Bipolar Generalized Fuzzy Optimization Model

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A Thesis
Submitted in the Partial Fulfilment of the
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IN
MATHEMATICS

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2023

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Dedication

I dedicate this thesis to my parents, **Shamim Akhtar** and **Khalil Ahmad Khan**. I have a special feeling of gratitude to them for being a continuous source of inspiration. Their countless prayers, encouragement, and support for my success have brought me to this stage. The fulfilment of this degree would not have been possible without them.

Acknowledgments

In the Name of **Allah**, the Merciful, the Beneficent. Praise be to the Lord of all worlds. Being a humble servant of His Domain, I would not have been able to accomplish this challenging task. I am grateful to the Almighty for giving me the strength and steadfastness to complete this research. Countless Peace and Mercy be upon The Last Prophet **Hazrat Muhammad (Peace be Upon Him)**, Who is the Bright Source of beneficial and fruitful knowledge for the whole universe.

I also offer my immense gratitude to my supervisor **Dr. Sajida Kousar**, for imparting her expertise and knowledge in this research. I am indebted to her for her constant supervision and guidance. She has been an ideal teacher, mentor, and thesis supervisor, offering advice and encouragement with a perfect blend of insight and knowledge. I am proud of my time working with her. I could not have undertaken this journey without her patience, guidance, kind behaviour, support and, above all, faith in me over the years.

I would like to extend appreciation and deepest gratitude for the help and support of my teachers, **Dr. Ambreen Afsar Khan**, **Dr. Khadija Maqbool** and **Dr. Maliha Rashid** for their assistance at every stage of my course work.

I want to convey my sincere and immeasurable appreciation to my **parents** for their prayers and encouragement that helped me to complete this work. This whole journey of mine was not an ordinary one. At many stages, I gave up and felt hopeless. I often lost faith in myself. I firmly believe that my parents' constant support, love, and prayers keep me motivated and make my journey possible. My accomplishments and success are because they believed in me. The deepest thanks to my **siblings**, who keep me grounded, remind me of what is important in life and are always supportive and kind.

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Preface

Though precision is an important aspect of many real life phenomena, it is advisable that we should notice and adjust the precision we look for with the uncertainty that exists. There remain a lot of situations in which the imprecision of humans is being relied upon. Initially, lack of precision was considered as an undesirable characteristic. However, with the passage of time and development in research methodologies, it was established that most physical applications and underlying models do not address the uncertainty in the information. Thus, to ensure the flow of information and fill this gap, fuzzy sets were developed.

The notion of fuzzy sets is something very unexpected and ordinary. Zadeh was the pioneer of this revolutionary concept. This idea takes the Boolean attribute of membership and non-membership of conventional sets to another level where every element of the universe under consideration operates under the influence of graded values within the closed interval [0 1] and hence generalizes the characteristic function to a membership function with no sharp boundaries. This generalization of the ordinary set is more natural and quite intuitive. Thus, fuzzy sets can be defined as a set of ideas and strategies aimed at providing an efficient framework for managing the imprecision and vagueness inherent in human perspectives and thought processes. The significance of the membership function has increased many-folds whenever dealing with random uncertainty and vague information.

Yin and Yang are the opposing polarities of Chinese medicine. The negative and feminine half of a system is called yin, while the positive or masculine side is called yang. It is believed that a person's physical and mental health, as well as the prosperity and stability of a social system, rely on their ability to coexist in peace, harmony, and equilibrium with their two sides. Although fuzzy logic offers one method for dealing with ambiguity and uncertainty and 2-valued

Boolean logic offers a theoretical foundation for digital computer technology, neither one has the representational or reasoning capabilities to directly model the coexistence and relationships of bipolar interactions. This is due to the fact that the logical values in the two classical logical models lie in the positive interval [0 1] and that they are unipolar models by nature, whereas the vast majority of human decision making is based on double sided or bipolar judgmental thinking with a positive and negative side. The two sides in decision and coordination are often, for example, collaboration and rivalry; effects and side effects; conflicting interests and common interests; probability and impossibility; likelihood and unlikelihood; feed-forward and feedback; etc. Thus, the presence of bipolarity as well as the demand for it coexist in nature and need to be investigated and unfolded.

Throughout the second half of the twentieth century, optimization discovered widespread applications in the investigation of physical and compound systems, planning, scheduling, and production systems, transportation problems, resource allocation issues in finance, asset assignment in monetary frameworks, building outlines, and engineering design. It was believed from the beginning of the application of optimization to these frameworks that specialists in the technical and natural models required decisions to be made in the presence of uncertainty. Uncertainty, for example, regulates the costs of fuels, the accessibility of power, demand and interest for synthetic compounds etc. In optimization with ambiguity, dealing with a large, uncertain space is one of the most difficult things to do. This is sometimes what drives large-scale optimization models. By including uncertainty in the mathematical programming frameworks, these models may be made better. As a result, fuzzy optimization may be seen as a realistic decision-making approach that has received high praise from several scholars. This kind of optimization, which models the inaccuracies using fuzzy relations and/or fuzzy variables, is relatively new in this field.

The current study is based on some modern techniques for dealing with optimization under the effects of uncertainty that fuzzy and bipolar fuzzy characteristics and variables have shown. It briefly reviews some known landscape, features and application of fuzzy optimization. Some fuzzy mathematical optimization methods and their generalisations will be investigated in the context of bipolarity. Some new computational algorithms to solve generalized bipolar fuzzy optimization problems are developed and illustrated with examples.

Chapter 1 is all about the emergence, existence, effects reasons, and causes of multi-valued logic, fuzzy and generalized fuzzy sets. A detailed discussion is carried out on the ambiguous nature of natural language and the limits of crisp characteristic function. The whole chain of human thought, beginning with the object and concluding with deduction and interpretation, is described in great detail. Additionally, we attempted to link various established theories, such as fuzzy set theory, theory of evidence, theory of information based on probability and theory of information based on possibility. An important historical event was the paradigm change from two-valued logic to many-valued logic. This procedure was not as simple as it could have been due to the lack of acceptance of logic other than two-valued logic. The contribution of Zadeh to the development of many-valued logic is described in depth, along with fundamental properties, definitions, set-theoretic operations, real-world examples of fuzzy, and a few refinements of fuzzy set.

Chapter 2 is based on historical background, literature review, fundamental concepts, set theoretic operations, algebraic operators, properties, and examples of bipolar fuzzy and bipolar neutrosophic sets.

A basic introduction, essential ideas, and problem formulation in optimization are all included in Chapter 3. A short discussion of several optimization techniques is also included. A foundation is built in order to comprehend bipolar fuzzy optimization. This chapter also includes a short discussion of already existing methodologies set in the backdrop of a fuzzy environment.

To the best of our knowledge, there is not one specific approach that can be used in order to compute the raking function of the bipolar neutrosophic numbers. Because of this deficiency, we are compelled to do research on bipolar neutrosophic numbers as well as the ranking of bipolar neutrosophic numbers. The first section of chapter 4 defines the bipolar neutrosophic number as a piece-wise continuous and monotonic function. The same part also defines triangular and trapezoidal bipolar neutrosophic numbers, as well as related arithmetic operations. In Section two, five different derivations are given for the ordering or debipolarizing bipolar neutrosophic numbers. The first two derivations are based on the score and accuracy functions of bipolar neutrosophic number, the third and fourth rely on the area and mean values of the respective memberships, and the last one is based on the centroid method. After providing only the formula line, the centroid method is left open for solution. In Section 3, the conclusion of the study is provided.

The application of the ranking function of the bipolar neutrosophic number, which was developed in chapter 4, serves as the foundation for chapter 5. The existing crisp model is used as the foundation for the development of a bipolar neutrosophic lp framework. The process of simplification and conversion is accomplished with the assistance of the ranking function of the bipolar neutrosophic number. The efficacy of the proposed model is proved by applying it to a real-life application based on the production of pulses in the upper Punjab region of Pakistan. Tables and graphs are used in the conclusion section to present the results. The efficiency of the strategy is demonstrated through a sensitivity analysis.

New and improved optimization models are required with each advancement and extension of the fuzzy system. Our motivation for Chapter 6 is to develop bipolar neutrosophic model that could be adequately utilized in the processes involving decision making. In most of these cases, situations arise when it is necessary to represent an unknown commodity with an ambiguous value or when the person responsible for making decisions must refrain from expressing their performance reviews. The development of the computational algorithm in a bipolar neutrosophic setting is the main objective of this chapter. We also intend to evaluate the effect of bipolar truth membership, bipolar indeterminacy membership and bipolar falsity membership in such optimization processes. Further, we have conducted a comparative study on intuitionistic fuzzy, neutrosophic fuzzy, refined neutrosophic fuzzy, and bipolar neutrosophic fuzzy optimization techniques. On the basis of the present study, conclusions have been drawn, and the future scope has been outlined. The contents of this chapter are published in **Optimization Theory Based on Neutrosophic and Plithogenic Sets. Academic Press, 2020. 289-314.**

In a typical real-life situation where fuzzy sets and generalizations are used, membership, non-membership, and indeterminacy grades are solely determined by expert opinions. For this purpose, one has to assign numerical values between 0 and 1 based on deterministic and quantitative attributes. This approach may fulfill the purpose theoretically, but it would be an unreliable judgment in real life. Wang presented the concept of interval-valued neutrosophic sets for a more realistic problem-solving approach. Most real-life optimization problems are fuzzy in nature. The loss of fuzzy information during the computational process is quite obvious in all the previously defined fuzzy optimization models. The interval-valued fuzzy set is a refinement that is defined to reduce such losses and reflect the fuzziness and uncertainty more efficiently and effectively. In chapter 7, the core concept is to modulate the magnitude of uncertainty by assigning intervals to the membership, non-membership, and indeterminacy degrees; otherwise, a rigid numerical value cant be considered realistic and error-free if assigned to merely an experts opinion. A computational algorithm is constructed in this chapter to solve a multi-objective linear programming problem in an interval-valued neutrosophic fuzzy environment. The model is then applied to some real-life problems like land use, car side impact, and production planning. The results are then compared with the numerous fuzzy optimization approaches to ensure the validity and efficiency of the newly established method. The contents of this chapter are published in **International Journal of Fuzzy Systems, 24(3), 1343-1355.**

The aim of Chapter 8 is to construct and crack a medicine supply-chain model in an interval-valued bipolar neutrosophic environment. This supply chain model is designed for a health-care system that is integrated as well as uncertain in terms of product complaints. In its modest form, a supply chain is the activities needed by an organisation to provide services or goods to buyers. Traditional medical health-care facilities are limited to drug companies, patients, and hospitals. Prescribed research includes a unified medicare framework that also includes the role of the public authority and the health-care department. The first part of the chapter provides a detailed description of the whole medical supply chain concept. In addition, objective functions and model constraints are briefly examined. Section two discusses the proposed approach and optimization algorithm based on the interval-valued bipolar neutrosophic set. The model presented in the first part serves as the basis for the numerical example presented in section three. A solution based on the recommended methodology is provided. Finally, the conclusion is included, and the results are discussed in detail. The observations are published in **Computers, Materials and Continua.73(3), 6207-6224, July 2022.**

Contents

List of Tables

List of Figures

List of Abbreviations

FS	Fuzzy set
IFS	Intuitionistic fuzzy set
NS	Neutrosophic set
$FVRNS$	Four-valued refined neutrosophic set
BF	Bipolar fuzzy set
BNS	Bipolar neutrosophic set
BNN	Bipolar neutrosophic number
$TBNN$	Triangular bipolar neutrosophic number
$TrBNN$	Trapezoidal bipolar neutrosophic number
OR	Operation research
LP	Linear programming
LPP	Linear programming problem
FLP	Fuzzy linear programming
NLP	Nonlinear programming
NLP	Neutrosophic linear programming
$BNLP$	Bipolar neutrosophic linear programming
$IFLP$	Intuitionistic fuzzy linear programming
FO	Fuzzy optimization

T	True grade of membership
F	False grade of membership
I	Indeterminate grade of membership
$C = T \wedge F$	Contradictory grade of membership
$G = T \vee F$	Grades of ignorance
IT	Indeterminacy leaning towards true
IF	Indeterminacy leaning towards false
IFO	Intuitionistic fuzzy optimization
$IVFO$	Interval-valued fuzzy optimization
BNO	Bipolar neutrosophic optimization
NSO	Neutrosophic optimization
$FVRNO$	Four-valued refined neutrosophic optimization
$IVNS$	Interval-valued neutrosophic set
$IVIFO$	Interval-valued intuitionistic fuzzy optimization
FR	Feasible region.

Chapter 1

The Concept of Fuzziness and Generalization

"So far as the laws of mathematics refer to reality, they are not certain and so far as they are certain they do not refer to reality."

Albert Einstein

Geometrie und Erfahrung, Lecture to Prussian Academy, 1921

In an epistemological context, the inverse of information can be considered as equivalent to uncertainty. There are many reasons behind the emergence of uncertainty. That can be from lack of knowledge, from imprecision, from various classes of randomness, from chance, lack of information, vagueness and complexity, as the information about any particular problem in such cases may be deficient, contradictory, imprecise, incomplete, fragmentary, unreliable, incomplete and vague. Practical frameworks throughout the world are excessively complicated and, due to natural subjectivity, handled with some degree of deliberation and abstraction. This deliberation is the final result of a compromise with the exactness and precision of the events. Without a doubt, such types of information can't tackle problems that require high accuracy, but relatively a few human issues require exactness and precision, for example, backing up a trailer, stopping a vehicle, exploring a vehicle among others on a turnpike or parkway, controlling traffic at crossing points, washing garments, making a decision about magnificent contenders, and acquiring a basic understanding of a complex system. Also, to describe the majority of physical procedures, we must rely on inadequate human thinking. Most of the time, this imprecision and error transmit extremely valuable information.

In daily life we come across many situations when we have to classify or dichotomize objects in to groups for an efficient analytical approach to compile data bases. Distinguished by certain pre-defined attributes, the elements are collected into different crisp sets and are defined as members, whereas the rejected elements from those sets are defined as non-members. But this ease is only available when we have a quantitative analysis at hand of the universe under consideration. If the information is communicated through natural language then a large amount of ambiguity is faced during the classification process due to the vague nature of natural language descriptions. The common adjectives like tall, lengthy, good, excellent, perfect, fast, swift, quick might hold different descriptive magnitude for every different person and thus data communication through natural language fails to set up a defined standard for further data classification into crisp sets. Thus, while processing data we have to face this bottle neck issue of natural language because for many real-world schemes and systems.

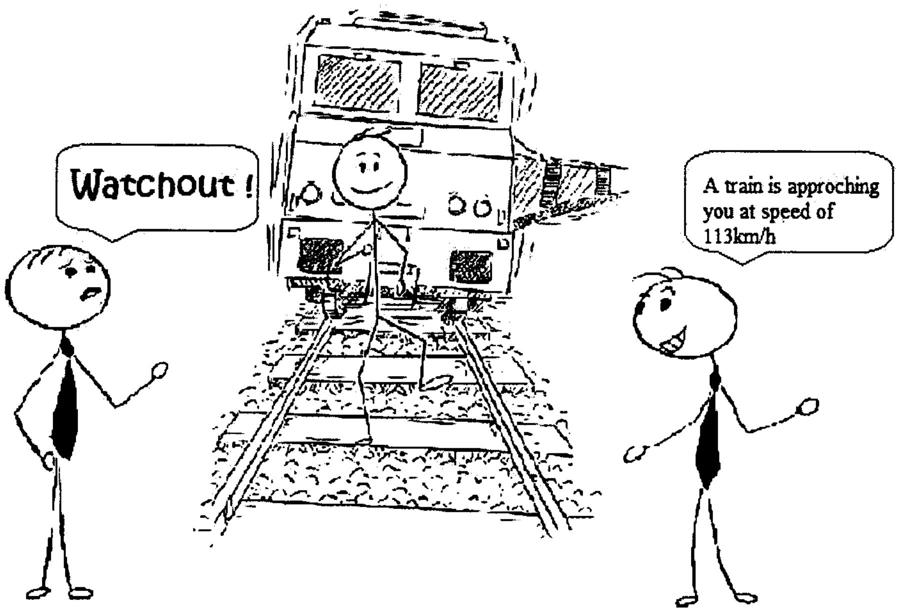

Figure 1.1: Effectiveness of fuzzy information

Figure 1.1 shows the effectiveness of fuzzy information. In the information world there are many forms of uncertainty and the general human reasoning process take cares of all of them. Our comprehension of most actual and physical processes depends to a great extent on uncertain human thinking. This uncertainty and imprecision, when compared with the exact amounts and quantities needed by computers, is in any case a type of material and evidence that can be very valuable to people. The capacity to insert such data and evidence in obstinate and difficult problems is the basis by which the efficiency of fuzzy logic is measured and assessed. Usually there maybe two main sources of information: human professionals and specialists who use natural languages to depict their insight and describe their knowledge (the subjective data); the other is through estimations and numerical models determined by physical laws (the objective data).

As we move into the era of information, to meet the accuracy in results, human information turns out to be progressively critical and becomes increasingly important. Computational techniques dependent on exact numerical and mathematical formulae are unfit for taking care of the genuine useful frameworks with a lot of subjective data. The simple way out to deal with them is to join this human information in a systematic manner along with other data got from exact empirical methods. But the important problem is: "How to transform or insert this human information and knowledge into the prevailing mathematical formulations." Thus special attention is paid to deal with uncertainty that arises from imprecision and ambiguity in human affairs. The human actions, activities and deeds are essentially characterized by their capability to notice and examine the world of objects and making deductions and inferences. This exercise works in two steps: observation i.e. perception, mental creation and the explanation. Perception and observation can only be liable of logical tests and investigation after being cast in a linguistic form. Shortly one can refer it as implication in the form of lingual depiction. In most of the cases, the content perceived is different in some way or other to the actual content of observation and the results or deductions come out to be an approximation with imprecision. In this way, the entire reasoning and inference process is a major source of impression and vagueness that can likewise be perceived as follows:

"There are differences between, what we think, what we want to say, what we think we say, what we say, what they want to hear, what they hear, what they want to understand, what they think they understand, and what they understand. That's why there are at least nine reasons for people to misunderstand each other". -Yager

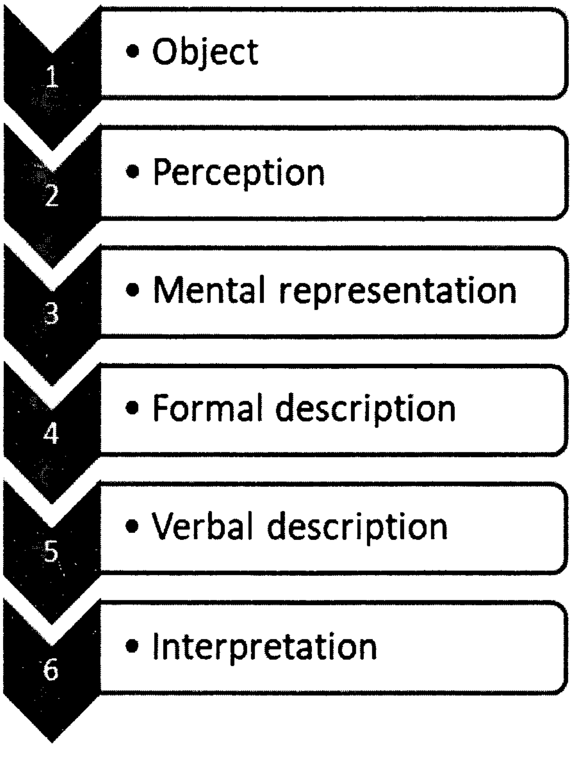

Figure 1.2: Flow chart

Figure 1.2 depicts the general human reasoning process from object to interpretation. In the past, numerous theories and hypothesis have been projected for managing uncertainty in an efficient manner. A few of these are rough set theory, probability theory, granular computing, fuzzy set theory and computing with words [1]. However, all these theories are affiliated with an intrinsic limitation and are deficient to deal with all aspects of uncertainty exclusively. There are two forms of uncertainty based classical theory of information in system science. First one is possibility based theory of information developed by Hartley [2] and the other is probability based theory of information established by Shannon [3]. The theory of probability is the investigation of laws and rules directing the phenomena of randomness while theory of possibility is an uncertainty theory dedicated to the treatment of inadequate information. Probability theory was the dominant theory for assessing uncertainty in scientific models from the late nineteenth to the late twentieth centuries. However, the continual growth of the uncertainty expression using probability theory was tested for the first time by Black [4] in 1937 with his investigations and work on vagueness, and then again in the following year by Zadeh [5] when he presented fuzzy sets. Dempster [6] built up a theory of evidence, which, unexpectedly, incorporated the absence of information, or an assessment of ignorance.

History has witnessed the unacceptability of fuzziness and uncertainty within the scientific community. Prior to the middle of the nineteenth century, this group saw uncertainty as an undesirable condition that must be safely ignored. Also from a researchers perspective, uncertainty was perceived as undesirable condition that needs to be avoided for getting correct results.

Zadeh was actually the first to demonstrate fuzzy sets as possibility distribution and also coined the name *Theory of possibility* [5]. Zadeh called the basic idea in logic that he had come up with "fuzzy set theory." His work not only influences and alters how scholars think about uncertainty, but it also challenges the notion that probability theory is the only approach to dealing with uncertainty. This belief has been affected and changed as a direct result of his work. Parallel to two-valued logic of probability theory he has established a possibility theory based on fuzzy sets. He was of the view that possibility is always related with some fuzziness either in the circumstantial way or in the set for which possibility is declared.

Thus the 20th century saw the main advancements of options in contrast to probability theory and to classical Aristotelian rationale as ideal models to address a bigger number of sorts of uncertainty than simply the random kind. Trends are now shifting towards the exploitation of tolerance for imprecision to acquire great results in the fields of science and engineering. Simple human-developed rules are behind the notion of fuzzy logic, and mathematical equivalents can be interpreted while dealing with problems using fuzzy systems for solutions [7]. Figure 1.3 shows us the complete information world.

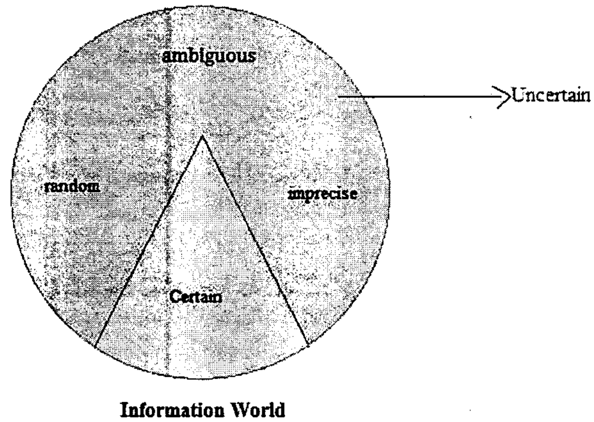

Figure 1.3: Information world

Developments and improvements in the fuzzy sets by Zadeh [5] actually provided foundation and pave the path for other researchers to constitute its theory. The fuzzy sets theory is, fundamentally, a theory of reviewed ideas and graded concepts, a theory where everything involves degree or, to put it metaphorically, everything has flexibility and elasticity. As clear from its name fuzzy sets are quite the opposite of crisp sets. This technique copes mathematically with the vagueness of determining boundaries by assigning grades of membership to the elements. These grades are mostly referred as partial membership functions because they represent how less or more the considered element deserves to be included in the fuzzy set. Mathematically, the most deserving or relevant element will have a degree of 1 and the least deserving or irrelevant element will have a degree of 0. Furthermore, the real numbers between 0 and 1 can accommodate and quantify more elements while assigning them membership grades. In both the fuzzy and crisp domains, the preceding figure demonstrates the same idea of relevancy and irrelevancy.

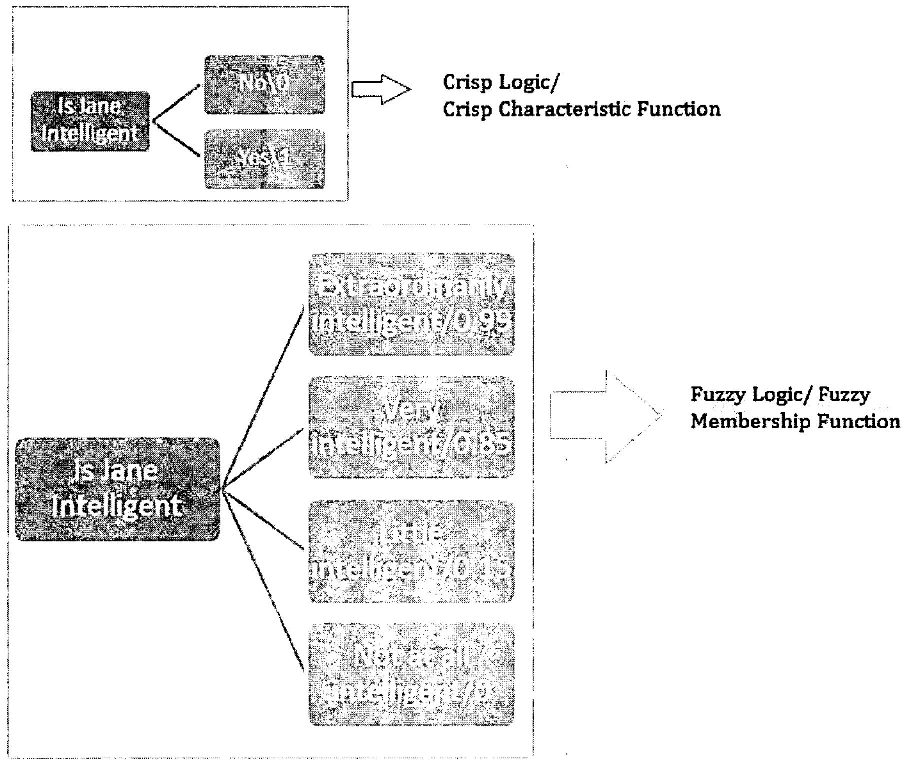

Figure 1.4: Crisp and fuzzy logic

The characteristic function of a crisp set determines character by values of 1 and 0 as full membership and full non-membership. Thus the degree of membership of a component belonging to classical set is bivalent as shown in figure 1.4. A component relates or doesn't relate to a set and it is not permitted to be included in a set and its reciprocal set at the same time. Classical set theory failed to handle many real life scenarios due to these limitations. No single value, such as 0 or 1, can be assigned to consistently varying events in a fuzzy set like those given in crisp sets and Boolean logic. Multiple values based on the exact answer or degree of approximation to the true value is used in fuzzy logic. These multiple values are based on the degree to which some value would belong to the given set of values. For instance, for a given set of values, certain element belonging to it can be true with certain level of precision ranging between 0 and 1. Transition from membership to non-membership in a fuzzy set is continuous unlike a discrete jump in a crisp set. The idea of embedding ordinary and conventional set theory into fuzzy sets is as normal, natural and expected as the mathematical embedding of the real numbers into complex plan. Hence it can be concluded that crisp sets are circumstantial cases of fuzzy sets but converse is not always true. Thus, the notion of fuzziness is one of enhancement and enrichment, not of replacement and a broader spectrum of application is covered under this term. Mathematical modulation, business, computer, control, medical and allied health sciences, social networking, psychology, statistics, image processing and signals, economics, fuzzy modeling, natural sciences and finance are currently using the fuzzy set theory abundantly.

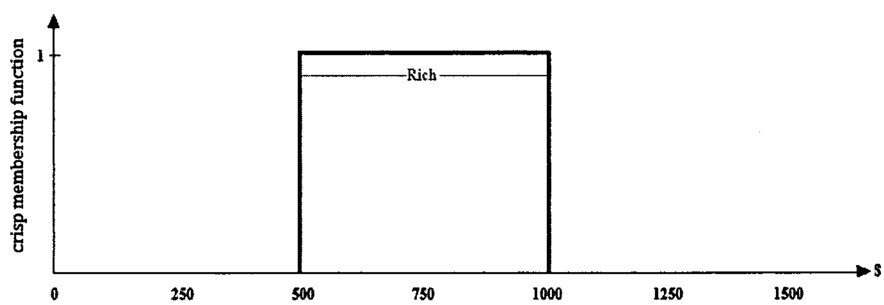

Figure 1.5: Rich people (crisp set)

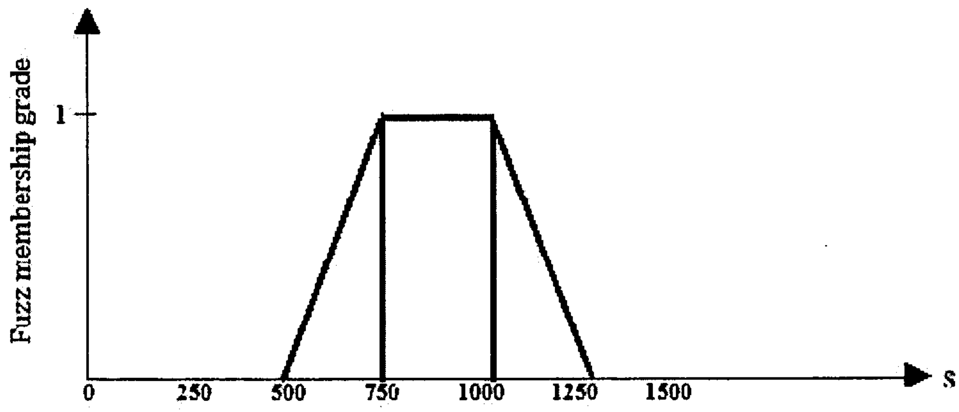

Figure 1.6: Rich people (Fuzzy set)

A pictographic depiction for rich people and hot weather is provided in Figure 1.5-1.8. In Figure 1.5 an interval of \$500 to \$1000 is considered as rich while other readings are not marked as rich. But in real life scenario one may merely find any difference in possessing \$499 or \$501. Same is the case with temperature. In Figure 1.7 an interval of 30°C to 50°C is considered as hot. Despite possessing sharp edges at the end point in crisp demonstration one cannot even distinguish between 30°C and 31°C in real scenario. Figure 1.8 and 1.9 shows clearly the distinction of utilizing fuzzy ideas rather than crisp strategies. It is very much clear that the partial membership grades make more sense than crisp membership function.

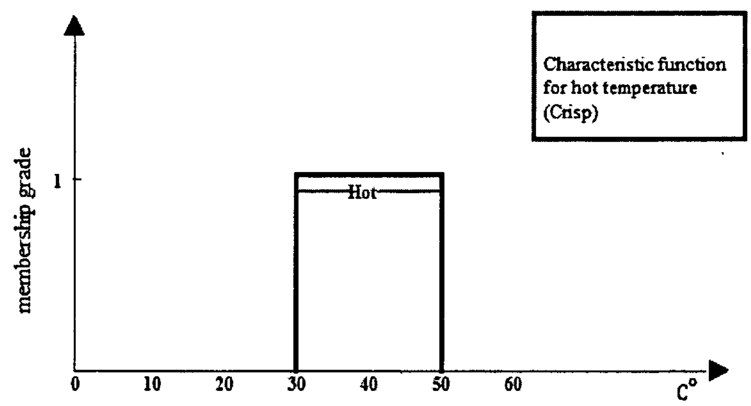

Figure 1.7: Hot temperature (crisp set)

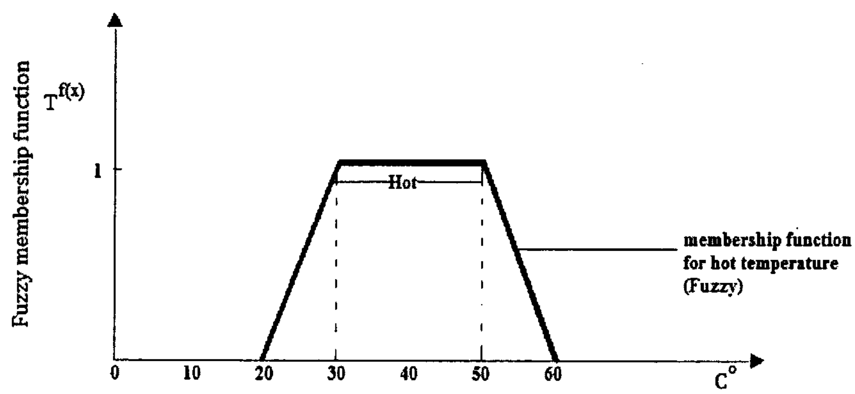

Figure 1.8: Hot temperature (fuzzy set)

1.1 Preliminaries

A fuzzy set is actually defined by its membership function which attempts to describe the ambiguity and vagueness of blur boundaries of the set. Mathematically, a membership function T_{A^F} of a fuzzy subset A^F is represented as $T_{A^F} : X \to [0,1]$, where X is the universe of discourse here. Corresponding to each element $x \in X$ there always exist a real number $T_{A^F}(x)$ in the interval $[0,1]$ and this value of $T_{A^F}(x)$ at x represents the membership grade of x in A^F. This grade compares to how much that entity is comparative or viable with the idea addressed by the fuzzy set.

A fuzzy set, thus, is a collection containing components that may have fluctuating grades of membership in the set. Grades of membership can be adjusted in the fuzzy sets to maximize the utility for a particular application and hence flexibility is gained but uniqueness is sacrificed. This concept is just the converse of crisp or classical set where elements belong to the collection only when their membership is full or 1, otherwise they do not belong to the set. In literature fuzzy sets are denoted in a number of ways [8]-[12]. The most common and preferred representation of fuzzy sets are ordered pairs. A fuzzy set A^F can be described as a set of ordered pairs of element x and membership grade $T_{A^F}(x)$ and is frequently written as

$$A^F = \{(x, T_{A^F}(x)| \, x \in X)\}. \tag{1.1.1}$$

Membership function of fuzzy set become identical with the characteristic function of crisp set when attained values 0 and 1. In other words A^F is no longer fuzzy subset, but an ordinary set A when the grade of membership $T_{A^F}(x)$ contains only two points 0 and 1. Thus grade of membership $T_{A^F}(x)$ is an obvious extension of characteristic function of a crisp set, because it can also takes values other than 0 and 1.

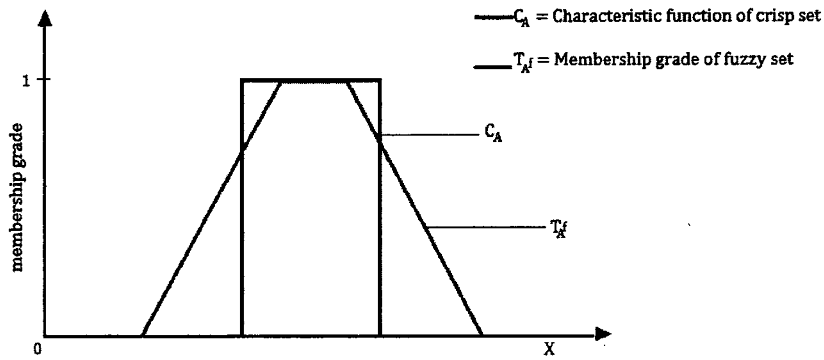

Figure 1.9: Fuzzy membership and crisp membership

Figure 1.9 clearly shows that in addition to the values 0 and 1, membership function of fuzzy set takes on different values between 0 and 1 as-well and hence is a tangible dilation of characteristic function of crisp set.
Zadeh [5] also proposed a notation for the representation of fuzzy sets when universe of discourse, X , is finite. This notation is as follows for a fuzzy setA^F

$$A^F = \left\{ \frac{T_{A^F}(x_1)}{x_1} + \frac{T_{A^F}(x_2)}{x_2} + \ldots + \frac{T_{A^F}(x_n)}{x_n} \right\} = \left\{ \sum_{i=1}^{n} \frac{T_{A^F}(x_n)}{x_n} \right\}. \quad (1.1.2)$$

Expression 1.1.2 indicates that the membership of element x_1 is $T_{A^F}(x_1)$, membership of element x_2 is $T_{A^F}(x_2)$, continuing in the similar manner, membership of element x_n is $T_{A^F}(x_n)$. Horizontal bar in each expression is not a quotient but used as a delimiter here. Similarly the summation symbol and addition symbol are not usual algebraic summation and algebraic addition but are collection or aggregation operators. When the universe of discourse, X, is infinite, then the fuzzy set A^F is denoted as:

$$A^F = \left\{ \int \frac{T_{A^F}(x_1)}{x_1} \right\}. \quad (1.1.3)$$

Similarly in Equation 1.1.3 the horizontal bar is again a delimiter not a quotient and integral sign is not usual algebraic integral but can be viewed as an extension of summation for continuous variables.

1.1.1 Basic Characteristics of Fuzzy Sets (Elementary Features)

Since all the information enclosed in a fuzzy set is described by its membership function therefore these sets can be portrayed in more detail by alluding to the features utilized in characterizing the grade of membership describing them [9] and [13]-[15].

1.1.1.1 Core

Core of a membership for some fuzzy set A^F is the area or region of the universe of discourse that exhibit a full and complete membership in the set A^F. Mathematically, core is defined as,

$$\text{Core}(A^F) = \{x \in X \,|\, T_{A^F}(x) = 1\}. \tag{1.1.4}$$

1.1.1.2 Support

Support of a membership for some fuzzy set A^F is the area or region of the universe of discourse that exhibit a nonzero membership in the set A^F. Mathematically, support is defined as,

$$\text{Support}(A^F) = \{x \in X \,|\, T_{A^F}(x) > 0\}. \tag{1.1.5}$$

1.1.1.3 Boundary

Boundaries of a membership grade for some fuzzy set A^F is the area or region of the universe of discourse comprises of those elements $x \in X$ having some partial degree of membership. Mathematically, Boundary is denoted as,

$$\text{Boundary}(A^F) = \{x \in X \,|\, 0 < T_{A^F}(x) < 1\}. \tag{1.1.6}$$

1.1.1.4 Height

Height of a fuzzy set A^F is the uppermost membership value of its membership function.

$$\text{Height}(A^F) =^{\sup}_{x \in X} T_{A^F}(x). \tag{1.1.7}$$

1.1.1.5 Normal

A fuzzy set A^F is said to be normal if maximum membership grade attained for at least one $x \in X$ is 1. That is height $(A^F) = 1$. For optimized fuzzy set, concept of normality is of immense importance.

Figure 1.10 shows the concepts of core, support, boundary and height graphically.

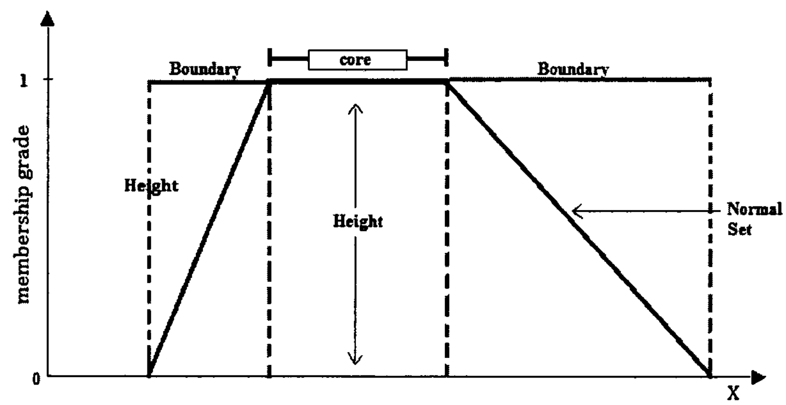

Figure 1.10: Core, support, boundary and height

1.1.1.6 α-Cut

The α-cut A^α of a fuzzy set A^F is an ordinary set satisfying the given condition,

$$A^\alpha = \{x \in X |\, T_{A^F}(x) \geq \alpha\} \quad , \alpha \in [0, 1]. \tag{1.1.8}$$

1.1.1.7 Strong α -Level Set

A strong α-level set ${A^F}_{+\alpha}$ of a fuzzy set A^F is denoted mathematically as,

$$ {A^F}_{+\alpha} = \{x \in X | T_{A^F}(x) > \alpha\} \quad , \alpha \in [0,1]. \tag{1.1.9} $$

1.1.1.8 Convex Fuzzy Set

Convexity of fuzzy set also depends on certain condition satisfied by membership grade. A fuzzy set A^F is said to be convex if its membership grade satisfies the conditions below.

$T_{A^F}(\lambda x_1 + (1-\lambda)x_2) \geq min(T_{A^F}(x_1), T_{A^F}(x_2))$ for all $x_1, x_2 \in A^F, \lambda \in [0,1]$

Geometrical Representation Let $t = (\lambda x_1 + (1-\lambda)x_2)$ for any two points x_1 and x_2 from set A^F and $\lambda \in [0,1]$. Then according to set condition t must lies between x_1 and x_2 and T_t also lies between T_{x_1} and T_{x_2}.

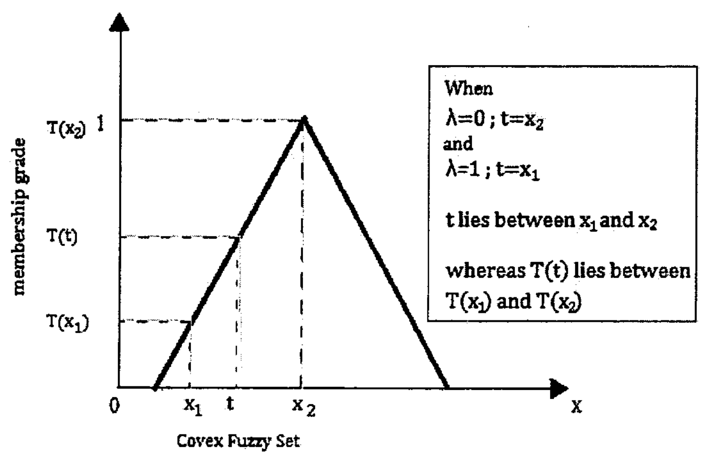

Figure 1.11: Convex fuzzy set

Condition $T_{A^F}(\lambda x_1 + (1-\lambda)x_2) \geq min(T_{A^F}(x_1), T_{A^F}(x_2))$ is satisfied in

Figure 1.11.

Example 1.1.1. From a universal set of ages ranging from three years till hundred years it is asked to make a subset ranking infant, toddlers, young, adult, old and very old.

Age	Infant	Toddlers	Young	Adult	Old	Very Old
3	0	0.9	0	0	0	0
5	0	0	0	0	0	0
10	0	0	0	0	0	0
15	0	0	0.2	0	0	0
25	0	0	0.8	0.8	0	0
35	0	0	1	0.9	0	0
45	0	0	0.6	1	0	0
55	0	0	0.5	1	0.9	0
65	0	0	0.1	0.9	1	0.1
75	0	0	0	0.9	1	0.2
85	0	0	0	0.8	1	0.6
90 above	0	0	0	0	1	0.9

Table 1.1: Fuzzy membership grades (ages)

Decision maker allocated the values to each age group based on his own knowledge, choice, experience, judgment and understanding, making the set fuzzy, Table 1.1. Each membership grade is picked from $[0, 1]$. It can easily be seen that despite 0 and 1 every age group may possess values between 0 and 1, depending on the intensity of belongingness.

Example 1.1.2. For given two fuzzy sets A^F and B^F:
$A^F = \{(x_1, 0.3), (x_2, 0.1), (x_3, 0.5), (x_4, 0.7), (x_5, 0.9)\}$

$B^F = \{(x_1, 0), (x_2, 0.4), (x_3, 0.6), (x_4, 0.8), (x_5, 1)\}$
certain characteristics of fuzzy sets are given below:

1. Support $(A^F) = \{x_1, x_2, x_3, x_4, x_5\}$; support$(B^F) = \{x_2, x_3, x_4, x_5\}$.

2. Height $A^F = 0.9$; height $B^F = 1$.

3. A^F is subnormal where as B^F is a normalized fuzzy set.

4. If $\alpha = 0.5$

$$A^F_{0.5} = \{x_3, x_4, x_5\} \quad ; \quad B^F_{0.5} = \{x_3, x_4, x_5\}$$
$$A^F_{0.5'} = \{x_4, x_5\} \quad ; \quad B^F_{0.5'} = \{x_3, x_4, x_5\}.$$

5.

$$\text{Core of} \quad A^F = \phi \quad ; \quad \text{Core of} \quad B^F = \{x_5\}.$$

1.1.2 Fuzzy Set Theoretic Operations

Set theoretic operations for fuzzy sets have a vast significance in many fields. Zadeh [5] himself proposed many of these operations involving fuzzy sets. Let X be a universal set and A^F, B^F and C^F are the three fuzzy sets defined on this universe.

1.1.2.1 Union

The union of two fuzzy sets A^F and B^F is denoted as $A^F \cup B^F$. The resultant fuzzy set is the smallest fuzzy set containing both A^F and B^F. Mathematically, union of two fuzzy sets is denoted as,

$$A^F \cup B^F = max\{(T_{A^F}(x), T_{B^F}(x)), \forall x \in X\}. \tag{1.1.10}$$

In fuzzy set theory the maximum of $\hat{a}$ and $\hat{b}$ is mostly expressed as,
maximum$(\hat{a}, \hat{b}) = \hat{a} \vee \hat{b}$.

Above equation can be expressed as,

$$A^F \cup B^F \Leftrightarrow T_{A^F \cup B^F}(x) = \{(T_{A^F}(x) \vee T_{B^F}(x)), \forall x \in X\}. \qquad (1.1.11)$$

1.1.2.2 Intersection

The intersection of two fuzzy sets A^F and B^F is denoted as $A^F \cap B^F$. The resultant fuzzy set is the largest fuzzy set which is contained in both A^F and B^F. Mathematically,

$$A^F \cap B^F = T_{A^F \cap B^F}(x) = min\{(T_{A^F}(x), T_{B^F}(x)), \forall x \in X\}. \qquad (1.1.12)$$

In the fuzzy set theory the minimum of $\hat{a}$ and $\hat{b}$ is mostly expressed as, minimum$(\hat{a}, \hat{b}) = \hat{a} \wedge \hat{b}$.
Above equation can be expressed as,

$$A^F \cap B^F \Leftrightarrow T_{A^F \cap B^F}(x) = \{(T_{A^F}(x) \wedge T_{B^F}(x)), \forall x \in X\}. \qquad (1.1.13)$$

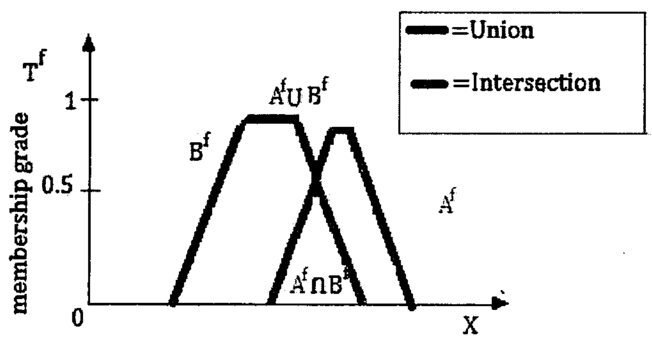

Figure 1.12: Fuzzy union and intersection

Figure 1.12 shows the union and intersection of fuzzy sets graphically.

1.1.2.3 Complement of Fuzzy Set

The complement of a fuzzy set A^F on the universe of discourse X, denoted as $(\overline{A^F})$, defined mathematically as,

$$(\overline{A^F}) = T_{\overline{A^F}}(x) = 1 - T_{A^F}(x), \quad \text{for all} \quad x \in X. \tag{1.1.14}$$

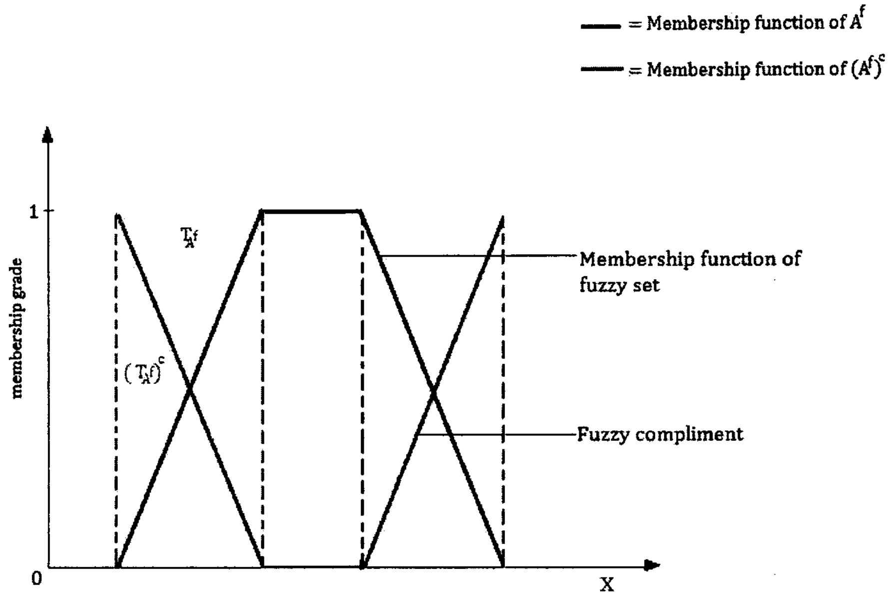

Figure 1.13: Fuzzy compliment

Complement of fuzzy set is shown graphically in Figure 1.13.

1.1.2.4 Equality of Two Fuzzy Sets

Two fuzzy sets A^F and B^F are said to be equal only if their membership functions are equal for all $x \in X$, where X is the universe of discourse here. Mathematically, this concept is expressed as,

$$A^F = B^F \Leftrightarrow T_{A^F}(x) = T_{B^F}(x), \quad \text{for all} \quad x \in X. \tag{1.1.15}$$

Note that a A^F is a subset of the universal set on which it is defined. Mathematically this concept is interpreted as,

$$A^F \subseteq X \Rightarrow T_{A^F}(x) \subseteq T_X(x). \tag{1.1.16}$$

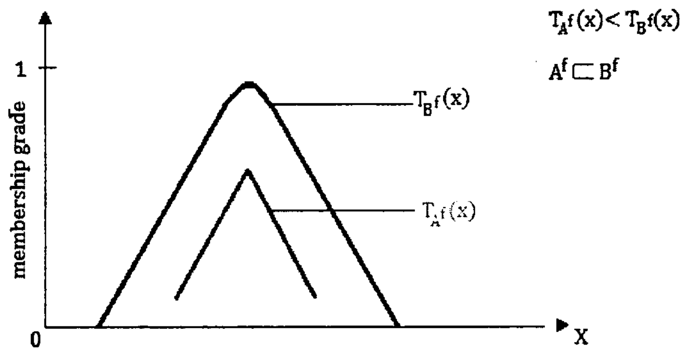

Figure 1.14: Proper fuzzy subset

Figure 1.14 gives graphic visuals of proper fuzzy subsets. Basic definitions suggested that it is feasible to broaden a significant number of the essential characters which hold for crisp sets to fuzzy sets. A few of such properties for intersection, union and complement which hold for fuzzy sets similar to crisp sets are given as,

1.1.2.5 Commutative Laws

$$A^F \cup B^F = B^F \cup A^F$$
$$A^F \cap B^F = B^F \cap A^F.$$

1.1.2.6 Associative Laws

$$A^F \cup (B^F \cup C^F) = (A^F \cup B^F) \cup C^F$$
$$A^F \cap (B^F \cap C^F) = (A^F \cap B^F) \cap C^F.$$

1.1.2.7 Distributive Laws

$$A^F \cup (B^F \cap C^F) = (A^F \cup B^F) \cap (A^F \cup C^F)$$
$$A^F \cap (B^F \cup C^F) = (A^F \cap B^F) \cup (A^F \cap C^F).$$

1.1.2.8 De Morgans Laws

$$\overline{A^F \cup B^F} = \overline{A^F} \cap \overline{B^F}$$
$$\overline{A^F \cap B^F} = \overline{A^F} \cup \overline{B^F}.$$

1.1.2.9 Involution

$$\overline{\overline{A^F}} = A^F.$$

The proofs of each of these properties are quiet straightforward. As mentioned earlier, almost all other operations defined on traditional crisp sets are valid for fuzzy sets with the exception of excluded middle axioms. As clear from the definition of fuzzy set an element $x \in A^F$ may also be an element of $\overline{A^F}$, hence disobeying axiom of contradiction which states that $\overline{A^F} \cap A^F = \varnothing$. Same is the case with axiom of excluded middle. In fuzzy case, $\overline{A^F} \cup A^F \neq X$. These are the only exceptions in terms of fuzzy sets.

1.2 Generalization of Fuzzy Set

Even though fuzzy sets have endless effects and applications, not all scientific and physical models use true grades of membership. A bibliometric analysis of fuzzy set is also proposed by Radu *et al.* [16]. There may arise situations where the degree of non-membership is also required with that of membership degree. Furthermore, from a designer or decision makers perspective input parameters of many problems may end-up yielding imprecise parameters with more than one type of uncertainty and many key governing factors. These factors may include the degree of acceptance for any parameter, rejection degree, or hesitancy[17], indeterminacy, neutrality, falsity [18] etc. Hence one can apply the fuzzy and generalized fuzzy system to the situations where uncertainty in the information lies either due to vagueness, or fuzziness, or impression, or due to indeterminacy, hesitancy, neutrality, bipolarity, or even when no information, as well as model, exists at all. This system also works so well in situations with continuously varying in-put labels. Rather this system is rich enough to tackle uncertainties contained not only in-puts but also in the out-put of the system. That distinction is achieved by inculcating automation in the circumstance. Hence by self-referent adjustments determined by specific guidelines make the fuzzy and generalized fuzzy system very successful.

1.2.1 Intuitionistic Fuzzy Set

In 1986 intuitionistic fuzzy sets (IFS) were introduced by Atanassov [19] as an extension of classical fuzzy set theory as it was proved to be more effective against ambiguity. Unlike the classical fuzzy set theory which only assigns membership grades, IFS also assigned non-membership grades to generic elements. Hence the truth and falsity grades of memberships in the IFS possess a sum that is less than or equal to one. Information available from both functions proves to be valuable in some scenarios. That is why IFS is found to be a more appropriate and relevant technique in the areas of science and technology. Microelectronics fault analysis, medical diagnosis and logic programming are few of those areas relevant to intuitionistic set theory.

An intuitionistic fuzzy set is a collection of triplets, given a universe X, which could handle membership and non-membership grades together. Hence can be represented as,

$$A^{IF} = \{(x, T_{A^{IF}}(x), F_{A^{IF}}(x) : x \in X)\}. \tag{1.2.1}$$

Where $T_{A^{IF}}(x), F_{A^{IF}}(x) \in [0,1]$ indicates truth and falsity grades with $0 \leqslant T_{A^{IF}}(x) + F_{A^{IF}}(x) \leqslant 1$.

1.2.1.1 Set Theoretic Operations of Intuitionistic Fuzzy Set

Basic model, properties, arithmetic operations, algebraic operators and relations over the intuitionistic set have been defined by Atanassov [19] in his paper. A few of them are given below.

Let A^{IF} , B^{IF} and C^{IF} be three intuitionistic fuzzy sets on a universe X.

1.2.1.2 Union

Union of any two intuitionistic fuzzy sets A^{IF} and B^{IF} is mathematically denoted as,

$$A^{IF} \cup B^{IF} = \{(x, max(T_{A^{IF}}(x), T_{B^{IF}}(x)), min(F_{A^{IF}}(x), F_{B^{IF}}(x)) : x \in X)\}.$$

1.2.1.3 Intersection

Intersection of any two intuitionistic fuzzy sets A^{IF} and B^{IF} is defined as,

$$A^{IF} \cap B^{IF} = \{(x, min(T_{A^{IF}}(x), T_{B^{IF}}(x)), max(F_{A^{IF}}(x), F_{B^{IF}}(x)) : x \in X)\}.$$

Above expressions of union and intersection can also be expressed as,

$$A^{IF} \cup B^{IF} = \{(x, T_{A^{IF}}(x) \vee T_{B^{IF}}(x), F_{A^{IF}}(x) \wedge F_{B^{IF}}(x) : x \in X)\},$$
$$A^{IF} \cap B^{IF} = \{(x, T_{A^{IF}}(x) \wedge T_{B^{IF}}(x), F_{A^{IF}}(x) \vee F_{B^{IF}}(x) : x \in X)\}.$$

1.2.1.4 Equality of Two Intuitionistic Fuzzy Sets

Intuitionistic fuzzy sets A^{IF} and B^{IF} are said to be equal iff,

$$A^{IF} = B^{IF} \Longleftrightarrow A^{IF} \subset B^{IF} \quad \text{and} \quad B^{IF} \subset A^{IF}.$$

A^{IF} is a subset of B^{IF} when the below condition is satisfied,

$$T_{A^{IF}}(x) \leq T_{B^{IF}}(x) \quad \text{and} \quad F_{A^{IF}}(x) \geq F_{B^{IF}}(x).$$

1.2.1.5 Compliment

The complement of a fuzzy set A^{IF} on the universe of discourse X, denoted as $\overline{A^{IF}}$ is defined as ,

$$\overline{A^{IF}} = \{(x, F^{IF}(x), T_{A^{IF}}(x) : x \in X)\}.$$

The basic set theoretic properties which holds for crisp and fuzzy set are also true for intuitionistic set. A few of such properties for intersection, union and complement which are valid for intuitionistic sets similar to crisp sets are commutative property, associative property, distributive laws, De Morgan's laws and involution.

Atanassov [20] himself have played a vital role in the development of theory based on intuitionistic sets. New and more generalized results were introduced by him in his paper written in the continuation of his introductory paper on IFS. Atanassov and Gargov [21]-[22] gave the idae of IVIFS in the significance of IVFS and proved the results for the relations, operators, operations and different properties for every two IVIFS. De *et al.* [23] suggested the use of fuzzy sets in medical analysis and procedures. Szmidt and Kacprzyk [24] presented the core idea that degree of non-membership in intuitionistic fuzzy sets may not always equals to one minus grade of membership. They have explained the concept of hesitation degree and following this basic line of reasoning defined Euclidean distance, hamming distance, normalized Euclidean distance and normalized hamming distance on intuitionistic sets.

1.2.2 Neutrosophic Sets

Developments in philosophy, human reasoning and language as well as their involvement in mathematical models have encouraged many to look for more advanced generalizations of fuzzy set. To model indeterminacy and inconsistency of human reasoning Smarandache [25] presented the idea of neutrosophic sets. As said by Smarandache *Neutrosophy is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra* [25]. Word neutrosophy is a combination of two words, Latin neuter which means neutral and Greek sophia which means wisdom and skills. Neutrosophy is actually *knowledge of neutral thoughts*, everyone experience in daily life. On broader spectrum neutrosophy is actually a view point which helps to the expansion of conventional theory of probability to neutrosophic probability theory, fuzzy sets to neutrosophic sets and fuzzy logic to neutrosophic logic. Hence new and emerging concept of neutrosophy is a better choice in many cases where new advancements and social as well as philosophical needs bind us to work in multi-disciplinary environment.

In real life there may exist such situations where together with membership and non-membership, indeterminacy grade is also required. Neutrosophic notion explains and deals with all three. Thus neutrosophic set is an advanced idea in which every element $x \in X$ to $\bar{A}^N$ has a true grade of membership $T_{\bar{A}^N}(x)$, non-membership grade $F_{\bar{A}^N}(x)$ and a degree of indeterminacy $I_{\bar{A}^N}(x)$, where each $T_{\bar{A}^N}(x), I_{\bar{A}^N}(x)$ and $F_{\bar{A}^N}(x)$ be subsets of $]0^-, 1^+[$ and independent of each other. The case when $T_{\bar{A}^N}(x), I_{\bar{A}^N}(x)$ and $F_{\bar{A}^N}(x) \in [0,1]$ referred to a class of neutrosophic sets named as single valued neutrosophic sets. On this specific case of neutrosophic set Haibin *et al.* [26] have characterized the set-theoretic operators which has a utilization in the engineering and scientific issues. Also for single valued neutrosophic set $T_{\bar{A}^N}(x), I_{\bar{A}^N}(x)$ and $F_{\bar{A}^N}(x)$ are not linked and related to one another with $0 \leq T_{\bar{A}^N}(x) + I_{\bar{A}^N}(x) + F_{\bar{A}^N}(x) \leq 3$.

1.2.2.1 Neutrosophic set Theoretic Operations

Let $\bar{A}^N$, $\bar{B}^N$ and $\bar{C}^N$ be three neutrosophic sets.

1.2.2.2 Union

Union of $\bar{A}^N$ and $\bar{B}^N$ is denoted as,

$$\begin{aligned}\bar{A}^N \cup \bar{B}^N =&\{(x, max(T_{\bar{A}^N}(x), T_{\bar{B}^N}(x)), max(I_{\bar{A}^N}(x), I_{\bar{B}^N}(x)),\\ &min(F_{\bar{A}^N}(x), F_{\bar{B}^N}(x)) : x \in X)\}.\end{aligned}$$

The union of two neutrosophic set $\bar{A}^N$ and $\bar{B}^N$ is again a neutrosophic set $\overline{U}^N$, denoted as,

$$\overline{U}^N = \bar{A}^N \cup \bar{B}^N.$$

Being a neutrosophic set this union is again defined in terms of its membership function. Thus the membership function of $\overline{U}^A$ is interpreted as:

$$T_{\overline{U}^N}(x) = \max(T_{\overline{A}^N}(x), T_{\overline{B}^N}(x)), \quad \text{or} \quad T_{\overline{U}^N}(x) = T_{\overline{A}^N}(x) \vee T_{\overline{B}^N}(x), \quad (1.2.2)$$

$$I_{\overline{U}^N}(x) = \max(I_{\overline{A}^N}(x), I_{\overline{B}^N}(x)) \quad \text{or} \quad I_{\overline{U}^N}(x) = I_{\overline{A}^N}(x) \vee I_{\overline{B}^N}(x), \quad (1.2.3)$$

$$F_{\overline{U}^N}(x) = \min(F_{\overline{A}^N}(x), F_{\overline{B}^N}(x)) \quad \text{or} \quad F_{\overline{U}^N}(x) = F_{\overline{A}^N}(x) \wedge F_{\overline{B}^N}(x). \quad (1.2.4)$$

$\forall \quad x \in X$

1.2.2.3 Intersection

Intersection of $\bar{A}^N$ and $\bar{B}^N$ is defined mathematically as,

$$\begin{aligned}\bar{A}^N \cap \bar{B}^N =&\{(x, min(T_{\bar{A}^N}(x), T_{\bar{B}^N}(x)), min(I_{\bar{A}^N}(x), I_{\bar{B}^N}(x)),\\ &max(F_{\bar{A}^N}(x), F_{\bar{B}^N}(x)) : x \in X)\}.\end{aligned}$$

Similarly, confluence of $\bar{A}^N$ and $\bar{B}^N$ is again a neutrosophic set which is defined by designing its grades of corresponding membership functions. Let $\overline{D}_N =$

$\bar{A}^N \cap \bar{B}^N$ denotes the intersection of two neutrosophic sets $\bar{A}^N$ and $\bar{B}^N$ defined as,

$$T_{\overline{D}^N}(x) = \min(T_{\overline{A}^N}(x), T_{\overline{B}^N}(x)) \quad \text{or} \quad T_{\overline{D}^N}(x) = T_{\overline{A}^N}(x) \wedge T_{\overline{B}^N}(x), \tag{1.2.5}$$

$$I_{\overline{D}^N}(x) = \min(I_{\overline{A}^N}(x), I_{\overline{B}^N}(x)) \quad \text{or} \quad I_{\overline{D}^N}(x) = I_{\overline{A}^N}(x) \wedge I_{\overline{B}^N}(x), \tag{1.2.6}$$

$$F_{\overline{D}^N}(x) = max(F_{\overline{A}^N}(x), F_{\overline{B}^N}(x)) \quad \text{or} \quad F_{\overline{D}^N}(x) = F_{\overline{A}^N}(x) \vee F_{\overline{B}^N}(x). \tag{1.2.7}$$

Where X be the universe of discourse and each $x \in X$.

1.2.2.4 Compliment

Compliment of $\overline{A}^N$ is denoted as $\overline{A}^{Nc}$. Mathematically defined as follow;

$$T_{\overline{A}^{Nc}}(x) = F_{\overline{A}^N}(x), \tag{1.2.8}$$

$$I_{\overline{A}^{Nc}}(x) = 1 - I_{\overline{A}^N}(x), \tag{1.2.9}$$

$$F_{\overline{A}^{Nc}}(x) = T_{\overline{A}^N}(x). \tag{1.2.10}$$

For all $x \in X$.

1.2.2.5 Containment

Containment of two neutrosophic sets again depends on how the membership functions are defined. $\overline{A}^N \subseteq \overline{B}^N$ if and only if

$$T_{\overline{A}^N} \leq T_{\overline{B}^N} \quad , \quad I_{\overline{A}^N} \leq I_{\overline{B}^N} \quad \text{and} \quad F_{\overline{A}^N} \geq F_{\overline{B}^N}.$$

Same as the fuzzy sets the union of two neutrosophic sets is the smallest neutrosophic set containing both $\overline{A}^N$ and $\overline{B}^N$, whereas, intersection of two neutrosophic sets is the largest neutrosophic set which is contained in both $\overline{A}^N$ and $\overline{B}^N$. Similar to crisp and fuzzy sets, neutrosophic sets also satisfy laws like commutativity, associativity, distributivity, absorption and idempotency, with the exception of the principle of excluded middle.

Samranche himself has worked a lot to establish the theory and build the foundation of neutrosophic sets. His first contribution, after defining neutrosophic sets, was to convert the philosophical aspect of standard or non-standard neutrosophic sets to the one technically fit for use in engineering and scientific scenarios.

In his work, Smarandache [18] presented a more vivid image of the notion that the neutrosophic set is a refinement of already exiting sets with uncertainties. The idea was supported by many examples from real life. There are circumstances in reality when the indeterminacy can be distinguished to be indeterminacy that has a greater amount of reality esteem than the false esteem yet cannot be named truth. Additionally, at times, the indeterminacy can be recognized to be indeterminacy that has a greater amount of the false incentive than reality esteem yet can't be delegated false. To give greater acceptability to indeterminacy, this sort of indeterminacy is grouped into two. Kandasamy and Mushtaq [27] has outlined doubled-valued neutrosophic sets with two distinct indeterminacy values. Indeterminacy inclining towards truth or indeterminacy inclining towards false makes the indeterminacy engaged with the situation to be more exact and precise. It gives a superior and detailed perspective of the current indeterminacy. A double-valued neutrosophic set $\overline{A}^{ND}$ in X is quantified by true grade of membership $T_{\overline{A}^{ND}}(x)$, indeterminacy inclining towards truth grade of membership $IT_{\overline{A}^{ND}}(x)$, indeterminacy inclining towards false membership grade $IF_{\overline{A}^{ND}}(x)$, and falsity membership grade $F_{\overline{A}^{ND}}(x)$. Different operators, associated properties and axioms are illustrated. Also, a clustering algorithm is constructed on the basis of generalized distance measure.

1.2.3 Refinements of Neutrosophic Set

Kandasamy and Smarandache [28] in their paper defined triple refined indeterminate neutrosophic set with three distinct indeterminacy values. They claimed that new set could handle incomplete and inconsistent information more efficiently and proved their results by an example based on personality classification. Smarandache [29] further extended the idea to the n-refinements and suggested that generalizations of NS can be attained by refining any of the true, false and indeterminacy membership. Another novel augmentation of the neutrosophic set was presented by Zhang *et al.* [30], which he named the interval-valued neutrosophic set. These sets are capable of handling fragmented information with greater effectiveness. Different operations and new refinements have been defined and by applying the aggregation operators a method for multi criteria decsion making was also developed. Smarandache [31] presented the idea of the degree of dependency and independence among the elements of NS.

Chapter 2

Bipolar Fuzzy Sets and Generalizations

Pivert and Bosc [32] stated that Bipolarity refers to the tendency of the human brain to settle on choices and reason based on certain negative and positive effects. Positive data indicates what is feasible, acceptable, permissible, achievable, desired, or considered appropriate and permitted. Whereas, negative statements express what is intolerable, prohibited, forbidden or rejected. Negative inclinations refers to constraints, since they determine which esteems, objects or values must be rejected (i.e., those that don't fulfill the constraints), while positive inclinations relate to wishes, as they indicate which articles are more required and anticipated than others (i.e., fulfill client wishes) without dismissing those that don't meet the desires. A wide assortment of human decision making, particularly multi-agent choice and decision examination, depends on bipolar or twofold judgmental reasoning on a negative side and a positive side. A few of them may include impact and symptom, fellowship and aggression, collaboration and competition, feedback and feed forward, companionship and enmity, mutual benefits and conflict benefits and many more. It is debated that positive and negative causal connections should not be covered in a summation if they are not neutralizing in the meantime, or not from same sources, or not through same ways. Boolean set and its augmentations are unipolar that cannot be straightforwardly used to characterize bipolar truth for conception. To evade the representational limitations of unipolar frameworks, a bipolar framework was proposed by Lee [33]-[34].

2.1 Bipolar Fuzzy Set

Let X indicate a universe of discourse. Then a bipolar fuzzy set defined on X, denoted as $BF(x)$ is a pair (T^+, T^-) , where $T^+ : X \longrightarrow [0,1]$ and $T^- : X \longrightarrow [-1,0]$ are any mappings (membership functions) and T^+ and T^- actually displays property and counter property.
There maybe conditions when $T^+ \neq 0$ and $T^- = 0$ or $T^+ = 0$ and $T^- \neq 0$ or an overlapping of membership function of property over counter property for an element x over some portion of X. For this situation $T^+ \neq 0$ and $T^- \neq 0$. $BF(x)$ is an expansion of fuzzy set whose grade of membership range between [-1,1]. Membership grade zero of a component implies that the component is irrelative to the comparing quality. Membership degree (0,1] of a component demonstrates that the component fairly fulfills the property and degree [-1,0) implies that the component somewhat fulfills the implicit counter property. It is very important to note that a bipolar fuzzy set capturing only T^+, i.e. positive property only is a special case when $BF(x)$ is actually behaving as fuzzy set.
As clear from the definition $BF(x)$ is a subset of the universe of discourse X. But for efficiency and simplicity, term subset is mostly omitted and bipolar fuzzy subset is referred only as bipolar fuzzy set.
When $X = \{x_1, x_2, ..., x_n\}$, then $BF(x)$ is expressed as,

$$\begin{aligned}\tilde{A}^{Bf} =&\{(x_1, T^+_{\tilde{A}^{Bf}}(x_1), T^-_{\tilde{A}^{Bf}}(x_1)), (x_2, T^+_{\tilde{A}^{Bf}}(x_2),\\ &T^-_{\tilde{A}^{Bf}}(x_2)), ..., (x_n, T^+_{\tilde{A}^{Bf}}(x_n), T^-_{\tilde{A}^{Bf}}(x_n))\}.\end{aligned}$$

Generally, when X is finite then $BF(x)$ is expressed as,

$$\tilde{A}^{Bf} = \frac{(T^{+}_{\tilde{A}^{Bf}}(x_1), T^{-}_{\tilde{A}^{Bf}}(x_1))}{x_1}, \frac{(T^{+}_{\tilde{A}^{Bf}}(x_2), T^{-}_{\tilde{A}^{Bf}}(x_2))}{x_2}, \ldots, \frac{(T^{+}_{\tilde{A}^{Bf}}(x_n), T^{-}_{\tilde{A}^{Bf}}(x_n))}{x_n}. \tag{2.1.1}$$

Or even more simply,

$$\tilde{A}^{Bf} = \sum_{i=1}^{n} \frac{(T^{+}_{\tilde{A}^{Bf}}(x_i), T^{-}_{\tilde{A}^{Bf}}(x_i))}{x_i}. \tag{2.1.2}$$

It is important to note that the operations, addition and summation, in the Equation 2.1.1 and 2.1.2 do not relate to the usual addition but actually the set theoretic or. On the other hand if X is infinite then frequently $\tilde{A}^{Bf}$ is expressed as,

$$\tilde{A}^{Bf} = \int_X \frac{(T^{+}_{\tilde{A}^{Bf}}(x), T^{-}_{\tilde{A}^{Bf}}(x))}{x}. \tag{2.1.3}$$

In Equation 2.1.3, "$\int$" can be considered as natural expression of summation.

2.1.1 Basic Characteristics of Bipolar Fuzzy Sets (Elementary Features)

Some basic definitions for bipolar fuzzy sets are as follow:

2.1.1.1 Support

Support of a bipolar fuzzy set $\tilde{A}^{Bf}$ on X, denoted as supp($\tilde{A}^{Bf}$) are those points in X at which $T^{+}_{\tilde{A}^{Bf}}(x)$ is positive and $T^{-}_{\tilde{A}^{Bf}}(x)$ is negative.

$$\text{supp}(\tilde{A}^{Bf}) = \{x \in X : T^{+}_{\tilde{A}^{Bf}}(x) > 0 \quad \wedge \quad T^{-}_{\tilde{A}^{Bf}}(x) < 0\}.$$

Sometime it is appropriate to mention only positive support or negative support of any bipolar fuzzy set $\tilde{A}^{Bf}$. Mathematically, positive and negative support are as follow,

$$\text{supp}^{+}(\tilde{A}^{Bf}) = \{x \in X : T^{+}_{\tilde{A}^{Bf}}(x) > 0\},$$
$$\text{supp}^{-}(\tilde{A}^{Bf}) = \{x \in X : T^{-}_{\tilde{A}^{Bf}}(x) < 0\}.$$

2.1.1.2 Height

On a nonempty set X height of $\tilde{A}^{Bf}$ is defined as the maximum value attained by the positive membership degree.

$$h(\tilde{A}^{Bf}) = \max\{T^{+}_{\tilde{A}^{Bf}}(x) \quad \text{for all} \quad x \in X\}.$$

2.1.1.3 Depth

In a similar manner depth of $\tilde{A}^{Bf}$ is defined as the minimum value attained by the negative membership degree, expressed mathematically as,

$$d(\tilde{A}^{Bf}) = \min\{T^{-}_{\tilde{A}^{Bf}}(x) \quad \text{for all} \quad x \in X\}.$$

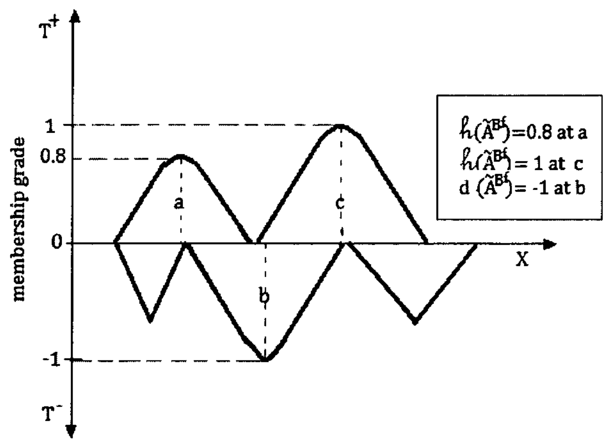

Figure 2.1: Height and depth of bipolar fuzzy set

Figure 2.1 represents the height and depth of bipolar fuzzy set.

2.1.1.4 Normalized Bipolar Fuzzy Set

If the depth of a bipolar fuzzy set $\tilde{A}^{Bf}$ is -1 and height is1 then the set is said to be normalized fuzzy set.

2.1.1.5 α^{+}-Level Set and α^{-}-Level Set

α^{+}-level set and α^{-}-level set of a bipolar fuzzy set are crisp sets defined mathematically as,

$$\tilde{A}^{Bf}_{\alpha^{+}} = \{x/T_{\tilde{A}^{Bf}}(x) \geq \alpha^{+}\} \quad \text{for all} \quad \alpha^{+} \in [0,1],$$

and

$$\tilde{A}^{Bf}_{\alpha^{-}} = \{x/T_{\tilde{A}^{Bf}}(x) \leq \alpha^{-}\} \quad \text{for all} \quad \alpha^{-} \in [-1,0].$$

strong α^{+}-level set is the case when $\alpha^{+} \in (0,1]$ and the weak α^{-}-level set when $\alpha^{-} \in [-1,0)$, mathematically,

$$\tilde{A}^{Bf}_{\alpha'} = \{x/T_{\tilde{A}^{Bf}}(x) > \alpha'\} \quad \text{for all} \quad \alpha' \in (0,1], \tag{2.1.4}$$

Equation 2.1.4 denotes strong α-level set.

$$\tilde{A}^{Bf}_{\alpha''} = \{x/T_{\tilde{A}^{Bf}}(x) < \alpha''\} \quad \text{for all} \quad \alpha'' \in [-1,0). \tag{2.1.5}$$

Equation 2.1.5 represents weak α-level set (Figure 2.2).

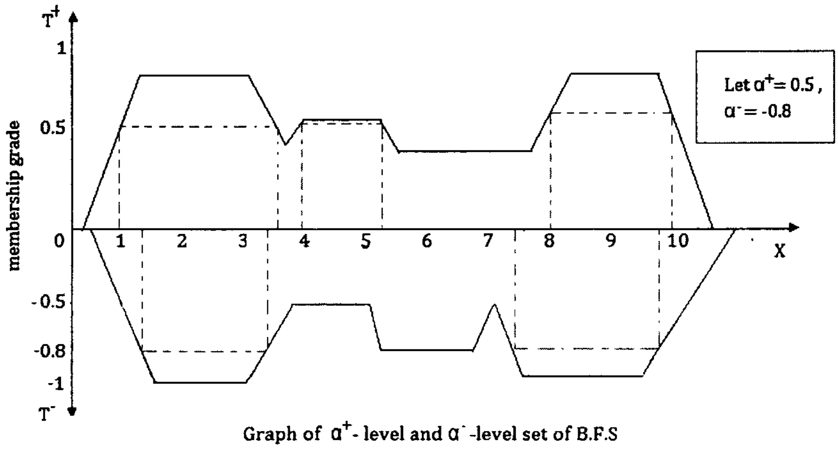

Figure 2.2: Graph of α^+-level and α^--level of bipolar fuzzy set

2.1.1.6 Upper Core of Bipolar Fuzzy Set

Upper core of bipolar fuzzy set $\tilde{A}^{Bf}$ are those points of $\tilde{A}^{Bf}$ whose positive membership grades attains full value, i.e. 1, mathematically expressed as,

$$\tilde{C}_{\tilde{A}^{Bf}} = \{x : T^+_{\tilde{A}^{Bf}}(x) = 1\}.$$

2.1.1.7 Lower Core of Bipolar Fuzzy Set

Lower core of bipolar fuzzy set $\tilde{A}^{Bf}$ are those elements of $\tilde{A}^{Bf}$ whose negative membership grades attains least value from the interval , i.e. -1, mathematically expressed as,

$$\bar{C}_{\tilde{A}^{Bf}} = \{x : T^-_{\tilde{A}^{Bf}}(x) = -1\}.$$

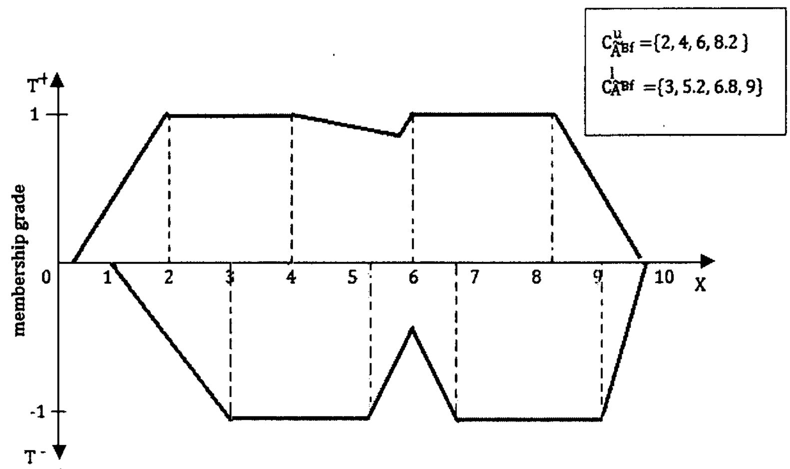

Figure 2.3: Upper core and lower core of bipolar fuzzy set

Figure 2.3 shows both upper core and lower core of bipolar fuzzy set.

2.1.1.8 Convex Bipolar Fuzzy Set

Convexity of bipolar fuzzy set also depends on certain conditions on membership grade given below,

$$T^{+}_{\tilde{A}^{Bf}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(T^{+}_{\tilde{A}^{Bf}}(x_1), T^{+}_{\tilde{A}^{Bf}}(x_2))$$

for all $x_1, x_2 \in \tilde{A}^{Bf}, \lambda \in [0, 1]$,

$$T^{-}_{\tilde{A}^{Bf}}((1+\lambda)x_1 - \lambda x_2) \leq max(T^{-}_{\tilde{A}^{Bf}}(x_1), T^{-}_{\tilde{A}^{Bf}}(x_2))$$

for all $x_1, x_2 \in \tilde{A}^{Bf}, \lambda \in [-1, 0]$.

Geometrical Representation : T^{+}**-convex** Let $t = (\lambda x_1 + (1-\lambda)x_2)$ for any particular points x_1 and x_2 from set $\tilde{A}^{Bf}$ and $\lambda \in [0, 1]$. Then $(\lambda x_1 + (1-\lambda)x_2)$ must lie between x_1 and x_2, Figure 2.4.

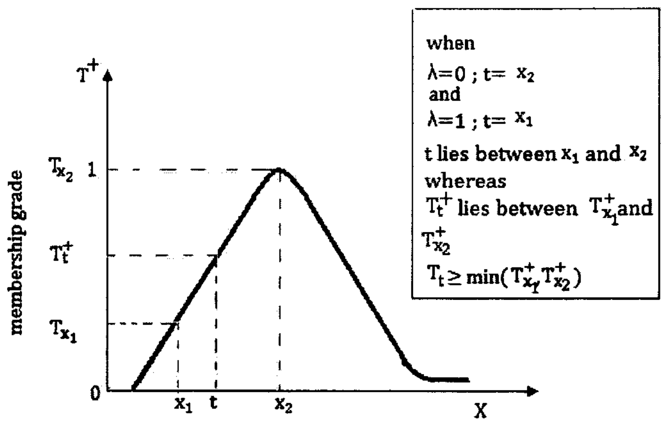

Figure 2.4: T^+-convex bipolar fuzzy set

Geometrical Representation : T^--convex Let $t = ((1-\lambda)x_1 - \lambda x_2)$ for all x_1 and x_2 belongs to $\tilde{A}^{Bf}$ and $\lambda \in [-1, 0]$. Then t must lie between x_1 and x_2, Figure 2.5.

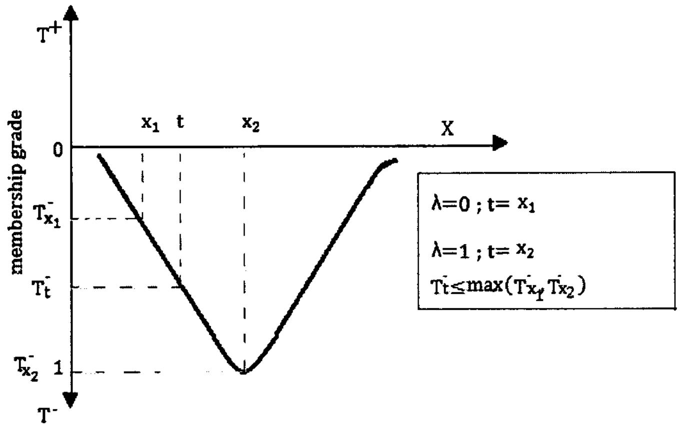

Figure 2.5: T^--convex bipolar fuzzy set

2.1.1.9 Equivalence of Two Bipolar Fuzzy Sets

Let $\tilde{A}^{Bf}$, $\tilde{B}^{Bf}$ and $\tilde{C}^{Bf}$ are three bipolar fuzzy sets. Equivalence of $\tilde{A}^{Bf}$ and $\tilde{B}^{Bf}$ also depends on the equivalence of their corresponding membership grades, i.e. $\tilde{A}^{Bf}$ and $\tilde{B}^{Bf}$ are equivalent, iff,

$$T^+_{\tilde{A}^{Bf}} = T^+_{\tilde{B}^{Bf}} \quad \text{and} \quad T^-_{\tilde{A}^{Bf}} = T^-_{\tilde{B}^{Bf}} \quad \text{for all} \quad x \in X.$$

2.1.1.10 Subset of Two Bipolar Fuzzy Sets

A $BF(x)$ named $\tilde{A}^{Bf}$ is a subset of another $BF(x)$ named $\tilde{B}^{Bf}$,(Figure 2.6), iff,

$$T^+_{\tilde{A}^{Bf}} \leq T^+_{\tilde{B}^{Bf}} \quad \text{and} \quad T^-_{\tilde{A}^{Bf}} \geq T^-_{\tilde{B}^{Bf}} \quad \text{for all} \quad x \in X.$$

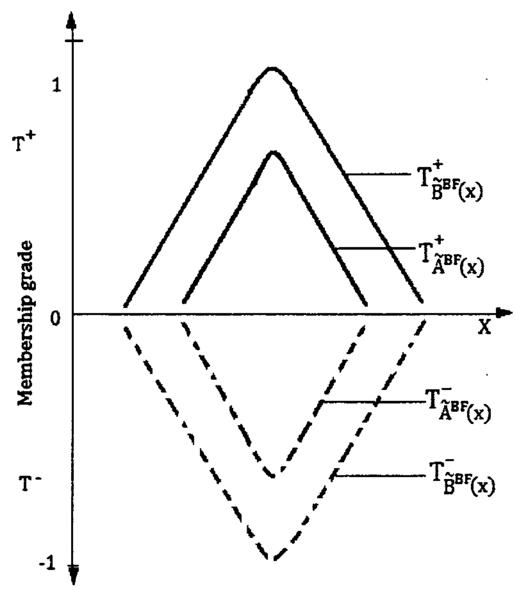

Figure 2.6: Bipolar fuzzy subset

Example 2.1.1. Suppose it is required to measure the effects and side effects of certain analgesics. Let their effects be marked as T^+ and side effects as T^-, whereas, $T^+ \in [0, 1]$ and $T^- \in [-1, 0]$. Table 2.1 is constructed on the basis of random study of different drugs.

Sample Analgesics $x \in X$	Generic Name	Membership grades in terms of effects and side effects (μ^+, μ^-)
x_1	Aspirin	$(x_1, 0.4, -0.2)$
x_2	Ibuprofen	$(x_2, 0.6, -0.4)$
x_3	Tramadol	$(x_3, 0.8, -0.3)$
x_4	Paracetamol	$(x_4, 0.7, -0.5)$
x_4	Panadol	$(x_5, 0.7, -0.6)$

Table 2.1: Bipolar fuzzy set

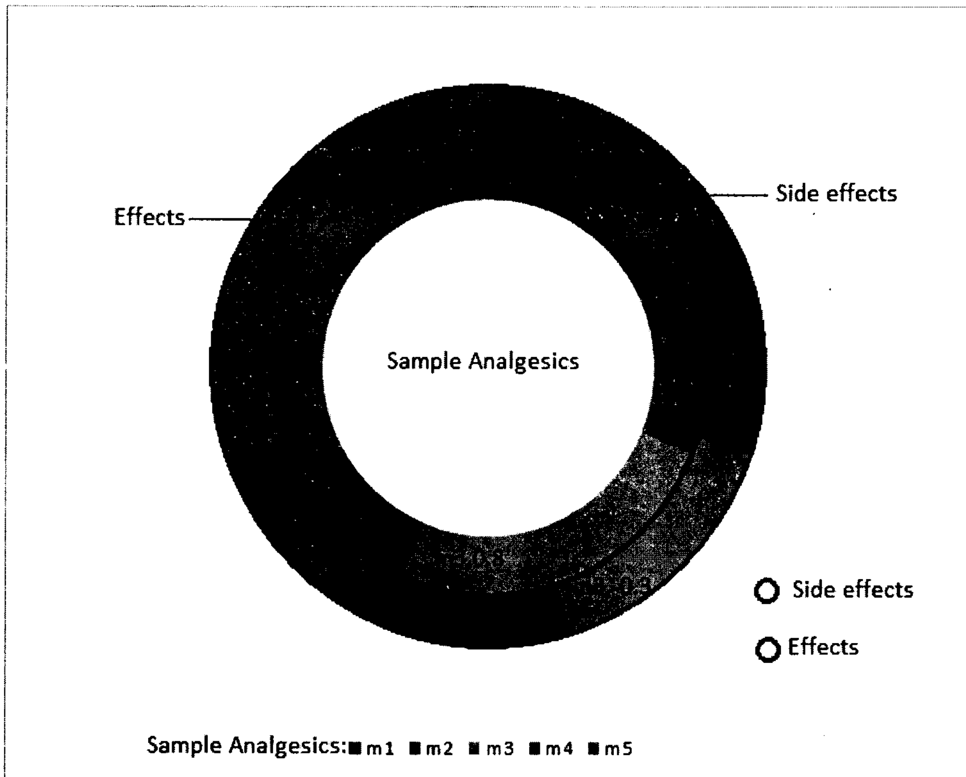

Figure 2.7: Comparison of effects and side effects of different pain-killer

Figure 2.7 provides a clear picture of effects and side effects of certain medicines.

Example 2.1.2. Let $\tilde{A}^{Bf}$ and $\tilde{B}^{Bf}$ be two BFs.

$$\tilde{A}^{Bf} = \{(x_1, 0, -0.1), (x_2, 0.1, -0.1), (x_3, 0.5, -0.4), (x_4, 0.7, -0.8), (x_5, 0.9, -0.9)$$

$$\tilde{B}^{Bf} = \{(x_1, 0.3, -0.2), (x_2, 0.4, -0.3), (x_3, 0.6, -0.5), (x_4, 0.8, -0.9), (x_5, 1, -1)\}.$$

Analysis of different characteristics of bipolar fuzzy sets using $\tilde{A}^{Bf}$ and $\tilde{B}^{Bf}$ are as fellow,

1. **Subset:** Clearly, $\tilde{A}^{Bf}$ is a subset of $\tilde{B}^{Bf}$, $\tilde{A}^{Bf} \subset \tilde{B}^{Bf}$.

2. **Support:**

$$\text{supp}^+(\tilde{A}^{Bf}) = \{x_2, x_3, x_4, x_5\},$$

$$\text{supp}^+(\tilde{B}^{Bf}) = \{x_1, x_2, x_3, x_4, x_5\},$$

$$\text{supp}^-(\tilde{A}^{Bf}) = \{x_1, x_2, x_3, x_4, x_5\},$$

$$\text{supp}^-(\tilde{B}^{Bf}) = \{x_1, x_2, x_3, x_4, x_5\}.$$

3. **Height:** $h(\tilde{A}^{Bf}) = 0.9$, $h(\tilde{B}^{Bf}) = 1$.

4. **Depth:**

$$d(\tilde{A}^{Bf}) = -0.9,$$

$$d(\tilde{B}^{Bf}) = -1.$$

5. **Normal:** $\tilde{A}^{Bf}$ is not normalized, whereas ,$\tilde{B}^{Bf}$ is a normalized bipolar fuzzy set.

6. If α^+ is 0.7, then,

$$\tilde{A}^{Bf}_{0.7} = \{x_4, x_5\},$$

$$\tilde{A}^{Bf}_{0.7'} = \{x_5\},$$

$$\tilde{B}^{Bf}_{0.7} = \{x_4, x_5\},$$

$$\tilde{B}^{Bf}_{0.7'} = \{x_4, x_5\}.$$

If α^- is -0.4, then,

$$\tilde{A}^{Bf}_{-0.4} = \{x_3, x_4, x_5\},$$

$$\tilde{A}^{Bf}_{-0.4''} = \{x_4, x_5\},$$

$$\tilde{B}^{Bf}_{-0.4} = \{x_3, x_4, x_5\},$$

$$\tilde{B}^{Bf}_{-0.4''} = \{x_3, x_4, x_5\}.$$

7. **Core:**

$$\text{Upper core of } \quad \tilde{A}^{Bf} = \tilde{C}_{\tilde{A}^{Bf}} = \phi,$$

$$\text{Upper core of } \quad \tilde{B}^{Bf} = \tilde{C}_{\tilde{B}^{Bf}} = \{x_5\},$$

2.1.2 Algebraic Operators and Properties of Bipolar Fuzzy Sets

2.1.2.1 Union

Union of two bipolar fuzzy sets,

$$\tilde{A}^{Bf} = \{(x, T^+_{\tilde{A}^{Bf}}, T^-_{\tilde{A}^{Bf}}) : \quad \forall \quad x \in X\},$$
$$\tilde{B}^{Bf} = \{(x, T^+_{\tilde{B}^{Bf}}, T^-_{\tilde{B}^{Bf}}) : \quad \forall \quad x \in X\}.$$

is denoted as $\tilde{U}^{Bf}(x) = \tilde{A}^{Bf}(x) \cup \tilde{B}^{Bf}(x)$ is also a bipolar fuzzy set whose membership grade is defined as,

$$\tilde{U}^{Bf}(x) = \{(x, \max(T^+_{\tilde{A}^{Bf}}(x), T^+_{\tilde{B}^{Bf}}(x)), \min(T^-_{\tilde{A}^{Bf}}(x), T^-_{\tilde{B}^{Bf}}(x))) : \forall \quad x \in X\}.$$

Thus;

$$T^+_{\tilde{U}^{Bf}}(x) = \max(T^+_{\tilde{A}^{Bf}}(x), T^+_{\tilde{B}^{Bf}}(x)),$$
$$T^-_{\tilde{U}^{Bf}}(x) = \min(T^-_{\tilde{A}^{Bf}}(x), T^-_{\tilde{B}^{Bf}}(x)).$$

2.1.2.2 Intersection

Intersection of two bipolar fuzzy sets;

$$\tilde{A}^{Bf} = \{(x, T^{+}_{\tilde{A}^{Bf}}, T^{-}_{\tilde{A}^{Bf}}) : \quad \forall \quad x \in X\},$$
$$\tilde{B}^{Bf} = \{(x, T^{+}_{\tilde{B}^{Bf}}, T^{-}_{\tilde{B}^{Bf}}) : \quad \forall \quad x \in X\},$$

is denoted as $\tilde{I}^{Bf}(x) = \tilde{A}^{Bf}(x) \cap \tilde{B}^{Bf}(x)$ is also a bipolar fuzzy set whose membership grade is defined as,

$$\tilde{I}^{Bf}(x) = \{(x, \min(T^{+}_{\tilde{A}^{Bf}}(x), T^{+}_{\tilde{B}^{Bf}}(x)), \max(T^{-}_{\tilde{A}^{Bf}}(x), T^{-}_{\tilde{B}^{Bf}}(x))) : \text{for all} \quad x \in X\}.$$

Thus;

$$T^{+}_{\tilde{I}^{Bf}}(x) = \min(T^{+}_{\tilde{A}^{Bf}}(x), T^{+}_{\tilde{B}^{Bf}}(x)),$$
$$T^{-}_{\tilde{I}^{Bf}}(x) = \max(T^{-}_{\tilde{A}^{Bf}}(x), T^{-}_{\tilde{B}^{Bf}}(x)).$$

2.1.2.3 Compliment

Compliment of a BFs,

$$\tilde{A}^{Bf} = \{(x, T^{+}_{\tilde{A}^{Bf}}, T^{-}_{\tilde{A}^{Bf}}) : \quad \text{for all} \quad x \in X\},$$

is also a BFs denoted as $\hat{C}^{Bf}$, where,

$$\hat{C}^{Bf} = \{x, (\{1^{+}\} - T^{+}_{\tilde{A}^{Bf}}), (\{1^{-}\} - T^{-}_{\tilde{A}^{Bf}})\}.$$

Thus;

$$T^{+}_{\hat{C}^{Bf}} = \{1^{+}\} - T^{+}_{\tilde{A}^{Bf}},$$
$$T^{-}_{\hat{C}^{Bf}} = \{1^{-}\} - T^{-}_{\tilde{A}^{Bf}}.$$

2.1.2.4 Involution

Involution of a bipolar fuzzy set,

$$\tilde{A}^{Bf} = \{(x, T^{+}_{\tilde{A}^{Bf}}, T^{-}_{\tilde{A}^{Bf}}) : \quad \text{for all} \quad x \in X\},$$

is denoted as $(\tilde{A}^{Bf})''$ and defined mathematically as,

$$(\tilde{A}^{Bf})'' = \{x, (\{1^+\} - T^+_{\tilde{C}^{Bf}}), (\{1^-\} - T^+_{\tilde{C}^{Bf}})\}.$$

Example 2.1.3. Given three bipolar fuzzy sets $\tilde{A}^{Bf}$, $\tilde{B}^{Bf}$ and $\tilde{C}^{Bf}$

$$\begin{aligned}
\tilde{A}^{Bf} &= \{(x_1, 0, -0.1), (x_2, 0.5, -0.4), (x_3, 0.7, -0.8)\}, \\
\tilde{B}^{Bf} &= \{(x_1, 0.3, -0.2), (x_2, 0.4, -0.3), (x_3, 0.6, -0.5)\}, \\
\tilde{C}^{Bf} &= \{(x_1, 0.1, -0.1), (x_2, 0.9, -0.9), (x_3, 1, -1)\}.
\end{aligned}$$

Union

$$\tilde{A}^{Bf} \cup \tilde{B}^{Bf} = \{(x_1, 0.3, -0.2), (x_2, 0.5, -0.4), (x_3, 0.7, -0.8)\}.$$

Intersection

$$\tilde{A}^{Bf} \cap \tilde{B}^{Bf} = \{(x_1, 0, -0.1), (x_2, 0.4, -0.3), (x_3, 0.6, -0.5)\}.$$

Compliment

$$(\tilde{A}^{Bf})' = \{(x_1, 1, -0.9), (x_2, 0.6, -0.6), (x_3, 0.3, -0.2)\}.$$

2.1.2.5 Properties Defined over Algebraic Operators of Bipolar Fuzzy Sets

Bipolar fuzzy set satisfies most of the algebraic laws and axioms. A few of them are given below.

Absorption Laws

$$\begin{aligned}
\tilde{A}^{Bf} \cup (\tilde{A}^{Bf} \cap \tilde{B}^{Bf}) &= \tilde{A}^{Bf}, \\
\tilde{A}^{Bf} \cap (\tilde{A}^{Bf} \cup \tilde{B}^{Bf}) &= \tilde{A}^{Bf}.
\end{aligned}$$

Commutative Laws

$$\tilde{A}^{Bf} \cup \tilde{B}^{Bf} = \tilde{B}^{Bf} \cup \tilde{A}^{Bf},$$

$$\tilde{A}^{Bf} \cap \tilde{B}^{Bf} = \tilde{B}^{Bf} \cap \tilde{A}^{Bf}.$$

Associative Laws

$$(\tilde{A}^{Bf} \cup \tilde{B}^{Bf}) \cup \tilde{C}^{Bf} = \tilde{A}^{Bf} \cup (\tilde{B}^{Bf} \cup \tilde{C}^{Bf}),$$

$$(\tilde{A}^{Bf} \cap \tilde{B}^{Bf}) \cap \tilde{C}^{Bf} = \tilde{A}^{Bf} \cap (\tilde{B}^{Bf} \cap \tilde{C}^{Bf}).$$

Distributive Laws

$$\tilde{A}^{Bf} \cup (\tilde{B}^{Bf} \cap \tilde{C}^{Bf}) = (\tilde{A}^{Bf} \cup \tilde{B}^{Bf}) \cap (\tilde{A}^{Bf} \cup \tilde{C}^{Bf}),$$

$$\tilde{A}^{Bf} \cap (\tilde{B}^{Bf} \cup \tilde{C}^{Bf}) = (\tilde{A}^{Bf} \cap \tilde{B}^{Bf}) \cup (\tilde{A}^{Bf} \cap \tilde{C}^{Bf}).$$

Idempotent

$$\tilde{A}^{Bf} \cup \tilde{A}^{Bf} = \tilde{A}^{Bf},$$

$$\tilde{A}^{Bf} \cap \tilde{A}^{Bf} = \tilde{A}^{Bf}.$$

De Morgan's law

$$(\tilde{A}^{Bf} \cup \tilde{B}^{Bf})' = (\tilde{A}^{Bf})' \cap (\tilde{B}^{Bf})',$$

$$(\tilde{A}^{Bf} \cap \tilde{B}^{Bf})' = (\tilde{A}^{Bf})' \cup (\tilde{B}^{Bf})'.$$

Involution Axiom

$$(\tilde{A}^{Bf})'' = \tilde{A}^{Bf}.$$

Identity Axioms

$$\tilde{A}^{Bf} \cup \phi = \tilde{A}^{Bf}; \tilde{A}^{Bf} \cup X = X,$$

$$\tilde{A}^{Bf} \cap \phi = \tilde{A}^{Bf}; \tilde{A}^{Bf} \cap X = X.$$

Axiom of Excluded Middle

$$(\tilde{A}^{Bf})' \cup \tilde{A}^{Bf} \neq X.$$

Axiom of Contradiction

$$(\tilde{A}^{Bf})' \cap \tilde{A}^{Bf} \neq \phi.$$

Many properties, such as associativity, commutativity, distributivity, De Morgan laws, absorption, idempotent, and identity, are satisfied by the union and intersection of bipolar fuzzy sets. involution is also valid for bipolar membership function. The membership function of a bipolar fuzzy set behaves almost in the same manner as the membership grade of fuzzy sets. For the same reason, unlike the characteristic function of a crisp set, it does not satisfy the axiom of contradiction and the axiom of excluded middle.

2.2 Bipolar Neutrosophic Set (BNS)

For a non-empty set X, a bipolar neutrosphic set [35] $\hat{A}^{BN}$ is defined as

$$\hat{A}^{BN} =< x, T^{+}_{\hat{A}^{BN}}(x), I^{+}_{\hat{A}^{BN}}(x), F^{+}_{\hat{A}^{BN}}(x), T^{-}_{\hat{A}^{BN}}(x), I^{-}_{\hat{A}^{BN}}(x), F^{-}_{\hat{A}^{BN}}(x) > .$$

Where the positive grades of membership $T^{+}_{\hat{A}^{BN}}(x), I^{+}_{\hat{A}^{BN}}(x), F^{+}_{\hat{A}^{BN}}(x)$ represents the truth, indeterminacy and falsity of a generic element analogous to set $\hat{A}^{BN}$ and $T^{-}_{\hat{A}^{BN}}(x), I^{-}_{\hat{A}^{BN}}(x), F^{-}_{\hat{A}^{BN}}(x)$ represents the truth, indeterminacy and falsity grades of a generic component to some implicit counter attribute analogous to a set $\hat{A}^{BN}$. Where,

$$T^{+}_{\hat{A}^{BN}}, I^{+}_{\hat{A}^{BN}}, F^{+}_{\hat{A}^{BN}} : X \rightarrow [0, 1],$$

$$T^{-}_{\hat{A}^{BN}}, I^{-}_{\hat{A}^{BN}}, F^{-}_{\hat{A}^{BN}} : X \rightarrow [-1, 0].$$

Theorem 2.2.1. *Let $\hat{A}^{BN}$ and $\hat{B}^{BN}$ be two BNS then $\hat{A}^{BN} \subseteq \hat{B}^{BN}$ iff,*

$$T^{+}_{\hat{A}^{BN}}(x) \leq T^{+}_{\hat{B}^{BN}}(x), I^{+}_{\hat{A}^{BN}}(x) \leq I^{+}_{\hat{B}^{BN}}(x), F^{+}_{\hat{A}^{BN}}(x) \geq F^{+}_{\hat{B}^{BN}}(x),$$

$$T^{-}_{\hat{A}^{BN}}(x) \geq T^{-}_{\hat{B}^{BN}}(x), I^{-}_{\hat{A}^{BN}}(x) \geq I^{-}_{\hat{B}^{BN}}(x), F^{-}_{\hat{A}^{BN}}(x) \leq F^{-}_{\hat{B}^{BN}}(x),$$

for all $x \in X$. [35]

Theorem 2.2.2. *Let $\hat{A}^{BN}$ and $\hat{B}^{BN}$ be two BNS. Then $\hat{A}^{BN} = \hat{B}^{BN}$ if and only if*

$$T^{+}_{\hat{A}^{BN}}(x) = T^{+}_{\hat{B}^{BN}}(x), I^{+}_{\hat{A}^{BN}}(x) = I^{+}_{\hat{B}^{BN}}(x), F^{+}_{\hat{A}^{BN}}(x) = F^{+}_{\hat{B}^{BN}}(x),$$

$$T^{-}_{\hat{A}^{BN}}(x) = T^{-}_{\hat{B}^{BN}}(x), I^{-}_{\hat{A}^{BN}}(x) = I^{-}_{\hat{B}^{BN}}(x), F^{-}_{\hat{A}^{BN}}(x) = F^{-}_{\hat{B}^{BN}}(x),$$

for all $x \in X$. [35]

Union, intersection and compliment of bipolar neutrosophic sets are already available in literature [35]. Since focus of current study is to develop algorithm for optimization problem, therefore according to requirement union , intersection and compliment of bipolar neutrosophic sets are redefined here.

2.2.1 Set Theoretic Operations of Bipolar Neutrosophic Sets

2.2.1.1 Union

Union of two bipolar neutrosophic sets
$\hat{A}^{BN} =< x, T^{+}_{\hat{A}^{BN}}(x), I^{+}_{\hat{A}^{BN}}(x), F^{+}_{\hat{A}^{BN}}(x), T^{-}_{\hat{A}^{BN}}(x), I^{-}_{\hat{A}^{BN}}(x), F^{-}_{\hat{A}^{BN}}(x) >$ and
$\hat{B}^{BN} =< x, T^{+}_{\hat{B}^{BN}}(x), I^{+}_{\hat{B}^{BN}}(x), F^{+}_{\hat{B}^{BN}}(x), T^{-}_{\hat{B}^{BN}}(x), I^{-}_{\hat{B}^{BN}}(x), F^{-}_{\hat{B}^{BN}}(x) >$ is a bipolar neutrosophic set $\hat{U}^{BN}$,where $\hat{U}^{BN} = \hat{A}^{BN} \cup \hat{B}^{BN}$,

$$T^{+}_{\hat{U}^{BN}}(x) = max(T^{+}_{\hat{A}^{BN}}(x), T^{+}_{\hat{B}^{BN}}(x)),$$

$$T^{-}_{\hat{U}^{BN}}(x) = min(T^{-}_{\hat{A}^{BN}}(x), T^{-}_{\hat{B}^{BN}}(x)),$$

$$I^{+}_{\hat{U}^{BN}}(x) = max(I^{+}_{\hat{A}^{BN}}(x), I^{+}_{\hat{B}^{BN}}(x)),$$

$$I^-_{\hat{U}^{BN}}(x) = min(I^-_{\hat{A}^{BN}}(x), I^-_{\hat{B}^{BN}}(x)),$$

$$F^+_{\hat{U}^{BN}}(x) = min(F^+_{\hat{A}^{BN}}(x), F^+_{\hat{B}^{BN}}(x)),$$

$$F^-_{\hat{U}^{BN}}(x) = max(F^-_{\hat{A}^{BN}}(x), F^-_{\hat{B}^{BN}}(x)).$$

2.2.1.2 Intersection

Intersection of two BNS $\hat{A}^{BN}$ and $\hat{B}^{BN}$
$\hat{A}^{BN} =< x, T^+_{\hat{A}^{BN}}(x), I^+_{\hat{A}^{BN}}(x), F^+_{\hat{A}^{BN}}(x), T^-_{\hat{A}^{BN}}(x), I^-_{\hat{A}^{BN}}(x), F^-_{\hat{A}^{BN}}(x) >$ and
$\hat{B}^{BN} =< x, T^+_{\hat{B}^{BN}}(x), I^+_{\hat{B}^{BN}}(x), F^+_{\hat{B}^{BN}}(x), T^-_{\hat{B}^{BN}}(x), I^-_{\hat{B}^{BN}}(x), F^-_{\hat{B}^{BN}}(x) >$ is a bipolar neutrosophic set $\hat{V}^{BN}$,where $\hat{V}^{BN} = \hat{A}^{BN} \cap \hat{B}^{BN}$,

$$T^+_{\hat{V}^{BN}}(x) = min(T^+_{\hat{A}^{BN}}(x), T^+_{\hat{B}^{BN}}(x)),$$

$$T^-_{\hat{V}^{BN}}(x) = max(T^-_{\hat{A}^{BN}}(x), T^-_{\hat{B}^{BN}}(x)),$$

$$I^+_{\hat{V}^{BN}}(x) = min(I^+_{\hat{A}^{BN}}(x), I^+_{\hat{B}^{BN}}(x)),$$

$$I^-_{\hat{V}^{BN}}(x) = max(I^-_{\hat{A}^{BN}}(x), I^-_{\hat{B}^{BN}}(x)),$$

$$F^+_{\hat{V}^{BN}}(x) = max(F^+_{\hat{A}^{BN}}(x), F^+_{\hat{B}^{BN}}(x)),$$

$$F^-_{\hat{V}^{BN}}(x) = min(F^-_{\hat{A}^{BN}}(x), F^-_{\hat{B}^{BN}}(x)).$$

2.2.1.3 Compliment

Compliment of a bipolar neutrosophic set
$\hat{A}^{BN} =< x, T^+_{\hat{A}^{BN}}(x), I^+_{\hat{A}^{BN}}(x), F^+_{\hat{A}^{BN}}(x), T^-_{\hat{A}^{BN}}(x), I^-_{\hat{A}^{BN}}(x), F^-_{\hat{A}^{BN}}(x) >$ is denoted by $\hat{C}^{BN}$, where,

$$T^+_{\hat{C}^{BN}}(x) = F^+_{\hat{A}^{BN}}(x),$$
$$I^+_{\hat{C}^{BN}}(x) = 1^+ - I^+_{\hat{A}^{BN}}(x),$$
$$F^+_{\hat{C}^{BN}}(x) = T^+_{\hat{A}^{BN}}(x),$$

$$
\begin{aligned}
T^-_{\hat{C}^{BN}}(x) &= F^-_{\hat{A}^{BN}}(x),\\
I^-_{\hat{C}^{BN}}(x) &= 1^- - I^-_{\hat{A}^{BN}}(x),\\
F^-_{\hat{C}^{BN}}(x) &= T^-_{\hat{A}^{BN}}(x).
\end{aligned}
$$

Example 2.2.3. Let $X = \{x_1, x_2\}$. Bipolar neutrosophic set $\hat{A}^{BN}$ and $\hat{B}^{BN}$ are defined as below;

$$
\begin{aligned}
\hat{A}^{BN} =& \{< x_1, 0.8, 0.3, 0.6, -0.3, -0.2, -0.02 >, < x_2, 0.4, 0.5, 0.1, -0.1, -0.7,\\
& -0.03 >\}\\
\hat{B}^{BN} =& \{< x_1, 0.2, 0.3, 0.4, -0.5, -0.033, -0.99 >, < x_2, 0.7, 0.8, 0.3, -0.08, -0.97,\\
& -0.5 >\}
\end{aligned}
$$

It can be easily verified that bipolar neutrosophic sets obey many of the properties like distributivity, associativity, absorption, idempotent, involution and De Morgan's laws, whereas, like fuzzy sets, they do not satisfy the principle of excluded middle.

Chapter 3

Optimization

The word optimization is derived from the same root as optimal, which implies the best. When someone optimizes anything, they're essentially making it better. Nature is always striving to be the best. A condition of low energy is the default state for physical systems. An isolated chemical system is comprised of molecules that react with one another till the cumulative energy of their electrons is reduced to its smallest value. Light rays move in pathways that are as short as possible in terms of travel time. People are always looking for ways to improve their status. In order to achieve a high rate of return, investors aim to construct portfolios that avoid excessive risk. When it comes to the operation and design of their manufacturing processes, manufacturers strive for the highest possible efficiency. Engineers modify parameters to improve the overall performance of their designs. Hence best can vary from problem to problem, situation to situation and individual to individual. As a result, optimization problems can be classified as either maximization or minimization problems. An optimization problem is actually minimizing or maximizing the value of some function in relation to a set, which is frequently a representation of the range of options accessible in a given situation. The function enables the comparison of different options in order to determine which might be the best.

Numerous disciplines have shown association with optimization which is one of the classical and historical fields of study. Commonly referred as mathematical programming, it involves collection of mathematical methods and principles that can find their applications in multiple fields for solving problems of great complexity. These disciplines may include engineering, architecture, transportation, nutrition, operations research, medicine, artificial intelligence, economics, management, biology and physics along with numerous others. These areas have shown interests in the structures, decisions, designs and ways of processing information in the phenomena of optimization. When a couple of restrictions or constraints are imposed, optimization seeks to identify the best possible way of solving any complex problem. It is not only efficient and resilient algorithms that make optimization possible, but also strong modeling techniques, thoughtful presentation of findings and results, and foolproof software that make optimization possible. Many contemporary applications of optimization problems in medical diagnostics, engineering, transportation infrastructure and synthetic biology are given in [36]-[44].

In our daily lives, all we require is the ability to make the correct decision at the appropriate time. The decisions we took were made with the goal of maximizing benefits while minimizing losses to the greatest extent possible. During World War II, a similar motivation prompted the development of optimization modeling. Decision variables and certain constraint equations are used in optimization to limit decision variables and help discover the best options within a specified range. Constraints are the best allocators of viable regions, but the main objective function that we must optimize determines the optimality of the task. Constraints are the best allocators of viable regions, but the main objective function that we must optimize determines the optimality of the task. Kantorovich [45] presented his study in 1939 with the goal of reducing government expenditures during wartime while increasing enemy losses. His work was based on general linear programming and the techniques used to solve it. He also collaborated with Koopmans, and in 1975, he was awarded the Nobel

Prize in Economic Sciences [46]. This invention aided various fields by offering the best outcomes, such as business, industry, architecture, engineering, construction, mathematics, and so on.

3.1 Problem Formulation in Optimization

Optimization is a useful technique in decision science as well as in the understanding of physical systems, among other applications. Prior to making use of this tool, it is necessary to first define an objective, quantitative measure that accurately reflects the system's performance under investigation. Time, profit, energy, or any other article or group of articles that can be depicted by a single number could all be seen as the ultimate objective. The achievement of the goal is dependent on specific features of the system, which are referred to as unknowns or variables. Our objective is to discover the values of the parameters that will maximize the achievement of the goals. Frequently, the variables are confined or restricted in some fashion. In mathematics, both the charge distribution in a molecule as well as the payment of interest charges on a mortgage cannot have a lower return. In the context of a problem, modelling is the process of defining the aims, goals, constraints, and variables of that issue. The creation of a suitable model is the first stage in the optimization process, and it is often the most crucial phase in the process. A model that is overly simplistic may fail to provide insightful information about a real-world problem. If the problem is incredibly complicated, it might be impossible to solve and simplify. Once the model has been developed, it can be solved using an optimization procedure, which is commonly done with the aid of a computer. There is no such thing as a universal optimization method; rather, it is a collection of tools and techniques, each one specific to a particular sort of optimization process. The user is frequently responsible for selecting the approach that is most appropriate for a particular application. This is a crucial decision because it may impact whether the problem is addressed quickly or slowly, as well as whether a solution is discovered at all.

We need to be able to determine whether or not an optimization strategy has been successful in locating a solution once it has been applied to the model [47]. In many circumstances, elegant mathematical formulas known as optimality criteria can be used to verify that the prevailing collection of parameters represents a solution to the model. Whereas if optimization criteria are not really met, they can give you valuable information about how to improve the existing solution estimation. Techniques like sensitivity analysis, which highlights the adaptability of the response to modifications in the data and the model, can help enhance the model. Interpreting the solution in relation to the implementation may also point to opportunities to improve or refine the model. If the model is modified in any way, the optimization model is therefore re-solved, and the procedure is then repeated.

3.1.1 Optimization Formulation in Mathematical Form

Mathematically, optimization is when a function is minimized or maximized with a set of constraints on the variables that make up the function [48]. Use the notation below:

- x is the parameter, which is an unknown vector of variables.
- an objective function $\ddot{f}$ is usually some scalar function of the variable x whose values we wish to optimize.
- constrained functions $(\hat{a}_i)$ are scalar functions of x that construct equalities and inequalities that must be satisfied by the unknown vector x.

The optimization issue can be expressed as follows using this notation:

$$\min \quad \ddot{f}(x,)$$

in accordance with,

$$\hat{a}_i(x) = 0, \qquad i \in \mathbf{E}, \qquad (3.1.1)$$
$$\hat{a}_i(x) \geq 0, \qquad i \in \mathbf{I}.$$

The set of indices **I** and **E** represent inequality and equality constraints respectively.

Example 3.1.1. A very simple illustration of an optimization issue that could occur in the manufacturing and transportation industries is as follows:
A chemical firm has three foundries, F_1, F_2 and F_3, as well as a seven retail stores, $R_1, R_2, ..., R_7$, Figure 3.1. Each foundry F_i has the capacity to generate a certain number of tones of a specific chemical substance every week, $\hat{b}_i$ is the plant's holding ability. Each R_j retail store has a predetermined weekly product requirement in terms of $\hat{c}_j$ tones. The cost of transporting one ton of goods from the foundry F_i to the retail outlet R_j is calculated as $\hat{a}_{ij}$. The challenge is to figure out the appropriate amount of merchandise that should be shipped from each plant to each outlet in order to meet all of the requirements while keeping costs as low as possible. The variables in the given model are $\breve{x}_{ij}$, where $i = 1, 2, 3$ *and* $j = 1, 2, \ldots, 7$, x_{ij} here is the quantity of tons of goods transported from the production plant F_i to the retail store, R_j.

$$\min \sum_{ij} \hat{a}_{ij}\breve{x}_{ij}. \qquad (3.1.2)$$

Capacity Constraint: $\sum_{j=1}^{7} \breve{x}_{ij} \leq \hat{b}_i, \quad i = 1, 2, 3$
Each factory F_i can produce $\hat{b}_i$ tons of a certain chemical product each week.
Demand Constraints: $\sum_{i=1}^{3} \breve{x}_{ij} \geq \hat{c}_j, \quad j = 1, 2, ..., 7 \breve{x}_{ij} \geq 0, \quad i = 1, 2, 3.$
$j = 1, 2, \ldots, 7.$

Each retail outlet R_j has a known weekly demand of $\hat{c}_j$ tons of the product.

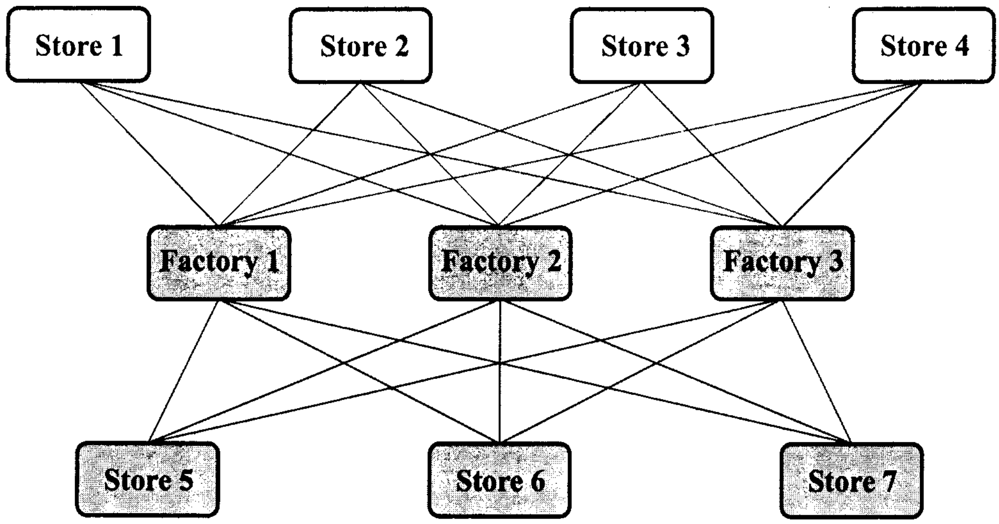

Figure 3.1: Goods transportation

3.2 Various Forms of Optimization Model

3.2.1 Linear Optimization

Model 3.1.2 is referred to as a linear programming model, because both the constraints and objective function are linear functions. Costs involved with manufacturing and storing the goods are also incorporated into a more realistic model. In practice, there may be volume discounts for transporting the commodities.

3.2.2 Non-linear Optimization

An optimization problem called nonlinear optimization [49], or nonlinear programming, is one where the goal or constraints are not linear functions.
Problem 3.1.2 will be converted into a non-linear problem if a small subscription fee is incorporated into the model; for example $\sum_{ij} \hat{a}_{ij}\sqrt{\rho + \breve{x}_{ij}}$, where $\rho > 0$ is a minimal registration fee. Because of the nonlinearity of the objective function, Problem 3.1.2 becomes a non-linear optimization problem. In

the natural sciences and engineering, nonlinear programming problems tend to develop spontaneously. Nevertheless, they are also becoming increasingly popular in business and the socioeconomic sciences.

3.2.3 Constrained and Unconstrained Optimization

There are several classifications of problems with the generic form 3.1.2, including the nature of constraints and objective function (convex, nonlinear and linear), the quantity of variables (small or large), the sharpness of the functions, and many others. There is a significant distinction among problems that have restrictions on the variables and problems that do not have constraints on the variables [48].

Approaches in which constraints contribute significantly to the solution, such as imposing resource constraints on a problem of economic significance or structural constraints in a design model, give birth to constrained optimization problems. Simple boundaries like $0 \leq \breve{x}_i \leq 50$, more general linear restrictions like $\sum_i \breve{x}_i \leq 1$, or nonlinear inequalities that describe intricate interactions between the variables are all examples of constraints. This makes the problem a constrained optimization problem, which is different from an unconstrained optimization problem.

Replacing $\mathbf{I} = \mathbf{E} = \phi$ in Equation 3.1.1 resulted in an unconstrained model of optimization. Many practical and real-world applications directly require these models. It may be permissible to disregard generic restrictions on variables in certain situations since they have no influence on the outcome and do not conflict with methods. Unconstrained optimization issues are often viewed as refinements of constrained optimization models in which the constraints are substituted by penalty factors that are inserted into the objective function and have the effect of preventing the constraints from being violated.

3.2.4 Convexity in Optimization

In optimization, the concept of convexity is vital. There are several applications for this feature in various practical areas such as estimation and signal processing, networks and communications, data analysis and modeling, electronic circuit design, finance, statistics and many other areas [48]. This feature is thus present in many practical issues, making them simpler to address both in theory and in reality. The word convex may be employed to describe both functions and sets. The convexity of a set $\mathbb{C}$ belongs to $\mathbb{R}^n$ is defined as the fact that the straight line segment joining any two locations in $\mathbb{C}$ falls wholly within $\mathbb{C}$. In formal terms, for $\mathbb{C}$ to be a convex set, we have $\rho\hat{x} + (1-\rho)\hat{y} \in \mathbb{C}$ for any two points $\hat{x}$ and $\hat{y} \in \mathbb{C}$ and $\rho \in [0, 1]$.

Likewise the function $\ddot{f}$ is convex if its domain $\mathbb{C}$ is a convex set and the following property holds for any two points x and y in $\mathbb{C}$.

$$\ddot{f}(\rho\hat{x} + (1-\rho)\hat{y}) \leq \rho f(\hat{x}) + (1-\rho) f(\hat{y}) \text{ and } \rho \in [0, 1].$$

Thus, a special class of mathematical optimization problems is known as convex optimization, which encompasses linear programming and least-squares issues. Problems involving linear programming and least-squares have very comprehensive theories. Identifying or expressing a problem in terms of a convex optimization problem has a number of significant advantages. One of the fundamental benefits is that the problem may therefore be cracked very consistently and effectively by utilizing interior-point methods or further specific approaches for convex optimization, which are extremely efficient and reliable. These approaches are trusted enough to be used in different analysis tools or in designs that are computer-aided, as well as in a reactive or autonomous control system that operates in real time.

Mathematically, a convex optimization problem can be expressed as,

$$\min \quad \ddot{f}(x),$$

in accordance with,

$$\ddot{f}_i(\breve{x}) \leq \hat{b}_i, \quad i = 1, 2, - - -, n$$

where functions $\ddot{f}_i(\breve{x}) : \mathbb{R}^{\kappa} \longrightarrow \mathbb{R}$ are convex, hence satisfy the below condition:

$$\ddot{f}_i(\alpha\breve{x} + \beta\breve{y}) \leq \alpha\ddot{f}_i(\breve{x}) + \beta\ddot{f}_i(\breve{y}),$$

here $\alpha + \beta \in \mathbb{R}$, also $\alpha \geq 0$, $\beta \geq 0$ with the condition on both as $\alpha + \beta = 1$ and all $x, y \in \mathbb{R}^{\kappa}$.

3.2.5 Local and Global Optimization

In local optimization, the compromise is to discontinue the search for the best x, which minimizes the goal across all possible locations. Instead, we seek a point that is just locally optimum, that is, the solution that minimizes the objective function among neighboring feasible sites but isn't certain to have a better objective value than all other probable feasible locations. Local optimization has received a lot of attention in general nonlinear programming research, and as a result, they are highly developed.

Local optimization methods are quick, can efficiently handle large-scale problems, and are broadly applicable [48]. Approaches of local optimization, as a result, are generally used in situations where identifying a good, if not the best, point is important. Local optimization, for example, might be utilized in an engineering design program to improve the effectiveness of a design that was initially created using manual or other design approaches.

The point with the lowest function value among all feasible points is referred to as the global solution. There are some situations where global solutions are required, yet these solutions are difficult to recognize and even more difficult to locate in many cases. Global optimization is used to solve problems with a small number of parameters when getting the true global solution is more important than computational speed and time, such as in financial modeling problems.

3.2.6 Deterministic and Stochastic Optimization

It is possible that the model for a given situation cannot be fully stated in a few optimization problems. This might be due to a variety of factors. Several of these include relying on parameters that are unknown at the time of formulation and the random character of available data when following any probability distribution. Many financial planning and economic models share this characteristic. They may be different in how they depend on future commodity prices, expected demand for a product, or expected interest rates, but in almost any application, randomness can happen on its own. Modelers might be able to get better results if they put more information about these factors into the model rather than simply using a best estimate for the ambiguous quantities. They may, for example, be aware of a variety of potential scenarios for the random demand as well as assessments of the likelihood of each scenario. These estimates of randomness are used by stochastic optimization techniques to come up with solutions that improve the estimated performance of the model [49]. So, when we have random variables that follow any probability distribution, stochastic optimization techniques can be used to find the best solution.

3.3 Fuzzy Optimization

Optimization methods and theory play an important role in the variety of fields to deal with different real-life as well as decision-making problems. Many different techniques have been introduced to deal with engineering optimization problems. In the last few years optimization methodology repeatedly be given attention because of the advancement in computing technologies and the growing dependency on optimization-based problems in real life. Many remarkable proposals have been introduced by many researchers. But, It has been observed that decision-making processes involve the collection of information from different sources, at any rate to a limited extent, fuzzy in nature. Therefore, a wide assortment of design and decision-making process often uses fuzzy set-theoretic concepts and fuzzy techniques are utilized as a helpful instrument to handle those circumstances which cannot be taken care of with established methods. Many concrete issues that might be thrown into an optimization setting are prevalent with imprecision sources. Most of the times it is not useful to choose exact conditions as many of these are picked up by estimation, or by analysts' perceptions. This demonstrates the persistent need to improve optimization models authenticity by making it possible to merge uncertainty into mathematical programming systems. This actually offers to move up to the domain of optimization under uncertainty.

Problems having a well-defined structure or configuration, frequently referred to as "hard systems," have long been addressed using conventional optimization approaches and techniques. Optimization problems of this kind are often well-stated, with exact objective functions and a precise set of constraints, so they can be solved mathematically with precision. It's unfortunate that real-world circumstances aren't always predictable. There are many kinds of uncertainty in economic, business, and social systems, such as the fact that events can't be predicted, system data that isn't accurate or clear, and ambiguities in language [50]. These uncertainties arise from a variety of sources, including measurement mistakes, a lack of statistical and historical data, insufficient reasoning, imperfect knowledge presentation, bias and subjectivity in human judgement, and so on.

We could use already established optimization methods with definite and well-defined values and constraints. Linear programming is the most abundant technique for applied optimization in real life problems. Yet if at any point we go over such a system where there is impression present in it, connecting the expression "imprecision" with "Fuzzy" we think of fuzzy optimization to be a final resort for our problem. Fuzzy optimization is a more recent approach to optimization under uncertainty in which imprecision is modelled by fuzzy relations and/or fuzzy parameters evolved from fuzzy sets. Fuzzy optimization techniques solve the systems involving ambiguity better than any stochastic optimization based on probability. Such systems prove fuzzy optimizations as an efficient and superior tool. Bellman and Zadeh [51] conflated the concepts of fuzziness and programming. They have proposed the fundamental notions of fuzzy objectives, fuzzy decision and fuzzy constraints.

In traditional approaches to optimization, it is thought that all parameters are fixed and well defined. However, these sorts of approximations are insufficient for addressing real-world situations when the majority of parameters are vague and implicit. The hypothesis of fuzzy linear programming was additionally improved by Tanaka and Asai [52]. They have highlighted the idea of a level set to outspread some of the standard results to problems comprising fuzzy objective functions and constraints. Zimmerman [53] analyzed such scenarios and developed the notion of fuzzy linear programming with multiple goal functions. One can view Zimmerman's work as a continuation of Bellman's. Tanaka and Asai [54] considered fuzzy numbers in the models of decision-making problems. Werner [55] in his work gave the idea of the formation of the fuzzy models and suggested the construction of membership functions under fuzzy constraints. Xu [56] has looked into the two-phase method for fuzzy programming of structures. Shih *et al.* [57] have created and proposed three alpha- level cuts approach for solving structural engineering problems with fuzzy properties. In 25-bar and 72-bar truss design problems, Shih and Lee [58] have exhibited an altered double cuts approach for extensive scale fuzzy optimization. It has been examined that the methodology is reliable for basic ideal structural design with fuzzy properties. Asimakopoulo et al. [59] have displayed another system for foreseeing the basic flashover voltage of contaminated covers in view of fuzzy logic.

From linear to nonlinear fuzzy programming, single-objective to multi-objective, integer to dynamic and possibilistic programming are just a few of the many approaches that researchers have developed over the years [60]-[76]. Numerous published applications of fuzzy optimization indicate the usefulness, effectiveness, supremacy, and significance of the topic [77]-[88].

3.3.1 Solution Methodology

Modeling a fuzzy problem and then using fuzzy optimization techniques are two activities that must be completed in order to successfully comprehend and solve a complicated problem in a fuzzy environment. The goal of fuzzy modelling is to make a good model based on a deep understanding of the problem and a close study of fuzzy data. The goal of fuzzy optimization is to solve the fuzzy model as efficiently as possible using optimization methods. Membership grades are used to formulate the fuzzy data. Each grade is set up so that it shows a real breach of the given inequalities. So basically, there are two main tasks: modelling and optimization, yet there are no clear lines between them. So, starting from modelling to ending in an optimal solution, the whole process can be divided into several complex stages. Although there is no hard and fast rule or set number of steps to follow, we may split the whole procedure into seven parts.

Stage 1: The first step might be thought of as the recognition stage, in which the goal function, constraints, and relationships between them are established.

Stage 2: The imprecise knowledge in the problem is used to construct fuzzy objectives, fuzzy coefficients, and fuzzy constraints. In a nutshell, it's referred to as fuzziness analysis.

Stage 3: In the third step, fuzzy goals, fuzzy coefficients, fuzzy constraints, and mathematical methods are used to make a fuzzy optimization model.

Stage 4: The fourth step is the construction of fuzzy membership. The main idea behind this process is the choice of the person making the decision or the kind of results they want to attain.

Stage 5: The fifth stage is the crisp interpretation of the fuzzy model. Here the fuzzy optimization model is transformed into an equivalent or approximating crisp optimization model.

Stage 6: The next thing to do is to find a solution to the precise optimization model by using the most suitable optimization strategies. For this, any relevant computer programme, method, or approach may be used.

Stage 7: The last phase in this series is to enhance the outcomes that have already been obtained. If the solution obtained is not acceptable, the fuzzy optimization system is refined again.

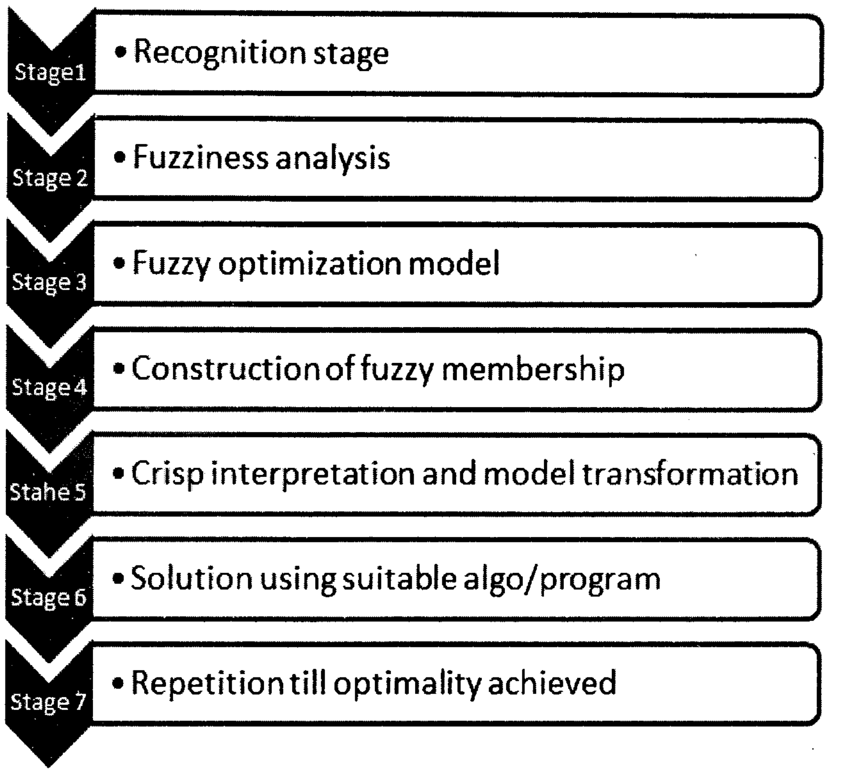

Figure 3.2: Steps to follow fuzzy optimization

3.3.2 Fuzzy and Generalized Fuzzy Optimization Formulation in Mathematical Form

Multi objective advancement and optimization problems occurred while dealing with optimization of non-commensurable, conflicting and multiple objective functions subject to certain conditions and circumstances.

3.3.2.1 General Multi-objective Optimization

A general representation of a multi objective optimization problem with q constraints, p objectives and n decision variables is as follows

$$\begin{aligned} \text{Minimize} \quad & \breve{f} = \{\breve{f}_1, \breve{f}_2, \cdots, \breve{f}_p\} \\ \text{so that} \quad & \breve{g}_j(x) \leq \breve{b}_j \qquad j = 1, 2, \cdots, q. \end{aligned} \tag{3.3.1}$$

3.3.2.2 Fuzzy Optimization

To handle uncertainties existed in the data fuzzy [53] optimization method is described as below:

$$\begin{aligned} \text{Maximize} \quad & \hat{\alpha} \\ \text{such that} \quad & T_k(x) \geq \hat{\alpha} \quad \text{for all } k = 1, 2, \cdots, p+q \\ \text{with} \quad & \breve{g}_j(x) \leq \breve{b}_j, \qquad j = 1, 2, \cdots, q \qquad x \geq 0 \end{aligned} \tag{3.3.2}$$

$\hat{\alpha}$ here represents the aspiration level where as $T_k(x)$ is the true value associated to the confluence of each objective and constraints.

3.3.2.3 Intuitionistic Fuzzy Optimization

For uncertain multi objective problems another approach known as intuitionistic fuzzy optimization [89] is used and is stated as,

$$\begin{aligned} Maximize \quad & \hat{\alpha} - \hat{\beta} \\ \text{subject to} \quad & T_k(x) \geq \hat{\alpha} \\ & F_k(x) \leq \hat{\beta} \\ \text{given that} \quad & \hat{\alpha} + \hat{\beta} \leq 1 \\ & \hat{\alpha} \geq \hat{\beta} \\ & \hat{\alpha}, \hat{\beta} \geq 0 \\ \breve{g}_j(x) \leq \breve{b}_j, \quad x \geq 0 & \qquad j = 1, 2, \cdots, q. \end{aligned}$$

$\hat{\alpha}$ and $\hat{\beta}$ here represents the aspiration level where as $T_k(x)$ and $F_k(x)$ are the true and false values associated to the confluence of each objective and constraints.

3.3.2.4 Neutrosophic Optimization

Neutrosophic Optimization [90] method for conflicting multi objective programming problem is given as

$$
\begin{aligned}
Maximize \quad & (\hat{\alpha} - \hat{\beta} + \hat{\sigma}) \\
\text{such that} \quad & T_k(x) \geq \hat{\alpha} \qquad k = 1, 2, \cdots, p+q. \\
& F_k(x) \leq \hat{\beta} \qquad k = 1, 2, \cdots, p+q. \\
& I_k(x) \geq \hat{\sigma} \qquad k = 1, 2, \cdots, p+q. \\
\text{with} \quad & \hat{\alpha} + \hat{\beta} + \hat{\sigma} \leq 3 \\
& \hat{\alpha} \geq \hat{\beta}, \hat{\alpha} \geq \hat{\sigma} \\
& \hat{\alpha}, \hat{\beta}, \hat{\sigma} \in [0, 1]. \\
& \breve{g}_j(x) \leq \breve{b}_j, \quad x \geq 0 \qquad j = 1, 2, \cdots, q.
\end{aligned}
$$

$\hat{\alpha}$, $\hat{\beta}$ and $\hat{\sigma}$ here represents the aspiration levels corresponding to true, false and indeterminate grades where as $T_k(x)$, $F_k(x)$ and $I_k(x)$ are the true, false and indeterminate values associated to the confluence of each objective and constraints.

3.3.2.5 Interval-valued Intuitionistic Optimization

Interval-valued intuitionistic [91] fuzzy optimization method is given as

$$
\begin{aligned}
Maximize \quad & \theta\hat{\alpha}+(1-\theta)\hat{\beta}-\theta\hat{\sigma}-(1-\theta)\hat{\gamma} \\
\text{given that} \quad & T_k^U(x) \geq \theta\hat{\alpha}+(1-\theta)\hat{\beta} \\
& T_k^L(x) \geq \hat{\alpha} \\
& F_k^U(x) \leq \theta\hat{\sigma}+(1-\theta)\hat{\gamma} \\
& F_k^L(x) \leq \hat{\sigma}\theta\hat{\alpha}+(1-\theta)\hat{\beta}-\theta\hat{\sigma}-(1-\theta)\hat{\gamma} \qquad \leq 1 \\
& \hat{\beta}+\hat{\gamma} \leq 1 \\
& \hat{\beta} \geq \hat{\alpha}, \hat{\gamma} \geq \hat{\sigma} \\
\breve{g}_j(x) \leq \breve{b}_j, \quad & x \geq 0 \qquad j=1,2,\cdots,q.
\end{aligned}
$$

here, $\theta \in [0,1]$ for all $k=1,2,\cdots,p+q$.
$\hat{\alpha}$, $\hat{\beta}$, $\hat{\sigma}$ and $\hat{\gamma}$ here represents the aspiration levels corresponding to true and false grades where as $T_k^L(x)$ and $F_k^L(x)$ are true lower and false lower grades and $T_k^U(x)$, $F_k^U(x)$ are true upper and false upper grades associated to the confluence of each objective and constraints.

Chapter 4

Bipolar Neutrosophic Numbers and Their Ranking Techniques

In everyday life, terms like "approximately six" or "around eight" are frequently used to represent a precise number in a vague but useful way. Such terms are typically ill-defined, if they are described at all. It is very much obvious that we all have a different interpretation of the phrase around six. Certainly, it would be extremely beneficial if such hazy concepts could be made more specific and operational. For this goal, well-defined mathematical conceptions can be introduced that represent the intuitive interpretation of the imprecise notions in a way that is satisfying to everyone. Fuzzy sets and emerging generalizations best fit this criteria.

The ranking function is a mechanism for arranging a fuzzy numbers that is both efficient and effective. Comparing and sorting fuzzy integers, or fuzzy sets, is always critical in real-world applications. When fuzzy sets and techniques are used to make mathematical equations and models, it is always hard to rank or compare fuzzy parameters or fuzzy variables. Each fuzzy set is mapped to the real line using a scoring function. Yager [92], Adamo [93] and Chang [94] were among the first few to use the ranking method for fuzzy numbers ordering. Degani and Bortolan [95] studied some approaches for scoring fuzzy subsets in their famous study. Chen [96] shows how to score fuzzy numbers using a maximising and a minimising set. Dubois and Prade [97] demonstrated how to calculate the fuzzy number mean value. Analysis of fuzzy numbers was conducted by Li and Lee [98] using the probability measure for fuzzy events. In the past, there have been various forms of ranking functions that could help people solve systems of linear equations with parameters that are fuzzy [99]-[110]. Nafei *et al.* [111] converted the neutrosophic model to its crisp counterpart using the ranking function and then utilized normal methods to achieve the solution. In a single-valued neutrosophic framework, Deli and Subas [112] employed a ranking algorithm to handle multi-criteria decision making issues. Darehmiraki [113] proposed a novel ranking approach for tackling the LP issue that incorporates neutrosophic numbers for every objective functions and constraint coefficients. To solve multi-criteria decision issues, Suresh *et al.* [114] suggested a ranking technique for trapezoidal neutrosophic numbers. Using the elimination area approach as a ranking method, Chakraborty *et al.* [115] suggest a debipolarization methodology for triangular and trapezoidal bipolar neutrosophic numbers (TrBNN).

4.1 Bipolar Neutrosophic Number

The current section is primarily concerned with developing the theory and mathematics of bipolar neutrosophic numbers, which will be used in the subsequent sections.

Definition 4.1.1. A bipolar neutrosophic number (BN-number) $\breve{A}^{Bn}$ defined on universe $\mathbf{Z}$, specified by points

$$\begin{aligned}\breve{A}^{Bn}(\mathfrak{z}) &= \langle T^{+}(\mathfrak{z}), T^{-}(\mathfrak{z}), I^{+}(\mathfrak{z}), I^{-}(\mathfrak{z}), F^{+}(\mathfrak{z}), F^{-}(\mathfrak{z}) \quad : \text{for all} \mathfrak{z} \in \mathbf{Z}\rangle \\ &= \langle [z_1, z_2, z_3, z_4; z_5, z_6, z_7, z_8]; [z_9, z_{10}, z_{11}, z_{12}; z_{13}, z_{14}, z_{15}, z_{16}]; \\ &\qquad [z_{17}, z_{18}, z_{19}, z_{20}; z_{21}, z_{22}, z_{23}, z_{24}]\rangle. \quad (4.1.1)\end{aligned}$$

With satisfaction degree (μ^{T^+}, μ^{T^-}), indeterminacy grades (ρ^{I^+}, ρ^{I^-}) and dis-satisfaction $(\sigma^{F^+}, \sigma^{F^-})$ degree obeying the given conditions.

1. μ^{T^+},ρ^{I^+},σ^{F^+} are functions which are piecewise continuous from $\mathbb{R}$ to $[0, 1]$ and μ^{T^-},ρ^{I^-},σ^{F^-} are functions which are piecewise continuous from $\mathbb{R}$ to $[-1, 0]$.

2. $\mu^{T^+} = 0$, for all $\mathfrak{z} \in (-\infty, z_1]$, and $\rho^{I^+} = 1 = \sigma^{F^+}$ for all $\mathfrak{z} \in (-\infty, z_9]$ and $(-\infty, z_{17}]$ respectively.

3. $\mu^{T^+} = 0$, for all $\mathfrak{z} \in [z_4, \infty)$, and $\rho^{I^+} = 1 = \sigma^{F^+}$ for all $\mathfrak{z} \in [z_{12}, \infty)$ and $[z_{20}, \infty)$ respectively.

4. $\mu^{T^-} = 0$, for all $\mathfrak{z} \in (-\infty, z_5]$, and $\mathfrak{z} \in [z_8, \infty)$,
$\rho^{I^-} = -1$, for all $\mathfrak{z} \in (-\infty, z_{13}]$, and $\mathfrak{z} \in [z_{16}, \infty)$,
$\sigma^{F^-} = -1$, for all $\mathfrak{z} \in (-\infty, z_{21}]$, and $\mathfrak{z} \in [z_{24}, \infty)$.

5. $\mu^{T^+} = 1$, for all $\mathfrak{z} \in [z_2, z_3]$, $\rho^{I^+} = 0$, for all $\mathfrak{z} \in [z_{10}, z_{11}]$ and $\sigma^{F^+} = 0$, for all $\mathfrak{z} \in [z_{18}, z_{19}]$.

6. $\mu^{T^-} = -1$, for all $\mathfrak{z} \in [z_6, z_7]$, $\rho^{I^-} = 0$, for all $\mathfrak{z} \in [z_{14}, z_{15}]$ and $\sigma^{F^-} = 0$, for all $\mathfrak{z} \in [z_{22}, z_{23}]$.

7. μ^{T^+} is strongly increasing on $[z_1, z_2]$ and strongly decreasing on $[z_3, z_4]$,
ρ^{I^+} is strongly decreasing on $[z_9, z_{10}]$ and strongly increasing on $[z_{11}, z_{12}]$,
σ^{F^+} is strongly decreasing on $[z_{17}, z_{18}]$ and strongly increasing on $[z_{19}, z_{20}]$.

8. μ^{T^-} is strongly decreasing on $[z_5, z_6]$ and strongly increasing on $[z_7, z_8]$,
ρ^{I^-} is strongly increasing on $[z_{13}, z_{14}]$ and strongly decreasing on $[z_{15}, z_{16}]$,
σ^{F^-} is strongly increasing on $[z_{21}, z_{22}]$ and strongly decreasing on $[z_{23}, z_{24}]$.

Whereas, the satisfaction degree (μ^{T^+}, μ^{T^-}), indeterminacy grades (ρ^{I^+}, ρ^{I^-})

and dissatisfaction degree $(\sigma^{F^+}, \sigma^{F^-})$ are given in a simpler manner as follows:

$$\mu^{T^+}_{\breve{A}^{Bn}}(\mathfrak{z}) = \begin{cases} \mu^{T^+}_L(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_1, z_2] \\ 1 & \text{if } \mathfrak{z} \in [z_2, z_3] \\ \mu^{T^+}_R(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_3, z_4] \\ 0 & \text{otherwise} \end{cases}$$

$$\mu^{T^-}_{\breve{A}^{Bn}}(\mathfrak{z}) = \begin{cases} \mu^{T^-}_L(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_5, z_6] \\ -1 & \text{if } \mathfrak{z} \in [z_6, z_7] \\ \mu^{T^-}_R(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_7, z_8] \\ 0 & \text{otherwise} \end{cases}$$

$$\rho^{I^+}_{\breve{A}^{Bn}}(\mathfrak{z}) = \begin{cases} \rho^{I^+}_L(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_9, z_{10}] \\ 0 & \text{if } \mathfrak{z} \in [z_{10}, z_{11}] \\ \rho^{I^+}_R(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_{11}, z_{12}] \\ 1 & \text{otherwise} \end{cases}$$

$$\rho^{I^-}_{\breve{A}^{Bn}}(\mathfrak{z}) = \begin{cases} \rho^{I^-}_L(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_{13}, z_{14}] \\ 0 & \text{if } \mathfrak{z} \in [z_{14}, z_{15}] \\ \rho^{I^-}_R(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_{15}, z_{16}] \\ -1 & \text{otherwise} \end{cases}$$

$$\sigma^{F^+}_{\breve{A}^{Bn}}(\mathfrak{z}) = \begin{cases} \sigma^{F^+}_L(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_{17}, z_{18}] \\ 0 & \text{if } \mathfrak{z} \in [z_{18}, z_{19}] \\ \sigma^{F^+}_R(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_{19}, z_{20}] \\ 1 & \text{otherwise} \end{cases}$$

$$\sigma^{F^-}_{\breve{A}^{Bn}}(\mathfrak{z}) = \begin{cases} \sigma^{F^-}_L(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_{21}, z_{22}] \\ 0 & \text{if } \mathfrak{z} \in [z_{22}, z_{23}] \\ \sigma^{F^-}_R(\mathfrak{z}) & \text{if } \mathfrak{z} \in [z_{23}, z_{24}] \\ -1 & \text{otherwise.} \end{cases}$$

Where,

$$\begin{array}{lcl}
\mu_L^{T^+}(\mathfrak{z}) : [z_1, z_2] \longrightarrow [0,1] & ; & \mu_R^{T^+}(\mathfrak{z}) : [z_3, z_4] \longrightarrow [0,1] \\
\mu_L^{T^-}(\mathfrak{z}) : [z_5, z_6] \longrightarrow [-1,0] & ; & \mu_R^{T^-}(\mathfrak{z}) : [z_7, z_8] \longrightarrow [-1,0] \\
\rho_L^{I^+}(\mathfrak{z}) : [z_9, z_{10}] \longrightarrow [0,1] & ; & \rho_R^{I^+}(\mathfrak{z}) : [z_{11}, z_{12}] \longrightarrow [0,1] \\
\rho_L^{I^-}(\mathfrak{z}) : [z_{13}, z_{14}] \longrightarrow [-1,0] & ; & \rho_R^{I^-}(\mathfrak{z}) : [z_{15}, z_{16}] \longrightarrow [-1,0] \\
\sigma_L^{F^+}(\mathfrak{z}) : [z_{17}, z_{18}] \longrightarrow [0,1] & ; & \sigma_R^{F^+}(\mathfrak{z}) : [z_{19}, z_{20}] \longrightarrow [0,1] \\
\sigma_L^{F^-}(\mathfrak{z}) : [z_{21}, z_{22}] \longrightarrow [-1,0] & ; & \sigma_R^{F^-}(\mathfrak{z}) : [z_{23}, z_{24}] \longrightarrow [-1,0].
\end{array}$$

In the above equations , the subscripts "L" and "R" shows nothing but the left and right side of the respective membership grades.

Definition 4.1.2. A trapezoidal bipolar neutrosophic number, denoted by $\breve{A}^{trBn}$ and specified by points,

$$\breve{A}^{trBn} = \langle T^+, T^-, I^+, I^-, F^+, F^- \rangle = \langle [\mathfrak{a}, b, c, d; e, f, g, h]; [i, j, k, l; m, n, o, p] \\ ; [q, r, s, t; u, v, w, \hat{x}] \rangle. \quad (4.1.2)$$

Thus the mathematical interpretation of $\breve{A}^{trBn}$ using membership functions is as follows:

$$T^+{}_{\breve{A}^{trBn}}(\mathfrak{z}) = \begin{cases} \frac{\mathfrak{z}-\mathfrak{a}}{b-\mathfrak{a}} & if\ \mathfrak{z} \in [\mathfrak{a}, b] \\ 1 & if\ \mathfrak{z} \in [b, c] \\ \frac{d-\mathfrak{z}}{d-c} & if\ \mathfrak{z} \in [c, d] \\ 0 & otherwise \end{cases}$$

$$T^-{}_{\breve{A}^{trBn}}(\mathfrak{z}) = \begin{cases} \frac{e-\mathfrak{z}}{f-e} & if\ \mathfrak{z} \in [e, f] \\ -1 & if\ \mathfrak{z} \in [f, g] \\ \frac{\mathfrak{z}-h}{h-g} & if\ \mathfrak{z} \in [g, h] \\ 0 & otherwise \end{cases}$$

$$I^+{}_{\breve{A}^{trBn}}(\mathfrak{z}) = \begin{cases} \frac{j-\mathfrak{z}}{i-j} & if\ \mathfrak{z} \in [i, j] \\ 0 & if\ \mathfrak{z} \in [j, k] \\ \frac{\mathfrak{z}-k}{l-k} & if\ \mathfrak{z} \in [k, l] \\ 1 & otherwise \end{cases}$$

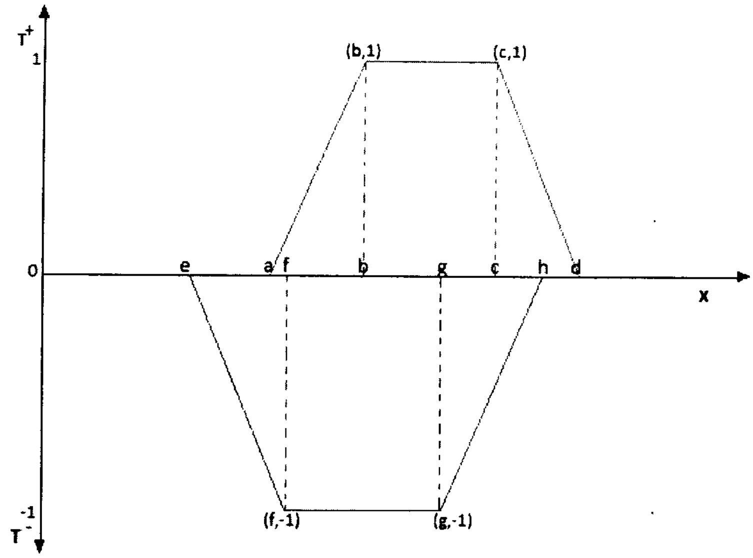

Figure 4.1: Truth membership of trapezoidal bipolar fuzzy number

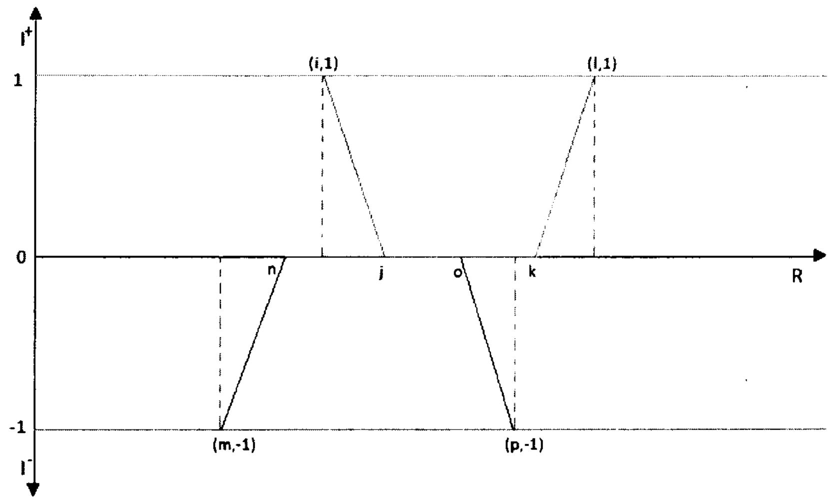

Figure 4.2: Indeterminacy of trapezoidal bipolar fuzzy number

$$I^{-}_{\breve{A}^{trBn}}(\mathfrak{z}) = \begin{cases} \frac{\mathfrak{z}-n}{n-m} & if\ \mathfrak{z} \in [m,n] \\ 0 & if\ \mathfrak{z} \in [n,o] \\ \frac{o-\mathfrak{z}}{p-o} & if\ \mathfrak{z} \in [o,p] \\ -1 & otherwise \end{cases}$$

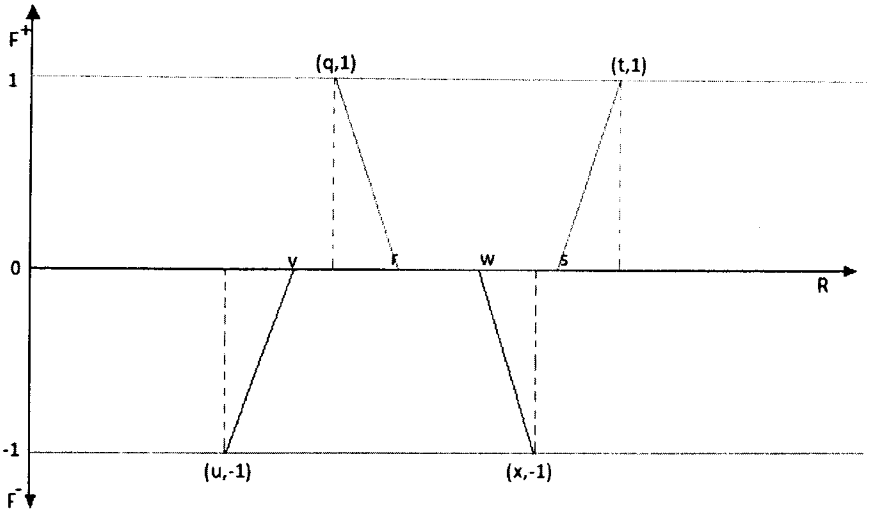

Figure 4.3: Falsity of trapezoidal bipolar fuzzy number

$$F^{+}{}_{\breve{A}^{trBn}}(\mathfrak{z}) = \begin{cases} \frac{r-\mathfrak{z}}{r-q} & if\ \mathfrak{z} \in [q,r] \\ 0 & if\ \mathfrak{z} \in [r,s] \\ \frac{\mathfrak{z}-s}{t-s} & if\ \mathfrak{z} \in [s,t] \\ 1 & otherwise \end{cases}$$

$$F^{-}{}_{\breve{A}^{trBn}}(\mathfrak{z}) = \begin{cases} \frac{\mathfrak{z}-v}{v-u} & if\ \mathfrak{z} \in [u,v] \\ 0 & if\ \mathfrak{z} \in [v,w] \\ \frac{w-\mathfrak{z}}{\hat{x}-w} & if\ \mathfrak{z} \in [w,\hat{x}] \\ -1 & otherwise \end{cases}$$

Definition 4.1.3. A triangular bipolar neutrosophic number $\breve{A}^{TBN}$ can be defined in terms of its membership function.

$$\breve{A}^{TBN} = \langle T^{+}_{\breve{A}^{TBN}}, T^{-}_{\breve{A}^{TBN}}, I^{+}_{\breve{A}^{TBN}}, I^{-}_{\breve{A}^{TBN}}, F^{+}_{\breve{A}^{TBN}}, F^{-}_{\breve{A}^{TBN}} \rangle = \langle [\mathfrak{a}, b, c, ; d, e, f]; \\ [g, h, i; j, k, l]; [m, n, o; p, q, r] \rangle. \quad (4.1.3)$$

$$T^{+}{}_{\breve{A}^{TBN}}(\mathfrak{z}) = \begin{cases} \frac{\mathfrak{z}-\mathfrak{a}}{b-\mathfrak{a}} & if\ \mathfrak{z} \in [\mathfrak{a},b] \\ \frac{c-\mathfrak{z}}{c-b} & if\ \mathfrak{z} \in [b,c] \\ 0 & otherwise \end{cases}$$

$$T^{-}{}_{\breve{A}^{TBN}}(\mathfrak{z}) = \begin{cases} \frac{-(\mathfrak{z}-d)}{(e-d)} & if\ \mathfrak{z} \in [d,e] \\ \frac{-(f-\mathfrak{z})}{(f-e)} & if\ \mathfrak{z} \in [e,f] \\ 0 & otherwise \end{cases}$$

$$I^{+}{}_{\breve{A}^{TBN}}(\mathfrak{z}) = \begin{cases} \frac{h-\mathfrak{z}}{h-g} & if\ \mathfrak{z} \in [g,h] \\ \frac{\mathfrak{z}-h}{i-h} & if\ \mathfrak{z} \in [h,i] \\ 1 & otherwise \end{cases}$$

$$I^{-}{}_{\breve{A}^{TBN}}(\mathfrak{z}) = \begin{cases} \frac{-(k-\mathfrak{z})}{(k-j)} & if\ \mathfrak{z} \in [j,k] \\ \frac{-(\mathfrak{z}-k)}{(l-k)} & if\ \mathfrak{z} \in [k,l] \\ -1 & otherwise \end{cases}$$

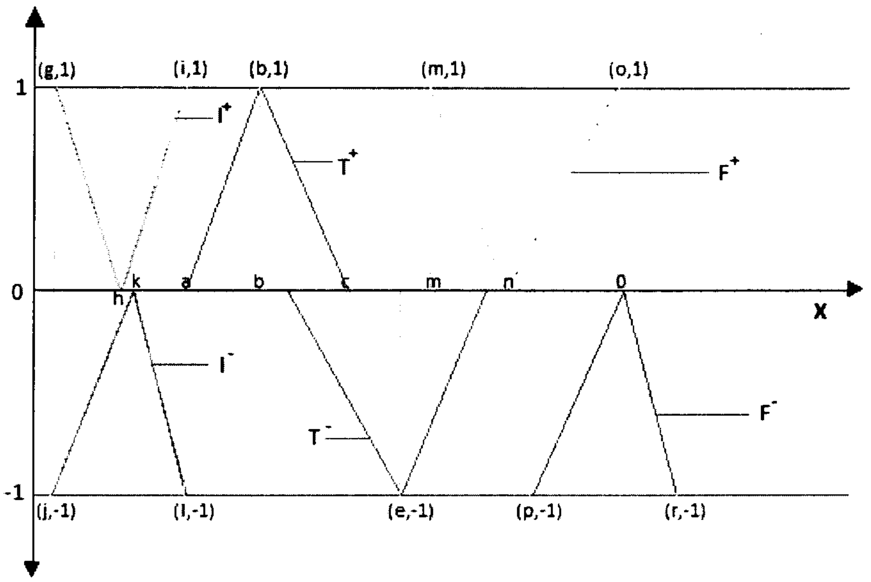

Figure 4.4: Triangular bipolar neutrosophic number

$$F^{+}{}_{\breve{A}^{TBN}}(\mathfrak{z}) = \begin{cases} \frac{n-\mathfrak{z}}{n-m} & if\ \mathfrak{z} \in [m,n] \\ \frac{\mathfrak{z}-n}{o-n} & if\ \mathfrak{z} \in [n,o] \\ 1 & otherwise \end{cases}$$

$$F^{-}{}_{\breve{A}^{TBN}}(\mathfrak{z}) = \begin{cases} \frac{-(q-\mathfrak{z})}{(q-p)} & if\ \mathfrak{z} \in [p,q] \\ \frac{-(\mathfrak{z}-q)}{(r-q)} & if\ \mathfrak{z} \in [q,r] \\ -1 & otherwise. \end{cases}$$

With each $T^{+}{}_{\breve{A}^{TBN}}, I^{+}{}_{\breve{A}^{TBN}}, F^{+}{}_{\breve{A}^{TBN}} \in [0,1]$ and $T^{-}{}_{\breve{A}^{TBN}}, I^{-}{}_{\breve{A}^{TBN}}, F^{-}{}_{\breve{A}^{TBN}} \in [-1,0]$.

Let $\breve{A}_1{}^{TBN}$ and $\breve{A}_2{}^{TBN}$ be two triangular bipolar neutrosophic numbers,

$$\breve{A}_1{}^{TBN} = \langle [\mathfrak{a}_1, b_1, c_1, ; d_1, e_1, f_1]; [g_1, h_1, i_1; j_1, k_1, l_1]; [m_1, n_1, o_1; p_1, q_1, r_1] \rangle;$$
$$\breve{A}_2{}^{TBN} = \langle [\mathfrak{a}_2, b_2, c_2, ; d_2, e_2, f_2]; [g_2, h_2, i_2; j_2, k_2, l_2]; [m_2, n_2, o_2; p_2, q_2, r_2] \rangle.$$

Example 4.1.4. There must be a situation in which the voltage supply is unbalanced in order to claim that the "voltage supply is ideal." As a result, a bipolar neutrosophic number can represent bipolarity. When we say "excellent or stable voltage supply," we can say that voltage ranges from 0 to 460 can be called "ideal or even."

A TBN-number can be defined as follows:

$$\breve{A}^{TBN} = [\langle 190, 220, 240; 130, 160, 200 \rangle; \langle 280, 290, 300; 90, 110, 120 \rangle; \langle 350, 400, 450; 10, 20, 30 \rangle].$$

A triangular bipolar neutrosophic number is said to be non-negative if and only if all the components of that number are non-negative and similarly two TBN-numbers are said to be equal if and only if their respective elements are equal.

4.1.1 Arithmetic Operations

Let $\breve{A}_1{}^{TBN}$ and $\breve{A}_2{}^{TBN}$ be two triangular bipolar neutrosophic numbers,

$$\breve{A}_1{}^{TBN} = \langle [\mathfrak{a}_1, b_1, c_1, ; d_1, e_1, f_1]; [g_1, h_1, i_1; j_1, k_1, l_1]; [m_1, n_1, o_1; p_1, q_1, r_1] \rangle;$$
$$\breve{A}_2{}^{TBN} = \langle [\mathfrak{a}_2, b_2, c_2, ; d_2, e_2, f_2]; [g_2, h_2, i_2; j_2, k_2, l_2]; [m_2, n_2, o_2; p_2, q_2, r_2] \rangle.$$

4.1.1.1 Addition

$$\breve{A}_1{}^{TBN} \oplus \breve{A}_2{}^{TBN} = \langle [\mathfrak{a}_1, b_1, c_1, ; d_1, e_1, f_1]; [g_1, h_1, i_1; j_1, k_1, l_1]; [m_1, n_1, o_1; p_1, q_1, r_1] \rangle \oplus \langle [\mathfrak{a}_2, b_2, c_2, ; d_2, e_2, f_2]; [g_2, h_2, i_2; j_2, k_2, l_2]; [m_2, n_2, o_2; p_2, q_2, + r_2] \rangle = \langle [\mathfrak{a}_1 + \mathfrak{a}_2, b_1 + b_2, c_1 + c_2, ; d_1 + d_2, e_1 + e_2, f_1 + f_2]; [g_1 + g_2, h_1 h_2, i_1 + i_2; j_1 + j_2, k_1 + k_2, l_1 + l_2]; [m_1 + m_2, n_1 + n_2, o_1 + o_2; p_1 + p_2 , q_1 + q_2, r_1 + r_2] \rangle. \tag{4.1.4}$$

4.1.1.2 Subtraction

$$-\breve{A}_2{}^{TBN} = \langle [-c_2, -b_2, -\mathfrak{a}_2, ; -f_2, -e_2, -d_2]; [-i_2, -h_2, -g_2; -l_2, -k_2, -j_2]; [-o_2, -n_2, -m_2; -r_2, -q_2, -p_2] \rangle$$

$$\breve{A}_1{}^{TBN} \ominus \breve{A}_2{}^{TBN} = \langle [\mathfrak{a}_1, b_1, c_1, ; d_1, e_1, f_1]; [g_1, h_1, i_1; j_1, k_1, l_1]; [m_1, n_1, o_1; p_1 , q_1, r_1] \rangle \oplus \langle [-c_2, -b_2, -\mathfrak{a}_2, ; -f_2, -e_2, -d_2]; [-i_2, -h_2, -g_2; -l_2, -k_2, -j_2]; [-o_2, -n_2, -m_2; -r_2, -q_2, -p_2] \rangle. \tag{4.1.5}$$

$$\breve{A}_1{}^{TBN} \ominus \breve{A}_2{}^{TBN} = \langle [\mathfrak{a}_1 - c_2, b_1 - b_2, c_1 - \mathfrak{a}_2, ; d_1 - f_2, e_1 - e_2, f_1 - d_2]; [g_1 - i_2, h_1 - h_2, i_1 - g_2; j_1 - l_2, k_1 - k_2, l_1 - j_2]; [m_1 - o_2, n_1 - n_2, o_1 - m_2; p_1 - r_2, q_1 - q_2, r_1 - p_2] \rangle \tag{4.1.6}$$

4.1.1.3 Multiplication

$$\kappa \breve{A}_1{}^{TBN} = \begin{cases} \langle [\kappa\mathfrak{a}, \kappa b, \kappa c, ; \kappa d, \kappa e, \kappa f]; [\kappa g, \kappa h, \kappa i; \kappa j, \kappa k, \kappa l]; [\kappa m, \kappa n, \kappa o; ; \kappa p, \\ \kappa q, \kappa r] \rangle & \mathit{for}\ \kappa \geq 0 \\ \langle [\kappa c, \kappa b, \kappa\mathfrak{a}, ; \kappa f, \kappa e, \kappa d]; [\kappa i, \kappa h, \kappa g; \kappa l, \kappa k, \kappa j]; [\kappa o, \kappa n, \kappa m; \kappa r, \\ \kappa q, \kappa p] \rangle & \mathit{for}\ \kappa < 0 \end{cases}$$

Multiplication by components is preferred because all elements of triangular bipolar neutrosophic numbers are taken to be positive in this study. Therefore,

$$\breve{A}_1{}^{TBN} \otimes \breve{A}_2{}^{TBN} = \langle [\mathfrak{a}_1\mathfrak{a}_2, b_1b_2, c_1c_2, ; d_1d_2, e_1e_2, f_1f_2]; [g_1g_2, h_1h_2, i_1i_2; j_1j_2, k_1k_2, l_1l_2]; [m_1m_2, n_1n_2, o_1o_2; p_1p_2, q_1q_2, r_1r_2] \rangle. \tag{4.1.7}$$

Example 4.1.5. Let $\breve{A}^{TBN}$ and $\breve{B}^{TBN}$ be two TBN-numbers.

$\breve{A}^{TBN} = [\langle 190, 220, 240; 130, 160, 200\rangle; \langle 280, 290, 300; 90, 110, 120\rangle; \langle 350, 400, 450; 10, 20, 30\rangle]$

$\breve{B}^{TBN} = [\langle 170, 200, 230; 110, 140, 170\rangle; \langle 290, 295, 300; 110, 120, 130\rangle; \langle 250, 280, 310; 80, 90, 100\rangle]$

Addition : $\breve{A}^{TBN} \oplus \breve{B}^{TBN} = [\langle 360, 420, 470; 240, 300, 370\rangle; \langle 570, 585, 600; 200, 230, 250\rangle; \langle 600, 680, 760; 90, 110, 130\rangle]$

Subtraction: $-\breve{A}^{TBN} = [\langle -240; -220, -190; -200, -160, -130\rangle; \langle -300, -290, -280; -120, -110, -90\rangle; \langle -450, -400, -350; -30, -20, -10\rangle]$

Scalar multiplication: $k \otimes \breve{A}^{TBN} = [\langle k \otimes 190, k \otimes 220, k \otimes 240; k \otimes 130, k \otimes 160, k \otimes 200\rangle; \langle k \otimes 280, k \otimes 290, k \otimes 300; k \otimes 90, k \otimes 110, k \otimes 120\rangle; \langle k \otimes 350, k \otimes 400, k \otimes 450; k \otimes 10, k \otimes 20, k \otimes 30\rangle]$.

Proposition 4.1.6. *The right membership grades* $T^{+R}_{\breve{A}^{TBN}}, I^{+R}_{\breve{A}^{TBN}}, F^{+R}_{\breve{A}^{TBN}}, T^{-R}_{\breve{A}^{TBN}}, I^{-R}_{\breve{A}^{TBN}}, F^{-R}_{\breve{A}^{TBN}}$ *and the left membership grades* $T^{+L}_{\breve{A}^{TBN}}, I^{+L}_{\breve{A}^{TBN}}, F^{+L}_{\breve{A}^{TBN}}, T^{-L}_{\breve{A}^{TBN}}, I^{-L}_{\breve{A}^{TBN}}, F^{-L}_{\breve{A}^{TBN}}$ *are invertible.*

Proof of above proposition is quite straightforward from the definition of bipolar neutrosophic numbers.

Since the left truth membership grade of $\breve{A}^{TBN}$ denoted as,

$T^{+L}_{\breve{A}^{TBN}} : [\mathfrak{a}, b] \longrightarrow [0,1]$, is strictly increasing and continuous, Section 4.1. Therefore, the inverse function of $T^{+L}_{\breve{A}^{TBN}}$ exists.

Let $G^{+L}_{\breve{A}^{TBN}} : [0,1] \longrightarrow [\mathfrak{a}, b]$ be the inverse of $T^{+L}_{\breve{A}^{TBN}}$.

Similarly, inverse of $T^{+R}_{\breve{A}^{TBN}} : [b,c] \longrightarrow [0,1]$ is $G^{+R}_{\breve{A}^{TBN}} : [0,1] \longrightarrow [b,c]$.

Again, $T^{-L}_{\breve{A}^{TBN}} : [d,e] \longrightarrow [-1,0]$, is strictly decreasing and continuous, Section 4.1, therefore, the inverse function of $T^{-L}_{\breve{A}^{TBN}}$ exists.

Let $G^{-L}_{\breve{A}^{TBN}} : [-1,0] \longrightarrow [d,e]$ be the inverse of $T^{-L}_{\breve{A}^{TBN}}$.

continuing in the same manner,

$$\begin{aligned}
&\textit{inverse of } T^{-R}_{\breve{A}^{TBN}} : [e,f] \longrightarrow [-1,0] \textit{ is } G^{-R}_{\breve{A}^{TBN}} : [-1,0] \longrightarrow [e,f],\\
&\textit{inverse of } I^{+L}_{\breve{A}^{TBN}} : [g,h] \longrightarrow [0,1] \textit{ is } J^{+L}_{\breve{A}^{TBN}} : [0,1] \longrightarrow [g,h],\\
&\textit{inverse of } I^{+R}_{\breve{A}^{TBN}} : [h,i] \longrightarrow [0,1] \textit{ is } J^{+R}_{\breve{A}^{TBN}} : [0,1] \longrightarrow [h,i],\\
&\textit{inverse of } I^{-L}_{\breve{A}^{TBN}} : [j,k] \longrightarrow [-1,0] \textit{ is } J^{-L}_{\breve{A}^{TBN}} : [-1,0] \longrightarrow [j,k],\\
&\textit{inverse of } I^{-R}_{\breve{A}^{TBN}} : [k,l] \longrightarrow [-1,0] \textit{ is } J^{-R}_{\breve{A}^{TBN}} : [-1,0] \longrightarrow [k,l],\\
&\textit{inverse of } F^{+L}_{\breve{A}^{TBN}} : [m,n] \longrightarrow [0,1] \textit{ is } M^{+L}_{\breve{A}^{TBN}} : [0,1] \longrightarrow [m,n],\\
&\textit{inverse of } F^{+R}_{\breve{A}^{TBN}} : [n,o] \longrightarrow [0,1] \textit{ is } M^{+R}_{\breve{A}^{TBN}} : [0,1] \longrightarrow [n,o],\\
&\textit{inverse of } F^{-L}_{\breve{A}^{TBN}} : [p,q] \longrightarrow [-1,0] \textit{ is } M^{-L}_{\breve{A}^{TBN}} : [-1,0] \longrightarrow [p,q],\\
&\textit{inverse of } F^{-R}_{\breve{A}^{TBN}} : [q,r] \longrightarrow [-1,0] \textit{ is } M^{-R}_{\breve{A}^{TBN}} : [-1,0] \longrightarrow [q,r].
\end{aligned}$$

Proposition 4.1.7. *The functions* $G^{+L}_{\breve{A}^{TBN}}, G^{+R}_{\breve{A}^{TBN}}, G^{-L}_{\breve{A}^{TBN}}, G^{-R}_{\breve{A}^{TBN}}, J^{+L}_{\breve{A}^{TBN}}, J^{+R}_{\breve{A}^{TBN}}, J^{-L}_{\breve{A}^{TBN}}, J^{-R}_{\breve{A}^{TBN}}, M^{+L}_{\breve{A}^{TBN}}, M^{+R}_{\breve{A}^{TBN}}, M^{-L}_{\breve{A}^{TBN}}, M^{-R}_{\breve{A}^{TBN}}$ *are also monotonic and continuous.*

Proposition 4.1.8. *The functions* $G^{+L}_{\breve{A}^{TBN}}, G^{+R}_{\breve{A}^{TBN}}, G^{-L}_{\breve{A}^{TBN}}, G^{-R}_{\breve{A}^{TBN}}, J^{+L}_{\breve{A}^{TBN}}, J^{+R}_{\breve{A}^{TBN}}, J^{-L}_{\breve{A}^{TBN}}, J^{-R}_{\breve{A}^{TBN}}, M^{+L}_{\breve{A}^{TBN}}, M^{+R}_{\breve{A}^{TBN}}, M^{-L}_{\breve{A}^{TBN}}, M^{-R}_{\breve{A}^{TBN}}$ *are integrable.*

4.2 Ordering/Ranking of Bipolar Neutrosophic Numbers

let $\mathbb{N}$ be a class of BN-numbers. Ranking function $\mathfrak{R}$ is defined so that it maps each bipolar neutrosophic number in $\mathbb{N}$ to a real number $\mathbb{R}$.

$\mathfrak{R} : \mathbb{N} \longrightarrow \mathbb{R}$. Let $\breve{A}^{TBN}$ and $\breve{B}^{TBN}$ be two bipolar neutrosophic numbers in $\mathbb{N}$.

Then $\mathfrak{R}$ must satisfy below conditions:

$a : \mathfrak{R}(\breve{A}^{TBN}) < \mathfrak{R}(\breve{B}^{TBN})$, $\breve{A}^{TBN} < \breve{B}^{TBN}$.

$b : \mathfrak{R}(\breve{A}^{TBN}) > \mathfrak{R}(\breve{B}^{TBN})$, $\breve{A}^{TBN} > \breve{B}^{TBN}$.

$c : \mathfrak{R}(\breve{A}^{TBN}) = \mathfrak{R}(\breve{B}^{TBN})$, $\breve{A}^{TBN} = \breve{B}^{TBN}$.

A bipolar neutrosophic number can be ranked or ordered in a number of ways. A few derivations will be constructed in this section.

4.2.1 First Derivation

The following is one possible definition of the scoring function of the bipolar neutrosophic number;

$$S(\breve{A}^{TBN}) = s(T_{\breve{A}^{TBN}}) + s(F_{\breve{A}^{TBN}}) + s(I_{\breve{A}^{TBN}}). \tag{4.2.1}$$

Where,

$$\begin{aligned} s(T_{\breve{A}^{TBN}}) =& [\ddot{I}^L(T^+_{\breve{A}^{TBN}}) + \ddot{I}^R(T^+_{\breve{A}^{TBN}})] + [\ddot{I}^L(T^-_{\breve{A}^{TBN}}) + \ddot{I}^R(T^-_{\breve{A}^{TBN}})], \\ s(I_{\breve{A}^{TBN}}) =& [\ddot{I}^L(I^+_{\breve{A}^{TBN}}) + \ddot{I}^R(I^+_{\breve{A}^{TBN}})] - [\ddot{I}^L(I^-_{\breve{A}^{TBN}}) + \ddot{I}^R(I^-_{\breve{A}^{TBN}})], \\ s(F_{\breve{A}^{TBN}}) =& [\ddot{I}^L(F^+_{\breve{A}^{TBN}}) + \ddot{I}^R(F^+_{\breve{A}^{TBN}})] - [\ddot{I}^L(F^-_{\breve{A}^{TBN}}) + \ddot{I}^R(F^-_{\breve{A}^{TBN}})]. \end{aligned} \tag{4.2.2}$$

And,

$$\ddot{I}^L(T^+_{\breve{A}^{TBN}}) = \int_0^1 G^{+L}_{\breve{A}^{TBN}}(y)dy \; ; \; \ddot{I}^R(T^+_{\breve{A}^{TBN}}) = \int_0^1 G^{+R}_{\breve{A}^{TBN}}(y)dy, \tag{4.2.3}$$

$$\ddot{I}^L(T^-_{\breve{A}^{TBN}}) = \int_{-1}^0 G^{-L}_{\breve{A}^{TBN}}(y)dy \; ; \; \ddot{I}^R(T^-_{\breve{A}^{TBN}}) = \int_{-1}^0 G^{-R}_{\breve{A}^{TBN}}(y)dy,$$

$$\ddot{I}^L(I^+_{\breve{A}^{TBN}}) = \int_0^1 J^{+L}_{\breve{A}^{TBN}}(y)dy \; ; \; \ddot{I}^R(I^+_{\breve{A}^{TBN}}) = \int_0^1 J^{+R}_{\breve{A}^{TBN}}(y)dy,$$

$$\ddot{I}^L(I^-_{\breve{A}^{TBN}}) = \int_{-1}^0 J^{-L}_{\breve{A}^{TBN}}(y)dy \; ; \; \ddot{I}^R(I^-_{\breve{A}^{TBN}}) = \int_{-1}^0 J^{-R}_{\breve{A}^{TBN}}(y)dy,$$

$$\ddot{I}^L(F^+_{\breve{A}^{TBN}}) = \int_0^1 M^{+L}_{\breve{A}^{TBN}}(y)dy \; ; \; \ddot{I}^R(F^+_{\breve{A}^{TBN}}) = \int_0^1 M^{+R}_{\breve{A}^{TBN}}(y)dy,$$

$$\ddot{I}^L(F^-_{\breve{A}^{TBN}}) = \int_{-1}^0 M^{-L}_{\breve{A}^{TBN}}(y)dy \; ; \; \ddot{I}^R(F^-_{\breve{A}^{TBN}}) = \int_{-1}^0 M^{-R}_{\breve{A}^{TBN}}(y)dy.$$

Using Equation 4.2.3 in Equation 4.2.2, we get:

$$s(T_{\breve{A}^{TBN}}) = \int_0^1 \mathfrak{a} + (b - \mathfrak{a}).ydy + \int_0^1 c - (c - b).ydy + \int_{-1}^0 d - (e - d).ydy + \int_{-1}^0 f + (f - e).ydy \tag{4.2.4}$$

$$s(T_{\breve{A}^{TBN}}) = \frac{\mathfrak{a} + 2b + c + d + 2e + f}{2} \tag{4.2.5}$$

Similarly,

$$s(F_{\breve{A}^{TBN}}) = \frac{m + 2n + o - p - 2q - r}{2}, \tag{4.2.6}$$

$$s(I_{\breve{A}^{TBN}}) = \frac{g + 2h + i - j - 2k - l}{2}. \tag{4.2.7}$$

using Equation 4.2.5, 4.2.6 and Equation 4.2.7 in Equation 4.2.1, we get;

$$S(\breve{A}^{TBN}) = \frac{\mathfrak{a}+2b+c+d+2e+f+g+2h+i-j-2k-l+m+2n+o-p-2q-r}{2}. \tag{4.2.8}$$

we will use this score function (4.2.1) to define ranking function of bipolar triangular neutrosophic numbers. Let the ranking function of bipolar neutrosophic number is defined by using "mean-max method also sometimes termed as middle of maxima",

$$\Re(\breve{A}^{TBN}) = \frac{S(\breve{A}^{TBN})}{6}. \tag{4.2.9}$$

Thus, the final form of Equation 4.2.9 is;

$$\Re(\breve{A}^{TBN}) = \frac{a+2b+c+d+2e+f+g+2h+i-j-2k-l+m+2n+o-p-2q-r}{12}. \tag{4.2.10}$$

4.2.1.1 Special Case 1

If point "b" is considered as the highest point of all positive membership grades as well as the lowest point of all negative grades of membership then $b = e = h = k = n = q$, in such case Equation 4.2.10 is transformed into a new equation;

$$\Re(\breve{A}^{TBN}) = \frac{a+4b+c+d+f+g+i-j-l+m+o-p-r}{12}. \tag{4.2.11}$$

4.2.2 Second Derivation

If Equation 4.2.1 is defined in a different manner as below;

$$S(\breve{A}^{TBN}) = s(T_{\breve{A}^{TBN}}) - s(F_{\breve{A}^{TBN}}) - s(I_{\breve{A}^{TBN}}). \tag{4.2.12}$$

In this scenario Equations 4.2.10 and 4.2.11 are transformed into a new ranking function.

Transformed form of Equation 4.2.10:

$$\Re(\breve{A}^{TBN})) = \frac{a+2b+c+d+2e+f-g-2h-i+j+2k+l-m-2n}{12} \\ \frac{-o+p+2q+r}{12}. \tag{4.2.13}$$

Transformed form of Equation 4.2.11:

$$\Re(\breve{A}^{TBN}) = \frac{a+4b+c+d+f-g-i+j+l-m-o+p+r}{12}. \tag{4.2.14}$$

4.2.3 Third Derivation

Ranking of a bipolar neutrosophic number can also be done employing area and mean values of the respective membership grades.

Let ranking function of a BNN $\breve{A}^{TBN}$ is represented mathematically as follows;

$$\begin{aligned}\Re(\breve{A}^{TBN}) &= [m(T^{+}_{\breve{A}^{TBN}}) + \zeta(T^{+}_{\breve{A}^{TBN}})] + [m(T^{-}_{\breve{A}^{TBN}}) + \zeta(T^{-}_{\breve{A}^{TBN}})] \\ &+ [m(I^{+}_{\breve{A}^{TBN}}) + \zeta(I^{+}_{\breve{A}^{TBN}})] - [m(I^{-}_{\breve{A}^{TBN}}) + \zeta(I^{-}_{\breve{A}^{TBN}})] \\ &+ [m(F^{+}_{\breve{A}^{TBN}}) + \zeta(F^{+}\breve{A}^{TBN})] - [m(F^{-}_{\breve{A}^{TBN}}) + \zeta(F^{-}_{\breve{A}^{TBN}})].\end{aligned} \tag{4.2.15}$$

In Equation 4.2.15, $m(T^{+}_{\breve{A}^{TBN}}), m(T^{-}_{\breve{A}^{TBN}}), m(I^{+}_{\breve{A}^{TBN}}), m(I^{-}_{\breve{A}^{TBN}}), m(F^{+}_{\breve{A}^{TBN}})$ and $m(F^{-}_{\breve{A}^{TBN}})$ denote the mean of respective elements of membership grades and $\zeta(T^{+}_{\breve{A}^{TBN}}), \zeta(T^{-}_{\breve{A}^{TBN}})$,
$\zeta(I^{+}_{\breve{A}^{TBN}}), \zeta(I^{-}_{\breve{A}^{TBN}}), \zeta(F^{+}_{\breve{A}^{TBN}})$ and $\zeta(F^{-}_{\breve{A}^{TBN}})$ represent the areas of the respective members.

Considering Equation 4.2.15,

$$\begin{aligned}\Re(\breve{A}^{TBN}) &= [m(T^{+}_{\breve{A}^{TBN}}) + \int_x T^{+}_{\breve{A}^{TBN}}(x).dx \quad] \quad +[m(T^{-}_{\breve{A}^{TBN}}) + \mid \int_x T^{-}_{\breve{A}^{TBN}}(x).dx \mid] \\ &+ [m(I^{+}_{\breve{A}^{TBN}}) + \int_x I^{+}_{\breve{A}^{TBN}}(x).dx \quad] - \quad [m(I^{-}_{\breve{A}^{TBN}}) + \mid \int_x I^{-}_{\breve{A}^{TBN}}(x).dx \mid] \\ &+ [m(F^{+}_{\breve{A}^{TBN}}) + \int_x F^{+}_{\breve{A}^{TBN}}(x).dx \quad] - [m(F^{-}_{\breve{A}^{TBN}}) + \mid \int_x F^{-}_{\breve{A}^{TBN}}(x).dx \mid].\end{aligned} \tag{4.2.16}$$

Whereas,

$$\begin{aligned}(T^{+}_{\breve{A}^{TBN}}) &= \frac{a+b+c}{3} \quad ; \quad \int_x T^{+}_{\breve{A}^{TBN}}(x).dx = \int_a^b \frac{x-a}{b-a}.dx + \int_b^c \frac{c-x}{c-b}.dx \\ m(T^{-}_{\breve{A}^{TBN}}) &= \frac{d+e+f}{3} \quad ; \quad \mid \int_x T^{-}_{\breve{A}^{TBN}}(x).dx \mid = \int_d^e \frac{x-d}{e-d}.dx + \int_e^f \frac{f-x}{f-e}.dx \\ m(I^{+}_{\breve{A}^{TBN}}) &= \frac{g+h+i}{3} \quad ; \quad \int_x I^{+}_{\breve{A}^{TBN}}(x).dx = \int_g^h \frac{h-x}{h-g}.dx + \int_h^i \frac{x-h}{i-h}.dx \\ m(I^{-}_{\breve{A}^{TBN}}) &= \frac{j+k+l}{3} \quad ; \quad \mid \int_x I^{-}_{\breve{A}^{TBN}}(x).dx \mid = \int_j^k \frac{k-x}{k-j}.dx + \int_k^l \frac{x-k}{l-k}.dx \\ m(F^{+}_{\breve{A}^{TBN}}) &= \frac{m+n+o}{3} \quad ; \quad \int_x F^{+}_{\breve{A}^{TBN}}(x).dx = \int_m^n \frac{n-x}{n-m}.dx + \int_n^o \frac{x-n}{o-n}.dx \\ m(F^{-}_{\breve{A}^{TBN}}) &= \frac{p+q+r}{3} \quad ; \quad \mid \int_x F^{-}_{\breve{A}^{TBN}}(x).dx \mid = \int_p^q \frac{q-x}{q-p}.dx + \int_q^r \frac{x-q}{r-q}.dx.\end{aligned} \tag{4.2.17}$$

Use all the values from Equation 4.2.17 in 4.2.15 and after all necessary simplification the resultant function is given as;

$$\Re(\breve{A}^{TBN}) = \frac{-a + 2b + 5c - d + 2e + 5f - m}{6} \\ \frac{+2n + 5o + p - 2q - 5r - g + 2h + 5i + j - 2k - 5l}{6}. \qquad (4.2.18)$$

4.2.4 Fourth Derivation

Let ranking function of a BN-number $\breve{A}^{TBN}$ is defined as follows;

$$\begin{aligned}\Re(\breve{A}^{TBN}) = {} & [m(T^{+}_{\breve{A}^{TBN}}) \times \zeta(T^{+}_{\breve{A}^{TBN}})] + [m(T^{-}_{\breve{A}^{TBN}}) \times \zeta(T^{-}_{\breve{A}^{TBN}})] \\ & + [m(I^{+}_{\breve{A}^{TBN}}) \times \zeta(I^{+}_{\breve{A}^{TBN}})] - [m(I^{-}_{\breve{A}^{TBN}}) \times \zeta(I^{-}_{\breve{A}^{TBN}})] \\ & + [m(F^{+}_{\breve{A}^{TBN}}) \times \zeta(F^{+}\breve{A}^{TBN})] - [m(F^{-}_{\breve{A}^{TBN}}) \times \zeta(F^{-}_{\breve{A}^{TBN}})].\end{aligned} \qquad (4.2.19)$$

Substituting necessary values and after required simplification Equation 4.2.19 takes the form;

$$\begin{aligned}\Re(\breve{A}^{TBN}) = {} & [\frac{a+b+c}{3} \times \frac{c-a}{2}] - [\frac{d+e+f}{3} \times \frac{f-d}{2}] + \\ & [\frac{g+h+i}{3} \times \frac{i-g}{2}] - [\frac{j+k+l}{3} \times \frac{l-j}{2}] + \\ & [\frac{m+n+o}{3} \times \frac{o-m}{2}] - [\frac{p+q+r}{3} \times \frac{r-p}{2}].\end{aligned} \qquad (4.2.20)$$

4.2.5 Fifth Derivation

Centroid method can also be employed to rank bipolar neutrosophic numbers. Mathematically;

$$\begin{aligned}\Re(\breve{A}^{TBN}) = {} & \frac{\int_x x.T^{+}_{\breve{A}^{TBN}}(x).dx}{\int_x T^{+}_{\breve{A}^{TBN}}(x).dx} + \frac{\int_x x.T^{-}_{\breve{A}^{TBN}}(x).dx}{\int_x T^{-}_{\breve{A}^{TBN}}(x).dx} \\ & + \frac{\int_x x.I^{+}_{\breve{A}^{TBN}}(x).dx}{\int_x I^{+}_{\breve{A}^{TBN}}(x).dx} - \frac{\int_x x.I^{-}_{\breve{A}^{TBN}}(x).dx}{\int_x I^{-}_{\breve{A}^{TBN}}(x).dx} \\ & + \frac{\int_x x.F^{+}_{\breve{A}^{TBN}}(x).dx}{\int_x F^{+}_{\breve{A}^{TBN}}(x).dx} - \frac{\int_x x.F^{-}_{\breve{A}^{TBN}}(x).dx}{\int_x F^{-}_{\breve{A}^{TBN}}(x).dx}.\end{aligned} \qquad (4.2.21)$$

Expression 4.2.21 can be solved for trapezoidal as well as triangular BN-numbers to obtain final value of ranking function.

Example 4.2.1. Let $\breve{A}^{TBN}$ and $\breve{B}^{TBN}$ be two TBN-numbers.

$$\begin{aligned}\breve{A}^{TBN} =&[\langle 190,220,240;130,160,200\rangle;\langle 280,290,300;90,110,120\rangle;\langle 350,400,450;\\&10,20,30\rangle]\\\breve{B}^{TBN} =&[\langle 170,200,230;110,140,170\rangle;\langle 290,295,300;110,120,130\rangle;\langle 250,280310;\\&80,90,100\rangle]\end{aligned}$$

using any of the ranking notation expressed in Equation 4.2.10, 4.2.13, 4.2.18 and Equation 4.2.20, we get the same results, i.e. $\mathfrak{R}(\breve{A}^{TBN} > \mathfrak{R}(\breve{B}^{TBN})$, thus $\breve{A}^{TBN} > \breve{B}^{TBN}$.

4.3 Conclusion

The BN-number is a significant theme for exhibiting uncertain data, as it demonstrates the presence of six disjunctive segments and allows analysts a wide range of applications in specific domains. A thorough discussion has taken place on the subject of BN-numbers. The definition is followed by a number of results and propositions that are comprehensive in their scope. The linear membership function of the BN-number has been created and several new ideas have been extended as a result of this construction. In a neutrosophic environment, ranking neutrosophic numbers in general and bipolar neutrosophic number in particular is an essential part of decision making. There is no way to put bipolar neutrosophic numbers in a sequence because they are represented by possibility distributions. The way possibility distributions are used to show neutrosophic numbers and bipolar neutrosophic numbers, it is possible for them to overlap with each other, making it impossible to put them in order in a way that makes sense. It is a fact that bipolar neutrosophic numbers are generally partial orders and as a result, they cannot be compared in the same way as the real numbers, which can be compared in a linear order. Up-till now, no model for ranking BN-numbers has been proposed in the literature. In order to rank bipolar neutrosophic quantities, we devised a few techniques for converting BN-numbers to real numbers. These numbers can thus be compared by establishing a suitable ordering function. This ranking function allocates a real number to each bipolar neutrosophic number that exhibits natural order. Future work may include the development of solutions procedures for addressing linear programming problems in a bipolar neutrosophic environment, using an appropriate ranking function.

Chapter 5

Bipolar Neutrosophic Linear Programming: An Application of the Ranking of Bipolar Neutrosophic Numbers

Linear programming (LP) is becoming increasingly important in various domains of science and industry. Because of their wide range of applications in operation research (OR) approaches, LP models have been applied in both crisp and fuzzy environments in recent decades. In this chapter, we will construct a bipolar neutrosophic linear programming (BNLP) model from the existing crisp model. Different types of bipolar NLP's are discussed here along with the new approach of solving BNLP employing the appropriate ranking function of bipolar neutrosophic numbers. Using the ranking function, the bipolar neutrosophic linear programming may be turned into an improved crisp linear programming. The practical formulation of the model is carried out through the application and data specifically gathered from the Punjab province for maximization of pulses production. A sensitivity analysis is used to compare and analyze the results, along with graphs and tables.

The simplest mathematical programming model is the linear programming model. LP is a powerful mathematical technique for allocating constrained resources to complex problems. LP is a way to figure out how to maximize or minimize a linear function when it has to meet certain constraints. It is a common strategy that is used in a variety of sectors, such as manufacturing, finance, medical field, marketing, and environmental sustainability [116]-[120]. In a crisp environment, mathematical models that have objective functions and constraints do not have the capability to incorporate uncertainty. In real-world scenario, LP models incorporate variables whose values are determined by professionals and decision-makers, but neither the professionals nor the decision-makers are aware of the precise and accurate crisp values. In a fuzzy environment, such circumstances that emerge in real-world settings are tackled using fuzzy linear programming (FLP), and the underlying problems become FLP problems [121]-[124].

The classical LP is typically insufficient for solving real-world problems, whereas the FLP has gained prominence in recent decades among the scientific community. Bellman and Zadeh [51] are the first to consider the FLP problem in detail. According to Lotfi *et al.* [125] no one has solved FLP problems with equality constraints, so they propose using an existing method to handle these problems. Zimmerman [53] proposed a multi-objective function-based strategy for solving the fuzzy linear programming (FLP) problem. This technique has been adopted by many scholars for the solution of FLPs, including Ebrahimnejad *et al.* [126], Campos and Verdegay [127], Ganesan and Veeramani [128], among others. FLP has grown in popularity as fuzzy sets have progressed and become more generalized. Hussein *et al.* [129] investigated a neutrosophic LP problem in which they used neutrosophic set parameters to transform the neutrosophic model into the equivalent crisp. Abdel-Basset *et al.*[130] presented a method for tackling the LP problem using trapezoidal neutrosophic numbers for all parameters. The complete neutrosophic LP problem was studied by Nafei *et al.* [131], in which all parameters are expressed as triangular interval-valued neutrosophic numbers. They converted the neutrosophic model to its crisp counterpart using the ranking function and then utilized normal methods to achieve the solution.

5.1 Bipolar Neutrosophic Linear Programming Model

Bipolar neutrosophic linear programming (BNLP) can be constructed from crisp LP by shifting any of the component from crisp number to bipolar neutrosophic number.

5.1.1 Linear Programming Model

Generally an LP can be formulated as,

$$\text{Maximize or Minimize} \quad d_{ij} f(\mathfrak{z}_{ij}), \tag{5.1.1}$$

given set of constraints,

$$\mathfrak{e}_{ij}\mathfrak{z}_{ij} <, >, = b_j \quad i = 1, 2, \ldots, p \ and \ j = 1, 2, \ldots, q. \tag{5.1.2}$$

The decision variable is denoted as $\mathfrak{z}_{ij}$ whereas, $f(\mathfrak{z}_{ij})$ depicts objective function, d_{ij} and $\mathfrak{e}_{ij}$ are the coefficients of the objective function and constraints respectively, q implicates the number of constraints. This modeling reflects a crisp design of the problem, but for the most useful application of this system in everyday issues various bipolar neutrosophic models can be fabricated by using the concepts of bipolar neutrosophic numbers.

5.1.2 Bipolar Neutrosophic Linear Programming Model

This model improves the intended requirements by thoroughly assessing the problem specifications using bipolar neutrosophic numbers, which are generalizations of fuzzy numbers. LP prescribed in Equation 5.1.1 and Equation

5.1.2 can take any form depending on the component of LP to be considered as bipolar neutrosophic number. This section comprises of the different forms of BNLP on the basis of the conversion of crisp component of model to bipolar neutrosophic number.

5.1.2.1 Bipolar NLP of Type-1

BNLP of type-1 is the case in which objective functions' coefficients are represented as bipolar neutrosophic number and the rest of the components are real entries.

$$\text{Max or Min} \quad \check{\check{d}}_{ij} f(\mathfrak{z}_{ij}) \tag{5.1.3}$$

subject to the constraints,

$$\mathfrak{e}_{ij}\mathfrak{z}_{ij} \lesssim, \gtrsim, \approx b_j \quad i = 1, 2, \ldots, p \ \textit{and} \ j = 1, 2, \ldots, q. \tag{5.1.4}$$

Where all $\mathfrak{z}_{ij}$ should be positive. In Equation 5.1.3 all $\check{\check{d}}_{ij}$ are bipolar neutrosophic numbers.

5.1.2.2 Bipolar NLP of Type-2

Type-2 BNLP is the case when all the components of LP are real with an exemption of coefficients of the variables in the constraints.

$$\text{Max or Min} \quad d_{ij} f(\mathfrak{z}_{ij}),$$

subject to the constraints,

$$\check{\check{\mathfrak{e}}}_{ij}\mathfrak{z}_{ij} \lesssim, \gtrsim, \approx b_j \quad i = 1, 2, \ldots, p \ \textit{and} \ j = 1, 2, \ldots, q. \tag{5.1.5}$$

Where all $\mathfrak{z}_{ij}$ should be positive. In problem 5.1.5 all $\check{\mathfrak{e}}_{ij}$ are bipolar neutrosophic numbers.

5.1.2.3 Bipolar NLP of Type-3

Type-3 BNLP is the case when both $\check{\check{\mathfrak{e}}}_{ij}$ and $\check{\check{b}}_j$ of constraint equations are bipolar neutrosophic numbers.

$$\text{Maximize or Minimize} \quad d_{ij} f(\mathfrak{z}_{ij}),$$

subject to the constraints,

$$\check{\mathfrak{e}}_{ij}\mathfrak{z}_{ij} \lesssim, \gtrsim, \approx \check{b}_j \quad i = 1, 2, \ldots, p \ and \ j = 1, 2, \ldots, q. \tag{5.1.6}$$

Where all $\mathfrak{z}_{ij}$ should be positive.

5.1.2.4 Total Bipolar Neutrosophic Linear Programming

Total BNLP is the case when all coefficients, parameters and variables used to model LP are bipolar neutrosophic numbers.

$$\text{Maximize or Minimize } \check{d}_{ij}f(\check{\mathfrak{z}}_{ij}),$$

subject to,

$$\check{\mathfrak{e}}_{ij}\check{\mathfrak{z}}_{ij} \lesssim, \gtrsim, \approx \check{b}_j \quad i = 1, 2, \ldots, p \ and \ j = 1, 2, \ldots, q. \tag{5.1.7}$$

In the given case $\check{d}_{ij}$, $\check{\mathfrak{z}}_{ij}$, $\check{\mathfrak{e}}_{ij}$ and $\check{b}_j$ are bipolar neutrosophic numbers.
These are a few of many forms and decision makers can choose any type of bipolar neutrosophic linear programming problem depending on the type of solution they are interested in and the type of model that best fits their needs.

5.2 Proposed Approach

Present work is based on type-3 BNLP, Section 5.1.2.3, Equation 5.1.6 .
Thus the Equations 5.1.1 and 5.1.2 takes the form,

$$\text{Max or Min} \quad d_{ij}f(\mathfrak{z}_{ij})$$

subject to the constraints,

$$\check{\mathfrak{e}}_{ij}\mathfrak{z}_{ij} \lesssim, \gtrsim, \approx \check{b}_j \quad i = 1, 2, \ldots, p \ and \ j = 1, 2, \ldots, q \tag{5.2.1}$$

In problem 5.2.1 all $\check{\mathfrak{e}}_{ij}$ and $\check{b}_i$ are bipolar neutrosophic numbers and non-negativity of the decision variable should be preserved here. The arithmetic operations of bipolar neutrosophic numbers, described in Chapter 5, Section 4.1.1, are used for the simplification purpose. Suitable ranking function, Section 4.2, will be used to simplify things even more. Ordering of bipolar neutrosophic number transform the constraint Equation 5.2.1 into crisp form. Resultant crisp model can easily be solved through any LP solver (excel solver or matlab algorithms)

5.2.1 Methodology

$$\text{Maximize } \sum_{j=1}^{p} D_j \mathfrak{Z}_j.$$

$$\sum_{j=1}^{p} \tilde{\tilde{A}}_{ij}^{n} \otimes \mathfrak{Z}_j \lesssim, \gtrsim, \approx \tilde{\tilde{B}}_i^{n} \ \forall i = 1, 2, \ldots, q. \tag{5.2.2}$$

Where all $\mathfrak{Z}_j's$ are non-negative and $\tilde{\tilde{A}}_{ij}^{n}$ and $\tilde{\tilde{B}}_i^{n}$ are triangular bipolar neutrosophic numbers.

Step 1: Assume;

$$\begin{aligned}\tilde{\tilde{A}}_{ij}^{n} =& [\mathfrak{a}_{ij,1}, \mathfrak{a}_{ij,2}, \mathfrak{a}_{ij,3}; \acute{\mathfrak{a}}_{ij,1}, \acute{\mathfrak{a}}_{ij,2}, \acute{\mathfrak{a}}_{ij,3}]; [\mathfrak{a}_{ij,4}, \mathfrak{a}_{ij,5}, \mathfrak{a}_{ij,6}; \acute{\mathfrak{a}}_{ij,4}, \acute{\mathfrak{a}}_{ij,5}, \acute{\mathfrak{a}}_{ij,6}]\\ & [\mathfrak{a}_{ij,7}, \mathfrak{a}_{ij,8}, \mathfrak{a}_{ij,9}; \acute{\mathfrak{a}}_{ij,7}, \acute{\mathfrak{a}}_{ij,8}, \acute{\mathfrak{a}}_{ij,9}]\\ \tilde{\tilde{B}}_i^{n} =& [b_{ij,1}, b_{ij,2}, b_{ij,3}; \acute{b}_{ij,1}, \acute{b}_{ij,2}, \acute{b}_{ij,3}]; [b_{ij,4}, b_{ij,5}, b_{ij,6}; \acute{b}_{ij,b}, \acute{b}_{ij,5}, \acute{b}_{ij,6}]\\ & [b_{ij,7}, b_{ij,8}, b_{ij,9}; \acute{b}_{ij,7}, \acute{b}_{ij,8}, \acute{b}_{ij,9}]\end{aligned}$$

the BNLP 5.2.2 can be reshaped as follows;

$$\text{Maximize } \sum_{j=1}^{p} D_j \mathfrak{Z}_j. \tag{5.2.3}$$

subject to;

$$\sum_{j=1}^{p} [[\mathfrak{a}_{ij,1}, \mathfrak{a}_{ij,2}, \mathfrak{a}_{ij,3}; \acute{\mathfrak{a}}_{ij,1}, \acute{\mathfrak{a}}_{ij,2}, \acute{\mathfrak{a}}_{ij,3}]; [\mathfrak{a}_{ij,4}, \mathfrak{a}_{ij,5}, \mathfrak{a}_{ij,6}; \acute{\mathfrak{a}}_{ij,4}, \acute{\mathfrak{a}}_{ij,5}, \acute{\mathfrak{a}}_{ij,6}][\mathfrak{a}_{ij,7}, \mathfrak{a}_{ij,8},$$

$$\mathfrak{a}_{ij,9}; \acute{\mathfrak{a}}_{ij,7}, \acute{\mathfrak{a}}_{ij,8}, \acute{\mathfrak{a}}_{ij,9}]] \otimes X_j \lesssim [b_{ij,1}, b_{ij,2}, b_{ij,3}; \acute{b}_{ij,1}, \acute{b}_{ij,2}, \acute{b}_{ij,3}]; [b_{ij,4}, b_{ij,5}, b_{ij,6}; \acute{b}_{ij,b},$$

$$\acute{b}_{ij,5}, x\acute{b}_{ij,6}]; [b_{ij,7}, b_{ij,8}, b_{ij,9}; \acute{b}_{ij,7}, \acute{b}_{ij,8}, \acute{b}_{ij,9}] \quad i = 1, 2, \ldots, q. \tag{5.2.4}$$

Step 2: Employing the product of bipolar neutrosophic number, Section 4.1.1, and applying the assumption below;

$$\sum_{j=1}^{p}[[\mathfrak{a}_{ij,1},\mathfrak{a}_{ij,2},\mathfrak{a}_{ij,3};\acute{\mathfrak{a}}_{ij,1},\acute{\mathfrak{a}}_{ij,2},\acute{\mathfrak{a}}_{ij,3}];[\mathfrak{a}_{ij,4},\mathfrak{a}_{ij,5},\mathfrak{a}_{ij,6};\acute{\mathfrak{a}}_{ij,4},\acute{\mathfrak{a}}_{ij,5},\acute{\mathfrak{a}}_{ij,6}][\mathfrak{a}_{ij,7},\mathfrak{a}_{ij,8},\mathfrak{a}_{ij,9};$$
$$\acute{\mathfrak{a}}_{ij,7},\acute{\mathfrak{a}}_{ij,8},\acute{\mathfrak{a}}_{ij,9}]]\otimes X_j\approx\sum_{j=1}^{p}[[\hat{\mathfrak{a}}_{ij,1},\hat{\mathfrak{a}}_{ij,2},\hat{\mathfrak{a}}_{ij,3};\hat{\hat{\mathfrak{a}}}_{ij,1},\hat{\hat{\mathfrak{a}}}_{ij,2},\hat{\hat{\mathfrak{a}}}_{ij,3}];[\hat{\mathfrak{a}}_{ij,4},\hat{\mathfrak{a}}_{ij,5},\hat{\mathfrak{a}}_{ij,6};\hat{\hat{\mathfrak{a}}}_{ij,4},$$
$$\hat{\hat{\mathfrak{a}}}_{ij,5},\hat{\hat{\mathfrak{a}}}_{ij,6}][\hat{\mathfrak{a}}_{ij,7},\hat{\mathfrak{a}}_{ij,8},\hat{\mathfrak{a}}_{ij,9};\hat{\hat{\mathfrak{a}}}_{ij,7},\hat{\hat{\mathfrak{a}}}_{ij,8},\hat{\hat{\mathfrak{a}}}_{ij,9}]] \tag{5.2.5}$$

$i = 1, 2, \ldots, q.$

The BNLP 5.2.3 and 5.2.4 will be transformed as;

$$\text{Maximize } \sum_{j=1}^{p} D_j \mathfrak{z}_j,$$

subject to;

$$\sum_{j=1}^{p}[[\hat{\mathfrak{a}}_{ij,1},\hat{\mathfrak{a}}_{ij,2},\hat{\mathfrak{a}}_{ij,3};\hat{\hat{\mathfrak{a}}}_{ij,1},\hat{\hat{\mathfrak{a}}}_{ij,2},\hat{\hat{\mathfrak{a}}}_{ij,3}];[\hat{\mathfrak{a}}_{ij,4},\hat{\mathfrak{a}}_{ij,5},\hat{\mathfrak{a}}_{ij,6};\hat{\hat{\mathfrak{a}}}_{ij,4},\hat{\hat{\mathfrak{a}}}_{ij,5},\hat{\hat{\mathfrak{a}}}_{ij,6}];[\hat{\mathfrak{a}}_{ij,7},\hat{\mathfrak{a}}_{ij,8},\hat{\mathfrak{a}}_{ij,9};$$
$$\hat{\hat{\mathfrak{a}}}_{ij,7},\hat{\hat{\mathfrak{a}}}_{ij,8},\hat{\hat{\mathfrak{a}}}_{ij,9}]]\approx\sum_{j=1}^{p}[b_{ij,1},b_{ij,2},b_{ij,3};\acute{b}_{ij,1},\acute{b}_{ij,2},\acute{b}_{ij,3}];[b_{ij,4},b_{ij,5},b_{ij,6};\acute{b}_{ij,b},\acute{b}_{ij,5},\acute{b}_{ij,6}];$$
$$[b_{ij,7},b_{ij,8},b_{ij,9};\acute{b}_{ij,7},\acute{b}_{ij,8},\acute{b}_{ij,9}]$$

$i = 1, 2, \ldots, q.$

step 3: Employing ranking function $\mathfrak{R}$, BNLP will be transformed as:

$$Maximize \sum_{j=1}^{p} D_j \mathfrak{z}_j,$$

subject to,

$$\mathfrak{R}\sum_{j=1}^{p}[[\hat{\mathfrak{a}}_{ij,1},\hat{\mathfrak{a}}_{ij,2},\hat{\mathfrak{a}}_{ij,3};\hat{\hat{\mathfrak{a}}}_{ij,1},\hat{\hat{\mathfrak{a}}}_{ij,2},\hat{\hat{\mathfrak{a}}}_{ij,3}];[\hat{\mathfrak{a}}_{ij,4},\hat{\mathfrak{a}}_{ij,5},\hat{\mathfrak{a}}_{ij,6};\hat{\hat{\mathfrak{a}}}_{ij,4},\hat{\hat{\mathfrak{a}}}_{ij,5},\hat{\hat{\mathfrak{a}}}_{ij,6}];[\hat{\mathfrak{a}}_{ij,7},\hat{\mathfrak{a}}_{ij,8},$$
$$\hat{\mathfrak{a}}_{ij,9};\hat{\hat{\mathfrak{a}}}_{ij,7},\hat{\hat{\mathfrak{a}}}_{ij,8},\hat{\hat{\mathfrak{a}}}_{ij,9}]]\approx\mathfrak{R}\sum_{j=1}^{p}[[b_{ij,1},b_{ij,2},b_{ij,3};\acute{b}_{ij,1},\acute{b}_{ij,2},\acute{b}_{ij,3}];[b_{ij,4},b_{ij,5},b_{ij,6};\acute{b}_{ij,b},$$
$$\acute{b}_{ij,5},\acute{b}_{ij,6}];[b_{ij,7},b_{ij,8},b_{ij,9};\acute{b}_{ij,7},\acute{b}_{ij,8},\acute{b}_{ij,9}]].\ i=1,2,\ldots,q.$$

Step 4: Applying the linearity property of ranking function [132], that is, $\Re\sum_{i=1}^{n}\breve{\mathfrak{S}}_i = \sum_{i=1}^{n}\Re\breve{\mathfrak{S}}_i$, where $\breve{\mathfrak{S}}_i$ is a bipolar neutrosophic number. The resultant crisp LP model is as follows;

$$\text{Maximize } \sum_{j=1}^{p} D_j\mathfrak{Z}_j,$$

subject to,

$$\sum_{i=1}^{q}\sum_{j=1}^{p}\Re[[\hat{\mathfrak{a}}_{ij,1}, \hat{\mathfrak{a}}_{ij,2}, \hat{\mathfrak{a}}_{ij,3}; \hat{\hat{\mathfrak{a}}}_{ij,1}, \hat{\hat{\mathfrak{a}}}_{ij,2}, \hat{\hat{\mathfrak{a}}}_{ij,3}]; [\hat{\mathfrak{a}}_{ij,4}, \hat{\mathfrak{a}}_{ij,5}, \hat{\mathfrak{a}}_{ij,6}; \hat{\hat{\mathfrak{a}}}_{ij,4}, \hat{\hat{\mathfrak{a}}}_{ij,5}, \hat{\hat{\mathfrak{a}}}_{ij,6}];$$

$$[\hat{\mathfrak{a}}_{ij,7}, \hat{\mathfrak{a}}_{ij,8}, \hat{\mathfrak{a}}_{ij,9}; \hat{\hat{\mathfrak{a}}}_{ij,7}, \hat{\hat{\mathfrak{a}}}_{ij,8}, \hat{\hat{\mathfrak{a}}}_{ij,9}]]$$

$$\approx \sum_{i=1}^{q}\sum_{j=1}^{p}\Re[[b_{ij,1}, b_{ij,2}, b_{ij,3}; \acute{b}_{ij,1}, \acute{b}_{ij,2}, \acute{b}_{ij,3}]; [b_{ij,4}, b_{ij,5}, b_{ij,6}; \acute{b}_{ij,4}, \acute{b}_{ij,5}, \acute{b}_{ij,6}];$$

$$[b_{ij,7}, b_{ij,8}, b_{ij,9}; \acute{b}_{ij,7}, \acute{b}_{ij,8}, \acute{b}_{ij,9}]]. \tag{5.2.6}$$

Step 5: Use any appropriate ranking function derived in Section 4.2 and perform necessary simplifications to solve Equation 5.2.6.

We have used ranking function derived in Section 4.2.3 in our application here. The particular ranking function used to solve above problem is given as;

$$\Re(\breve{A}^{TBN}) = \frac{-\mathfrak{a} + 2b + 5c - d + 2e + 5f - m}{6}$$
$$\frac{+2n + 5o + p - 2q - 5r - g + 2h + 5i + j - 2k - 5l}{6}$$

After necessary simplification the final LP would be of the form;

$$\text{Maximize } \sum_{j=1}^{p} D_j\mathfrak{Z}_j.$$

subject to,

$$\sum_{i=1}^{q}\sum_{j=1}^{p}\frac{1}{6}[-\hat{\mathfrak{a}}_{ij,1} + 2\hat{\mathfrak{a}}_{ij,2} + 5\hat{\mathfrak{a}}_{ij,3} - \hat{\hat{\mathfrak{a}}}_{ij,1} + 2\hat{\hat{\mathfrak{a}}}_{ij,2} + 5\hat{\hat{\mathfrak{a}}}_{ij,3} - \hat{\mathfrak{a}}_{ij,4} + 2\hat{\mathfrak{a}}_{ij,5} + 5\hat{\mathfrak{a}}_{ij,6} +$$

$$\hat{\hat{\mathfrak{a}}}_{ij,4} - 2\hat{\hat{\mathfrak{a}}}_{ij,5} - 5\hat{\hat{\mathfrak{a}}}_{ij,6} - \hat{\mathfrak{a}}_{ij,7} + 2\hat{\mathfrak{a}}_{ij,8} + 5\hat{\mathfrak{a}}_{ij,9} + \hat{\hat{\mathfrak{a}}}_{ij,4} - 7\hat{\hat{\mathfrak{a}}}_{ij,8} - 5\hat{\hat{\mathfrak{a}}}_{ij,9}].$$

$$\approx \sum_{i=1}^{q}\sum_{j=1}^{p}\frac{1}{6}[-\hat{b}_{ij,1} + 2\hat{b}_{ij,2} + 5\hat{b}_{ij,3} - \hat{\hat{b}}_{ij,1} + 2\hat{\hat{b}}_{ij,2} + 5\hat{\hat{b}}_{ij,3} - \hat{b}_{ij,4} + 2\hat{b}_{ij,5} + 5\hat{b}_{ij,6} +$$

$$\hat{\hat{b}}_{ij,4} - 2\hat{\hat{b}}_{ij,5} - 5\hat{\hat{b}}_{ij,6} - \hat{b}_{ij,7} + 2\hat{b}_{ij,8} + 5\hat{b}_{ij,9} + \hat{\hat{b}}_{ij,4} - 7\hat{\hat{b}}_{ij,8} - 5\hat{\hat{b}}_{ij,9}]. \tag{5.2.7}$$

Step 6 : Solve crisp LP 5.2.7 by any appropriate algorithm or excel solver to find the optimal value of function.

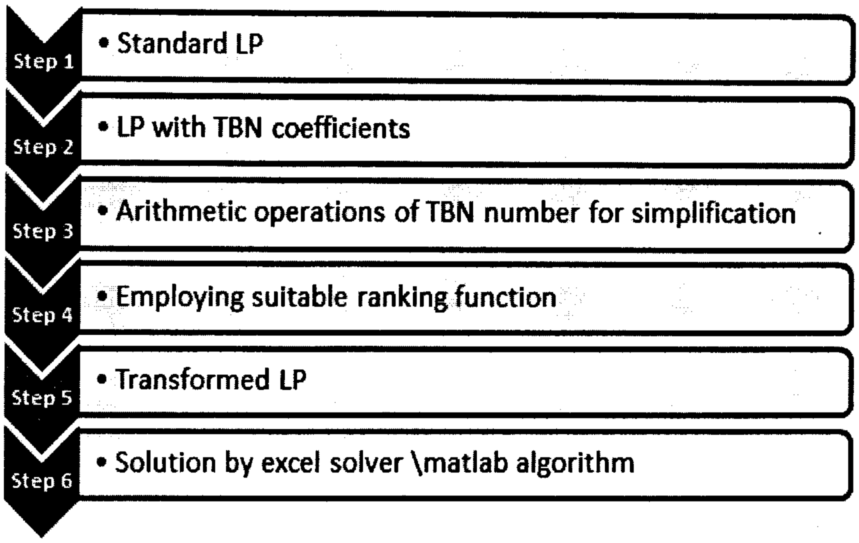

Figure 5.1: Flow chart bipolar neutrosophic linear programming

To confirm the existence of optimal solution of BNLP 5.2.3 and 5.2.4 we impose the below condition;

Remark 5.2.1. The solution of BNLP

$$\text{Maximize or Minimize} \quad \sum_{j=1}^{p} D_j 3_j,$$

subject to,

$$\sum_{j=1}^{p}\sum_{i=1}^{q} \tilde{E}_{ij}^{n} \otimes 3_j \approx, \lesssim, \gtrsim \tilde{B}_i \quad \forall \quad 3_j \geq 0, \tag{5.2.8}$$

where $\tilde{E}_{ij}^{n}$ and $\tilde{B}_i$ are bipolar neutrosophic numbers, exist when the solution of associated crisp LP,

$$\text{Max}\backslash Min\ Z = \sum_{j=1}^{p} C_j X_j,$$

subject to,

$$\sum_{j=1}^{m}\sum_{i=1}^{n} d_{ij} = w_i.$$

Where,

$$d_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} \Re[[\hat{\mathfrak{a}}_{ij,1}, \hat{\mathfrak{a}}_{ij,2}, \hat{\mathfrak{a}}_{ij,3}; \hat{\hat{\mathfrak{a}}}_{ij,1}, \hat{\hat{\mathfrak{a}}}_{ij,2}, \hat{\hat{\mathfrak{a}}}_{ij,3}]; [\hat{\mathfrak{a}}_{ij,4}, \hat{\mathfrak{a}}_{ij,5}, \hat{\mathfrak{a}}_{ij,6}; \hat{\hat{\mathfrak{a}}}_{ij,4}, \hat{\hat{\mathfrak{a}}}_{ij,5}, \hat{\hat{\mathfrak{a}}}_{ij,6}];$$
$$[\hat{\mathfrak{a}}_{ij,7}, \hat{\mathfrak{a}}_{ij,8}, \hat{\mathfrak{a}}_{ij,9}; \hat{\hat{\mathfrak{a}}}_{ij,7}, \hat{\hat{\mathfrak{a}}}_{ij,8}, \hat{\hat{\mathfrak{a}}}_{ij,9}]]$$

$$w_i = \sum_{i=1}^{n} \Re[[b_{ij,1}, b_{ij,2}, b_{ij,3}; \acute{b}_{ij,1}, \acute{b}_{ij,2}, \acute{b}_{ij,3}]; [b_{ij,4}, b_{ij,5}, b_{ij,6}; \acute{b}_{ij,4}, \acute{b}_{ij,5}, \acute{b}_{ij,6}];$$
$$[b_{ij,7}, b_{ij,8}, b_{ij,9}; \acute{b}_{ij,7}, \acute{b}_{ij,8}, \acute{b}_{ij,9}]].$$

exists.

Thus optimality of BNLP depends entirely on the existence of the optimal solution of associated crisp LPP.

5.3 Application

Advancements, techniques, and strategies used to improve agriculture and many other sectors in a productive way provide a window into the history of human civilisation. As a result of changes in the agricultural area, sustainable agriculture has evolved into a much more advanced form that effects the ecosystem, society, atmosphere and the economy. Population growth and globalization are causing natural resources to be depleted, which in turn is causing food shortages and scarcity. In addition to this, both COVID-19 and acute changes in climate have a negative effect on agricultural productivity, which makes the overall demand for food go up. The system for producing food must be completely altered, equipped with endurance, versatility, and great variety to combat various circumstances and conditions like these.

Rice, wheat, sugarcane, cotton, and maize are some of the most significant crops grown in Pakistan, which is a developing nation with an average income. Other important crops include vegetables, fruits, oilseeds, and pulses. Pulses have a very minimal impact on the environment, both in terms of the amount of greenhouse gases they produce and the amount of water they require. Additionally, pulses can be stored for extended periods of time without losing any nutritional value. Current study aims to maximize pulses production in upper punjab belt of Pakistan. A specific application of linear programming is constructed, which is generalised in this model with capital, labor, and available resources.

The maximization LP model can be constructed as;

$$\text{Maximize } (plp)(\mathfrak{z}) = \sum_{m=1}^{p}\sum_{n=1}^{q} C_{mn}\mathfrak{Z}_{mn}. \tag{5.3.1}$$

Model constraints:

$$\begin{aligned}
\sum \mathcal{W}_i^h R_i &\leq \mathcal{MD}_i,\\
\sum \mathcal{F}_i R_i &= 0,\\
\sum \mathcal{P}_i R_i &= 0,\\
\sum \mathcal{E}_i R_i &= 0,\\
\sum \mathcal{G}_i R_i - \mathcal{S}p_i &= 0,\\
\sum_{m=1}^{p}\sum_{n=1}^{q} D_{mn}\mathfrak{Z}_{mn} &= a_m,\\
\sum_{i=1}^{g} \mathcal{A}_i^f &\leq \mathcal{L}_t^c,\\
\mathcal{A}_i^f &\leq \mathcal{L}_t^{cG},\\
\mathcal{A}_i^f &\leq \mathcal{L}_t^{cA},
\end{aligned}$$

$$\mathcal{A}_i^f \leq \mathcal{L}_t^{cC},$$
$$\mathcal{A}_i^f \leq \mathcal{L}_t^{cAl},$$
$$\mathcal{A}_i^f \leq \mathcal{L}_t^{c\mathcal{P}}.$$

Where $\mathfrak{Z}_{mn} \geqslant 0$ is the non-negativity condition for all $m = 1,2,3...,p$; $n = 1,2,3,...,q$.

Here the list of indices are as bellow;

R_i = area under cultivation for individual pulses.

$\mathcal{MD}_i$ = days-man for workers.

$\mathcal{F}_i$ = amount of fertilizers required per hector.

$\mathcal{P}_i$ = pesticides needed per hector.

$\mathcal{E}_i$ = total estimated expenditure per hector.

$\mathcal{G}_i$ = production per hector in kg.

$\mathcal{S}p_i$ = Market sales price per kg of yield.

plp = production of pulses.

$\mathfrak{Z}_{mn}$ = indicates different activities, as for example, leveling, harvesting, cutting, thinning, pruning, sales, etc.

C_{mn} = different objectives of the coefficients which includes profit of product, variable's current market price, etc.

D_{mn} = constraints coefficients which includes used resources and investment cost per unit of agricultural production.

a_m = total quantity, volume, and units of supplies per hector available.

g = number of pulses.

$\mathcal{A}_i^f$ = total area accessible for pulses cultivation.

$\mathcal{L}_t^c$ = total area under cultivation.

$\mathcal{L}_t^{cG}$ = total land for the cultivation of gram.

$\mathcal{L}_t^{cA}$ = total land for the cultivation of mung bean.

$\mathcal{L}_t^{cC}$ = total land for the cultivation of chickpea.

$\mathcal{L}_t^{c\mathcal{P}}$ = total land for the cultivation of mash bean .

$\mathcal{L}_t^{cAl}$ = total land for the cultivation of lentil.

$\mathcal{W}_i^h$ = per day working required for each crops.

Objective function:

$$Max\ \mathbb{Z}_{plp} = -110\mathfrak{x}_6 + \mathfrak{x}_7 + 160\mathfrak{x}_9 + 120\mathfrak{x}_{10} + 150\mathfrak{x}_{11} + 150\mathfrak{x}_{12} + 200\mathfrak{x}_{13} \quad (5.3.2)$$

subject to the constraints:

$$\begin{aligned}
\mathfrak{x}_1 + \mathfrak{x}_2 + \mathfrak{x}_3 + \mathfrak{x}_4 + \mathfrak{x}_5 &= 120,\\
\mathfrak{x}_1 &\leq 30,\\
\mathfrak{x}_2 &\leq 30,\\
\mathfrak{x}_3 + \mathfrak{x}_4 + \mathfrak{x}_5 &\leq 60,\\
-300\mathfrak{x}_1 - 250\mathfrak{x}_2 - 200\mathfrak{x}_3 - 210\mathfrak{x}_4 - 230\mathfrak{x}_5 + \mathfrak{x}_6 &= 0,\\
3000\mathfrak{x}_1 + 2850\mathfrak{x}_2 + 2900\mathfrak{x}_3 + 3100\mathfrak{x}_4 + 3050\mathfrak{x}_5 - \mathfrak{x}_7 &= 0,\\
-6.1\mathfrak{x}_1 - 3.5\mathfrak{x}_2 - 4.5\mathfrak{x}_3 - 3.2\mathfrak{x}_4 - 3.5\mathfrak{x}_5 + \mathfrak{x}_8 &\leq 20.5,\\
13700\mathfrak{x}_1 - \mathfrak{x}_9 &= 0,\\
17100\mathfrak{x}_1 - \mathfrak{x}_{10} &= 0,\\
25000\mathfrak{x}_1 - \mathfrak{x}_{11} &= 0,\\
81600\mathfrak{x}_1 - \mathfrak{x}_{12} &= 0,\\
52800\mathfrak{x}_1 - \mathfrak{x}_{13} &= 0. \qquad (5.3.3)
\end{aligned}$$

In this model to incorporate ambiguity associated with real life scenario we will convert the coefficients of constraints to the bipolar neutrosophic numbers. The transformed model in bipolar neutrosophy, where all the constraints coefficients are bipolar neutrosophic in nature are as below:

$$\text{Maximize } \mathbb{Z}_{frp} = -110\mathfrak{x}_6 + \mathfrak{x}_7 + 160\mathfrak{x}_9 + 120\mathfrak{x}_{10} + 150\mathfrak{x}_{11} + 150\mathfrak{x}_{12} + 200\mathfrak{x}_{13} \quad (5.3.4)$$

subjected to bipolar neutrosophic constraints

$$\langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_1\rangle \oplus \langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_2\rangle \oplus \langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_3\rangle \oplus \langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_4\rangle \oplus \langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_5\rangle = \langle(70,120,145;45,70,120);(20,70,120;45,95,120);(70,120,170;120,145,170)\rangle. \tag{5.3.5}$$

$$\langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_1\rangle \leq \langle(20,30,35;15,20,30);(10,20,30;15,25,30);(20,30,40;30,35,40)\rangle. \tag{5.3.6}$$

$$\langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_2\rangle \leq \langle(20,30,35;15,20,30);(10,20,30;15,25,30);(20,30,40;30,35,40)\rangle. \tag{5.3.7}$$

$$\langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_3\rangle \oplus \langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_4\rangle \oplus \langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_5\rangle \leq \langle(40,60,70;30,40,60);(20,40,60;30,50,60);(40,60,80;60,70,80)\rangle. \tag{5.3.8}$$

$$\langle((13650,13700,13725;13625,13650,13700);(13600,13650,13700;13625,13675,13700);(13650,13700,13750;13700,13725,13750)) \otimes \tilde{\mathfrak{x}}_1\rangle \ominus \langle((0.50,1,1.25;0.25,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50)) \otimes \tilde{\mathfrak{x}}_9\rangle = ((-4,0,2;-6,-4,0);(-8,-4,0;-6,-2,0);(-4,0,4;0,2,4)). \tag{5.3.9}$$

$$\langle(((17050, 17100, 17125; 17025, 17050, 17100); (17000, 17050, 17100; 17025, 17075, 17100); (17050, 17100, 17150; 17100, 17125, 17150)) \otimes \tilde{\mathfrak{x}_1}\rangle \ominus \langle(((0.50, 1, 1.25; 0.25, 0.5, 1); (0, 0.50, 1; 0.25, 0.75, 1); (0.50, 1, 1.50; 1, 1.25, 1.50))\otimes \tilde{\mathfrak{x}_{10}}\rangle = ((-4, 0, 2; -6, -4, 0); (-8, -4, 0; -6, -2, 0); (-4, 0, 4; 0, 2, 4)). \quad (5.3.10)$$

$$\langle-((250, 300, 325; 225, 250, 300); (200, 250, 300; 225, 275, 300); (250, 300, 350; 300, 325, 350)) \otimes \tilde{\mathfrak{x}_1}\rangle \ominus \langle(((200, 250, 275; 175, 200, 250); (150, 200, 250; 175, 225, 250); (200, 250, 300; 250, 275, 300)) \otimes \tilde{\mathfrak{x}_2}\rangle \ominus \langle(((150, 200, 225; 125, 150, 200); (100, 150, 200; 125, 175, 200); (150, 200, 250; 200, 225, 250)) \otimes \tilde{\mathfrak{x}_3}\rangle \ominus \langle(((160, 210, 235; 135, 160, 210); (110, 160, 210; 135, 185, 210); (160, 210, 260; 210, 235, 260)) \otimes \tilde{\mathfrak{x}_4}\rangle \ominus \langle(((180, 230, 255; 155, 180, 230); (130, 180, 230; 155, 205, 230); (180, 230, 280; 230, 255, 280)) \otimes \tilde{\mathfrak{x}_5}\rangle \oplus \langle(((0.50, 1, 1.25; 0.25, 0.5, 1); (0, 0.50, 1; 0.25, 0.75, 1); (0.50, 1, 1.50; 1, 1.25, 1.50)) \otimes \tilde{\mathfrak{x}_6}\rangle = ((-4, 0, 2; -6, -4, 0); (-8, -4, 0; -6, -2, 0); (-4, 0, 4; 0, 2, 4)) \quad (5.3.11)$$

$$\langle(((2950, 3000, 3025; 2925, 2950, 3000); (2900, 2950, 3000; 2925, 2975, 3000); (2950, 3000, 3050; 3000, 3025, 3050)) \otimes \tilde{\mathfrak{x}_1}\rangle \oplus \langle(((2800, 2850, 2875; 2775, 2800, 2850); (2750, 2800, 2850; 2775, 2825, 2850); (2800, 2850, 2900; 2850, 2875, 2900)) \otimes \tilde{\mathfrak{x}_2}\rangle \oplus \langle(((2850, 2900, 2925; 2825, 2850, 2900); (2800, 2850, 2900; 2850, 2875, 2900); (2850, 2900, 2950; 2900, 2925, 2950)) \otimes \tilde{\mathfrak{x}_3}\rangle \oplus \langle(((3050, 3100, 3125; 3025, 3050, 3100); (3000, 3050, 3100; 3025, 3075, 3100); (3050, 3100, 3150; 3100, 3125, 3150)) \otimes \tilde{\mathfrak{x}_4}\rangle \oplus \langle(((3000, 3050, 3075; 2975, 3000, 3050); (2950, 3000, 3050; 2975, 3025, 3050); (3000, 3050, 3100; 3050, 3075, 3100)) \otimes \tilde{\mathfrak{x}_5}\rangle \ominus \langle(((0.50, 1, 1.25; 0.25, 0.5, 1); (0, 0.50, 1; 0.25, 0.75, 1); (0.50, 1, 1.50; 1, 1.25, 1.50)) \otimes \tilde{\mathfrak{x}_7}\rangle = ((-4, 0, 2; -6, -4, 0); (-8, -4, 0; -6, -2, 0); (-4, 0, 4; 0, 2, 4)). \quad (5.3.12)$$

$$\langle(-((4.1,6.1,7.1;3.1,4.1,6.1);(2.1,4.1,6.1;3.1,5.1,6.1);(4.1,6.1,8.1;6.1, 7.1,8.1)))\otimes \tilde{x_1}\rangle \ominus \langle(-((2.5,3.5,4;2,2.5,3.5);(1.5,2.5,3.5;2,3,3.5);(2.5, 3.5,4.5;3.5,4,5.4)))\otimes \tilde{x_2}\rangle \ominus \langle(((2.5,4.5,5.5;1.5,2.5,4.5);(0.5,2.5,4.5;1.5, 3.5,4.5);(2.5,4.5,6.5;4.5,5.5,6.5))\otimes \tilde{x_3}\rangle \ominus \langle(((2.2,3.2,3.7;1.7,2.2,3.2);(1.2, 2.2,3.2;1.7,2.7,3.2);(2.2,3.2,4.2;3.2,3.7,4.2))\otimes \tilde{x_4}\rangle \ominus \langle(((2.5,3.5,4;2,2.5,3.5); (1.5,2.5,3.5;2,3,3.5);(2.5,3.5,4.5;3.5,4,5.4))\otimes \tilde{x_5}\rangle \oplus \langle(((0.50,1,1.25;0.25,0.5,1) ;(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50))\otimes \tilde{x_8}\rangle \leq ((13.5,20.5,24;10, 13.5,20.5);(6.5,13.5,20.5;10,17,20.5);(13.5,20.5,27.5;20.5,24,27.5). \tag{5.3.13}$$

$$\langle(((24950,25000,25025;24925,24950,25000);(24900,24925,25000;24925,24975 ,25000);(24950,25000,25050;25000,25025,25050))\otimes \tilde{x_1}\rangle \ominus \langle(((0.50,1,1.25;0.25 ,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50))\otimes \tilde{x_{11}}\rangle((-4,0,2;-6 =,-4,0);(-8,-4,0;-6,-2,0);(-4,0,4;0,2,4)). \tag{5.3.14}$$

$$\langle(((81550,81600,81625;81525,81550,81600);(81500,81550,81600;81550,81575 ,81600);(81550,81600,81650;81600,81625,81650))\otimes \tilde{x_1}\rangle \ominus \langle(((0.50,1,1.25;0.25 ,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50))\otimes \tilde{x_{12}}\rangle = ((-4,0,2; -6,-4,0);(-8,-4,0;-6,-2,0);(-4,0,4;0,2,4)). \tag{5.3.15}$$

$$\langle(((52750,52800,52825;52725,52750,52800);(52700,52750,52800;52725,52775 ,52800);(52750,52800,52850;52800,52825,52850))\otimes \tilde{x_1}\rangle \ominus \langle(((0.50,1,1.25;0.25 ,0.5,1);(0,0.50,1;0.25,0.75,1);(0.50,1,1.50;1,1.25,1.50))\otimes \tilde{x_{13}}\rangle = ((-4,0,2; -6 \tag{5.3.16}$$

$$,-4,0);(-8,-4,0;-6,-2,0);(-4,0,4;0,2,4)). \tag{5.3.17}$$

Importantly, all neutrosophic numbers are determined according to the criteria specified in Figure 5.2

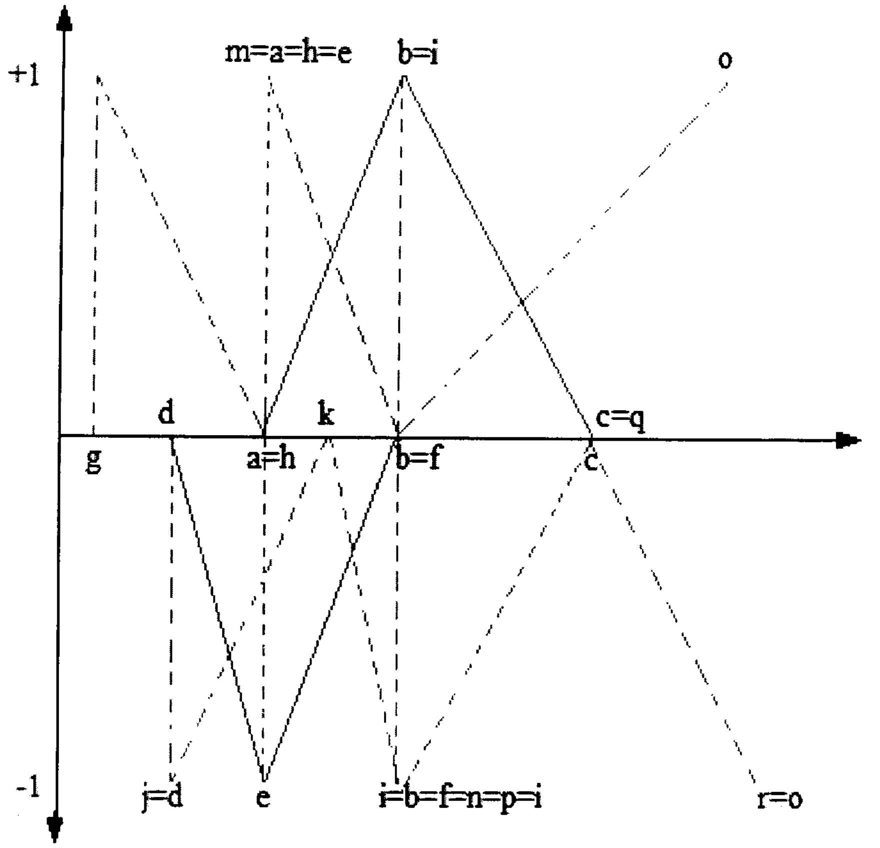

Figure 5.2: Membership grades of bipolar neutrosophic set

Now to solve this model we require the arithmetic operations defined in Chapter 4, Section 4.1.1 and suitable accuracy function derived in Section 4.2 of bipolar neutrosophic numbers.

Follow all the steps provided in Section 5.2.1 and use matlab or excel solver to attain the final solution for maximum pulses yield. Here we have used excel solver for final results and the attained objective function value is 897489684.5 kg.

Variables	Final value	Reduced cost	Objective coefficient	Allowable increase	Allowable decrease
x_1	29.05719332	0	0	1E+30	125232.0532
x_2	15.72295431	0	0	125232.0532	1E+30
x_3	6.65723E-06	0	0	7689587.749	1E+30
x_4	73.33332352	0	0	1E+30	1476856.075
x_5	3.15279E-06	0	0	1476856.075	1E+30
x_6	25825.08916	0	-110	118811.468	3671.304871
x_7	326536.579	0	1	30874.22546	921.8239251
x_8	0	0	0	0	1E+30
x_9	360807.8727	0	160	1E+30	10.0853965
x_{10}	243649.9631	0	120	8.081333591	1E+30
x_{11}	0	-339.47691	150	339.4769065	1E+30
x_{12}	5420244.35	0	150	1E+30	19.98115847
x_{13}	0	-30.877821	200	30.8778219	1E+30

Table 5.1: Sensitivity report(variable)

5.4 Conclusion

This chapter uses the idea of the ranking of bipolar neutrosophic numbers in LP models of the relevant area as its major point of emphasis. Different types of bipolar neutrosophic linear programmes may have different solutions to linear programming problems. Similarly, it is quite possible that each selection of a different ranking function may end up with a different conclusion. Since, as far as we know, there is no strict rule for employing any of the derived ranking functions, it is expected that the behaviour of the LP solution and

Constraint	Final value	Shadow price	RHS	Allowable Increase	Allowable decrease
1	-971.61908	0	34	1E+30	1005.619083
2	0.332999429	-72.453924	0.333	796772.0252	1E+30
3	0.332999614	-54.340443	0.333	538052.2135	1E+30
4	0.333	-221.65326	0.333	3668193.607	0.333
5	0.332991421	-67.925554	0.333	11969525.6	1E+30
6	0.333	-104.55003	0.333	6780686.955	0.333
7	260.83	830193.4306	260.83	29.446	34.72097851
8	64.167	56709.71026	64.167	34.72097851	29.446
9	34.721	0	64.167	1E+30	29.446
10	110	6157378.457	110	23.58441687	20.00135851
11	0.333	-49.812072	0.333	1E+30	57029.5444
12	0.332999483	-0.4528370	0.333	721090.7274	1E+30

Table 5.2: Sensitivity analysis (constraints)

the sensitivity of variables may vary depending on the ranking function. Future research may involve an examination of the consequences and effects that various ranking functions may have on the solution. In the current study, we have only focused on the BNLP of type 3 and the ranking function derived in Section 4.2.3. This type of triangular bipolar neutrosophic linear programming was shown to be the most effective way for dealing with the particular issue based on a comparison of a few methods, statistics, and post-optimality (sensitivity) analysis. Within an acceptable distance, the feasible area for optimum manufacturing in a bipolar neutrosophic environment stays viable and ideal. In order to determine the optimum pulses production, we constrained the model with parameters such as the kinds of fertilizers and their costs, the number of labour hours that were available, and the average yield. Different activities such as cultivation, field preparation, selection of cite, bedding and sowing, trimming of vines and fruit trees, thinning, pollination, harvesting, handling of crops, irrigation, fertilisation, soil management, preservation and pest control are taken into account. Figure 5.3-5.13 graphically show the results attained by our proposed model. The graphs show that even a small adjustment in activities may lead to a significant increase in the output. The results should be compared to see how well the optimization model in a bipolar neutrosophic environment correlates with those of linear programming in a crisp environment. Figure 5.3 shows that proposed model help us attain the maximum pulses yield with the available resources.

Sensitivity analysis is done with the parameters of the goal function and the constraints values on the RHS. The amount of the objective coefficient that may vary before the best solution changes is specified by the allowable increase and decrease values. Here the analysis provided in Table 5.1 and 5.2 shows that the best solution is still possible within the given ranges of variables and parameters. Table 5.1 comprises the coefficient limitations for each variable in the form of allowable decrease and increase. For example, the solution is still the best for the range covered by the limit of the coefficient of x_7 with original value 1, which is between 30874.22546 and 921.8239251. The field with the value 1E + 30 in the permitted increase or decrease column indicates that the decrease or increase of that particular variable is unrestricted. Similarly each constraint's range is shown in Table 5.2 along with a shadow increase, which is only applicable within certain ranges. A modification on a constraint's right-hand side immediately alters the feasible area, which may have an impact on the optimality of the solution. Table 5.2 shows that our feasibility area stays the same even if the constraints alters as long as they stay within the range that is allowed. According to the above discussion, this ranking function bipolar neutrosophic LP model offers a viable, flexible, and optimum solution employing the original data.

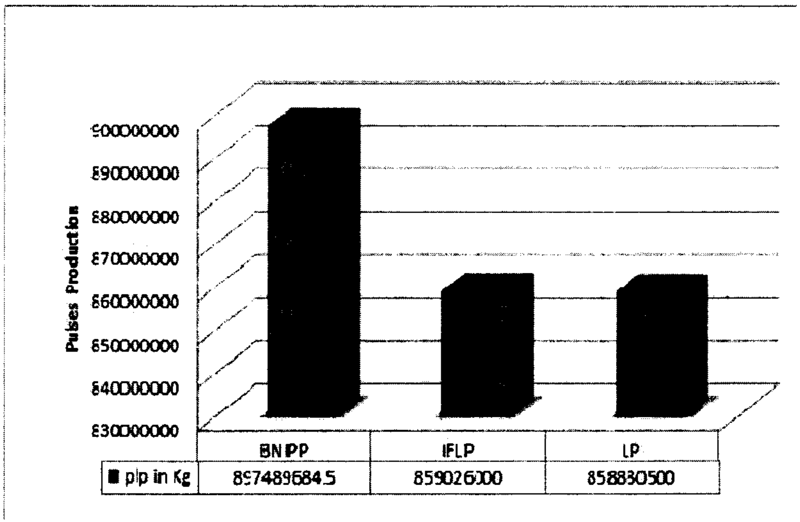

Figure 5.3: Comparison of objectives

Graphs comparing various activities are shown below.

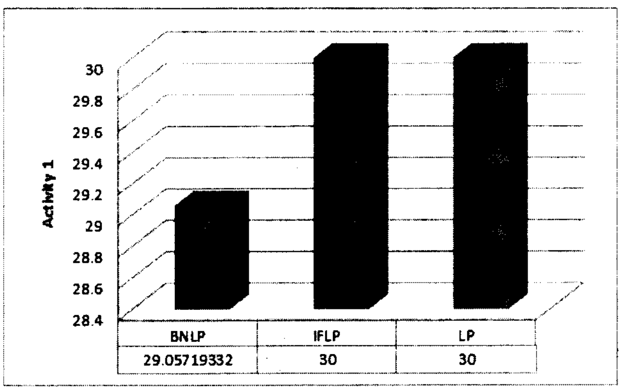

Figure 5.4: Comparison of activity 1 (x_1)

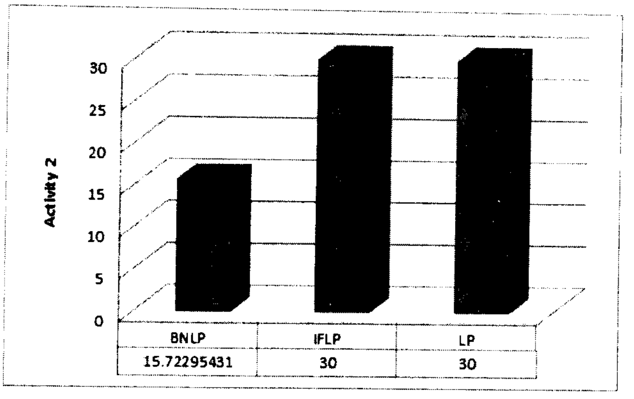

Figure 5.5: Comparison of activity 2 (x_2)

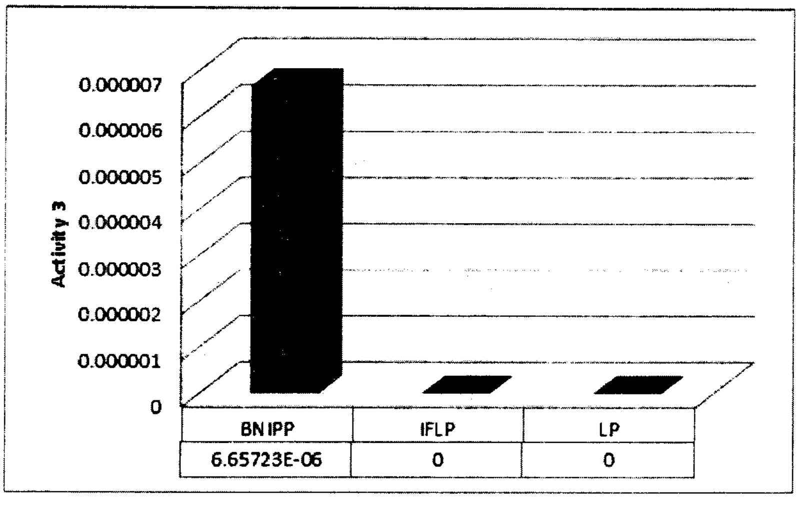

Figure 5.6: Comparison of activity 3 (x_3)

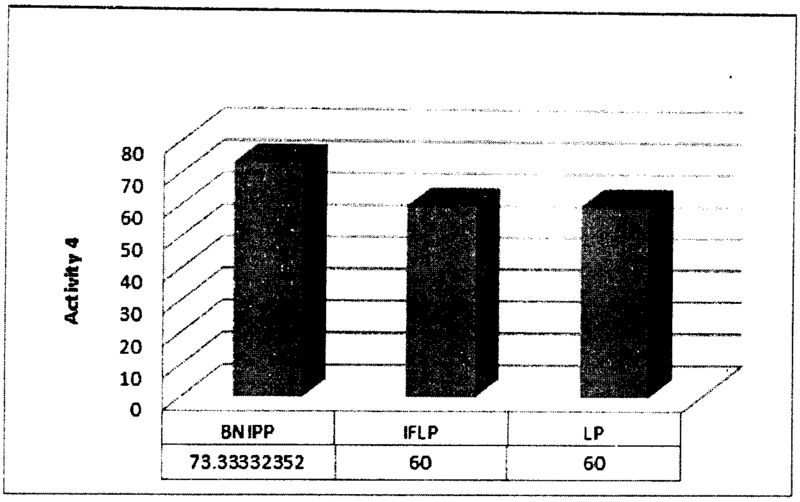

Figure 5.7: Comparison of activity 4 (x_4)

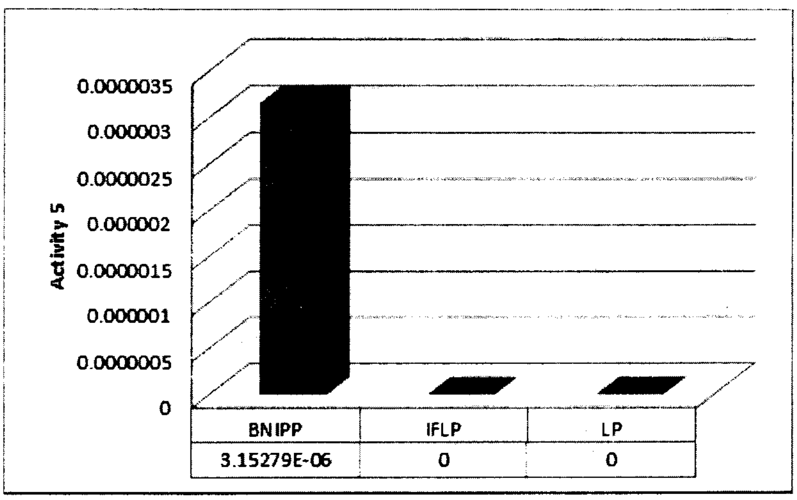

Figure 5.8: Comparison of activity 5 (x_5)

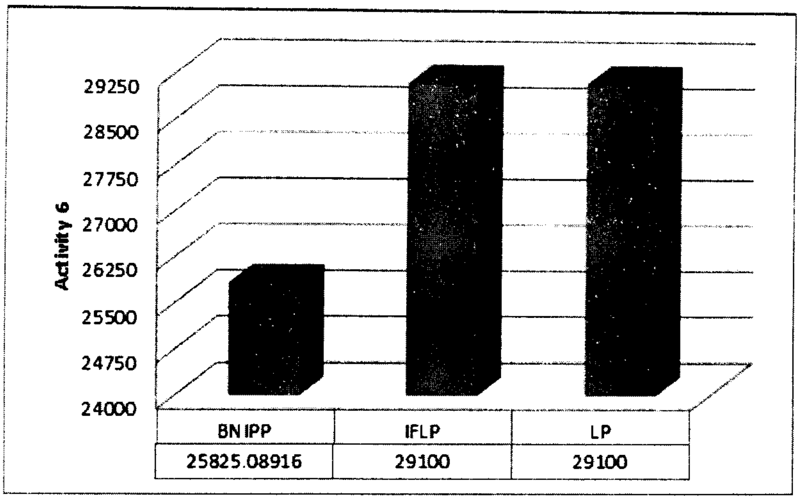

Figure 5.9: Comparison of activity 6 (x_6)

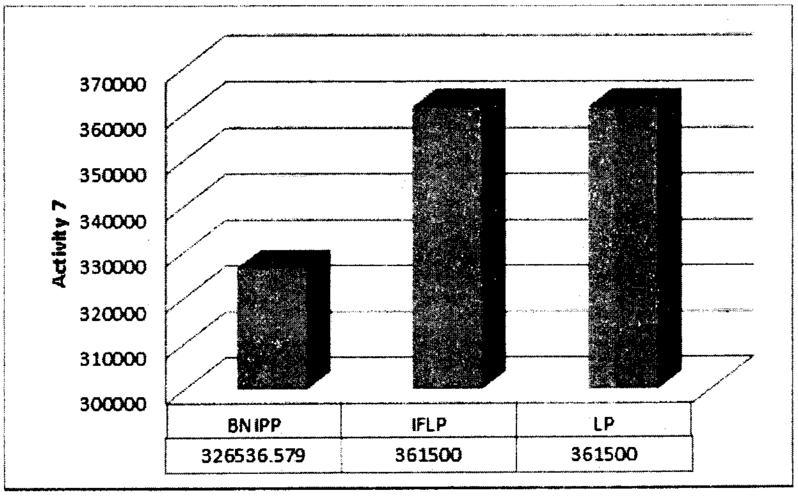

Figure 5.10: Comparison of activity 7 (x_7)

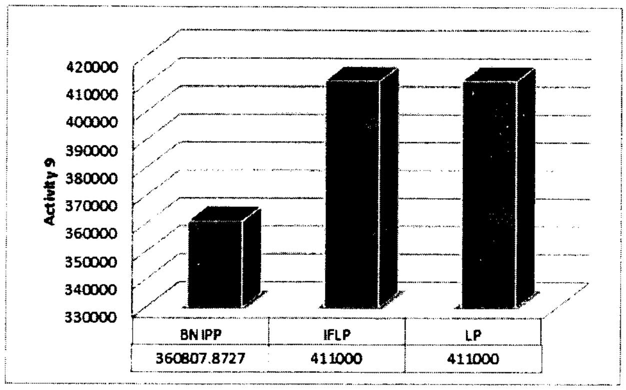

Figure 5.11: Comparison of activity 9 (x_9)

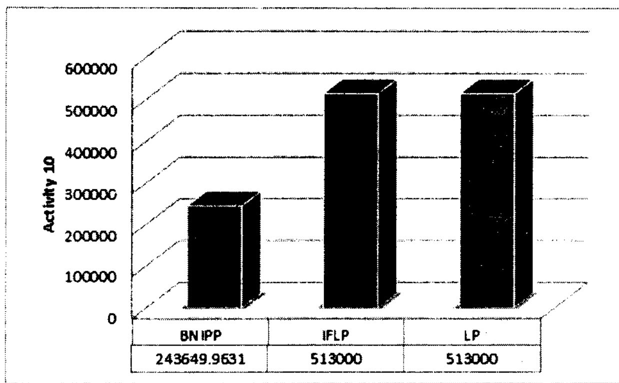

Figure 5.12: Comparison of activity 10 (x_{10})

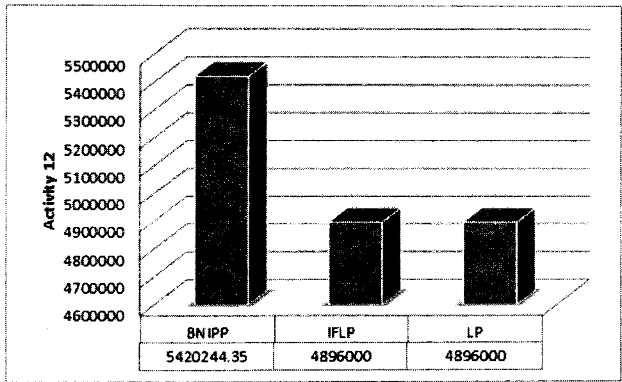

Figure 5.13: Comparison of activity 12 (x_{12})

Chapter 6

Multi-objective Non-linear Bipolar Neutrosophic Optimization and its Comparison with Existing Technique

The current study is based on some standard ways to deal with optimization when uncertainty is present in both the neutrosophic framework and the bipolar neutrosophic framework. Throughout the second half of the twentieth century, optimization discovered widespread applications in the investigation of physical and compound systems, planning scheduling and production systems, transportation problems, resource and allocation issues in finance, asset assignment in monetary frameworks, building outlines and engineering design. It was realised from the beginning of the application of optimization to these systems that professionals in technical and natural domain frameworks must make decisions in the presence of uncertainty. Uncertainty, for example, regulates the costs of fuels, the accessibility of power, demand and interest for synthetic compounds etc. Optimization with uncertainty is hard because you have to deal with a big, fuzzy field, which sometimes requires big optimization models. By including uncertainty in the mathematical programming frameworks, these models may be enhanced. As a result, optimization in fuzzy environments and generalisations may be seen as a practical choice that has received high praise from a number of scholars. In this area, where improved neutrosophic and bipolar neutrosophic parameters and relations are used to describe uncertainty, extended neutrosophic optimization and bipolar neutrosophic optimization are relatively new approaches. In this work, we have constructed an optimization algorithm based on a bipolar neutrosophic set to solve multi-objective non-linear optimization issues. To demonstrate the effectiveness and applicability of the recommended technique, a case study has been provided, and a comparison between the newly proposed and existing techniques has been conducted. On the basis of the findings of the comparative research, a conclusion has been reached.

6.1 Bipolar Neutrosophic Optimization Technique

Consider a nonlinear multi-objective optimization problem

$$Minimize\ \{\hat{f}_i(x)\} \qquad i = 1, 2, \ldots, p,$$

such that,

$$\hat{g}_j(x) \leq \hat{b}_j \qquad j = 1, 2, \ldots, q.$$

$\hat{f}_i(x)$ represents objective functions, x are decision variables, $\hat{g}_j(x)$ represents the constraint functions, p and q represents the number of objective and constraints respectively.

Now $\hat{D}$, for BN confluence of objectives and constraints is denoted as

$$\hat{D} = (\cap_{k=1}^{p}\hat{O}_k) \cap (\cap_{j=1}^{q}\hat{L}_j) = \{(x, T_{\hat{D}}^{+}, T_{\hat{D}}^{-}, I_{\hat{D}}^{+}, I_{\hat{D}}^{-}, F_{\hat{D}}^{+}, F_{\hat{D}}^{-})\},$$

where,

$$T^{+}_{\hat{D}}(x) = min(T^{+}_{\hat{O}_1}(x), T^{+}_{\hat{O}_2}(x), \ldots, T^{+}_{\hat{O}_p}(x); T^{+}_{\hat{L}_1}(x), T^{+}_{\hat{L}_2}(x), \ldots, T^{+}_{\hat{L}_q}(x)),$$

$$T^{-}_{\hat{D}}(x) = max(T^{-}_{\hat{O}_1}(x), T^{-}_{\hat{O}_2}(x), \ldots, T^{-}_{\hat{O}_p}(x); T^{-}_{\hat{L}_1}(x), T^{-}_{\hat{L}_2}(x), \ldots, T^{-}_{\hat{L}_q}(x)),$$

$$I^{+}_{\hat{D}}(x) = min(I^{+}_{\hat{O}_1}(x), I^{+}_{\hat{O}_2}(x), \ldots, I^{+}_{\hat{O}_p}(x); I^{+}_{\hat{L}_1}(x), I^{+}_{\hat{L}_2}(x), \ldots, I^{+}_{\hat{L}_q}(x)),$$

$$I^{-}_{\hat{D}}(x) = max(I^{-}_{\hat{O}_1}(x), I^{-}_{\hat{O}_2}(x), \ldots, I^{-}_{\hat{O}_p}(x); I^{-}_{\hat{L}_1}(x), I^{-}_{\hat{L}_2}(x), \ldots, I^{-}_{\hat{L}_q}(x)),$$

$$F^{+}_{\hat{D}}(x) = max(F^{+}_{\hat{O}_1}(x), F^{+}_{\hat{O}_2}(x), \ldots, F^{+}_{\hat{O}_p}(x); F^{+}_{\hat{L}_1}(x), F^{+}_{\hat{L}_2}(x), \ldots, F^{+}_{\hat{L}_q}(x)),$$

$$F^{-}_{\hat{D}}(x) = min(F^{-}_{\hat{O}_1}(x), F^{-}_{\hat{O}_2}(x), \ldots, F^{-}_{\hat{O}_p}(x); F^{-}_{\hat{L}_1}(x), F^{-}_{\hat{L}_2}(x), \ldots, F^{-}_{\hat{L}_q}(x)).$$

$\forall \;\; x \in X$. Whereas $T^{+}_{\hat{D}}$, $T^{-}_{\hat{D}}$, $I^{+}_{\hat{D}}$, $I^{-}_{\hat{D}}$,$F^{+}_{\hat{D}}$,$F^{-}_{\hat{D}}$ represents true positive, true negative, positive indeterminacy, negative indeterminacy, positive falsity and negative falsity grade of BN decision set. Now the bipolar neutrosophic optimization will be remodeled as,

$$Max\, \alpha^{+}, \quad Max\, \gamma^{+}, \quad Min\, \beta^{+}, \quad Min\, \alpha^{-}, \quad Min\, \gamma^{-}, \quad Max\, \beta^{-},$$

such that;

$$\begin{gathered}
T^{+}_{\hat{O}_k}(x) \geq \alpha^{+}, \quad T^{+}_{\hat{L}_j}(x) \geq \alpha^{+}, \\
T^{-}_{\hat{O}_k}(x) \leq \alpha^{-}, \quad T^{-}_{\hat{L}_j}(x) \leq \alpha^{-}, \\
I^{+}_{\hat{O}_k}(x) \geq \delta^{+}, \quad I^{+}_{\hat{L}_j}(x) \geq \delta^{+}, \\
I^{-}_{\hat{O}_k}(x) \leq \delta^{-}, \quad I^{-}_{\hat{L}_j}(x) \leq \delta^{-}, \\
F^{+}_{\hat{O}_k}(x) \leq \beta^{+}, \quad F^{+}_{\hat{L}_j}(x) \leq \beta^{+}, \\
F^{-}_{\hat{O}_k}(x) \geq \beta^{-}, \quad F^{-}_{\hat{L}_j}(x) \geq \beta^{-}, \\
\alpha^{+} \geq \beta^{+}, \quad \alpha^{+} \geq \gamma^{+} \quad \alpha^{-} \leq \beta^{-}, \quad \alpha^{-} \leq \gamma^{-}, \\
\alpha^{+} + \beta^{+} + \gamma^{+} + \alpha^{-} + \beta^{-} + \gamma^{-} \leq 3, \\
\alpha^{+},\ \beta^{+},\ \gamma^{+},\ \in [0,1], \\
\alpha^{-},\ \beta^{-},\ \gamma^{-},\ \in [-1,0], \\
\hat{g}_j(x) \leq \hat{b}_j, \quad x \geq 0, \quad j = 1,2,\ldots,q.
\end{gathered}$$

6.1.1 Computational Algorithm

6.1.1.1 Step 1.

Take the first objective function as a single objective function w.r.t the given constraints and solve it to find values of decision variables.

6.1.1.2 Step 2.

Values of remaining objectives are computed by using the values of the decision variables.

6.1.1.3 Step 3.

Repeat the above two steps to get the values of the remaining objective functions.

$$\begin{bmatrix} \hat{f}_1^*(x^1) & \hat{f}_2(x^1) & \cdots & \hat{f}_p(x^1) \\ \hat{f}_1(x^2) & \hat{f}_2^*(x^2) & \cdots & \hat{f}_1(x^2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{f}_1(x^r) & \hat{f}_2(x^r) & \cdots & \hat{f}_p^*(x^r) \end{bmatrix}$$

.

6.1.1.4 Step 4.

Find $\hat{L}_p^{T^+}$,$\hat{L}_p^{T^-}$ and $\hat{U}_p^{T^+}$,$\hat{U}_p^{T^-}$ corresponding to each objective $\hat{f}_k(x)$.

$$\hat{U}_p^{T^+} = max\{\hat{f}_p(x^r)\} \qquad \text{and} \qquad \hat{L}_p^{T^+} = min\{\hat{f}_p(x^r),\}$$

$$\hat{U}_p^{T^-} = min\{\hat{f}_p(x^r)\} \qquad \text{and} \qquad \hat{L}_p^{T^+} = max\{\hat{f}_p(x^r),\}$$

whereas, $r = 1, 2, ..., p$.

$\hat{U}_p^{F+}$, $\hat{U}_p^{F-}$ and $\hat{L}_p^{F^+}$, $\hat{L}_p^{F^-}$ for non-membership grades of objectives are,

$$\hat{U}_p^{F^+} = \hat{U}_p^{T^+}, \hat{U}_p^{F^-} = \hat{U}_p^{T^-} \qquad \text{and} \qquad \hat{L}_p^{F^+} = \hat{L}_p^{T^+} + t(\hat{U}_p^{T^+} - \hat{L}_p^{T^+}),$$

$$\hat{L}_p^{F^-} = \hat{L}_p^{T^-} + t(\hat{U}_p^{T^-} - \hat{L}_p^{T^-}).$$

Upper $\hat{U}_p^{I^+}$, $\hat{U}_p^{I^-}$ and lower $\hat{L}_p^{I^+}$, $\hat{L}_p^{I^-}$ bounds for indeterminacy membership of objectives are,

$$\hat{L}_p^{I^+} = \hat{L}_p^{T^+}, \hat{L}_p^{I^-} = \hat{L}_p^{T^-} \quad \text{and} \quad \hat{U}_p^{I^+} = \hat{L}_p^{T^+} + s(\hat{U}_p^{T^+} - \hat{L}_p^{T^+}),$$

$$\hat{U}_p^{I^-} = \hat{L}_p^{T^-} + s(\hat{U}_p^{T^-} - \hat{L}_p^{T^-}).$$

where t, s, $\in (0,1)$.

6.1.1.5 Step 5.

In this step, we will define membership functions for bipolar neutrosophic set as:

$$T^+{}_p(\hat{f}_p(x)) = \begin{cases} 1 & \hat{f}_p(x) \leq \hat{L}_p^{T^+} \\ \frac{\hat{U}_p^{T^+} - f_p(x)}{\hat{U}_p^{T^+} - \hat{L}_p^{T^+}} & \hat{L}_p^{T^+} \leq \hat{f}_p(x) \leq \hat{U}_p^{T^+} \\ 0 & \hat{f}_p(x) \geq \hat{U}_p^{T^+} \end{cases}$$

$$T^-{}_p(\hat{f}_p(x)) = \begin{cases} -1 & \hat{f}_p(x) \geq \hat{L}_p^{T^-} \\ \frac{\hat{U}_p^{T^-} - f_p(x)}{\hat{U}_p^{T^-} - \hat{L}_p^{T^-}} & \hat{L}_p^{T^-} \geq \hat{f}_p(x) \geq \hat{U}_p^{T^-} \\ 0 & \hat{f}_p(x) \leq \hat{U}_p^{T^-} \end{cases}$$

$$I^+{}_p(\hat{f}_p(x)) = \begin{cases} 1 & \hat{f}_p(x) \leq \hat{L}_p^{I^+} \\ \frac{\hat{U}_p^{I^+} - \hat{f}_p(x)}{\hat{U}_p^{I^+} - \hat{L}_p^{I^+}} & \hat{L}_p^{I^+} \leq \hat{f}_p(x) \leq \hat{U}_p^{I^+} \\ 0 & \hat{f}_p(x) \geq \hat{U}_p^{I^+} \end{cases}$$

$$I^-{}_p(\hat{f}_p(x)) = \begin{cases} -1 & \hat{f}_p(x) \geq \hat{L}_p^{I^-} \\ \frac{\hat{U}_p^{I^-} - \hat{f}_p(x)}{\hat{U}_p^{I^-} - \hat{L}_p^{I^-}} & \hat{L}_p^{I^-} \geq \hat{f}_p(x) \geq \hat{U}_p^{I^-} \\ 0 & \hat{f}_p(x) \leq \hat{U}_p^{I^-} \end{cases}$$

$$
F^{+}{}_{p}(\hat{f}_p(x)) = \begin{cases} 0 & \hat{f}_p(x) \le \hat{L}_p^{F^+} \\ \frac{\hat{f}_p(x) - \hat{L}_p^{F^+}}{\hat{U}_p^{F^+} - \hat{L}_p^{F^+}} & \hat{L}_p^{F^+} \le \hat{f}_p(x) \le \hat{U}_p^{F^+} \\ 1 & \hat{f}_p(x) \ \ge \hat{U}_p^{F^+} \end{cases}
$$

$$
F^{-}{}_{p}(\hat{f}_p(x)) = \begin{cases} 0 & \hat{f}_p(x) \ge \hat{L}_p^{F^-} \\ \frac{\hat{f}_p(x) - \hat{L}_p^{F^-}}{\hat{U}_p^{F^-} - \hat{L}_p^{F^-}} & \hat{L}_p^{F^-} \ge \hat{f}_p(x) \ge \hat{U}_p^{F^-} \\ -1 & \hat{f}_p(x) \ \le \hat{U}_p^{F^-} \end{cases}
$$

6.1.1.6 Step 6.

The equivalent nonlinear problem can be given as:

Max $\alpha^+ - \alpha^- - \beta^+ + \beta^- + \gamma^+ - \gamma^-$, such that,

$$T^{+}{}_{p}(\hat{f}_p(x)) \ge \alpha^+,$$

$$T^{-}{}_{p}(\hat{f}_p(x)) \le \alpha^-,$$

$$I^{+}{}_{p}(\hat{f}_p(x)) \ge \gamma^+,$$

$$I^{-}{}_{p}(\hat{f}_p(x)) \le \gamma^-,$$

$$F^{+}{}_{p}(\hat{f}_p(x)) \le \beta^+,$$

$$F^{-}{}_{p}(\hat{f}_p(x)) \ge \beta^-,$$

with

$$\alpha^+ \ge \beta^+, \quad \alpha^+ \ge \gamma^+ \quad \alpha^- \le \beta^-, \quad \alpha^- \le \gamma^-$$

$$\alpha^+ + \beta^+ + \gamma^+ + \alpha^- + \beta^- + \gamma^- \le 3,$$

$$\alpha^+, \ \beta^+, \ \gamma^+, \ \in [0,1],$$

$$\alpha^-, \ \beta^-, \ \gamma^-, \ \in [-1,0],$$

$$\hat{g}_j(x) \le \hat{b}_j, \quad x \ge 0, \quad j = 1, 2, \ldots, q.$$

Which corresponds to non-linear programming as:

$$Max \qquad \alpha^+ - \alpha^- - \beta^+ + \beta^- + \gamma^+ - \gamma^-,$$

such that,

$$\hat{f}_p(x) + (\hat{U}_p^{T^+} - \hat{L}_p^{T^+}).\alpha^+ \le \hat{U}_p^{T^+},$$

$$\hat{f}_p(x) + (\hat{U}_p^{T^-} - \hat{L}_p^{T^-}).\alpha^- \le \hat{U}_p^{T^-},$$

$$\hat{f}_p(x) + (\hat{U}_p^{I^+} - \hat{L}_p^{I^+}).\gamma^+ \le \hat{U}_p^{I^+},$$

$$\hat{f}_p(x) + (\hat{U}_p^{I^-} - \hat{L}_p^{I^-}).\gamma^- \le \hat{U}_p^{I^-},$$

$$\hat{f}_p(x) - (\hat{U}_p^{F^+} - \hat{L}_p^{F^+})\ .\beta^+ \le \hat{L}_p^{F^+},$$

$$\hat{f}_p(x) - (\hat{U}_p^{F^-} - \hat{L}_p^{F^-})\ .\beta^+ \le \hat{L}_p^{F^-},$$

for $p = 1, 2, \ldots, k$.

$$\alpha^+ \ge \beta^+, \quad \alpha^+ \ge \gamma^+ \quad \alpha^- \le \beta^-, \quad \alpha^- \le \gamma^-,$$

$$\alpha^+ + \beta^+ + \gamma^+ + \alpha^- + \beta^- + \gamma^- \le 3,$$

$$\alpha^+,\ \beta^+,\ \gamma^+,\ \in [0,1],$$

$$\alpha^-,\ \beta^-,\ \gamma^-,\ \in [-1,0],$$

$$\hat{g}_j(x) \le \hat{b}_j, \quad x \ge 0, \quad j = 1, 2, \ldots, q.$$

Example 6.1.1. Min $f_1(x_1, x_2) = {x_1}^{-1}{x_2}^{-2}$,
Min $f_2(x_1, x_2) = 2{x_1}^{-2}{x_2}^{-3}$.
such that,

$$x_1 + x_2 \le 1.$$

Step 1

Consider the first objective function to be a single objective function subject to the stated constraints and solve it to determine the values of the decision variables, we get the value of $x_1 = 0.33$, $x_2 = 0.67$, $(f_1)_1 = 6.75$.

Step 2

We compute the second outstanding objective function, $(f_2)_1 = 60.78$, using these decision variables.

Step 3

The next two steps will be repeated for subsequent objective functions.

Step 4

Ideal solution is given as

$$= \begin{bmatrix} 6.75 & 60.78 \\ 6.94 & 57.87 \end{bmatrix}$$

Determine all the bounds that corresponds to each goal $f_p(x)$.

$$\begin{aligned}
\hat{U}_1^{T^+} &= \hat{L}_1^{T^-} = 6.94, \\
\hat{U}_1^{T^-} &= \hat{L}_1^{T^+} = 6.75, \\
\hat{L}_1^{I^+} &= 6.75,\ \hat{L}_1^{I^-} = 6.94, \\
\hat{U}_1^{I^+} &= 6.75 + s(0.19) = 6.826, \\
\hat{U}_1^{I^-} &= 6.94 + s(-0.19) = 6.864, \\
\hat{L}_1^{F^+} &= 6.75 + t(0.19) = 6.807, \\
\hat{L}_1^{F^-} &= 6.94 + t(-0.19) = 6.883, \\
\hat{U}_1^{F^+} &= 6.94,\ \hat{U}_1^{F^-} = 6.75, \\
\hat{U}_2^{T^+} &= \hat{L}_2^{T^-} = 60.78, \\
\hat{U}_2^{T^-} &= \hat{L}_2^{T^+} = 57.87, \\
\hat{L}_2^{I^+} &= 57.87, \hat{L}_2^{I^-} = 60.78, \\
\hat{U}_2^{I^+} &= 57.87 + s(2.91) = 59.034, \\
\hat{U}_2^{I^-} &= 60.78 + s(-2.91) = 59.616, \\
\hat{L}_2^{F^+} &= 57.87 + t(2.91) = 58.743, \\
\hat{L}_2^{F^-} &= 60.78 + t(-2.91) = 59.907. \\
\hat{U}_2^{F^+} &= 60.78,\ \hat{U}_2^{F^-} = 57.87
\end{aligned}$$

Where t, s, $\in (0,1)$, take $t = 0.3$, and $s = 0.4$,

Step 5.

$$T^{+}{}_1({x_1}^{-1}{x_2}^{-2}) = \begin{cases} 1 & {x_1}^{-1}{x_2}^{-2} \le 6.75 \\ \frac{6.94-{x_1}^{-1}{x_2}^{-2}}{6.94-6.75} & 6.75 \le {x_1}^{-1}{x_2}^{-2} \le 6.94 \\ 0 & {x_1}^{-1}{x_2}^{-2} \ \ge 6.94 \end{cases}$$

$$T^{+}{}_2(2{x_1}^{-2}{x_2}^{-3}) = \begin{cases} 1 & 2{x_1}^{-2}{x_2}^{-3} \le 57.87 \\ \frac{60.78-2{x_1}^{-2}{x_2}^{-3}}{60.78-57.87} & 57.87 \le 2{x_1}^{-2}{x_2}^{-3} \le 60.78 \\ 0 & {x_1}^{-1}{x_2}^{-2} \ \ge 60.78 \end{cases}$$

$$T^{-}{}_1({x_1}^{-1}{x_2}^{-2}) = \begin{cases} 1 & {x_1}^{-1}{x_2}^{-2} \ge 6.94 \\ \frac{6.75-{x_1}^{-1}{x_2}^{-2}}{6.75-6.94} & 6.94 \ge {x_1}^{-1}{x_2}^{-2} \ge 6.75 \\ 0 & {x_1}^{-1}{x_2}^{-2} \ \le 6.75 \end{cases}$$

$$T^{-}{}_2(2{x_1}^{-2}{x_2}^{-3}) = \begin{cases} 1 & 2{x_1}^{-2}{x_2}^{-3} \ge 60.78 \\ \frac{57.87-2{x_1}^{-2}{x_2}^{-3}}{57.87-60.78} & 57.87 \ge 2{x_1}^{-2}{x_2}^{-3} \ge 57.87 \\ 0 & {x_1}^{-1}{x_2}^{-2} \ \ge 57.87 \end{cases}$$

$$I^{+}{}_1({x_1}^{-1}{x_2}^{-2}) = \begin{cases} 1 & {x_1}^{-1}{x_2}^{-2} \le 6.75 \\ \frac{6.826-{x_1}^{-1}{x_2}^{-2}}{6.826-6.75} & 6.75 \le {x_1}^{-1}{x_2}^{-2} \le 6.826 \\ 0 & {x_1}^{-1}{x_2}^{-2} \ \ge 6.826 \end{cases}$$

$$I^{+}{}_2(2{x_1}^{-2}{x_2}^{-3}) = \begin{cases} 1 & 2{x_1}^{-2}{x_2}^{-3} \le 57.87 \\ \frac{59.034-2{x_1}^{-2}{x_2}^{-3}}{59.034-57.87} & 57.87 \le 2{x_1}^{-2}{x_2}^{-3} \le 59.034 \\ 0 & 2{x_1}^{-2}{x_2}^{-3} \ \ge 59.034 \end{cases}$$

$$I^{-}{}_1(x_1{}^{-1}x_2{}^{-2}) = \begin{cases} 1 & x_1{}^{-1}x_2{}^{-2} \geq 6.94 \\ \frac{6.864 - x_1{}^{-1}x_2{}^{-2}}{6.826 - 6.75} & 6.94 \geq x_1{}^{-1}x_2{}^{-2} \geq 6.864 \\ 0 & x_1{}^{-1}x_2{}^{-2} \leq 6.864 \end{cases}$$

$$I^{-}{}_2(2x_1{}^{-2}x_2{}^{-3}) = \begin{cases} 1 & 2x_1{}^{-2}x_2{}^{-3} \geq 60.78 \\ \frac{59.616 - 2x_1{}^{-2}x_2{}^{-3}}{59.616 - 60.78} & 60.78 \geq 2x_1{}^{-2}x_2{}^{-3} \geq 59.616 \\ 0 & 2x_1{}^{-2}x_2{}^{-3} \leq 59.616 \end{cases}$$

$$F^{+}{}_1(x_1{}^{-1}x_2{}^{-2}) = \begin{cases} 0 & x_1{}^{-1}x_2{}^{-2} \leq 6.807 \\ \frac{x_1{}^{-1}x_2{}^{-2} - 6.807}{6.94 - 6.807} & 6.807 \leq x_1{}^{-1}x_2{}^{-2} \leq 6.94 \\ 1 & x_1{}^{-1}x_2{}^{-2} \geq 6.94 \end{cases}$$

$$F^{+}{}_2(2x_1{}^{-2}x_2{}^{-3}) = \begin{cases} 0 & 2x_1{}^{-2}x_2{}^{-3} \leq 58.743 \\ \frac{2x_1{}^{-2}x_2{}^{-3} - 58.743}{60.78 - 58.743} & 58.743 \leq 2x_1{}^{-2}x_2{}^{-3} \leq 60.78 \\ 1 & 2x_1{}^{-2}x_2{}^{-3} \geq 60.78 \end{cases}$$

$$F^{-}{}_1(x_1{}^{-1}x_2{}^{-2}) = \begin{cases} 0 & x_1{}^{-1}x_2{}^{-2} \geq 6.883 \\ \frac{x_1{}^{-1}x_2{}^{-2} - 6.883}{6.75 - 6.883} & 6.883 \geq x_1{}^{-1}x_2{}^{-2} \geq 6.75 \\ 1 & x_1{}^{-1}x_2{}^{-2} \leq 6.75 \end{cases}$$

$$F^{-}{}_2(2x_1{}^{-2}x_2{}^{-3}) = \begin{cases} 0 & 2x_1{}^{-2}x_2{}^{-3} \geq 59.907 \\ \frac{2x_1{}^{-2}x_2{}^{-3} - 59.907}{57.87 - 59.907} & 59.886 \geq 2x_1{}^{-2}x_2{}^{-3} \geq 57.87 \\ 1 & 2x_1{}^{-2}x_2{}^{-3} \leq 57.87 \end{cases}$$

Step 6.

Nonlinear programming problem in bipolar neutrosophic is,

$$Max \alpha^{+} - \alpha^{-} - \beta^{+} + \beta^{-} + \gamma^{+} - \gamma^{-},$$

such that,

$$x_1{}^{-1}x_2{}^{-2} + (0.19)\alpha^+ \leq 6.94,$$

$$2x_1{}^{-2}x_2{}^{-3} + (2.91)\alpha^+ \leq 60.78,$$

$$x_1{}^{-1}x_2{}^{-2} + (-0.19)\alpha^- \leq 6.75,$$

$$2x_1{}^{-2}x_2{}^{-3} + (-2.91)\alpha^- \leq 57.87,$$

$$x_1{}^{-1}x_2{}^{-2} + (0.076)\gamma^+ \leq 6.826,$$

$$2x_1{}^{-2}x_2{}^{-3} + (1.164)\gamma^+ \leq 59.034,$$

$$x_1{}^{-1}x_2{}^{-2} + (-0.076)\gamma^- \leq 6.864,$$

$$2x_1{}^{-2}x_2{}^{-3} + (-1.164)\gamma^- \leq 59.616,$$

$$x_1{}^{-1}x_2{}^{-2} - (0.133)\beta^+ \leq 6.807,$$

$$2x_1{}^{-2}x_2{}^{-3} - (2.037)\beta^+ \leq 58.743,$$

$$x_1{}^{-1}x_2{}^{-2} - (-0.133)\beta^- \leq 6.883,$$

$$2x_1{}^{-2}x_2{}^{-3} - (-2.037)\beta^- \leq 59.907.$$

$$x_1 + x_2 \leq 1$$

$$0 \leq \alpha^+ \leq 1,$$

$$0 \leq \beta^+ \leq 1,$$

$$0 \leq \gamma^+ \leq 1,$$

$$-1 \leq \alpha^- \leq 0,$$

$$-1 \leq \beta^- \leq 0,$$

$$-1 \leq \gamma^- \leq 0.$$

$$\alpha^+ \geq \beta^+, \quad \alpha^+ \geq \gamma^+ \quad \alpha^- \leq \beta^-, \quad \alpha^- \leq \gamma^-,$$

$$\alpha^+ + \beta^+ + \gamma^+ + \alpha^- + \beta^- + \gamma^- \leq 3.$$

Comparisons between the suggested optimal solution and existing methodologies are shown below.

Table 6.1: Comparison of optimal solutions

Optimization techniques	optimal Decision variables x_1^*, x_2^*	optimal Objective functions f_1^*, f_2^*	sum of Optimal objective values $f_1^* + f_2^*$
IFO	0.3659009, 0.6356811	6.797078, 58.79110	65.588178
NSO	0.3635224, 0.6364776	6.790513, 58.68732	65.487833
BNO	0.373367, 0.638284	6.574097696, 55.17168667	61.74578437

Table 6.2: Percentage gap w.r.t. bipolar neutrosophic optimization

Optimization techniques	objective function f_1^*	objective function f_2^*	cumulative percentage gap
IFO	3.280531781%	6.156396682%	9.436928464%
NSO	3.187024368%	6.00646389%	9.193488257%
FVRNO	3.270308779%	5.853553601%	9.123862379%
BNO	0%	0%	0%

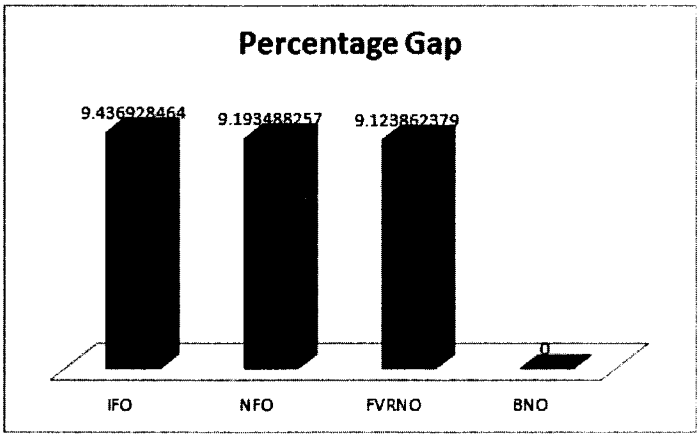

Figure 6.1: Percentage gap w.r.t. bipolar neutrosophic optimization

6.2 Application of Bipolar Neutrosophic in Riser Design

Multi-objective nonlinear problem is [90]

Minimize

$$v_r(\hat{d},\hat{h}) = \frac{\pi \hat{d}^2 \hat{h}}{4},$$

Minimize

$$s_t(\hat{d},\hat{h}) = \frac{\hat{d}\hat{h}}{4\hat{h}+2\hat{d}},$$

such that,

$$\frac{48}{19}\hat{d}^{-1} + \frac{24}{19}\hat{h}^{-1} \leq 1$$

Here pay-off matrix is

$$= \begin{bmatrix} 42.75642 & 0.631579 \\ 12.510209 & 0.6315786 \end{bmatrix}$$

Table 6.3: Comparison of optimal solutions

Optimization techniques	optimal decision variables d^*, h^*	optimal Objective functions v_r^*, S_t^*	cumulative Percentage gap
NSO	3.152158, 3.152158	24.60870, 0.6315787	82.73991921%
FVRNS	2.979, 3.50964	24.44968309, 0.5228508083	20.28175084%
BNO	2.51653, 2.51541	12.5163487817, 0.419359426	0%

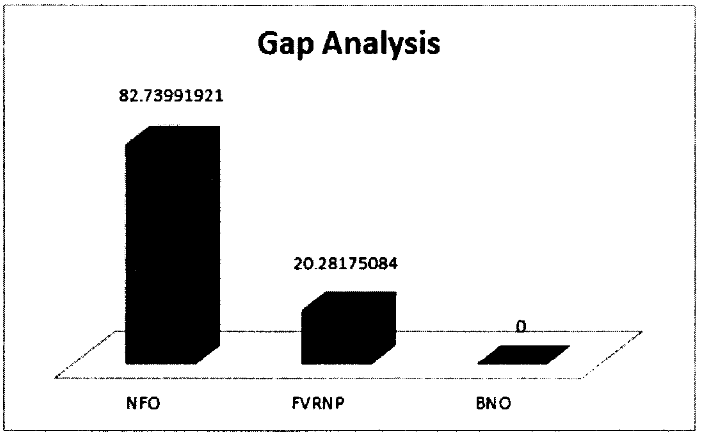

Figure 6.2: Comparison of proposed methodology with percentage gap

6.3 Conclusion

In this chapter computational algorithms based on bipolar neutrosophic sets have been developed to find the best possible solution to the given nonlinear multi-objective optimization models. The proposed technique is very simple and user-friendly. From the results, it can be seen that with each extension of the neutrosophic set, we get improvements in results. Also, the comparative study shown in Table 6.1-6.3 and Figure 6.1-6.2 suggested that each extension of neutrosophic optimization techniques provide improved results with respect to intuitionistic and neutrosophic optimization techniques, but the bipolar neutrosophic optimization technique provides the most optimum results. Therefore, bipolar neutrosophic optimization is a more effective, reliable and comprehensive method for addressing nonlinear multi-objective optimization problems in an uncertain environment. In the future, we may look for those improvements in neutrosophic sets, which we could use to develop new optimization algorithms with effective, improved, efficient, and reliable results when dealing with uncertainty.

Chapter 7

Multi-Objective Interval Valued Neutrosophic Optimization with Application

The information in this chapter was taken from real-time applications in an interval neutrosophic system. The suggested method is based on the idea that the decision made is the result of how constraints and goals interact, and membership functions are set by the decision-aspirations. Since the present work examines interval neutrosophic optimization, the aspiration values for IVN truth, IVN indeterminacy, and IVN falsehood exclusively rely on the confluence rules and arithmetic of interval-neutrosophic sets. The authenticity, relevance, and validity of the suggested method are supported by a number of real-world examples, and a comparison of the newly established and existing approaches has been conducted. The inference that has been made in light of the relevant analysis and findings not only provides new avenues for study in optimization but also in data processing, statistics, and decision-making.

7.1 Interval-valued Neutrosophic Set

Consider a universal set X. An interval valued neutrosophic subset (IVNS) $\breve{A}^{IN}$ of X [133], is

$$\breve{A}^{IN} = \left\langle \left(x, \left[T^L_{\breve{A}^{IN}}(x), T^U_{\breve{A}^{IN}}(x)\right], \left[I^L_{\breve{A}^{IN}}(x), I^U_{\breve{A}^{IN}}(x)\right], \left[F^L_{\breve{A}^{IN}}(x), F^U_{\breve{A}^{IN}}(x)\right]\right)\right\rangle.$$

Here,

$$\left[T^L_{\breve{A}^{IN}}(x), T^U_{\breve{A}^{IN}}(x)\right], \left[I^L_{\breve{A}^{IN}}(x), I^U_{\breve{A}^{IN}}(x)\right] \text{ and } \left[F^L_{\breve{A}^{IN}}(x), F^U_{\breve{A}^{IN}}(x)\right],$$

are intervals of membership, indeterminacy and non-membership functions whose images are contained in $[0, 1]$. IVNS is an advancement of neutrosophic, intuitionistic fuzzy and crisp set as shown in the following figure.

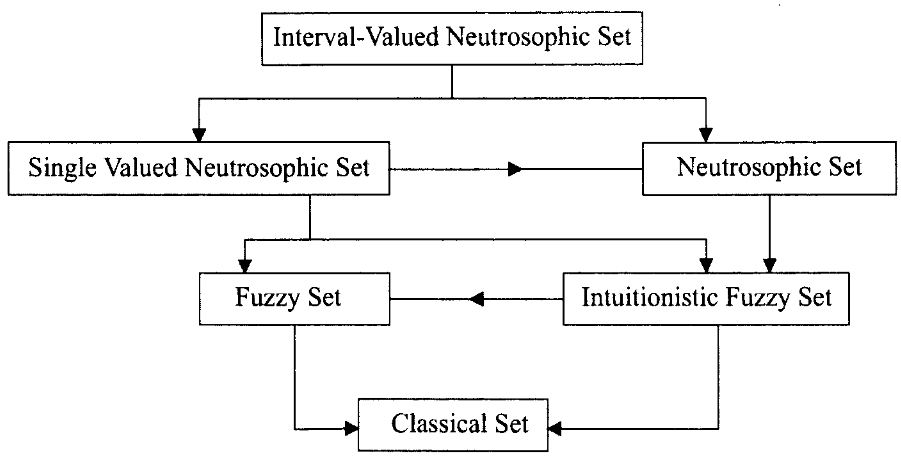

Figure 7.1: Pictorial description of interval valued neutrosophic set

An inter-valued neutrosophic set $\breve{A}^{IN}$ is in $\breve{B}^{IN}$, $\breve{A}^{IN} \subseteq \breve{B}^{IN}$ iff for any x in X:

$$\begin{aligned}
T^L_{\breve{A}^{IN}}(x) \le T^L_{\breve{B}^{IN}}(x), \quad & T^U_{\breve{A}^{IN}}(x) \le T^U_{\breve{B}^{IN}}(x) \\
I^L_{\breve{A}^{IN}}(x) \le I^L_{\breve{B}^{IN}}(x), \quad & I^U_{\breve{A}^{IN}}(x) \le I^U_{\breve{B}^{IN}}(x) \\
F^L_{\breve{A}^{IN}}(x) \ge F^L_{\breve{B}^{IN}}(x), \quad & F^U_{\breve{A}^{IN}}(x) \ge F^U_{\breve{B}^{IN}}(x).
\end{aligned}$$

Two sets $\breve{A}^{IN}$ and $\breve{B}^{IN}$ in an IVN framework are equal iff $\breve{A}^{IN} \subseteq \breve{B}^{IN}$ and $\breve{B}^{IN} \subseteq \breve{A}^{IN}$.

Complement of an IVNS

$$\breve{A}^{IN} = \left\langle \left(x, \left[T^L_{\breve{A}^{IN}}(x), T^U_{\breve{A}^{IN}}(x) \right], \left[I^L_{\breve{A}^{IN}}(x), I^U_{\breve{A}^{IN}}(x) \right], \left[F^L_{\breve{A}^{IN}}(x), F^U_{\breve{A}^{IN}}(x) \right] \right) \right\rangle,$$

is defined by

$$\breve{A}^{IN} = \left\langle \left(x, \left[F^L_{\breve{A}^{IN}}(x), F^U_{\breve{A}^{IN}}(x) \right], \left[I^L_{\breve{A}^{IN}}(x), I^U_{\breve{A}^{IN}}(x) \right], \left[T^L_{\breve{A}^{IN}}(x), T^U_{\breve{A}^{IN}}(x) \right] \right) \right\rangle.$$

The maximum of an IVNS is $\langle [1,1], [0,0], [0,0] \rangle$ and minimum is $\langle [0,0], [0,0], [1,1] \rangle$.

Union of the two IVNS $\breve{A}^{IN}$ and $\breve{B}^{IN}$, denoted as

$$\breve{C}^{IN} = \breve{A}^{IN} \cup \breve{B}^{IN}.$$

With lower and upper membership, indeterminacy and nonmembership grades defined as:

$$\begin{aligned}
T^L_{\breve{C}^{IN}} &= \max\{T^L_{\breve{A}^{IN}}, T^L_{\breve{B}^{IN}}\}, \quad T^U_{\breve{C}^{IN}} = \max\{T^U_{\breve{A}^{IN}}, T^U_{\breve{B}^{IN}}\} \\
I^L_{\breve{C}^{IN}} &= \max\{I^L_{\breve{A}^{IN}}, I^L_{\breve{B}^{IN}}\}, \quad I^U_{\breve{C}^{IN}} = \max\{I^U_{\breve{A}^{IN}}, I^U_{\breve{B}^{IN}}\} \\
F^L_{\breve{C}^{IN}} &= \min\{F^L_{\breve{A}^{IN}}, F^L_{\breve{B}^{IN}}\}, \quad F^U_{\breve{C}^{IN}} = \min\{F^U_{\breve{A}^{IN}}, F^U_{\breve{B}^{IN}}\}.
\end{aligned}$$

Intersection of the two IVNS $\breve{A}^{IN}$ and $\breve{B}^{IN}$ is denoted as,

$$\breve{C}^{IN} = \breve{A}^{IN} \cap \breve{B}^{IN}.$$

With lower and upper membership, indeterminacy and nonmembership grades defined as:

$$\begin{aligned}
T^L_{\breve{C}^{IN}} &= \min\{T^L_{\breve{A}^{IN}}, T^L_{\breve{B}^{IN}}\}, \quad T^U_{\breve{C}^{IN}} = \min\{T^U_{\breve{A}^{IN}}, T^U_{\breve{B}^{IN}}\} \\
I^L_{\breve{C}^{IN}} &= \min\{I^L_{\breve{A}^{IN}}, I^L_{\breve{B}^{IN}}\}, \quad I^U_{\breve{C}^{IN}} = \min\{I^U_{\breve{A}^{IN}}, I^U_{\breve{B}^{IN}}\} \\
F^L_{\breve{C}^{IN}} &= \max\{F^L_{\breve{A}^{IN}}, F^L_{\breve{B}^{IN}}\}, \quad F^U_{\breve{C}^{IN}} = \max\{F^U_{\breve{A}^{IN}}, F^U_{\breve{B}^{IN}}\}.
\end{aligned}$$

Theorem 7.1.1. *Let $\breve{A}^{IN}$ and $\breve{B}^{IN}$ be the two IVNS then their intersection $\breve{A}^{IN} \cap \breve{B}^{IN}$ is the largest IVNS which is contained in both $\breve{A}^{IN}$ and $\breve{B}^{IN}$ and union $\breve{A}^{IN} \cup \breve{B}^{IN}$ is the smallest IVNS containing both $\breve{A}^{IN}$ and $\breve{B}^{IN}$.*

7.2 Development of Proposed Approach

Given an optimization problem

$$\text{Minimize } \{f_i(x)\} \qquad i = 1, 2, \ldots, p \tag{7.2.1}$$

such that;

$$g_j(x) \le b_j \qquad j = 1, 2, \ldots, q.$$

Here, $f_i(x)$ represents objective functions, $x = (x_1, x_2, \ldots, x_n)$ is the n-tuple of decision variables and $g_j(x)$ represents constraint functions. Suppose FR represent the feasible region obtained for the above problem using any suitable optimization technique. Ultimate aim is to remodel the region FR by means of interval valued neutrosophic scheme. It is proceeded as:

$$D = (\cap_{i=1}^{p} O_i) \cap (\cap_{j=1}^{q} C_j) = \{(x, [T_D^L, T_D^U], [I_D^L, I_D^U], [F_D^L, F_D^U]) : x \in FR\},$$

where,

$$\begin{aligned}
\left[T_D^L, T_D^U\right] &= \left[\min\left\{T_{O_i}^L(x); T_{C_j}^L(x)\right\}, \min\left\{T_{O_i}^U(x); T_{C_j}^U(x)\right\}\right] \\
\left[I_D^L, I_D^U\right] &= \left[\min\left\{I_{O_i}^L(x); I_{C_j}^L(x)\right\}, \min\left\{I_{O_i}^U(x); I_{C_j}^U(x)\right\}\right] \\
\left[F_D^L, F_D^U\right] &= \left[\max\left\{F_{O_i}^L(x); F_{C_j}^L(x)\right\}, \max\left\{F_{O_i}^U(x); F_{C_j}^U(x)\right\}\right].
\end{aligned}$$

With $1 \le i \le p, 1 \le j \le q$. Here $[T_D^L, T_D^U]$, $[I_D^L, I_D^U]$, $[F_D^L, F_D^U]$ are interval-valued grades of acceptance, indeterminacy and falsity of interval-valued neutrosophic decision.

In this case, the decision is to escalate rejection spectrum while keeping the acceptance and indeterminacy range minimum. Hence, the upper and lower membership functions are defined as follow:

$$T_k^U(f_k(x)) = \begin{cases} 1 & f_k(x) \le \hat{L}_k^T \\ \frac{\hat{U}_k^T - f_k(x)}{\hat{U}_k^T - \hat{L}_k^T} & \hat{L}_k^T \le f_k(x) \le \hat{U}_k^T \\ 0 & f_k(x) \ge \hat{U}_k^T \end{cases}$$

$$T_k^L(f_k(x)) = \begin{cases} 1 & f_k(x) \leq \hat{L}_k^T \\ \eta\frac{\hat{U}_k^T - f_k(x)}{\hat{U}_k^T - \hat{L}_k^T} & \hat{L}_k^T \leq f_k(x) \leq \hat{U}_k^T \\ 0 & f_k(x) \ \geq \hat{U}_k^T \end{cases}$$

$$I_k^U(f_k(x)) = \begin{cases} 1 & f_k(x) \leq \hat{L}_k^I \\ \frac{\hat{U}_k^I - f_k(x)}{\hat{U}_k^I - \hat{L}_k^I} & \hat{L}_k^I \leq f_k(x) \leq \hat{U}_k^I \\ 0 & f_k(x) \ \geq \hat{U}_k^I \end{cases}$$

$$I_k^L(f_k(x)) = \begin{cases} 1 & f_k(x) \leq \hat{L}_k^I \\ \eta\frac{\hat{U}_k^I - f_k(x)}{\hat{U}_k^I - \hat{L}_k^I} & \hat{L}_k^I \leq f_k(x) \leq \hat{U}_k^I \\ 0 & f_k(x) \ \geq \hat{U}_k^I \end{cases}$$

$$F_k^U(f_k(x)) = \begin{cases} 0 & f_k(x) \leq \hat{L}_k^F \\ \frac{f_k(x) - \hat{L}_k^F}{\hat{U}_k^F - \hat{L}_k^F} & \hat{L}_k^F \leq f_k(x) \leq \hat{U}_k^F \\ 1 & f_k(x) \ \geq \hat{U}_k^F \end{cases}$$

$$F_k^L(f_k(x)) = \begin{cases} 0 & f_k(x) \leq \hat{L}_k^F \\ \eta\frac{f_k(x) - \hat{L}_k^F}{\hat{U}_k^F - \hat{L}_k^F} & \hat{L}_k^F \leq f_k(x) \leq \hat{U}_k^F \\ 1 & f_k(x) \ \geq \hat{U}_k^F. \end{cases}$$

Where $0 \leq k \leq p+q$, $0 \leq \eta \leq 1$.

Corresponding to each objective function $T_i^L(f_i(x))$ is lower truth functions and $T_i^U(f_i(x))$ is upper truth functions . Then the truth membership function T_D over the domain D is given by

$$T_D(x^*) = \left[\max\left\{\min_{1\leq k\leq p+q} T_k^L(f_k(x))\right\}, \max\left\{\min_{1\leq k\leq p+q} T_k^U(f_k(x))\right\}\right]. \qquad (7.2.2)$$

Replace $\min_{1\leq k\leq p+q} T_k^L(f_k(x))$ and $\min_{1\leq k\leq p+q} T_k^U(f_k(x))$ by auxiliary variables a and $\theta a + (1-\theta)b$, then Equation 7.2.2 is converted into conventional linear programming problem, i.e,

$$\text{Maximize } \ \theta a + (1-\theta)b, \qquad (7.2.3)$$

subject to,

$$T_k^U(f_k(x)) \geq \theta a + (1-\theta)b,$$

$$T_k^L(f_k(x)) \geq a,$$

where $1 \leq k \leq p+q$ and $0 \leq a, b, \theta \leq 1$ with $b \geq a$.

Similar fashion will be adopted for indeterminacy grade and continuing in same manner, $F_D(x^*)$ can be interpreted as:

$$F_D(x^*) = \left[\min\left\{\max_{1\leq k\leq p+q} F_k^L(f_k(x))\right\}, \min\left\{\max_{1\leq k\leq p+q} F_k^U(f_k(x))\right\}\right], \qquad (7.2.4)$$

end up yielding

$$\text{minimize} \quad \theta c + (1-\theta)d, \qquad (7.2.5)$$

subject to,

$$\begin{aligned} F_k^U(f_k(x)) &\leq \theta c + (1-\theta)d, \\ F_k^L(f_k(x)) &\leq c, \end{aligned}$$

where $1 \leq k \leq p+q$ and $0 \leq c, d, \theta \leq 1$ with $d \geq c$.

Combining all expression of interval-valued true, false and indeterminate grades of membership Expression 7.2.1 is converted into the following multi-objective programming problem.

$$\text{Maximize} \quad \theta a + (1-\theta)b - \theta c - (1-\theta)d + \theta e + (1-\theta)f, \qquad (7.2.6)$$

such that,

$$\begin{aligned} T_k^U(f_k(x)) &\geq \theta a + (1-\theta)b, \\ T_k^L(f_k(x)) &\geq a, \\ F_k^U(f_k(x)) &\leq \theta c + (1-\theta)d, \\ F_k^L(f_k(x)) &\leq c, \\ I_k^U(f_k(x)) &\geq \theta e + (1-\theta)f, \\ I_k^L(f_k(x)) &\geq e, \end{aligned}$$

$$\theta a + (1-\theta)b + \theta c + s(1-\theta)d + \theta e + (1-\theta)f \leq 3,$$

$$b \geq a \quad d \geq c \quad f \geq e$$

$$0 \leq \theta \leq 1 \qquad a, b, c, d, e, f \in [0,1]$$

$$g_j(x) \leq b_j \qquad j = 1, 2, 3, \ldots, q, \quad k = 1, 2, \ldots, p+q.$$

Expression 7.2.6 represents the generic form of interval-valued neutrosophic optimization problem. The following cases of neutrosophic optimization problem, interval intuitionistic fuzzy optimization problem and intuitionistic fuzzy optimization problem can be attained with this expression.

If objective function needs to be maximized, then the range of truth and indeterminacy must increase while keeping the range of falsity minimum. The choice of higher the better will effect the membership functions. For instance, the upper truth membership function will be

$$T_k^U(f_k(x)) = \begin{cases} 0 & f_k(x) \leq \hat{L}_k^T \\ \frac{f_k(x)-\hat{L}_k^T}{\hat{U}_k^T-\hat{L}_k^T} & \hat{L}_k^T \leq f_k(x) \leq \hat{U}_k^T \\ 1 & f_k(x) \geq \hat{U}_k^T \end{cases}$$

objective functions will be maximized in the interval valued neutrosophic environment in the similar manner as described for minimization case.

Remark 7.2.1. The interval valued neutrosophic optimization is a generalization of previously defined optimization techniques. By assigning zero value to the auxiliary variables in different combinations one can deduce fuzzy, intuitionistic , interval-valued intuitionistic and neutrosophic linear programming from the newly established optimization technique.

7.2.1 Solution Procedure

From the set of p objectives, solve the multi-objective problem (1) as a single objective by taking one of the objectives, while neglecting others. Find all respective optimal solutions x^k for p different objectives. By using these optimal solution, figure out each objective value. Formulate a pay-off matrix by using these objective values at each optimal solution (optimal point).

$$\begin{bmatrix} f_1^*(x^1) & f_2(x^1) & \cdots & f_p(x^1) \\ f_1(x^2) & f_2^*(x^2) & \cdots & f_1(x^2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x^r) & f_2(x^r) & \cdots & f_p^*(x^r) \end{bmatrix}$$

Next step is to find lower and upper bounds of true, falsity and indeterminate grades of membership corresponding to each objective.

$$
\begin{aligned}
U_k^T &= \max\{f_k(x^r)\} \quad && \text{and} \quad L_k^T = \min\{f_k(x^r)\}, \quad \text{where} \quad r = 1, 2, \dots p \\
U_k^F &= U_k^T \quad && \text{and} \quad L_k^F = L_k^T + v(U_k^T - L_k^T) \\
L_k^I &= L_k^T \quad && \text{and} \quad U_k^I = L_k^T + w(U_k^T - L_k^T).
\end{aligned}
$$

Where $w, v \in (0, 1)$. Since the main focus of current work is to cover such problems, where objectives are inconsistent in nature and it is almost impossible to designate exactly a number to true, false and indeterminate grades of membership. Hence, while taking care of the fact that decision maker here want to maximize rejection range, while keeping the acceptance and indeterminacy range minimum, define interval valued neutrosophic grades of membership, non-membership and indeterminacy as in Equation 7.2.6. By substitution the values of membership functions the minimization problem can be transformed into corresponding lp problem, given as

$$
\text{maximize} \quad \theta a + (1 - \theta)b - \theta c - (1 - \theta)d + \theta e + (1 - \theta)f,
$$

such that,

$$
\begin{aligned}
f_k(x) + (\theta a + (1-\theta)b)(U_k^T - L_k^T) &\leq U_k^T \\
f_k(x) + \frac{a}{\eta}(U_k^T - L_k^T) &\leq U_k^T \\
f_k(x) + (\theta c + (1-\theta)d)(U_k^I - L_k^I) &\leq U_k^I \\
f_k(x) + \frac{c}{\eta}(U_k^I - L_k^I) &\geq L_k^I \\
f_k(x) - (\theta e + (1-\theta)f)(U_k^F - L_k^F) &\leq L_k^F \\
f_k(x) - \frac{e}{\eta}(U_k^F - L_k^F) &\leq L_k^F \\
\theta a + (1-\theta)b + \theta c + (1-\theta)d + \theta e + (1-\theta)f &\leq 3 \\
b \geq a, \quad d \geq c, \quad f &\geq e \\
0 \leq \theta &\leq 1 \\
a, b, c, d, e, f &\in [0,1] \\
0 \leq \eta &\leq 1 \\
g_j(x) &\leq b_j \\
j = 1, 2, 3, ..., q \qquad i = 1, 2, ..., p + q.
\end{aligned}
$$

7.3 Application

In the following cases newly established mathematical model will be applied and comparison of the results will be presented with some well-known optimization techniques.

7.3.1 Production Planning Problem

Consider a construction, planning and manufacturing problem studied by Bharati and Singh [134], where a cluster of six machines are dedicated to deliver three items. The present limit portfolio and separate innovative prerequisites and the important information are given in Table 7.1.

Table 7.1: Parameters values

Machine Type	Machine hours	Unit prices	Products x_1, x_2, x_3
Milling	1400	0.75	12, 17, 0
Lather	1000	0.60	3,9,8
Grinder	1750	0.35	10,13,15
Jig Saw	1325	0.50	6,0,16
Drill Press	900	1.15	0,12,7
Band Saw	1075	0.65	9.5, 9.5, 4
Total capacity cost	4658.75		

Let x_1, x_2 and x_3 indicate three items, at that point the total scientific detailing of the previously quoted problem as a multi-objective programming problem. Now multi-objective linear problem is given as

$$\text{Maximize } f_1(x) = 50x_1 + 100x_2 + 17.5x_3$$

$$\text{Maximize } f_2(x) = 92x_1 + 75x_2 + 50x_3$$

$$\text{Maximize } f_3(x) = 25x_1 + 100x_2 + 75x_3$$

such that

$$\begin{aligned} 12x_1 + 17x_2 &\leq 1400 \\ 3x_1 + 9x_2 + 8x_3 &\leq 1000 \\ 10x_1 + 13x_2 + 15x_3 &\leq 1750 \\ 6x_1 + 16x_3 &\leq 1325 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Pay-off matrix of above problem is

$$\begin{bmatrix} 8041 & 10,020.33 & 9319.25 \\ 5452.63 & 10,950.59 & 5903.00 \\ 7983.60 & 10,056.99 & 9355.90 \end{bmatrix}$$

Next step is to find the lower and upper bounds of grade of membership of each objective function.

$$U_1^T = \max\{f_1(x^r)\} = 8041, \quad U_2^T = \max\{f_2(x^r)\} = 10950.59$$

$$U_3^T = \max\{f_3(x^r)\} = 9355.90, \quad L_1^T = \min\{f_1(x^r)\} = 5452.63$$

$$L_2^T = \min\{f_2(x^r)\} = 10020.33, \quad L_3^T = \min\{f_3(x^r)\} = 5903.00$$

$$U_1^F = U_1^T = 8041, \quad U_2^F = U_2^T = 10950.59, \quad U_3^F = U_3^T = 9355.90$$

$$L_1^I = L_1^T = 5452.63, \quad L_2^I = L_2^T = 10020.33, \quad L_3^I = L_3^T = 5903.00$$

$$L_k^F = L_k^T + v(U_k^T - L_k^T)$$

$$L_1^F = 5452.63 + 0.2(8041 - 5452.63) = 5970.304$$

$$L_2^F = 10020.33 + 0.2(10950.59 - 10020.33) = 10206.382$$

$$L_3^F = 5903 + 0.2(9355.90 - 5903) = 6593.58$$

$$U_k^I = L_k^T + w(U_k^T - L_k^T)$$

$$U_1^I = 5452.63 + 0.3(8041 - 5452.63) = 6229.141$$

$$U_2^I = 10020.33 + 0.3(10950.59 - 10020.33) = 10299.408$$

$$U_3^I = 5903 + 0.0.3(9355.90 - 5903) = 6938.87$$

To construct the membership function, use the above mentioned values of the bounds for each objective function.

By using the grade of membership, Interval-valued neutrosophic optimization technique is converted into lp problem as

$$
\begin{aligned}
\text{maximize} \quad & \theta a + (1-\theta)b - \theta c - (1-\theta)d + \theta e + (1-\theta)f \text{ such that} \\
& f_k(x) - (\theta a + (1-\theta)b)(U_k^T - L_k^T) \geq L_k^T \\
& f_1(x) - (\theta a + (1-\theta)b)(2588.37) \geq 5452.63 \\
& f_2(x) - (\theta a + (1-\theta)b)(930.26) \geq 10020.33 \\
& f_3(x) - (\theta a + (1-\theta)b)(3452.9) \geq 5903.00 \\
& f_k(x) - \frac{a}{\eta}(U_k^T - L_k^T) \geq L_k^T \\
& f_1(x) - \frac{a}{\eta}(2588.37) \geq 5452.63 \\
& f_2(x) - \frac{a}{\eta}(930.26) \geq 10020.33 \\
& f_3(x) - \frac{a}{\eta}(3452.9) \geq 5903.00 \\
& f_k(x) - (\theta c + (1-\theta)d)(U_k^I - L_k^I) \geq L_k^I \\
& f_1(x) - (\theta c + (1-\theta)d)(776.511) \geq 5452.63 \\
& f_2(x) - (\theta c + (1-\theta)d)(279.078) \geq 100200.33 \\
& f_3(x) - (\theta c + (1-\theta)d)(1035.87) \geq 5903.00 \\
& f_k(x) - \frac{c}{\eta}(U_k^I - L_k^I) \geq L_k^I \\
& f_1(x) - \frac{c}{\eta}(776.511) \geq 5452.63 \\
& f_2(x) - \frac{c}{\eta}(279.078) \geq 100200.33 \\
& f_3(x) - \frac{c}{\eta}(1035.87) \geq 5903.00 \\
& f_k(x) + (\theta e + (1-\theta)f)(U_k^F - L_k^F) \geq U_k^F \\
& f_1(x) + (\theta e + (1-\theta)f)(2070.696) \geq 8041 \\
& f_2(x) + (\theta e + (1-\theta)f)(744.208) \geq 10950.59 \\
& f_3(x) + (\theta e + (1-\theta)f)(2762.32) \geq 9355.90 \\
& f_k(x) + \frac{e}{\eta}(U_k^F - L_k^F) \geq U_k^F \\
& f_1(x) + \frac{e}{\eta}(2070.696) \geq 8041 \\
& f_2(x) + \frac{e}{\eta}(744.208) \geq 10950.59 \\
& f_3(x) + \frac{e}{\eta}(2762.32) \geq 9355.90
\end{aligned}
\tag{7.3.1}
$$

$$\theta a + (1-\theta)b + \theta c + (1-\theta)d + \theta e + (1-\theta)f \leq 3$$

$$b \geq a, \qquad d \geq c, \qquad f \geq e$$

$$0 \leq \theta \leq 1$$

$$a, b, c, d, e, f \in [0, 1]$$

$$0 \leq \eta \leq 1$$

$$12x_1 + 17x_2 \leq 14003x_1 + 9x_2 + 8x_3 \quad \leq 1000$$

$$10x_1 + 13x_2 + 15x_3 \leq 1750$$

$$6x_1 + 16x_3 \leq 1325$$

$$x_1, x_2, x_3 \geq 0$$

$$j = 1, 2, 3, ..., q \qquad i = 1, 2, ..., p + q.$$

Table 7.2 provides a comparison between different fuzzy optimization techniques used to attain optimality. As clear from this table a slight diversion from the objective f_2 improves both the objectives f_1 and f_3, when handle the problem with Interval valued neutrosophic optimization technique. The ultimate gap analysis provided in Table 7.3.

Table 7.2: Comparison of fuzzy optimization techniques for production planning

Optimization techniques	f_1^*	f_2^*	f_3^*
FO	6826.7920	10514.1757	8060.7275
IFO	7217.9710	10359.7261	8498.5925
IVFO	7545.54725	10521.7632	8427.2375
NSO	7674.13075	10722.5858	8509.59
IVNO	7856.42575	10261.3862	9247.7475

Table 7.3: Percentage gap of different fuzzy optimization techniques for production planning

Optimization Techniques	f_1^*	f_2^*	f_3^*	Cumulative % Gap
FO	15.08224873	1.982182017	14.72596611	31.79039686
IFO	8.845349337	3.502599359	8.815047904	21.1629966
IVFO	4.120025887	1.908640179	9.736405317	15.76507138
NSO	2.37544819	0	8.674419097	11.04986729
IVNO	0	4.494515566	0	4.494515566

In Figure 7.2 the comparison of values of the objective functions and cumulative percentage gap obtained from five different fuzzy optimization techniques are plotted. It is observed that the cumulative percentage gap of IVNSO is smaller than the other four values. Which indicates the efficiency of current technique over the existing in production and planning problems.

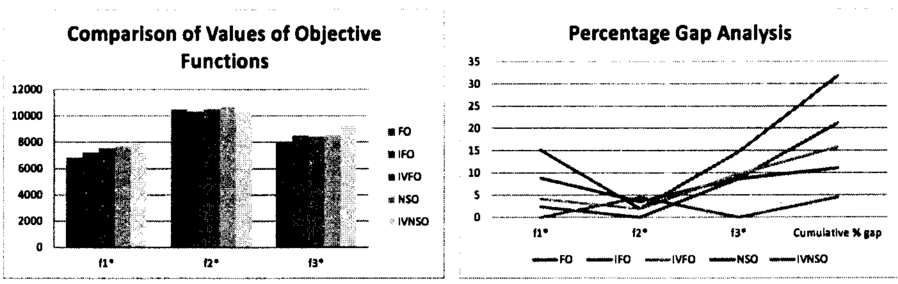

Figure 7.2: Production planning

7.3.2 Car-side Impact

Car-side impact problem was first discussed by Jain and Deb [135]. They have talked about the issue by characterizing transformative many-objective optimization approach utilizing non-overwhelmed categorizing approach based on reference point. This problem plans to limit the load of the vehicle, the pubic power experienced by a traveler and the normal speed of the V-Pillar liable to resist the effect load. These three goals are clashing; a few limitations counting as body load, complex design and development, pubic power and speed, and plan factors of vehicle transform it into a mind boggling optimization issue in engineering design. The numerical plan is given as:

$$\begin{aligned}
\text{Minimize } f_1(z) =& 1.98 + 4.9z_1 + 6.67z_2 + 6.98z_3 + 4.01z_4 + 1.78z_5 + \\
& 0.00001z_6 + 2.73z_7 \\
\text{Minimize } f_2(z) =& F \\
\text{Minimize } f_3(z) =& 0.5(V_{MBP} + V_{FD})
\end{aligned} \tag{7.3.2}$$

such that

$$\begin{aligned}
c_1(z) =& 1.16 - 0.3717z_2z_4 - 0.0092928z_3 \leq 1 \\
c_2(z) =& 0.261 - 0.0159z_1z_2 - 0.06486z_1 - 0.019z_2z_7 + 0.0144z_3z_5 + 0.0154464z_6 \\
& \leq 0.32 \\
c_3(z) =& 0.214 + 0.00817z_5 - 0.045195z_1 - 0.0135168z_1 + 0.03099z_2z_6 - 0.018z_2z_7 \\
& + 0.007176z_3 + 0.023232z_3 - 0.00364z_5z_6 - 0.018{z_2}^2 \leq 0.32 \\
c_4(z) =& 0.74 - 0.61z_2 - 0.031296z_3 - 0.013872z_7 + 0.227{z_2}^2 \leq 0.32 \\
c_5(z) =& 28.98 + 3.818z_3 - 4.2z_1z_2 + 1.27296z_6 - 2.68065z_7 \leq 32 \\
c_6(z) =& 33.86 + 2.95z_3 - 5.057z_1z_2 - 3.795z_2 - 3.4431z_7 + 1.45728 \leq 32 \\
c_7(z) =& 46.36 - 9.9z_2 - 4.4505z_1 \leq 32 \\
c_8(z) =& F = 4.72 - 0.5z_4 - 0.19z_2z_3 \leq 4 \\
c_9(z) =& V_{MBP} = 10.58 - 0.674z_1z_2 - 0.67275z_2 \leq 9.9 \\
c_{10}(z) =& V_{FD} = 16.45 - 0.489z_3z_7 - 0.843z_5z_6 \leq 15.7 \\
& 0.5 \leq z_1 \leq 1.5, \qquad 0.45 \leq z_2 \leq 1.35 \\
& 0.5 \leq z_3 \leq 1.5, \qquad 0.5 \leq z_4 \leq 1.5 \\
& 0.875 \leq z_5 \leq 2.625, \qquad 0.4 \leq z_6 \leq 1.2 \\
& 0.4 \leq z_7 \leq 1.2
\end{aligned}$$

where f_1, f_2, f_3 are objective functions; z_i's are decision variables and c_i's are constraints.

$$\begin{bmatrix} 23.5857 & 4.000 & 12.5065 \\ 39.9449 & 3.5853 & 11.4189 \\ 41.7890 & 3.7073 & 10.6106 \end{bmatrix}$$

Comparative study of different fuzzy optimization models are provided in below tables.

Table 7.4: Comparison of fuzzy optimization techniques for car-side impact

Optimization Techniques	f_1^*	f_2^*	f_3^*
Intuitionistics Fuzzy Optimization(IFO)	39.2892	4.0190	10.6106
Neutrosophic Optimization(NSO)	30.8193	4.1787	11.0095
Interval Valued Neutrosophic Optimization(IVNO)	30.1712	3.7841	11.4511

Table 7.5: Percentage gap of fuzzy optimization techniques for car-side impact

Optimization Techniques	f_1^*	f_2^*	f_3^*	Cumulative % gap
IFO	23.20739542	5.844737497	0	29.05213291
NSO	2.102903051	9.443128246	3.623234479	15.16926578
IVNO	0	0	7.33990621	7.33990621

The problem under consideration has three objectives that need to be minimized. From Table 7.4 it is clear that the values of the first two objectives (that are, the load of the vehicle and the pubic power experienced by a traveler) obtained from interval valued neutrosophic optimization method are minimum than those calculated from intuitionistics fuzzy and neutrosophic optimization

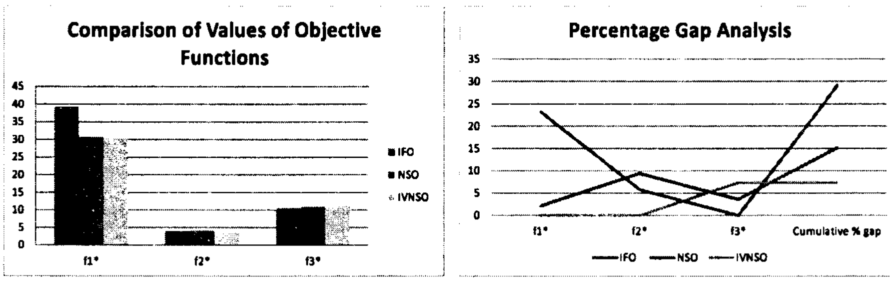

Figure 7.3: Car-side impact

models. Whereas, the value of third objective (that is, the normal speed of the V-Pillar) is slightly higher than the other two values. But the cumulative percentage gap analysis of the three techniques in Table 7.5 indicates that IVNSO is more effective to minimizes the three essential factors of car-side impact. The comparison of methodologies is elaborated in Figure 7.3.

7.3.3 Land Use Planning

Land use planning is probably the biggest public-sector problem as it involves more conflict and clash between various perspectives and interests. That is the reason a multi-target approach was opted when government authorities in DuPage County, Illinois, which is a fast developing rural territory close to Chicago, tried to build an arrangement controlling utilization of its undeveloped area [136]. No single paradigm completely catches the propriety of relegating undeveloped sections of land to a given use. Problem consist of five different objectives.

1. Compatibility: an indicator of the unity between each possible use in a zone and the existing uses in and around the sector.
2. Transportation: the time acquired in making trips generated by the land use to/from large scale passage and auto links.
3. Tax load: the proportion of extra annual running cost for government

work associated with the use versus high in the property tax assessment base.

4. Environmental impact: the relative deterioration of the climate due to inordinate and enormous the land use.

5. Facilities: the fixed, one time expense for hospitals, schools and other community facilities to reinforce the land use.

Mathematical model for these objectives is defined as:

$$Objective1 : \text{maximize} \sum_{k=1}^{M}\sum_{p=1}^{N} c_{k,p}x_{k,p}$$

$$Objective2 : \text{minimize} \left\{ \sum_{k=1}^{M}\sum_{p=1}^{N} t_{k,p}x_{k,p}, \sum_{k=1}^{M}\sum_{p=1}^{N} r_{k,p}x_{k,p}, \sum_{k=1}^{M}\sum_{p=1}^{N} e_{k,p}x_{k,p}, \sum_{k=1}^{M}\sum_{p=1}^{N} f_{k,p}x_{k,p} \right\}$$

subject to

$$\sum_{k=1}^{M}\sum_{p=1}^{N} x_{k,p} = b_p \qquad p = 1, 2, \ldots, N$$

$$\sum_{k=1}^{M}\sum_{p=1}^{N} x_{k,p} \geq l_k \qquad k = 1, 2, \ldots, M$$

$$\sum_{k=1}^{M}\sum_{p=1}^{N} x_{k,p} \leq u_k \qquad k = 1, 2, \ldots, M$$

$$x_{k,p} \geq O_p \qquad p = 1, 2, \ldots, N$$

$$x_{k,p} \geq s_k x_{1,p} + d_k x_{2,p} \qquad k = 3, 6, 7; \quad p = 1, 2, \ldots, N$$

$$x_{k,p} \geq o \qquad p = 1, 2, \ldots, N \quad k = 1, 2, \ldots, M.$$

Then the pay-off matrix is

$$\begin{bmatrix} 431006 & 133172 & 22550 & 17889.74 & 869694 \\ 406704.3537 & 127134 & 33547.78114 & 18920.53476 & 1543491.014 \\ 384138 & 131672 & 16986.00001 & 20202.14 & 1089398 \\ 405236.0005 & 136165.9999 & 31179.99949 & 17087.50002 & 1117869.994 \\ 422891.9996 & 133309.9999 & 23409.99999 & 17994.24003 & 820398.0011 \end{bmatrix}$$

The values of the objective functions are calculated by using current technique and the already existing interval valued intuitionistic fuzzy optimization(IVIFO). These values are presented in the following table.

Table 7.6: Comparison of fuzzy optimization techniques for land use planning

Optimization techniques	f_1^*	f_2^*	f_3^*	f_4^*	f_5^*
IVIFO	398450.62	131418.23	27000.30	18965.56	1234660.68
IVNO	402620.37	132053.59	20364.32	17963.07	1023282.38

In the given problem the compatibility need to be maximized, which is actually unity between each possible unit, while maintaining minimum possible values for transportation time, tax load, involvement of different environmental impact factors and controlling different expenses dedicated to provide miscellaneous facilities. Interval valued neutrosophic technique suggested that a slightly conciliation by authorities on an increase in transportation time end up yielding maximum compatibility and optimal figures for tax load, controlled environmental impacts and least expenses as compared to the other technique mention in the Table 7.6.

7.4 Conclusion

The search of an appropriate, advanced and effectual optimization model that dealt with uncertain real life problems in an accurate manner is the main rational behind the formulation of interval valued neutrosophic optimization model. In an interval valued fuzzy set the membership function is presented in the form of interval thus; make more degree of freedom for uncertainties and ultimately more wide uncertainty modelling can be provided. Membership/truth, non-membership/falsity and indeterminacy functions are the key components of a neutrosophic set that makes it more realistic. The combination of these two fuzzy generalizations result into conversion of membership/truth, non-membership/falsity and indeterminacy degrees in terms of intervals which influence the feasible region by proving more degree of freedom to incorporate imprecision. Main focus of current work is to cover such problems, where objectives are conflicting in nature and it is almost impossible to designate exactly a number to true, false and indeterminate grades of membership. The optimization method discussed here is based on the remodeling of the feasible region obtained by solving the ordinary optimization problem in crisp environment, the minimum and maximum values of the objective functions will be used to calculate upper and lower bounds for truth, falsity and indeterminacy grades, which are used for the formulation of membership functions that corresponds to interval valued neutrosophic objective function and constraints. The choice of lower the better for minimization and higher the better for maximization will effect the construction of membership function and in return the inequality sign in constraint but the interval valued neutrosophic objective will remain the same for both the cases. In application section we discussed the cases where objective functions need to be minimized(Car-side Impact), maximized(Production Planning) and both maximized and minimized(Land Use Planning). The comparison of different fuzzy optimization techniques are presented in Table 7.2, 7.4 and 7.6. The percentage gap analysis of different fuzzy optimization methods presented in Table 7.3 and 7.5 depicts that the cumulative percentage gap of interval valued neutrosophic optimization is smallest than any other method and this comparison is evident enough to guarantee the supremacy of newly established methodology.

Urban planning is a matter of great concern for developing countries with the objective to improve and uplift life style of population by providing all the essential facilities like affordable and attractive housing, education, hospitals, transportation, water and energy resources and healthy environment subject to funding and budget, land availability and geographical conditions. The defined model can help the urban planner in decision making by reducing the loss of information. In search of better trading routes growing world economies invested heavily on the construction of road and rail networks, infrastructure and maritime transportation. Maritime transport accounts for roughly 80% of international trade [139]. International trade depends on the social and economic stability of the exporters and importers. Maritime accidents due to climatic changes or human error like the recent Suez Canal incident cause huge economic damages. The uncertainties associated with transshipment makes the transportation problem fuzzy in nature. Maximize the profit and minimize the travelling cost and time are the major objectives subjected to demand, supply, routing, load management, ship capacity, fuel, labor, distance and time constraints. The newly established model can be applied to maritime transportation to achieve the said objectives. Interval valued neutrosophic optimization model is applicable not only in urban planning, engineering, transshipment but also in all those problems where decisions variables are large in number, the objectives are conflicting and subjected to various constraints.

Chapter 8

Bipolar Interval-valued Neutrosophic Optimization Model of Integrated Healthcare System

Bipolar Interval-valued neutrosophic set [137] is another generalization of fuzzy set, neutrosophic set, bipolar fuzzy set and bipolar neutrosophic set and thus when applied to the optimization problem handles uncertain data more efficiently and flexibly. Current work is an effort to design a flexible optimization model in the backdrop of interval-valued bipolar neutrosophic sets. Bipolar interval-valued neutrosophic membership grades are picked so that they indicate the restriction of the plausible infringement of the inequalities given in the problem. To prove the adequacy and effectiveness of the method a unified system of sustainable medical healthcare supply chain model with an uncertain figure of product complaints is used. Time, quality and cost are considered as satisfaction level to choose best supplier for medicine procurement. The proposed model ensures 99% satisfaction for cost reduction, 63% satisfaction for the quality of product and 64% satisfaction for total time taken in medicine supply chain.

8.1 Mathematical Modelling

The supply chain refers to the resources necessary to provide services and products to a wide variety of consumers. Strategic sourcing in healthcare is often a highly complicated and fragmented operation. This process and the allocation of resources become more complex if the available data contains fuzziness. Recent enriched technological and innovative, organizational and financial progressions in health-care organizations have given expanded admittance of dealing with patients [138]-[143]. According to a delineation by the World Bank, the world fatality rate has decreased to 7.75 per thousand persons in 2014 from 17.8 in 1960 [144]. Many dynamic factors, such as improved disease management, health-care facilities, and drug accessibility, have contributed to this decrease in the expiry rate. Regardless of these improvements, advances in the foundations of medical services, supply chain management and infrastructure are unavoidable. Thus, for patients' safety and recovery, access to the suitable medicine with the perfect and balanced formula for the exact patient at the exact time is needed.

Current work aims to construct and crack a medicine supply-chain model in an interval-valued bipolar neutrosophic environment. This supply chain model is designed for a healthcare system that is integrated and uncertain in terms of product complaints. In its most modest form, a supply chain is the set of activities needed by the association to provide services or goods to the buyers. Conventional medical health-care facilities are restricted to drug organizations, patients and hospitals. Prescribed studies comprise a unified medical care framework, which likewise incorporates the roles of public authorities and healthcare departments.

8.1.1 Problem Description

For the proposed study, a set of contractors: a health division and a network of clinics and hospitals have been considered. The network of clinics and hospitals needs to choose the most appropriate suppliers for the medicines based on the business triad of cost, time and quality.

Table 8.1: Abbreviation and notations

$\tilde{h}$	hospitals index/clinics in the proposed integrated framework	$\tilde{h} = 1, 2, \ldots, \tilde{H}$
$\tilde{m}$	reference for drugs	$\tilde{m} = 1, 2, \ldots, \tilde{M}$
$\tilde{d}$	reference for dealers	$\tilde{d} = 1, 2, \ldots, \tilde{D}$
$\tilde{t}$	reference for time period	$\tilde{t} = 1, 2, \ldots, \tilde{T}$
$\tilde{o}$	reference for objectives	$\tilde{o} = 1, 2, \ldots, \tilde{O}$

Table 8.2: Parameters

PC_{md}	price of medicine $\tilde{m}$ supplied by the dealer $\tilde{d}$($)
CL	cost of labor ($)
CE	cost of energy ($)
$CEH_{\tilde{m}}$	carbon emissions cost for handling the medicine $\tilde{m}$
$CEt_{\tilde{m}}$	carbon emissions tax for transportation of medicine $\tilde{m}$
CT	cost of transportation $(\frac{\$}{km})$
$\tilde{S}_{\tilde{d}\tilde{h}}$	distance between dealer $\tilde{d}$ and hospital $\tilde{h}$
$CQI_{\tilde{m}\tilde{d}\tilde{t}}$	cost of quality inspection of medicine $\tilde{m}$ supplied by dealer $\tilde{d}$ in time period $\tilde{t}$
$CQ_{\tilde{m}\tilde{d}}$	complaints regarding quality of medicine $\tilde{m}$ provided by dealer $\tilde{d}$ in the last year
$US_{\tilde{m}\tilde{d}}$	number of units sold in the last year of medicine $\tilde{m}$ by dealer $\tilde{d}$
$TM_{\tilde{m}\tilde{d}}$	production time of batch of drug $\tilde{m}$ by dealer $\tilde{d}$
$SB_{\tilde{m}\tilde{d}}$	size of batch of medicine $\tilde{m}$ by dealer $\tilde{d}$
AS	average speed $(\frac{km}{hr})$
$QIT_{\tilde{m}\tilde{d}\tilde{t}}$	inspection time of quality in health department of medicine $\tilde{m}$ supplied by dealer $\tilde{d}$ in time period $\tilde{t}$
$MC_{\tilde{d}\tilde{m}}$	manufacturing capacity of dealer $\tilde{d}$ for medicine$\tilde{m}$
$qal_{\tilde{m}\tilde{d}}$	quality acceptance limit of medicine $\tilde{m}$ manufactured by supplier $\tilde{d}$
QAL	standard acceptance limit of quality
$\tilde{\delta}$	fuzzy deviation variable

Table 8.3: Decision variable

$\tilde{Q}_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}}$	Quantity of medicine $\tilde{m}$ provided by dealer $\tilde{d}$ to hospital $\tilde{h}$ in time period $\tilde{t}$
$\tilde{M}_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}}$	$= \begin{cases} 1, & \text{if medicine } \tilde{m} \text{ is provided by dealer } \tilde{d} \text{ to hospital } \tilde{h} \text{ in time period } \tilde{t}; \\ 0, & \text{otherwise.} \end{cases}$

8.1.1.1 Model Assumption

1: The production limit of each dealer and price of specific medicine remains the same for the whole year.

2: In each period quality assurance cost remains the same.

3: The number of complaints regarding quality of product received by the manufacturer is uncertain.

4: The quality of units of medicine delivered over in the last year is known.

5: Labor cost remains the same throughout the year.

6: Energy cost, carbon emission cost for handling and carbon emission tax for transportation of medicines are known and fixed.

7: It is an integrated supply chain model in which network of hospitals bares the cost of transportation.

8.1.2 Objective Functions

The description of each objective function and constraints used to model the problem mathematically are as follow:

Cost of the medicine supply chain: There are numerous factors effecting cost of medicine supply chain. At various level of this network different types of costs are involved. However,this particular model is bound to the following costs.

$$\text{Minimize } F_{cost} = \sum_{\tilde{m}=1}^{\tilde{M}} \sum_{\tilde{d}=1}^{\tilde{D}} \sum_{\tilde{h}=1}^{\tilde{H}} \sum_{\tilde{t}=1}^{\tilde{T}} (PC_{\tilde{m}\tilde{d}} + CL + CE + CEH_{\tilde{m}}) X \tilde{Q}_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}}$$

$$\sum_{\tilde{m}=1}^{\tilde{M}} \sum_{\tilde{d}=1}^{\tilde{D}} \sum_{\tilde{h}=1}^{\tilde{H}} \sum_{\tilde{t}=1}^{\tilde{T}} ((CEt_{\tilde{m}} + CT) \times S_{\tilde{d}\tilde{h}} + CQI_{\tilde{m}\tilde{d}\tilde{t}}) \times M_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}.}$$

The first term in the above equation depicts the cost of production, labour, energy and carbon emission handling. Labour cost, carbon emission, handling cost and energy cost will remain the same throughout the specific period. The second term comprises of transportation constraints and carbon emissions for transportation, and the cost of quality inspection done by health care department.

Total time: Three types of times are involved in medicine supply chain. First is the manufacturing time, second is the quality inspection time by the health care authorities and lastly the transportation time.

$$\text{Minimize } F_{time} = \sum_{\tilde{m}=1}^{\tilde{M}} \sum_{\tilde{d}=1}^{\tilde{D}} \sum_{\tilde{h}=1}^{\tilde{H}} \sum_{\tilde{t}=1}^{\tilde{T}} (\frac{TM_{\tilde{m}\tilde{d}}}{SB_{\tilde{m}\tilde{d}}} \times \tilde{Q}_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}}) + \sum_{\tilde{m}=1}^{\tilde{M}} \sum_{\tilde{d}=1}^{\tilde{D}} \sum_{\tilde{h}=1}^{\tilde{H}}$$
$$\sum_{\tilde{t}=1}^{\tilde{T}} (\frac{S_{\tilde{d}\tilde{h}}}{AS} + QIT_{\tilde{m}\tilde{d}\tilde{t}}) \times M_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}}$$

Here, the first term is the time taken by the suppliers or dealers in the manufacturing and second term is the sum of transportation time and time taken by health care department for quality inspection.

Quality involved in medicine supply chain: There are also many parameters to measure the quality level in the medicine supply chain; however the prescribed model rely on the complaints launched by the customers to maintain the quality level of suppliers. Mathematically described below:

$$\text{Minimize } F_{quality} = \sum_{\tilde{m}=1}^{\tilde{M}} \sum_{\tilde{d}=1}^{\tilde{D}} \sum_{\tilde{h}=1}^{\tilde{H}} \sum_{\tilde{t}=1}^{\tilde{T}} (\frac{CQ_{\tilde{m}\tilde{d}}}{US_{\tilde{m}\tilde{m}}} \times (1 \text{ Millions })) \times M_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}}.$$

Since number of complaints by the consumers is taken as quality assurance parameter; therefore, quality function is also minimized in the model.These complaints aren't very likely to be true because the suppliers don't know for sure what goods have quality complaints. Thus, fuzzy theory is used to handle the uncertainty associated with the model.

The process to determine the deviation is the fuzzy variable associated with quality function and comprises of three steps. In the first step, any of the fuzzy membership grades like trapezoidal or triangular can be taken. Afterwards, in the second step, fuzzification technique of conversion of a crisp function into a fuzzy function will be applied and at the last step defuzzification process will be followed. Again for defuzzification any of the already developed techniques like centroid method,first or last maxims, weighted average method, signed-distance method, center of largest area, and many more. On defuzzified quality complaints function using signed-distance method is given below as;

$$(F_{quality})^{crip} = \sum_{\tilde{m}=1}^{\tilde{M}} \sum_{\tilde{d}=1}^{\tilde{D}} \sum_{\tilde{h}=1}^{\tilde{H}} \sum_{\tilde{t}=1}^{\tilde{T}} ((\frac{4cq_{\tilde{m}\tilde{d}+(\tilde{\delta}_2-\tilde{\delta}_1)}}{4US_{\tilde{m}\tilde{d}}} \times (1 \text{ Millions })) \times M_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}}.$$

Constraints: All the above stated objectives are subjected to the constraints given below:

Demand Constraints:

$$\sum_{\tilde{d}=1}^{\tilde{D}} Q_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}} = \tilde{D}_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}} \qquad \forall \tilde{m}\ ; \quad \forall \tilde{h}\ ; \quad \forall \tilde{t}$$

This constraints limit the purchase by putting sum of quantity of medicine provided by dealers equals to the demand generated by health care units.

Quality Constraints:

$$qal_{\tilde{m}\tilde{d}} \times M_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}} \leq QAL \qquad \forall \tilde{m}\ ; \quad \forall \tilde{d}\ ; \quad \forall \tilde{h}\ ; \quad \forall \tilde{t}$$

Quality constraints guarantees the standard acceptance quality limit.

Capacity Constraints

$$Q_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}} \leq MC_{\tilde{m}\tilde{d}} \times M_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}} \qquad \forall \tilde{m}\ ; \quad \forall \tilde{d}\ ; \quad \forall \tilde{h}\ ; \quad \forall \tilde{t}$$

Capacity constraint shows that the manufacturing capacity of dealers for any specific medicine should be greater than or equal to the quantity of medicine provided by dealer.

Non-negativity Constraints:

$$Q_{\tilde{m}\tilde{d}\tilde{h}\tilde{t}} \geq 0.$$

This constraint ensures that the quantity must always be positive.

Binary Variable:

$$M_{\tilde{m}\bar{d}\bar{h}\hat{t}} \in \{0,1\}$$

The value of binary variable is 1 when the quantity is released by the dealer and otherwise 0.

8.2 Proposed Approach

Multi-objective optimization problems occurred while dealing with optimization of non-commensurable, conflicting and multiple objective functions subject to certain conditions and circumstances. A general representation of a multi objective optimization problem with q constraints, p objectives and n decision variables is as follows.

$$\text{Minimize}\{\breve{f}_i(x)\} \quad i = 1, 2, \cdots, p, \tag{8.2.1}$$

such that,

$$\breve{g}_j(x) \leq b_j \quad j = 1, 2, \cdots, q.$$

Where "x" is the representation of decision variables, $\breve{g}_j(x)$ represents the constraints and $\breve{f}_j(x)$ are the objective function q and p are the number of constraints and objective functions respectively.

Problem 8.2.1 is reconsidered in a fuzzy as environment as

Find x such that, $\hat{Z}_k(x) \gtrsim \hat{g}_k, \quad k = 1, 2, \cdots, p$

and $\hat{g}_j(x) \geq \hat{b}_j, \quad j = 1, 2, \cdots, q \;\; x \geq 0.$

Where $\hat{g}_k$ denotes the goals and decision maker wants to minimize all objective functions, where as $\gtrsim$ is the bipolar interval-valued neutrosophic inequality.

To build up bipolar interval-valued membership functions of different target capacities we could initially acquire the table of positive solutions. Lower and upper bound of each membership grade is acquired by using this positive solution table.

Angelov [89] extended the idea to use Bellman and Zadeh's [51] Max-min operator to solve multi objective linear programming problem. The confluence $\breve{D}$ of bipolar interval-valued objectives and bipolar neutrosophic constraints is referred as bipolar interval-valued decision. Taking intersection is the main step of optimization in fuzzy environment. Main focus of this work is to develop a technique to simplify a multi objective optimization problem by means of bipolar interval-valued neutrosophic scheme.

$$\begin{aligned}\breve{D} &= (\bigcap_{k=1}^{p} \breve{O}_k) \bigcap (\bigcap_{j=1}^{q} \breve{C}_j) \\ &= \{(x, [T_{\breve{D}}^{L^+}, T_{\breve{D}}^{U^+}], [T_{\breve{D}}^{L^-}, T_{\breve{D}}^{U^-}], [I_{\breve{D}}^{L^+}, I_{\breve{D}}^{U^+}], [I_{\breve{D}}^{L^-}, I_{\breve{D}}^{U^-}], [F_{\breve{D}}^{L^+}, F_{\breve{D}}^{U^+}], [F_{\breve{D}}^{L^-}, F_{\breve{D}}^{U^-}]) \\ &: x \in X\}. \end{aligned} \tag{8.2.2}$$

Where,

$$\begin{aligned}
[T_{\breve{D}}^{L^+}, T_{\breve{D}}^{U^+}] &= [\min\{T_{\breve{O}_i}^{L^+}(x); T_{\breve{C}_j}^{L^+}(x)\}, \min\{T_{\breve{O}_i}^{U^+}(x); T_{\breve{C}_j}^{U^+}(x)\}] \\
[T_{\breve{D}}^{L^-}, T_{\breve{D}}^{U^-}] &= [\max\{T_{\breve{O}_i}^{L^-}(x); T_{\breve{C}_j}^{L^-}(x)\}, \max\{T_{\breve{O}_i}^{U^-}(x); T_{\breve{C}_j}^{U^-}(x)\}] \\
[I_{\breve{D}}^{L^+}, I_{\breve{D}}^{U^+}] &= [\min\{I_{\breve{O}_i}^{L^+}(x); I_{\breve{C}_j}^{L^+}(x)\}, \min\{I_{\breve{O}_i}^{U^+}(x); I_{\breve{C}_j}^{U^+}(x)\}] \\
[I_{\breve{D}}^{L^-}, I_{\breve{D}}^{U^-}] &= [\max\{I_{\breve{O}_i}^{L^-}(x); I_{\breve{C}_j}^{L^-}(x)\}, \max\{I_{\breve{O}_i}^{U^-}(x); I_{\breve{C}_j}^{U^-}(x)\}] \\
[F_{\breve{D}}^{L^+}, F_{\breve{D}}^{U^+}] &= [\max\{F_{\breve{O}_i}^{L^+}(x); F_{\breve{C}_j}^{L^+}(x)\}, \max\{F_{\breve{O}_i}^{U^+}(x); F_{\breve{C}_j}^{U^+}(x)\}] \\
[F_{\breve{D}}^{L^-}, F_{\breve{D}}^{U^-}] &= [\min\{F_{\breve{O}_i}^{L^-}(x); F_{\breve{C}_j}^{L^-}(x)\}, \min\{F_{\breve{O}_i}^{U^-}(x); F_{\breve{C}_j}^{U^-}(x)\}].
\end{aligned}$$

Where $T^{L^+}, T^{U^+}, I^{L^+}, I^{U^+}, F^{L^+}, F^{U^+} : X \to [0,1]$ and $T^{L^-}, T^{U^-}, I^{L^-}, I^{U^-}, F^{L^-}, F^{U^-} : X \to [-1, 0]$ are membership grades associated with bipolar interval-valued decision and $1 \le i \le p,\ 1 \le j \le q$.

Transformed bipolar interval-valued neutrosophic optimization problem is given by;

$$\text{Maximize } \theta a^{+} + (1-\theta)b^{+}$$

$$\text{Maximize } \theta e^{+} + (1-\theta)f^{+}$$

$$\text{Maximize } \theta c^{-} + (1-\theta)d^{-}$$

$$\text{Minimize } \theta a^{-} + (1-\theta)b^{-}$$

$$\text{Minimize } \theta e^{-} + (1-\theta)f^{-}$$

$$\text{Minimize } \theta c^{+} + (1-\theta)d^{+}$$

Such that

$$T_{\breve{D}}^{U^+}(x) \geq \theta a^{+} + (1-\theta)b^{+}$$

$$T_{\breve{D}}^{L^+}(x) \geq a^{+}$$

$$T_{\breve{D}}^{U^-}(x) \leq \theta a^{-} + (1-\theta)b^{-}$$

$$T_{\breve{D}}^{L^-}(x) \leq a^{-}$$

$$I_{\breve{D}}^{U^+}(x) \geq \theta e^{+} + (1-\theta)f^{+}$$

$$I_{\breve{D}}^{L^+}(x) \geq e^{+}$$

$$I_{\breve{D}}^{U^-}(x) \leq \theta e^{-} + (1-\theta)f^{-}$$

$$I_{\breve{D}}^{L^-}(x) \leq e^{-}$$

$$F_{\breve{D}}^{U^+}(x) \leq \theta c^{+} + (1-\theta)d^{+}$$

$$F_{\breve{D}}^{L^+}(x) \leq c^{+}$$

$$F_{\breve{D}}^{U^-}(x) \geq \theta c^{-} + (1-\theta)d^{-}$$

$$F_{\breve{D}}^{L^-}(x) \geq c^{-}$$

with $a^{+}, b^{+}, c^{+}, d^{+}, e^{+}, f^{+} \in [0,1]$ and $a^{-}, b^{-}, c^{-}, d^{-}, e^{-}, f^{-} \in [-1,0]$.

$$a^{+} + b^{+} + c^{+} + d^{+} + e^{+} + f^{+} + a^{-} + b^{-} + c^{-} + d^{-} + e^{-} + f^{-} \leq 3$$

Also $a^{+} \geq c^{+}$; $a^{+} \geq e^{+}$; $b^{+} \geq d^{+}$; $b^{+} \geq f^{+}$; $a^{-} \leq c^{-}$; $a^{-} \leq e^{-}$; $b^{-} \leq d^{-}$; $b^{-} \leq f^{-}$;

where $0 \leq \theta \leq 1$ $0 \leq \eta \leq 1$

$$\breve{g}_j(x) \leq b_j \qquad j = 1, 2, \cdots, q.$$

8.2.1 Solution Methodology

In this model, bipolar interval-valued neutrosophic optimization technique is employed to solve multi-objective and multi-period problem. Following steps are followed to solve prescribed model.

8.2.1.1 Step 1

Table of positive solutions is attained by solving each objective function from set of k objectives as a single objective (subject to given set of constraints).

8.2.1.2 Step 2

Lower and upper bound of every objective function is attained by using below given relations

$$\begin{aligned}
U_p^{T+} &= \max\{f_p(x*)\} \quad ; L_p^{T+} = \min\{f_p(x^8)\} \\
U_p^{T^-} &= \min\{f_p(x*)\} \quad ; L_p^{T^-} = \max\{f_p(x^8)\} \\
U_p^{F^+} &= U_p^{T^+} \quad ; L_p^{F^+} = L_p^{T^+} + t(U_p^{T^+} - L_p^{T^+}) \\
U_p^{F^-} &= U_p^{T^-} \quad ; L_p^{F^+} = L_p^{T^-} + t(U_p^{T^-} - L_p^{T^-}) \\
L_p^{I^+} &= L_p^{T^+} \quad ; U_p^{I^+} = L_p^{T^+} + s(U_p^{T^+} - L_p^{T^+}) \\
L_p^{I^-} &= L_p^{T^-} \quad ; U_p^{I^-} = L_p^{T^-} + s(U_p^{T^-} - L_p^{T^-}).
\end{aligned}$$

Where $s, t \in (0, 1)$ for all p.

8.2.1.3 Step 3

Possible membership grades for bipolar interval valued neutrosophic linear programming problem is constructed as below.

$$T_p^{\breve{U}^+}(\breve{f}_p(x)) = \begin{cases} 1, & \breve{f}_p(x) \le \breve{L}_p^{T^+}; \\ \frac{\breve{U}_p^{T^+} - \breve{f}_p(x)}{\breve{U}_p^{T^+} - \breve{L}_p^{T^+}}, & \breve{L}_p^{T^+} \le \breve{f}_p(x) \le \breve{U}_p^{T^+}; \\ 0, & \breve{f}_p(x) \ge \breve{U}_p^{T^+}. \end{cases}$$

$$T_p^{\breve{L}^+}(\breve{f}_p(x)) = \begin{cases} 1, & \breve{f}_p(x) \leq \breve{L}_p^{T^+}; \\ \eta_p \frac{\breve{U}_p^{T^+} - \breve{f}_p(x)}{\breve{U}_p^{T^+} - \breve{L}_p^{T^+}}, & \breve{L}_p^{T^+} \leq \breve{f}_p(x) \leq \breve{U}_p^{T^+}; \\ 0, & \breve{f}_p(x) \geq \breve{U}_p^{T^+}. \end{cases}$$

$$T_p^{\breve{U}^-}(\breve{f}_p(x)) = \begin{cases} -1, & \breve{f}_p(x) \geq \breve{L}_p^{T^-}; \\ \frac{\breve{f}_p(x) - \breve{U}_p^{T^-}}{\breve{L}_p^{T^-} - \breve{U}_p^{T^-}}, & \breve{U}_p^{T^-} \leq \breve{f}_p(x) \leq \breve{L}_p^{T^-}; \\ 0, & \breve{f}_p(x) \leq \breve{U}_p^{T^-}. \end{cases}$$

$$T_p^{\breve{L}^-}(\breve{f}_p(x)) = \begin{cases} -1, & \breve{f}_p(x) \geq \breve{L}_p^{T^-}; \\ \eta_p \frac{\breve{f}_p(x) - \breve{U}_p^{T^-}}{\breve{L}_p^{T^-} - \breve{U}_p^{T^-}}, & \breve{U}_p^{T^-} \leq \breve{f}_p(x) \leq \breve{L}_p^{T^-}; \\ 0, & \breve{f}_p(x) \leq \breve{U}_p^{T^-}. \end{cases}$$

$$I_p^{\breve{U}^+}(\breve{f}_p(x)) = \begin{cases} 1, & \breve{f}_p(x) \leq \breve{L}_p^{T^+}; \\ \frac{\breve{U}_p^{T^+} - \breve{f}_p(x)}{\breve{U}_p^{T^+} - \breve{L}_p^{T^+}}, & \breve{L}_p^{T^+} \leq \breve{f}_p(x) \leq \breve{U}_p^{T^+}; \\ 0, & \breve{f}_p(x) \geq \breve{U}_p^{T^+}. \end{cases}$$

$$I_p^{\breve{L}^+}(\breve{f}_p(x)) = \begin{cases} 1, & \breve{f}_p(x) \leq \breve{L}_p^{T^+}; \\ \eta_p \frac{\breve{U}_p^{T^+} - \breve{f}_p(x)}{\breve{U}_p^{T^+} - \breve{L}_p^{T^+}}, & \breve{L}_p^{T^+} \leq \breve{f}_p(x) \leq \breve{U}_p^{T^+}; \\ 0, & \breve{f}_p(x) \geq \breve{U}_p^{T^+}. \end{cases}$$

$$I_p^{\breve{U}^-}(\breve{f}_p(x)) = \begin{cases} -1, & \breve{f}_p(x) \geq \breve{L}_p^{I^-}; \\ \frac{\breve{f}_p(x) - \breve{U}_p^{I^-}}{\breve{L}_p^{I^-} - \breve{U}_p^{I^-}}, & \breve{U}_p^{I^-} \leq \breve{f}_p(x) \leq \breve{L}_p^{I^-}; \\ 0, & \breve{f}_p(x) \leq \breve{U}_p^{T^-}. \end{cases}$$

$$I_p^{\breve{L}^-}(\breve{f}_p(x)) = \begin{cases} -1, & \breve{f}_p(x) \geq \breve{L}_p^{I^-}; \\ \eta_p \frac{\breve{f}_p(x) - \breve{U}_p^{I^-}}{\breve{L}_p^{I^-} - \breve{U}_p^{I^-}}, & \breve{U}_p^{I^-} \leq \breve{f}_p(x) \leq \breve{L}_p^{I^-}; \\ 0, & \breve{f}_p(x) \leq \breve{U}_p^{T^-}. \end{cases}$$

$$F_p^{\breve{U}^+}(\breve{f}_p(x)) = \begin{cases} 0, & \breve{f}_p(x) \leq \breve{L}_p^{F^+}; \\ \frac{\breve{f}_p(x) - \breve{U}_p^{F^+}}{\breve{U}_p^{F^+} - \breve{L}_p^{F^+}}, & \breve{L}_p^{F^+} \leq \breve{f}_p(x) \leq \breve{U}_p^{F^+}; \\ 1, & \breve{f}_p(x) \geq \breve{U}_p^{F^+}. \end{cases}$$

$$F_p^{\check{L}^+}(\check{f}_p(x)) = \begin{cases} 0, & \check{f}_p(x) \leq \check{L}_p^{F^+}; \\ \eta_p \frac{\check{f}_p(x) - \check{L}_p^{F^+}}{\check{U}_p^{F^+} - \check{L}_p^{F^+}}, & \check{L}_p^{F^+} \leq \check{f}_p(x) \leq \check{U}_p^{F^+}; \\ 1, & \check{f}_p(x) \geq \check{U}_p^{T^+}. \end{cases}$$

$$F_p^{\check{U}^-}(\check{f}_p(x)) = \begin{cases} 0, & \check{f}_p(x) \geq \check{L}_p^{F^-}; \\ \frac{\check{L}_p^{F^-} - \check{f}_p(x)}{\check{L}_p^{F^-} - \check{U}_p^{F^-}}, & \check{U}_p^{F^-} \leq \check{f}_p(x) \leq \check{L}_p^{F^-}; \\ -1, & \check{f}_p(x) \leq \check{U}_p^{F^-}. \end{cases}$$

$$F_p^{\check{L}^-}(\check{f}_p(x)) = \begin{cases} 0, & \check{f}_p(x) \geq \check{L}_p^{F^-}; \\ \eta_p \frac{\check{L}_p^{T^-} - \check{f}_p(x)}{\check{L}_p^{F^-} - \check{U}_p^{F^-}}, & \check{U}_p^{F^-} \leq \check{f}_p(x) \leq \check{L}_p^{F^-}; \\ -1, & \check{f}_p(x) \leq \check{U}_p^{F^-}. \end{cases}$$

Where $0 \leq \eta_p \leq 1$.

8.2.1.4 Step 4

Transformed bipolar interval-valued neutrosophic optimization problem is given by Maximize $\theta a^+ + (1-\theta)b^+ + e^+ + (1-\theta)f^+ + c^- + (1-\theta)d^- - a^- + (1-\theta)b^- - e^- + (1-\theta)f^- - c^+ + (1-\theta)d^+$, Such that

$$T_{\check{D}}^{U^+}(x) \geq \theta a^+ + (1-\theta)b^+$$
$$T_{\check{D}}^{L^+}(x) \geq a^+$$
$$T_{\check{D}}^{U^-}(x) \leq \theta a^- + (1-\theta)b^-$$
$$T_{\check{D}}^{L^-}(x) \leq a^-$$
$$I_{\check{D}}^{U^+}(x) \geq \theta e^+ + (1-\theta)f^+$$
$$I_{\check{D}}^{L^+}(x) \geq e^+$$
$$I_{\check{D}}^{U^-}(x) \leq \theta e^- + (1-\theta)f^-$$
$$I_{\check{D}}^{L^-}(x) \leq e^-$$
$$F_{\check{D}}^{U^+}(x) \leq \theta c^+ + (1-\theta)d^+$$
$$F_{\check{D}}^{L^+}(x) \leq c^+$$
$$F_{\check{D}}^{U^-}(x) \geq \theta c^- + (1-\theta)d^-$$
$$F_{\check{D}}^{L^-}(x) \geq c^-$$

with $a^+, b^+, c^+, d^+, e^+, f^+ \in [0,1]$ and $a^-, b^-, c^-, d^-, e^-, f^- \in [-1,0]$.

$$a^+ + b^+ + c^+ + d^+ + e^+ + f^+ + a^- + b^- + c^- + d^- + e^- + f^- \leq 3$$

Also $a^+ \geq c^+$; $a^+ \geq e^+$; $b^+ \geq d^+$; $b^+ \geq f^+$; $a^- \leq c^-$; $a^- \leq e^-$; $b^- \leq d^-$; $b^- \leq f^-$;

where $0 \leq \theta \leq 1$ $0 \leq \eta \leq 1$

$$\breve{g}_j(x) \leq b_j \qquad j = 1, 2, \cdots, q.$$

Figure 8.1 graphically depicts the complete solution methodology.

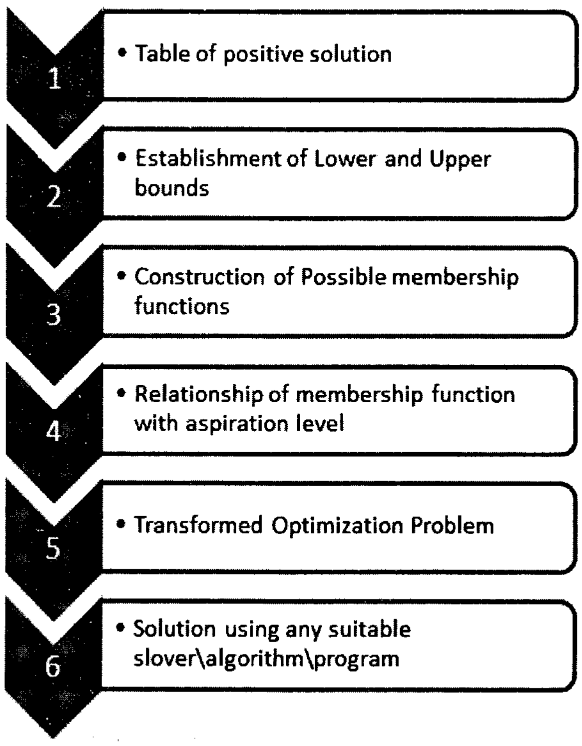

Figure 8.1: Flow chart of proposed approach

8.3 Application and Computational Results

This part portrays the use and efficacy of the prescribed model and approach. The concept and technique of finding a solution are further shown and validated using a real-world integrated medical supply chain application.

Numerical Example: To tackle the prescribed model, we assume an example that is just hypothetical. Three medical healthcare units are situated in the central city and are responsible for providing medical care facilities to the nearby regions and surrounding areas (see Figure 8.2). A large number of patients visit these healthcare units daily, but randomly. As these patients come with diverse complaints and different diseases, therefore, hospitals need to buy various types of medicines on a regular basis. This particular example necessitates the purchase of medicines for a three-month period, and hospitals are eager to purchase four medications. Within a 100-kilometer radius, four different dealers can be found. The procurement division of the healthcare department is in charge of purchasing medicines. Swift delivery, high quality and low cost are the parameters set by healthcare systems in the evaluation of the best supplier. It is decided to seek help from the healthcare department, which has a system of conducting external audits for the evaluation of medicines based on quality, expenditure and delivery time.

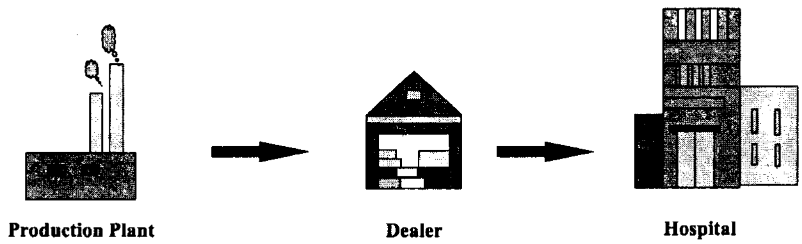

Figure 8.2: Medical healthcare supply chain network

Hospital administration buys those medicines approved by the healthcare department, which means that the hospitals require verification and approval from the department to purchase medicines.

Table 8.4: Demand of buyers in each period (kgs)

Time period	hospitals	Medicines			
		m=1	m=2	m=3	m=4
t=1	h=1	18146	15648	15467	12786
	h=2	17423	12796	13454	13899
	h=3	12378	17584	13477	16581
t=2	h=1	18569	17852	18526	16545
	h=2	15488	18524	13692	12332
	h=3	14588	17650	17414	14520
t=3	h=1	16540	13214	16589	12542
	h=2	15465	12111	19545	18541
	h=3	15647	16587	12548	16298

Table 8.5: Production capacity of suppliers (kgs)

	$\check{d} = 1$	$\check{d} = 2$	$\check{d} = 3$	$\check{d} = 4$
$\check{m} = 1$	13255	23852	13217	11748
$\check{m} = 2$	17852	22155	16257	14855
$\check{m} = 3$	15425	12352	18542	23659
$\check{m} = 4$	18564	12511	15212	12144

Table 8.6: Production time of suppliers (hrs/batch)

	$\check{d} = 1$	$\check{d} = 2$	$\check{d} = 3$	$\check{d} = 4$
$\check{m} = 1$	8	6	7	8
$\check{m} = 2$	9	7	10	8
$\check{m} = 3$	9	4	8	7
$\check{m} = 4$	8	7	9	7

Table 8.7: Batch size of suppliers (kgs)

	$\check{d}=1$	$\check{d}=2$	$\check{d}=3$	$\check{d}=4$
$\check{m}=1$	1124	960	1277	783
$\check{m}=2$	859	1844	652	1233
$\check{m}=3$	759	1247	1127	1025
$\check{m}=4$	751	1050	1241	659

Table 8.8: Product price

	$\check{d}=1$	$\check{d}=2$	$\check{d}=3$	$\check{d}=4$
$\check{m}=1$	5	4	3	4
$\check{m}=2$	4	6	3	4
$\check{m}=3$	6	4	5	3
$\check{m}=4$	4	4	3	2

Table 8.9: Distance between suppliers and buyers

	$\check{d}=1$	$\check{d}=2$	$\check{d}=3$	$\check{d}=4$
$\check{h}=1$	82	78	51	79
$\check{h}=2$	74	64	65	97
$\check{h}=3$	65	45	74	52

Table 8.10: Number of units sold in last year

	$\check{d}=1$	$\check{d}=2$	$\check{d}=3$	$\check{d}=4$
$\check{m}=1$	125632	384521	525415	175421
$\check{m}=2$	221217	136516	185421	186544
$\check{m}=3$	194529	146520	201452	134527
$\check{m}=4$	356426	125243	375423	215246

Table 8.11: Number of complaints regarding quality per million kg of products sold in last year

	$\breve{d}=1$	$\breve{d}=2$	$\breve{d}=3$	$\breve{d}=4$
$\breve{m}=1$	124	230	312	393
$\breve{m}=2$	246	360	386	161
$\breve{m}=3$	119	131	185	361
$\breve{m}=4$	371	346	257	121

Table 8.12: AQL level of suppliers

	$\breve{d}=1$	$\breve{d}=2$	$\breve{d}=3$	$\breve{d}=4$
$\breve{m}=1$	3	3.5	1.2	2.3
$\breve{m}=2$	2.5	1.8	3.9	4.1
$\breve{m}=3$	2.5	3.2	1.6	4.4
$\breve{m}=4$	1.6	3.4	4	2.8

Expected demand by the hospitals for each medicine for the coming three months is provided in Table 8.4 whereas capacity, production time and batch size of each month of each dealer for every medicine is given in Table 8.5, 8.6, and Table 8.7. Whereas the production price and number of units sold in the last year are enlisted in Table 8.8, and Table 8.10. Table 8.9 exhibits the distance among the dealers and health care units. Quota of each medicine sold by each dealer in the past one year and customers complaints are evaluated by using the data for past year for each dealer is presented in Table 8.12, and Table 8.11.

Standard acceptance level for this particular model is 3.4. Inspection cost paid by the hospitals for external audits is assumed to be 6$ per hour and inspection time fixed for each audit is 3 hrs. Energy cost is presumed to be .004$ per kg, whereas labour cost is fixed as 0.003$. Transportation cost is decided to be 0.004\$ per km and average speed of the vehicles for transportation is 100 km/hr. Cost of handling carbon emission and carbon emission tax for transportation are 0.002\$ per kg and 0.06\$ per km respectively.

8.4 Solution and Analysis Based on Numerical Results

Bipolar interval-valued neutrosophic optimization technique is applied to solve the numerical example. Step by step solution procedures are discussed in Section 8.2 Payoff table is constructed as per the procedure discussed in the same section.

$$POF = \begin{bmatrix} 5979360 & 87953.31 & 4521.66 \\ 6587.59 & 47085.01 & 5049.49 \\ 5106.57 & 89361.07 & 3140.66 \end{bmatrix}$$

The proposed technique includes the conversion of multi-objective problems to single objectives. Step 2 is applied to determine the lower and upper boundaries of each objective from the payoff matrix. Step 3 is followed to construct whole sets of membership grades. And lastly, the optimality of each objective function is attained by following the procedure explained in step 4. The numerical findings were made on a personal computer with 4 GB RAM and a 2.50GHz processor using MATLAB (R2017a).

Table 8.13: Levels of satisfaction and function values.

Objective function	Satisfaction Level w.r.t proposed approach	Level of Satisfaction w.r.t fuzzy optimization	Objective function value
Cost	99%	85%	5427
Quality	64%	58%	3797
Time	64%	62%	62789.92

Table 8.13 shows the optimal function and the level of satisfaction attained for each objective. The level of satisfaction scales the relative rate of divergence of attained function values from the function values at higher and lower bounds. Closer the optimum value to the lower bound higher will be the satisfaction level. It is merely possible to achieve 100% satisfaction due to the dependency of lower and upper bounds on other objectives and constraints. Table 8.13 shows a 99% satisfaction is achieved for cost, 63% for quality and 64% for time. Percentage and priority of these goals and objectives differs from corporation to corporation. In setting the goals top level administration plays a significant role. For the given model, with conflicting objectives, bipolar interval-valued neutrosophic optimization still provides good results and better level of satisfactions are achieved.

8.4.1 Decision Variables

Table 8.14 and Table 8.15 shows the values of decision variables in each period with respect to each medicine and supplier. Values of binary variable provided in Table 8.14 clearly show the most suitable supplier for specific medicine in some particular time period to each hospital. Acquiring the specified objectives while satisfying the conflicting constraint actually proves the success of any optimization method. Demand constraint is the primary constraint in the given model. From Table 8.4 the request of medicine m=2 from the hospital h=2 in time span t=3 is "12111" and comparing this value with Table 8.15 shows that medicine m=2 clearly meets the demand. Additionally the procurement department can easily find out the most suitable supplier for medicine purchase. In a similar manner demand constraint of all three time periods can be verified and best supplier can be selected.

Table 8.14: Value of binary alternatives

		$\check{t}$=1			$\check{t}=2$			$\check{t}=3$		
		$\check{h}=1$	$\check{h}=2$	$\check{h}=3$	$\check{h}=1$	$\check{h}=2$	$\check{h}=3$	$\check{h}=1$	$\check{h}=2$	$\check{h}=3$
$\check{m}=1$	$\check{d}=1$	1	1	0	1	1	1	1	1	1
	$\check{d}=2$	0	0	0	0	0	0	0	0	0
	$\check{d}=3$	1	1	1	1	1	1	1	1	1
	$\check{d}=4$	0	0	0	0	0	0	0	0	0
$\check{m}=2$	$\check{d}=1$	0	0	0	0	0	0	0	1	0
	$\check{d}=2$	1	1	1	1	1	1	1	0	1
	$\check{d}=3$	0	0	0	0	0	0	0	0	0
	$\check{d}=4$	0	0	0	0	0	0	0	0	0
$\check{m}=3$	$\check{d}=1$	0	1	1	0	1	0	0	0	1
	$\check{d}=2$	0	1	1	0	1	1	0	1	1
	$\check{d}=3$	1	0	0	1	0	1	1	1	0
	$\check{d}=4$	0	0	0	0	0	0	0	0	0
$\check{m}=4$	$\check{d}=1$	1	1	1	1	1	1	1	1	1
	$\check{d}=2$	0	0	0	0	0	0	0	0	0
	$\check{d}=3$	0	0	0	0	0	0	0	0	0
	$\check{d}=4$	0	0	0	0	0	0	0	0	0

Table 8.15: Amount of medicines provided by the suitable dealer in every time span

		$\check{t}$=1			$\check{t}=2$			$\check{t}=3$		
		$\check{h}=1$	$\check{h}=2$	$\check{h}=3$	$\check{h}=1$	$\check{h}=2$	$\check{h}=3$	$\check{h}=1$	$\check{h}=2$	$\check{h}=3$
$\check{m}=1$	$\check{d}=1$	4929	4206	0	5352	2271	1371	3323	2248	2430
	$\check{d}=2$	0	0	0	0	0	0	0	0	0
	$\check{d}=3$	13217	13217	12378	13217	13217	13217	13217	13217	13217
	$\check{d}=4$	0	0	0	0	0	0	0	0	0
$\check{m}=2$	$\check{d}=1$	0	0	0	0	0	0	0	12111	0
	$\check{d}=2$	15648	12796	17584	17852	18524	17650	13214	0	16587
	$\check{d}=3$	0	0	0	0	0	0	0	0	0
	$\check{d}=4$	0	0	0	0	0	0	0	0	0
$\check{m}=3$	$\check{d}=1$	0	1102	1125	0	1340	0	0	0	196
	$\check{d}=2$	0	12352	12352	0	12352	12352	0	12352	12352
	$\check{d}=3$	15467	0	0	18526	0	5062	16589	7193	0
	$\check{d}=4$	0	0	0	0	0	0	0	0	0
$\check{m}=4$	$\check{d}=1$	12786	13899	16581	16545	12332	14520	12542	18541	16298
	$\check{d}=2$	0	0	0	0	0	0	0	0	0
	$\check{d}=3$	0	0	0	0	0	0	0	0	0
	$\check{d}=4$	0	0	0	0	0	0	0	0	0

8.5 Conclusion

Selection of the most appropriate supplier in the procurement process of a healthcare system is always very challenging. Current work is an effort to model an integrated multi-objective and multi-period medicine supply chain problem to evaluate suppliers on the account of less number of quality complaints against per million units release of medicine, production, transportation and external audit cost and time taken for manufacturing and logistics. Current model is also taking care of the environmental aspect and effect of structured network. To minimize the effects of greenhouse gases a carbon emission cost and carbon emission tax for transportation of medicines are also infused in the prescribed model. To ensure the maintenance of quality this model also incorporates QAL constraint and significant input of health care department. On time delivery is also assured in the given model by minimizing manufacturing time, delivery time and quality inspection time. For illustration purpose a numerical model of three health care units, four dealers, and four medicines for three months of planning time period is considered. Bipolar interval-valued neutrosophic optimization is employed to attain optimality of the proposed model and the results achieved are quite satisfactory. Proposed approach not only convert this complicated multi-objective model into single objective but also provided 99% satisfaction for cost reduction, 63% satisfaction for the quality of product and 64% satisfaction for total time taken. To purchase different medicines procurement department can use the data of decision variables to select best dealer for each hospital. Future line of business may involve the purchase of life saving medical equipment from national and international markets, making it a global supply chain model. Since the environmental impact is also taken into account in the proposed model so in future it can be modified and upgraded for the management of any complex sustainable supply chain network.

Bibliography

[1] M. Sakawa, *"Genetic Algorithms and Fuzzy Multi-objective Optimization"*, Springer Science+Business Media, Inc., New York, 2002.

[2] R. V. L. Hartley, *Transmission of information*, Bell System Technical Journal. **7** (1928), no. 3, 535–563.

[3] C. E. Shannon, *The mathematical theory of communication*, Mobile Computing and Communications Review. **1** (2001), no. 5, 3–55.

[4] M. Black, *Vagueness: an exercise in logical analysis*, Philosophy of Science. **4** (1937), no. 4, 427–455.

[5] L. Zadeh, *Fuzzy sets*, Information and Control. **8** (1965), no. 3, 338–353.

[6] A. P. Dempster, *Upper and lower probabilities induced by a multivalued mapping*, Annals of Mathematical Statistics. **2** (1967), no. 38, 325–339.

[7] T. J.Ross, *"Fuzzy Logic With Engineering Application"*, Jhon Wiley and Sons., Hoboken, New Jersey, U.S., 2005.

[8] O. Castillo, P. Melin, J. Kacprzyk, and W. Pedrycz, *Type-2 fuzzy logic: theory and applications*, in IEEE international conference on granular, Fremont, CA, USA. (2007), 145–145.

[9] G. J. Klir and B. Yuan, *Fuzzy sets and fuzzy logic: theory and applications.*, Possibility Theory versus Probability Theory. **32** (1996), no. 2, 207–208.

[10] J. M. Mendel, *"Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions"*, Prentice-Hall, Upper-Saddle River, NJ., United States, 2001.

[11] J. M. Mendel, *"Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions"*, Prentice-Hall, Upper-Saddle River, NJ., United States, 2001.

[12] W. Pedrycz and F. Gomide, *"An Introduction to Fuzzy Sets: Analysis and Design"*, The MIT Press, MA., Massachusetts, US, 1998.

[13] D. Dubois and H. Prade, *"Fuzzy Sets and Systems: Theory and Applications"*, Academic Press, NY., USA, 1980.

[14] K. H. Lee, *"First Course on Fuzzy Theory and Applications, Advances in Soft Computing"*, Springer, Verlag, Berlin., Heidelberg, 2005.

[15] H. J. Zimmermann and D. Dubois, *"Fuzzy Set Theory and Its Applications"*, Academic Publishers, kluwer., Boston, 1991.

[16] V. Radu, A. I. Tabirca, F. Radu, S. I. Saplacan, and R. Lile, *Bibliometric analysis of fuzzy logic research in international scientific databases*, International Journal of Computers, Communications and Control. **16** (2021), no. 1.

[17] V. Torra, *Hesitant fuzzy sets*, International Journal of Intelligent Systems **25** (2010), no. 3, 529–539.

[18] F. Smarandache, *Neutrosophic set-a generalization of the intuitionistic fuzzy set*, Journal of Defense Resources Management, JoDRM **1** (2010), no. 1, 107–116.

[19] K.T. Atanassove, *Intuitionistic fuzzy ets*, Fuzzy Sets and System. **20** (1986), no. 4, 87–96.

[20] K.T. Atanassove, *More on intuitionistic fuzzy Set*, Fuzzy Sets and System. **33** (1989), no. 1, 37–46.

[21] K.T. Atanassove and G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and System. **31** (1989), no. 3, 343–349.

[22] K.T. Atanassove, *Operators over interval valued intuitionistic fuzzy sets*, Fuzzy Sets and System. **64** (1994), no. 2, 159–174.

[23] S. K. De, R. Biswas, and A. R. Roy, *An application of intuitionistic fuzzy sets in medical diagnosis*, Fuzzy Sets and System. **117** (2001), no. 2, 209–213.

[24] E. Szmidt and J. Kacprzyk, *Distances between intuitionistic fuzzy sets*, Fuzzy Sets and System. **114** (2000), no. 3, 505–518.

[25] F. Smarandache, *"A Unifying Field in Logics: Neutrosophic Logic. In Philosophy"*, American Research Press, Santa Fe , New Mexico., U.S, 1999.

[26] W. Haibin, F. Smarandache, Y. Zhang, and R. Sunderraman, *"Single Valued Neutrosophic Sets"*, Infinite Study., 2010.

[27] I. Kandasamy and Q. Mushtaq, *Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm*, Journal of Intelligent Systems. **27** (2018), no. 2, 163–182.

[28] I. Kandasamy and F. Smarandache, *Triple refined indeterminate neutrosophic sets for personality classification.*, IEEE Symposium Series on Computational Intelligence (SSCI). (2016), 1–8.

[29] F. Smarandache, *n-Valued refined neutrosophic logic and its applications to physics*, Infinite Study. **4** (2013), 143–146.

[30] H. Y. Zhang, J. Q. Wang, and X. H. Chen, *Interval neutrosophic sets and their application in multicriteria decision making problems*, Infinite Study. **28** (2008), no. 2, 363–370.

[31] F. Smarandache, *Degree of dependence and independence of the sub components of fuzzy set and neutrosophic set*, Neutrosophic Sets and Systems **11** (2016), no. 3, 95–97.

[32] P. Bosc and O. Pivert, *On a fuzzy bipolar relational algebra*, Information Sciences. **219** (2013), 1–16.

[33] K. M. Lee, *Bipolar-valued fuzzy sets and their operations*, In Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand. (2000), 307–312.

[34] K. J. Lee, *Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras*, Bulletin of the Malaysian Mathematical Sciences Society. **32** (2009), no. 3.

[35] I. Deli, M. Ali, and F. Smarandache, *Bipolar neutrosophic sets and their application based on multi-criteria decision making problems*, In IEEE International Conference on Advanced Mechatronic Systems (ICAMechS), Beijing, China. **2** (2015), 249–254.

[36] D. Van, T. Savage, P. Petsagkourakis, N. Shah, and E. A. D. R. Chanona, *Data-driven optimization for process systems engineering applications*, Chemical Engineering Science. **248** (2022), 117–135.

[37] V. Guarrasi, N. C. Damico, R. Sicilia, E. Cordelli, and P. Soda, *Pareto optimization of deep networks for COVID-19 diagnosis from chest X-rays*, Pattern Recognition. **121** (2022), 108–242.

[38] M. Alweshah, M. A. Alkhalaileh, A. Betar, A. A. Bakar, and P. Soda, *Coronavirus herd immunity optimizer with greedy crossover for feature selection in medical diagnosis*, Knowledge-Based Systems. **235** (2022), 107–629.

[39] V. I. Khabarov and O. G. Khabarova, *Efficiency of labor resources use and issues of transport supply and demand development in large cities and agglomerations*, Transportation Research Procedia. **61** (2022), 308–313.

[40] M. Shehab, L. Abualigah, H. A. Hamad, H. Alabool, and M. Alshinwan, *Moth-flame optimization algorithm: variants and applications*, Neural Computing and Applications. **32** (2020), no. 7, 9859–9884.

[41] L. Abualigah, *Group search optimizer: a nature-inspired meta-heuristic optimization algorithm with its results, variants, and applications*, Neural Computing and Applications. **33** (2020), no. 7, 1–24.

[42] G. Naseri and M. A. Koffas, *Application of combinatorial optimization strategies in synthetic biology*, Nature Communications. **11** (2020), no. 1, 1–14.

[43] C. N. Wang, T. M. Le, and H. K. Nguyen, *Application of optimization to select contractors to develop strategies and policies for the development of transport infrastructure*, Mathematics. **7** (2019), no. 1, 98–112.

[44] Al. Qaness, A. A. Mohammed, M. A. Ewees, H. Fan, and M. A. A. Aziz, *Optimization method for forecasting confirmed cases of COVID-19 in China*, Journal of Clinical Medicine. **9** (2020), no. 3, 674–689.

[45] L. V. Kantorovich, *The mathematical method of production planning and organization*, Management Science. **6** (1939), no. 4, 363–422.

[46] D. Du and P. M. Pardalos, *"Handbook of combinatorial optimization"*, Germeny, 1998.

[47] S. Wright and J. Nocedal, *Numerical optimization*, Springer Science. **35** (1995), no. 1, 67–68.

[48] S. Boyd, S. P. Boyd, and L. Vandenberghe, *"Convex Optimization"*, Cambridge university press., New York, 2004.

[49] L. Jorge and J. W. Stephen, *"Numerical optimization"*, Springer Science+Business Media, Inc., New York, 2006.

[50] S. French, *Uncertainty and imprecision: Modelling and analysis*, Journal of the Operational Research Society. **46** (1995), no. 1, 70–79.

[51] R. E. Bellman and L. A. Zadeh, *Decision-making in a fuzzy environment*, Management Science. **17** (1970), no. 4, B–141.

[52] H. K. Tanaka and K. Asai, *On fuzzy mathematical programming*, Journal of Cybernetics. **3** (1973), 37–46.

[53] H. J. Zimmermann, *Fuzzy programming and linear programming with several objective functions*, Fuzzy Sets and Systems. **1** (1978), no. 1, 45–55.

[54] H. K. Tanaka and K. Asai, *Fuzzy linear programming problems with fuzzy numbers*, Fuzzy Sets and Systems **13** (1984), no. 1, 1–10.

[55] B. Werners, *An interactive fuzzy programming system*, Fuzzy Sets and Systems. **23** (1987), no. 1, 131–147.

[56] C. Xu, *Fuzzy optimization of structures by the two phase method*, Computer and Structure. **31** (1989), no. 4, 575-580.

[57] C. J. Shih, C. C. Chi, and J. H. Hsiao, *Alternative level cuts methods for optimum structural design with fuzzy resources*, Computers and Structures. **81** (2003), no. 28, 2579-2587.

[58] C. J. Shih and H. W. Lee, *Modified double-cuts approach in 25-bar and 72-bar fuzzy truss optimization*, Computers and Structures. **84** (2006), no. 29, 2100–2104.

[59] G. E. Asimakopoulou, V. T. Kontargyri, G. J. Tsekouras, C. N. Elias, F. E. Asimakopoulou, and I. A. Stathopulos, *A fuzzy logic optimization methodology for the estimation of the critical flashover voltage on insulators*, Electric Power Systems Research. **81** (2011), no. 2, 580–588.

[60] S. Fang, C. F. Hu, H. F. Wang, and S. Y. Wu, *Linear programming with fuzzy coefficients in constraints*, Computers and Mathematics with Applications. **37** (1999), no. 10, 63–76.

[61] S. Fang and G. Li, *Solving fuzzy relation equations with a linear objective function*, Fuzzy Sets and Systems. **103** (1999), no. 1, 107–113.

[62] H. Hamacher and H. J. Zimmermann, *Sensitivity analysis in fuzzy linear programming*, Fuzzy Sets and Systems. **1** (1978), no. 4, 269–281.

[63] S. Han and S. Fujii, *One machine scheduling problem with fuzzy due dates*, European Journal of Operational Research. **79** (1994), no. 1, 1–12.

[64] H. Ishibuchi, N. Yamamoto, T. Murata, and H. Tanaka, *Genetic algorithms and neighborhood search algorithms for fuzzy flow-shop scheduling problems*, Fuzzy Sets and Systems. **67** (1994), no. 1, 81–100.

[65] H. Rommelfanger, *Fuzzy linear programming and applications*, European Journal of Operations Research. **92** (1996), no. 3, 512–527.

[66] J. Ramk and H. Rommelfanger, *Fuzzy mathematical programming based on some new inequality relations*, Fuzzy Sets and Systems. **81** (1996), no. 1, 77–87.

[67] D. Wang, *An inexact approach for linear programming problems with fuzzy objective and resources*, Fuzzy Sets and Systems. **89** (1997), no. 1, 61–68.

[68] D. Wang and S. C. Fang, *A genetic-based approach for aggregate production planning in fuzzy environment*, IEEE Trans. on SMC (Part A). **12** (1997), no. 7, 636–645.

[69] M. Sakawa and H. Yano, *An interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters*, Fuzzy Sets and Systems. **30** (1989), no. 3, 221–238.

[70] J. Kacprzyk and A. O. Esogbue, *Fuzzy dynamic programming: Main developments and applications*, Fuzzy Sets and Systems. **81** (1996), no. 1, 31–46.

[71] Y. J. Lai and C. L. Hang, *A new approach to some possibilistic linear programming problems*, Fuzzy Sets and Systems. **49** (1992), no. 1, 121–133.

[72] J. Ramik and J. Rimanek, *Inequality relation between fuzzy numbers and it's use in fuzzy optimization*, Fuzzy Sets and Systems. **16** (1985), no. 3, 123–133.

[73] H. Rommelfanger, R. Hanuscheck, and J. Wolf, *Linear programming with fuzzy objective*, Fuzzy Sets and Systems. **29** (1989), no. 2, 31–48.

[74] J. Tang and D. Wang, *A non-symmetric model for fuzzy nonlinear programming problems with penalty coefficients*, Computers and Operations Research. **24** (1997), no. 8, 717–725.

[75] J. Tang, D. Wang, R. Y. Fung, and D. Wang, *Model and method based on GA for non-linear programming problems with fuzzy objective and resources*, International Journal of Systems Science. **29** (1998), no. 8, 907–913.

[76] J. F. C. Trappey, C. R. Liu, and T. C. Chang, *Fuzzy non-linear programming: Theory and application in manufacturing*, International Journal of Production Research. **26** (1988), no. 5, 957–985.

[77] S. Chanas, *The use of parametric programming in fuzzy linear programming*, Fuzzy Sets and Systems. **1** (1983), no. 3, 229–241.

[78] J. L. Verdegay, *A dual approach to solve the fuzzy linear programming problem*, Fuzzy Sets and Systems. **14** (1986), no. 2, 131–141.

[79] C. Carlsson and P. Korhonen, *A parametric approach to fuzzy linear programming*, Fuzzy Sets and Systems. **20** (1986), no. 1, 17–30.

[80] L. Campos, *Fuzzy linear programming models to solve fuzzy matrix games*, Fuzzy Sets and Systems. **32** (1989), no. 3, 275–289.

[81] M. K. Luhandjula, *Fuzzy optimization: an appraisal*, Fuzzy Sets and Systems. **30** (1989), no. 3, 257–282.

[82] H. Jain and K. Deb, *An evolutionary many-objective optimization algorithm using reference-point based non-dominated sorting approach, part II: Handling constraints and extending to an adaptive approach*, IEEE Transactions on Evolutionary Computation. **18** (2013), no. 4, 602–622.

[83] S. M. Guu and Y. K. Wu, *Multiple objective optimization for systems with additionmin fuzzy relational inequalities*, Fuzzy Optimization and Decision Making. **18** (2019), no. 4, 529–544.

[84] A. Zhou and Y. Wang, *Objective extraction via fuzzy clustering in evolutionary many-objective optimization*, Information Sciences. **509** (2020), 343–355.

[85] A. Ghodousian, *Optimization of linear problems subjected to the intersection of two fuzzy relational inequalities defined by Dubois-Prade family of t-norms*, Information Sciences. **503** (2019), 291–303.

[86] N. V. Sahinidis, *Optimization under uncertainty: state-of-the-art and opportunities*, Computers and Chemical Engineering,. **28** (2004), no. 6, 971–983.

[87] S. Chakrabortty, M. Pal, and P. K. Nayak, *Intuitionistic fuzzy optimization technique for Pareto optimal solution of manufacturing inventory models with shortages*, European Journal of Operational Research. **228** (2013), no. 2, 381–387.

[88] S. P. Wan, F. Wang, G. L. Xu, J. Y. Dong, and J. Tang, *An intuitionistic fuzzy programming method for group decision making with interval-valued fuzzy preference relations*, Fuzzy Optimization and Decision Making. **16** (2017), no. 3, 269–295.

[89] P. P. Angelov, *Optimization in an intuitionistic fuzzy environment*, Fuzzy Sets and Systems. **86** (1997), no. 3, 299-306.

[90] P. Das and T. K. Roy, *Multi-objective non-linear programming problem based on neutrosophic optimization technique and its application in riser design problem*, Neutrosophic Sets and Systems. **9** (2015), no. 1, 15-21.

[91] S. K. Bharati and S. R. Singh, *Interval-valued intuitionistic fuzzy linear programming problem*, New Mathematics and Natural Computation **16** (2020), no. 01, 53-71.

[92] R. R. Yager, *Ranking fuzzy subsets over the unit interval*, In IEEE conference on decision and control including the 17th symposium on adaptive processes . (1979), 1435–1437.

[93] J. M. Adamo, *Fuzzy decision trees*, Fuzzy Sets and Systems. **4** (1980), no. 3, 207–219.

[94] W. Chang, *Ranking of fuzzy utilities with triangular membership functions*, International Conference on Policy anal and Information Systems. (1981), 263–271.

[95] G. Bortolan and R. Degani, *A review of some methods for ranking fuzzy subsets*, Fuzzy Sets and Systems. **15** (1985), no. 01, 01–19.

[96] S. H. Chen, *Ranking fuzzy numbers with maximizing set and minimizing set*, Fuzzy Sets and Systems. **17** (1985), no. 02, 113–129.

[97] D. Dubois and H. Prade, *The mean value of a fuzzy number*, Fuzzy Sets and Systems. **2** (2014), no. 1, 35–50.

[98] E. S. Lee and R. J. Li, *Comparison of fuzzy numbers based on the probability measure of fuzzy events*, Computers and Mathematics with Applications. **15** (1988), no. 10, 887–896.

[99] T. S. Liou and M. J. J. Wang, *Ranking fuzzy numbers with integral value*, Fuzzy Sets and Systems. **50** (1992), no. 3, 247–255.

[100] C. H. Cheng and R. J. Li, *A new approach for ranking fuzzy numbers by distance method*, Fuzzy Sets and Systems. **95** (1998), no. 3, 307–317.

[101] H. K. Lee and J. H. Lee, *A method for ranking fuzzy numbers and its application to decision-making*, IEEE Transactions on Fuzzy Systems. **7** (1999), no. 6, 677–685.

[102] X. Liu, *Measuring the satisfaction of constraints in fuzzy linear programming*, Fuzzy Sets and Systems. **122** (2001), no. 02, 263–275.

[103] T. C. Chu and C. T. Tsao, *Ranking fuzzy numbers with an area between the centroid point and original point*, Computers and Mathematics with Applications. **43** (2002), no. 2, 111–117.

[104] B. Asady and A. Zendehnam, *Ranking fuzzy numbers by distance minimization*, Applied Mathematical Modelling. **31** (2007), no. 11, 2589–2598.

[105] T. Allahviranloo, F. H. Lotfi, M. K. Kiasary, N.A. Kiani, and L. Alizadeh, *Solving fully fuzzy linear programming problem by the ranking function*, Applied mathematical sciences. **2** (2008), no. 1, 19–32.

[106] P. Rao and N. R. Shankar, *Ranking fuzzy numbers with a distance method using circum-center of centroids and an index of modality*, Advances in Fuzzy Systems. (2011).

[107] A. Karpagam and P. Sumathi, *New approach to solve fuzzy linear programming problems by the ranking function*, Bonfring International Journal of Data Mining. **4** (2014), no. 4, 22–25.

[108] I. H. Alkanani and F. A. Adnan, *Ranking function methods for solving fuzzy linear programming problems*, Mathematical Theory and Modeling. **4** (2014), no. 4, 65–72.

[109] T. L. Nguyen, *Methods in ranking fuzzy numbers: a unified index and comparative reviews*, Fuzzy Calculus Theory and Its Applications. **4** (2017).

[110] M. A. Mehmood, M. Akram, M. G. Alharbi, and S. Bashir, *Solution of fully bipolar fuzzy linear programming models*, Generalised Fuzzy Models Applied to Logical Algebras and Intelligent Systems in Engineering (2021).

[111] A. Nafei, M. Arif, W. Yuan, and H. Nasseri, *A new method for solving interval neutrosophic linear programming problems*, Gazi University Journal of Science. **33** (2020), no. 4, 796–808.

[112] I. Deli and Y. Subas, *A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems*, International Journal of Machine Learning and Cybernetics. **8** (2017), no. 4, 1309–1322.

[113] M. Darehmiraki, *"Optimization theory based on Neutrosophic and Plithogenic Sets "*, Academic Press., New York, 2020.

[114] M. Suresh, K. A. Prakash, and S. Vengataasalam, *Multi-criteria decision making based on ranking of neutrosophic trapezoidal fuzzy numbers*, Granular Computing. **6** (2021), no. 4, 943–952.

[115] A. Chakraborty, S. P. Mondal, S. Alam, and A. Dey, *Classification of trapezoidal bipolar neutrosophic number, debipolarization technique and its execution in cloud service based MCGDM problem*, Complex and Intelligent Systems. **7** (2021), no. 1, 100–111.

[116] D. B. Field, *Linear programming: out of the classroom and into the woods*, Journal of Forestry. **75** (1977), no. 6, 330–334.

[117] G. W. Dean, Y. Isyar, and C. V. Moore, *Programming model for evaluating economic and financial feasibility of irrigation projects with extended development periods*, Water Resources Research. **9** (1973), no. 3, 546–555.

[118] C. Granja, L. B. Almada, F. Janela, J. Seabra, and A. Mendes, *An optimization based on simulation approach to the patient admission scheduling problem using a linear programming algorithm*, Journal of biomedical informatics, **52** (2014), no. 4, 427–437.

[119] A. lafsson and S. J. Wright, *Linear programming formulations and algorithms for radiotherapy treatment planning*, Optimization Methods and Software. **21** (2006), no. 2, 201–231.

[120] P. Kotler, *The use of mathematical models in marketing*, Journal of Marketing. **27** (1963), no. 4, 31–41.

[121] S. H. Amin, J. Razmi, and G. Zhang, *Supplier selection and order allocation based on fuzzy SWOT analysis and fuzzy linear programming*, Expert Systems with Applications. **38** (2011), no. 1, 334–342.

[122] L. H. Chen and W. C. Ko, *Fuzzy linear programming models for new product design using QFD with FMEA*, Mathematical Problems in Engineering Applied Mathematical Modeling. **33** (2009), no. 2, 633–647.

[123] D. Peidro, J. Mula, and M. Jimenez, *A fuzzy linear programming based approach for tactical supply chain planning in an uncertainty environment*, European Journal of Operational Research. **205** (2010), no. 1, 65–80.

[124] A. Rong and R. Lahdelma, *Fuzzy chance constrained linear programming model for optimizing the scrap charge in steel production*, European Journal of Operational Research. **186** (2008), no. 3, 953–964.

[125] F. H. Lotfi, T. Allahviranloo, M. A. Jondabeha, and L. A. Zadeh, *Solving a fully fuzzy linear programming using lexicography method and fuzzy approximate solution*, Applied Mathematical Modelling. **33** (2009), no. 7, 3151–3156.

[126] A. Ebrahimnejad, S. H. Nasseri, F. H. Lot, and M. Soltanifar, *A primal dual method for linear programming problems with fuzzy variables*, European Journal of Industrial Engineering. **4** (2010), no. 2, 189–209.

[127] K. Campos and P. Veeramani, *Fuzzy linear programs with trapezoidal fuzzy numbers*, Annals of Operational Research. **143** (2006), no. 1, 305–315.

[128] K. Ganesan and P. Veeramani, *Fuzzy linear programs with trapezoidal fuzzy numbers*, Annals of Operational Research. **143** (2006), no. 1, 305–315.

[129] A. N. Hussein, M. Mohamed, M. Abdel-Basct, and F. Smarandache, *Fuzzy linear programs with trapezoidal fuzzy numbers*, Infinity Series. (2018), 1–15.

[130] M. Abdel-Basset, M. Gunasekaran, M. Mohamed, and F. Smarandache, *A novel method for solving the fully neutrosophic linear programming problem*, Neural Computing and Applications. **31** (2019), no. 5, 1595–1605.

[131] A. Nafei, W. Yuan, and H. Nasseri, *A new method for solving interval neutrosophic linear programming problems*, Gazi University Journal of Science **33** (2020), no. 4, 796–808.

[132] M. A. Mehmood, M. G. Alharbi, and S. Bashir, *Solution of fully bipolar fuzzy linear programming models*, Solution of Fully Bipolar Fuzzy Linear Programming Models. (2021).

[133] H. Wang, F. Smarandache, R. Sunderraman, and Y. Q. Zhang, *"Interval Neutrosophic Sets and Logic: Theory and Applications in Computing"*, Infinity Study., New York, 2005.

[134] S. K. Bharati and S. R. Singh, *Solution of multiobjective linear programming problems in interval-valued intuitionistic fuzzy environment*, Soft Computing. **23** (2019), no. 1, 77–84.

[135] H. Jain and K. Deb, *An evolutionary many-objective optimization algorithm using reference-point based nondominated sorting approach, part II: handling constraints and extending to an adaptive approach,* IEEE Transactions on Evolutionary Computation. **18** (2013), no. 4, 602–622.

[136] J. M. Templeton, *Land use planning tools in Illinois: preventing or promoting sprawl. Suburban Sprawl,* Private Decisions and Public Policy. (2002), 62–87.

[137] I. Deli, S. Yusuf, F. Smarandache, and M. Ali, *Interval Valued Bipolar Neutrosophic Sets and their Application in Pattern Recognition,* Informatics and Systems. **33** (2022), 100622-100641.

[138] P. D. Singh, G. Dhiman, and R. Sharma, *Internet of Things for sustaining a smart and secure healthcare system,* IEEE World Congress on Computational Intelligence, Vancouver, Canada (2016).

[139] S.N. Sirimanne, J. Hoffman, W. Juan, R. Asariotis, M. Assaf, G. Ayala, H. Benamara, D. Chantrel, and A. Premti, *Review of maritime transport,* In United Nations Conference on trade and Development, Geneva, Switzerland. (2019).

[140] M. Navascus, C. Budroni, and Y. Guryanova, *Disease control as an optimization problem,* Informatics and Systems. **16** (2021), no. 6, 0257958–0257975.

[141] M. Beaulieu, O. Bentahar, and Y. Guryanova, *Digitalization of the healthcare supply chain: A roadmap to generate benefits and effectively support healthcare delivery,* Technological Forecasting and Social Change. **167** (2021), 120717–120727.

[142] D. Friday, D. A. Savage, S. A. Melnyk, N. Harrison, and S. Ryan, *A collaborative approach to maintaining optimal inventory and mitigating stock-out risks during a pandemic: capabilities for enabling health-care supply chain resilience,* Journal of Humanitarian Logistics and Supply Chain Management. **11** (2021), no. 2, 248–271.

[143] J. C. Ho, M. K. Shalishali, T. L. B. Tseng, and D. S. Ang, *A collaborative approach to maintaining optimal inventory and mitigating stock-out risks during a pandemic: capabilities for enabling health-care supply chain resilience*, The Coastal Business Journal **8** (2022), no. 1, 2–17.

[144] *http://data.worldbank.org/indicator/SP.DYN.CDRT.IN.*