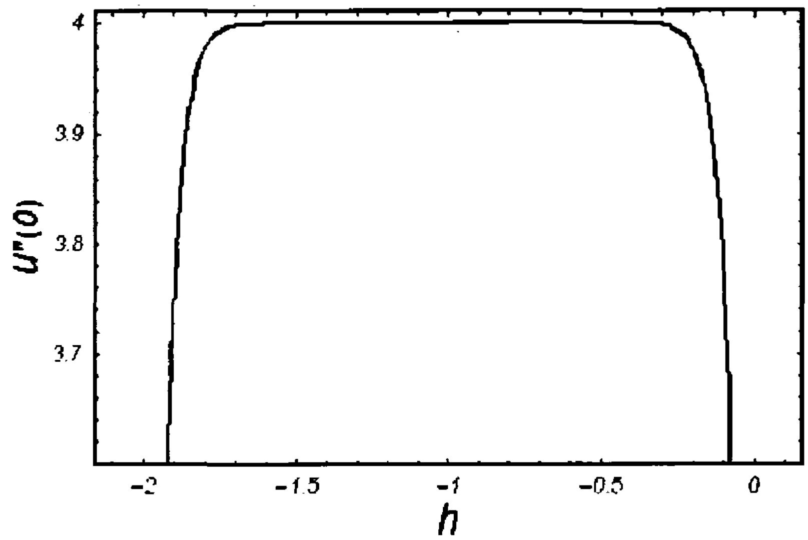

Fig. 9.1 : $\hbar$−curve for 20th order approximation at $\varrho = 1$.

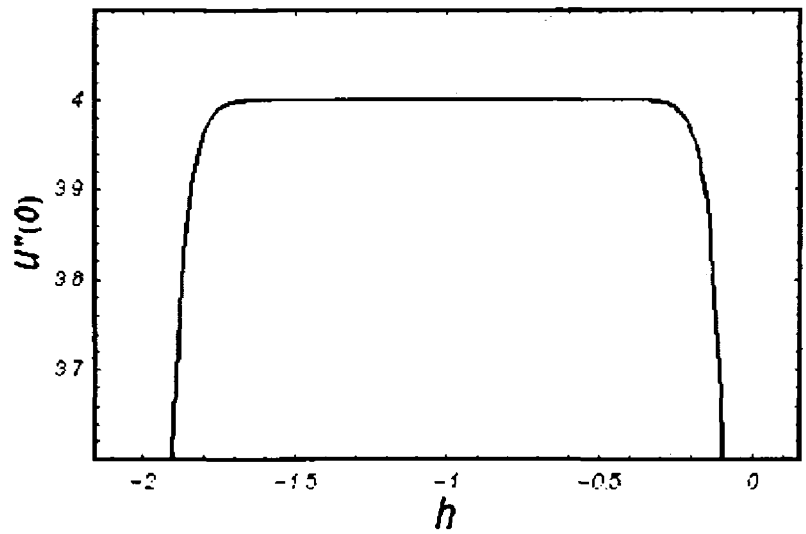

Fig. 9.2: $\hbar$−curve for 20th order approximation at $\varrho = 2$.

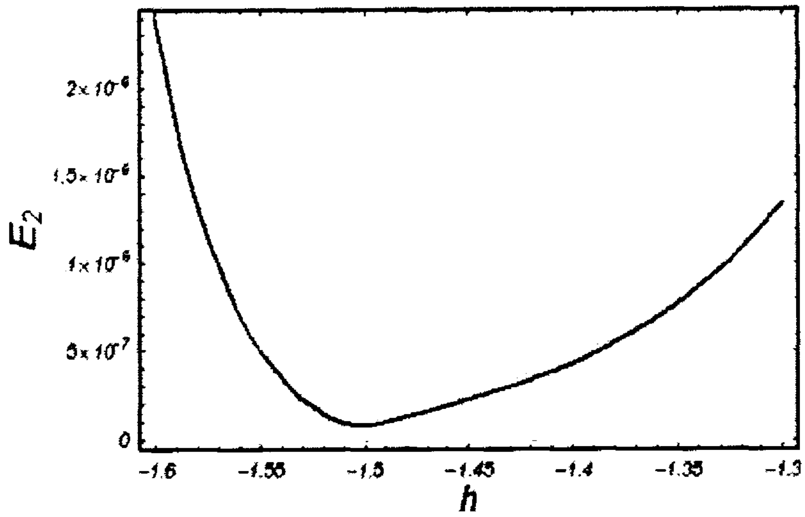

Fig. 9.3 : Error of norm 2 for the 20th-order approximation by HAM or $u(x)$ per $\hbar$.

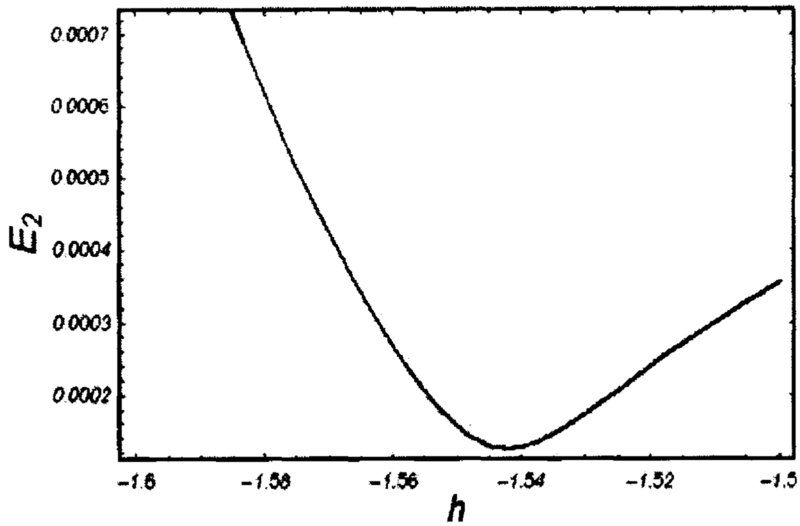

Fig. 9.4 : Error of norm 2 for the $20th$-order approximation by HAM for $u(x)$ per $\hbar$.

9.4 Numerical Results

HAM solution is compared with other analytic method used in [86]. In Table 9.1 and 9.3 comparison of the values of $u(x)$ by different methods and HAM at $\varrho = 1$ and 2. Tables 9.2 and 9.4 comparison of the values of $u'(x)$ by different methods and HAM at $\varrho = 1$ and 2 is given. Tables 9.1 – 9.4 show that the HAM solution presents a better approximation as compared with ADM solution, HPM solution and Legendre tau solution. Tables 9.5 – 9.7 give comparison of the error of $u(x)$ by different methods at $\varrho = 1$ and 2. Tables 9.6 – 9.8 Compare the values of $u'(x)$ by different methods and HAM at $\varrho = 1$ and 2.

Table 9.1

x	analytic continuation	ADM and HPM	Legendre Tau method	HAM solution
0.05	1.0038	1.00377556945843	1.0376658057109	1.00378
0.10	1.0152	1.01524353738588	1.01525769105649	1.01524
0.15	1.0347	1.03470887678813	1.03470091241366	1.03471
0.20	1.0626	1.06261465111813	1.06259867821323	1.06261
0.25	1.0996	1.09956760325147	1.09959144661382	1.09957
0.30	1.1464	1.14637603460243	1.14638760701734	1.14638
0.35	1.2041	1.20410448048055	1.20407307763244	1.20410
0.40	1.2742	1.27415228539083	1.27413995233202	1.27415
0.45	1.3584	1.35836736627797	1.35840243876556	1.35837
0.50	1.4592	1.45921344816914	1.45923019232761	1.45921
0.55	1.5800	1.58002119375708	1.57998509852588	1.58002
0.60	1.7254	1.72537554656527	1.72535519560916	1.72538
0.65	1.9017	1.90173288879669	1.90176944030828	1.90173
0.70	2.1184	2.11844346203985	2.118446103714198	2.11844
0.75	2.3895	2.38952666502192	2.38948818144855	2.38953
0.80	2.737	2.73693554820900	2.73693515523338	2.73694
0.85	3.197	3.19700338869069	3.19703536410817	3.19701
0.90	3.834	3.83438275510100	3.83437315572520	3.8344
0.95	4.776	4.77593656643791	4.77624693047256	4.77623

Table 9.2

x	analytic continuation	ADM and HPM.	Legendre Tau method	HAM
0.05	0.1516	0.15163005612500	0.151522147109079	0.15163
0.10	0.3081	0.30809994093753	0.30839616285499	0.3081
0.15	0.4720	0.47198211410548	0.47120076741792	0.471982
0.20	0.6463	0.6625891667356	0.64686068663295	0.646259
0.25	0.8345	0.83453560799690	0.83505827505242	0.834536
0.30	1.0413	1.04132442194996	1.04041642453365	1.04132
0.35	1.2724	1.27244080661556	1.27203305549665	1.27244
0.40	1.5256	1.53557674523997	1.53660136211238	1.53558
0.45	1.8412	1.84115754906510	1.84162918219226	1.84116
0.50	2.2037	2.20366299100167	2.20258598447253	2.20366
0.55	2.6440	2.64373198583305	2.64313353371108	2.64373
0.60	3.1920	3.191637681455640	3.19276051287306	3.19164
0.65	3.8930	3.89326306291223	3.89390846667811	3.89327
0.70	4.8210	4.82087325717188	4.81961487256922	4.82088
0.75	6.0940	6.09365539825298	6.093280837663830	6.09366
0.80	7.9190	7.91963993781202	7.92110979463905	7.91965
0.85	10.700	10.68878979691819	10.68810973968357	10.6889
0.90	15.200	15.20291903194554	15.20358105190098	15.2039
0.95	23.300	23.32488089779040	23.34147297218442	23.341

Table 9.3

x	Analytic continuation	ADM and HPM.	Legendre Tau method	HAM solution
0.05	1.0050	1.00502714606259	1.00452901168885	1.00503
0.10	1.0203	1.02026919477424	1.02104733101283	1.02027
0.15	1.0461	1.04609205682859	1.04568805007883	1.04609
0.20	1.0830	1.08304976093849	1.08211622538865	1.08305
0.25	1.1319	1.13192491551257	1.13321491002347	1.13192
0.30	1.1938	1.19379024536885	1.19453158450896	1.19379
0.35	1.2701	1.27009977531541	1.26838341456581	1.2701
0.40	1.3628	1.36282365175205	1.36199227404374	1.36282
0.45	1.4746	1.47464972088494	1.41656054759296	1.47465
0.50	1.6093	1.60929103954460	1.61041647339453	1.60929
0.55	1.7720	1.77196801040459	1.77002375632266	1.77197
0.60	1.9702	1.97019074865113	1.96881771762475	1.97019
0.65	2.2151	2.21508321179155	2.21705133168309	2.21508
0.70	2.5237	2.52374246288910	2.52502186102312	2.52374
0.75	2.9237	2.92371780534123	2.92156709341536	2.92372
0.80	3.4622	3.46222343428592	3.46189169209309	3.46223
0.85	4.2270	4.22716380111867	4.22917124265622	4.22718
0.90	5.4020	5.40230384073914	5.40095659634246	5.40256
0.95	6.9250	7.44220956078449	7.44903796388528	7.4479

Table 9.4

x	Analytic continuation.	ADM and HPM.	Legendre Tau method	HAM solution
0.05	0.2018	0.2017565935233	0.19506737519209	0.201756
0.10	0.4091	0.40913700000763	0.4266978440918	0.409136
0.15	0.6256	0.62561099607917	0.58180200663215	0.625611
0.20	0.8553	0.85528574376796	0.88690901421489	0.855286
0.25	1.1033	1.10326643289016	1.13471732639540	1.10327
0.30	1.3761	1.137614962029666	1.32713207266902	1.37615
0.35	1.6827	1.68273330750550	1.65624172867177	1.68273
0.40	2.0351	2.03508180897498	2.09055954859542	2.03508
0.45	2.4502	2.45018266771106	2.48155989614621	2.45018
0.50	2.9526	2.95262120064461	2.89480407392915	2.95262
0.55	3.5790	3.57907120593636	3.53916714774305	3.57907
0.60	4.3860	4.38617741400896	4.4459235753419	4.38618
0.65	5.4650	5.46512495162840	5.5097378842758	5.46514
0.70	6.9700	6.97028178853458	6.90225094640852	6.9703
0.75	9.1800	9.17992302915627	9.14737093027648	9.17995
0.80	12.640	12.637802335521595	12.72413192501885	12.6379
0.85	18.50	18.52647814752540	18.49529693564642	18.5272
0.90	29.80	29.82591522501663	29.80537398992756	29.8414
0.95	56.10	55.74810648174442	56.15249937871179	56.1052

Table 9.5

x	Analytic continuation & ADM and HPM	Analytic continuation & Legendre Tau method	Analytic continuation & HAM at $\hbar = -1.49$
0.05	0.033737603	0.033737603	$1.99243E-05$
0.10	$4.28855E-05$	$5.68273E-05$	$3.94011E-05$
0.15	$8.57909E-06$	$8.81815E-07$	$9.66464E-06$
0.20	$1.3788E-05$	$1.24392E-06$	$9.41088E-06$
0.25	$2.94623E-05$	$7.77863E-06$	$9.41088E-06$
0.30	$2.09049E-05$	$1.08103E-05$	$1.74459E-05$
0.35	$3.72102E-06$	$2.23589E-05$	0.000000
0.40	$3.74467E-05$	$4.71258E-05$	$3.92403E-05$
0.45	$2.40236E-05$	$1.79532E-06$	$2.20848E-05$
0.50	$9.21612E-06$	$2.0691E-05$	$6.85307E-06$
0.55	$1.34138E-05$	$9.43131E-06$	$1.26582E-05$
0.60	$1.41726E-05$	$2.59675E-05$	$1.15915E-05$
0.65	$1.72944E-05$	$3.65149E-05$	$1.57754E-05$
0.70	$2.05164E-05$	$2.17635E-05$	$1.88822E-05$
0.75	$1.11592E-05$	$4.94604E-06$	$1.25549E-05$
0.80	$2.35483E-05$	$2.36919E-05$	$2.19218E-05$
0.85	$2.35483E-05$	$1.10617E-05$	$3.12793E-06$
0.90	$9.98318E-05$	$9.7328E-05$	0.00010433
0.95	$1.32817E-05$	$5.17024E-05$	$4.81575E-05$

Table 9.6

x	Analytic continuation & ADM and HPM	Analytic continuation & Legendre Tau method	Analytic continuation & HAM at $\hbar = -1.49$
0.05	0.000198259	0.000513541	0.000197889
0.10	$1.91699E-07$	0.000961256	0
0.15	$3.78938E-05$	0.001693289	$3.81356E-05$
0.20	0.025203724	0.000867533	$6.3438E-05$
0.25	$4.26699E-05$	0.000668993	$4.31396E-05$
0.30	$2.34533E-05$	0.000848531	$1.92068E-05$
0.35	$3.20706E-05$	0.000288388	$3.14367E-05$
0.40	0.006539555	0.007211171	0.006541689
0.45	$2.30561E-05$	0.000233099	$2.1725E-05$
0.50	$1.6794E-05$	0.000505521	$1.81513E-05$
0.55	0.000101367	0.00032771	0.000102118
0.60	0.000113508	0.000238256	0.000112782
0.65	$6.75733E-05$	0.000233359	$6.93553E-05$
0.70	$2.62897E-05$	0.000287311	$2.48911E-05$
0.75	$5.65477E-05$	0.000118012	$5.57926E-05$
0.80	$8.08104E-05$	0.000266422	$8.20811E-05$
0.85	0.001047683	0.001111239	0.001037383
0.90	0.000192042	0.000235596	0.000256579
0.95	0.00106785	0.001779956	0.001759657

Table 9.7

x	Analytic continuation & ADM and HPM	Analytic continuation & Legendre Tau method	Analytic continuation & HAM at $\hbar = -1.54$
0.05	$2.7011E-05$	0.000468645	$2.98507E-05$
0.10	$3.01923E-05$	0.000732462	$2.94031E-05$
0.15	$7.59313E-06$	0.000393796	$9.55932E-06$
0.20	$4.59473E-05$	0.000816043	$4.61681E-05$
0.25	$2.20121E-05$	0.001161684	$1.76694E-05$
0.30	$8.17108E-06$	0.00061282	$8.37661E-06$
0.35	$1.76903E-07$	0.001351536	0.0000000
0.40	$1.73553E-05$	0.000592696	$1.46757E-05$
0.45	$3.37182E-05$	0.039359455	$3.39075E-05$
0.50	$5.56792E-06$	0.000693763	$6.21388E-06$
0.55	$1.80528E-05$	0.001115262	$1.693E-05$
0.60	$4.69564E-06$	0.000701595	$5.07563E-06$
0.65	$7.57898E-06$	0.000880923	$9.02894E-06$
0.70	$1.68256E-05$	0.000523779	$1.58497E-05$
0.75	$6.09E-06$	0.000729523	$6.84065E-06$
0.80	$6.76861E-06$	$8.90497E-05$	$8.66501E-06$
0.85	$3.87512E-05$	0.00051366	$4.25834E-05$
0.90	$5.6246E-05$	0.000193151	0.000103665
0.95	0.074687301	0.075673352	0.075509025

Table 9.8

x	Analytic continuation & ADM and HPM	Analytic continuation & Legendre Tau method	Analytic continuation & HAM at $\hbar = -1.54$
0.05	0.000215097	0.033362858	0.000218038
0.10	$9.04425E-05$	0.043015996	$8.7998E-05$
0.15	$1.75769E-05$	0.07000958	$1.75831E-05$
0.20	$1.66681E-05$	0.03695664	$1.63685E-05$
0.25	$3.04243E-05$	0.028475778	$2.71912E-05$
0.30	0.17330502	0.03558457	$3.63346E-05$
0.35	$1.97941E-05$	0.015723701	$1.78285E-05$
0.40	$8.93864E-06$	0.02725151	$9.82753E-06$
0.45	$7.07383E-06$	0.012798913	$8.1626E-06$
0.50	$7.18033E-06$	0.019574587	$6.77369E-06$
0.55	$1.98955E-05$	0.011129604	$1.95585E-05$
0.60	$4.04501E-05$	0.013662466	$4.10397E-05$
0.65	$2.2864E-05$	0.008186255	$2.56176E-05$
0.70	$4.04288E-05$	0.009720094	$4.30416E-05$
0.75	$8.38462E-06$	0.003554365	$5.44662E-06$
0.80	0.000173866	0.006656007	0.000166139
0.85	0.001431251	0.00025422	0.00147027
0.90	0.000869638	0.000180335	0.001389262
0.95	0.006272612	0.000935818	$9.26916E-05$

9.5 Conclusion

The following features were noted in this study:

- As we used a Taylor's series expansion, the solution obtained is valid in the interval (0,1).
- HPM is the special case of HAM at $\hbar = -1$ in Eq. (9.13) one can easily get HPM solution.
- The given comparison in Tables $9.1 - 9.4$ indicate that HAM solution is better than ADM solution, HPM solution, and Legendre Tau solution.

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Appendix

Appendix A

$$
\begin{aligned}
A_{11} &= 24C_1(P - M^2 + B_r(\gamma-1)^3 + G_r(\gamma-1)^3 + 3M^2\gamma - 2P\gamma - 3M^2\gamma^2 \\
&\quad + P\gamma^2 + -4\gamma\ln 2 + \ln 4 + \gamma^2\ln 4 + \Lambda\ln 4) + 2c^3C_1\Lambda(81\ln 2 - 56 - \\
&\quad 6\gamma(54\ln 2 - 37) + 6\gamma^2(72\ln 2 - 49) - 64\gamma^3(\ln 8 - 2)) + 36c^2C_1\Lambda \\
&\quad (-6 - 8\gamma(\ln 8 - 2) + 2\gamma^2(\ln 256 - 5) + \ln 512) + 24c(\gamma+1)^3 + 2cC_1 \\
&\quad (M^2(\gamma-1)^3 - 6(P(\gamma-1)^2(2\gamma-1) + 2(1 + 6\Lambda + \gamma^2(3 - 11\ln 2) \\
&\quad - 9\Lambda\ln 2 - \ln 8 + \gamma^3(-1 + \ln 16) + \gamma(-3 + 6\Lambda(\ln 4 - 1) + \ln 1024))) ,
\end{aligned}
$$

$$
\begin{aligned}
A_{12} &= 6c^3C_1\Lambda(38 - 27\ln 2 + \gamma(3(-49 + 36\ln 2) + (\gamma 189 - 144\ln 2 + \\
&\quad 16\gamma(\ln 16 - 5)))) - 36c(\gamma-1)^3 cC_1(-M^2(\gamma-1)^2(13\gamma - 9) + \\
&\quad 2(P(\gamma-1)^2(-13 + 22\gamma) + 6\gamma^2(9 - 22\ln 2) - 3(-1 + 6\Lambda(\ln 2 - 1) \\
&\quad + \ln 4) + \gamma(20\ln 2 - 9 + 6\Lambda(\ln 16 - 3) + \gamma^3(\ln 256 - 3)) - 36c^2C_1 \\
&\quad \Lambda(-11 + \gamma 28 - 17\gamma + 16\gamma\ln 2 - 4\ln 64 + \ln 512) - 4C_1(-11M^2 + \\
&\quad 9P + 11B_r(\gamma-1)^3 + 11G_r(\gamma-1)^3 + \gamma(9P(\gamma-2) + M^2(31 + \\
&\quad \gamma(9\gamma - 29))) + 12((\gamma-2) + \Lambda)\ln 2 + \ln 4096) - 3C_1(-2 + \\
&\quad c(-3 + 4\gamma))(4(\gamma-1)^2 + (-2 + c(4\gamma-3))^2\Lambda)\ln r,
\end{aligned}
$$

$$
\begin{aligned}
A_{13} &= 6(\gamma-1)((\gamma-1)(2C_1(P + M^2(\gamma-2) + 2B_r(\gamma-1) + 2G_r(\gamma-1) \\
&\quad + c(2(\gamma-1) + C_1(3P - 2 + 2\gamma + 4P\gamma + M^2(-2 + 3\gamma)))) + \\
&\quad 3cC_1(-2 + c(-3 + 4\gamma))^2\Lambda) ,
\end{aligned}
$$

$$
\begin{aligned}
A_{14} &= -2C_1(\gamma-1)^2 - 2B_r - 2G_r - M^2(2 + 3c) + 2c(P + (2B_r + G_r \\
&\quad + 2cM^2 - cP)\gamma + 9c^2(c(4\gamma - 3) - 2)\Lambda),
\end{aligned}
$$

$$A_{15} = cC_1(\gamma-1)^3(M^2 + 8c^2\Lambda) ,$$

$$
\begin{aligned}
A_{21} = {} & \frac{1}{1440N_b\alpha(-1+\gamma)^5}(-120N_b\alpha(-1+\gamma)^2(c^3+C_2\Lambda(-2(-1+\gamma)(28 \\
& +\gamma(-83+64\gamma))+(-3+4\gamma)^3\ln 8)-12C_2(M^2-P+B_r(-1+\gamma)^3 \\
& +G_r(-1+\gamma)^3+\ln[4]+(\gamma^2+\Lambda)\ln[4]-\gamma(P(-2+\gamma)+M^2(3+ \\
& (-3+\gamma)\gamma)+\ln[16]))+c(-12(-1+\gamma)^3+C_2(M^2(-1+\gamma)^3-6(-2 \\
& +P(-1+\gamma)^2(-1+2\gamma)-12\Lambda+2\gamma(3-3\gamma+\gamma^2+6\Lambda-(10-11\gamma+ \\
& 4\gamma^2+12\Lambda)\ln[2])+6\Lambda\ln[8]+\ln[64])))-18c^2C_2\Lambda(-6+2\gamma(8-5\gamma \\
& +4(-3+2\gamma)\ln[2])+\ln[512]))+C_1^2(60G_rN_b(-1+\gamma)^3(2(40+M^2 \\
& -5P)\alpha+2N_b(-1+\gamma)^2+\alpha(2(40+M^2-5P)(-2+\gamma)\gamma+3(60+c \\
& (180+137c))\Lambda+6c\gamma(-120+c(-182+121\gamma))\Lambda-120\ln[2]-6(20 \\
& (-2+\gamma)\gamma+11(2+c(3-4\gamma))^2\Lambda)\ln[2]))-\alpha(-60B_r(-1+\gamma)^2(N_b(60 \\
& (-1+\gamma)^2+2M^2(-1+\gamma)^2-10P(-1+\gamma)^2+3(60+c(180+137c- \\
& 4(60+91c)\gamma+242c\gamma^2))\Lambda-22(4(-1+\gamma)^2+3(2+3c-4c\gamma)^2\Lambda) \\
& \ln[2])-8N_t(-1+\gamma)^2(-5+\ln[256]))+N_b(30c^4\Lambda^2(4(-1+\gamma) \\
& (-2886+\gamma(12320+\gamma(-17521+8303\gamma)))-18(-3+4\gamma)(-449 \\
& +2\gamma(888+\gamma(-1161+500\gamma)))\ln[2]+135(3-4\gamma)^4\ln[2]^2)+ \\
& 3c^5\Lambda^2(-4(-1+\gamma)(8832+\gamma(-49758+\gamma(105437+\gamma(-99493+ \\
& 35252\gamma))))+90(3-4\gamma)^2(-91+2\gamma(181+\gamma(-239+104\gamma)))\ln[2] \\
& -135(-3+4\gamma)^5\ln[2]^2)-10c^3\Lambda((-1+\gamma)(-9P(-1+\gamma)(-5+6\gamma)(33 \\
& +2\gamma(-52+41\gamma))+8(-1+\gamma)^2(364+\gamma(-1052+769\gamma))+24(1831 \\
& +\gamma(-5201+3694\gamma))\Lambda)+18((-1+\gamma)(458+P(3-4\gamma)^3(-13+22\gamma) \\
& -2\gamma(903+2\gamma(-591+256\gamma)))12(-434+\gamma(1692+\gamma(-2181+928\gamma))) \\
& \Lambda)\ln[2]+\ln[2]))+60c^2\Lambda(2M^2(-1+\gamma)^2(339-1078\gamma+1054\gamma^2- \\
& 294\gamma^3++6(-3+4\gamma)(27+2\gamma(-25+9\gamma))\ln[2])+3(P(-1+\gamma)^2(-223 \\
& +632\gamma-442\gamma^2+(-3+4\gamma)(-53+80\gamma)\ln[4])+12(2\gamma(-6(9+23\Lambda) \\
& +(165+326\Lambda)\ln[2]-18(7+10\Lambda)\ln[2]+120C_1N_b\alpha(-1+\gamma)^3(12B_r \\
& (-1+\gamma)+c^3\Lambda(2(-1+\gamma)(112+\gamma(-305+211\gamma))-3(3-4\gamma)^2+ \\
& (-12+13\gamma)\ln[2])+12((P+G_r(-1+\gamma)+M(-1+\gamma))(-1+\gamma)^2- \\
& 2(3+\gamma^2(9-20\ln[2])-6\Lambda(-2+\ln 8)+\gamma(-9+19\ln 2+3\Lambda(-4+ \\
& \ln 128))+\gamma^3(-3+\ln 128))-\ln 4096))\,,
\end{aligned}
$$

$$
\begin{aligned}
A_{23} = & \frac{1}{96N_b\alpha(-1+\gamma)^4}(24N_b\alpha(-1+\gamma)^2(-(-1+\gamma)(2C_2(P+M^2(-2+\gamma) \\
& -2B_r(-1+\gamma)-2G_r(-1+\gamma))+c(2-2\gamma+C_2(2+3P-2\gamma-4P\gamma \\
& +M^2(-2+3\gamma)))))+3cC_2(-2+c(-3+4\gamma))^2\Lambda)+24C_1N_b\alpha \\
& (-1+\gamma)^2((-1+\gamma)(2(P+M^2(-2+\gamma)+2G_r(-1+\gamma))+4B_r(-1+\gamma) \\
& +c(-7P\gamma+6(-1+P+\gamma)+M^2(-4+5\gamma)))+6c(-2+c(-3+4\gamma)) \\
& (-2+c(-6+7\gamma))\Lambda)+C_1^2(8G_rN_b(-1+\gamma)^2(6N_b(-1+\gamma)^2+\alpha(6M^2 \\
& (-1+\gamma)^2-11P(-1+\gamma)^2+6(4+2(2+3c)(3+10c)\Lambda-\ln 4+\gamma(-8 \\
& -c(70+149c)\Lambda+\gamma(4+92c^2\Lambda-\ln 4)+\ln 16)))+\alpha(8B_r(-1+\gamma)^2 \\
& (N_b(24+6M^2-11P)(-1+\gamma)^2+6N_b(-2+c(-3+4\gamma))(-6+c(-20 \\
& +23\gamma))\Lambda-24N_t(-1+\gamma)^2\ln 2)+N_b(144c^4\Lambda^2(299-1179\gamma+1542\gamma^2 \\
& -668\gamma^3+4(-3+4\gamma)^2\ln 2)-36c^5(-3+4\gamma)\Lambda^2(157-54\ln 2+2\gamma \\
& (-309+108\ln 2+\gamma(405-144\ln 2+16\gamma(-11+\ln 16))))+c^3\Lambda(4M^2 \\
& (299-1502\gamma+2814\gamma^2-2327\gamma^3+716\gamma^4-3(-1+\gamma)(-3+4\gamma)^3\ln 2) \\
& +3(P(-1+\gamma)(761-108\ln 2+4\gamma(-758+108\ln 2+\gamma(997-144\ln 2+ \\
& 16\gamma(-27+\ln 16))))-24(-45-564\Lambda+\gamma^3(-345-924\Lambda+292\ln 2+ \\
& 384\Lambda\ln 2)-12\gamma(-17-121\Lambda+(7+24\Lambda)\ln 4)+9\ln 16+8\gamma^4(-9+ \\
& \ln 256))))+6c^2\Lambda(-P(-1+\gamma)(-749+1912\gamma-1196\gamma^2+12(3-4\gamma)^2 \\
& \ln 2)+48(-3\gamma(13+\gamma(-14+5\gamma)+22\Lambda)+4\gamma(10+\gamma(-11+4\gamma)+8\Lambda) \\
& \ln 2+6(2+9\Lambda-\ln 4)-6\Lambda\ln 16)+4M^2(-1+\gamma)(\gamma(551-72\ln 2+\gamma \\
& (-497+120\gamma+48\ln 2))+3(-61+\ln 512)))+2c(2M^4(-1+\gamma)^4+2 \\
& (P^2(-1+\gamma)^3(-13+22\gamma)+24(1-4\gamma^3+\gamma^4+9\Lambda(1+2\Lambda)+\gamma^2(6+9\Lambda \\
& -12\Lambda\ln 2)-6\Lambda(1+\Lambda)\ln 4+2\gamma(-2+3\Lambda(-3+\ln 16)))+3P(-1+\gamma)(\gamma^2 \\
& (51-44\ln 2)+3(5+69\Lambda-4(1+3\Lambda)\ln 2)+8\gamma(-6+\ln 32+\Lambda(-30+ \\
& \ln 64))+2\gamma^3(-9+\ln 256)))-M^2(-1+\gamma)(P(-1+\gamma)^2(-21+37\gamma)- \\
& 2+c(-3+4\gamma))(P(-1+\gamma)+6c(2+c(3-4\gamma))\Lambda)(4(-1+\gamma)^2\Lambda)\ln r) \;,
\end{aligned}
$$

$$A_{24} = \frac{1}{14\alpha N_b(-1+\gamma)^3}(-12C_2N_b\alpha(-1+\gamma)^2(-2B_r-2G_r+(2+3c)M^2$$
$$-2cP+2(B_r+G_r+c(-2M^2+P))\gamma+9c^2(-2+c(-3+4\gamma))\Lambda)-$$
$$12C_1N_b\alpha(-1+\gamma)^2(-2(1+3c)M^2+4cP+2B_r(-1+\gamma)+2G_r$$
$$(-1+\gamma)+c(7M^2\gamma-4P\gamma+27c(-2-4c+5c\gamma)\Lambda))+C_1^2(-G_rN_b$$
$$(-1+\gamma)(36N_b(-1+\gamma)^2+\alpha(16+44M^2(-1+\gamma)^2-48P(-1+\gamma)^2$$
$$+9(12+c(132+215c))\Lambda+4\gamma(\gamma(4+639c^2\Lambda-12\ln 2)-4(2+$$
$$9c(9+31c)\Lambda-6\ln 2))-48\ln 2))+\alpha(B_r(-1+\gamma)(-4N_b(9+11M^2$$
$$-12P)(-1+\gamma)^2-9N_b(12+c(132+215c-16(9+31c)\gamma+284c\gamma^2))$$
$$\Lambda+8N_t(-1+\gamma)^2(5+\ln 4096))+N_b(-M^4(-1+\gamma)^2(-44+36\gamma+c$$
$$(-9+13\gamma))+M^2(6c^2\Lambda(519-1410\gamma+1100\gamma^2-198\gamma^3-6(3-4\gamma)^2$$
$$\ln 2)+c^3\Lambda(1254-4833\gamma+6165\gamma^2-2597\gamma^3+6(-3+4\gamma)^3+\ln 2)+$$
$$4(3P(-1+\gamma)^2(-7+2\gamma)+2(2+11\Lambda+\gamma^2(2-6\ln 2)-18\ln 2+\gamma^3$$
$$+(-17+24\ln 2)-9\Lambda(-41+\ln 64)+\gamma(-47+60\ln 2+6\Lambda(-85+$$
$$\ln 4096)))3(8P^2(-1+\gamma)^2+4cP(-1+\gamma)(5(-1+\gamma)^2+P(-3+(7$$
$$-4\gamma)\gamma)+57\Lambda)+54c^4\Lambda^2(175-464\gamma+306\gamma^2-(3-4\gamma)^2\ln 4)+$$
$$9c^5\Lambda^2(535-54\ln 2-6\gamma(355-36\ln 2+\gamma(-471+208\gamma+48\ln 2))$$
$$+32\gamma^3\ln 16)+3c^3\Lambda(P(-1+\gamma)(197-526\gamma+348\gamma^2)+12(\gamma^2(53-$$
$$22\ln 2)+\gamma(-49+20\ln 2+6\Lambda(-37+\ln 16))-3(-5+\ln 4+\Lambda(-57+$$
$$\ln 64))+\gamma^3(-19+\ln 256)))-12c^2\Lambda(P(-1+\gamma)(-66+85\gamma)+6(-4$$
$$+\Lambda(-17+\ln 4)+\gamma^2(-4+\ln 4)-\gamma(-8+\ln 16))+\ln 4096))))-3C_1^2$$
$$\alpha(8G_rN_b(-1+\gamma)^2+16B_rN_t(-1+\gamma)^3+36c^3N_b(-1+\gamma)^2(-2-3c+$$
$$4c\gamma)\Lambda+9c^2N_b(-2-3c+4c\gamma)^3\Lambda^2+M^2N_b(-2-3c+4c\gamma)(4(-1+\gamma)^2$$
$$+(2+3c-4c\gamma)^2+\Lambda))\ln r),$$

$$A_{25} = \frac{1}{144\alpha(-1+\gamma)^2}6cC_2\alpha(-1+\gamma)^2+(-M^2+8c^2\Lambda)+12cC_1\alpha(-1+\gamma)^2$$
$$(M^2+16c^2\Lambda)+C_1^2(2(6B+7c)M^2\alpha+6G_r(-1+\gamma)((N_b+2M^2\alpha-P\alpha)$$
$$(-1+\gamma)+12c\alpha(-2+c(-7+8\gamma))\Lambda)+\alpha(6P(-B_r+cP)(-1+\gamma)^2+3M^4$$
$$(-1+\gamma)(-4-2c+2\gamma+3c\gamma)+c(-1+\gamma)(c(342P+c(171P(3-4\gamma)+$$
$$208(-1+\gamma)))+72B_r(-2-7c+8c\gamma))\Lambda+588c^3(2+c(3-4\gamma))^2\Lambda^2+$$
$$M^2(2(6B+7c)(-2+\gamma)\gamma-6P(-1+\gamma)(-2-3c+4c\gamma)+$$
$$3c(68+c(300+201c-16(26+33c)\gamma+8(6+43c)\gamma^2))\Lambda))),$$

$$A_{26} = \frac{1}{240(\gamma-1)}(C_1^2(M^2(2B_r+2G_r+M^2(2+3c)-4cP-2(B_r+G_r$$
$$+2c(M^2-P))\gamma)-2c^2(-42(2+3c)M^2+53cP+45B_r(\gamma-1)$$
$$+45G_r(\gamma-1)+c(168M^2-53P)\gamma)\Lambda-645c^4(-2+c(-3+4\gamma))\Lambda^2,$$

$$A_{27} = \frac{1}{720} c C_1^2 \left(M^2 + 8c^2 \Lambda\right) \left(M^2 + 72c^2 \Lambda\right),$$

$$\begin{aligned}
B_{11} \;=\;& -48C_1(B_r + G_r - M^2) + 18c^2 C_1 \Lambda(-24 + 27\gamma + 40\gamma^2 - 44\gamma^3 - 2 \\
& (-3 + 4\gamma)(19 + 4\gamma(-7 + 2\gamma)) \ln[2]) + c^3 C_1 \Lambda(-456 + 1475\gamma - \\
& 1284\gamma^2 - 132\gamma^3 + 400\gamma^4 + 6(3 - 4\gamma)^2(27 + 4\gamma(-10 + 3\gamma)) \ln[2]) \\
& +24C_1(8(B_r + G_r) \ln[2] - 8M^2 \ln[2] + 4(1 + P) \ln[2] + \gamma(-2 + \\
& 5B_r + 5G_r - 3M^2 - 2P - \Lambda - 2(4 + 14B_r + 14G_r - 12M^2 + 5P) \ln[2]) \\
& +\gamma^4(B_r + G_r + 4(B_r + G_r) \ln[2] - 2M^2(1 + \ln[2])) + \Lambda \ln[4] - \gamma^3 \\
& (2 + B_r + G_r - 5M^2 + 2P + (10B_r + 10G_r - 6M^2 + P) \ln[4]) + \gamma^2 \\
& (4 - 3B_r - 3G_r - 2M^2 + 4P + (18B_r + 18G_r - 13M^2 + 4P) \ln[4] \\
& + \ln[16])) - 4c(-6(-1 + \gamma)^3(-2 + 3\gamma) + C_1(M^2(-1 + \gamma)^2(-12 + \\
& 22\gamma - 7\gamma^2 + 6(-2 + \gamma)(-2 + 3\gamma) \ln[2]) - 3(8\gamma^4(2 + \ln[2]) - 2\gamma^3 \\
& (23 + 28 \ln[2]) - \gamma(14 + 33\Lambda + 96(1 + \Lambda) \ln[2]) + (14 + 33\Lambda) \ln[4] \\
& +2P(-1 + \gamma)^2(-2 + \gamma(2 + \gamma - 11 \ln[2] + \gamma \ln[16]) + \ln[64]) + 4\gamma^3 \\
& (11 + 29 \ln[2] + \Lambda(9 + \ln[64]))))) + 6(8C_1(-1 + \gamma)^3(-2 + c(-3 + 4\gamma)) \\
& +C_1(-1 + \gamma)(-2 + c(-3 + 4\gamma))^3 \Lambda) \ln[r],
\end{aligned}$$

$$\begin{aligned}
B_{12} \;=\;& 72C_1(B_r + G_r - M^2) + c^3 C_1 \Lambda(748 - 2751\gamma + 3360\gamma^2 - 1360\gamma^3 - 6 \\
& (3 - 4\gamma)^2(27 + 4\gamma(-10 + 3\gamma)) \ln[2]) + 18c^2 C_1(-3 + 4\gamma)\Lambda(-12 + \\
& 38 \ln[2] + \gamma(5 - 56 \ln[2] + 8\gamma(1 + \ln[4]))) + 4c(-6(-1 + \gamma)^2 + \\
& (-3 + 4\gamma) + C_1(M^2(-1 + \gamma)^2(\gamma(41 + 18\gamma(-1 + \ln[2]) - 48 \ln[2]) \\
& +4(-5 + \ln[64])) - 3(8\gamma^4(2 + \ln[2]) - 2\gamma^3(23 + 28 \ln[2]) - \\
& \gamma(14 + 33\Lambda + 96(1 + \Lambda) \ln[2]) + 2P(-1 + \gamma)^2(-3 + 4\gamma)(1 + \gamma \ln[2] \\
& - \ln[4]) + (14 + 33\Lambda) \ln[4] + 4\gamma^2(11 + 29 \ln[2] + \Lambda(9 + \ln[64]))) \\
& -24C_1(-8M^2 \ln[2] + \gamma(-2 + 9B_r + 9G_r - 6M^2 - 2P - \Lambda - 2(4 + 14B_r + \\
& 14G_r - 12M^2 + 5P) \ln[2]) + \Lambda \ln[4] - \gamma^3(2 - 3B_r - 3G_r + 20(B_r + G_r) \\
& \ln[2] + P(2 + \ln[4]) - 4M^2(1 + \ln[8])) + (1 + P) \ln[16] + \gamma^4 \\
& (-M^2(2 + \ln[4]) + (B_r + G_r) \ln[16]) + \gamma^2(4 - 9B_r - 9G_r + M^2(1 - \\
& 26 \ln[2]) + 4P(1 + \ln[4]) + \ln[16] + 9(B_r + G_r) \ln[16]) + (B_r + G_r) \\
& \ln[256]) + 24C_1(-1 + \gamma)^2(4B_r(-1 + \gamma)^2 - (-1 + \gamma)(2(P + M^2 \\
& (-2 + \gamma) - 2G_r(-1 + \gamma)) + c(4 + 3P - 4(1 + P)\gamma + M^2(-2 + 3\gamma))) \\
& +3c(-2 + c(-3 + 4\gamma))^2 \Lambda) \ln[r],
\end{aligned}$$

$$\begin{aligned}
B_{13} \;=\;& -12(-1 + \gamma)^3(2C_1(M^2 + B_r(-1 + \gamma) + G_r(-1 + \gamma)) + c(2 + C_1(M^2 \\
& (3 - 4\gamma) + 2P(-1 + \gamma)) - 2\gamma) - 18c^2 C_1 \Lambda + 9c^3 C_1(-3 + 4\gamma)\Lambda),
\end{aligned}$$

$$B_{14} = 4cC_1(\gamma - 1)^4 \left(M^2 + 8c^2 \Lambda\right),$$

$$B_{21} = \frac{C_1(2+C_1)(-2+c(-3+4\gamma))(-8(-1+\gamma)^2-(-2+c(-3+4\gamma))^2\Lambda}{16(-1+\gamma)^3},$$

$$\begin{aligned}
B_{22} \;=\; & \frac{1}{1152N_b\alpha(-1+\gamma)^7}(1152C_1^2G_rNb^2-4\gamma(3(377+750\ln[2])+4\gamma(-293\\
&-348\ln[2]+36\gamma(3+\ln[4])120N_b\alpha(-1+\gamma)^2(c^3+C_2\Lambda(-12(-1+\gamma)(281\\
&+\gamma(-79+156\gamma))+(-3+4\gamma)^5\ln 8)-12C_2(M^2-P+B_r(-1+\gamma)^3+G_r\\
&(-1+\gamma)^3+\ln 4+(\gamma^2+\Lambda)\ln 4-\gamma(P(-2+\gamma)+M^2(3+(-3+\gamma)\gamma)+\ln\\
&+c(-12(-1+\gamma)^3+C_2(M^2(-1+\gamma)^3-6(-2+P(-1+\gamma)^2(-1+2\gamma)\\
&-12\Lambda+2\gamma(3-3\gamma+\gamma^2+6\Lambda-(10-11\gamma+4\gamma^2+12\Lambda)\ln 2)+6\Lambda\ln\\
&8+\ln 64)))-18c^2C_2\Lambda(-6+2\gamma(8-5\gamma+4(-3+2\gamma)\ln[2])+\ln[512]))\\
&+C_1^2(60G_rN_b(-1+\gamma)^3(2(40+M^2-5P)\alpha+2N_b(-1+\gamma)^2+\alpha(2(40+\\
&M^2-5P)(-2+\gamma)\gamma+3(60+c(180+137c))\Lambda+6c\gamma(-120+c(-182+\\
&121\gamma))\Lambda-120\ln[2]-2160\Lambda\ln[2]+\gamma(-58-123\Lambda-288(1+\Lambda)\ln[2])\\
&+4\gamma^5(6(20(-2+\gamma)\gamma+11(2+c(3-4\gamma))^2\Lambda)\ln[2]))-\alpha(-60B_r(-1+\gamma)^2\\
&(N_b(6380(-1+\gamma)^2+2M^2(-1+\gamma)^2-10P(-1+\gamma)^2+3(60+c(180+\\
&137c-4(60+91c)\gamma+242c\gamma^2))\Lambda-22(4(-1+\gamma)^2+3(2+3c-4c\gamma)^2\\
&\Lambda)\ln 2)-8N_t(-1+\gamma)^2(-5+\ln 256))+N_b(30\Lambda^2(4(-1+\gamma)(-2886+\gamma\\
&(12320+\gamma(-17521+8303\gamma)))-18(-3+4\gamma)(-449+2\gamma(888+\gamma\\
&(-1161+500\gamma)))\ln 2+135(3-4\gamma)^4\ln 2^2)+3c^5\Lambda^2(-4(-1+\gamma)(8832+\\
&\gamma(-49758+\gamma(105437+\gamma(-99493+35252\gamma))))+90(3-4\gamma)^2(-91+\\
&2\gamma(181+\gamma(-239+104\gamma)))\ln 2-135(-3+4\gamma)^5\ln 2^2)-10c^3\Lambda((-1+\gamma)\\
&(-9P(-1+\gamma)(-5+6\gamma)(33+2\gamma(-52+41\gamma))+8(-1+\gamma)^2(364+\gamma\\
&(-1052+769\gamma))+24(1831+\gamma(-5201+3694\gamma))\Lambda)+18((-1+\gamma)(458\\
&+P(3-4\gamma)^3(-13+22\gamma)-2\gamma(903+2\gamma(-591+256\gamma)))12(-434+\gamma\\
&(1692+\gamma(-2181+928\gamma)))\Lambda)\ln[2]+\ln[2]))+60c^2\Lambda(2M^2(-1+\gamma)^2(339-\\
&1078\gamma+1054\gamma^2-294\gamma^3++6(-3+4\gamma)(27+2\gamma(-25+9\gamma))\ln[2])+3(P\\
&(-1+\gamma)^2(-223+632\gamma-442\gamma^2+(-3+4\gamma)(-53+80\gamma)\ln[4])+12(2\gamma\\
&(-6(9+23\Lambda)+(165+326\Lambda)\ln[2]-18(7+10\Lambda)\ln[2]+120C_1N_b\alpha\\
&(-1+\gamma)^3(12B_r(-1+\gamma)+c^3\Lambda(2(-1+\gamma)(112+\gamma(-305+211\gamma))-\\
&3(3-4\gamma)^2+(-12+13\gamma)\ln[2])+12((P+G_r(-1+\gamma)+M(-1+\gamma))\\
&(-1+\gamma)^2-2(3+\gamma^2(9-20\ln[2])-6\Lambda(-2+\ln[8])+\gamma(-9+19\ln[2]\\
&+3\Lambda(-4+\ln[128]))+\gamma^3(-3+\ln[128]))+\ln[4096]))),
\end{aligned}$$

$$
\begin{aligned}
B_{23} \quad = \quad & \frac{1}{384N_b\alpha(-1+\gamma)^7}(124N_b\alpha(-1+\gamma)^2+P(-1+\gamma)^2(1-\gamma(3+\ln[4]) \\
& +\ln[16])(-1+\gamma)(2C_2(P+M^2(-2+\gamma)-2B_r(-1+\gamma) \\
& -2G_r(-1+\gamma))+c(2-2\gamma+C_2(2+3P-2\gamma-4P\gamma+M^2(-2+3\gamma)))) \\
& +3cC_2(-2+c(-3+4\gamma))^2\Lambda)+24C_1N_b\alpha(-1+\gamma)^2((-1+\gamma)(2(P+M^2 \\
& (-2+\gamma)+2G_r(-1+\gamma))+14B_r(-1+\gamma)+c(-7P\gamma+6(-1+P+\gamma)+ \\
& M^2(-4+5\gamma)))+6c(-2+c(-3+4\gamma))(-2+c(-6+7\gamma))\Lambda)+C_1^2(8G_r \\
& N_b(-1+\gamma)^2(6N_b(-1+\gamma)^2+\alpha(6M^2(-1+\gamma)^2-17P(-1+\gamma)^2+6(4 \\
& +2(2+3c)(3+10c)\Lambda-\ln 4+\gamma(-8-c(70+149c)\Lambda+\gamma(4+92c^2\Lambda \\
& -\ln 4)+\ln 16)))+\alpha(8B_r(-1+\gamma)^2(N_b(24+6M^2-11P)(-1+\gamma)^2+ \\
& 6N_b(-2+c(-3+4\gamma))(-6+c(-20+23\gamma))\Lambda-24N_t(-1+\gamma)^2\ln 2)+ \\
& N_b(144c^4\Lambda^2(299-1179\gamma+1542\gamma^2-668\gamma^3+4(-3+4\gamma)^2\ln 2)- \\
& 36c^5(-3+4\gamma)\Lambda^2(157-54\ln 2+2\gamma(-309+108\ln 2+P(-1+\gamma)^2(1 \\
& -\gamma(3+\ln[4])+\ln[16])+3\gamma^4\Lambda(7+\ln[64])+\gamma(405-144\ln 2+16\gamma \\
& (-11+\ln 16))))+c^3\Lambda(4M^2(299-1502\gamma+2814\gamma^2-2327\gamma^3+716\gamma^4 \\
& -3(-1+\gamma)(-3+4\gamma)^3\ln 2)+3(P(-1+\gamma)(761-108\ln 2+4\gamma(-758+ \\
& 108\ln 2+\gamma(997-144\ln 2+16\gamma(-27+\ln 16))))-24(-45-564\Lambda+ \\
& \gamma^3(-345-924\Lambda+292\ln 2+384\Lambda\ln 2)-12\gamma(-17-121\Lambda+(7+24\Lambda) \\
& \ln 4)+9\ln 16+8\gamma^4(-9+\ln 256))))+6c^2\Lambda(-P(-1+\gamma)(-749+1912\gamma \\
& -1196\gamma^2+12(3-4\gamma)^2\ln 2)+48(-3\gamma(13+\gamma(-14+5\gamma)+22\Lambda)+ \\
& 4\gamma(10+\gamma(-11+4\gamma)+8\Lambda)\ln 2+6(2+9\Lambda-\ln 4)-6\Lambda\ln 16)+4M^2 \\
& (-1+\gamma)(\gamma(551-72\ln 2+\gamma(-497+120\gamma+48\ln 2))+3(-61+\ln 512))) \\
& +2c(2M^4(-1+\gamma)^4+2(P^2(-1+\gamma)^3(-13+22\gamma)+24(1-4\gamma^3+\gamma^4+ \\
& 9\Lambda(1+2\Lambda)+\gamma^2(6+9\Lambda-12\Lambda\ln 2)-6\Lambda(1+\Lambda)\ln 4+2\gamma(-2+3\Lambda(-3 \\
& +\ln 16)))+3P(-1+\gamma)(\gamma^2(51-44\ln 2)+3(5+69\Lambda-4(1+3\Lambda)\ln 2) \\
& +8\gamma(-6+\ln 32+\Lambda(-30+\ln 64))+2\gamma^3(-9+\ln 256)))-M^2(-1+\gamma) \\
& (P(-1+\gamma)^2(-21+37\gamma)-2+c(-3+4\gamma))(P(-1+\gamma)+6c(2+c \\
& (3-4\gamma))\Lambda)(4(-1+\gamma)^2\Lambda)\ln r),
\end{aligned}
$$

$$
\begin{aligned}
B_{24} = {} & \frac{1}{96Nb\alpha(-1+\gamma)^4}(24C_1N_b\alpha(-1+\gamma)^3((-2P(-1+\gamma)+M^2-1+2\gamma)+ \\
& (45+16c^2(-1+\gamma)-108\gamma)\Lambda)-24C_2Nb\alpha(-1+\gamma)^3(-2B_r-2G_r+ \\
& (2+3c)M^2-2cP+2(B_r+G_r+c(-2M^2+P))\gamma+9c^2(-2+c(-3+ \\
& 4\gamma))\Lambda)-12C_1N_b\alpha(-1+\gamma)^2(-2(1+3c)M^2+4cP+2B_r(-1+\gamma)+ \\
& 2G_r(-1+\gamma)+c(7M^2\gamma-4P\gamma+27c(-2-4c+5c\gamma)\Lambda))+C_1^2(-G_rN_b \\
& (-1+\gamma)(36N_b(-1+\gamma)^2+\alpha(16+44M^2(-1+\gamma)^2-48P(-1+\gamma)^2 \\
& +9(12+c(132+215c))\Lambda+4\gamma(\gamma(4+639c^2\Lambda-12\ln 2)-4(2+ \\
& 9c(9+31c)\Lambda-6\ln 2))-48\ln 2))+16\gamma^3(-40+3\Lambda(-751+348\ln[2])) \\
& +\gamma(-640+9\Lambda(-2413+2016\ln[2])-32\gamma^3(-5+18\Lambda(-19+\ln[64]))) \\
& +9\alpha(B_r(-1+\gamma)(-4N_b(9+11M^2-12P)(-1+\gamma)^2-9N_b(12+c(132 \\
& +215c-16(9+31c)\gamma+284c\gamma^2))\Lambda+8N_t(-1+\gamma)^2(5+\ln 4096)) \\
& +N_b(-M^4(-1+\gamma)^2(-23+36\gamma+c(-9+13\gamma))+M^2(6c^2\Lambda(215 \\
& -1410\gamma+1100\gamma^2-198\gamma^3-6(3-4\gamma)^2\ln 2)+c^3\Lambda(1254-4833\gamma \\
& +6165\gamma^2-2597\gamma^3+6(-3+4\gamma)^3+\ln 2)+4(3P(-1+\gamma)^2(-7+2\gamma) \\
& +2(2+11\Lambda+\gamma^2(2-6\ln 2)-18\ln 2+\gamma^3(-17+24\ln 2)-9\Lambda(-41+ \\
& \ln 64)+\gamma(-47+58\ln 2+6\Lambda(-32+\ln 16)))))+3(8P^2(-1+\gamma)^2+ \\
& 4cP(-1+\gamma)^2(5(-1+\gamma)^2+P(-3+(7-4\gamma)\gamma)+5\Lambda)+54c^4\Lambda^2(175 \\
& -464\gamma+306\gamma^2-(3-4\gamma)^2\ln 4)+9c^5\Lambda^2(535-54\ln 2-6\gamma(355- \\
& 36\ln 2+\gamma(-471+208\gamma+48\ln 2))+32\gamma^3\ln 16)+3c^3\Lambda(P(-1+\gamma) \\
& (32-926\gamma+36\gamma^2)+12(\gamma^2(53-22\ln 2)+\gamma(-49+20\ln 2+6\Lambda(-37+ \\
& \ln 16))-3(-5+\ln 4+\Lambda(-57+\ln 64))+\gamma^3(-19+\ln 256)))-12c^2\Lambda \\
& (P(-1+\gamma)(-66+85\gamma)+6(-4+\Lambda(-17+\ln 4)+\gamma^2(-4+\ln 4)-\gamma(-8+ \\
& \ln 16))+\ln 4096))))-3C_1^2\alpha(8G_rN_b(-1+\gamma)^2+16B_rN_t(-1+\gamma)^3+ \\
& 36c^3N_b(-1+\gamma)^2(-2-3c+4c\gamma)^3\Lambda+9c^2N_b(-2-3c+4c\gamma)^3\Lambda^2+ \\
& PN_b(3+4c\gamma)(4(-1+\gamma)^2+(3-4c\gamma)^2+\Lambda))\ln r),
\end{aligned}
$$

$$
\begin{aligned}
B_{25} = {} & \frac{1}{24\alpha(-1+\gamma)}6cC_1\alpha(-1+\gamma)(-M^2-24\Lambda)+2cC_2\alpha(-1+\gamma) \\
& (M^2-98\Lambda)+C_1^2(2G_r(6B+7c)M^2\alpha+64G_r(-1+\gamma)((N_b+ \\
& 12M^2\alpha-P\alpha)(-1+\gamma)+12c\alpha(-2+c(-7+8\gamma))\Lambda)+\alpha(6P(-B_r \\
& +cP)(-1+\gamma)^2+13M^3(-1+\gamma)(-4-2c+2\gamma+3c\gamma)+(-1+\gamma)^3 \\
& (c(342B_r+c(154P(3-4\gamma)+134(-1+\gamma)^2))+12B_r(-7-17c+ \\
& 83c\gamma))\Lambda+588c^3(2+c(3-4\gamma))+3c(68+c(300+201c-16 \\
& (26+33c)\gamma+8(6+43c)\gamma^2))\Lambda))),
\end{aligned}
$$

$$
B_{26} = \frac{1}{144}cC_1^2\left(M^2-45\Lambda\right)\left(M^2-8c^2\Lambda\right),
$$