DE-NOISING OF IMAGES BASED ON SPARSE SIGNAL THEORY

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Hayatullah

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ABSTRACT

During data acquisition and transmission, it may be corrupted by different types of noises. These noises may be AWGN, Poisson, Rician or any combination of the additive and multiplicative noises. In case of compressed sensing, the noise caused by under sampling is completely de-noised by the recovery techniques used for recovery of the under sampled data. However, if the signal (or image) is also having additive or multiplicative noise, then it cannot be recovered only by the compressed sensing recovery techniques. In the first part of this work, Wavelet based soft, hard, garrote and Logarithmic technique (proposed) is used for de-noising of images, corrupted by noise during under-sampling in the transform domain. These techniques are applied to under-sampled Shepp-Logan Phantom images. Experimental results show that the proposed method is 7-10% better in PSNR values than the existing classical techniques. The second part of this work recovers and de-noises the under-sampled phantom image. The phantom image is corrupted by aliasing (Gaussian noise) and salt & pepper noise. This work is based on the recovery and de-noising of under sampled and noisy image through the classical and proposed thresholding techniques and compute their correlations with the original image. By applying the thresholding techniques only, gives the correlation values close to the noisy image. However, by applying median filter in sequence with the thresholding techniques, gives 30-35% better results. So, the second part of this work recovers the sparse under-sampled images by the combination of shrinkage functions and median filtering.

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Chapter 1

INTRODUCTION

Modern research in the field of optimization has made it easy to solve the constrained optimization and inverse problems efficiently in less time. Problems of these kinds are discussed in image processing such as image de-noising, de-blurring and restoration of the regularly and randomly under-sampled images etc. In most of the cases in image processing, we have access to under-sampled noisy versions of images [1] [2]. The inverse problem of reconstruction of signals (or images) from incomplete, faulty and under-sampled data is the field of interest from a few years [3]. We can exactly recover under-sampled, sparse signals using L1 optimization [4].The recent developments in the field of compressed sensing (CS) open a window towards the area of sparse signal processing. Signals have sparse representation in some transform domain i.e. Discrete Cosine transform (DCT), Fourier Transform (FT) and Wavelet Transform etc [5]. CS suggests that if the signal of interest is sparse (in some domain or in its own) it is possible under some assumptions to reconstruct the signal exactly with high probability with many fewer samples than the standard Shannon-Nyquist theory recommends [6]. CS has been applied to MRI (M.Lusting, D.L.Donoho and J.M.Pauly in 2007) and in particular techniques have been developed specifically for dynamic Magnetic Resonance Imaging (MRI) [7].

This work is divided into three models, sparse one dimensional signal recovery using the classical and new shrinkage functions. The recovery of sparse signal is an ill-posed problem. Take a sparse signal, under sample it in Fourier domain. Regular under-sampling leads to aliasing and random under-sampling offsets aliasing leads to noise. Then taking inverse Fourier transform, applying the thresholding functions in the signal domain. We iteratively apply hard, soft, garrote and proposed thresholding functions to recover the original signal. By random under –sampling this research turn an ill-conditioned problem to a sparse signal recovery problem [8].

The second contribution in this thesis is on the recovery and de-noising of sparse images having the aliasing noise due to random under-sampling. We have used Shepp-Logan Phantom image (of size 256 x 256) for testing the accuracy of our proposed technique. By applying the proposed, hard, and soft and garrote thresholding techniques iteratively in transform domain for the recovery of under sampled image. We observed that our proposed technique is better 7-10% in terms of PSNR. This suggests that the proposed function has more flexibility for the recovery of sparse signals (or images).

The third portion of this work have focused on the recovery and de-noising of the phantom images corrupted by under-sampling aliasing noise and impulsive noise. We have added salt and pepper noise to the original phantom image before making it under sampled. We apply the four thresholding techniques, which removes the noise produced due to aliasing but the impulsive noise cannot be removed significantly by using thresholding functions only. For that noise, it need the help of the existing de-noising and filtering techniques such as total variation de-noising (TVD) [9], Total Variation Filtering (TVF) [10], Filtered Variation De-noising (FVD) [11], mean filtering , Gaussian filtering and median filtering [12]. In our case we have used median filtering for the reduction of salt and pepper noise. It compares the results of the thresholding techniques with that of the median filtering, based on the correlation values. The median filtering produces 30-35% better results than that of the ordinary thresholding techniques.

1.1 Problem Statement

At its heart, signal and image processing prefers accurate processing of data. For the storage, transmission and acquisition of data we need a simple and reliable representations of the data. To represent the actual data with linear combinations of a few seed samples and avoid the complex calculations, computations and measurements leads to the field of sparse signal processing. Sparsity is used for signals and images compression [13], de-blurring [14]and source localization [15]. It has many applications in the field of detection and estimation, because some time we are only interested not in the full recovery of the signals but we just want sensing of the signals [16]. Sparsity and CS save the time in conventional in MRI. MRI scanners sample lines in spatial Fourier domain of the image and thus take the benefits of sparsity and CS [17].

However sparse signal processing is also of great importance in the field of signals (or images) recovery and signal (or images) de-noising. With the introduction of CS, sparse signal processing techniques are of more importance. Many researchers have use sparsity and CS in different contest for the signals recovery and de-noising problems. The signals that are sparse in some frequency domain such that wavelet and Fourier transform can be efficiently recovered as compare to the ordinary compression of data which leads to missing of information. DCT based sparsity is used for JPEG images and Wavelet for JPEG200 [18].

1.2 Proposed Work

Our focus is on the recovery of discrete time signals and images and also on the de-noising of images using wavelet based de-noising techniques and sparsity. This work will recover 1D discrete time sparse signal by means of the four thresholding techniques and also recover phantom image. In both cases this work compares the results of the four thresholding methods (i.e. soft, hard, garrote and logarithmic). For simple one dimensional signal recovery, it runs iterations for maximum error between the original and recovered signal and judge which method reduces the error first. In addition this work recovers the phantom images through the four thresholding techniques and compares their PSNR values. The Proposed technique produces best results among the four shrinkage functions. In the last part of this work, we de-noise phantom images through the four wavelet thresholding techniques along with median filtering. The parameter for quality checking is the correlation between the original and de-noised images. The images recovered through the wavelet based thresholding functions produce low value as compared to the median filtering used after thresholding. Only median filtering cannot recover the sparse signal but just can remove the impulsive noise. This work uses median filter in parallel with the wavelet based thresholding to recover and de-noise the sparse signals and images.

1.3 Thesis Organization

After introduction of the thesis, problem statement and proposed work, in chapter 2, literature review is given. In chapter 2, we simply take an overview of the classical and new techniques applied for signal (or image) sampling, recovery, de-noising, de-blurring, sparse signal processing, compress sensing, theoretical and mathematical description of sparse signal processing, CS and wavelet transform. Chapter 3 is dedicated to the discrete time sparse signal recovery problem, soft, hard, garrote and proposed thresholding techniques. In chapter 4, this research explains the new work. Applying the four thresholding techniques to phantom, sparse images for recovery and compare their results based on PSNR values. In the same chapter, the four thresholding techniques are used for de-noising of phantom image having salt and pepper noise. In chapter 5, results, conclusions, future work is discussed followed by references.

Chapter 2

LITERATURE REVIEW

2.1 De-noising Techniques

Noise reflects the meaning of unwanted signal in communication, signal and image processing. Noise is an information bearing signal, which shows the property of the system, e.g. heat from a laptop suggests that it is processing something heavy or its inner cooling fan is not working. Noise in communication and signal processing occurs due to different reasons such as due to the processing system and acquisition etc. Image restoration in image processing attempts to recover a full fledge de-noised version of the degraded, blurred and noisy images.

Images are often degraded by noise. Noise can occur during image capture, transmission etc. Noise removal is an important task in image processing. In general the results of the noise removal have a strong influence on the quality of the image processing techniques. Several techniques for noise removal are well established in image processing. The nature of the noise removal problem depends on the type of the noise corrupting the image. In the field of image noise reduction, several linear and nonlinear filtering methods have been proposed. De-noising of image is very important and inverse problem of image processing which is useful in the areas of image mining, image segmentation, pattern recognition and an important preprocessing technique to remove the noise from the naturally corrupted image by different types of noises. The wavelet techniques are very effective to remove the noise. This work reviews on noises like Salt & Pepper noise, aliasing noise (Gaussian noise) etc. and various techniques available for de-noising the image.

De-noising is a challenging problem for researchers from many years. Different de-noising techniques are used. Total variation de-noising (TVD) is a classical de-noising method is in use since 1990. It was first used by Usher and Fatemi. According to this method, difference of the neighboring pixels is calculated and then tries to minimize the "$l1$" and "$l2$" norms of the difference vector. The discretize Total Variation (TV) function of x is given by

$$TV(x) = \sqrt{(x[n] - x[n+1])^2} \quad (1)$$

Where $x[n]$ is the n^{th} pixel of the signal. The term $x[n] - x[n+1]$ is called the discrete gradient and a closed approximation of the high pass filter.

If "x" is the observed signal, corrupted by Additive White Gaussian Noise (AWGN) "n", then the noisy signal is given by

$$y = x + n, \qquad x, y, n \in R^n \quad (2)$$

TVD recover "x" by minimizing the function

$$J(x) = ||y - x||_2^2 + \lambda\, ||TV(x)||_1 \quad (3)$$

Where "λ" is the regularization parameter, which controls fluctuation in the signal [19].

If we filter the signal with Haar high pass filter and calculate $l1$ and $l2$ norm of the filtered signal, it reflects the same meaning as TVD.

In the light of the last paragraph an easy version of the Total Variation is the Filtered Variation (FV) method used for de-noising and sparse signal processing. According to this method, if the original signal is **x**, take transform (D) of it. Apply filter (H) on it and calculate the $l1$ or $l2$ norms of it or mathematically

$$FV_p(x) = ||HDx||_p \qquad p = 1,2 \quad (4)$$

Where "p" represents the order of norm, employed in the above equation. From simple TVD, it is better to use FV because of its application as to provide sparsity along with less mathematical complexity [20].

Besides the TVD and FV, there are many other techniques for signal (or image) de-noising including the whole family of filtering techniques, such as linear and non-linear filtering. Linear filters are also known as convolution filters as they can be represented using a matrix multiplication. Thresholding is an example of nonlinear operations, as is the median filtering. The linear filtering includes mean filtering [21], averaging filtering [22] etc. linear filtering replaces each pixel by linear combination of its neighboring pixels.

Wavelet based de-noising is the most efficient technique for signal (or image) de-noising due to its simplicity and sparsity applications. Most of the signal noise can be suppressed using wavelet based de-noising. Wavelet based de-noising is used in different forms, one of them is wavelet based thresholding. Wavelet based de-noising is easily applicable and most of the under-sampling noise can be removed through this method.

2.2 Wavelet Transform (WT)

A wavelet is a small function used to divide a given function or continuous-time signal into different scale components. Wavelets are mathematical functions that cup up data into different frequency components, and then study each component with a resolution matched to its scale. Wavelet transformation decomposes a signal into a set of basis functions. These basis functions are called wavelets.

The first wavelet related theory was given by Alfred Haar in 1909 in his dissertation "On the Orthogonal Function Systems" for his Doctoral Degree.

Wavelet is a relatively new theory it has enjoyed a tremendous attention and success over the last decade, and for a good reason. Almost all signals encountered in practice call for a time-frequency analysis and wavelets provide a very simple and efficient way to perform such an analysis. Still there is a lot to discover in this new theory, due to the infinite variety of non-stationary signals encountered in real life [23].

Wavelets are obtained from a single prototype wavelet ψ (t) called mother wavelet by dilations and shifting.

In general we use a scaling function to derive more wavelet basis i.e.

$$\Psi_{a,b}(t) = \left(\frac{1}{\sqrt{a}}\right) \psi\left(\frac{t-b}{a}\right) \quad (5)$$

Where "a" is the scaling parameter and b is the shifting parameter

2.2.1 Advantages of Wavelet Transform

i) Wavelet transform provides a way for analyzing waveforms in both frequency and time.

ii) Accurately represent the functions that have discontinuities and sharp peaks.

iii) Accurately deconstructing and reconstructing finite, non-periodic and non-stationary signals.

iv) WT allows signal to be stored more efficiently than by Fourier transform.

2.2.2 Limitations of Fourier Transform (FT)

i) WT and FT are different in time but same in frequency representation.

ii) FT only gives what frequency components exist in a signal.

iii) FT does not tell at what time the frequency components occur.

iv) Time-Frequency representation is needed in most cases.

v) WT gives both details i.e. in time and in frequency.

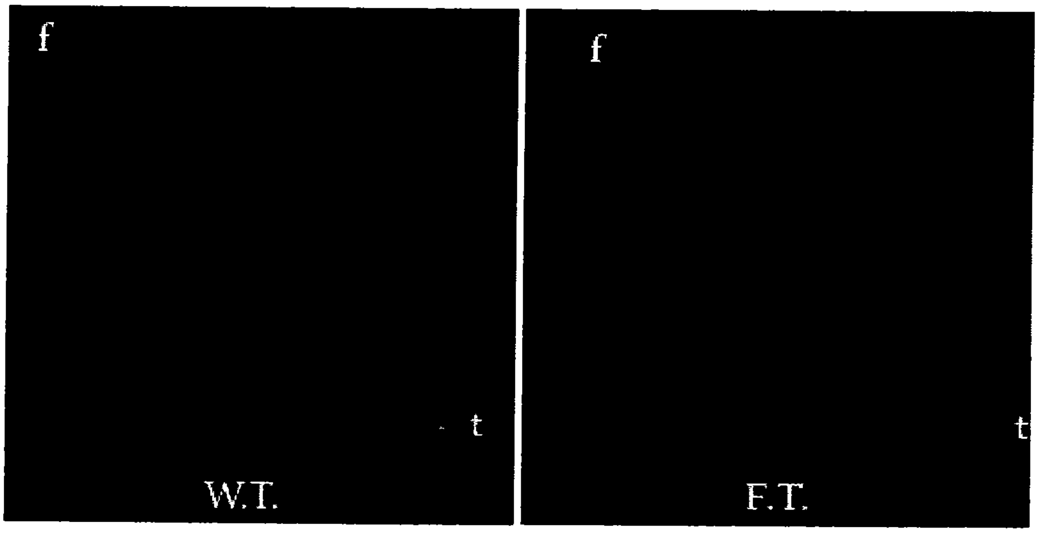


Figure 1: WT & FT, Frequency and Time representations

2.2.3 Types of WT

1) Continuous Wavelet Transform (CWT)

2) Discrete Wavelet Transform (DWT)

2.2.3.1 Continuous Wavelet Transform (CWT)

We construct different wavelet basis functions from a single scaling function $\phi(x)$. By translating and scaling this function we construct the other wavelet basis function. i.e. $\phi(cx - d)$. By increasing the value of "c", shrinks the function and by giving positive value to "d", we shift it to right.

The prototype wavelet can be given as a linear combination of the scaling function. The scaled and translated function $\Psi_{j,k}(x)$ can be given as

$$\Psi_{c,d} = \frac{1}{\sqrt{|c|}}\Psi\left(\frac{x-d}{c}\right) \quad (6)$$

Where c and d belongs to R, but $c \neq 0$

$$W(c,d)[f(x)] = \int_{-\infty}^{\infty} \psi_{c,d}^{*}\, f(x)dx = \psi_{c,d}|f(x) \quad (7)$$

Where $< \psi_{c,d} | f(x) >$ denotes the inner product or in other words it denotes the projection of $f(x)$ the scaling function "$\psi_{c,d}$" [24].

2.2.3.2 Discrete Wavelet Transform (DWT)

The main difference in CWT and DWT is the values, we assign to "c" and "d". In CWT there is no bound on the values of "c" and "d" while in DWT, we select the values from the set of integers.

$$\Psi_{c,d} = a_0^{\frac{j}{2}} \Psi\left(a_0^j - k\right) \tag{8}$$

Where $j, k \epsilon Z$, and a_0 is most of the time consider as 2.

A simple unit scaling function is given by

$$\psi(x) = \begin{Bmatrix} \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \end{Bmatrix} \tag{9}$$

In the discrete signal case we compute the Discrete Wavelet Transform by successive low pass and high pass filtering of the discrete time-domain signal. This is called the Mallat algorithm of Mallat-tree decomposition. The recently developed JPEG2000 standard is based on DWT while the first JPEG standard is based on DCT. The wavelet transform can be used to create smaller and smaller summary images, thus resulting in a Multi-resolution Analysis (MRA). In this type we do two things, one filtering and the other sampling using two types of filters, high and low pass filters.

Extension to 2D is very simple. 2D discrete wavelet transform (1D DWT applied alternatively to vertical and horizontal direction line by line) converts images into "sub-bands" Upper left is the DC coefficient, Lower right are higher frequency sub-bands [25].

The main story of wavelet starts from the introduction of Haar wavelet transform. It is the simplest wavelet transform for coarse to fine study of the images.

2.2.3.2.1 Haar Wavelet

The Haar wavelets are defined as:

$$\psi(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

The scaling function for Haar wavelet is given by

$$\phi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

It can be give $\psi(x)$ in terms of $\phi(x)$ as

$$\psi(x) = \phi(2x) - \phi(2x-1) \quad (12)$$

The image decomposition process is given as

Scale1:

In this section 4 sub-bands of image are made i,e. LL_1 , HL_1 , LH_1 and HH_1. Each coefficient has a 2x2 area in the original image. Low frequencies $0 < \omega < \frac{\pi}{2}$ and high frequencies are $\frac{\pi}{2} < \omega < \pi$.

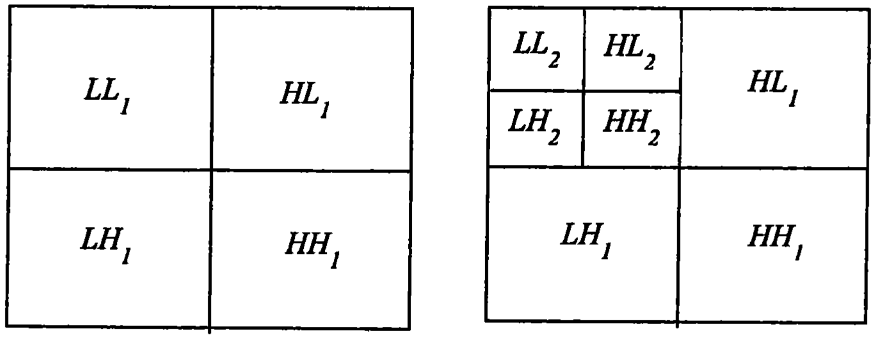


Figure 2: Image compression levels

Scale2:

Again 4 sub-bands are LL_2 , HL_2 , LH_2 and HH_2 , each coefficient take a 2x2 area in scale1 image. Low frequencies $0 < \omega < \frac{\pi}{4}$ and high frequencies $\frac{\pi}{4} < \omega < \frac{\pi}{2}$; the scale1 and scale2 is shown in figure 2.

The coefficients are parent, children, descendants and ancestors. Descendants are the corresponding coefficients at finer and ancestors are the corresponding coefficients at coarser. This is shown in figure 3.

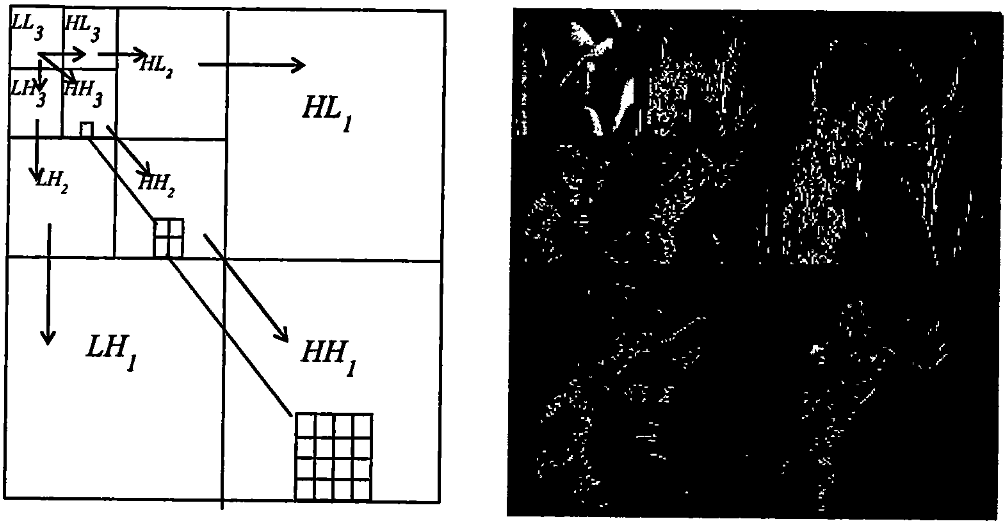


Figure 3: Image compression

2.2.3.2.2 Daubechies Wavelet

Another, type of wellknown wavelet is Daubechies wavelet. In this work we have used Daubechies4 wavelet.

In 1985 Stephen Mallat discovered a way to compare filtered compression and orthonormal wavelet based compression. Later Ingrid Daubechies developed a new method for discrete representations of the continuous wavelet transform called wavelet frames [26].

In wavelet transform, two types of filters are needed for a function. The high pass corresponds to the prototype function while the low pass filter corresponds to scaling function. The scaling function gives the resolution information of the wavelet function.

For the Daubechies4 scaling function wavelet coefficients are given by

$$\begin{Bmatrix} c_0 = \frac{1+\sqrt{3}}{4} \\ c_1 = \frac{3+\sqrt{3}}{4} \\ c_2 = \frac{3-\sqrt{3}}{4} \\ c_3 = \frac{-(\sqrt{3}-1)}{4} \end{Bmatrix} \qquad (13)$$

The corresponding prototype function has a recursive relation with scaling function, i.e.

$$\psi(x) = -c_3\phi(2x) + c_2\phi(2x-1) - c_1\phi(2x-2) + c_0\phi(2x-3) \qquad (14)$$

Chapter 3

COMPRESSED SENSING

3.1 Motivation

Practically in the vast majority of applications data acquisition is based on Shannon/Nyquist sampling theorem that requires sampling rate at least twice the messages signal bandwidth in order to achieve exact recovery and prevent aliasing [27]. This requirement is not practical for video industry since the signal bandwidth is very wide and the technology is not feasible to achieve necessary rates in order to satisfy the Shannon/Nyquist sampling theorem. There is a significant class of signals (pictures for example) that are compressible, not all the data is necessary to transmit in order to get 'good enough' representation of the original message. Practical solution introduces lossy compression processing at the source level.

Compressed sensing (CS), also known as compressed sensing, compressive sampling has opened a new era of signal processing. Compressed sensing is new method to capture and represent compressible signals at the rate well below Nyquist's rate. It uses random measurement matrix, preserves the signal structure (length or the sparse vectors is conserved) and reconstruct the signal from the projections using optimization process ($l1$ *norm*) [28].

Compressed sensing is a new theory of sampling. According to the old sampling theory, huge information is acquired by sampling, but most of it is waste. E.g. sample at pixels ($1000 \times 1000 = 10^6$), we will get raw image. According to compressed sensing, compute wavelet coefficients. They decay fast. Retain only few largest (100) pixels and discard the rest as waste.

Compressed sensing is the paradigm-busting field in the field of mathematics and engineering that reshaping the way people work with large data sets. Only nine years old, CS has already inspired more than a thousand papers and pulled in millions of dollars in federal grants. In 2006, Candes work on the topic was rewarded with the $ 500000 Waterman Prize, the highest honor bestowed by the national Science Foundation. It is not hard to see why? Imagine MRI machines that take seconds to produce images that used to take up to an hour, military software that is vastly better at intercepting an antagonist's communications and sensors that can analyze distant interstellar radio waves. Suddenly, data becomes easier to gather, manipulate, and interpret.

3.1.1 Digital Image

An image is a two-dimensional function, f(x, y), where x and y are spatial coordinates. When x, y and the amplitude values of f are all finite, discrete quantities, we call the image a digital image.

3.1.2 Digital Image Processing

It refers to processing digital images by means of a digital computer. The digital image is composed of finite number of elements, each of which has a particular location and values. These elements are referred to as picture elements, image elements and pixels.

3.1.3 Image Compression

Digital images usually require a very large number of bits, this cause critical problem for digital image data transmission and storage. It is the Art and Science of reducing the amount of data required to represent an image. It is one of the most useful and commercially successful technologies in the field of Digital Image Processing (DIP).

3.2 Compressed Sensing Theory

There are two types of compression

1. Lossless

   This is digitally identical to the original image and only achieves a modest amount of compression.

2. Lossy

This type of compression only discards components of the signal that are known to be redundant. Signal is therefore changed from input.

The graphical representation and flow chart of Image compression is given below in figure 4 followed by Lossy compression figure 5.

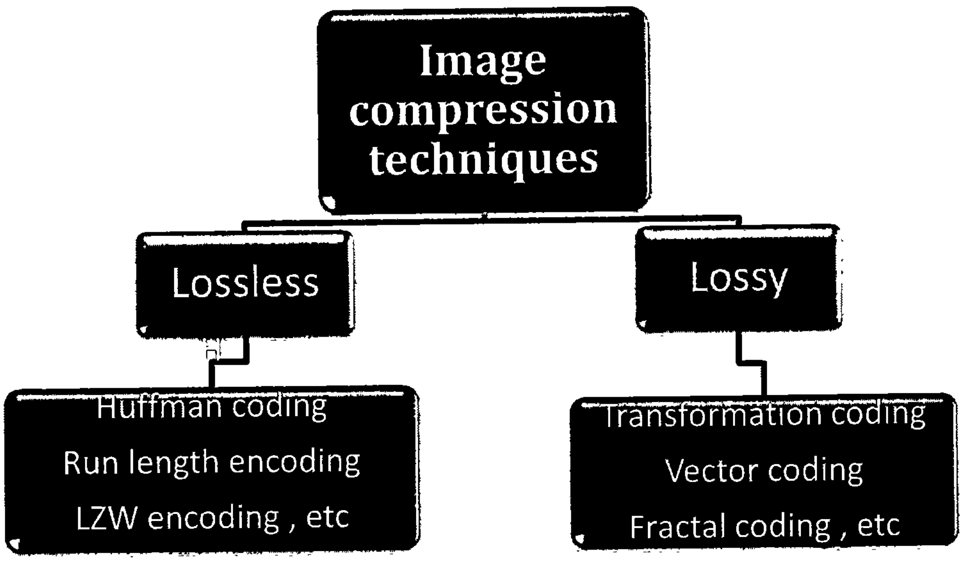


Figure 4: Image compression techniques

Lossy compression is further extended to its components, as shown in figure 6.

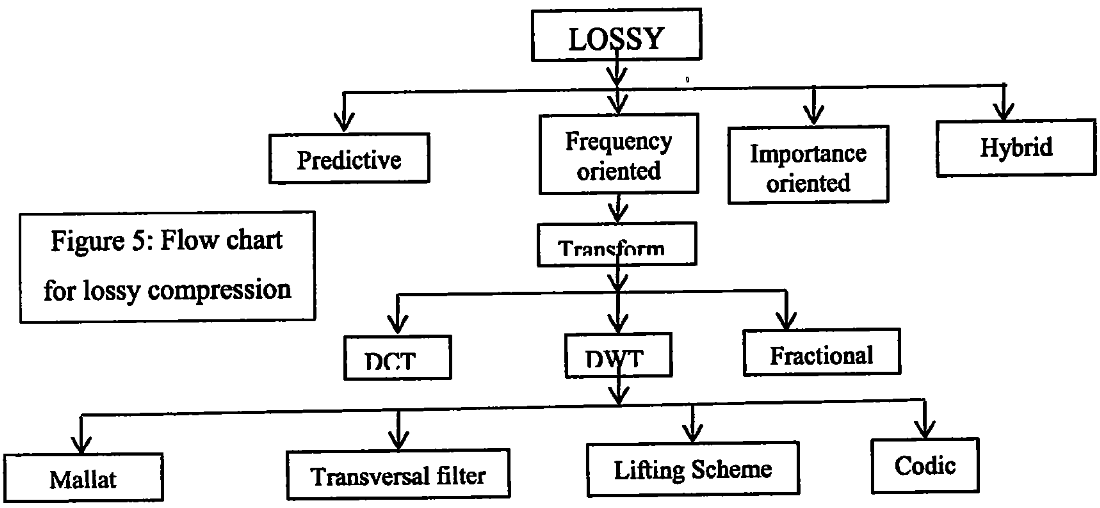


Figure 5: Flow chart for lossy compression

3.3 Compressed Sensing Mathematically

Let x is a signal of dimensions Nx1 is K sparse in a basis or dictionary "Ψ_{NxN}", then it is possible to fully recover the signal x by a measurement "y_{Mx1}" taken by "Φ_{MxN}", where M<<N under some conditions [29]; i.e.

For

$$x = \sum_{i=1}^{N} v_i \psi_i \tag{15}$$

$$\text{Take} \quad y = \Phi x \tag{16}$$

Where ψ_i are basis vectors of orthonormal Ψ_{NxN} , and vector v_i has only K non-zero elements, K<<N and measurement matrix Φ satisfies Restricted Isometric Property (RIP) [30], where $M \geq cKlog\left(\frac{N}{K}\right) << N$

3.3.1 Stepwise Explanation of the Method

According to conventional coding method, compress/ reconstruct the signal through discarding sufficiently small coefficients.

$$x = \Psi v \approx \sum_{i \in K \, largest} v_i \psi_i \tag{17}$$

CS theorems state that, for signals that are sparse in some domain, they can be fully reconstructed using only few designed measurements. Number of measurements depends on sparsity. In order to recover the sparse signal, the measurements should be taken from a measurement matrix whose product with the basis satisfies RIP [31].

3.3.2 Restricted Isometric Property (RIP)

A matrix "A" is said to satisfy the RIP of order K with isometry constant "δ_k" , which is not too close to one such that

$$(1 - \delta_k)||v||_2^2 \leq ||Av||_2^2 \leq (1 + \delta_k)||v||_2^2 \tag{18}$$

In other words, *"A"* approximately preserves the Euclidean length for K-sparse signals and all subsets of K columns, taken from *"A"* are nearly orthogonal.

It can be explained as; the order of $K(K \leq M)$, and $0 < \delta_k < 1$, for A to obey RIP of order K, no K-sparse signal can lie in the null space of A. (If it did then we would have $||Av||_2^2 = 0$, and that obviously does not preserve the squared magnitude of the vector v)

3.3.3 The Measurement Matrix

Key Problem:

How to design the measurement matrix Φ such that $A = \Phi\Psi$ satisfies RIP of order 2K?

However in practice, there is no computationally feasible way to check RIP property for a given matrix Φ. i.e. Require $\binom{N}{K}$ checks for all non-zero K elements.

Solution:

Pick random matrices which are usually incoherent with any fix basis Ψ.

Examples:

1) Random Gaussian
2) Random Bernoulli
3) Hadamard Matrix
4) Random Sparse

Hadamard Matrices are given by

$$H_1 = [1], \qquad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix}$$

Hadamard Matrices exist of orders 1,2 and 4 or of a multiple of 4. Hadamard matrices of interest, we can define codes based on the rows of the Hadamard matrix.

3.3.4 CS Recovery

Reconstruction or decoding Problem:

The formulation is given by

$$y = \Phi\Psi v \tag{19}$$

Where "Φ" is called the measurement (or sensing) matrix. "Ψ" is assumed to having sparsifying basis and "v" is the target sparse representation. Such that $\dim(y) << \dim(x)$, this is an ill-posed problem in general.

3.3.5 CS Recovery Algorithms

Signal recovery algorithm aims to find signal's sparse solution.

1) $L_2 - norm$ (energy) minimization:

$$\min||v||_2 \quad \text{Subject to} \quad Av = y \tag{20}$$

$$v = (A^T A)^{-1} A^T y \tag{21}$$

Equation (21) is called the pseudo inverse of (20). The pseudo-inverse is a closed form but is not a sparse solution.

2) $L_0 - norm$ minimization:

$$\min||v||_0 \quad \text{subject to} \quad Av = y \tag{22}$$

Through, $L_0 - norm$, we can exactly recover the K-sparse signal with high probability from $M = K + 1$ measurements but solving it is NP-complete. It need exhaust search, all $\binom{N}{K}$ positions non-zero coefficients.

As signal reconstruction algorithm tries always to find the sparsest solution. That's why this work uses $L_1 - norm$ minimization.

3) $L_1 - norm$ minimization:

$$\min||v||_0 \quad \text{subject to} \quad Av = y \tag{23}$$

Through, $L_1 - norm$, we can exactly recover K-sparse signal with high probability using only $M \geq cKlog\left(\frac{N}{K}\right) \ll N$ measurements (Mild Oversampling)

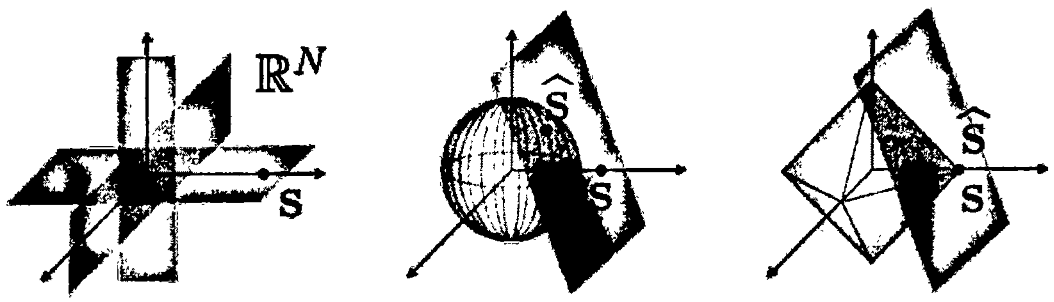


Figure 6: Graphical representation of L_0, L_2 *and* $L_1 - norms$

$L_1 - norm$ Minimization is using different techniques, such as Basis Pursuits (BP) [32], Matching Pursuits (MP) [33] etc. This work does $L_1 - norm$ minimization using iterative wavelet based thresholding.

3.3.6 Wavelet Based Image Compression and De-noising

Wavelet based thresholding techniques consist of some classical and the recently introduced shrinkage functions. D.L. Donoho used hard shrinkage function for the first time. According to this rule the coefficients below the threshold level are discarded while keep the remaining unchanged. Another form of hard shrinkage function was proposed later by D.L. Donoho in 1995, called soft shrinkage function [34]. Later on a new shrinkage function named garrote, was examined by Hong-Ye Gao in his work [35]. Although these all shrinkage functions produce good results but still there are some problems which could not be handled through the classical shrinkage functions. A new proposed shrinkage function with some modifications called logarithmic shrinkage function, is used in this work for images recovery and de-noising.

Many researchers for de-noising and regularization use the Tychonov penalty to estimate the signal from degraded data. Actually an attempt is made to solve the optimization problem,

$$\text{argmin} \frac{1}{2} ||x-y||_2^2 + \lambda \frac{1}{2} ||x||_2^2 \quad (24)$$

This equation has a close form solution as under.

$$\hat{x} = \frac{1}{1+\lambda} y \quad (25)$$

Where "x" is the original signal, "$\hat{x}$" is the estimated signal,"y" is the noisy signal and "λ" is the regularization parameter.

This work compares the four thresholding techniques for recovery of under-sampled sparse images and also uses these techniques in combination with median filtering to recover the sparse images, degraded by salt and pepper noise. In the first portion of this work, take the wavelet transform of under-sampled Shepp-Logan Phantom image, apply the four thresholding techniques individually and then recover the image by taking the Inverse Wavelet Transform of the thresholded image. Proposed shrinkage function shines with the best results and gives high PSNR values as compare to hard, garrote and soft shrinkage functions. In the second part of this work, take Shepp-Logan Phantom image and add salt and pepper (impulsive) noise to it. Taking the wavelet transform of the noisy image, threshold it and then iteratively recover the sparse image through hard, soft, garrote and proposed shrinkage functions. Through these techniques this work can recover the sparse images but cannot satisfactorily de-noise them, when they are corrupted by impulsive noise, that's why this work applies median filtering in addition with the thresholding techniques to get better results.

3.3.6.1 Wavelet Based Thresholding

Basically it solves the following optimization problem for de-noising and regularization in the first portion of this work using $L_1 - norm$ minimization.

$$\text{argmin} \frac{1}{2} ||x-y||_2^2 + \lambda \frac{1}{2} ||x||_1 \quad (26)$$

This applies iterative hard, garrote and soft shrinkage functions.

Let $\hat{X}_u$ is the under-sampled Fourier transform of "$\hat{x}$". At the start it is supposed that $\hat{X}_0 = Y$.

1. Compute the inverse Fourier Transform of $\hat{X}$ as $\hat{x}_i = F'(\hat{X}_i)$

2. Perform Wavelet transform to obtain sub-bands coefficients.

3. Threshold all low frequency sub-band coefficients using certain shrinkage function in the Wavelet domain $\hat{x}_t$= Thresh $(W * \hat{x}_i, \lambda)$

4. Compute the inverse Wavelet transform to recover the noisy image $\hat{x}_i = W' * \hat{x}_i$

5. Compute the Fourier transform $\hat{X}_i = F(\hat{x}_i)$

6. Enforce data consistency in the transform domain $X = X(Y == 0) + Y$

7. Run the iterations until $||\hat{x}_{i+1} - \hat{x}_i|| < \varepsilon$

Where "ε" is stopping criteria.

This work applies the same procedure for garrote and soft shrinkage functions. The above method is a Projection onto Convex Sets (POCS) type algorithm. Apply twenty iterations to force the data consistency and get the satisfactory results. Compare the thresholding techniques by means of PSNR.

3.3.6.2 Thresholding Techniques

This chapter will discuss the classical shrinkage functions. The proposed shrinkage function is discussed in the next chapter.

3.3.6.2.1 Hard Thresholding

Hard thresholding works on the principle of keep or kill rule. It removes the wavelet coefficients whose absolute values are below the threshold value and keep the remaining. It doesn't change the values of coefficients above the threshold value [36]. It is formulated as:

$$x_{ht} = \begin{cases} 0 & \text{if } |x| \le \lambda \\ x & \text{if } |x| > \lambda \end{cases} \quad (27)$$

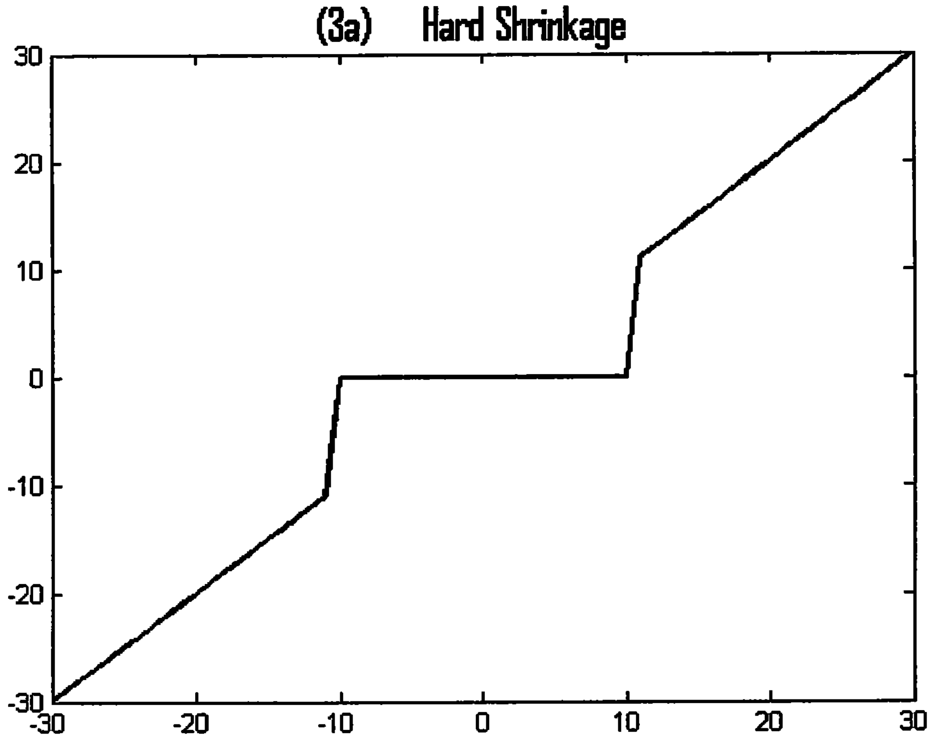


Figure 7: Hard shrinkage function

3.3.6.2.2 Garrote Thresholding

Garrote thresholding adopts a modest way between hard and soft thresholding, and is a good compromise between hard and soft thresholding .It is more flexible than hard threshold and continuous like soft threshold, therefore it is more stable than hard threshold and soft threshold [37]. It is represented as:

$$x_{gt} = \begin{cases} 0 & \text{if } |x| \le \lambda \\ x - \frac{\lambda 2}{x} & \text{if } |x| > \lambda \end{cases} \qquad (28)$$

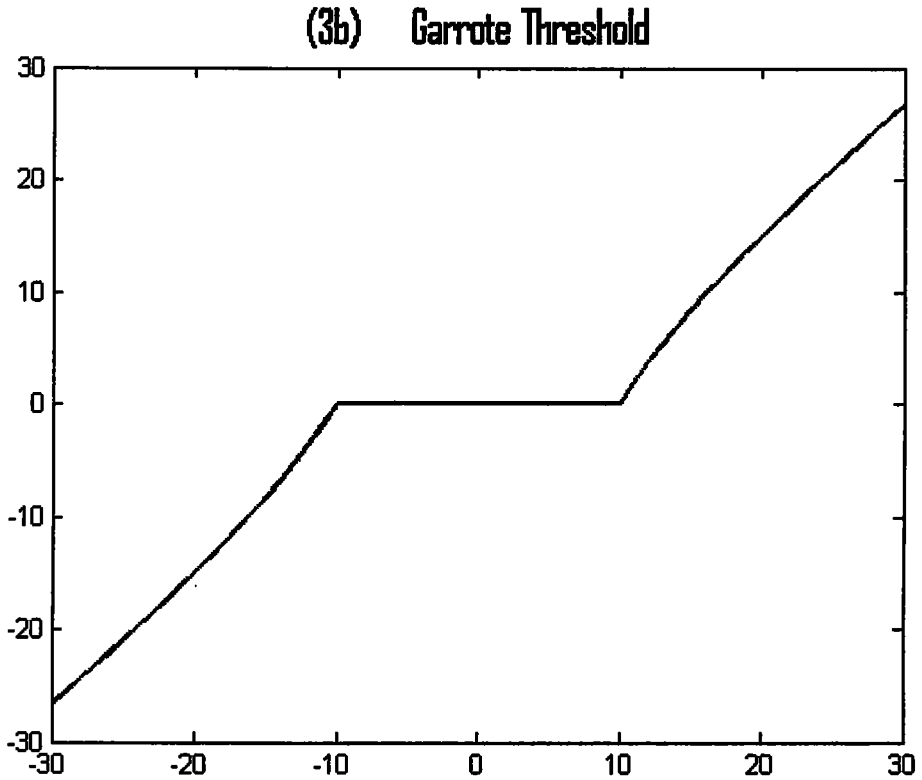


Figure 8: Garrote shrinkage function

3.3.6.2.3 Soft Thresholding

Actually soft threshold is an extension of hard threshold. It is continuous shrinkage function, produces better results than hard and garrote threshold in our experiments [38]. Soft shrinkage function is formulated as:

$$x_{st} = \begin{cases} 0 & \text{if } |x| \leq \lambda \\ x\frac{|x|-\lambda}{|x|} & \text{if } |x| > \lambda \end{cases} \tag{29}$$

Equation (29) is compact form of soft shrinkage function.

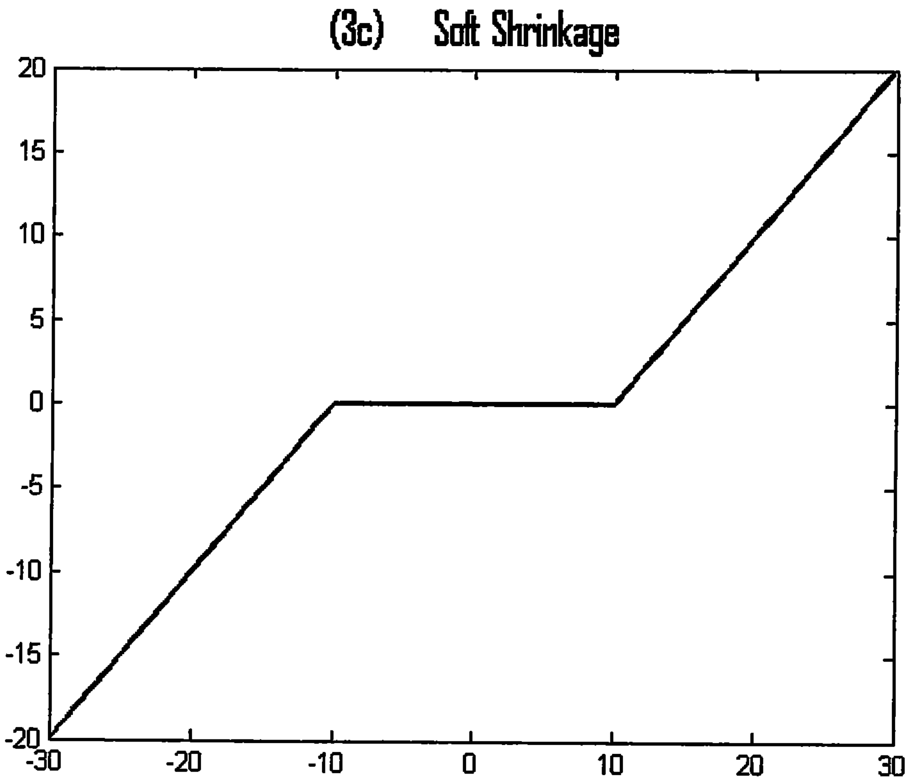


Figure 9: Soft shrinkage function

3.3.6.3 Proof of Soft Shrinkage Function

Taking the following problem

$$\text{argmin } ||x - b||_2^2 + \lambda||x||_1 \qquad (30)$$

As it is known that it has three unique solutions

$$\text{argmin } ||x - b||_2^2 + \lambda||x||_1 \qquad \text{when } x > 0 \qquad (31)$$

$$\text{argmin } ||x - b||_2^2 - \lambda||x||_1 \qquad \text{when } x < 0 \qquad (32)$$

$$\text{argmin } ||x - b||_2^2 - \lambda||x||_1 \qquad \text{when } x = 0 \qquad (33)$$

Differentiating these three equations with respect to x,

Differentiating (31);

$$\frac{d(||x-b||_2^2+\lambda||x||_1)}{dx}=0$$

$$\Rightarrow \frac{d(x^2-2bx+b^2+\lambda x)}{dx}=0$$

$$\Rightarrow 2x-2b+\lambda=0$$

$$\Rightarrow x=b-\frac{\lambda}{2}$$

According to (31), $x > 0$, so

$$b-\frac{\lambda}{2}>0$$

$$\Rightarrow b>\frac{\lambda}{2} \tag{34}$$

Differentiating (32),

$$\frac{d(||x-b||_2^2-\lambda||x||_1)}{dx}=0$$

$$\Rightarrow \frac{d(x^2-2bx+b^2-\lambda x)}{dx}=0$$

$$\Rightarrow 2x-2b-\lambda=0$$

$$\Rightarrow x=b+\frac{\lambda}{2}$$

According to (32), $x < 0$, so

$$b+\frac{\lambda}{2}<0$$

$$\Rightarrow b<-\frac{\lambda}{2} \tag{35}$$

Differentiating (33),

$$\frac{d(||x-b||_2^2)}{dx}=0$$

$$\Rightarrow \frac{d(x^2 + b^2 - 2bx)}{dx} = 0$$

$$\Rightarrow 2x - 2b = 0$$

$$\Rightarrow 2b = 2x$$

$$\Rightarrow b = x$$

From (33), we have, $x = 0$; so

$$\Rightarrow \; b = 0 \tag{36}$$

Soft thresholding operator becomes;

$$S_\lambda(b) = \begin{cases} b - \frac{\lambda}{2}, & \text{if } b > \frac{\lambda}{2} \\ b + \frac{\lambda}{2}, & \text{if } b < -\frac{\lambda}{2} \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

Compact form of (37) in terms of x is given by,

$$S_\lambda(x) = \begin{cases} 0 & \text{if } |x| \leq \lambda \\ x\frac{|x|-\lambda}{|x|} & \text{if } |x| > \lambda \end{cases} \tag{38}$$

Chapter 4

PROPOSED WORK

4.1 Overview

Although the shrinkage functions discussed in chapter 3, produce good results but still there are some problems which could not be handled through the classical shrinkage functions. A new proposed shrinkage function with some modifications called logarithmic shrinkage function is used in this work for images recovery and de-noising.

The first portion of this research applies the proposed shrinkage function called logarithmic shrinkage function to the phantom image. The quality factor for checking the performance is Peak Signal-to-Noise Ratio (PSNR). The proposed function produces better PSNR values than the other three classical shrinkage functions.

4.2 Proposed Method

This research proposes a new shrinkage function, the Logarithmic Shrinkage Function (LSF). Repeating the procedure in chapter two for wavelet based thresholding using LSF. The experimental results show that the proposed method is 7-10% better in PSNR values than the classical methods. The graphs of PSNR are given in results section.

The previous three techniques are used continuously from a few decades. This is a new shrinkage function. It produces the best results due to the usage of "log" function. The mathematical form of the proposed function is given by

$$x_{pt} = \begin{cases} 0 & \text{if } |x| \leq \lambda \\ |x| - s\log\left(1 + \frac{|x|}{s}\right) & \text{if } |x| > \lambda \end{cases} \quad (39)$$

The values of "s" are small positive constants. Equation (26) is, an optimization problem and the thresholding functions try to find the global minima. In the case of logarithmic shrinkage function, it finds local minima instead of global minima. This is an iterative $l1$-norm approach. Due to direct relation with $l_1 - norm$ it produces better results in recovery than soft thresholding and is also better in performance than hard shrinkage thresholding due to the discontinuity of hard shrinkage function. $l_1 - norm$ is a pointy function and having a great ability to provide the sparse solution. In case of $l_2 - norm$, it can find pseudo inverse but having no sparsity while in case of $l_0 - norm$, it can recover the exact sparse signal but solving it, is NP-complete. It need exhaust search, all $\binom{N}{K}$ are non-zero coefficients.

Logarithmic function is the extension of exponential function. The exponential functions have derivatives exist at origin. That's why the proposed function produces better results for sparse signals recovery. For more flexibility in (39), it can select two different values of thresholding for positive and negative values of "x". The derivative of the exponential function is given below.

4.2.1 Derivative of Exponential Function

Let $$f(x) = a^x \quad (40)$$

From the basics of derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad (41)$$

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \quad (42)$$

$$f'(x) = \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \quad (43)$$

$$f'(x) = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} \quad (44)$$

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h} \qquad \text{such that } a^x \text{ is constant w.r.t "h"} \tag{45}$$

If $x = 0$ then $f(0) = a^0 = 1$, So $f'(0) = \lim_{h \to 0}(a^h - 1)/h$

Or $$f'(x) = f'(0)a^x \tag{46}$$

The original shape of the log thresholding function is given below,

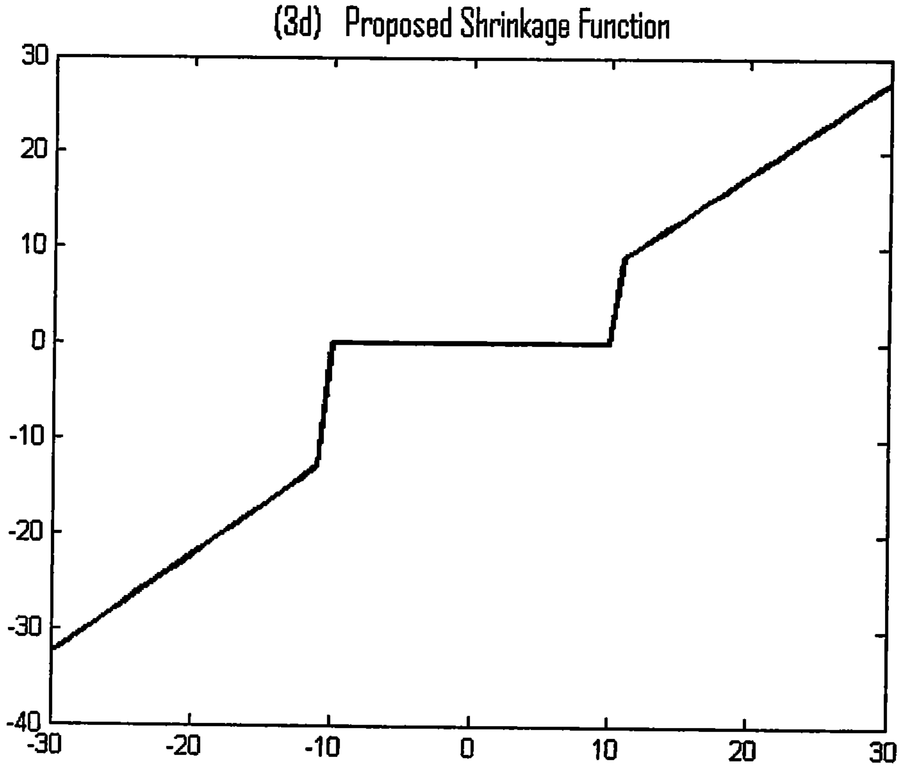


Figure 10: Proposed shrinkage function

4.3 Proposed Model for Noisy Images

In addition with the sparse images recovery (section 3.3.6, section 4.2), in this section, we de-noise phantom image by means of wavelet based thresholding and median filtering having salt and pepper noise. Because only thresholding is very less effective in case of impulsive noise and give the correlation values of the recovered images closed to the noisy image. In signal and image processing median filter is frequently used for de-noising of signals and images because of its edge preserving properties. But in this work uses thresholding technique in addition with median filter to recover and de-noised sparse images. It applies wavelet based thresholding techniques, by means of which it recovers the sparse images and then apply median filter to eliminate the salt and pepper noise from the images. Flow chart of the technique for sparse and noisy images recovery and de-noising is given in Figure12.

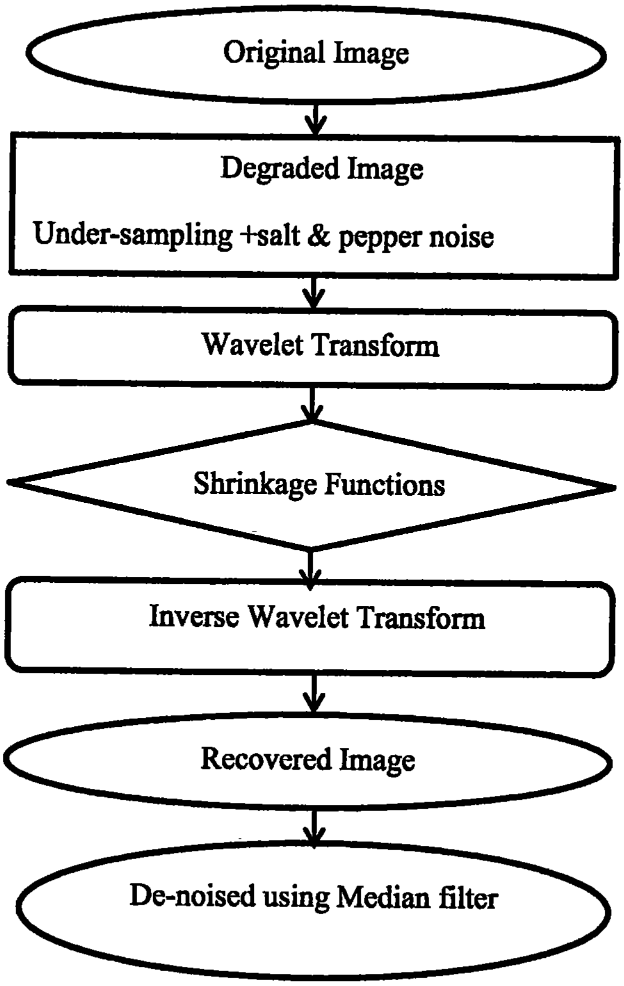


Figure 11: Flow chart of the proposed de-noising method

4.4 Basic Terms

4.4.1 Peak Signal-to-Noise Ratio (PSNR)

Mathematically PSNR is given by;

$$\mathrm{PSNR} = 10\log_{10}\mathrm{Max}/\mathrm{MSE} \quad (47)$$

And MSE is defined as

$$\mathrm{MSE} = \sum_{M,N}[I_1(m,n) - I_2(m,n)]^2 \quad (48)$$

Where M, N are the number of rows and columns in the input images.

4.4.2 Correlation

It is the statistical measure that indicates the extent to which two or more images fluctuate together. A positive correlation indicates the extent to which those variables increase or decrease in parallel, a negative correlation indicates the extent to which one variable increase as the other decreases.

Mathematically

$$F \circ I(x) = \sum_{i=-N}^{N} F(i)I(x+i) \quad (49)$$

For this notation, index F from –N to N.

$$F \circ I(x,y) = \sum_{j=-N}^{N}\sum_{i=-N}^{N} F(i)I(x+i) \quad (50)$$

4.4.3 Median Filtering

The average filtering leads to blur edges and details in an image and are not effective in case of salt & pepper (impulsive) noise. It is type of non-linear filtering. The gray level of each pixel is replaced by the median of its neighbor. It produces good results at de-noising (salt & pepper noise / impulsive noise). It takes the median value instead of the average or weighted average of pixels in the window.

Median sort all the pixels in an increasing order, take the middle one. The window structure for median filtering does not need to be a square. Special shapes can preserve line structures.

Chapter 5

DISCUSSIONS

5.1 Results

This work presents four thresholding techniques for signal and image de-noising using wavelet transform. In case of one dimensional signal, take a discrete time sparse signal having five non-zero samples out of 128. The magnitude of the samples is 1,2,3,4 and 5. Normalize it. In the next step add Gaussian noise of density 0.05 to the original signal. Under-sample the signal as uniform and random individually. Apply the soft shrinkage function to uniform and random under-sampled signal.

Repeat above procedure for hard and garrote. Draw the iteration versus max ($|x - x_e|$). The garrote shrinkage function decays very fast. The graphical results are shown in figure13 and 14.

Next, take Shepp-Logan Phantom image of size 256 x 256. Individually apply the four shrinkage functions on Shepp-Logan Phantom. Performance of the techniques is judged by the PSNR values and correlation values.

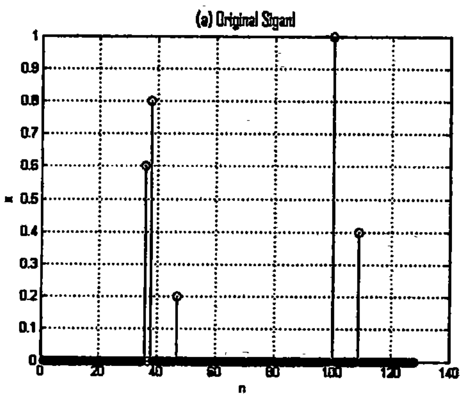
PSNR of the noisy Shepp-Logan Phantom image is 68dB. The PSNR value achieved by hard, garrote and soft shrinkage functions are 81.5dB, 83.4dB, and 85.5dB respectively, while the PSNR value achieved by the proposed technique is 87.0dB, which is greater than the other three techniques.

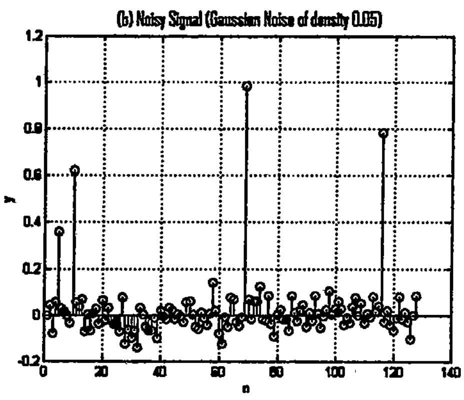
For phantom image, hard threshold gives satisfactory results at "$\lambda = 0.35$", garrote threshold is comparatively better at $\lambda = 0.20$ from hard while soft threshold shows the best results at "$\lambda = 0.070$". The proposed shrinkage function gives 87.0 dB, which is the best of all. The values of PSNR and λ for hard, garrote, soft and proposed methods are given in table.1.

From the behavior of shrinkage functions it is clear that the proposed shrinkage function is the best among the other shrinkage functions, followed by soft, garrote and hard. Different values of "s" can be selected as $o < s < 1$. This is the additional parameter along with the "log function" which makes the proposed technique more flexible among the others. The justification is as;

The original and recovered images through the thresholding techniques are given in Figure15. Table.1 shows the PSNR values of the four shrinkage functions. The PSNR curves are given in Figure16.

In the second portion of our work median filtering is applied in parallel with the thresholding. First apply the shrinkage functions on the sparse noisy images through which the images are recovered but still having the impulsive "salt and pepper" noise. Then apply median filtering to eliminate the salt and pepper noise. This hybrid technique is very effective in case of proposed thresholding, and gives high correlation values as compared to the other three thresholding techniques. The noisy and de-noised images are given in Figure17. Evaluated the images based on the correlation values. The correlations values of images, de-noised through these techniques and through median filtering are given in Table.2





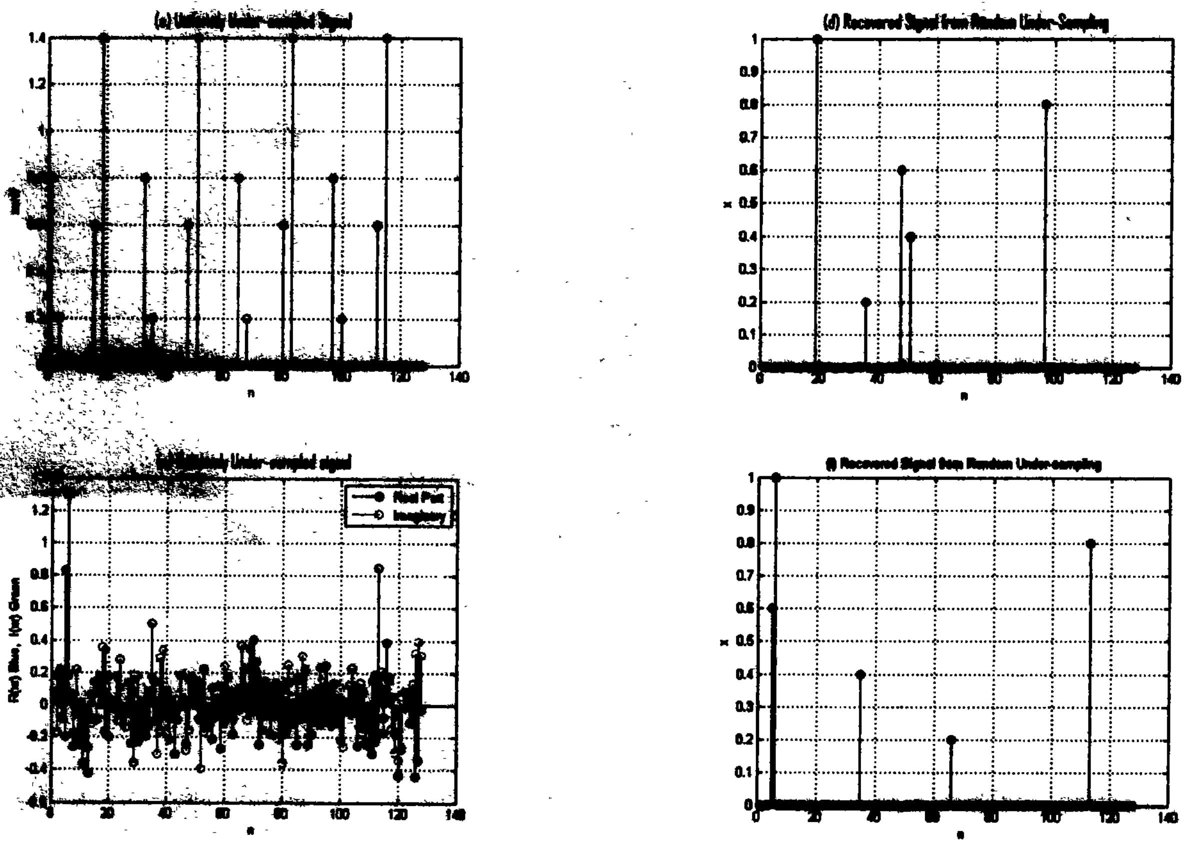


Figure 12: Original, Noisy, under-sampled and recovered signals

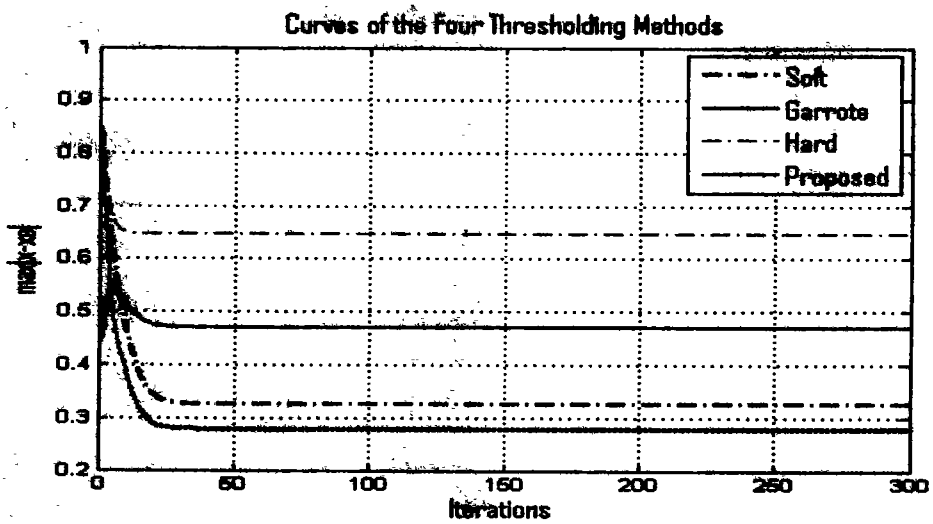
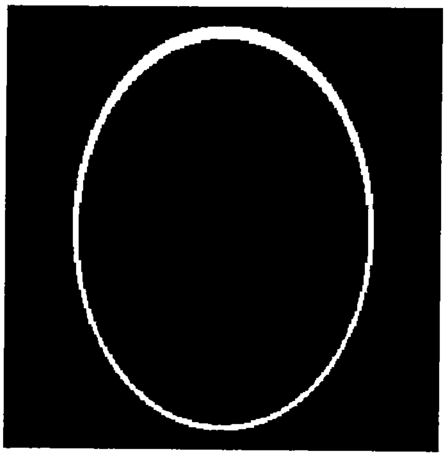
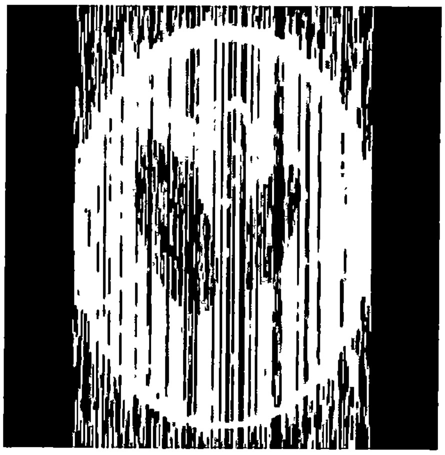


Figure 13: Errors versus iterations

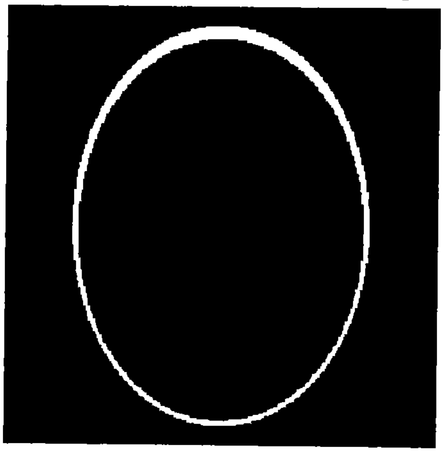
(a) Original Image



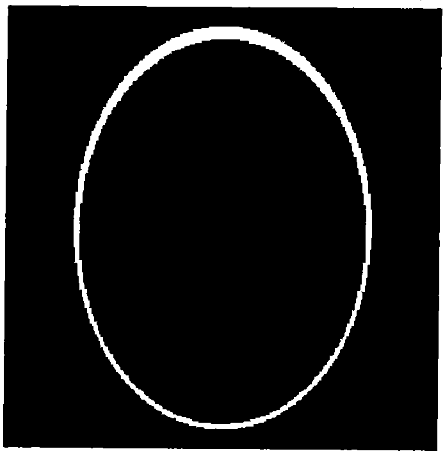
(b) Noisy Image



(c) De-noised Image, Hard Thresholding



(d) De-noised Image, Garrote Thresholding



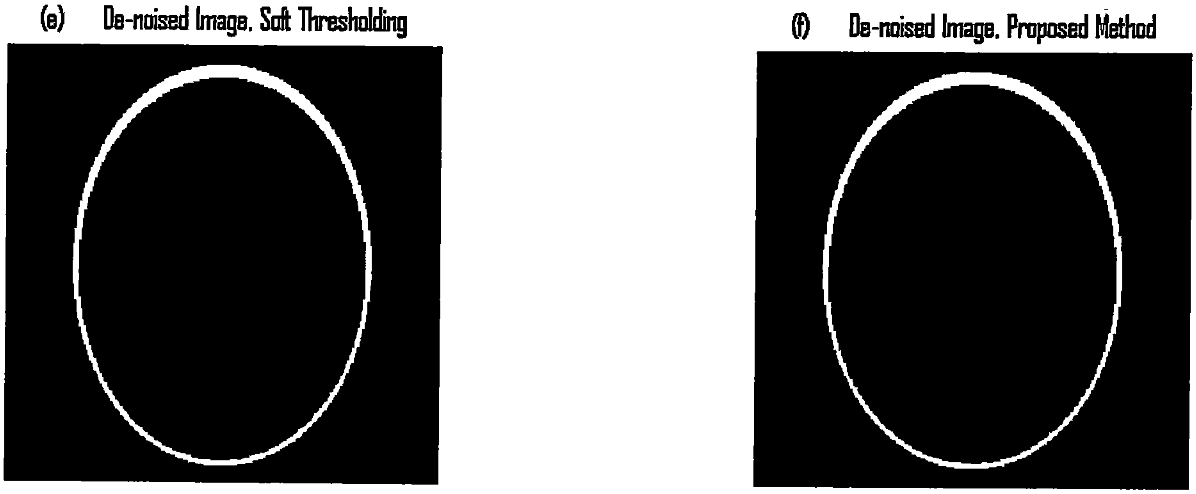


Figure 14: Original Phantom, Noisy and De-noised Images

Table 1: Under-sampling in frequency domain leads to Gaussian noise

Shepp-Logan Phantom Image		
Shrinkage Functions	λ	PSNR values
Hard	0.35	81.6
Garrote	0.20	83.4
Soft	0.07	85.5
Proposed	0.10	87.0

Table 2: Correlation values of the noise and recovered images with the original image

Salt & Pepper Noise	Noise Density	0.10
Correlations of the Noisy and De-noised Images with the Original Image.	Noisy	0.6884
	Soft	0.6966
	Soft+Median	0.9761
	Hard	0.6865
	Hard+Median	0.9681
	Garrote	0.6786
	Grrote+Median	0.9551
	Proposed	0.7046
	Proposed+Median	0.9853

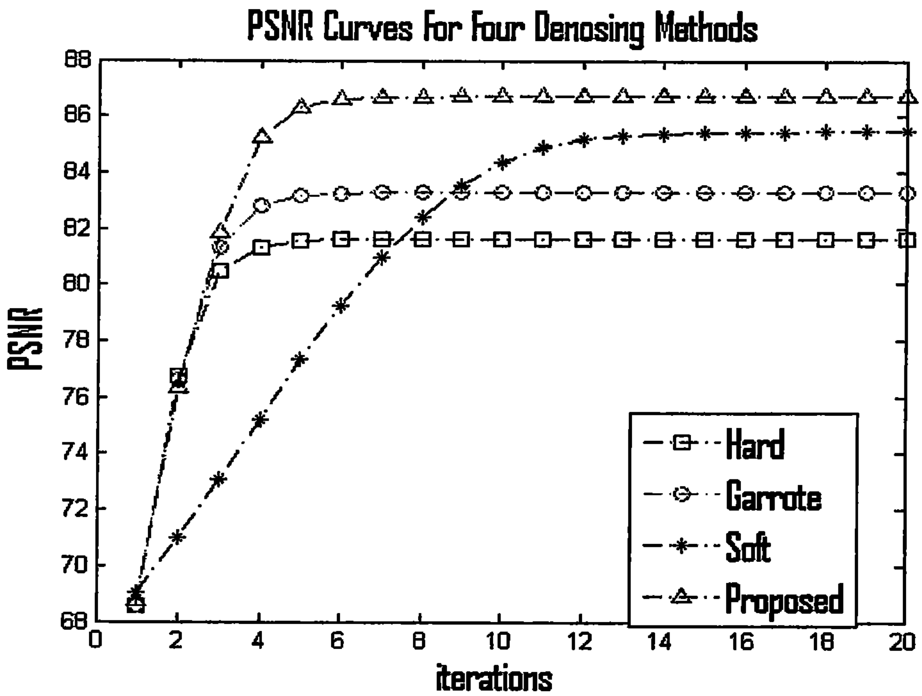
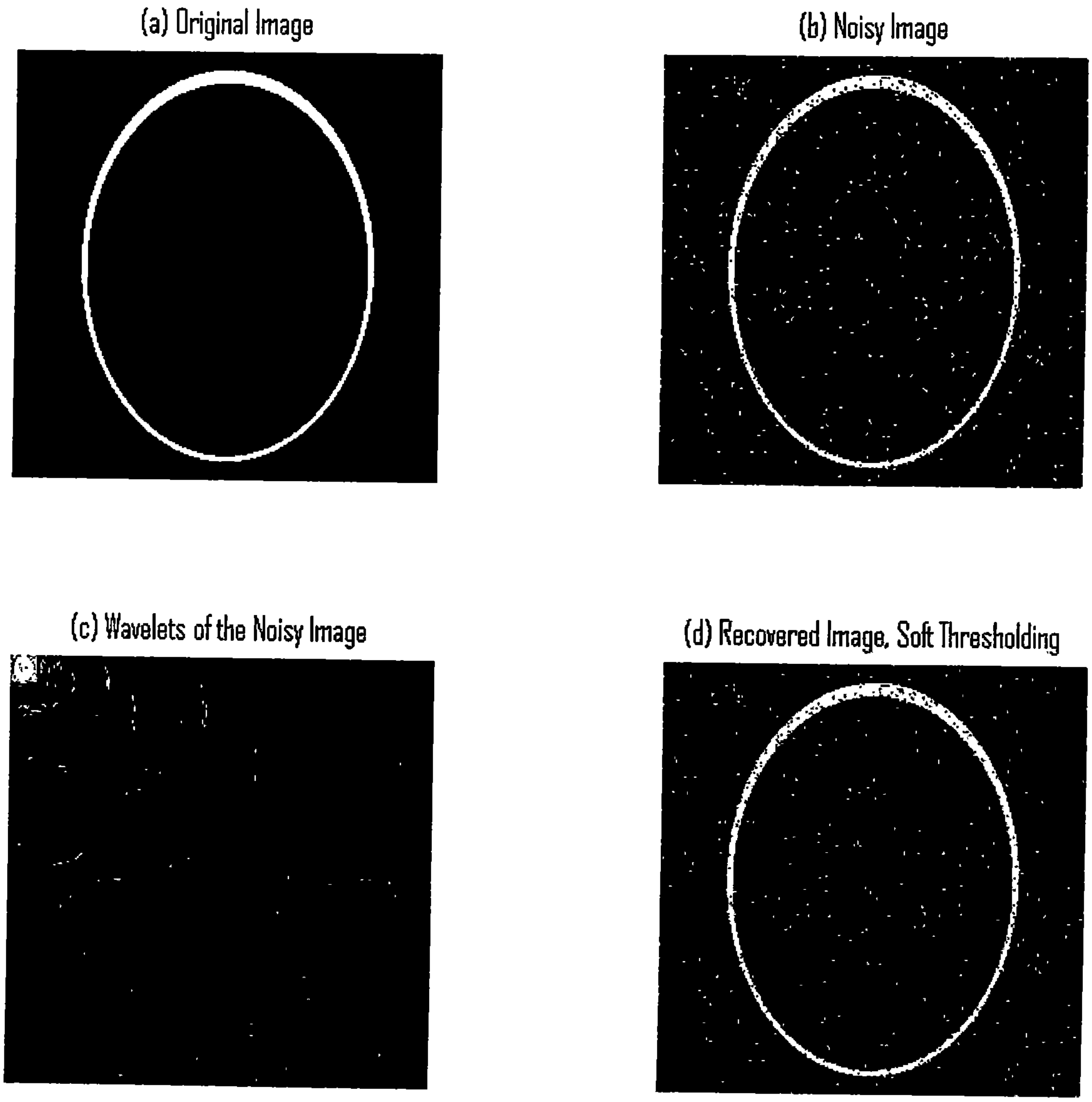
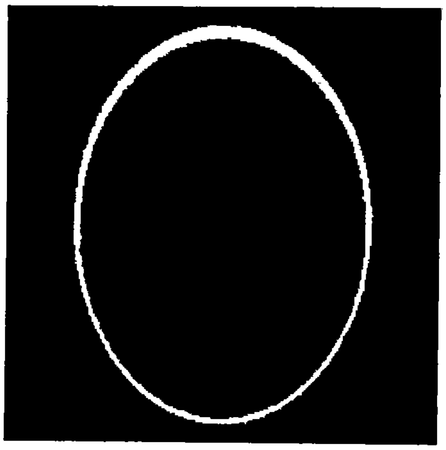


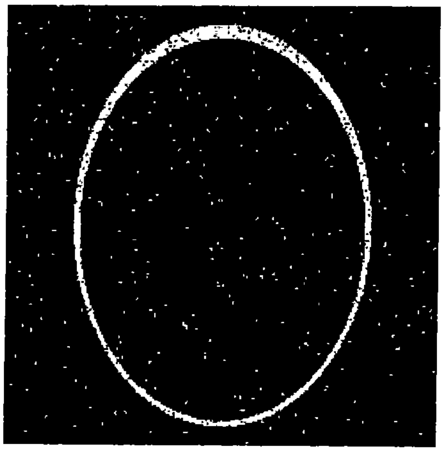
Figure 15: PSNR curves for recovery of phantom image by the four thresholding techniques


(a) Original Image
(b) Noisy Image
(c) Wavelets of the Noisy Image
(d) Recovered Image, Soft Thresholding

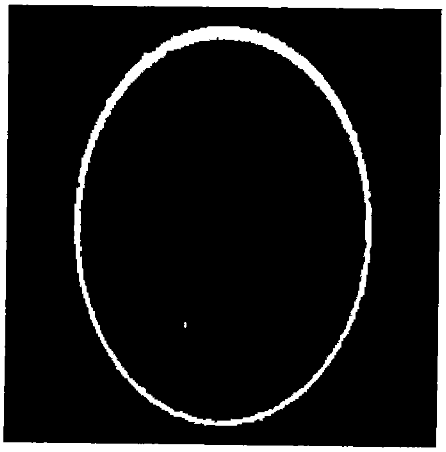
(e) De-noised Image, Median filter+Soft



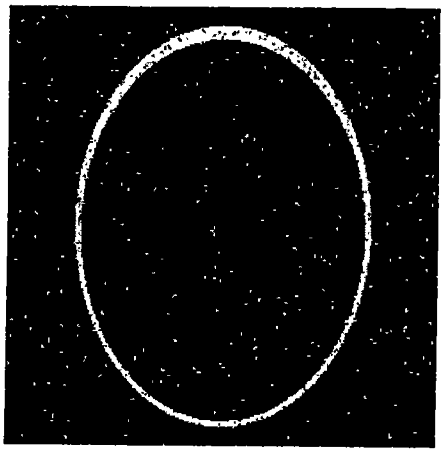
(f) Recovered Image, Hard thresholding



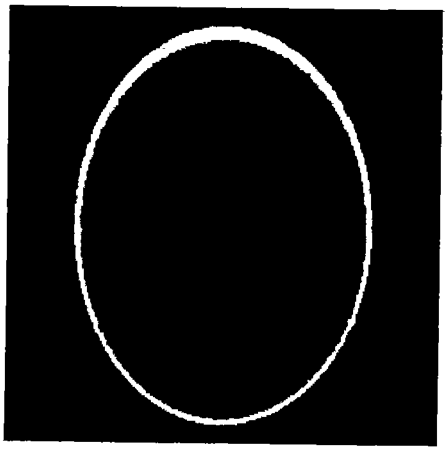
(g) De-noised Image, Median+Hard



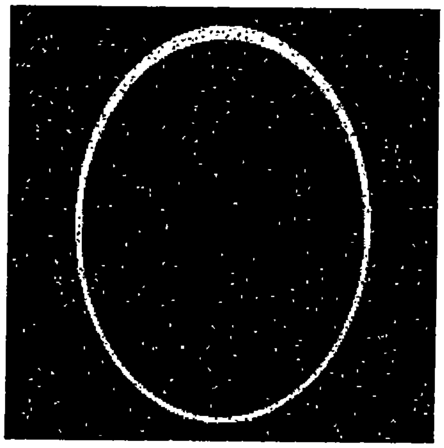
(h) Recovered Image, Garrote Thresholding



(i) De-noised Image, Median+Garrote



(j) Recovered Image, Proposed Shrinkage



(k) De-noised Image, Median+Proposed

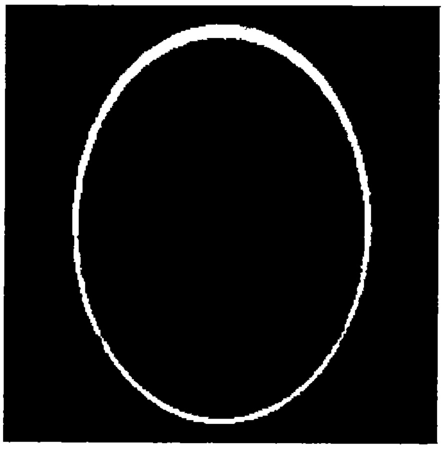


Figure 16: Original, Noisy, wavelets of the noisy and de-noised images.

5.2 Conclusions

This work addresses the signal and image de-noising problem and compares the performance of the three existing shrinkage functions with that of the proposed logarithmic shrinkage function. Apply the four shrinkage functions for recovery to the noisy under-sampled (uniformly and randomly) signal. Draw the curves of max $|x - xe|$ versus iterations. The graph of logarithmic shrinkage function decays very quickly and closed to 0.1 followed by the soft (closed to 0.2), garrote (closed to 0.45) and hard (closed 0.55) shrinkage functions.

The four shrinkage functions are then applied for recovery of images having under-sampling noise (in transform domain). Test the Shepp-Logan Phantom (256 x 256) images in the experiments. The proposed threshold technique removes the under-sampling (aliasing and Gaussian) noise significantly and shows the best results as compared to hard, garrote and soft threshold functions. The proposed technique produces 7 to 10% better results than the existing classical shrinkage functions.

In the second part of this work, the de-noising results of the four thresholding techniques are compared with that of the median filtering. Use median filtering for de-noising of sparse and noisy images, recovered through soft, hard, garrote and proposed shrinkage functions. These four thresholding techniques remove the under-sampling noise from the images but are not significant in the case of impulsive noise, that's why the use median filtering in sequence with the thresholding techniques for sparse images corrupted by salt and pepper noise is proposed. The thresholding techniques in sequence with the median filtering produce 30 to 35% better results than produced by ordinary thresholding techniques.

5.3 Applications

1) FBI Fingerprint Compression

A single fingerprint is about 700,000 pixels, and requires about 0.6Mbytes.

Figure 17: Finger prints and compression

2) Pattern Recognition

Wavelets are widely used in the field of pattern recognition (especially fractal patterns) due to their ability to zoom on finer patterns as well as view the entire global trend.

3) Edge recognition

   Wavelets can be used to separate out the edge of images and the greatest application of this property is in the field of finger print recognition.

4) Scientific data analysis

Not only can wavelets de-noise and compress data sets but it can also predict the time varying patterns in a data set. It is greatly used now a day in scientific data analysis. The applications of wavelet transform in the field of science and engineering are many and may are rapidly evolving. These small waves have u-shared a tsunami of change in various fields.

5) CS is used for signal and images recovery.
6) CS is used for signal and images de-noising.
7) CS is used in machine learning.
8) CS is used in statistical signal processing.
9) CS is used in histogram maintenance.
10) CS is used in dimension reduction and embedding.
11) CS is used for medical imaging.
12) CS is used for video processing.
13) CS is used for video sampling.
14) CS is used analog to information conversion.
15) CS is used in computational biology.
16) CS is used in geophysical data analysis.
17) CS is used in hyper-spectral imaging.
18) CS is used in compressive radar.
19) CS is used in astronomy.
20) CS is used in compressive system identification and dynamical systems.
21) CS is used in communication.
22) CS is used in detection and estimation.
23) CS is used in surface metrology.
24) CS is used in acoustics, audio and speech processing.
25) CS is used in remote sensing.
26) CS is used in computer engineering.
27) CS is used in computer graphics.
28) CS is used in robotics and control.
29) CS is used in optics and holography.

5.4 Future Work

1) Face recognition through sparse representations.

2) Low-rank matrix recovery through convex optimization.

3) The method used in this work can be applied to noisy image having salt and pepper noise, Gaussian noise, pink noise etc.

4) This work can be extended for video de-noising.

5) This work can be applied to real signal and images.

6) Satellite images can also be recovered and de-noised through this method.

7) These methods are also applicable to hand writing.

8) Apply the techniques to CT, Ultrasound and range data etc.

9) Cognitive Radio can use CS to utilize efficient utilization of channel.

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