Generalizations of Intuitionistic Fuzzy Sets and their Comparative Study

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Fuzzy sets
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IN

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Dedicated

To

My

Beloved Father, Zafeer Ullah Khan

My Mother

My brothers, sisters and other family members

My Friends, Students, Well wishers

My respected teachers

And my Mentor Dr. Tahir Mahmood

Declaration

I hereby, declare, that this thesis neither as a whole nor as a part thereof has been copied out from any source. It is further declared that I have prepared this thesis entirely on the basis of my personal efforts made under the sincere guidance of my kind supervisor. No portion of the work, presented in this thesis, has been submitted in the support of any application for any degree or qualification of this or any other institute of learning.

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Kifayat Ullah

Contents

0. Research Profile

1. Mahmood T., **Ullah K.**, Khan Q. and Jan N. "An Approach Towards Decision Making and Medical Diagnosis Problems Using the Concept of Spherical Fuzzy Sets." Neural Computing and Applications. 2018, 31 (11), 7041-7053.
2. **Ullah K.**, Mahmood T. and Jan N. Similarity Measures for T-Spherical Fuzzy Sets with Applications in Pattern Recognition. Symmetry. 2018, 10(6), 193-207.
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5. **Ullah K.**, Mahmood T. and Garg, H. Evaluation of the Performance of Search and Rescue Robots Using T-spherical Fuzzy Hamacher Aggregation Operators. International Journal of Fuzzy Systems, 2020.
6. **Ullah K.**, Hassan N., Mahmood T. Jan N. and Hassan M. Evaluation of Investment Policy Based on Multi-Attribute Decision-Making Using Interval Valued T-Spherical Fuzzy Aggregation Operators. Symmetry, 2019, 11 (3), 357-380.

0.1. Introduction

In classical set theory, an element either belongs to a set or does not belong to a set and there is no third possibility. This type of framework can assign only two numeric values to describe the membership grade of a certain element i.e. 0 or 1. There are many phenomena that cannot be described using classical set theory for instance consider the concepts of *height*, *intelligence* and *age* etc. All these concepts cannot be described using classical set theory which leads Zadeh [1] to develop the idea of fuzzy set (FS) theory in 1965. Zadeh introduced the idea of a membership grade (MG) of an element of a set which is defined by a membership function on a unit interval [0, 1]. This idea of membership function by Zadeh can be used to describe the uncertainty lies in human opinion hence provides solution to many real-life problems such as decision making [2], medical diagnosis (MD) [3] and pattern recognition [4].

Human opinion about a certain thing or phenomena is not always unidirectional while the framework of FS only describe the MG of an uncertain event hence not providing any information about the non-membership grade of the event. This leads Atanassov [5] to propose the concept of intuitionistic fuzzy set (IFS) and intuitionistic fuzzy number (IFN). Atanassov's concept of IFS is based on a MG and a non-membership grade (NG) denoted by ş and d' respectively with a constraint that their sum must not exceeds 1 i.e. $\text{ş}um(\text{ş}, \text{d}') \in [0, 1]$. Further, the term $1 - \text{ş}um(\text{ş}, \text{d}')$ is referred to as hesitancy degree. This concept of IFS is likely to model any uncertain event

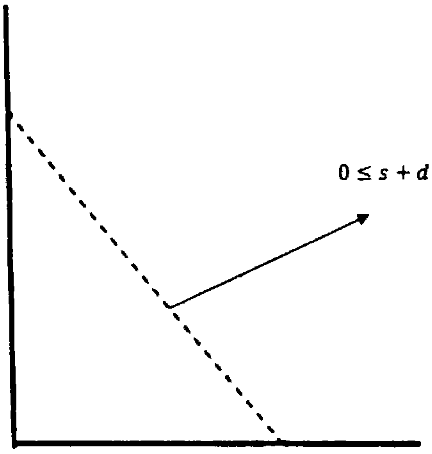

Figure 1 (Space of all possible IFNs)

with the help of MG and NG but in some cases, it has some certain limitations. For instance, if the MG and NG are assigned the values $(0.8, 0.5)$ i.e. $ş = 0.8$ and $d' = 0.5$ then $şum(0.8, 0.5) = 1.3 \notin [0, 1]$. This means that the duplet $(0.8, 0.5)$ cannot be considered as an IFN. The space of all possible IFNs is depicted in Figure 1.

As the framework of IFS has its limitations in assigning the MG and NG because their sum exceeds from $[0, 1]$ in many cases. To overcome this situation, Yager [6, 7] developed the idea of Pythagorean FS (PyFS) and consequently Pythagorean fuzzy number (PyFN) which increased the range of Atanassov's IFS. A PyFN is also based on a MG and non-membership grade (NG) denoted by ş and d' respectively with a constraint that the sum of their squares must not exceeds 1 i.e. $şum(ş^2, d'^2) \in [0, 1]$. Further, the term $\sqrt{1 - şum(ş^2, d'^2)}$ is referred to as hesitancy degree. If we observe the duplets which cannot be considered as an IFN i.e. $(0.8, 0.5)$, then it can be a PyFN because $şum(0.8^2, 0.5^2) = 0.89 \in [0, 1]$. This shows that the range of PyFS is greater than that of IFS as demonstrated in Figure 2.

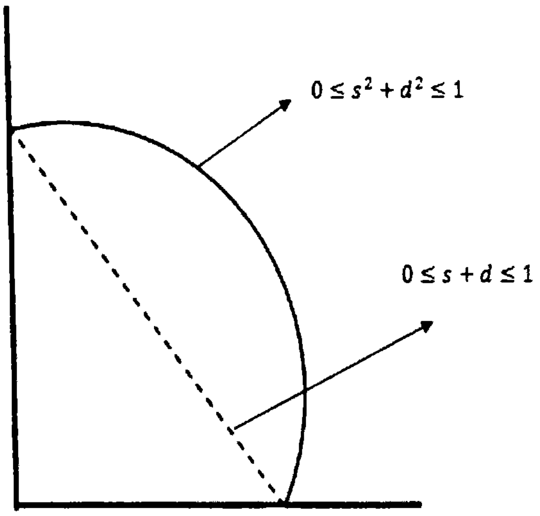

Figure 2 (Space of all possible IFNs and PyFNs)

Figure 2 shows that Yager's PyFS significantly improved the limitations that occured in Atanassov's IFS but still there exist some duplets that cannot be categorized as IFN or PyFN. For instance, the duplet $(0.9, 0.6)$ is neither an IFN nor a PyFN because $\text{ş}um(0.9, 0.6) = 1.5 \notin [0, 1]$ and $\text{ş}um(0.9^2, 0.6^2) = 1.17 \notin [0, 1]$. If we observe the geometrical position of the duplet $(0.9, 0.6)$, its just outside the range of PyFNs which is shown in Figure 3.

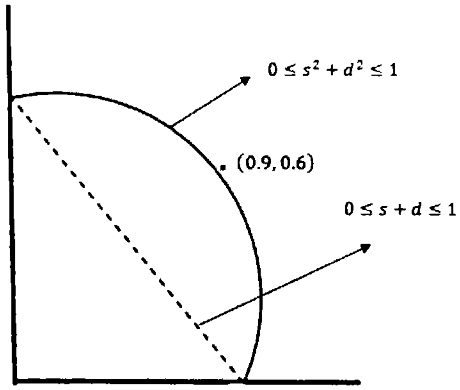

Figure 3 (Limitation of Space of IFNs and PyFNs)

This leads Yager [8] to introduce the idea of q-rung ortho pair FS (q-ROFS) and consequently q-rung ortho pair fuzzy number (q-ROFN) which take over any kind of duplet with a variable parameter n where $n \in \mathbb{Z}^+$. For instance, consider the duplet $(0.9, 0.6)$ which is neither an IFN nor a PyFN, but it can be a q-ROFN for $n = 3$ i.e. $\text{ş}um\ (0.9^3, 0.6^3) = 0.945 \in [0, 1]$. Similarly, for every duplet of the form $(\text{ş}, \text{d}')$, there exists an $n \in \mathbb{Z}^+$ such that $\text{ş}um(\text{ş}^n, \text{d}'^n) \in [0, 1]$. This clearly shows the superiority of q-ROFS over IFS and PyFS which is geometrically demonstrated in Figure 4.

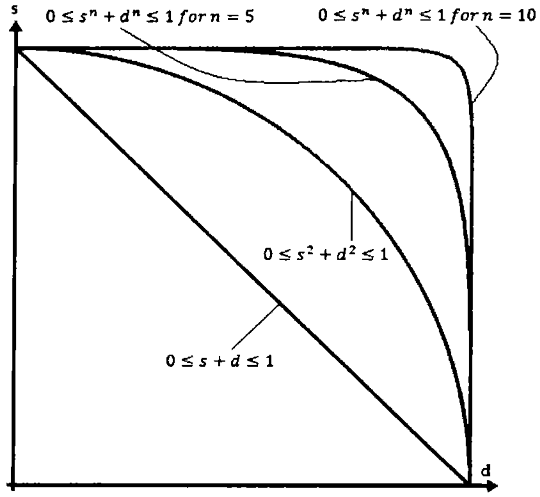

Figure 4 (space of q-ROFNs in comparison with space of PyFNs and IFNs)

The concepts of IFS, PyFS and q-ROFS discussed only two aspects of the human opinion about an uncertain event i.e. yes or no type of aspects denoted by the MG and NG. However, human opinion cannot be restricted to yes and no type, but it has some sort of abstinence and refusal degree also as suggested by Cuong [9]. For example consider the phenomenon of voting where one can vote in favor of someone or vote against someone or abstain to vote or refuse to vote. Cuong [9, 10] proposed the concept of picture fuzzy set (PFS) and consequently picture fuzzy number (PFN) where three types of membership grades are utilized to model human opinion of yes, abstain and no type. These three types of grades include MG, abstinence grade (AG) and NG denoted by ş, i and d' respectively with a restriction that their sum must not exceeds 1 i.e. şum(ş, i, d') $\in [0, 1]$. Further, the term $1 -$ şum(ş, i, d') is termed as refusal grade (RG) denoted by r. Cuong's concept of PFS is a generalization of the Atanassov's IFS and deals with uncertain and imprecise information in a more flexible way than IFS, PyFS or q-ROFS does. The space of all PFNs is geometrically shown in Figure 5.

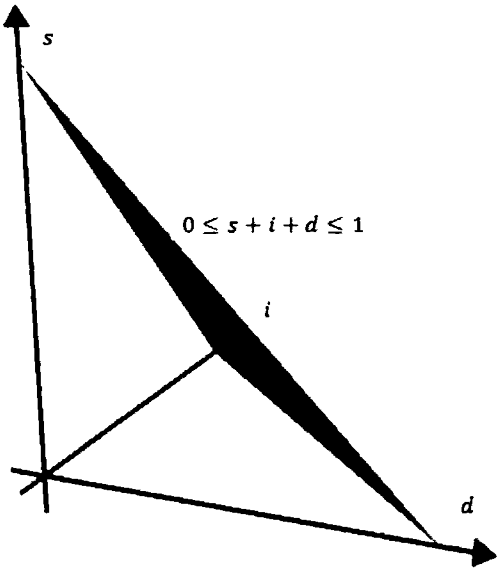

Figure 5 (space of all PFNs)

Yager [6, 8] generalizes the framework of IFS to PyFS and q-ROFS while Cuong [9, 10] conceived the idea of PFS from IFS and all these have significance when it comes to their viability. There is another generalization of IFS known as single valued neutrosophic set (SVNS) proposed by Wang et al. [11] as a generalization of neutrosophic set (NS) proposed by Smarandache [12]. A SVNS described the imprecision of an uncertain event with the help of a MG, neutral value and a NG denoted by $ş, i$ and d' with the restriction that $0 \leq şum(ş, i, \text{d'}) \leq 3$. Allowing the sum of all three membership grades between 0 and 3, a SVNS allows the decision makers to assign the values of $ş, i$ and d' independently from $[0, 1]$.

In the concept of Zadeh's FS, a MG is assigned to describe the uncertainty of an event or object from $[0, 1]$ interval. By doing so, there is a possibility of losing some information which leads us to the concept of interval valued FS (IVFS) [13]. In IVFS, the MG is in the form of closed subinterval of $[0, 1]$ which decreases the chances of losing information. Using the same approach, Atanassov and Gargov [14] extended the idea of IFS to interval valued IFS (IVIFS) by describing the MG and NG in terms of closed subintervals of $[0, 1]$ which decreases the loss of information as compared to IFS. Due to the significance of expressing the MG and NG in terms of interval instead of crisp number, the concept of PyFS and q-ROFS are also extended to interval valued PyFS (IVPyFS) and interval valued q-RSFS (IVq-ROFS) by Peng and Yang [15] and Joshi et al. [16] respectively. This concept is adapted in forthcoming fuzzy frameworks as well and the concept of interval valued PFS (IVPFS) is proposed by Cuong [10] where the MG, AG and NG are described in the form of closed subintervals of $[0, 1]$. Similarly, describing the MG, neutral value and NG using closed subintervals in NSs, the idea of interval valued NS (IVNS) is proposed by Wang et al. [17].

So far, we have discussed the origin of several fuzzy extensions and their structures. When it comes to viability, almost every fuzzy framework has been extensively used in several practical situation including MD, multi-attribute decision making (MADM), pattern recognition and clustering. The problem of MD is discussed in several fuzzy frameworks on a wide range. Fuzzy relations and their compositions are the tools that are used in the process of MD. Yao and Yao [18] studied the problem of MD using fuzzy inference system where the concept of composition of fuzzy relations has been utilized. Rakityanskaya and Rotshtein [19] modified the compositional rule proposed by Zadeh based on fuzzy relations to deal with MD. Samuel and Balamurugan [20] extended the idea of fuzzy MD proposed by [21] to intuitionistic fuzzy environment and used max-min composition of IFS to deal with MD problems. De et al. [22] investigated the problem of MD using intuitionistic fuzzy relations and compositions. Thong [23] proposed a recommender system in intuitionistic fuzzy environment to diagnose a medical issue using few symptoms. Kumar and Krishnan [24] discussed the MD problem using similarity measures (SMs) of IFSs. Garg [25] studied the MD and pattern recognition problems using correlation coefficients of PyFSs. Xiao and Ding [26] investigated the divergence measures of PyFSs in studying MD problems. Some similarity measures of q-ROFS are also utilized in examining MD problems by Wang et al. [27]. The concept PFSs is also successfully utilized in MD problems by Wang et al. [28] where they proposed some picture fuzzy relations and developed their compositions for this purpose.

Another useful application of FS theory and its generalizations is handling MADM problems where human opinion may have several aspects. The widely used tools for handling MADM problem under uncertainty are aggregation operators (AOs) and a large variety of such AOs have been introduced in the past few years. These AOs include weighted averaging aggregation (WAA) operators, weighted geometric aggregation (WGA), Einstein aggregation (EA) operators, Hamacher aggregation (HA) operators, power aggregation (PA) operators etc. Theory of WAA and WGA operators of IFSs have been developed by Xu [29] and Xu and Yager [30] which were further applied in MADM problems. To improve the work of [29, 30], some generalized WAA, induced generalized WAA and induced generalized WGA operators for IFSs are proposed by [31, 32, 33] and their applications in MADM problems have been demonstrated. The concept of WAA and WGA operators for IVIFSs are proposed by Wang [34] and Wei and Wang [35] respectively while some generalized WAA and generalized WGA operators for IVIFSs are developed by Yu [36] and Xu and Cai [37] respectively. Some other useful tools of aggregation of IFSs and IVIFSs along with their applications were also constructed [38-43]. For PyFSs, Rahman et al. [44] and Peng and Yang [45] developed the Pythagorean fuzzy WAA and WGA operators. The WAA and WGA operators for IVPyFSs are proposed by [46, 47] which were further applied in MADM problems. Some recent advancement on the theory of aggregation of PyFSs and IVPyFSs and their applications in MADM are discussed [48, 49]. Due to its enhanced and diverse structure, PFSs have gained attention and some WAA as well as WGA operators in picture fuzzy environment are developed by [50, 51].

The conept of Hamacher aggregation (HA) operators is also an important topic that is widly discussed in fuzzy frameworks and utilized in MADM problems. HA operators are based on Hamacher t-norm and t-conorm [52]. Huang [53] introduced the notion of HA operators in intuitionistic fuzzy settings based on Hamacher t-norm and t-conorm and investigated their applicability in MADM problems. Wu and Wei [54] proposed Pythagorean fuzzy HA operators for MADM while Gao [55] developed prioritized Pythagorean HA operators for MADM problems. HA operators for interval valued IFSs are developed by Liu [56] and some entropy-based HA operators for IFSs are proposed by Garg [57]. Wei [58, 59] proposed picture fuzzy HA operators and Pythagorean fuzzy Hamacher power AOs for solving MADM problems. For some other recent work on HA operators and their application, one is referred to [60-63].

Similarity measures (SMs) or correlation coefficients (CCs) and distance measures (DMs) are some tools that remaiend under considerration in every fuzzy framework especially when it comess to study pattern recognition problems, MADM problems and cluster analysis. A SM or CC shows the similaity degree of two objects that how similar they are based on some some attributes. Similarly, a DM is a function that is used to compute the dissimilaity degree of two objects. Dengfeng and Chuntian [64] proposed some novel SMs in intuitionistic fuzzy setting and examined their applicability in pattern recognition. Huang et al. [65] used the concept of Hausdorff distance to propose some SMs for IFS and utilized the proposed SMs in pattern recognition. Liang and Shi [66] and Vlachos and Sergiadis [67] contributed by introducing some new SMs in the environment of IFSs. Ye [68] developed cosine SMs for IFSs based on cosine rule and studied the viability of new cosine SMs in pattern recognition. Szmidt and Kacprzyk [69] used the concept of SMs of IFSs for medical diagnosis problem solving and Xu [70] utilizes the SMs of IFSs in solving MADM problems. An overview of SMs of IFSs is carried out by Xu and Chen [71] where they proposed some continuous similarity and distance measures. Xu and Cai 72] proposed some DMs, SMs, and correlation of IFNs for intuitionistic fuzzy information aggregation. The concepts of SMs and DMs is also rich in the framework of PFSs as Wei [73] developed three kinds of SMs for PFNs and utilized them the recognition of building materials. Son [74] proposed some picture fuzzy DMs and studied their applicability in clustering. A CC in intuitionistic fuzzy setting for cluster analysis is proposed by Gerstenkorn and Manko [75] which was extended by Sing [76] to the environment of PFS utilizing it in cluster analysis and showing the advantages of picture fuzzy CC over intuitionistic fuzzy CC. For a complete understanding of SMs, DMs, some CCs, AOs and their applications in MADM. Pattern recognition and clustering, the readers are further referred to [77-97].

0.2. Chapter Wise Study

This section aims to provide a thorough chapter wise study of our accomplished work. A short description containing key findings of each chapter is provided as follows.

Chapter 1

In this chapter, we aim to discuss the basic definition of FS, IFS and their several extensions including PyFS, q-ROFS, PFS and SVNS. Some basic notions and terms of several existing fuzzy frameworks are discussed including the basic operations of sum, product, scalar multiplications and power operations. For, ranking purpose, some score and accuracy functions proposed in each fuzzy environment are also observed. We analyze the importance of expressing the grades of memberships using closed subintervals and recalled the definitions of IVIFS, IVPyFS, IVq-ROFS, IVPFS and IVNS. The rest of the related previous study is discussed in respective chapters.

Chapter 2

In this chapter, we observe that the concept of PFS is limited in structure and that there are many triplets of the form (ş, i, d') which cannot be considered as PFNs. To resolve this issue, we proposed the concept of spherical fuzzy set (SFS) and consequently T-spherical fuzzy set (TSFS) and prove their superiority over the existing ideas with the help of some example and geometrical demonstration. The concepts of relations and their compositions in spherical fuzzy environment are proposed and their properties are investigated. The max-min composition of spherical fuzzy relations (SFRs) is defined to solve a MD problem. We further developed some new arithmetic operations for SFSs and

TSFSs based on which some WGA operators of TSFSs are also developed and utilized in MADM problems.

Chapter 3

In this chapter, we discussed the limitations of some existing SMs of IFSs and PFSs and proposed some new SMs in T-spherical fuzzy (TSF) environment. The significance of newly developed SMs over the existing SMs is shown using some remarks. The newly proposed SMs are applied to a problem of building material recognition where the data involved is of the kind that it cannot be solved by the existing SMs. A comparative study of the proposed and existing SMs is established showing the diversity of newly proposed SMs.

Chapter 4

In this chapter, we proposed some new CCs in TSF environment by showing the limitations of the existing SMs of IFSs and PFSs. The validity of the proposed new CCs is examined and some of their properties are studied. The newly developed CCs are applied in a clustering algorithm to obtain some useful results. A MADM problem is also solved using newly established CCs of TSFSs and a comparison of the results obtained is established. The limitation of the existing CCs of IFSs and PFSs are stated and the advantages of new defined CCs are demonstrated.

Chapter 5

In this chapter, some WAA and WGA operators of TSFSs are defined keeping the limitations of the WAA and WGA operators of IFSs, PyFSs, q-ROFSs and PFSs. The validity of newly proposed WAA and WGA operators of TSFSs is investigated using mathematical induction and some basic properties of aggregation are examined. The superiority and generalization of the proposed WAA and WGA operators of TSFSs over existing AOs is shown and a MADM problem is solved where the information provided cannot be processed by the existing WAA and WGA operators. A comparative study of the new and previous study is established to demonstrate any advantages or disadvantages.

Chapter 6

This chapter aims to draw attention towards the HA operators of IFS, PyFS and PFSs and investigated their non-viability in some cases. Then some new HA operators are developed in TSF environment that generalizes all existing HA operators. These newly developed HA operators include TSF Hamacher weighted averaging (TSFHWA) operator, TSF Hamacher ordered weighted averaging (TSFHOWA) operator, TSF Hamacher hybrid averaging (TSFHHA) operator, TSF Hamacher weighted geometric (TSFHWG) operator, TSF Hamacher ordered weighted geometric (TSFHOWG) operator and TSF Hamacher hybrid geometric (TSFHHG) operator. The validity of newly proposed HA operators is checked using induction method and some properties of the aggregation are studied. A MADM problem for search and rescue robots is solved using the proposed HA operators and a comparative study is established with existing literature.

Chapter 7

As suggested in literature review sections that expressing the membership grades in terms of a crisp value from [0, 1] may cause some loss of information that can be covered up by expressing the membership degrees in the form of a closed subinterval of [0, 1]. Adapting this approach, in this chapter, the concept of interval valued TSFS (IVTSFS) is introduced where the MG, AG, NG and RG are expressed in terms of a closed subinterval of [0, 1]. An IVTSFS generalized the concepts of IFS, IVIFS, PyFS, IVPyFS, q-ROFS, IVq-ROFS, PFS, IVPFS, SFS and interval valued SFS (IVSFS). Then some WAA and WGA operators in interval valued TSF environment are developed and their properties are investigated. A problem concerning the evaluation of investment policy is solved using a MADM technique using WAA and WGA operators of IVTSFSs. A comparative study of new proposed work is established with existing work.

Chapter 8

This chapter aims to provide a comparative study of two major generalizations of IFSs known as SVNS and TSFS. We critically examined the framework of SVNS in comparison to TSFS and looked for possible advantages and disadvantages of both SVNS and TSFS. To do so, we take few tools that are already developed in single valued neutrosophic (SVN) environment and TSF environment and apply them to some data of imprecise information. The results obtained using tools of SVNS and TSFS are critically observed and some major conclusions are drawn.

Chapter 1

Preliminaries

The aim of this chapter is to study all the relevant terms and notions of FSs, IFSs, PyFSs, q-ROFSs, PFSs and SVNSs. We discussed the basic operations of these fuzzy frameworks along with the sum, product, scalar multiplication and power operations in each environment. The concept of score and accuracy functions for comparison purpose are also discussed. Furthermore, we investigated the definitions of the IVIFS, IVPyFS, IVq-ROFS and IVPFS. Note that in our onward work, we denote the term "*precede* by $\preccurlyeq$, *exceed* by $\succcurlyeq$, *inclusion* by $\subseteq$, max by $\vee$ and min by $\wedge$.

1.1. Fuzzy Set

Zadeh [1] gave the concept of FS back in 1965 to discuss the fuzziness of an element in terms of membership grades in $[0,1]$ interval. Before Zadeh's work, an element can either belongs to a set or does not belongs to a set. Zadeh's framework of FS helps to model several measures that could not be dealt with classical set theory such as height, age, beauty etc. We start from the main definition of FS along with its basic operations.

1.1.1. Definition [1]

On a set X, a FS is of the shape $I = \{(\varkappa, \c{s}) : \c{s}: X \to [0,1], \varkappa \in X\}$. Here, $\c{s}$ denotes the MG of the element $\varkappa \in X$. The larger the MG of an element, the more that element said to satisfy the property of the FS and vice versa.

Zadeh also defined the basic operations of FSs in his historical paper [1] which are described as:

1.1.2. Definition [5]

Let $I = (\varkappa, \varsigma(\varkappa)), I_1 = (\varkappa, \varsigma_1(\varkappa))$ and $I_2 = (\varkappa, \varsigma_2(\varkappa))$ be three fuzzy numbers. Then

1. $I_1 \subseteq I_2$ iff $\varsigma_1 \preccurlyeq \varsigma_2$
2. $I_1 = I_2$ iff $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$
3. $I_1 \cup I_2 = (\curlyvee (\varsigma_1, \varsigma_2))$
4. $I_1 \cap I_2 = (\curlywedge (\varsigma_1, \varsigma_2))$
5. $I^c = 1 - \varsigma$

1.2. Intuitionistic Fuzzy Set

The concept of IFS was proposed by Atanassov [5] in which he extended the concept of Zadeh's [1] FS by introducing a NG along with MG with a condition that their sum must not exceed 1.

1.2.1. Definition [5]

On a set X, an IFS is of the shape $I = \{(\varkappa, (\varsigma, d')) : 0 \leq \varsigma um(\varsigma(\varkappa), d'(\varkappa)) \leq 1\}$. Further, $r(\varkappa) = 1 - \varsigma um(\varsigma, d')$ represents the hesitancy degree of $\varkappa \in X$ and the pair (ς, d') is termed as an IFN.

The basic set-theoretic operations of union, intersection, inclusion and complement of IFNs were also proposed by Atanassov [5] which are given as follows.

1.2.2. Definition [5]

Let $I = (\varsigma, d'), I_1 = (\varsigma_1, d'_1)$ and $I_2 = (\varsigma_2, d'_2)$ be three IFNs. Then

1. $I_1 \subseteq I_2$ iff $\varsigma_1 \preccurlyeq \varsigma_2, d'_1 \succcurlyeq d'_2$

2. $I_1 = I_2$ iff $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$

3. $I_1 \cup I_2 = \left(\curlyvee (ş_1, ş_2), \curlywedge (d'_1, d'_2)\right)$

4. $I_1 \cap I_2 = \left(\curlywedge (ş_1, ş_2), \curlyvee (d'_1, d'_2)\right)$

5. $I^c = (d', ş)$

The sum, product, scalar multiplication and power operations of IFNs are given as:

1.2.3. Definition [29, 30]

Let $I_1 = (ş_1, d'_1)$ & $I_2 = (ş_2, d'_2)$ be two IFNs and $\lambda > 0$. Then

1. $I_1 \oplus I_2 = (ş_1 + ş_2 - ş_1.ş_2, d'_1.d'_2)$.
2. $I_1 \otimes I_2 = (ş_1.ş_2, d'_1 + d'_2 - d'_1.d'_2)$
3. $\lambda . I = \left(1 - (1 - ş)^\lambda, (d')^\lambda\right)$.
4. $I^\lambda = \left((ş)^\lambda, 1 - (1 - d')^\lambda\right)$.

To rank two IFNs, a score function is usually used and in case when score function could not differentiate between two IFNs, an accuracy function is introduced. The concept of score and accuracy functions of IFNs are defined as:

1.2.4. Definition [29]

Let $I = (ş, d')$ be an IFN. Then the score value of I is defined as $SC(I) = ş - d'$ and $SC(I) \in [-1, 1]$.

In view of the score function, for two INFs $I_1 = (ş_1, d'_1)$ and $I_2 = (ş_2, d'_2)$, we have

- I_1 is superior to I_2 if $SC(I_1) > SC(I_2)$.
- I_1 is inferior to I_2 if $SC(I_1) < SC(I_2)$.

In case when, $SC(I_1) = SC(I_2)$. Then two IFNs can be distinguished from each other by the help of accuracy function which is defined as:

1.2.5. Definition [29]

Let $I = (\text{ş}, \text{d'})$ be an IFN. Then the accuracy value of I is defined as $AC(I) = \text{ş} + \text{d'}$ and $AC(I) \in [0, 1]$.

In view of the accuracy function, for two INFs $I_1 = (\text{ş}_1, \text{d'}_1)$ and $I_2 = (\text{ş}_2, \text{d'}_2)$, we have

- I_1 is superior to I_2 if $AC(I_1) > AC(I_2)$.
- I_1 is inferior to I_2 if $AC(I_1) < AC(I_2)$.
- I_1 is similar to I_2 if $AC(I_1) = AC(I_2)$.

Atanassov [14] improved the idea of IFS to IVIFS by expressing the MG and NG in terms of closed subintervals of $[0, 1]$ instead of crisp numbers.

1.2.6. Definition [14]

On a set X, an IVIFS is of the shape $I = \left\{\left(\varkappa, ([\text{ş}^l, \text{ş}^u], [\text{d'}^l, \text{d'}^u])\right) : 0 \leq \text{ş}um(\text{ş}^u(\varkappa), \text{d'}^u(\varkappa)) \leq 1\right\}$. Further, $r(\varkappa) = [r^l, r^u] = \left[1 - \text{ş}um\left(\text{ş}^l(\varkappa), \text{d'}^l(\varkappa)\right), 1 - \text{ş}um(\text{ş}^u(\varkappa), \text{d'}^u(\varkappa))\right]$ represents the hesitancy degree of $\varkappa \in X$ and the pair $([\text{ş}^l, \text{ş}^u], [\text{d'}^l, \text{d'}^u])$ is termed as an interval valued IFN (IVIFN).

All other basics operations of IVIFSs are defined analogously and one is referred to [14, 34-37] for a better understanding of these concepts.

The concept of Atanassov's IFS [5] provides a lesser range for selecting the MG and NG by a restriction. To deal with such situation, the concept of PyFS [6] was introduced which gave relatively larger range.

1.3. Pythagorean Fuzzy Set

The concept of PyFS was proposed by Yager [6] to increase the range for selecting MG and NG. PyFS allows the sum of squares of the MG and NG to be in $[0, 1]$.

1.3.1. Definition [6]

On a set X, a PyFS is of the shape $I = \{(\varkappa, (\c{s}, \mathrm{d'})) : 0 \leq \c{s}um(\c{s}^2(\varkappa), \mathrm{d'}^2(\varkappa)) \leq 1\}$. Further, $r(\varkappa) = \sqrt{1 - \c{s}um(\c{s}^2(\varkappa), \mathrm{d'}^2(\varkappa))}$ represents the hesitancy degree of $\varkappa \in X$ and the pair $(\c{s}, \mathrm{d'})$ is termed as a PyFN.

The basic set-theoretic operations of union, intersection, inclusion and complement of PyFNs were proposed by Peng and Yang [15] which are given as follows.

1.3.2. Definition [15]

Let $I = (\c{s}, \mathrm{d'}), I_1 = (\c{s}_1, \mathrm{d'}_1)$ and $I_2 = (\c{s}_2, \mathrm{d'}_2)$ be three PyFNs. Then

1. $I_1 \subseteq I_2$ iff $\c{s}_1 \preccurlyeq \c{s}_2, \mathrm{d'}_1 \succcurlyeq \mathrm{d'}_2$
2. $I_1 = I_2$ iff $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$
3. $I_1 \cup I_2 = (\curlyvee (\c{s}_1, \c{s}_2), \curlywedge (\mathrm{d'}_1, \mathrm{d'}_2))$
4. $I_1 \cap I_2 = (\curlywedge (\c{s}_1, \c{s}_2), \curlyvee (\mathrm{d'}_1, \mathrm{d'}_2))$
5. $I^c = (\mathrm{d'}, \c{s})$

The sum, product, scalar multiplication and power operations of PyFNs are given as:

1.3.3. Definition [15]

Let $I_1 = (\S_1, \text{d'}_1)$ & $I_2 = (\S_2, \text{d'}_2)$ be two PyFNs and $\lambda > 0$. Then

1. $I_1 \oplus I_2 = \left(\sqrt{\S_1^2 + \S_2^2 - \S_1^2 . \S_2^2}, \text{d'}_1 . \text{d'}_2\right)$.
2. $I_1 \otimes I_2 = \left(\S_1 . \S_2, \sqrt{\text{d'}_1^2 + \text{d'}_2^2 - \text{d'}_1^2 . \text{d'}_2^2}\right)$
3. $\lambda . I = \left(\sqrt{1 - (1 - \S^2)^\lambda}, (\text{d'})^\lambda\right)$.
4. $I^\lambda = \left((\S)^\lambda, \sqrt{1 - (1 - \text{d'}^2)^\lambda}\right)$.

To rank two PyFNs, a score function is usually used and in case when score function could not differentiate between two PyFNs, an accuracy function is introduced. The concept of score and accuracy functions of PyFNs are defined as:

1.3.4. Definition [15]

Let $I = (\S, \text{d'})$ be a PyFN. Then the score value of I is defined as $SC(I) = \S^2(\varkappa) - \text{d'}^2(\varkappa)$ and $SC(I) \in [-1, 1]$.

In view of the score function, for two PyFNs $I_1 = (\S_1, \text{d'}_1)$ and $I_2 = (\S_2, \text{d'}_2)$, we have

- I_1 is superior to I_2 if $SC(I_1) > SC(I_2)$.
- I_1 is inferior to I_2 if $SC(I_1) < SC(I_2)$.

In case when, $SC(I_1) = SC(I_2)$. Then two PyFNs can be distinguished from each other by the help of accuracy function which is defined as:

1.3.5. Definition [15]

Let $I = (ş, d')$ be a PyFN. Then the accuracy value of I is defined as $AC(I) = ş^2(\varkappa) + d'^2(\varkappa)$ and $AC(I) \in [0, 1]$.

In view of the accuracy function, for two PyFNs $I_1 = (ş_1, d'_1)$ and $I_2 = (ş_2, d'_2)$, we have

- I_1 is superior to I_2 if $AC(I_1) > AC(I_2)$.
- I_1 is inferior to I_2 if $AC(I_1) < AC(I_2)$.
- I_1 is similar to I_2 if $AC(I_1) = AC(I_2)$.

Peng and Yang [15] improved the idea of PyFS to IVPyFS by expressing the MG and NG in terms of closed subintervals of $[0, 1]$ instead of crisp numbers and is defined as follows:

1.3.6. Definition [15]

On a set X, an IVPyFS is of the shape $I = \left\{\left(\varkappa, ([ş^l, ş^u], [d'^l, d'^u])\right): 0 \leq şum\left((ş^u(\varkappa))^2, (d'^u(\varkappa))^2\right) \leq 1\right\}$. Further, $r(\varkappa) = [r^l(\varkappa), r^u(\varkappa)] = \left[1 - şum\left((ş^l(\varkappa))^2, (d'^l(\varkappa))^2\right), 1 - şum\left((ş^u(\varkappa))^2, (d'^u(\varkappa))^2\right)\right]$ represents the hesitancy degree of $\varkappa \in X$ and the pair $([ş^l, ş^u], [d'^l, d'^u])$ is termed as an interval valued PyFN (IVPyFN).

All other basics operations of IVPyFSs are defined analogously and one is referred to [15, 46, 47] for a better picture of these notions.

The concept of PyFS [6] provides comparatively larger range than Atanassov's IFS [5] for selecting the MG and NG but still it is not enough to make every pair $(ş, d')$ a PyFN. To improve the concept of PyFS, another concept of q-ROFS [7] is introduced which increased the range of IFS as well as PyFS significantly.

1.4. q-Rung Orthopair Fuzzy Set

The concept of q-ROFS was proposed by Yager [8] to further increase the range for selecting MG and NG. A q-ROFS allows the sum of q^{th} power of the MG and NG to be in $[0, 1]$ which generalizes both, the framework of IFSs and PyFSs.

1.4.1. Definition [8]

On a set X, a q-ROFS is of the shape $I = \{(\varkappa, (\c{s}, \text{d}')) : 0 \leq \c{s}um(\c{s}^q(\varkappa), \text{d}'^q(\varkappa)) \leq 1, q \in \mathbb{Z}^+\}$. Further, $r(\varkappa) = \sqrt[q]{1 - \c{s}um(\c{s}^q(\varkappa), \text{d}'^q(\varkappa))}$ represents the hesitancy degree of $\varkappa \in X$ and the pair $(\c{s}, \text{d}')$ is termed as a q-ROFN.

The sum, product, scalar multiplication and power operations of q-ROFNs are given as:

1.4.2. Definition [85]

Let $I_1 = (\c{s}_1, \text{d}'_1)$ & $I_2 = (\c{s}_2, \text{d}'_2)$ be two q-ROFNs and $\lambda > 0$. Then

1. $I_1 \oplus I_2 = \left(\sqrt[q]{\c{s}_1^q + \c{s}_2^q - \c{s}_1^q . \c{s}_2^q}, \text{d}'_1 . \text{d}'_2\right)$.
2. $I_1 \otimes I_2 = \left(\c{s}_1 . \c{s}_2, \sqrt[q]{\text{d}'^q_1 + \text{d}'^q_2 - \text{d}'^q_1 . \text{d}'^q_2}\right)$
3. $\lambda . I = \left(\sqrt[q]{1 - (1 - \c{s}^q)^\lambda}, (\text{d}')^\lambda\right)$.
4. $I^\lambda = \left((\c{s})^\lambda, \sqrt[q]{1 - (1 - \text{d}'^q)^\lambda}\right)$.

To rank two q-ROFNs, a score function is usually used and in case when score function could not differentiate between two q-ROFNs, an accuracy function is introduced. The concept of score and accuracy functions of q-ROFNs are defined as:

1.4.3. Definition [85]

Let $I = (\c{s}, d')$ be a q-ROFN. Then the score value of I is defined as $SC(I) = \c{s}^q(\varkappa) - d'^q(\varkappa)$ and $SC(I) \in [-1, 1]$.

In view of the score function, for two q-ROFNs $I_1 = (\c{s}_1, d'_1)$ and $I_2 = (\c{s}_2, d'_2)$, we have

- I_1 is superior to I_2 if $SC(I_1) > SC(I_2)$.
- I_1 is inferior to I_2 if $SC(I_1) < SC(I_2)$.

In case when, $SC(I_1) = SC(I_2)$. Then two q-ROFNs can be distinguished from each other by the help of accuracy function which is defined as:

1.4.4. Definition [85]

Let $I = (\c{s}, d')$ be a q-ROFN. Then the accuracy value of I is defined as $AC(I) = \c{s}^q(\varkappa) + d'^q(\varkappa)$ and $AC(I) \in [0, 1]$.

In view of the accuracy function, for two q-ROFNs $I_1 = (\c{s}_1, d'_1)$ and $I_2 = (\c{s}_2, d'_2)$, we have

- I_1 is superior to I_2 if $AC(I_1) > AC(I_2)$.
- I_1 is inferior to I_2 if $AC(I_1) < AC(I_2)$.
- I_1 is similar to I_2 if $AC(I_1) = AC(I_2)$.

Joshi et al. [16] improved the idea of q-ROFS to IVq-ROFS by expressing the MG and NG in terms of closed subintervals of [0, 1] instead of crisp numbers and defined it as follows:

1.4.5. Definition [16]

On a set X, an IVq-ROFS is of the shape $I = \left\{\left(\varkappa, ([\c{s}^l(\varkappa), \c{s}^u(\varkappa)], [d'^l(\varkappa), d'^u(\varkappa)])\right) : 0 \leq \c{s}um((\c{s}^u)^q, (d'^u)^q) \leq 1, q \in \mathbb{Z}^+\right\}$. Further, $r(\varkappa) = [r^l(\varkappa), r^u(\varkappa)] = [1 - \c{s}um((\c{s}^l)^q, (d'^l)^q), 1 - \c{s}um((\c{s}^u)^q, (d'^u)^q)]$ represents the hesitancy degree of $\varkappa \in X$ and the pair $([\c{s}^l, \c{s}^u], [d'^l, d'^u])$ is termed as an interval valued q-ROFN (IVq-ROFN).

All other basics operations of IVq-ROFSs are defined analogously and one is referred to [15, 46, 47] for a better understanding of these notions.

The frameworks of IFSs, PyFSs and q-ROFSs proposed by Atanassov [5], and Yager [6, 8] respectively and their generalized versions where MG and NG are expressed in interval as defined in [14-16] can deal with only two aspects of human opinion i.e. yes and no or favor and disfavor. In real life problems, such as decision making, pattern recognition and clustering, a human opinion have more than two aspects and along with a MG and a NG there is an AG and a RG as well. To face such problems, Cuong introduced the notion of PFSs which takes four types of functions denoting MG, AG, NG and RG to model an uncertain situation. We discussed some aspects of PFSs in the following section.

1.5. Picture Fuzzy Set

The concept of PFS was proposed by Cuong [9] in which he extended the concept of Atanassov's IFS [5] by introducing an AG grade along with a MG and a NG with a condition that their sum must not exceeds 1.

1.5.1. **Definition** [9]

On a set X, a PFS is of the shape $I = \{(\varkappa, (\text{ş}, i, \text{d}')) : 0 \leq \text{ş}um(\text{ş}(\varkappa), i(\varkappa), \text{d}'(\varkappa)) \leq 1\}$. Further, $r(\varkappa) = 1 - \text{ş}um(\text{ş}(\varkappa), i(\varkappa), \text{d}'(\varkappa))$ represents the RG of $\varkappa \in X$ and the triplet $(\text{ş}, i, \text{d}')$ is termed as a PFN.

The basic set-theoretic operations of union, intersection, inclusion and complement of PFNs were also proposed by Cuong [9] which are given as follows.

1.5.2. **Definition** [9]

Let $I = (\text{ş}, i, \text{d}'), I_1 = (\text{ş}_1, i_1, \text{d}'_1)$ and $I_2 = (\text{ş}_2, i_2, \text{d}'_2)$ be three PFNs. Then

1. $I_1 \subseteq I_2$ iff $\text{ş}_1 \preccurlyeq \text{ş}_2, i_1 \preccurlyeq i_2, \text{d}'_1 \succcurlyeq \text{d}'_2$.
2. $I_1 = I_2$ iff $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$.
3. $I_1 \cup I_2 = \left(\curlyvee(\text{ş}_1, \text{ş}_2), \curlywedge(i_1, i_2), \curlywedge(\text{d}'_1, \text{d}'_2)\right)$.
4. $I_1 \cap I_2 = \left(\curlywedge(\text{ş}_1, \text{ş}_2), \curlywedge(i_1, i_2), \curlyvee(\text{d}'_1, \text{d}'_2)\right)$.
5. $I^c = (\text{d}', i, \text{ş})$.

The sum, product, scalar multiplication and power operations of PFNs are investigated by three different groups. Wang et al. [51] proposed the product and power operations for PFNs while Garg [50] defined the sum, product, scalar multiplication and power operations of PFNs in a different way than Wang et al. [51]. However, Wei [83] also investigated these basic operations and applied them in MADM problems.

The picture fuzzy operations proposed by Wang et al. [51] are given by:

1.5.3. Definition [51]

Let $I_1 = (ş_1, i_1, d'_1)$ & $I_2 = (ş_2, i_2, d'_2)$ be two PFNs and $\lambda > 0$. Then

1. $I_1 \otimes I_2 = \left((ş_1 + i_1).(ş_2 + i_2) - i_1.i_2, i_1.i_2, 1 - (1 - d'_1)(1 - d'_2)\right)$.
2. $I^\lambda = \left((ş + i)^\lambda - i^\lambda, i^\lambda, 1 - (1 - d')^\lambda\right)$.

The basic operations of PFNs proposed by Garg [50] and Wei [83] are defined as:

1.5.4. Definition [50, 83]

Let $I_1 = (ş_1, i_1, d'_1)$ & $I_2 = (ş_2, i_2, d'_2)$ be two PFNs and $\lambda > 0$. Then

1. $I_1 \oplus I_2 = (ş_1 + ş_2 - ş_1.ş_2, i_1.i_2, d'_1.d'_2)$.
2. $I_1 \otimes I_2 = (ş_1.ş_2, i_1 + i_2 - i_1.i_2, d'_1 + d'_2 - d'_1.d'_2)$
3. $\lambda.I = \left(1 - (1 - ş)^\lambda, (i)^\lambda, (d')^\lambda\right)$.
4. $I^\lambda = \left((ş)^\lambda, 1 - (1 - i)^\lambda,\ 1 - (1 - d')^\lambda\right)$.

To rank two PFNs, a score function is usually used and in case when score function could not differentiate between two PFNs, an accuracy function is introduced. The concept of score function of a PFN defined by Wang et al. [51] and Wei [83] is given by:

1.5.5. Definition [51, 83]

Let $I = (ş, i, d')$ be a PFN. Then the score value of I is defined as $SC(I) = ş(\varkappa) - d'(\varkappa)$ and $SC(I) \in [-1, 1]$.

The score function for PFNs proposed by Garg [50] is given as:

1.5.6. Definition [50]

Let $I = (\c{s}, i, d')$ be a PFN. Then the score value of I is defined as $SC(I) = \c{s}(\varkappa) - i(\varkappa) - d'(\varkappa)$ and $SC(I) \in [-1, 1]$.

In view of these score functions, for two PNFs $I_1 = (\c{s}_1, i_1, d'_1)$ and $I_2 = (\c{s}_2, i_2, d'_2)$, we have

- I_1 is superior to I_2 if $SC(I_1) > SC(I_2)$.
- I_1 is inferior to I_2 if $SC(I_1) < SC(I_2)$.

In case when, $SC(I_1) = SC(I_2)$. Then two PFNs can be distinguished from each other by the help of accuracy function which is defined as:

1.5.7. Definition [50, 51, 83]

Let $I = (\c{s}, i, d')$ be a PFN. Then the accuracy value of I is defined as $AC(I) = \c{s}(\varkappa) + i(\varkappa) + d'(\varkappa)$ and $AC(I) \in [0, 1]$.

In view of the accuracy function, for two PNFs $I_1 = (\c{s}_1, i_1, d'_1)$ and $I_2 = (\c{s}_2, i_2, d'_2)$, we have

- I_1 is superior to I_2 if $AC(I_1) > AC(I_2)$.
- I_1 is inferior to I_2 if $AC(I_1) < AC(I_2)$.
- I_1 is similar to I_2 if $AC(I_1) = AC(I_2)$.

Keeping in mind the significance of IVIFS [14], IVPyFS [15] and IVq-ROFS [16], Cuong [9] also improved the idea of PFS to develop IVPFS by expressing the MG, AG and NG in terms of closed subintervals of $[0, 1]$ instead of crisp numbers.

1.5.8. Definition [14]

On a set X, an IVPFS is of the shape $I = \{(\varkappa, ([ş^l, ş^u], [i^l, i^u], [d'^l, d'^u])) : 0 \leq şum(ş^u, i^u, d'^u) \leq 1\}$. Further, $r(\varkappa) = [r^l, r^u] = [1 - şum(ş^l(\varkappa), i^l(\varkappa), d'^l(\varkappa)), 1 - şum(ş^u(\varkappa), i^u(\varkappa), d'^u(\varkappa))]$ represents the RG of $\varkappa \in X$ and the triplet $([ş^l, ş^u], [i^l, i^u], [d'^l, d'^u])$ is termed as an interval valued PFN (IVPFN).

All other basics operations of IVPFSs are defined analogously and one is referred to [14] for a better understanding of these concepts.

Another interesting and widely used framework that generalizes the framework of IFS is NS theory proposed by Smarandache [12] which is further extended to propose the idea of SVNS by Wang et al. [17]. A SVNS also consider the neutral membership degree of human opinion and did not restrict the MG, neutral MG and NG at all. According to SVNS, the sum of MG, neutral MG and NG must be less than or equal to 3 hence providing a smooth platform to assign MG, neutral MG and NG without fear of exceeding from 1 like in IFS and PFS. Some basic notions of SVNSs are discussed the following section.

1.6. Single Valued Neutrosophic Set

The concept of SVNS was proposed by Wang et al. [17] in which he extended the concept of Atanassov's IFS [5] by introducing a neutral MG along with a MG and a NG with a condition that their sum must not exceeds 3.

1.6.1. Definition [17]

On a set X, an SVNS is of the shape $I = \{(\varkappa, (ş, i, d')) : 0 \leq şum(ş(\varkappa), i(\varkappa), d'(\varkappa)) \leq 3\}$.

Further, the triplet $(ş, i, d')$ is termed as a SVNN.

The sum, product, scalar multiplication and power operations of SVNNs are investigated by Ye [77]. Operations proposed by Ye have some shortcoming suggested by Zhang et al. [91] in interval neutrosophic settings and are defined as follows:

1.6.2. Definition [91]

Let $I_1 = (\c{s}_1, i_1, d'_1)$ & $I_2 = (\c{s}_2, i_2, d'_2)$ be two SVNNs and $\lambda > 0$. Then

1. $I_1 \oplus I_2 = (\c{s}_1 + \c{s}_2 - \c{s}_1.\c{s}_2, i_1.i_2, d'_1.d'_2)$.
2. $I_1 \otimes I_2 = (\c{s}_1.\c{s}_2, i_1 + i_2 - i_1.i_2, d'_1 + d'_2 - d'_1.d'_2)$
3. $\lambda.I = \left(1 - (1 - \c{s})^{\lambda}, (i)^{\lambda}, (d')^{\lambda}\right)$.
4. $I^{\lambda} = \left((\c{s})^{\lambda}, 1 - (1 - i)^{\lambda}, 1 - (1 - d')^{\lambda}\right)$.

To rank two SVNNs, a score function is usually used and in case when score function could not differentiate between two PFNs, an accuracy function is introduced. The concept of score function of a SVNN defined by Sahin [89] which was further improved by Nancy and Garg [90] and is described as:

1.6.3. Definition [90]

Let $I = (\c{s}, i, d')$ be a SVNN. Then the score value of I is defined as

$$SC(I) = \frac{1 + \c{s} - 2i - d'}{2}$$

and $SC(I) \in [-1, 1]$.

The score function for SVNNs proposed by Garg [90] is given as:

1.6.4. Definition [90]

Let $I = (\text{ş}, i, \text{d'})$ be a SVNN. Then the score value of I is defined as

$$SC(I) = \frac{1 + (\text{ş} - 2i - \text{d'})(2 - \text{ş} - \text{d'})}{2}$$

and $SC(I) \in [-1, 1]$.

Chapter 2

An Approach Towards Decision Making Using the Concept of Spherical and T-Spherical Fuzzy Sets[1]

Human opinion cannot be restricted to yes or no as depicted by conventional FS [1] and IFS [5], PyFS [6] and q-ROFS [8] but it can be yes, abstain, no and refusal as explained by PFS [9]. In this chapter, the concept of spherical fuzzy set (SFS) and T-spherical fuzzy set (TSFS) is introduced as a generalization of FS, IFS, PyFS, q-ROFS and PFS. The novelty of SFS and TSFS is shown by examples and graphical comparison with early established concepts. Some operations of SFSs and TSFSs along with spherical fuzzy relations (SFRs) are defined and related results are conferred. Medical diagnosis and decision-making problems are discussed in the environment of SFSs and TSFSs as practical applications.

2.1. Spherical Fuzzy Set

This section is based on the description of the novel concept of SFSs such as the definition of SFS, its importance and novelty and its graphical comparison with early established concepts.

[1] Work of this chapter has been published in the following paper:
Mahmood T., **Ullah K.**, Khan Q. and Jan N. An Approach Towards Decision Making and Medical Diagnosis Problems Using the Concept of Spherical Fuzzy Sets. Neural Computing and Applications. 2018, 31 (11), 7041-7053. https://doi.org/10.1007/s00521-018-3521-2

2.1.1. Definition

For any universal set X, a SFS is of the form $I = \{(\varkappa, (ş, i, d')) \; \forall \; \varkappa \in X\}$. Here $ş, i$ and d' are mappings from $X \to [0,1]$ denoting the MG, AG and NG respectively provided that $0 \leq şum\left(ş^2(\varkappa), i^2(\varkappa), d'^2(\varkappa)\right) \leq 1$ and $r(\varkappa) = \sqrt{1 - şum\left(ş^2(\varkappa), i^2(\varkappa), d'^2(\varkappa)\right)}$ is known as the RD of $\varkappa$ in I. The triplet $(ş, i, d')$ is considered as a spherical fuzzy number (SFN).

SFSs have its importance in a situation where opinion is not only restricted to yes or no, but some sort of abstinence or refusal aspects are involved. A good example of SFS could be decision making where human opinion about a candidate could possibly be in the form of yes, abstain, no or refusal. Another example could be of voting where vote could be of four types i.e. vote in favor or vote against or refuse to vote or abstain.

A question arises that why we need SFS or what are the limitations of PFS that leads us to the concept of SFS?

The main drawback of PFS is the constraint on it i.e. $0 \leq şum\left(ş(\varkappa), i(\varkappa), d'(\varkappa)\right) \leq 1$ as this condition does not allow the decision makers to assign membership values of their own choice. The decision makers are somehow bounded in a specific range. For example, if we choose $ş = 0.8, i = 0.2$ and $d' = 0.4$. Then in this case $0 \leq şum\,(ş, i, d') = 1.4 \nleq 1$. But by squaring $ş, i$ and d' and the sum becomes less than or equal to one i.e. $0 \leq şum\,((0.8)^2, (0.2)^2, (0.4)^2) = \leq 1$. This shows that SFS generalizes FS, IFS, PyFS and PFS.

To understand why this new structure is named as SFS, consider ş, i and d' represent the MG, AG and NG of a SFS respectively such that $0 \leq şum\ (ş^2, i^2, d'^2) \leq 1$. Now if we observe the given condition, we may get the equation $0 \leq şum\ (ş^2, i^2, d'^2) \leq 1$ which represents the space occupied by a part of unit sphere as shown in Figure 6.

If we look at the condition of PFS i.e. $0 \leq şum\ (ş, i, d') \leq 1$. This represents a space under the curve $ş = 1 - (i + d')$ where the values of ş, i and d' are taken from $[0, 1]$. For comparison purpose, the space of all PFNs it is depicted in Figure 7.

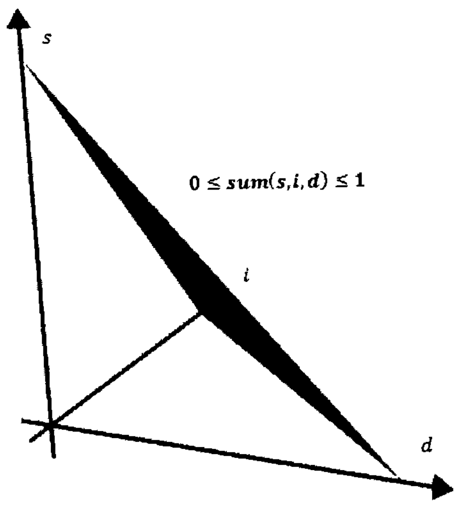

Figure 7 ((Space of all PFNs))

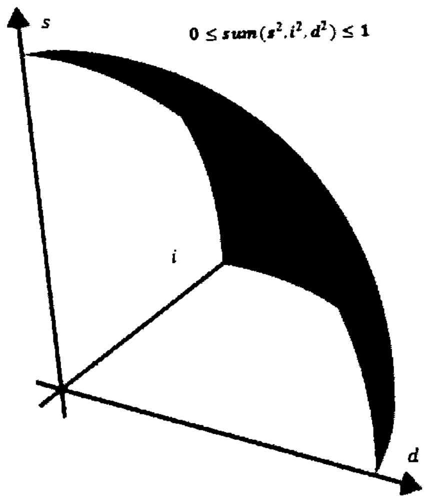

Figure 6 (Space of all SFNs)

This comparison is similar to that of IFSs and PyFSs as shown in Figure 2 but in that cases their planes are 2-dimensional while here we have 3-dimensional space due to the number of components in SFS.

The following theorem explained the generalization SFS over PFS more clearly.

2.1.2. Theorem

The space of SFSs is greater than that of space of PFSs.

Proof: This result can be proved in two parts.

1. Every PFN is also a SFN. For example, let $(\c{s}, i, \mathrm{d}')$ be a PFN with $\c{s}, i$ and' $\mathrm{d}' \in [0, 1]$ such that $0 \leq \c{s}um(\c{s}, i, \mathrm{d}') \leq 1$. Then it is obvious that $\c{s}um(\c{s}^2, i^2, \mathrm{d}'^2) \in [0, 1]$ as $\c{s}^2 \leq \c{s}, i^2 \leq i$ and $\mathrm{d}'^2 \leq \mathrm{d}'$.
2. A SFN may not necessarily be a PFN. For example, if $\c{s} = 0.8, i = 0.3$ and $\mathrm{d}' = 0.6$ then obviously $\c{s}um(\c{s}^2, i^2, \mathrm{d}'^2) \in [0, 1]$ but $0 \leq \c{s}um(\c{s}, i, \mathrm{d}') = 1.7 \nleq 1$.

Thus, the space of SFSs is larger than space of PFSs. While working in spherical fuzzy environment, one may have much more choices of assigning values to $\c{s}$, i and d' from $[0, 1]$ due to which ground of SFNs is of more value than PFNs.

2.2. Operations on Spherical Fuzzy Sets

This section is about some operations on SFSs such as inclusion, union, intersection, complement and their properties. We also defined the concept of relations in spherical fuzzy environment and studied their composition. In our further discussion X plays the role of universal set and $I = (\c{s}, i, \mathrm{d}'), I_1 = (\c{s}_1, i_1, \mathrm{d}'_1)$ and $I_2 = (\c{s}_2, i_2, \mathrm{d}'_2)$ shall denote three SFNs.

2.2.1. Definition

Let I, I_1 and I_2 be three SFNs. Then

1. $I_1 \subseteq I_2$ iff $\c{s}_1 \preccurlyeq \c{s}_2, i_1 \preccurlyeq i_2, \mathrm{d}'_1 \succcurlyeq \mathrm{d}'_2$.
2. $I_1 = I_2$ iff $I_1 \subseteq I_2$ and $I_2 \subseteq I_1$.
3. $I_1 \cup I_2 = \left(\curlyvee (\c{s}_1, \c{s}_2), \curlywedge (i_1, i_2), \curlywedge (\mathrm{d}'_1, \mathrm{d}'_2)\right)$.
4. $I_1 \cap I_2 = \left(\curlywedge (\c{s}_1, \c{s}_2), \curlywedge (i_1, i_2), \curlyvee (\mathrm{d}'_1, \mathrm{d}'_2)\right)$.
5. $I^c = (\mathrm{d}', i, \c{s})$.

2.2.2. Proposition

Let I, I_1 and I_2 be three SFNs. Then

1. $I_1 \subseteq I_2$ & $I_2 \subseteq I_3$ implies that $I_1 \subseteq I_3$.
2. $I_1 \cup I_2 = I_2 \cup I_1$ & $I_1 \cap I_2 = I_2 \cap I_1$.
3. $I_1 \cup (I_2 \cup I_3) = (I_1 \cup I_2) \cup I_3$ & $I_1 \cap (I_2 \cap I_3) = (I_1 \cap I_2) \cap I_3$.
4. $I_1 \cup (I_2 \cap I_3) = (I_1 \cup I_2) \cap (I_1 \cup I_3)$ & $I_1 \cap (I_2 \cup I_3) = (I_1 \cap I_2) \cup (I_1 \cap I_3)$.
5. De-Morgan's laws hold for I_1 and I_2.

Proof:

(1). $I_1 \subseteq I_2$ implies $\varsigma_1 \leqslant \varsigma_2, i_1 \leqslant i_2, d'_1 \geqslant d'_2$ for all $\varkappa \in X$ and

$I_2 \subseteq I_3$ implies $\varsigma_2 \leqslant \varsigma_3, i_2 \leqslant i_3, d'_2 \geqslant d'_3$ for all $\varkappa \in X$. Now

$\varsigma_1 \leqslant \varsigma_2$ and $\varsigma_2 \leqslant \varsigma_3$ implies $\varsigma_1 \leqslant \varsigma_3$.

$i_1 \leqslant i_2$ and $i_2 \leqslant i_3$ implies $i_1 \leqslant i_3$.

$d'_1 \geqslant d'_2$ and $d'_2 \geqslant d'_3$ implies $d'_1 \geqslant d'_3$.

Hence $I_1 \subseteq I_3$.

The remaining parts of the proposition holds i.e. parts 2, 3, 4, 5 holds trivially.

Now, we defined the distance function for SFNs followed by an example for illustration purpose.

2.2.3. Definition

For two SFNs $I_1 = (\c{s}_1, i_1, d'_1)$ and $I_2 = (\c{s}_2, i_2, d'_2)$, the normalized hamming distance is denoted and defined as:

$$HD(I_1, I_2) = \frac{1}{m}\sum_{i=1}^{m}(|\c{s}_1^2(\varkappa_i) - \c{s}_2^2(\varkappa_i)| + |i_1^2(\varkappa_i) - i_2^2(\varkappa_i)| + |d'^2_1(\varkappa_i) - d'^2_2(\varkappa_i)|)$$

2.2.4. Example

Let I_1 and I_2 be two SFNs in $X = \{\varkappa_1, \varkappa_2, \varkappa_3\}$ i.e. $I_1 = \{\varkappa_1, (0.7, 0.4, 0.2), \varkappa_2, (0.5, 0.3, 0.7), \varkappa_3, (0.2, 0.8, 0.3)\}$ & $I_2 = \{\varkappa_1, (0.7, 0.6, 0.0), \varkappa_2, (0.2, 0.4, 0.5), \varkappa_3, (0.6, 0.2, 0.3)\}$.

Then $HD(A, B) = 0.1666667$.

Comparison rules in FS theory have always importance especially in decision making and some other practical problems. The comparison rules enable us to differentiate between two fuzzy number or sometime tells us the strength between a pair of relation i.e. how strongly two variables are related. Here, we define the score and accuracy function for SFNs.

2.2.5. Definition

Let I be a SFN. Then the score value of I is defined as $SC(I) = \c{s}^2(\varkappa) - d'^2(\varkappa)$ where $SC(I) \in [-1, 1]$. In case, when score function could not differentiate between two SFNs, the concept of accuracy function is used. The accuracy function of a SFN I is defined as $AC(I) = \c{s}^2(\varkappa) + i^2(\varkappa) + d'^2(\varkappa)$ and $AC(I) \in [0, 1]$.

Based on above definition, for two SFNs I_1 and I_2, we have

- I_1 is superior to I_2 if $SC(I_1) > SC(I_2)$.
- I_1 is inferior to I_2 if $SC(I_1) < SC(I_2)$.

If $SC(I_1) = SC(I_2)$ for two SFNs. Then

- I_1 is superior to I_2 if $AC(I_1) > AC(I_2)$.
- I_1 is inferior to I_2 if $AC(I_1) < AC(I_2)$.
- I_1 is similar to I_2 if $AC(I_1) = AC(I_2)$.

2.3. Relations on Spherical Fuzzy Sets

A quality work is done in [19-22, 28] so far on theory of relations. These ideas are extensively used in practical work such as in the process of medical diagnosis etc. This section deals with SFRs, their properties and compositions of the defined relations. The defined composition is then applied to a problem of medical diagnosis using spherical fuzzy information.

To proceed with our study of SFRs and their compositions, first we define the concept of Cartesian product for two SFSs I_1 and I_2.

2.3.1. Definition

Let X_1 and X_2 be two universal sets. Then the Cartesian product of two SFSs I_1 and I_2 is of the form:

$$I_1 \times_1 I_2 = \left\{\left((\varkappa, y), \left(\S_1(\varkappa).\S_2(y), i_1(\varkappa).i_2(y), d'_1(\varkappa).d'_2(y)\right)\right) : \varkappa \in X_1, y \in X_2\right\}.$$

$$I_1 \times_2 I_2 = \left\{\left((\varkappa, y), \left(\S_1(\varkappa)\wedge\S_2(y), i_1(\varkappa)\wedge i_2(y), d'_1(\varkappa)\vee d'_2(y)\right)\right) : \varkappa \in X_1, y \in X_2\right\}.$$

These two definitions are the direct extensions of the Cartesian product of FSs, IFSs and PFSs.

2.3.2. Definition

A SFR R on $X \times Y$ is a spherical fuzzy subset of $X \times Y$ i.e. $R = \{((\varkappa, y), (\c{s}, i, \mathrm{d}')) : \varkappa \in X \,\&\, y \in Y\}$ provided that the values of $\c{s}(\varkappa, y), i(\varkappa, y)$, $\mathrm{d}'(\varkappa, y)$ are form $[0, 1]$ and $0 \leq \c{s}um(\c{s}^2(\varkappa, y), i^2(\varkappa, y), \mathrm{d}'^2(\varkappa, y)) \leq 1$ for all $(\varkappa, y) \in X \times Y$. Moreover, the set of all spherical fuzzy relations on $X \times Y$ is denoted by $SFR(X \times Y)$.

The inverse of a SFR can be obtained using the following definition.

2.3.3. Definition

For any SFR R on $X \times Y$, we defined R^{-1} on $Y \times X$ (an inverse relation of R) as $\c{s}_{R^{-1}}(y, \varkappa) = \c{s}_R(\varkappa, y)$, $i_{R^{-1}}(y, \varkappa) = i_R(\varkappa, y)$ and $\mathrm{d}'_{R^{-1}}(y, \varkappa) = \mathrm{d}'_R(\varkappa, y)$ for all $(\varkappa, y) \in X \times Y$.

The SFRs preserve some basic properties of the relations which are described in following definition.

2.3.4. Definition

Let R_1 & R_2 be two SFRs on $X \times Y$. Then for all $(\varkappa, y) \in X \times Y$

1. $R_1 \leq R_2$ if $\c{s}_{R_1}(\varkappa, y) \leq \c{s}_{R_2}(\varkappa, y), i_{R_1}(\varkappa, y) \leq i_{R_2}(\varkappa, y)$ and $\mathrm{d}'_{R_1}(\varkappa, y) \geq \mathrm{d}'_{R_2}(\varkappa, y)$.
2. $R_1 \vee R_2 = \{((\varkappa, y), \c{s}_{R_1}(\varkappa, y) \vee \c{s}_{R_2}(\varkappa, y), i_{R_1}(\varkappa, y) \wedge i_{R_2}(\varkappa, y), \mathrm{d}'_{R_1}(\varkappa, y) \wedge \mathrm{d}'_{R_2}(\varkappa, y)) : \varkappa \in X, y \in Y\}$.

3. $R_1 \wedge R_2 = \Big\{\Big((\varkappa, y), \varsigma_{R_1}(\varkappa, y) \wedge \varsigma_{R_2}(\varkappa, y), i_{R_1}(\varkappa, y) \wedge i_{R_2}(\varkappa, y), \text{d'}_{R_1}(\varkappa, y) \vee \text{d'}_{R_2}(\varkappa, y)\Big) : \varkappa \in X, y \in Y\Big\}$.
4. $R^c = \{(\varkappa, y), \text{d'}(\varkappa, y), i(\varkappa, y), \varsigma(\varkappa, y) : \varkappa \in X, y \in Y\}$.

2.3.5. Proposition

For three SFRs R, R_1, R_2 the following properties holds:

1. $R_1 \leq R_2 \Rightarrow R_1^{-1} \leq R_2^{-1}$.
2. $(R^{-1})^{-1} = R$.
3. $(R_1 \wedge R_2)^{-1} = R_1^{-1} \wedge R_2^{-1}$.
4. $R \wedge (R_1 \vee R_2) = (R \wedge R_1) \vee (R \wedge R_2)$.
5. $R \vee (R_1 \wedge R_2) = (R \vee R_1) \wedge (R \vee R_2)$.
6. $R_1 \wedge R_2 \leq R_1$.
7. $R_1 \leq R_1 \vee R_2$.
8. $R_1 \leq R \; \& \; R_2 \leq R \;\Rightarrow\; R_1 \vee R_2 \leq R$.
9. $R \leq R_1 \; \& \; R \leq R_2 \Rightarrow R \leq R_1 \wedge R_2$.

Proof: Trivial.

In our next study, different types of compositions of SFRs are discussed. Work on the composition of fuzzy relations, intuitionistic fuzzy relations and picture fuzzy relations have been of great importance [20-22, 28].

2.3.6. Definition

Let $R_1, R_2 \in SFR(X \times Y)$ be two SFRs. Then *max-min* composed relation $R_2 C R_1 \in SFR(X \times Z)$ is defined as:

$R_2CR_1 = \{(\varkappa, z), ş_{R_2CR_1}(\varkappa, z), i_{R_2CR_1}(\varkappa, z), d'_{R_2CR_1}(\varkappa, z): \varkappa \in X, z \in Z\}$ where

$ş_{R_2CR_1}(\varkappa, z) = \bigvee_y \{(ş_{R_1}(\varkappa, y) \wedge ş_{R_2}(y, z))\}.$

$i_{R_2CR_1}(\varkappa, z) = \bigwedge_y \{(i_{R_1}(\varkappa, y) \wedge i_{R_2}(y, z))\}.$

$d'_{R_2CR_1}(\varkappa, z) = \bigwedge_y \{(d'_{R_1}(\varkappa, y) \vee d'_{R_2}(y, z))\}.$

2.3.7. Definition

The affiliation degree or score value of two components $(\varkappa, y)$ of a SFR $R_1: X \to Y$ can be defined as: $SC^*(\varkappa, y) = ş^2_{R_1}(\varkappa, y) - d'^2_{R_1}(\varkappa, y).r^2_{R_1}(\varkappa, y)$. Where $r^2_{R_1}(\varkappa, y) = 1 - \left(ş^2_{R_1}(\varkappa, y) + i^2_{R_1}(\varkappa, y) + d'^2_{R_1}(\varkappa, y)\right)$.

2.4. T-Spherical Fuzzy Set

In this chapter, the idea of SFS is successfully presented and its comparison with PFS is established. The aim of introducing SFS is to enlarge the space PFS as it has its limitations in assigning MG, AG and NG to elements of a set. It is described geometrically that how SFS increased the range for assigning MG, AG and NG.

Our theory is not limited to SFSs despite it provides a better range for assigning MG, AG and NG but it has its limitations too. If a situation is considered where the triplet $(ş, i, d')$ has values from $[0, 1]$ such as $ş = 0.99, i = 0.57$ & $d' = 0.49$. In such case, the sum of components i.e. $şum(ş, i, d')$ exceeds $[0, 1]$ while if SFSs are considered then $şum\ (ş^2, i^2, d'^2) = 1.5259$ which exceeds $[0, 1]$ too. So, our claim is that the framework of PFSs and SFSs are not enough to deal with such situations. However, if the power on constraints is raised to n where $n \in \mathbb{Z}^{+}$ then we can assign any value of our choice to ş, i, d' in the interval $[0,1]$. In the current example if n is taken as 6. Then $şum(ş^{6}, i^{6}, d'^{6}) = 0.986556 \in [0,1]$. The choice of n is up to decision makers and it may affect the results in aggregation process. In view of this example, we propose the concept of TSFS as follows:

2.4.1. Definition

For any universal set X, a TSFS is of the form $I = \{(\varkappa, (ş, i, d')) \ \forall \varkappa \in X\}$. Here ş, i and d' are mappings from $X \rightarrow [0,1]$ denoting the MG, AG and NG respectively provided that for some least $n \in \mathbb{Z}^{+}$, $0 \leq şum\left(ş^{n}(\varkappa), i^{n}(\varkappa), d'^{n}(\varkappa)\right) \leq 1$ and $r(\varkappa) = \sqrt[n]{1 - şum\left(ş^{n}(\varkappa), i^{n}(\varkappa), d'^{n}(\varkappa)\right)}$ is known as the RD of $\varkappa$ in I. The triplet $(ş, i, d')$ is considered as a T-spherical fuzzy number (TSFN).

Obviously, this concept of TSFS provided the decision makers with an opportunity to get values for ş, i, d' from anywhere in the interval $[0,1]$ with no fear of limitation. A graphical

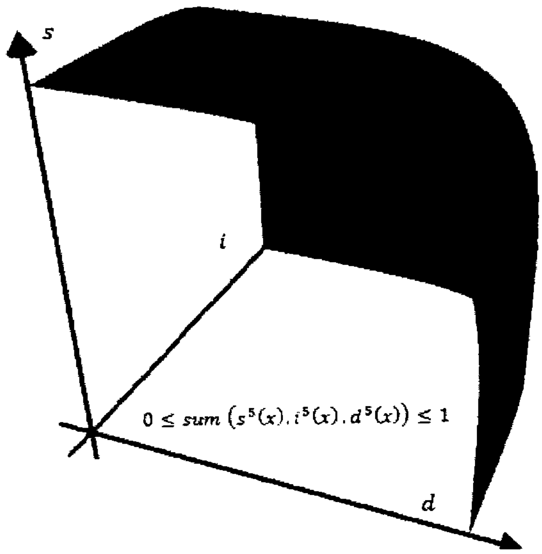

Figure 9 (Space of all TSFNs for n=5)

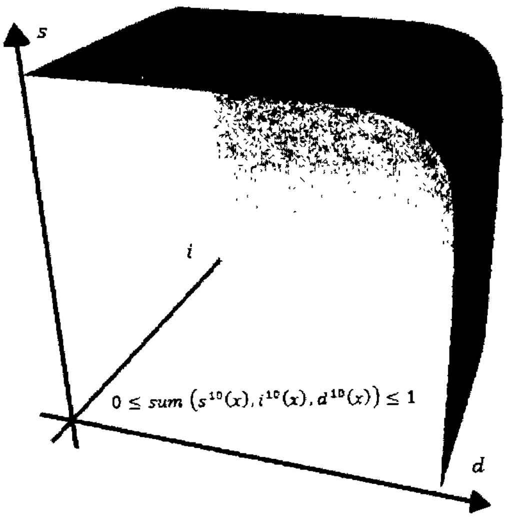

Figure 8 (Space of all TSFNs for n=10)

representation of the range space of TSFSs is depicted in Figure 8 and Figure 9 which clearly shows the superiority of TSFSs over existing fuzzy frameworks for assigning grades of memberships.

The graphs in in Figure 8 and Figure 9 represent space of TSFNs for $n = 10$ and $n = 5$ which is larger than that of the space of PFNs and SFNs depicted in in Figure 5 and Figure 6.

The following theorem support our claim of generalizations of TSFSs over PFSs and SFSs.

2.4.2. Theorem

The space of TSFNs is greater than that of space of SFNs.

Proof: This result can be proved similarly as Theorem 2.1.2.

2.4.3. Theorem

The space of TSFNs is greater than that of PFNs.

Proof: This is straightforward by transitivity of statements of Theorem 2.1.2 and Theorem 2.4.2.

2.4.4. Remark

The results stated in Section 2.2 and Section 2.3 can be extended to TSF environment trivially.

From now onward throughout our thesis, we'll establish results for TSFSs and the same could be obtained for SFSs by placing $n = 2$.

2.5. Aggregation Operators for TSFSs

The aim of this section is to introduce some AOs for TSFSs can consequently for SFSs. For this purpose, we first observe that the existing AOs proposed for PFS has some limitations as these cannot be applied to any kind of information and are restricted for a certain range of triplets. For this purpose, we study the WGA operators proposed Wang et al. [51] as follows:

2.5.1. Definition [51]

For some PFNs $I_j = (j = 1, 2, 3, \dots m)$, the picture fuzzy weighted geometric (PFWG) operator is defined as

$$PFWG(I_1, I_2, \dots I_m) = \prod_{j=1}^{m} I_j^{w_j}$$

$$= \left(\prod_{j=1}^{m} (ş_j + i_j)^{w_j} - \prod_{j=1}^{m} i_j^{w_j}, \prod_{j=1}^{m} i_j^{w_j}, 1 - \prod_{j=1}^{m} (1 - d'_j)^{w_j} \right)$$

The WGA operator proposed in Definition 2.5.12.5.1 cannot be applied to triplets for which $şum(ş, i, d')$ exceeds 1 hence providing a lesser range for assigning values to $ş, i,$ and d'. Therefore, in this section, we propose new operations which leads us to setup new WGA operators for TSFSs.

2.5.2. Definition

For three TSFNs I, I_1 and I_2 and for $\lambda > 0$. We define

1. $I_1 . I_2 = \left((ş_1 + i_1)(ş_2 + i_2) - i_1 . i_2, i_1 . i_2, \sqrt[n]{1 - (1 - d'^n_1)(1 - d'^n_2)} \right).$

2. $I_1^{\lambda} = \left((ş_1 + i_1)^{\lambda} - i_1^{\lambda}, i_1^{\lambda}, \sqrt[n]{1 - (1 - d'^{n}_1)^{\lambda}}\right).$

2.5.3. Remark

The operations defined in Definition 2.5.2 reduces to spherical fuzzy environment by taking $n = 2$.

The above operations are generalizations of the operations of PFSs defined in [51]. Also, if we assume $i_1 = i_2 = 0$. Then the above operations become operations of IFS as explained in [30].

In order to rank two TSFNs, we extended the definition of score and accuracy function of SFSs discussed in Definition 2.2.5 to TSF environment and is given by:

2.5.4. Definition

Let $I = (ş, i, d')$ be a TSFN. Then the score value of I is defined as $SC(I) = ş^{n}(\varkappa) - d'^{n}(\varkappa)$ and $SC(I) \in [-1, 1]$. The accuracy value of I is defined as $AC(I) = ş^{n}(\varkappa) + i^{n}(\varkappa) + d'^{n}(\varkappa)$ and $AC(I) \in [0, 1]$. Based on these two rules, for two TSFNs I_1 and I_2:

- I_1 is superior to I_2 if $SC(I_1) > SC(I_2)$.
- I_1 is inferior to I_2 if $SC(I_1) < SC(I_2)$.

If $SC(I_1) = SC(B)$ for two TSFNs. Then

- I_1 is superior to I_2 if $AC(I_1) > AC(I_2)$.
- I_1 is inferior to I_2 if $AC(I_1) < AC(I_2)$.
- I_1 is similar to I_2 if $AC(I_1) = AC(I_2)$.

Now we propose WGA operator for TSFNs and investigate its validity using mathematical induction.

2.5.5. Definition

For TSFNs $I_j = (j = 1, 2, 3, \dots m)$, the T-spherical fuzzy weighted geometric (TSFWG) operator is defined as

$$TSFWG(I_1, I_2, \dots I_m) = \prod_{j=1}^{m} I_j^{w_j}$$

where $w = (w_1, w_2, \dots w_m)^t$ be the weighted vector of $I_j = (j = 1, 2, 3, \dots m)$ and $w_j > 0$ and $\sum_{j=1}^{m} w_j = 1$.

Based on Definition 2.5.2, the following result can be obtained.

2.5.6. Theorem

The aggregated value of a collection of TSFNs $I_j (j = 1, 2, 3 \dots m)$ using TSFWG operator is also a T-SFN and

$$TSFWG(I_1, I_2, \dots I_m) = \left(\prod_{j=1}^{m} (\varsigma_j + i_j)^{w_i} - \prod_{j=1}^{m} i_j^{w_i}, \prod_{j=1}^{m} i_j^{w_i}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - d_j^n)^{w_i}} \right)$$

Proof: We prove this result using mathematical induction. For $m = 2$, consider

$$I_1^{w_1} = \left((\varsigma_1 + i_1)^{w_1} - i_1^{w_1}, i_1^{w_1}, \sqrt[n]{1 - (1 - d_1^n)^{w_1}} \right)$$

$$I_2^{w_2} = \left((\varsigma_2 + i_2)^{w_2} - i_2^{w_2}, i_2^{w_2}, \sqrt[n]{1 - (1 - d_2^n)^{w_2}} \right)$$

Then

$$I_1^{w_1}.I_2^{w_2} = \begin{pmatrix} \left((ş_1+i_1)^{w_1}+i_1^{w_1}\right)\left((ş_2+i_2)^{w_2}+i_2^{w_2}\right) - i_1^{w_1}.i_1^{w_1}, \\ i_1^{w_1}.i_1^{w_1}, \sqrt[n]{1-(1-d_1'^{n})^{w_1}.(1-d_2'^{n})^{w_2}} \end{pmatrix}$$

Thus, result holds true for $m = 2$. Further, assume that the result holds for $m = k$ i.e.

$$TSFWG(I_1, I_2, \dots I_k) = \left(\prod_{j=1}^{k} (ş_j + i_j)^{w_i} - \prod_{j=1}^{k} i_j^{w_i}, \prod_{j=1}^{k} i_j^{w_i}, \sqrt[n]{1 - \prod_{j=1}^{k} (1 - d_j'^{n})^{w_i}} \right)$$

Now we prove the result for $m = k + 1$

$$TSFWG(I_1, I_2, \dots I_{k+1}) = \prod_{j=1}^{k+1} I_j^{w_j} = \prod_{j=1}^{k} I_j^{w_j} \otimes I_{k+1}^{w_{k+1}}$$

$$= \begin{pmatrix} \left(\prod_{j=1}^{k} (ş_j + i_j)^{w_i} - \prod_{j=1}^{k} i_j^{w_i}, \prod_{j=1}^{k} i_j^{w_i}, \sqrt[n]{1 - \prod_{j=1}^{k} (1 - d_j'^{n})^{w_i}} \right) \otimes \\ \left((ş_1 + i_1)^{w_1} - i_1^{w_1}, i_1^{w_1}, \sqrt[n]{1 - (1 - d_1'^{n})^{w_1}} \right) \end{pmatrix}$$

$$TSFWG(I_1, I_2, \dots I_{k+1}) = \left(\prod_{j=1}^{k+1} (ş_j + i_j)^{w_i} - \prod_{j=1}^{k+1} i_j^{w_i}, \prod_{j=1}^{k+1} i_j^{w_i}, \sqrt[n]{1 - \prod_{j=1}^{k+1} (1 - d_j'^{n})^{w_i}} \right)$$

2.6. Applications

In this section, some real-life problems in spherical and TSF environments are discussed including medical diagnosis and decision-making problems.

2.6.1. Applications in Medical Diagnosis

Theory on fuzzy relations is of great importance when it comes to their practical applications [19-22, 28]. Before, the problem of medical diagnosis is investigated using intuitionistic fuzzy relations [22] and picture fuzzy relations [28]. An application of SFRs in medical diagnosis is discussed in this section. A "spherical medical knowledge" is defined as spherical fuzzy relation $R: P \rightarrow D$ which described the affiliation of a patient to a diagnosis where P represents the set of patients and D is the set of numerous diagnosis. Moreover Q is the set of symptoms that a patient may suffer from. An algorithm for the process of medical diagnosis based on spherical fuzzy information, followed by a flowchart in Figure 10, is described as follows:

2.6.1.1. Algorithm:

1. Determination of symptoms.
2. Establish a relation R_1 $(P \rightarrow Q)$.
3. Establish a relation R_2 $(Q \rightarrow D)$.
4. Find a composed relation R $(P \rightarrow D)$.
5. Find the affiliation of a patient P_i to a diagnosis D_i using Definition 2.3.7.

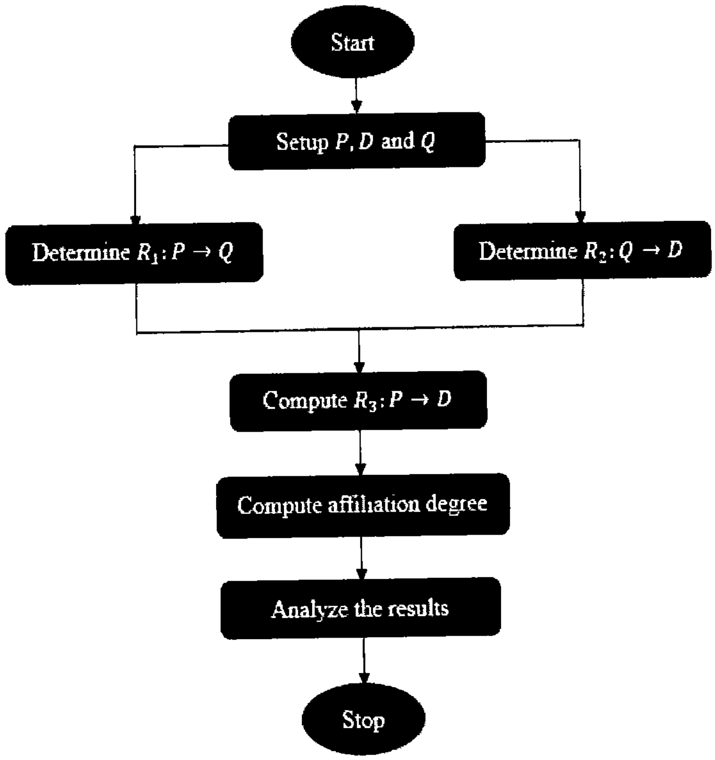

Figure 10 (Flow chart of spherical medical knowledge)

An example to illustrate the above algorithm is solved as follow:

2.6.1.2. Example

Let the set of patients be $P = \{P_1, P_2, P_3, P_4\}$, to be diagnosed with respect to set of symptoms $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5\}$ and $D = \{D_1, D_2, D_3, D_4, D_5\}$ is the set of diagnosis.

- A hypothetical relation R_1 $(P \to Q)$ is given in Table 1. This information described the relationship of symptoms and patients in the form of SFNs.

R_1	Q_1	Q_2	Q_3	Q_4	Q_5
P_1	$(0.7, 0.1, 0.4)$	$(0.5, 0.1, 0.8)$	$(0.4, 0.6, 0.5)$	$(0.8, 0, 0.4)$	$(0, 0.8, 0.5)$
P_2	$(0.9, 0.1, 0)$	$(0.5, 0, 0.2)$	$(0, 0.6, 0.8)$	$(0.3, 0.3, 0.3)$	$(0.5, 0.5, 0.7)$
P_3	$(0.4, 0.4, 0.4)$	$(0.2, 0.1, 0.2)$	$(0.7, 0.4, 0.2)$	$(0, 0.8, 0.4)$	$(0.7, 0.3, 0.3)$
P_4	$(0, 0.1, 0.6)$	$(0.5, 0.5, 0.5)$	$(0.9, 0.4, 0.1)$	$(1, 0, 0)$	$(0.2, 0.5, 0.7)$

Table 1 (R_1 from P to Q)

- A hypothetical relation R_2 $(Q \rightarrow D)$ is given in Table 2. This relation described that up to how much extent the symptoms Q needs the diagnosis D.

R_2	D_1	D_2	D_3	D_4	D_5
Q_1	$(0.4, 0.3, 0.4)$	$(0.5, 0.5, 0.3)$	$(0.6, 0, 0.1)$	$(0.4, 0.3, 0.7)$	$(0.3, 0.3, 0.6)$
Q_2	$(0.7, 0.1, 0.2)$	$(0, 0, 0.5)$	$(0.5, 0.1, 0.6)$	$(0.4, 0.6, 0.6)$	$(0.1, 0.3, 0.7)$
Q_3	$(0.4, 0.2, 0.7)$	$(0.2, 0, 0.8)$	$(0.5, 0.2, 0.4)$	$(1, 0, 0)$	$(0.1, 0.4, 0.6)$
Q_4	$(0, 0, 1)$	$(0.45, 0.4, 0)$	$(0.9, 0, 0.1)$	$(0.1, 0.6, 0.7)$	$(0.7, 0.2, 0.1)$
Q_5	$(0.7, 0.1, 0.4)$	$(0.41, 0.3, 0.6)$	$(0.5, 0.5, 0.5)$	$(0.76, 0.3, 0.2)$	$(0.2, 0, 0.81)$

Table 2 (R_2 from Q to D)

- The $max - min$ composed relation R $(P \rightarrow D)$ is given in Table 3. For this purpose, we used Definition 2.3.6.

R	D_1	D_2	D_3	D_4	D_5
P_1	$(0.5, 0, 0.4)$	$(0.5, 0, 0.4)$	$(0.8, 0, 0.4)$	$(0.4, 0, 0.5)$	$(0.7, 0, 0.4)$
P_2	$(0.5, 0, 0.2)$	$(0.5, 0, 0.3)$	$(0.6, 0, 0.1)$	$(0.5, 0, 0.6)$	$(0.3, 0, 0.3)$
P_3	$(0.4, 0, 0.2)$	$(0.41, 0, 0.4)$	$(0.5, 0, 0.1)$	$(0.7, 0, 0.2)$	$(0.3, 0, 0.4)$
P_4	$(0.5, 0, 0.4)$	$(0.45, 0, 0)$	$(0.9, 0, 0.1)$	$(0.9, 0, 0.1)$	$(0.7, 0, 0.1)$

Table 3 ($R = R_1 C R_2$)

- The degree of affiliation (using Definition 2.3.6) between patient P_i with a diagnosis D_i is calculated in Table 4.

R	D_1	D_2	D_3	D_4	D_5
P_1	0.194304	0.194304	0.6336	0.072975	0.4704
P_2	0.229836	0.210796	0.356031	0.195244	0.029484
P_3	0.1344	0.095868	0.244524	0.481164	0
P_4	0.194304	0.2025	0.809676	0.809676	0.4875

Table 4 (Affiliation strength of patents and diagnosis)

The analysis of Table 4 shows that the affiliation degree of (P_4, D_3) and (P_4, D_4) are equal and highest among all other score values which indicates that P_4 is a patient of D_3 & D_4 both strongly. Further, P_1 is considerably a patient of D_3 or D_5. P_3's position is uncertain to diagnose but to some extent P_3 has some germs of D_4. In this analysis, usually if score values are greater than 5 then there are more chances that a patient is suffering from a specific diagnosis. A score value of less than 5 mean the chances of patient P suffering from diagnosis D is relatively less.

Now, the limitations of previous study and advantages of the studying the medical diagnosis problem in spherical fuzzy environment over previous models are stated as follows.

- In FS theory, there exist only membership grades and in IFS theory only membership and non-membership grades exist. Therefore, those fuzzy frameworks cannot deal with the information where abstinence and refusal degrees are involved.
- The space PFSs is limited and due to this reason decision makers are restricted in a certain range and are unable to assign values to three components by their own choice.
- The medical diagnosis models proposed inn spherical fuzzy environment produce better results as the chance of information loss is reduced by taking abstinence and refusal degrees into account.

Hence our proposed framework of SFSs has an advantage over FS, IFS and PFS as it is closer to human nature and is more flexible in assigning values to its membership components. The results could be more improved if we use TSFSs instead SFSs. Further, these results are valid in space of TSFSs also.

2.6.2. Application in Multi-Attribute Decision Making

In this subsection, we aim to use TSFNs and their AOs in MADM process. As TSFS is a generalization of FS, IFS, PyFS, q-ROFS, PFS and SFS and could deal with real life problems more effectively than the existing concepts. A medical diagnosis problem is effectively solved by using SFRs. Now to show the effectiveness of TSFNs, a MADM problem is solved using aggregation of TSFNs.

2.6.2.1. Algorithm:

Let $A = \{A_1, A_2, A_3, \dots A_n\}$ be a finite set of alternatives and let $E = \{E_1, E_2. E_3, \dots E_n\}$ be the set of evaluations based on given attributes and $E_i = e(A_i), i = 1, 2, 3, \dots n$. Each E_i is a TSFN i.e. $e(A_i) = (ş(A_i), i(A_i), d'(A_i))$ provided that $ş^n(A_i) + i^n(A_i) + d'^n(A_i)$ where $n \in \mathbb{Z}^+$. The decision-making problem involves the ranking of alternatives and selection of best alternative against given attributes. The decision-making method has the following steps.

1. Formation of decision matrix i.e. decision makers assigned some values to each alternative under given attributes in the form of TSFNs.
2. The aggregation of information provided by decision makers in decision matrix.
3. Comparison of aggregated data based on score and accuracy values.
4. Arrangement of aggregated data for the selection of best alternative.

2.6.2.2. Example

A firm needs a manager for their sale/purchase department. The firm advertised the vacant post and several candidates have applied. After initial screening, three candidates have been selected. To find the best of them, the firm needs to evaluate the candidates based on some attributes. The governing board of the firm have the task to evaluate the three selected candidates based on the attributes set E and the weight vector w of each attribute is given below.

$E = \{Communication\ şkillş, Reşponşibility, Creativity\}$.

The wright vector of attributes is $w = (0.2, 0.35, 0.45)^t$

The decision makers anonymously gave their opinion in the form of TSFNs which are given in Table 5.

Step 1: Information provided by decision makers is provided in Table 5.

	E_1	E_2	E_3
A_1	$(0.8, 0.5, 0.4)$	$(0.7, 0.4, 0.4)$	$(0.3, 0.5, 0.4)$
A_2	$(0.9, 0.2, 0.4)$	$(0.6, 0.3, 0.2)$	$(0.4, 0.1, 0.7)$
A_3	$(0.5, 0.5, 0.5)$	$(0.8, 0.2, 0.3)$	$(0.6, 0.4, 0.3)$

Table 5 (Decision Matrix)

It is clear from given data that all values of Table 5 are purely TSFNs for $n = 3$. This indicates that this type data cannot be handled by using SFSs or PFSs which shows the significance of using TSFSs over the existing concepts.

Step 2: [2] To aggregate the information provided in Table 5, TSFWG operator is used and the aggregated data is given below.

$A_1 = TSFWG(A_{11}, A_{12}, A_{13}) = (0.523087, 0.46243586, 0.4).$

$A_2 = TSFWG(A_{21}, A_{22}, A_{23}) = (0.550385, 0.16873238, 0.5702233).$

$A_3 = TSFWG(A_{31}, A_{32}, A_{33}) = (0.671843, 0.32815687, 0.3620078).$

Step 3: The score values of aggregated data are given as:

[2] We are thankful to Mr. Buoying Zhu (from Shandong University of Finance and Economics, China) for pointing out towards some errors (which are now corrected) in the published version of this paper[1] .

$SC(A_1) = (0.523087)^3 - (0.4)^3 = 0.079127.$

$SC(A_2) = (0.550385)^3 - (0.5702233)^3 = -0.01869.$

$SC(A_3) = (0.671843)^3 - (0.3620078)^3 = 0.255811.$

Step 4: Based on the score values obtained in step 3, we have

$SC(A_3) > SC(A_1) > SC(A_2)$ which shows that candidate A_3 is the best among the three candidates based on mentioned attributes.

The results of MADM using T-spherical fuzzy WGA operators are better than that of produced by WGA operators of IFSs, PyFSs, q-ROFSs and PFSs because of the diverse structure of TSFSs and its close affiliation to human nature. Furthermore, the data aggregated by using TSFWG operators cannot be aggregated by existing operators.

Chapter 3

Some Similarity Measures for T-Spherical Fuzzy Sets with Applications in Pattern Recognition[3]

In this chapter, some new improved SMs are developed that can deal with information involving human opinion based on MG, AG, NG and RG. First, in this chapter, some SMs in the frameworks of IFSs and PFSs are discussed. It is discussed that existing SMs have some limitations and could not be applied to problems where information involved opinion having MG, AG, NG and RG. Therefore, some SMs in the framework of TSFSs and consequently SFSs are proposed including cosine SMs, grey SMs, and set theoretic SMs. With the help of some results, it is proved that the proposed SMs are the generalizations of existing SMs. The newly defined SMs are subjected to a well-known problem of pattern recognition and the results are discussed. A comparative study of new and existing SMs are established and some advantages of the proposed work are discussed.

3.1. Some Similarity Measures and Their Drawbacks

In this section, we aim to provide some SMs developed in the environment of IFSs, PyFSs and PFSs. In view of some examples, we show the drawbacks of these SMs which leads us to develop some new SMs. In our next theory, X shall denote a universe of discourse.

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[3] Work of this chapter has been published in the following paper.

Ullah K., Mahmood T. and Jan N. Similarity Measures for T-Spherical Fuzzy Sets with Applications in Pattern Recognition. Symmetry. 2018, 10(6), 193-207. https://doi.org/10.3390/sym10060193

Cosine SM (CSM) for IFSs was proposed by Ye [68] which is given by:

3.1.1. Definition [68]

For two IFNs $P = (ş_P, d'_P)$ and $Q = (ş_Q, d'_Q)$, a CSM is defined as:

$$Ç^1_{IFS}(P, Q) = \frac{1}{m}\sum_{i=1}^{m} \frac{ş_P(\varkappa_i) \cdot ş_Q(\varkappa_i) + d'_P(\varkappa_i) \cdot d'_Q(\varkappa_i)}{\sqrt{ş^2_P(\varkappa_i) + d'^2_P(\varkappa_i)} \cdot \sqrt{ş^2_Q(\varkappa_i) + d'^2_Q(\varkappa_i)}}$$

Xu and Cai [72] proposed some set-theoretic SMs for IFSs as follows:

3.1.2. Definition [72]

For two IFNs $P = (ş_P, d'_P)$ and $Q = (ş_Q, d'_Q)$ with hesitancy degree r_P and r_Q respectively, a set-theoretic SM is defined as:

$$Ç^2_{IFS}(P, Q) = \frac{1}{m}\sum_{i=1}^{m} \frac{ş_P(\varkappa_i) \cdot ş_Q(\varkappa_i) + d'_P(\varkappa_i) \cdot d'_Q(\varkappa_i) + r_P(\varkappa_i) \cdot r_Q(\varkappa_i)}{max\left(ş^2_P(\varkappa_i) + d'^2_P(\varkappa_i) + r^2_P(\varkappa_i), ş^2_Q(\varkappa_i) + d'^2_Q(\varkappa_i) + r^2_Q(\varkappa_i)\right)}$$

Xu and Cai [72] also proposed grey SM in intuitionistic fuzzy settings which is defined as:

3.1.3. Definition [72]

For two IFNs $P = (ş_P, d'_P)$ and $Q = (ş_Q, d'_Q)$, the grey SM is defined as:

$$Ç^3_{IFS}(P, Q) = \frac{1}{3m}\sum_{i=1}^{m} \left(\frac{\Delta ş_{min} + \Delta ş_{max}}{\Delta ş_i + \Delta ş_{max}} + \frac{\Delta d'_{min} + \Delta d'_{max}}{\Delta d'_i + \Delta d'_{max}}\right)$$

where $\Delta\varsigma_i = |\varsigma_P(x_i) - \varsigma_Q(x_i)|$ and $\Delta d'_i = |d'_P(x_i) - d'_Q(x_i)|$. Further $\Delta\varsigma_{min} = min\{|\varsigma_P(x_i) - \varsigma_Q(x_i)|\}$ and $\Delta d'_{min} = min\{|d'_P(x_i) - d'_Q(x_i)|\}$ also $\Delta\varsigma_{max} = max\{|\varsigma_P(x_i) - \varsigma_Q(x_i)|\}$ and $\Delta d'_{max} = max\{|d'_P(x_i) - d'_Q(x_i)|\}$.

The concept of SMs proposed by Ye [68] and Xu and Cai [72] in intuitionistic fuzzy environment was extended to picture fuzzy settings by Wei [73] and are defined as:

3.1.4. Definition [73]

For two PFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$ on X, a CSM is defined as:

$$Ç^1_{PFS}(P,Q) = \frac{1}{m}\sum_{i=1}^{m} \frac{\varsigma_P(x_i)\cdot\varsigma_Q(x_i) + i_P(x_i)\cdot i_Q(x_i) + d'_P(x_i)\cdot d'_Q(x_i)}{\sqrt{\varsigma_P^2(x_i) + i_P^2(x_i) + d'^2_P(x_i)}\cdot\sqrt{\varsigma_Q^2(x_i) + i_Q^2(x_i) + d'^2_Q(x_i)}}$$

3.1.5. Definition [73]

For two PFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$ on X, a set-theoretic SM is defined as:

$$Ç^2_{PFS}(P,Q) = \frac{1}{m}\sum_{i=1}^{m} \frac{\varsigma_P(x_i)\cdot\varsigma_Q(x_i) + i_P(x_i)\cdot i_Q(x_i) + d'_P(x_i)\cdot d'_Q(x_i)}{max\left(\varsigma_P^2(x_i) + i_P^2(x_i) + d'^2_P(x_i), \varsigma_Q^2(x_i) + i_Q^2(x_i) + d'^2_Q(x_i)\right)}$$

3.1.6. Definition [73]

For two PFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$ on X, the grey SM is defined as:

$$Ç^3_{PFS}(P,Q) = \frac{1}{3m}\sum_{i=1}^{m}\left(\frac{\Delta\varsigma_{min} + \Delta\varsigma_{max}}{\Delta\varsigma_i + \Delta\varsigma_{max}} + \frac{\Delta i_{min} + \Delta i_{max}}{\Delta i_i + \Delta i_{max}} + \frac{\Delta d'_{min} + \Delta d'_{max}}{\Delta d'_i + \Delta d'_{max}}\right)$$

where $\Delta\varsigma_i = |\varsigma_P(\varkappa_i) - \varsigma_Q(\varkappa_i)|$, $\Delta i_i = |i_P(\varkappa_i) - i_Q(\varkappa_i)|$ and $\Delta d'_i = |d'_P(\varkappa_i) - d'_Q(\varkappa_i)|$. Further $\Delta\varsigma_{min} = min\{|\varsigma_P(\varkappa_i) - \varsigma_Q(\varkappa_i)|\}$, $\Delta i_{min} = min\{|i_P(\varkappa_i) - i_Q(\varkappa_i)|\}$ and $\Delta d'_{min} = min\{|d'_P(\varkappa_i) - d'_Q(\varkappa_i)|\}$ also $\Delta\varsigma_{max} = max\{|\varsigma_P(\varkappa_i) - \varsigma_Q(\varkappa_i)|\}$, $\Delta i_{max} = max\{|i_P(\varkappa_i) - i_Q(\varkappa_i)|\}$ and $\Delta d'_{max} = max\{|d'_P(\varkappa_i) - d'_Q(\varkappa_i)|\}$

The SMs [68, 72] and [73], discussed in this section, are limited and can handle the data provided in the framework of IFSs or PFSs respectively. The SMs proposed in [68, 72] does not take into account the abstinence and refusal degree while the SMs proposed in [73] are limited and can process PFNs from a certain range Therefore, in this chapter, some new SMs in the environment of TSFSs are proposed as a generalization of SMs defined in [68, 72] and [73].

3.2. T-Spherical Fuzzy Cosine Similarity Measures

In this section, some CSMs are developed in the environment of TSFSs keeping in mind the limitations of the SMs proposed by [68, 72] and [73]. It is proved that the SMs developed here are the generalizations of the existing SMs (See Definitions 3.1.1 and 3.1.4) and can be applied in situations where existing SMs fails.

3.2.1. Definition

For two TSFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$, a CSM is defined as:

$$\c{C}^1_{TSFS}(P, Q) = \frac{1}{m}\sum_{i=1}^{m} \frac{\varsigma_P^n(\varkappa_i)\cdot\varsigma_Q^n(\varkappa_i) + i_P^n(\varkappa_i)\cdot i_Q^n(\varkappa_i) + d'^n_P(\varkappa_i)\cdot d'^n_Q(\varkappa_i)}{\sqrt{\left(\varsigma_P^n(\varkappa_i)\right)^2 + \left(i_P^n(\varkappa_i)\right)^2 + \left(d'^n_P(\varkappa_i)\right)^2}\cdot\sqrt{\left(\varsigma_Q^n(\varkappa_i)\right)^2 + \left(i_Q^n(\varkappa_i)\right)^2 + \left(d'^n_Q(\varkappa_i)\right)^2}}$$

The following properties hold true for the CSM of three TSFNs $P = (\varsigma_P, i_P, d'_P)$, $Q = (\varsigma_Q, i_Q, d'_Q)$ and $R = (\varsigma_R, i_R, d'_R)$.

1. $0 \leq Ç^1_{TSFS}(P, Q) \leq 1$.
2. $Ç^1_{TSFS}(P, Q) = Ç^1_{TSFS}(Q, P)$.
3. $Ç^1_{TSFS}(P, Q) = 1$ if $P = Q$.
4. If $P \subseteq Q \subseteq R$. Then $Ç^1_{TSFS}(P, R) \leq Ç^1_{TSFS}(P, Q), Ç^1_{TSFS}(P, R) \leq Ç^1_{TSFS}(Q, R)$.

Proof: The proofs of first and second are obvious. To prove the third part, take $P = Q$ i.e. $\varsigma_P(\varkappa_i) = \varsigma_Q(\varkappa_i), i_P(\varkappa_i) = i_Q(\varkappa_i), d'_P(\varkappa_i) = d'_Q(\varkappa_i)$. Hence

$$Ç^1_{TSFS}(P, Q)$$

$$= \frac{1}{m}\sum_{i=1}^{m} \frac{\varsigma^n_P(\varkappa_i) \cdot \varsigma^n_P(\varkappa_i) + i^n_P(\varkappa_i) \cdot i^n_P(\varkappa_i) + d'^n_P(\varkappa_i) \cdot d'^n_P(\varkappa_i)}{\sqrt{(\varsigma^n_P(\varkappa_i))^2 + (i^n_P(\varkappa_i))^2 + (d'^n_P(\varkappa_i))^2} \cdot \sqrt{(\varsigma^n_P(\varkappa_i))^2 + (i^n_P(\varkappa_i))^2 + (d'^n_P(\varkappa_i))^2}}$$

$$= \frac{1}{m}\sum_{i=1}^{m} \frac{(\varsigma^n_P(\varkappa_i))^2 + (i^n_P(\varkappa_i))^2 + (d'^n_P(\varkappa_i))^2}{(\varsigma^n_P(\varkappa_i))^2 + (i^n_P(\varkappa_i))^2 + (d'^n_P(\varkappa_i))^2}$$

$$= 1$$

The fourth part is obvious as, geometrically, the angle of P, R is greater than that of P, Q and Q, R.

3.2.2. Definition

The DM of the angle between two TSFNs P and Q is defined as:

$$d'(P, Q) = arcco\varsigma(Ç^1_{TSFS}(P, Q))$$

The following properties hold true for the DM of three TSFNs $P = (\varsigma_P, i_P, d'_P)$, $Q = (\varsigma_Q, i_Q, d'_Q)$ and $R = (\varsigma_R, i_R, d'_R)$.

1. If $0 \leq \varsigma^{1}_{TSFS}(P,Q) \leq 1$. Then $d'(P,Q) \geq 0$.

2. If $\varsigma^{1}_{TSFS}(P,Q) = \varsigma^{1}_{TSFS}(Q,P)$. Then $d'(P,Q) = d'(Q,P)$.

3. If $\varsigma^{1}_{TSFS}(P,Q) = 1$ for $P = Q$. Then $d'(P,Q) = 0$.

4. If $P \subseteq Q \subseteq R$. Then $d'(P,R) \leq d'(P,Q) + d'(Q,R)$.

Proof: The proofs of Part (1) – (3) are obvious. To prove Part (4), let $P \subseteq Q \subseteq R$. Then the DMs of P, Q and R are:

$$d'\big(P(\varkappa_i), Q(\varkappa_i)\big) = arccoş\left(\varsigma^{1}_{TSFS}\big(P(\varkappa_i), Q(\varkappa_i)\big)\right)$$

$$d'(Q(\varkappa_i), R(\varkappa_i)) = arccoş\left(\varsigma^{1}_{TSFS}\big(Q(\varkappa_i), R(\varkappa_i)\big)\right)$$

$$d'(P(\varkappa_i), R(\varkappa_i)) = arccoş\left(\varsigma^{1}_{TSFS}\big(P(\varkappa_i), R(\varkappa_i)\big)\right)$$

where $i = 1, 2, 3, \dots$ m and

$$\varsigma^{1}_{TSFS}\big(P(\varkappa_i), Q(\varkappa_i)\big)$$

$$= \frac{1}{m}\sum_{i=1}^{m} \frac{ş^{n}_{P}(\varkappa_i) \cdot ş^{n}_{Q}(\varkappa_i) + i^{n}_{P}(\varkappa_i) \cdot i^{n}_{Q}(\varkappa_i) + d'^{n}_{P}(\varkappa_i) \cdot d'^{n}_{Q}(\varkappa_i)}{\sqrt{\big(ş^{n}_{P}(\varkappa_i)\big)^2 + \big(i^{n}_{P}(\varkappa_i)\big)^2 + \big(d'^{n}_{P}(\varkappa_i)\big)^2} \cdot \sqrt{\big(ş^{n}_{Q}(\varkappa_i)\big)^2 + \big(i^{n}_{Q}(\varkappa_i)\big)^2 + \big(d'^{n}_{Q}(\varkappa_i)\big)^2}}$$

If $P = [ş_P(\varkappa_i), i_P(\varkappa_i), d'_P(\varkappa_i)]$, $Q = [ş_Q(\varkappa_i), i_Q(\varkappa_i), d'_Q(\varkappa_i)]$ and $R = [ş_Q(\varkappa_i), i_Q(\varkappa_i), d'_Q(\varkappa_i)]$ are considered as three vectors in a plane such that $P(\varkappa_i) \subseteq Q(\varkappa_i) \subseteq R(\varkappa_i)$ and using triangular inequality, we have $d'\big(P(\varkappa_i), R(\varkappa_i)\big) \leq d'\big(P(\varkappa_i), Q(\varkappa_i)\big) + d'\big(Q(\varkappa_i), R(\varkappa_i)\big)$ and hence (4) holds true.

In the following, $w = (w_1, w_2, w_3 \dots w_m)^t$ represents a weight vector such that $w_i \in [0, 1]$ and $\sum_{i=1}^{m} w_i = 1$

3.2.3. Definition

For two TSFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$, a weighted CSM is defined as:

$$\text{W}\varsigma^1_{TSFS}(P,Q)$$

$$= \frac{1}{m}\sum_{i=1}^{m} w_i \frac{\varsigma^n_P(\varkappa_i)\cdot\varsigma^n_Q(\varkappa_i) + i^n_P(\varkappa_i)\cdot i^n_Q(\varkappa_i) + d'^n_P(\varkappa_i)\cdot d'^n_Q(\varkappa_i)}{\sqrt{(\varsigma^n_P(\varkappa_i))^2 + (i^n_P(\varkappa_i))^2 + (d'^n_P(\varkappa_i))^2}\cdot\sqrt{(\varsigma^n_Q(\varkappa_i))^2 + (i^n_Q(\varkappa_i))^2 + (d'^n_Q(\varkappa_i))^2}}$$

By taking $w_i = \frac{1}{m}$, Definition 3.2.3 reduced to Definition 3.2.1.

The following properties hold true for the weighted cosine similarity measures of three TSFNs $P = (\varsigma_P, i_P, d'_P)$, $Q = (\varsigma_Q, i_Q, d'_Q)$ and $R = (\varsigma_R, i_R, d'_R)$.

1. $0 \leq \text{W}\varsigma^1_{TSFS}(P,Q) \leq 1$.
2. $\text{W}\varsigma^1_{TSFS}(P,Q) = \text{W}\varsigma^1_{TSFS}(Q,P)$.
3. $\text{W}\varsigma^1_{TSFS}(P,Q) = 1$ iff $P = Q$.
4. If $P \subseteq Q \subseteq R$. Then $\text{W}\varsigma^1_{TSFS}(P,R) \leq \text{W}\varsigma^1_{TSFS}(P,Q)$, $\text{W}\varsigma^1_{TSFS}(P,R) \leq \text{W}\varsigma^1_{TSFS}(Q,R)$.

Proof: Proofs are straightforward.

3.3. T-spherical Fuzzy Set-Theoretic Similarity Measures

In this section, some set-theoretic SMs are developed in the environment of TSFSs keeping in mind the limitations of the set-theoretic SMs proposed by [72] and [73]. It is observed that for computing similarity index using set theoretic SM, Xu and Cai [72] include hesitancy degree. However, for the set-theoretic SMs of PFSs proposed by Sing [73], the refusal degree is being ignored. In this section, it is proved that the SMs developed here are the generalizations of the existing SMs (See Definitions 3.1.2 and 3.1.5) and can be applied in situations where the existing SMs fails.

3.3.1. Definition

For two TSFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$, a set-theoretic SM is defined as:

$$\text{Ç}^2_{TSFS}(P,Q) = \frac{1}{m}\sum_{i=1}^{m} \frac{\varsigma^n_P(x_i)\cdot\varsigma^n_Q(x_i) + i^n_P(x_i)\cdot i^n_Q(x_i) + d'^n_P(x_i)\cdot d'^n_Q(x_i)}{max\left(\left(\varsigma^n_P(x_i)\right)^2 + \left(i^n_P(x_i)\right)^2 + \left(d'^n_P(x_i)\right)^2, \left(\varsigma^n_Q(x_i)\right)^2 + \left(i^n_Q(x_i)\right)^2 + \left(d'^n_Q(x_i)\right)^2\right)}$$

3.3.2. Definition

For two TSFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$, a weighted set-theoretic SM is defined as:

$$W\text{Ç}^2_{TSFS}(P,Q) = \frac{1}{m}\sum_{i=1}^{m} w_i \frac{\varsigma^n_P(x_i)\cdot\varsigma^n_Q(x_i) + i^n_P(x_i)\cdot i^n_Q(x_i) + d'^n_P(x_i)\cdot d'^n_Q(x_i)}{max\left(\left(\varsigma^n_P(x_i)\right)^2 + \left(i^n_P(x_i)\right)^2 + \left(d'^n_P(x_i)\right)^2, \left(\varsigma^n_Q(x_i)\right)^2 + \left(i^n_Q(x_i)\right)^2 + \left(d'^n_Q(x_i)\right)^2\right)}$$

Definition 3.3.2 reduces to Definition 3.3.1 if we place $w_i = \frac{1}{m}$.

The following properties hold true for the set-theoretic SMs of two TSFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$.

1. $0 \leq \text{Ç}^2_{TSFS}(P,Q) \leq 1$.
2. $\text{Ç}^2_{TSFS}(P,Q) = \text{Ç}^2_{TSFS}(Q,P)$.
3. $\text{Ç}^2_{TSFS}(P,Q) = 1$ if $P = Q$.

The following three properties hold true for the weighted set-theoretic SM of two TSFNs $P = (\text{ş}_P, i_P, \text{d'}_P)$ and $Q = (\text{ş}_Q, i_Q, \text{d'}_Q)$.

1. $0 \leq \mathfrak{W}\text{Ç}^2_{TSFS}(P, Q) \leq 1$.
2. $\mathfrak{W}\text{Ç}^2_{TSFS}(P, Q) = \mathfrak{W}\text{Ç}^2_{TSFS}(Q, P)$.
3. $\mathfrak{W}\text{Ç}^2_{TSFS}(P, Q) = 1$ if $P = Q$.

If we include the refusal degree for computing set-theoretic SMs, the Definition 3.3.2 would take the following form:

3.3.3. Definition

For two TSFNs $P = (\text{ş}_P, i_P, \text{d'}_P)$ and $Q = (\text{ş}_Q, i_Q, \text{d'}_Q)$ with refusal degrees r_P and r_Q respectively, a weighted set-theoretic SM is defined as:

$$\mathfrak{W}\text{Ç}^3_{TSFS}(P, Q) = \frac{1}{m}\sum_{\iota=1}^{m} w_\iota \frac{\text{ş}^n_P(\varkappa_\iota)\cdot \text{ş}^n_Q(\varkappa_\iota) + i^n_P(\varkappa_\iota)\cdot i^n_Q(\varkappa_\iota) + \text{d'}^n_P(\varkappa_\iota)\cdot \text{d'}^n_Q(\varkappa_\iota) + r^n_P(\varkappa_\iota)\cdot r^n_Q(\varkappa_\iota)}{max\begin{pmatrix}\left(\text{ş}^n_P(\varkappa_\iota)\right)^2 + \left(i^n_P(\varkappa_\iota)\right)^2 + \left(\text{d'}^n_P(\varkappa_\iota)\right)^2 + \left(r^n_P(\varkappa_\iota)\right)^2, \\ \left(\text{ş}^n_Q(\varkappa_\iota)\right)^2 + \left(i^n_Q(\varkappa_\iota)\right)^2 + \left(\text{d'}^n_Q(\varkappa_\iota)\right)^2 + \left(r^n_Q(\varkappa_\iota)\right)^2\end{pmatrix}}$$

3.4. T-Spherical Fuzzy Grey Similarity Measures

In this section, some grey SMs are developed in the environment of TSFSs keeping in mind the limitations of the grey SMs proposed by [72] and [73]. It is proved that the SMs developed here are the generalizations of the existing SMs (See Definitions 3.1.3 and 3.1.6) and can be applied in situations where existing SMs fails.

3.4.1. Definition

For two TSFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$, the grey SM is defined as:

$$Ç^3_{TSFS}(P,Q) = \frac{1}{3m}\sum_{i=1}^{m}\left(\frac{\Delta\varsigma_{min} + \Delta\varsigma_{max}}{\Delta\varsigma_i + \Delta\varsigma_{max}} + \frac{\Delta i_{min} + \Delta i_{max}}{\Delta i_i + \Delta i_{max}} + \frac{\Delta d'_{min} + \Delta d'_{max}}{\Delta d'_i + \Delta d'_{max}}\right)$$

where $\Delta\varsigma_i = |\varsigma_P^n(\varkappa_i) - \varsigma_Q^n(\varkappa_i)|$, $\Delta i_i = |i_P^n(\varkappa_i) - i_Q^n(\varkappa_i)|$ and $\Delta d'_i = |d'^n_P(\varkappa_i) - d'^n_Q(\varkappa_i)|$. Further $\Delta\varsigma_{min} = min\{|\varsigma_P^n(\varkappa_i) - \varsigma_Q^n(\varkappa_i)|\}$, $\Delta i_{min} = min\{|i_P^n(\varkappa_i) - i_Q^n(\varkappa_i)|\}$ and $\Delta d'_{min} = min\{|d'^n_P(\varkappa_i) - d'^n_Q(\varkappa_i)|\}$ also $\Delta\varsigma_{max} = max\{|\varsigma_P^n(\varkappa_i) - \varsigma_Q^n(\varkappa_i)|\}$, $\Delta i_{max} = max\{|i_P^n(\varkappa_i) - i_Q^n(\varkappa_i)|\}$ and $\Delta d'_{max} = max\{|d'^n_P(\varkappa_i) - d'^n_Q(\varkappa_i)|\}$.

The following properties hold true for the Grey SM of two TSFNs $P = (\varsigma_P, i_P, d'_P)$, $Q = (\varsigma_Q, i_Q, d'_Q)$.

1. $0 \leq Ç^3_{TSFS}(P,Q) \leq 1$.
2. $Ç^3_{TSFS}(P,Q) = Ç^3_{TSFS}(Q,P)$.
3. $Ç^3_{TSFS}(P,Q) = 1$ if $P = Q$.

Whenever the weight of the elements is considered in a real-life phenomenon, attributes have different importance in different situations and therefore need to be weighted accordingly. Therefore, we developed weighted grey SM as follows:

3.4.2. Definition

For two TSFNs $P = (\varsigma_P, i_P, d'_P)$ and $Q = (\varsigma_Q, i_Q, d'_Q)$, the weighted grey SM is defined as:

$$\mathrm{W}\mathrm{Ç}^3_{TSFS}(P,Q) = \frac{1}{3}\sum_{i=1}^{m} w_i \left(\frac{\Delta ş_{min} + \Delta ş_{max}}{\Delta ş_i + \Delta ş_{max}} + \frac{\Delta i_{min} + \Delta i_{max}}{\Delta i_i + \Delta i_{max}} + \frac{\Delta d'_{min} + \Delta d'_{max}}{\Delta d'_i + \Delta d'_{max}}\right)$$

where $\Delta ş_i = \left|ş^n_P(\varkappa_i) - ş^n_Q(\varkappa_i)\right|$, $\Delta i_i = \left|i^n_P(\varkappa_i) - i^n_Q(\varkappa_i)\right|$ and $\Delta d'_i = \left|d'^n_P(\varkappa_i) - d'^n_Q(\varkappa_i)\right|$. Further $\Delta ş_{min} = min\{\left|ş^n_P(\varkappa_i) - ş^n_Q(\varkappa_i)\right|\}$, $\Delta i_{min} = min\{\left|i^n_P(\varkappa_i) - i^n_Q(\varkappa_i)\right|\}$ and $\Delta d'_{min} = min\{\left|d'^n_P(\varkappa_i) - d'^n_Q(\varkappa_i)\right|\}$ also $\Delta ş_{max} = max\{\left|ş^n_P(\varkappa_i) - ş^n_Q(\varkappa_i)\right|\}$, $\Delta i_{max} = max\{\left|i^n_P(\varkappa_i) - i^n_Q(\varkappa_i)\right|\}$ and $\Delta d'_{max} = max\{\left|d'^n_P(\varkappa_i) - d'^n_Q(\varkappa_i)\right|\}$.

The following properties hold true for the weighted grey similarity measure of two TSFNs $P = (ş_P, i_P, d'_P)$ and $Q = (ş_Q, i_Q, d'_Q)$.

1. $0 \leq \mathrm{W}\mathrm{Ç}^3_{TSFS}(P,Q) \leq 1$.
2. $\mathrm{W}\mathrm{Ç}^3_{TSFS}(P,Q) = \mathrm{W}\mathrm{Ç}^3_{TSFS}(Q,P)$.
3. $\mathrm{W}\mathrm{Ç}^3_{TSFS}(P,Q) = 1$ if $P = Q$.

3.5. Application in Pattern Recognition

The tools of SMs have applications in pattern recognition. In problems, the class of an unknown pattern or object is determined using some information measure tools and some preferences of decision makers. Here, we aim to apply the SMs proposed in T-Spherical Fuzzy environment to a problem of building material recognition.

3.5.1. Building Material Recognition

In this subsection, the T-spherical fuzzy SMs, developed in Section 3.2, 3.3 and 3.4 so far, are applied to a building material recognition problem where the class of an unknown building material is determined. The results obtained using the SMs of TSFSs are then analyzed for the advantages of proposed work and the limitations of existing work. To explain the phenomenon, an illustrative example adapted from Reference [73] is discussed.

3.5.1.1.Example

Consider four TSFNs $P_i (i = 1, 2, 3, 4)$ represent four known building materials and let $X = \{x_i : i = 1, 2, 3, ... 7\}$ be the collection of attributes having weights $w = (0.16, 0.12, 0.09, 0.18, 0.20, 0.10, 0.15)^T$ based on which building materials are categorized. We assume another unknown material P with the hypothetical information listed in Table 6. With the help of defined SMs for TSFSs, we shall identify the class of unknown building material from four materials denoted by $P_i (i = 1, 2, 3, 4)$. Using the recognition principle discussed in [68, 73], given below, the evaluation of the class of P to P_i is established.

$$k = arg \underset{1 \le i \le 4}{Max} \{WÇ_{TSFS}\}$$

	P_1	P_2	P_3	P_4	P
x_1	(0.56, 0.47, 0.22)	(0.81,0.3,0.37)	(0.43, 0.43, 0.55)	(0.57, 0.51, 0.39)	(0.34,0.56,0.78)
x_2	(0.11, 0.11, 0.11)	(0.59, 0.66,0.66)	(0.91, 0.34, 0.68)	(0.56,0.76,0.31)	(0.47, 0.38,0.84)
x_3	(0.35, 0.45, 0.61)	(0.42,0.56, 0.71)	(0.81, 0.41, 0.35)	(0.27,0.59, 0.72)	(0.55,0.44,0.65)
x_4	(0.33, 0.54,0.31)	(0.59,0.45,0.9)	(0.44,0.55,0.77)	(0.46, 0.46,0.45)	(0.76,0.46,0.85)
x_5	(0.35, 0.2, 0.64)	(0.16,0.33,0.42)	(0.55, 0.44, 0.29)	(0.57,0.66,0.91)	(0.13, 0.35,0.57)
x_6	(0.47, 0.37, 0.68)	(0.68, 0.46, 0.88)	(0.47, 0.66, 0.75)	(0.41,0.73, 0.41)	(0.24, 0.54, 0.45)
x_7	(0.78, 0.55,0.03)	(0.49, 0.54, 0.39)	(0.58,0.34, 0.23)	(0.21, 0.43, 0.13)	(0.82,0.46, 0.69)

Table 6 (Data on building materials)

All the numbers in Table 1 are purely TSFNs for $n = 4$ which means that the SMs of IFSs and PFSs could not handle such type of data as their structures are limited. Even SFSs could not handle of this type of data as in SFSs we have $n = 2$. This shows the significance of working in the environment of TSFSs. Now, the different SMs defined in Section 3.2, 3.3 and 3.4 are applied to the given data in Table 1 and the results are provided in Table 7.

Similarity Measures	(P_1, P)	(P_2, P)	(P_3, P)	(P_4, P)
WÇ^{1}_{TSFS}	0.612207	0.690072	0.64601	0.603693
WÇ^{2}_{TSFS}	0.3197	0.367149	0.26296	0.160122
WÇ^{3}_{TSFS}	0.762518	0.792809	0.796319	0.750893

Table 7 (Similarity measures of P_i with P)

Analyzing Table 7, it seems that material P_2 is close to P as the similarity index of $(P_2, P) = 0.69$ and $(P_2, P) = 0.79$ which is larger than the similarity index of all other pairs if we apply CSMs or set-theoretic SMs. However, if we apply grey SMs, it seems that the values of (P_2, P) and (P_3, P) can be considered nearly equal or the similarity of (P_3, P) is slightly higher than that of (P_2, P), so by grey SM, P has a relatively larger similarity index with the class of P_3. Thus, it is concluded that based on CSM or set-theoretic SM, the unknown material P belongs to the class of P_2 type material while based on grey SM, the material P belongs to P_3 type material. The choice of using any of the SM is up to decision makers.

3.6. Comparative Study and Advantages

The SMs developed in this chapter are the generalizations of the SMs proposed in [68, 72] and [73]. In the following remarks, we prove that the previously defined SMs becomes the special cases of the proposed SMs.

3.6.1. Remark

- By placing $n = 2$, Definitions 3.2.1 and 3.3.2 reduced to spherical fuzzy environment.
- By placing $n = 1$, Definitions 3.2.1 and 3.3.2 reduced to picture fuzzy environment [73].
- By placing $n = 1$ and $i = 0$, Definitions 3.2.1 and 3.3.2 reduced to intuitionistic fuzzy environment [73].

3.6.2. Remark

- By placing $n = 2$, Definitions 3.3.1 and 3.3.2 reduced to spherical fuzzy environment.
- By placing $n = 1$, Definitions 3.3.1 and 3.3.2 reduced to picture fuzzy environment [73].
- By placing$n = 1$ and $i = 0$, Definitions 3.3.1 and 3.3.2 reduced to intuitionistic fuzzy environment [73].

3.6.3. Remark

- By placing $n = 2$, Definitions 3.4.1 and 3.4.2 reduced to spherical fuzzy environment.

- By placing $n = 1$, Definitions 3.4.1 and 3.4.2 reduced to picture fuzzy environment [73].

- By placing $n = 1$ and $i = 0$, Definitions 3.4.1 and 3.4.2 reduced to intuitionistic fuzzy environment [73].

The main advantage of the SMs proposed in the environment of SFSs and TSFSs is that these SMs can handle the data provided in [68, 72] and [73]. However, none of the SMs of the IFSs, PyFSs, q-ROFSs and PFSs can handle the information given in the form of TSFNs.

Now, the building material recognition problem from Reference [73] is solved using SMs of TSFSs for $n = 1$.

3.6.4. Example

In this problem from [73], four building materials are denoted by $P_i(i = 1, 2, 3, 4)$. The weighted SMs of TSFSs defined in 3.2.3, 3.3.2 and 3.4.2 are applied on the data provided in Table 8 to evaluate the class of unknown building material P. The weight vector in this case is $(0.12, 0.15, 0.09, 0.16, 0.20, 0.10, 0.18)^T$.

	P_1	P_2	P_3	P_4	P
x_1	(0.17, 0.53, 0.13)	(0.51, 0.24, 0.21)	(0.31, 0.39, 0.25)	(1, 0, 0)	(0.91, 0.03, 0.05)
x_2	(0.10, 0.81, 0.05)	(0.62, 0.12, 0.07)	(0.60, 0.26, 0.11)	(1, 0, 0)	(0.78, 0.12, 0.07)
x_3	(0.53, 0.33, 0.09)	(1, 0, 0)	(0.91, 0.03, 0.02)	(0.85, 0.09, 0.05)	(0.90, 0.05, 0.02)
x_4	(0.89, 0.08, 0.03)	(0.13, 0.64, 0.21)	(0.07, 0.09, 0.07)	(0.74, 0.16, 0.1)	(0.68, 0.08, 0.21)
x_5	(0.42, 0.35, 0.18)	(0.03, 0.82, 0.13)	(0.04, 0.85, 0.10)	(0.02, 0.89, 0.05)	(0.05, 0.87, 0.06)
x_6	(0.08, 0.89, 0.02)	(0.73, 0.15, 0.08)	(0.68, 0.26, 0.06)	(0.08, 0.84, 0.06)	(0.13, 0.75, 0.09)
x_7	(0.33, 0.51, 0.12)	(0.52, 0.31, 0.16)	(0.15, 0.76, 0.07)	(0.16, 0.71, 0.05)	(0.15, 0.73, 0.08)

Table 8 (information about known and unknown patterns [73])

All data in Table 8 is in picture fuzzy environment, we can use the proposed T-spherical fuzzy SMs by taking $n = 1$ and the results are given in Table 9.

SMs	(P_1, P)	(P_2, P)	(P_3, P)	(P_4, P)
WÇ^1_{TSFS}	0.715235	0.763072	0.855508	0.993654
WÇ^2_{TSFS}	0.5556	0.65557	0.693305	0.919909
WÇ^3_{TSFS}	0.708292	0.785837	0.915441	0.941545

Table 9 (SMs of P_i with P)

The results obtained in Table 9 are similar to those obtained in [73] which strengthened our claim that the SMs of TSFSs can handle the data provided in the environment of PFSs and subsequently other existing fuzzy frameworks. Conversely, the SMs of IFSs and PFSs could not handle the data provided in TSF environment as a TSFN cannot be considered as an IFN or PFN in general.

The main two advantages of working in TSF environment are as follows:

- A TSF framework allows us to discuss the four aspects of uncertainty with the help of the MG, AG, NG and RG while IFS, PyFS and q-ROFS have only MG and NG. This means that the chances of the loss in information in previous study is more than the proposed study.

- Proposed SMs of TSFSs has no restriction for assigning values to MG, AG and NG while in the environment of IFSs, PyFSs and PFSs there are some restrictions in assigning membership values.

Chapter 4

Correlation Coefficients for T-Spherical Fuzzy Sets and Their Applications [4]

The objective of this chapter is to develop some correlation coefficients (CCs) for TSFSs due to the non-applicability of CCs of IFSs and PFSs in some certain circumstances. The validity of new CCs has been discussed and their significance is studied with the help of some results. A clustering and multi-attribute decision making algorithms have been proposed in the environment of TSFSs. To demonstrate the viability of proposed algorithms and CCs, two real life problems including a clustering problem and a MADM problem have been solved. A comparative study of newly presented and pre-existing literature is established showing the superiority of proposed work. Some advantages of the new CCs and the drawbacks of the previous study are demonstrated with the help of numerical examples.

4.1. Examining the Drawbacks in Existing Study

The goal of this section is to study some CCs in intuitionistic and picture fuzzy settings and point out their limitations. CCs are most useful for MADM, pattern recognition, clustering and medical diagnosis problems. For a detailed study about CCs, one is referred

[4] The work of this chapter is from following published paper:
Ullah K., *Garg H., Mahmood T., Jan N. and Ali Z. Correlation coefficients for T-spherical fuzzy sets and their applications in clustering and multi-attribute decision making. Soft Computing, 2019. https://doi.org/10.1007/s00500-019-03993-6

to [25, 72, 75, 76, 78, 79, 82]. Throughout this chapter, X shall denote a universe of discourse and $\varkappa_j \in X$ for $j = 1, 2, 3, \dots m$.

The concept of information energy (IE) of an IFS was given by Gerstenkorn and Manko [75] and is defined as:

4.1.1. Definition [75]

For an IFS $I = \{(\varkappa, (\text{ş}, \text{d'})) : \varkappa \in X\}$, the IE of I is defined as:

$$E_{IFS}(I) = \sum_{j=1}^{m} [\text{ş}^2(\varkappa_j) + \text{d'}^2(\varkappa_j)]$$

Gerstenkorn and Manko [75] used the above IE to define the correlation of two *IFS*s as follows:

4.1.2. Definition [75]

For two IFSs I_1 and I_2, the correlation between I_1 and I_2 is defined as:

$$C_{IFS}^{1}(I_1, I_2) = \sum_{j=1}^{m} [\text{ş}_1(\varkappa_j) \cdot \text{ş}_2(\varkappa_j) + \text{d'}_1(\varkappa_j) \cdot \text{d'}_2(\varkappa_j)]$$

Extending this idea of correlation, Gerstenkorn and Manko [75] proposed the concept of CC between two IFSs as follows:

4.1.3. Definition [75]

For two IFSs I_1 and I_2, the CC between I_1 and I_2 is defined as:

$$\mathcal{K}_{IFS}^{1}(I_1, I_2) = \frac{\sum_{j=1}^{m} [\text{ş}_1(\varkappa_j) \cdot \text{ş}_2(\varkappa_j) + \text{d'}_1(\varkappa_j) \cdot \text{d'}_2(\varkappa_j)]}{\sqrt{[(\sum_{j=1}^{m} [\text{ş}_1^2(\varkappa_j) + \text{d'}_1^2(\varkappa_j)])(\sum_{j=1}^{m} [\text{ş}_2^2(\varkappa_j) + \text{d'}_2^2(\varkappa_j)])]}}$$

This CC of IFS describe two aspects of human opinion i.e. yes or no denoted by MG and NG only. However, it fails whenever information has some abstinence and refusal degree such as in MADM an pattern recognition. To overcome this difficulty, Sing [76] introduced the idea of IE and CCs in picture fuzzy environment.

The concept of information energy (IE) of a PFS proposed by Sing [76] and is defined as:

4.1.4. Definition [76]

For a PFS $I = \{(\varkappa, (\varsigma, i, \text{d'})) : \varkappa \in X\}$, the IE of I is defined as:

$$E_{PFS}(I) = \sum_{j=1}^{m} [\varsigma^2(\varkappa_j) + i^2(\varkappa_j) + \text{d'}^2(\varkappa_j) + r^2(\varkappa_j)]$$

Sing [76] used the above IE to define the correlation of two *IFS*s as follows:

4.1.5. Definition [76]

For two PFSs I_1 and I_2, the correlation between I_1 and I_2 is defined as:

$$C^1_{PFS}(I_1, I_2) = \sum_{j=1}^{m} [\varsigma_1(\varkappa_j) \cdot \varsigma_2(\varkappa_j) + i_1(\varkappa_j) \cdot i_2(\varkappa_j) + \text{d'}_1(\varkappa_j) \cdot \text{d'}_2(\varkappa_j) + r_1(\varkappa_j) \cdot r_2(\varkappa_j)]$$

Extending this idea of correlation, Sing [76] proposed the concept of CC between two PFSs as follows:

4.1.6. Definition [76]

For two PFSs I_1 and I_2, the CCs between I_1 and I_2 is defined as:

$\mathcal{K}^1_{PFS}(I_1, I_2)$

$$= \frac{\sum_{j=1}^{m}\left[\varsigma_1(\varkappa_j)\cdot\varsigma_2(\varkappa_j) + i_1(\varkappa_j)\cdot i_2(\varkappa_j) + d'_1(\varkappa_j)\cdot d'_2(\varkappa_j) + r_1(\varkappa_j)\cdot r_2(\varkappa_j)\right]}{\sqrt{\left(\sum_{j=1}^{m}\left[\varsigma_1^2(\varkappa_j) + i_1^2(\varkappa_j) + d'^2_1(\varkappa_j) + r_1^2(\varkappa_j)\right]\right)\left(\sum_{j=1}^{m}\left[\varsigma_2^2(\varkappa_j) + i_2^2(\varkappa_j) + d'^2_2(\varkappa_j) + r_2^2(\varkappa_j)\right]\right)}}$$

This concept of CC for PFS is a generalization of the CC of IFS proposed by Gerstenkorn and Manko [75] and has the ability of describing abstinence and refusal degree along with MG and NG and hence reduced the chances of information loss. However, it is observed that under some certain circumstances, the defined CCs are unable to measures of degree of relationship between two PFSs. For instance, consider triplets of the form $I_1 = (0.5, 0.6, 0.8)$ and $I_2 = (0.3, 0.6, 0.9)$. These two triplets cannot be considered as PFSs as $0 \leq 0.5 + 0.6 + 0.8 = 1.9 \nleq 1$ and $0 \leq 0.3 + 0.6 + 0.9 = 1.8 \nleq 1$ and the above-mentioned CCs cannot be applied to compute their degree of correlation. Therefore, we aim to develop some improved more general CCs that can be used to compute the CC of such triplets.

4.2. New Correlation Coefficients

The aim of this section is to develop some new CCs to overcome the situations which were remained unsolved by the previous literature studied in [75, 76]. We propose the new type of IE, correlation and CCs in the environment of TSFSs as follows:

4.2.1. Definition

For a TSFS $I = \{(\varkappa, (\varsigma, i, d')): \varkappa \in X\}$, the IE of I is defined as:

$$E_{TSFS}(I) = \sum_{j=1}^{m}\left[\left(\varsigma^2(\varkappa_j)\right)^n + \left(i^2(\varkappa_j)\right)^n + \left(d'^2(\varkappa_j)\right)^n + \left(r^2(\varkappa_j)\right)^n\right]$$

4.2.2. Definition

For two TSFSs I_1 and I_2, the correlation between I_1 and I_2 is defined as:

$$C^1_{TSFS}(I_1, I_2) = \sum_{j=1}^{m} \begin{bmatrix} \mathfrak{s}_1^n(\varkappa_j) \cdot \mathfrak{s}_2^n(\varkappa_j) + i_1^n(\varkappa_j) \cdot i_2^n(\varkappa_j) + \\ \mathrm{d}_1'^n(\varkappa_j) \cdot \mathrm{d}_2'^n(\varkappa_j) + r_1^n(\varkappa_j) \cdot r_2^n(\varkappa_j) \end{bmatrix}$$

Extending this idea of correlation, we propose the concept of CC between two TSFSs as follows:

4.2.3. Definition

For two TSFSs I_1 and I_2, the CC between I_1 and I_2 is defined as:

$$\mathcal{K}^1_{TSFS}(I_1, I_2) = \frac{\sum_{j=1}^{m} \begin{bmatrix} \mathfrak{s}_1^n(\varkappa_j) \cdot \mathfrak{s}_2^n(\varkappa_j) + i_1^n(\varkappa_j) \cdot i_2^n(\varkappa_j) + \\ \mathrm{d}_1'^n(\varkappa_j) \cdot \mathrm{d}_2'^n(\varkappa_j) + r_1^n(\varkappa_j) \cdot r_2^n(\varkappa_j) \end{bmatrix}}{\sqrt{\begin{matrix} \left(\sum_{j=1}^{m} \left[\left(\mathfrak{s}_1^2(\varkappa_j)\right)^n + \left(i_1^2(\varkappa_j)\right)^n + \left(\mathrm{d}_1'^2(\varkappa_j)\right)^n + \left(r_1^2(\varkappa_j)\right)^n \right]\right) \cdot \\ \left(\sum_{j=1}^{m} \left[\left(\mathfrak{s}_2^2(\varkappa_j)\right)^n + \left(i_2^2(\varkappa_j)\right)^n + \left(\mathrm{d}_2'^2(\varkappa_j)\right)^n + \left(r_2^2(\varkappa_j)\right)^n \right]\right) \end{matrix}}}$$

The Definition 4.2.3 likely to satisfy the following properties:

1. $\mathcal{K}^1_{TSFS}(I_1, I_2) = \mathcal{K}^1_{TSFS}(I_2, I_1)$
2. $0 \le \mathcal{K}^1_{TSFS}(I_1, I_2) \le 1$
3. $\mathcal{K}^1_{TSFS}(I_1, I_2) = 1 \Leftrightarrow I_1 = I_2$

Proof:

1. From Definition 4.2.3, we have

$$\mathcal{K}^1_{TSFS}(I_1, I_2) = \frac{\sum_{j=1}^{m}\begin{bmatrix} \varsigma_1^n(\varkappa_j)\cdot\varsigma_2^n(\varkappa_j) + i_1^n(\varkappa_j)\cdot i_2^n(\varkappa_j) + \\ \mathrm{d'}_1^n(\varkappa_j)\cdot\mathrm{d'}_2^n(\varkappa_j) + r_1^n(\varkappa_j)\cdot r_2^n(\varkappa_j)\end{bmatrix}}{\sqrt{\begin{array}{c}\left(\sum_{j=1}^{m}\left[\left(\varsigma_1^2(\varkappa_j)\right)^n + \left(i_1^2(\varkappa_j)\right)^n + \left(\mathrm{d'}_1^2(\varkappa_j)\right)^n + \left(r_1^2(\varkappa_j)\right)^n\right]\right)\cdot \\ \left(\sum_{j=1}^{m}\left[\left(\varsigma_2^2(\varkappa_j)\right)^n + \left(i_2^2(\varkappa_j)\right)^n + \left(\mathrm{d'}_2^2(\varkappa_j)\right)^n + \left(r_2^2(\varkappa_j)\right)^n\right]\right)\end{array}}}$$

$$= \frac{\sum_{j=1}^{m}\begin{bmatrix} \varsigma_2^n(\varkappa_j).\varsigma_1^n(\varkappa_j) + i_2^n(\varkappa_j).i_1^n(\varkappa_j) + \\ \mathrm{d'}_2^n(\varkappa_j).\mathrm{d'}_1^n(\varkappa_j) + r_2^n(\varkappa_j).r_1^n(\varkappa_j)\end{bmatrix}}{\sqrt{\begin{array}{c}\left(\sum_{j=1}^{m}\left[\left(\varsigma_2^2(\varkappa_j)\right)^n + \left(i_2^2(\varkappa_j)\right)^n + \left(\mathrm{d'}_2^2(\varkappa_j)\right)^n + \left(r_2^2(\varkappa_j)\right)^n\right]\right)\cdot \\ \left(\sum_{j=1}^{m}\left[\left(\varsigma_1^2(\varkappa_j)\right)^n + \left(i_1^2(\varkappa_j)\right)^n + \left(\mathrm{d'}_1^2(\varkappa_j)\right)^n + \left(r_1^2(\varkappa_j)\right)^n\right]\right)\end{array}}}$$

$$= \mathcal{K}^1_{TSFS}(I_2, I_1)$$

2. For two TSFSs $I_1$and I_2, the inequality $\mathcal{K}^1_{TSFS}(I_1, I_2) \geq 0$ holds trivially by the Definition 4.2.3. Now, we show that $\mathcal{K}^1_{TSFS}(I_1, I_2) \leq 1$.

$$C^1_{TSFS}(I_1, I_2) = \sum_{j=1}^{m}\begin{bmatrix} \varsigma_1^n(\varkappa_j)\cdot\varsigma_2^n(\varkappa_j) + i_1^n(\varkappa_j)\cdot i_2^n(\varkappa_j) + \\ \mathrm{d'}_1^n(\varkappa_j)\cdot\mathrm{d'}_2^n(\varkappa_j) + r_1^n(\varkappa_j)\cdot r_2^n(\varkappa_j)\end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} \varsigma_1^n(\varkappa_1)\cdot\varsigma_2^n(\varkappa_1) + i_1^n(\varkappa_1)\cdot i_2^n(\varkappa_1) + \\ \mathrm{d'}_1^n(\varkappa_1)\cdot\mathrm{d'}_2^n(\varkappa_1) + r_1^n(\varkappa_1)\cdot r_2^n(\varkappa_1)\end{bmatrix} + \begin{bmatrix} \varsigma_1^n(\varkappa_2)\cdot\varsigma_2^n(\varkappa_2) + i_1^n(\varkappa_2)\cdot i_2^n(\varkappa_2) + \\ \mathrm{d'}_1^n(\varkappa_2)\cdot\mathrm{d'}_2^n(\varkappa_2) + r_1^n(\varkappa_2)\cdot r_2^n(\varkappa_2)\end{bmatrix} + \cdots + \\ \begin{bmatrix} \varsigma_1^n(\varkappa_m)\cdot\varsigma_2^n(\varkappa_m) + i_1^n(\varkappa_m)\cdot i_2^n(\varkappa_m) + \\ \mathrm{d'}_1^n(\varkappa_m)\cdot\mathrm{d'}_2^n(\varkappa_m) + r_1^n(\varkappa_m)\cdot r_2^n(\varkappa_m)\end{bmatrix}\end{bmatrix}$$

Using Cauchy Schwarz Inequality for $(\varkappa_1, \varkappa_2, \dots, \varkappa_m), (y_1, y_2, \dots, y_m) \in R^m$, we have

$$(\varkappa_1 y_1 + \varkappa_2 y_2 + \varkappa_3 y_3 + \cdots \varkappa_m y_m)^{\frac{1}{2}} \leq (\varkappa_1^2 + \varkappa_2^2 + \cdots + \varkappa_m^2).(y_1^2 + y_2^2 + \cdots + y_m^2).$$

Thus,

$$\left(C^1_{TSFS}(I_1, I_2)\right)^2$$

$$\leq \left[\begin{bmatrix} \begin{bmatrix} ş_1^n(\varkappa_1)\cdot ş_2^n(\varkappa_1)+i_1^n(\varkappa_1)\cdot i_2^n(\varkappa_1)+ \\ \text{d}_1'^n(\varkappa_1)\cdot \text{d}_2'^n(\varkappa_1)+r_1^n(\varkappa_1)\cdot r_2^n(\varkappa_1)\end{bmatrix}+\begin{bmatrix} ş_1^n(\varkappa_2)\cdot ş_2^n(\varkappa_2)+i_1^n(\varkappa_2)\cdot i_2^n(\varkappa_2)+ \\ \text{d}_1'^n(\varkappa_2)\cdot \text{d}_2'^n(\varkappa_2)+r_1^n(\varkappa_2)\cdot r_2^n(\varkappa_2)\end{bmatrix}+\cdots+ \\ \begin{bmatrix} ş_1^n(\varkappa_m)\cdot ş_2^n(\varkappa_m)+i_1^n(\varkappa_m)\cdot i_2^n(\varkappa_m)+ \\ \text{d}_1'^n(\varkappa_m)\cdot \text{d}_2'^n(\varkappa_m)+r_1^n(\varkappa_m)\cdot r_2^n(\varkappa_m)\end{bmatrix}\end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix} \begin{bmatrix} ş_1^n(\varkappa_1)\cdot ş_2^n(\varkappa_1)+i_1^n(\varkappa_1)\cdot i_2^n(\varkappa_1)+ \\ \text{d}_1'^n(\varkappa_1)\cdot \text{d}_2'^n(\varkappa_1)+r_1^n(\varkappa_1)\cdot r_2^n(\varkappa_1)\end{bmatrix}+\begin{bmatrix} ş_1^n(\varkappa_2)\cdot ş_2^n(\varkappa_2)+i_1^n(\varkappa_2)\cdot i_2^n(\varkappa_2)+ \\ \text{d}_1'^n(\varkappa_2)\cdot \text{d}_2'^n(\varkappa_2)+r_1^n(\varkappa_2)\cdot r_2^n(\varkappa_2)\end{bmatrix}+\cdots+ \\ \begin{bmatrix} ş_1^n(\varkappa_m)\cdot ş_2^n(\varkappa_m)+i_1^n(\varkappa_m)\cdot i_2^n(\varkappa_m)+ \\ \text{d}_1'^n(\varkappa_m)\cdot \text{d}_2'^n(\varkappa_m)+r_1^n(\varkappa_m)\cdot r_2^n(\varkappa_m)\end{bmatrix}\end{bmatrix}\right]$$

$$= \begin{bmatrix} \left(\sum_{j=1}^{m}\left[\left(ş_1^2(\varkappa_j)\right)^n+\left(i_1^2(\varkappa_j)\right)^n+\left(\text{d}_1'^2(\varkappa_j)\right)^n+\left(r_1^2(\varkappa_j)\right)^n\right]\right)\times \\ \left(\sum_{j=1}^{m}\left[\left(ş_2^2(\varkappa_j)\right)^n+\left(i_2^2(\varkappa_j)\right)^n+\left(\text{d}_2'^2(\varkappa_j)\right)^n+\left(r_2^2(\varkappa_j)\right)^n\right]\right)\end{bmatrix}$$

$$= E_{TSFS}(I_1).E_{TSFS}(I_2)$$

Therefore, $\left(C_{TSFS}^1(I_1,I_2)\right)^2 \leq E_{TSFS}(I_1).E_{TSFS}(I_2)$, which implies that $\mathcal{K}_{TSFS}^1(I_1,I_2) \leq 1$.

Hence, $0 \leq \mathcal{K}_{TSFS}^1(I_1,I_2) \leq 1$.

3.

Using the Definition of TSFS (Definition 2.4.1), we have $I_1 = I_2 \Leftrightarrow ş_1^n(\varkappa_j) = ş_2^n(\varkappa_j), i_1^n(\varkappa_j) = i_2^n(\varkappa_j)$ and $\text{d}_1'^n(\varkappa_j) = \text{d}_2'^n(\varkappa_j)$ where $\varkappa_j \in X$. Hence $\mathcal{K}_{TSFS}^1(I_1,I_2) = 1$.

4.2.4. Remark

If we set $n = 2$ in Definition 4.2.3. Then it reduces to the CC of SFSs.

4.2.5. Example

Consider $I_1 = (<\varkappa_1, 0.7,0.8,0.9>, <\varkappa_2, 0.7,0.6,0.4>, <\varkappa_3, 0.5,0.4,0.5>)$ and $I_2 = (<\varkappa_1, 0.7,0.6,0.5>, <\varkappa_2, 0.9,0.8,0.9>, <\varkappa_3, 0.4,0.7,0.5>)$ be two TSFSs. Clearly I_1 and

I_2 are PFss as the sum of all three grades in I_1 and I_2 exceeds 1. So, the CC for PFSs defined in Definition 4.1.6 could not be used to compute the CC of I_1 and I_2. On the other hand, the new correlation proposed Definition 4.2.3 can compute the correlation of I_1 and I_2. Further note that I_1 and I_2 are TSFSs for $n = 6$. First, using Definition 4.2.1, the information energy of I_1 and I_2 are computed as $E_{TSFS}(I_1) = 0.38$ and $E_{TSFS}(I_2) = 0.66$. Then, using Definition 4.2.2, we compute $C^1_{TSFS}(I_1, I_2) = 0.109$. Finally, the CC between the TSFSs I_1 and I_2 is computed by using Definition 4.2.3 given by $\mathcal{K}^1_{TSFS}(I_1, I_2) = 0.22$.

Now, using the IE and correlation (see Definition 4.2.1 and 4.2.2), we propose the following CC for TSFSs.

4.2.6. Definition

For two TSFSs I_1 and I_2, the CC between I_1 and I_2 is defined as:

$$\mathcal{K}^2_{TSFS}(I_1, I_2) = \frac{\sum_{j=1}^{m}\begin{bmatrix} \mathfrak{s}_1^n(\varkappa_j)\cdot \mathfrak{s}_2^n(\varkappa_j) + i_1^n(\varkappa_j)\cdot i_2^n(\varkappa_j) + \\ \mathrm{d'}_1^n(\varkappa_j)\cdot \mathrm{d'}_2^n(\varkappa_j) + r_1^n(\varkappa_j)\cdot r_2^n(\varkappa_j) \end{bmatrix}}{max\begin{pmatrix} \left(\sum_{j=1}^{m}\left[\left(\mathfrak{s}_1^2(\varkappa_j)\right)^n + \left(i_1^2(\varkappa_j)\right)^n + \left(\mathrm{d'}_1^2(\varkappa_j)\right)^n + \left(r_1^2(\varkappa_j)\right)^n\right]\right)\cdot \\ \left(\sum_{j=1}^{m}\left[\left(\mathfrak{s}_2^2(\varkappa_j)\right)^n + \left(i_2^2(\varkappa_j)\right)^n + \left(\mathrm{d'}_2^2(\varkappa_j)\right)^n + \left(r_2^2(\varkappa_j)\right)^n\right]\right) \end{pmatrix}}$$

The Definition 4.2.6 likely to satisfy the following properties:

6. $\mathcal{K}^2_{TSFS}(I_1, I_2) = \mathcal{K}^2_{TSFS}(I_2, I_1)$
7. $0 \leq \mathcal{K}^2_{TSFS}(I_1, I_2) \leq 1$
8. $\mathcal{K}^2_{TSFS}(I_1, I_2) = 1 \Leftrightarrow I_1 = I_2$

Proof: The proof of 6 and 8 is obvious. To prove 7, it is obvious that $\mathcal{K}^2_{TSFS}(I_1, I_2) \geq 0$. To show $\mathcal{K}^2_{TSFS}(I_1, I_2) \leq 1$, as

$$\left(C^1_{TSFS}(I_1, I_2)\right) \leq \sqrt{E_{TSFS}(I_1).E_{TSFS}(I_2)}$$

Hence

$$\left(C^1_{TSFS}(I_1, I_2)\right) \leq max\left(E_{TSFS}(I_1), E_{TSFS}(I_2)\right)$$

Therefore, $\mathcal{K}^2_{TSFS}(I_1, I_2) \leq 1$.

Whenever we deal with real life problems, the weight of experts plays an essential role such as in MADM problems. For such kinds of scenarios, we developed some weighted CCs. In our further study, $w = (w_1, w_2, \dots, w_m)^t$ is considered as a weight vector such that $w_j > 0$ where$j \in \{1, 2, 3, 4, \dots m\}$ and $\sum_{j=1}^{m} w_j = 1$.

The weighted CC corresponding to Definition 4.2.64.2.3 is defined as

4.2.7. Definition

For two TSFSs I_1 and I_2, the weighted CC between I_1 and I_2 is defined as:

$$\mathcal{K}^3_{TSFS}(I_1, I_2) = \frac{\sum_{j=1}^{m} w_j \begin{bmatrix} \text{ş}_1^n(\varkappa_j) \cdot \text{ş}_2^n(\varkappa_j) + i_1^n(\varkappa_j) \cdot i_2^n(\varkappa_j) + \\ \text{d}'^n_1(\varkappa_j) \cdot \text{d}'^n_2(\varkappa_j) + r_1^n(\varkappa_j) \cdot r_2^n(\varkappa_j) \end{bmatrix}}{\sqrt{\begin{array}{c}\left(\sum_{j=1}^{m} w_j \left[\left(\text{ş}_1^2(\varkappa_j)\right)^n + \left(i_1^2(\varkappa_j)\right)^n + \left(\text{d}'^2_1(\varkappa_j)\right)^n + \left(r_1^2(\varkappa_j)\right)^n\right]\right) \cdot \\ \left(\sum_{j=1}^{m} w_j \left[\left(\text{ş}_2^2(\varkappa_j)\right)^n + \left(i_2^2(\varkappa_j)\right)^n + \left(\text{d}'^2_2(\varkappa_j)\right)^n + \left(r_2^2(\varkappa_j)\right)^n\right]\right)\end{array}}}$$

The Definition 4.2.7 likely to satisfy the following properties:

1. $\mathcal{K}^3_{TSFS}(I_1, I_2) = \mathcal{K}^3_{TSFS}(I_2, I_1)$
2. $0 \leq \mathcal{K}^3_{TSFS}(I_1, I_2) \leq 1$
3. $\mathcal{K}^3_{TSFS}(I_1, I_2) = 1 \Leftrightarrow I_1 = I_2$

Proof: Straightforward

The weighted CC corresponding to Definition 4.2.74.2.6 is defined as

4.2.8. Definition

For two TSFSs I_1 and I_2, the weighted CC between I_1 and I_2 is defined as:

$$\mathcal{K}_{TSFS}^4(I_1, I_2) = \frac{\sum_{j=1}^{m} w_j \begin{bmatrix} \text{ş}_1^n(\varkappa_j) \cdot \text{ş}_2^n(\varkappa_j) + i_1^n(\varkappa_j) \cdot i_2^n(\varkappa_j) + \\ \text{d'}_1^n(\varkappa_j) \cdot \text{d'}_2^n(\varkappa_j) + r_1^n(\varkappa_j) \cdot r_2^n(\varkappa_j) \end{bmatrix}}{max \begin{pmatrix} \left(\sum_{j=1}^{m} w_j \left[\left(\text{ş}_1^2(\varkappa_j)\right)^n + \left(i_1^2(\varkappa_j)\right)^n + \left(\text{d'}_1^2(\varkappa_j)\right)^n + \left(r_1^2(\varkappa_j)\right)^n\right]\right) \cdot \\ \left(\sum_{j=1}^{m} w_j \left[\left(\text{ş}_2^2(\varkappa_j)\right)^n + \left(i_2^2(\varkappa_j)\right)^n + \left(\text{d'}_2^2(\varkappa_j)\right)^n + \left(r_2^2(\varkappa_j)\right)^n\right]\right) \end{pmatrix}}$$

The Definition 4.2.8 likely to satisfy the following properties:

1. $\mathcal{K}_{TSFS}^4(I_1, I_2) = \mathcal{K}_{TSFS}^4(I_2, I_1)$
2. $0 \leq \mathcal{K}_{TSFS}^4(I_1, I_2) \leq 1$
3. $\mathcal{K}_{TSFS}^4(I_1, I_2) = 1 \Leftrightarrow I_1 = I_2$

Proof: Straightforward

4.2.9. Remark

If we set $n = 2$, then the weighted CCs proposed in Definition 4.2.7 and 4.2.8 reduced to the CCs for SFSs.

4.2.10. Remark

The Definition 4.2.7 and 4.2.8 reduces to Definition 4.2.3 and 4.2.6 respectively if we assume $w = (\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m})^t$.

4.3. Applications

In this section, our aim is to use the CCs introduced in TSF environment in clustering and MADM problems. We studied algorithms for clustering and MADM and solved numerical problems to illustrate the algorithms.

4.3.1. Clustering

A clustering algorithm based on intuitionistic fuzzy CC is introduced by Gerstenkorn and Manko [75] which was further extended by Sing [76] to picture fuzzy environment. Sing [76] used CCs of PFNs instead of CCs of IFNs which obviously enhanced the results as a PFS can model human opinion better than IFS. As we have studied in Section 4.1 that in some cases, the CCs of IFNs and PFNs are unable to apply to practical problems and further the loss of information is more in intuitionistic and picture fuzzy environment. We used the algorithm proposed by Gerstenkorn and Manko [75] using CCs of TSFSs and solved a clustering problem.

4.3.1.1. Definition

For a collection of TSFNs I_j, $D = (\mathcal{K}_{jk})_{m\times m}$ be a matrix of CCs where $\mathcal{K}_{jk} = \mathcal{K}(A_j, A_k)$ is a CC between (A_j, A_k) satisfying

1. $0 \leq \mathcal{K}(A_j, A_k) \leq 1$
2. $\mathcal{K}(A_j, A_k) = 1$
3. $\mathcal{K}(A_j, A_k) = \mathcal{K}(A_k, A_j)$

4.3.1.2. Definition [75]

For a matrix of CCs $D = (\mathcal{K}_{jk})_{m\times m}$ if $D^2 = D \circ D = (\dot{\mathcal{K}}_{jk})_{m\times m}$. Then D^2 is known as the composition matrix of D and $\dot{\mathcal{K}}_{jk}$ is defined as:

$$\dot{\mathcal{K}}_{jk} = \max_{l}\{min(\mathcal{K}_{jl}, \mathcal{K}_{lk})\}$$

4.3.1.3. Theorem [75]

For a correlation matrix D and for $l_1, l_2 \in \mathbb{Z}$, the composite matrix $D^{l_1 l_2} = D^{l_1} \circ D^{l_2}$ is also a correlation matrix.

4.3.1.4. Definition [75]

For a matrix of CCs $D = (\mathcal{K}_{jk})_{m\times m}$ such that $D^2 \subseteq D$, D is considered as an equivalent correlation matrix. Further $D^2 \subseteq D$ means $\max_{l}\{min(\mathcal{K}_{jl}, \mathcal{K}_{lk})\} \leq \mathcal{K}_{jk}$.

4.3.1.5. Theorem [75]

Let $D = (\mathcal{K}_{jk})_{m\times m}$ be a matrix of correlation. Then after a finite times composition i.e. $D \to D^2 \to D^4 \to \cdots D^{2k} \to \cdots$ there must exist some $k \in \mathbb{Z}$ such that $D^{2k} = D^{2(k+1)}$ where D^{2k} is an equivalent correlation matrix.

4.3.1.6. Definition [75]

For a matrix of correlation coefficients $D = (\mathcal{K}_{jk})_{m\times m}$, we say $D_\sigma = (\sigma\mathcal{K}_{jk})_{m\times m}$ is the σ −cutting matrix of D where $\sigma \in [0,1]$ and

$$\sigma\mathcal{K}(A_j, A_k) = \begin{cases} 0 & if \;\; \mathcal{K}_{jk} \leq \sigma \\ 1 & if \;\; \mathcal{K}_{jk} \geq \sigma \end{cases}$$

Next, we present the algorithm for solving clustering problem under the TSF environment as follows. The various steps summarized in it are given as below followed by a flow chart in Figure 11.

Step 1. Let A_i be a collection of TSFNs. Then using Definition 4.2.7 or Definition 4.2.8, the matrix of CCs i.e. $D = (\mathcal{K}_{jk})_{m\times m}$ is obtained.

Step 2. Check, if $D^2 \subseteq D$ i.e. if D is an equivalent matrix. If not, construct the finite time composition to get the equivalent matrix D^{2^l} unless $D^{2^l} = D^{2^{l+1}}$

Step 3. Using Definition 4.3.1.6, an $\sigma - cutting$ matrix is established for classification of the TSFNs. If all the elements of the j^{th} line and the corresponding element of the k^{th} line in D_σ are the same. Then the TSFNs are considered as of the same type. Using this principle, all TSFNs are classified.

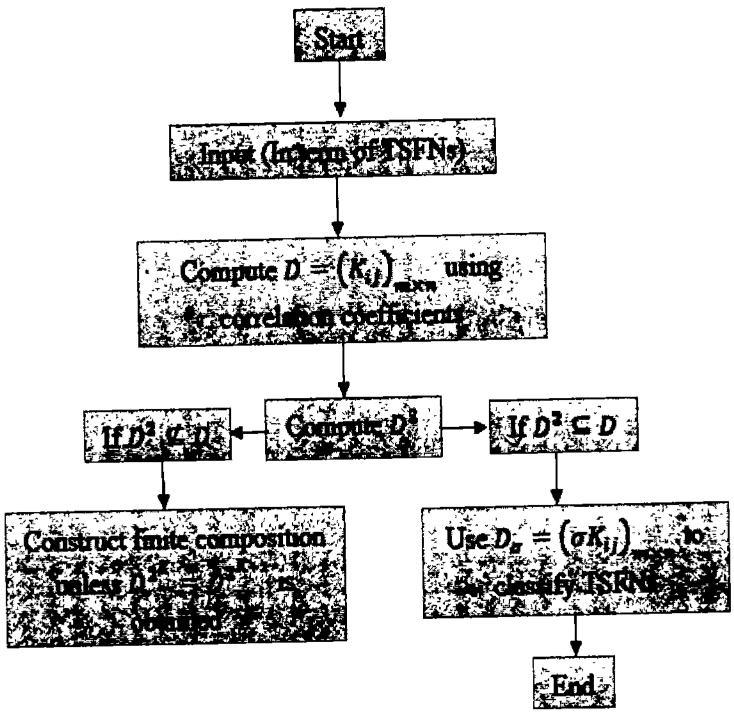

Figure 11 (Flowchart of the clustering algorithm)

The following Example 4.3.1.7 is to demonstrate the above defined clustering algorithm.

4.3.1.7. Example

Consider a car company needed to classify their cars based on some attribute. In this example (adapted from [76]), four cars are taken, and we need to classify these four cars based on three attributes. Let $A = \{A_1, A_2, A_3, A_4\}$ denotes the set of four cars under the attribute set $C = \{c_1, c_2, c_3\}$ where c_1, c_2 and c_3 denote the *fuel economy*, *safety* and *price of the car* respectively. Let $w = \{0.5,0.3,0.2\}^t$ be the weight vector and the evaluation of experts on each car are listed in Table 10. The data clearly indicate the MG, AG and the NG of the cars A_i under the attributes c_i. Moreover, note that every entry provided in Table 10 is purely a TSFN for $n = 3$.

	c_1	c_2	c_3
A_1	$(0.67, 0.59, 0.46)$	$(0.93, 0.46, 0.46)$	$(0.46, 0.10, 0.71)$
A_2	$(0.74, 0.46, 0.50)$	$(0.10, 0.89, 0.46)$	$(0.67, 0.59, 0.74)$
A_3	$(0.59, 0.56, 0.46)$	$(0.90, 0.46, 0.10)$	$(0.89, 0.46, 0.00)$
A_4	$(0.90, 0.40, 0.20)$	$(0.50, 0.50, 0.40)$	$(0.60, 0.50, 0.40)$

Table 10 (Decision Matrix based on evaluation of cars)

Clearly, the data provided in Table 10 could not be handled using the approach proposed in [75] and [76]. The correlations of IFSs are based on MG and NG only and therefore unable to process information provided in the form of TSFNs. Similarly, the correlation of PFSs has a specific constraint on its membership grades and therefore unable to process the data provided in the form of TSFNs. For example, if we consider the value of A_1 under c_1 which is $(0.67, 0.59, 0.46)$. So $0 \leq$ şum $(0.67, 0.59, 0.46) = 1.72 \nleq 1$. Now, the algorithm proposed for clustering is applied on data provided in Table 10 and the stepwise computations are discussed below.

Step 1: Using Definition 4.2.7 on the data provided in Table 10, the correlation matrix is constructed after computing CCs as follows:

$$D = \begin{bmatrix} 1 & .4455 & .8227 & .5329 \\ .4455 & 1 & .4574 & .6466 \\ .8227 & .4574 & 1 & .4498 \\ .5329 & .6466 & .4498 & 1 \end{bmatrix}$$

Step 2: Construction of equivalent correlation matrix.

$$D^2 = D \circ D = \begin{bmatrix} 1 & .5329 & .8227 & .5329 \\ .5329 & 1 & .4574 & .6466 \\ .8227 & .4574 & 1 & .5329 \\ .5329 & .6466 & .5329 & 1 \end{bmatrix}$$

$$D^4 = D^2 \circ D^2 = \begin{bmatrix} 1 & .5329 & .8227 & .5329 \\ .5329 & 1 & .5329 & .6466 \\ .8227 & .5329 & 1 & .5329 \\ .5329 & .6466 & .5329 & 1 \end{bmatrix}$$

and

$$D^8 = D^4 \circ D^4 = \begin{bmatrix} 1 & .5329 & .8227 & .5329 \\ .5329 & 1 & .5329 & .6466 \\ .8227 & .5329 & 1 & .5329 \\ .5329 & .6466 & .5329 & 1 \end{bmatrix}$$

Obviously $D^8 = D^4$. Therefore, D^4 is the equivalent correlation matrix in this case.

Step 3: Using Definition 4.3.1.6, the $\sigma - cutting$ matrix is obtained based on which the following classifications are established.

1. If $0 \leq \sigma \leq 0.5329$. Then all A_1, A_2, A_3 and A_4 of the same type i.e. we have

   $\{A_1, A_2, A_3, A_4\}$.

2. If $0.5329 \leq \sigma \leq 0.6466$. Then the classifications are.

   $\{A_2\}, \{A_1, A_3, A_4\}$.

3. If $0.6466 \leq \sigma \leq 0.8227$. Then the cars are classified into three types.

$\{A_1, A_3\}, \{A_2\}, \{A_4\}$.

4. If $0.8227 \leq \sigma \leq 1$. Then the cars are classified into four types.

$\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}$.

The results obtained clearly indicated the effectiveness of correlation of TSFSs as all the four cars are classified in four types which occur very rarely in cluster analysis. This type of problems was discussed in intuitionistic and picture fuzzy environments [75, 76] but the loss of information is more likely to occur in those cases.

4.3.2. Multi-Attribute Decision Making

MADM is one of the most discussed topics in fuzzy mathematics. Several approaches based on aggregation theory or distance/ similarity measures have been adapted so far in several extensions of FSs. Some valuable work on MADM process in various fuzzy environments could be found in [25-31, 64, 65, 73, 75, 76, 80-82, 86, 87]. In this subsection, the method of MADM is established in the environment of TSFSs based on the proposed CCs. To demonstrate the approach and to show its effectiveness, a numerical example is presented.

In MADM, the selection of best alternative has been carried out among a list of alternatives $A_j (j = 1,2,3, \dots m)$ under some attributes $C_k (k = 1,2,3, \dots m)$ using the CCs of TSFNs. Here, w denotes the weight vector showing the weight of each attribute. The detailed steps of algorithm for MADM are as follows:

Step 1: This step involves the formation of decision matrix i.e. the decision makers provide their information about alternatives under attributes in the form of TSFNs.

Step 2: The attributes under consideration are not always of benefit type. To make all the attributes of benefit type, the decision matrix provided in Step 1 is normalized. In normalization, the cost type attributes are converted into benefit type by taking their complement (see Definition 2.2.1).

Step 3: This step involves the computation of CC of all TSFNs provided in the decision matrix with $B = (1, 0, 0)$ which is considered as an ideal value for an alternative. The alternatives whose degree of CC is largest will be considered as best.

Step 4: In this step, the CCs are ranked in order to get the best alternative.

To demonstrate the steps of proposed algorithm, an example is solved using new CCs of TSFSs. Here, we also declared that the CCs of IFSs and PFSs developed in [75] and [76] are unable to solve this type of problem while the proposed new CCs can solve these problems conveniently with no information loss.

4.3.2.1. Example

Islamabad is the capital of Pakistan and is considered as one of the most beautiful cities. There are several parks and picnic points in Islamabad which a number of people visit on daily basis. The administration of Islamabad is governed by Metropolitan Corporation of Islamabad (MCI). The MCI decided to renovate all the parks and picnic points to maintain their beauty. To do so, MCI needs some private contractors to get hired. After some initial screening MCI selected 4 private firms for further selection. The four firms include A_1: *Arish Associates*, A_2: *Nauman Estate and Builders*, A_3: *Areva Engineering, Construction and Interiors* and A_4: *The Wow Architects*. The experts of MCI set 5 attributes for the selection of best firm/company. These 5 attributes include C_1: *Cost*, C_2: *Previous* *performance*, C_3: *Time constraint*, C_4: *Quality Assurance* and C_5: *Quantity of labor*. Further the weight vector of attributes is given as $w = (0.3, 0.15, 0.25, 0.2, 0.1)^T$. The decision-making panel of MCI has provided their information in the form of TSFNs provided in Table 11. The detailed steps of decision-making process are as follows:

Step 1. Opinion of decision makers in the form of TSFNs is provided in Table 11.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.1,0.7,0.4)	(0.5,0.8,0.9)	(0.8,0.8,0.8)	(0.6,0.7,0.8)	(0.3,0.5,0.7)
A_2	(0.2,0.7,0.6)	(0.6,0.7,0.8)	(0.3,0.7,0.7)	(0.1,0.7,0.9)	(0.4,0.6,0.8)
A_3	(0.5,0.6,0.6)	(0.5,0.6,0.7)	(0.5,0.7,0.1)	(0.9,0.6,0.2)	(0.5,0.6,0.9)
A_4	((0.5,0.6,0.8))	(0.8,0.7,0.4)	(0.8,0.7,0.3)	(0.6,0.6,0.1)	(0.8,0.4,0.4)

Table 11 (Decision Matrix)

Note that all the values in Table 11 are purely TSFNs for $n = 5$ which means that neither the operators of IFSs, PyFSs, q-ROFSs and nor of PFSs can deal with such type of information.

Step 2. The criteria C_1 is of cost type. Therefore, the values under C_1 are converted into benefit type by taking complement of TSFNs. The normalized decision matrix is obtained in Table 12.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.4,0.7,0.1)	(0.5,0.8,0.9)	(0.8,0.8,0.8)	(0.6,0.7,0.8)	(0.3,0.5,0.7)
A_2	(0.6,0.7,0.2)	(0.6,0.7,0.8)	(0.3,0.7,0.7)	(0.1,0.7,0.9)	(0.4,0.6,0.8)
A_3	(0.6,0.6,0.5)	(0.5,0.6,0.7)	(0.5,0.7,0.1)	(0.9,0.6,0.2)	(0.5,0.6,0.9)
A_4	((0.8,0.6,0.5))	(0.8,0.7,0.4)	(0.8,0.7,0.3)	(0.6,0.6,0.1)	(0.8,0.4,0.4)

Table 12: (Normalized decision matrix of the alternatives)

Step 3. The CC of each TSFN provided in Table 12 is computed with $(1,0,0)$ using Definition 4.2.7 and are listed in Table 13.

	$\mathcal{W}\mathcal{K}^3_{TSFS}(A_1, B)$	$\mathcal{W}\mathcal{K}^3_{TSFS}(A_2, B)$	$\mathcal{W}\mathcal{K}^3_{TSFS}(A_3, B)$	$\mathcal{W}\mathcal{K}^3_{TSFS}(A_4, B)$
Values	0.1982	0.0883	0.3927	0.8310

Table 13: Correlation degrees

Now the correlation of each TSFN given in Table 12 is computed using Definition 4.2.8 and is provided in Table 14.

	$\mathcal{W}\mathcal{K}^4_{TSFS}(A_1, B)$	$\mathcal{W}\mathcal{K}^4_{TSFS}(A_1, B)$	$\mathcal{W}\mathcal{K}^4_{TSFS}(A_1, B)$	$\mathcal{W}\mathcal{K}^4_{TSFS}(A_1, B)$
Values	0.0703	0.0282	0.1208	0.2525

Table 14: Correlation degrees

Step 4: Both, analysis of Table 13 and Table 14 leads us to the same ranking which is given as:

$$A_2 < A_1 < A_3 < A_4$$

Therefore, A_4 is the best choice i.e. the *Wow Architect* is the best according to the evaluation of decision makers based on CCs of TSFNs. Further note that due to complex structure of TSFNs, the CCs of IFSs and PFSs cannot be applied to this type of problem.

4.4. Comparative Study and Advantages

In this section, we discussed the generalizations of the proposed CCs over the existing CCs. The following two remarks show the generalization of new CCs of TSFSs over the CCs of IFSs and PFSs.

4.4.1. Remark

If we replace the value of n by 1. Then the Definition 4.2.7 and Definition 4.2.8 reduced to CCs of PFSs proposed in [76].

4.4.2. Remark

If we replace the value of n by 1 and set $i_1 = i_2 = 0$. Then, the Definition 4.2.7 and Definition 4.2.8 reduced to CCs of IFSs proposed in [75].

Now we show that the problems solved in [76] and [92] can be solved using the CCs of TSFSs. However, the CCs of IFSs [75] and PFSs [76] are unable to handle the data provided in the environment of TSFSs as discussed in Example 4.3.1.7 and 4.3.2.1.

4.4.3. Example

In this example, we take the data about cars from [76] and apply the CCs of TSFSs on it. The information about cars by the experts in [76] is given in the below Table 15. The calculations are demonstrated stepwise below. Moreover, because a PFS is a special case of TSFS for $n = 1$. So, in solving this example, we take the value of $n = 1$.

	c_1	c_2	c_3
A_1	(0.3, 0.2, 0.1)	(0.8, 0.1, 0.1)	(0.1, 0.0, 0.9)
A_2	(0.4, 0.1, 0.5)	(0.0, 0.7, 0.1)	(0.3, 0.2, 0.4)
A_3	(0.2, 0.6, 0.1)	(0.9, 0.1, 0.0)	(0.7, 0.1, 0.2)
A_4	(0.7, 0.3, 0.0)	(0.5, 0.1, 0.3)	(0.6, 0.0, 0.3)

Table 15 ((Decision matrix based on car data set))

We applied the algorithm proposed in Section 4.3.1 for clustering and stepwise computations are discussed below.

Step 1: Using Definition 4.2.7 on the data provided in Table 15, the correlation matrix is constructed after computing CCs as follows:

$$D = \begin{bmatrix} 1 & .4787 & .8176 & .7453 \\ .4787 & 1 & .4046 & .5580 \\ .8176 & .4646 & 1 & .8042 \\ .7453 & .5580 & .8042 & 1 \end{bmatrix}$$

Step 2: Construction of equivalent correlation matrix.

$$D^2 = D \circ D = \begin{bmatrix} 1 & .5580 & .8176 & .8042 \\ .5580 & 1 & .5580 & .5580 \\ .8176 & .5580 & 1 & .8042 \\ .8042 & .5580 & .8042 & 1 \end{bmatrix}$$

$$D^4 = D^2 \circ D^2 = \begin{bmatrix} 1 & .5580 & .8176 & .8042 \\ .5580 & 1 & .5580 & .5580 \\ .8176 & .5580 & 1 & .8042 \\ .8042 & .5580 & .8042 & 1 \end{bmatrix}$$

and

$$D^8 = D^4 \circ D^4 = \begin{bmatrix} 1 & .5580 & .8176 & .8042 \\ .5580 & 1 & .5580 & .5580 \\ .8176 & .5580 & 1 & .8042 \\ .8042 & .5580 & .8042 & 1 \end{bmatrix}$$

Obviously $D^8 = D^4$. Therefore, D^4 is the equivalent correlation matrix in this case.

Step 3: Using Definition 4.3.1.6, the $\sigma - cutting$ matrix is obtained based on which the following classifications are established.

1. If $0 \leq \sigma \leq 0.5580$. Then all A_1, A_2, A_3 and A_4 of the same type i.e. we have

   $\{A_1, A_2, A_3, A_4\}$.

2. If $0.5580 \leq \sigma \leq 0.8042$. Then the classifications are.

   $\{A_2\}, \{A_1, A_3, A_4\}$.

3. If $0.8042 \leq \sigma \leq 0.8176$. Then the cars are classified into three types.

   $\{A_1, A_3\}, \{A_2\}, \{A_4\}$.

4. If $0.8176 \leq \sigma \leq 1$. Then the cars are classified into four types.

   $\{A_1\}, \{A_2\}, \{A_3\}, \{A_4\}$.

The clustering results obtained here are the same as obtained in [76]. Hence our claim that CC of TSFSs generalizes CC of PFSs holds true. Similarly, by taking $i = 0$ and $n = 1$ the proposed CC can be applied to the problems discussed in the environment of IFS in [75]. These examples prominently show the significance of CCa of TSFSs. On the other hand, the CCs IFSs and PFSs cannot be applied to problems involving TSFNs studied in Example 4.3.1.7.

Now we prove the effectiveness of proposed CCs in MADM problem where a problem in the environment of PFSs is solved using proposed CCs. The problem discussed in [92] is solved here using proposed CCs of TSFSs for $n = 1$.

4.4.4. Example

This problem is taken from [92] where the selection of best policy for upcoming financial year has been discussed using SM of PFSs. We prove the usefulness of our proposed work by solving this problem using CCs of TSFSs. This problem is analogous to Example 4.3.2.1 except the weight vector is taken as $(0.2, 0.3, 0.1, 0.4)^t$ and the decision matrix is given in Table 16.

Step 1. Opinion of decision makers in the form of TSFNs is provided in Table 16.

	C_1	C_2	C_3	C_4
A_1	(0.2,0.1,0.6)	(0.5,0.3,0.1)	(0.5,0.1,0.3)	(0.4,0.3,0.2)
A_2	(0.1,0.4,0.4)	(0.6,0.3,0.1)	(0.5,0.2,0.2)	(0.2,0.1,0.7)
A_3	(0.3,0.2,0.2)	(0.6,0.2,0.1)	(0.4,0.1,0.3)	(0.3,0.3,0.4)
A_4	((0.3,0.1,0.6))	(0.1,0.2,0.6)	(0.1,0.3,0.5)	(0.2,0.3,0.2)

Table 16 (Decision matrix)

Step 2. To make all the attributes are of benefit type, the normalized decision matrix is obtained in Table 17.

	C_1	C_2	C_3	C_4
A_1	(0.6,0.1,0.2)	(0.5,0.3,0.1)	(0.5,0.1,0.3)	(0.4,0.3,0.2)
A_2	(0.4,0.4,0.1)	(0.6,0.3,0.1)	(0.5,0.2,0.2)	(0.2,0.1,0.7)
A_3	(0.2,0.2,0.3)	(0.6,0.2,0.1)	(0.4,0.1,0.3)	(0.3,0.3,0.4)
A_4	((0.6,0.1,0.3))	(0.1,0.2,0.6)	(0.1,0.3,0.5)	(0.2,0.3,0.2)

Table 17 (Normalized decision matrix)

Step 3. The correlation of each TSFN provided in Table 17 are computed with ideal value $B = (1, 0, 0)$ using Definition 4.2.7 for $n = 1$ and are listed in Table 18.

	$\mathcal{WK}^3_{TSFS}(A_1, B)$	$\mathcal{WK}^3_{TSFS}(A_2, B)$	$\mathcal{WK}^3_{TSFS}(A_3, B)$	$\mathcal{WK}^3_{TSFS}(A_4, B)$
Values	0.6621	0.8531	0.6757	0.3136

Table 18 (Correlation degrees)

Now the correlation of each TSFN given in Table 17 is computed using Definition 4.2.8 for $n = 1$ and is provided in Table 19.

	$\mathcal{WK}^4_{TSFS}(A_1, A)$	$\mathcal{WK}^4_{TSFS}(A_2, A)$	$\mathcal{WK}^4_{TSFS}(A_3, A)$	$\mathcal{WK}^4_{TSFS}(A_4, A)$
Values	0.0256	0.1008	0.0300	0.0196

Table 19 ((Correlation degrees))

Step 4: Analysis of Table 18 and Table 19 leads us to the following ranking.

$$A_4 < A_1 < A_3 < A_2$$

Therefore, A_2 is the best strategy i.e. the investment plan of *Russia* is the best according to the evaluation of decision makers based on CCs of TSFNs.

The advantages of proposed CCs is that these can be applied to problems in the environment of SFSs, PFSs, q-ROPFSs, PyFSs and IFSs. In Example 4.4.3, the defined CCs have been applied to a problem involving PFNs from [76] and obtained results are the same as in [76]. Similarly, the defined CCs are applied to MADM problem studied in [92]. On the other hand, neither the correlation of IFSs proposed in [75] nor the correlation of PFSs proposed in [76] could be applied to problems presented in Example 4.3.1.7 and 4.3.2.1. All this shows the effectiveness and superiority of proposed work over the previous literature.

Chapter 5

Averaging and Geometric Aggregation Operators of T-Spherical Fuzzy Sets [5]

The aim of this chapter is to develop some new set theoretic operations for TSFSs including algebraic sum, product etc. Based on new operations some averaging and geometric AOs including T-spherical fuzzy weighted averaging (TSFWA) operator and weighted geometric operator abbreviated as TSFWG, T-spherical fuzzy ordered weighted averaging (TSFOWA) operator and ordered weighted geometric operator abbreviated as TSFOWG, T-spherical fuzzy hybrid averaging (TSFHA) operator and hybrid geometric abbreviated as TSFHG operator are developed. The monotonicity, idempotency and boundedness of the defined operators are investigated, and their fitness is validated using induction method. Numerical examples are given to support the applicability of the new AOs. As an application, MADM process is briefly demonstrated in the environment of TSFSs and numerically explained with the help of new AOs. The new proposed work and the existing literature is compared numerically demonstrating the drawbacks of the existing work and the significance of the new operations. Some advantages of proposed work over existing work are also studied.

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5.1. Previous Study and its Drawbacks

In this section, we listed all the previous study on averaging and geometric AOs in several fuzzy frameworks. We also described the reasons for the failure of these operators in several circumstances. Note that throughout this chapter, $w = (w_1, w_2, w_{3,} \dots w_m)^t$ shall denote the weight vector of ş, i, d' and r where $j = 1, 2, 3, \dots m$ such that $w_j > 0$ and $\sum_{j=1}^{m} w_j = 1$.

The aggregation theory of IFSs was started by Xu [29] and Xu and Yager [30] and are defined as:

5.1.1. Definition

For some IFNs I_j, the IFWA and IFWG operators are of the following form

$$IFWA(I_1, I_2, I_3 \dots I_m) = \left(1 - \prod_{j=1}^{m} (1 - ş_j)^{w_j}, \prod_{j=1}^{m} (d'_j)^{w_j}\right)$$

$$IFWG(I_1, I_2, I_3 \dots I_m) = \left(\prod_{j=1}^{m} (ş_j)^{w_j}, 1 - \prod_{j=1}^{m} (1 - d'_j)^{w_j}\right)$$

Intuitionistic fuzzy WAA and WGA operators discussed the MG and NG of an uncertain event only but in a restricted range. Whenever we have the information of the form $(0.8, 0.5)$, the applicability of the above defined IFWA and IFWG operators became a challenge to aggregate such information.

Keeping the limitations of IFWA and IFWG operators in mind, Rahman et al. [44] and Peng and Yang [45] proposed WGA and WAA operators in Pythagorean fuzzy environment respectively which are given as:

5.1.2. Definition

For some PyFNs I_j, the PyFWA and PyFWG operators are of the following form

$$PyFWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt[2]{1 - \prod_{j=1}^{m} (1 - \text{ş}_j^2)^{w_j}}, \prod_{j=1}^{m} (\text{d}'_j)^{w_j} \right)$$

$$PyFWG(I_1, I_2, I_3 \dots I_m) = \left(\prod_{j=1}^{m} (\text{ş}_j)^{w_j}, \sqrt[2]{1 - \prod_{j=1}^{m} (1 - \text{d}'^2_j)^{w_j}} \right)$$

The operations proposed by Rahman et al. [44] and Peng and Yang [45] still had applicability issues which leads Liu and Wang [85] to develop WAA and WGA operators in q-rung orthopair fuzzy environment defined as:

5.1.3. Definition

For some q-ROFNs I_j, the q-ROFWA and q-ROFWG operators are of the following form

$$q - ROFWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} (1 - \text{ş}_j^n)^{w_j}}, \prod_{j=1}^{m} (\text{d}'_j)^{w_j} \right)$$

$$q - ROFWG(I_1, I_2, I_3 \dots I_m) = \left(\prod_{j=1}^{m} (\text{ş}_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - \text{d}'^n_j)^{w_j}} \right)$$

The main advantage of the q-ROFWA and q-ROFWG operators is that these operators can aggregate any kind of information without the barrier of any restricted range. However, as suggested by Cuong [9] that duplets from intuitionistic fuzzy, Pythagorean fuzzy and q-rung orthopair fuzzy environments only discussed the MG and NG of an uncertain event. In real life, human opinion has abstinence and refusal degree as well. Cuong's [9] concept of PFS provided a better platform for working with uncertainties and the WAA and WGA operators in picture fuzzy environment were discussed by three different researchers [50, 51, 83].

The WGA operators proposed Wang et al. [51] in picture fuzzy environment is defined as follows:

5.1.4. Definition [51]

For some PFNs I_j, the PFWG operator is of the form

$$PFWG(I_1, I_2, \dots I_m) = \left(\prod_{j=1}^{m} (\varsigma_j + i_j)^{w_j} - \prod_{j=1}^{m} i_j^{w_j}, \prod_{j=1}^{m} i_j^{w_j}, 1 - \prod_{j=1}^{m} (1 - d'_j)^{w_j} \right)$$

Garg [50] proposed the concept of WAA operators for PFSs defined as:

5.1.5. Definition [51]

For some PFNs I_j, the PFWA operator is of the form

$$PFWA(I_1, I_2, \dots I_m) = \left(1 - \prod_{j=1}^{m} \left(1 - (\varsigma_j)\right)^{w_j}, \prod_{j=1}^{m} (i_j)^{w_j}, \prod_{j=1}^{m} (d'_j)^{w_j} \right)$$

Wei [83] proposed both WAA and WGA operators for PFSs independently in the same year and are defined as:

5.1.6. Definition [51]

For some PFNs I_j, the PFWA and PFWG operators are of the form

$$PFWA(I_1, I_2, \dots I_m) = \left(1 - \prod_{j=1}^{m}\left(1 - (\c{s}_j)\right)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \prod_{j=1}^{m}(d'_j)^{w_j}\right)$$

$$PFWG(I_1, I_2, \dots I_m) = \left(\prod_{j=1}^{m}(\c{s}_j)^{w_j}, 1 - \prod_{j=1}^{m}\left(1 - (i_j)\right)^{w_j}, 1 - \prod_{j=1}^{m}\left(1 - (d'_j)\right)^{w_j}\right)$$

The WGA operator proposed in Definition 5.1.4, 5.1.5 and 5.1.6 cannot be applied to triplets for which $\c{s}um(\c{s}, i, d')$ exceeds 1 hence providing a lesser range for assigning values to MG, AG and NG. Therefore, in the next section, we propose some new operations which leads us to setup new WAA and WGA operators that can overcome the drawbacks of the previous study.

5.2. T-Spherical Fuzzy Operational Laws

The aim of this section is to develop some new operational laws in TSF environment. The new operational laws are the generalization of the operational laws of IFS, PyFS, q-ROFSs and PFSs studied in [29, 30, 44, 45, 85, 50, 51, 83]. The TSF operational laws along with some sub sequential results are proposed as follows:

5.2.1. Definition

Consider three TSFNs I, I_1 and I_2 for some $n \in \mathbb{Z}^+$ and let $\lambda > 0$. Then:

1. $I_1 \oplus I_2 = \left(\sqrt[n]{ş_1^n + ş_2^n - ş_1^n . ş_2^n}, i_1 . i_2, d'_1 . d'_2\right)$.

2. $I_1 \otimes I_2 = \left(ş_1 . ş_2, i_1 . i_2, \sqrt[n]{d'^n_1 + d'^n_2 - d'^n_1 . d'^n_2}\right)$

3. $\lambda . I = \left(\sqrt[n]{1 - (1 - ş^n)^\lambda}, (i)^\lambda, (d')^\lambda\right)$.

4. $I^\lambda = \left((ş)^\lambda, (i)^\lambda, \sqrt[n]{1 - (1 - d'^n)^\lambda}\right)$

5.2.2. Remark

Obviously $I_1 \oplus I_2$, $I_1 \otimes I_2$, $\lambda . I$ and I^λ are TSFNs.

5.2.3. Remark

The operations proposed in Definition 5.2.1 reduced to spherical fuzzy environment for $n = 2$ as follows:

1. $I_1 \oplus I_2 = \left(\sqrt{ş_1^2 + ş_2^2 - ş_1^2 . ş_2^2}, i_1 . i_2, d'_1 . d'_2\right)$

2. $I_1 \otimes I_2 = \left(ş_1 . ş_2, i_1 . i_2, \sqrt{d'^2_1 + d'^2_2 - d'^2_1 . d'^2_2}\right)$

3. $\lambda . I = \left(\sqrt{1 - (1 - ş^2)^\lambda}, (i)^\lambda, (d')^\lambda\right)$

4. $I^\lambda = \left((ş)^\lambda, (i)^\lambda, \sqrt{1 - (1 - d'^2)^\lambda}\right)$

5.2.4. Remark

- The operations proposed in Definition 5.2.1 reduced to picture fuzzy environment for $n = 1$.
- The operations proposed in Definition 5.2.1 reduced to q-rung orthopair fuzzy environment for $i = i_1 = i_2 = 0$.
- The operations proposed in Definition 5.2.1 reduced to Pythagorean fuzzy environment for $n = 2$ and $i = i_1 = i_2 = 0$.

- The operations proposed in Definition 5.2.1 reduced to intuitionistic fuzzy environment for $n = 1$ and $i = i_1 = i_2 = 0$.

The basic properties of the new operational laws of TSFSs are examined in the following theorem.

5.2.5. Theorem For three TSFNs I, I_1 and I_2 and for $\lambda, \lambda_1, \lambda_2 > 0$, the following holds:

1. $I_1 \oplus I_2 = I_2 \oplus I_1$
2. $I_1 \otimes I_2 = I_2 \otimes I_1$
3. $\lambda(I_1 \oplus I_2) = \lambda I_1 \oplus \lambda I_2$
4. $(I_1 \otimes I_2)^\lambda = {I_1}^\lambda \otimes {I_2}^\lambda$
5. $\lambda_1 I \oplus \lambda_2 I = (\lambda_1 + \lambda_2) I$
6. $I^{\lambda_1} \otimes I^{\lambda_2} = I^{\lambda_1 + \lambda_2}$
7. $(I^c)^\lambda = (\lambda I)^c$
8. $\lambda(I^c) = \left(I^\lambda\right)^c$
9. ${I_1}^c \oplus {I_2}^c = (I_1 \otimes I_2)^c$
10. ${I_1}^c \otimes {I_2}^c = (I_1 \oplus I_2)^c$

Proof: The proof for result 1, 3, 5, 7 and 9 are provided below. The remaining results could be proved analogously. Let $I_1 = (ş_1, i_1, d'_1)$ and $I_2 = (ş_2, i_2, d'_2)$ and $\lambda, \lambda_1, \lambda_2 > 0$. Then

1. $I_1 \oplus I_2 = \left(\sqrt[n]{ş_1^n + ş_2^n - ş_1^n . ş_2^n}, i_1 . i_2, d'_1 . d'_2\right)$

$$= \left(\sqrt[n]{ş_2^n + ş_1^n - ş_2^n . ş_1^n}, i_2 . i_1, d'_2 . d'_1\right)$$

$$= I_2 \oplus I_1$$

3. $\lambda(I_1 \oplus I_2) = \lambda\left(\sqrt[n]{ş_1^n + ş_2^n - ş_1^n.ş_2^n}, i_1.i_2, \text{d'}_1.\text{d'}_2\right)$

$$= \left(\sqrt[n]{1 - \left(1 - \left(\sqrt[n]{ş_1^n + ş_2^n - ş_1^n.ş_2^n}\right)^n\right)^\lambda}, (i_1.i_2)^\lambda, (\text{d'}_1.\text{d'}_2)^\lambda\right)$$

$$= \left(\sqrt[n]{1 - \left(1 - (ş_1^n + ş_2^n - ş_1^n.ş_2^n)\right)^\lambda}, (i_1)^\lambda(i_2)^\lambda, (\text{d'}_1)^\lambda(\text{d'}_2)^\lambda\right) \quad \dots (1')$$

$\lambda I_1 \oplus \lambda I_2 = \lambda(ş_1, i_1, \text{d'}_1) \oplus \lambda(ş_2, i_2, \text{d'}_2)$

$$= \left(\sqrt[n]{1 - (1 - ş_1^n)^\lambda}, i_1, \text{d'}_1\right) \oplus \left(\sqrt[n]{1 - (1 - ş_2^n)^\lambda}, i_2, \text{d'}_2\right)$$

$$= \left(\begin{array}{c}\sqrt[n]{\left(\sqrt[n]{1 - (1 - ş_1^n)^\lambda}\right)^n + \left(\sqrt[n]{1 - (1 - ş_2^n)^\lambda}\right)^n - \left(\sqrt[n]{1 - (1 - ş_1^n)^\lambda}\right)^n . \left(\sqrt[n]{1 - (1 - ş_2^n)^\lambda}\right)^n}, \\ (i_1)^\lambda(i_2)^\lambda, (\text{d'}_1)^\lambda(\text{d'}_2)^\lambda\end{array}\right)$$

$$= \left(\begin{array}{c}\sqrt[n]{(1 - (1 - ş_1^n)^\lambda) + (1 - (1 - ş_2^n)^\lambda) - (1 - (1 - ş_1^n)^\lambda).(1 - (1 - ş_2^n)^\lambda)}, \\ (i_1)^\lambda(i_2)^\lambda, (\text{d'}_1)^\lambda(\text{d'}_2)^\lambda\end{array}\right)$$

$$= \left(\sqrt[n]{1 - (1 - ş_1^n)^\lambda.(1 - ş_2^n)^\lambda}, (i_1)^\lambda(i_2)^\lambda, (\text{d'}_1)^\lambda(\text{d'}_2)^\lambda\right)$$

$$= \left(\sqrt[n]{1 - \left(1 - (ş_1^n + ş_2^n - ş_1^n.ş_2^n)\right)^\lambda}, (i_1)^\lambda(i_2)^\lambda, (\text{d'}_1)^\lambda(\text{d'}_2)^\lambda\right) \quad \dots (2')$$

11. Using (1′) and (2′), we have $\lambda(I_1 \oplus I_2) = \lambda I_1 \oplus \lambda I_2$

5. $\lambda_1 I \oplus \lambda_2 I$

$$= \left(\sqrt[n]{1 - (1 - ş_1^n)^{\lambda_1}}, (i_1)^{\lambda_1}, (\text{d'}_1)^{\lambda_1}\right) \oplus \left(\sqrt[n]{1 - (1 - ş_1^n)^{\lambda_2}}, (i_1)^{\lambda_2}, (\text{d'}_1)^{\lambda_2}\right)$$

$$= \left(\sqrt[n]{\left(\sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda_1}}\right)^n + \left(\sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda_2}}\right)^n - \left(\sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda_1}}\right)^n . \left(\sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda_2}}\right)^n}, \atop (i_1)^{\lambda_1}(i_1)^{\lambda_2}, (d'_1)^{\lambda_1}(d'_1)^{\lambda_2} \right)$$

Proceeding as we did in Proof of 3, we have,

$$= \left(\sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda}.(1-\text{ş}_1^n)^{\lambda}},\ (i_1)^{\lambda_1}(i_1)^{\lambda_2}, (d'_1)^{\lambda_1}(d'_1)^{\lambda_2}\right)$$

$$= \left(\sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda_1}.(1-\text{ş}_1^n)^{\lambda_2}}, (i_1)^{\lambda_1}(i_1)^{\lambda_2}, (d'_1)^{\lambda_1}(d'_1)^{\lambda_2}\right)$$

$$= \left(\sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda_1+\lambda_2}}, (i_1)^{\lambda_1+\lambda_2}, (d'_1)^{\lambda_1+\lambda_2}\right)$$

$$= (\lambda_1 + \lambda_2)I$$

7. $(I^c)^{\lambda} = ((\text{ş}_1, i_1, d'_1)^c)^{\lambda} = (d'_1, i_1, \text{ş}_1)^{\lambda}$

$$= \left((d'_1)^{\lambda}, (i_1)^{\lambda}, \sqrt[n]{1-(1-\text{ş}_1^n)^{\lambda}}\right)$$

$$= (\lambda I)^c$$

9. $I_1^c \oplus I_2^c = (d'_1, i_1, \text{ş}_1) \oplus (d'_2, i_2, \text{ş}_2)$

$$= \left(\sqrt[n]{d'^n_1 + d'^n_2 - d'^n_1 . d'^n_2}, i_1 . i_2, \text{ş}_1 . \text{ş}_2\right) \quad ...(3')$$

$$(I_1 \otimes I_2)^c = \left(\text{ş}_1 . \text{ş}_2, i_1 . i_2, \sqrt[n]{d'^n_1 + d'^n_2 - d'^n_1 . d'^n_2}\right)^c$$

$$= \left(\sqrt[n]{d'^n_1 + d'^n_2 - d'^n_1 . d'^n_2}, i_1 . i_2, \text{ş}_1 . \text{ş}_2\right) \quad ...(4')$$

From (3′) and (4′), we have $I_1^c \oplus I_2^c = (I_1 \otimes I_2)^c$

5.3. T-Spherical Fuzzy Averaging Aggregation Operators

In this section, the WAA operators for TSFSs are developed including TSFWA operator, TSFOWA operator and TSFHA operator. We investigated the basic properties of these operators like monotonicity, idempotency and boundedness etc. The fitness of these AOs is validated using mathematical induction.

5.3.1. Definition

For some TSFNs I_j, the TSFWA operator is a mapping defined as:

$$TSFWA(I_1, I_2, I_3 \dots I_m) = \sum_{j=1}^{m} w_j I_j$$

5.3.2. Theorem

The aggregated value of some TSFNs I_j using TSFWA operator is a TSFN and is given by:

$$TSFWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} (1 - \varsigma_j^n)^{w_j}}, \prod_{j=1}^{m} (i_j)^{w_j}, \prod_{j=1}^{m} (d'_j)^{w_j} \right)$$

Proof: Using mathematical induction,

For $m = 2$

$w_1 I_1 = \left(\sqrt[n]{1 - (1 - \varsigma_1^n)^{w_1}}, (i_1)^{w_1}, (d'_1)^{w_1}\right)$ and

$w_2 I_2 = \left(\sqrt[n]{1 - (1 - \varsigma_2^n)^{w_2}}, (i_2)^{w_2}, (d'_2)^{w_2}\right)$

$w_1 I_1 \oplus w_2 I_2 = \left(\sqrt[n]{1 - (1 - \varsigma_1^n)^{w_1}}, (i_1)^{w_1}, (d'_1)^{w_1}\right) \oplus \left(\sqrt[n]{1 - (1 - \varsigma_2^n)^{w_2}}, (i_2)^{w_2}, (d'_2)^{w_2}\right)$

$$= \left(\sqrt[n]{\left(\sqrt[n]{1-(1-\S_1^n)^{w_1}}\right)^n + \left(\sqrt[n]{1-(1-\S_2^n)^{w_2}}\right)^n - \sqrt[n]{1-(1-\S_1^n)^{w_1}}.\sqrt[n]{1-(1-\S_2^n)^{w_2}}}, \atop (i_1)^{w_1}.(i_2)^{w_2}, (d'_1)^{w_1}.(d'_2)^{w_2} \right)$$

$$= \left(\sqrt[n]{1-(1-\S_1^n)^{w_1}.(1-\S_2^n)^{w_2}}, (i_1)^{w_1}.(i_2)^{w_2}, (d'_1)^{w_1}.(d'_2)^{w_2}\right)$$

$$= \left(\sqrt[n]{1-(1-\S_1^n)^{w_1}.(1-\S_2^n)^{w_2}}, (i_1)^{w_1}.(i_2)^{w_2}, (d'_1)^{w_1}.(d'_2)^{w_2}\right)$$

$$= \left(\sqrt[n]{1-\prod_{j=1}^{2}(1-\S_j^n)^{w_j}}, \prod_{j=1}^{2}(i_j)^{w_j}, \prod_{j=1}^{2}(d'_j)^{w_j} \right)$$

Assume that result is true for $m = k$ i.e.

$$TSFWA(I_1, I_2, I_3 \dots I_k) = \left(\sqrt[n]{1-\prod_{j=1}^{k}(1-\S_j^n)^{w_j}}, \prod_{j=1}^{k}(i_j)^{w_j}, \prod_{j=1}^{k}(d'_j)^{w_j} \right)$$

To prove this result for $m = k + 1$. Consider

$$TSFWA(I_1, I_2, I_3 \dots I_k, I_{k+1}) = \sum_{j=1}^{k+1} w_j I_j = \sum_{j=1}^{k} w_j I_j \oplus w_{k+1} I_{k+1}$$

$$= \left(\sqrt[n]{1-\prod_{j=1}^{k}(1-\S_j^n)^{w_j}}, \prod_{j=1}^{k}(i_j)^{w_j}, \prod_{j=1}^{k}(d'_j)^{w_j} \right) \oplus \left(\left(\sqrt[n]{1-(1-\S_2^n)^{w_{k+1}}}\right), (i_2)^{w_{k+1}}, (d'_2)^{w_{k+1}} \right)$$

Proceeding like we did in **Step 1**.

$$TSFWA(I_1, I_2, I_3 \dots I_k, I_{k+1}) = \left(\sqrt[n]{1-\prod_{j=1}^{k+1}(1-\S_j^n)^{w_j}}, \prod_{j=1}^{k+1}(i_j)^{w_j}, \prod_{j=1}^{k+1}(d'_j)^{w_j} \right)$$

Hence the result holds for $m = k + 1$.

The following Theorem is based on the basic properties of aggregation.

5.3.3. Theorem

1. (Idempotency)

   If for all $j = 1, 2, 3, \dots m$, $I_j = I = (\text{ş}, i, \text{d'})$. Then $TSFWA(I_1, I_2, I_3 \dots I_m) = I$

2. (Boundedness)

   If $I^- = \left(\min_j \text{ş}_j, \max_j i_j, \max_j \text{d'}_j\right)$

   and $I^+ = \left(\max_j \text{ş}_j, \min_j i_j, \min_j \text{d'}_j\right)$. Then

$$I^- \leq TSFWA(I_1, I_2, I_3 \dots I_m) \leq I^+$$

3. (Monotonicity)

   Let $I_j = \left(\text{ş}_{I_j}, i_{I_j}, \text{d'}_{I_j}\right)$ and $P_j = \left(\text{ş}_{P_j}, i_{P_j}, \text{d'}_{P_j}\right)$ be two TSFNs such that $I_j \leq P_j \; \forall j$.

   Then

$$TSFWA(I_1, I_2, I_3 \dots I_m) \leq TSFWA(P_1, P_2, P_3 \dots P_m)$$

4. (Shift Invariance)

   For another TSFN $P = (\text{ş}_P, i_P, \text{d'}_P)$

$$TSFWA(I_1 + P, I_2 + P, I_3 + P, \dots I_m + P) \leq TSFWA(I_1, I_2, I_3 \dots I_m) \oplus P$$

5. (Homogeneity)

   For $\lambda > 0$, $TSFWA(\lambda I_1, \lambda I_2, \lambda I_3 \dots \lambda I_m) = \lambda . TSFWA(I_1, I_2, I_3 \dots I_m)$

The following example illustrates the applicability of TSFWA operators.

5.3.4. Example

Consider for $n = 2$, we have three TSFNs $I_1 = (0.67, 0.34, 0.58)$, $I_2 = (0.43, 0.59, 0.31)$ and $I_3 = (0.78, 0.63, 0.48)$ and $w = (0.5, 0.3, 0.2)^t$ be the weight vector. Then

$TSFWA(I_1, I_2, I_3)$

$$= \begin{pmatrix} \sqrt{1-((1-0.67^2)^{0.5})((1-0.43^2)^{0.3})((1-0.78^2)^{0.2})}, \\ (0.34^2)^{0.5}(0.59^2)^{0.3}(0.63^2)^{0.2}, (0.58^2)^{0.5}(0.31^2)^{0.3}(0.2^2)^{0.2} \end{pmatrix}$$

$TSFWA(I_1, I_2, I_3) = (0.64898, 0.453799, 0.374583)$

In TSFWA operators, the weight is applied to TSFNs. Sometimes in MADM problems, when we need to weight the ordered position of the TSFN, then we need to develop TSFOWA operator and when we need to weight the TSFNs as well as its ordered position we use the concept of TSFHA operator. Therefore, we developed TSFOWA operators and TSFHA operators and explained them with the help of numerical examples.

5.3.5. Definition

For some TSFNs I_j, the TSFOWA operator is a mapping defined as:

$$TSFOWA(I_1, I_2, I_3 \dots I_m) = \sum_{j=1}^{m} w_j I_{\sigma(j)}$$

Where $I_{\sigma(j)}$ is the j^{th} largest of TSFNs I_j.

5.3.6. Theorem

The aggregated value of some TSFNs I_j using TSFOWA operator is a TSFN and is given by:

$$TSFOWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \check{s}^{n}_{\sigma(j)}\right)^{w_j}}, \prod_{j=1}^{m} \left(i_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{m} \left(d'_{\sigma(j)}\right)^{w_j} \right)$$

Proof: Similarly

5.3.7. Remark

Some Special Cases of TSFOWA operator are as follows:

1. For $w = (1, 0, 0, \dots 0)^t$, $TSFOWA(I_1, I_2, I_3 \dots I_m) = max\{I_1, I_2, I_3 \dots I_m\}$
2. For $w = (0, 0, 0, \dots 1)^t$, $TSFOWA(I_1, I_2, I_3 \dots I_m) = min\{I_1, I_2, I_3 \dots I_m\}$
3. For $w_j = 1$ or 0, $TSFOWA(I_1, I_2, I_3 \dots I_m) = I_{\sigma(j)}$ where $I_{\sigma(j)}$ is the j^{th} largest of TSFNs I_j.

5.3.8. Example

We solve Example 5.3.4 using TSFOWA operators. To do so, first we need to calculate the score values of TSFNs to arrange them in ordered positions and then simply aggregate them using TSFOWA operator. The score values are: $\check{S}(I_1) = 0.3045$, $\check{S}(I_2) = 0.0888$, $\check{S}(I_3) = 0.37$. Based on score values, the new ordered position of TSFNSs are:

$I_{\sigma(1)} = I_3 = (0.78, 0.63, 0.48)$

$I_{\sigma(2)} = I_1 = (0.67, 0.34, 0.58)$

$I_{\sigma(3)} = I_2 = (0.43, 0.59, 0.31)$

$TSFOWA(I_3, I_1, I_2) = (0.705422, 0.516755, 0.410042)$

5.3.9. Definition

For some TSFNs I_j, the TSFHA operator is a mapping defined as:

$$TSFHA(I_1, I_2, I_3 \dots I_m) = \sum_{j=1}^{m} w_j \dot{I}_{\sigma(j)}$$

Where $\dot{I}_{\sigma(j)}$ is the j^{th} largest of TSFNs $\dot{I}_j$ and $\dot{I}_j = m\omega_j I_j$ such that m is the number of TSFNs and $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^I$ is the weight vector of I_j.

While applying TSFHA operator, we first determine $\dot{I}_j = m\omega_j I_j$ using the weight vector $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^I$. Then the weighted TSFNs $\dot{I}_j$ are rearranged where $\dot{I}_{\sigma(j)}$ is the j^{th} largest of TSFNs $\dot{I}_j$. Finally, the TSFHA operator is used to aggregate the TSFNs $\dot{I}_j$. Using the basic operations of TSFNs, the following theorem is proposed.

5.3.10. Theorem

The aggregated value of some TSFNs I_j using TSFHA operator is a TSFN and is given by:

$$TSFHA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \dot{\$}_{\sigma(j)}^{n}\right)^{w_j}}, \prod_{j=1}^{m} \left(\dot{i}_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{m} \left(\dot{d}_{\sigma(j)}\right)^{w_j} \right)$$

Proof: Similar

5.3.11. Example

Consider for $n = 2$, we have three TSFNs $I_1 = (0.67, 0.34, 0.58)$, $I_2 = (0.43, 0.59, 0.31)$ and $I_3 = (0.78, 0.63, 0.48)$ and $\omega = (0.4, 0.35, 0.25)^t$ be the weight vector of given TSFNs while $w = (0.5, 0.3, 0.2)^t$ is the aggregated associated weighted vector. Then

$$\dot{I}_1 = 3\omega_1 I_1 = 3 \times 0.4\, I_1 = \left(\sqrt[2]{1-(1-0.67^2)^{3\times 0.4}}, (0.34^2)^{3\times 0.4}, (0.58^2)^{3\times 0.4}\right)$$

$= (0.71, 0.27, 0.52)$

Similarly, $\dot{I}_2 = (0.44, 0.57, 0.29)$ and $\dot{I}_3 = (0.71, 0.71, 0.58)$. Now we use score function to compute the ordered position of $\dot{I}_j$ as follows:

$\check{S}(\dot{I}_1) = 0.2337$, $\check{S}(\dot{I}_2) = 0.1095$, $\check{S}(\dot{I}_3) = 0.1677$. Based on score values, the new ordered position of TSFNSs are:

$\dot{I}_{\sigma(1)} = \dot{I}_1 = (0.7, 0.08, 0.27)$

$\dot{I}_{\sigma(2)} = \dot{I}_3 = (0.7, 0.5, 0.33)$

$\dot{I}_{\sigma(3)} = \dot{I}_2 = (0.44, 0.33, 0.08)$

$TSFHA(\dot{I}_1, \dot{I}_3, \dot{I}_2) = (0.65286, 0.409917, 0.446074)$

5.3.12. Theorem

If we assume the weight vector w as $\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$. Then the TSFHA operator reduced to TSFWA operator.

Proof: As $\dot{I}_j = m\omega_j I_j$ and $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$. So $w_j \dot{I}_j = \omega_j I_j$ and

$$TSFHA(I_1, I_2, I_3 \dots I_m) = \sum_{j=1}^{m} w_j \dot{I}_{\sigma(j)} = \sum_{j=1}^{m} \omega_j I_j = TSFWA(I_1, I_2, I_3 \dots I_m)$$

5.3.13. Theorem

If we assume the weight vector ω as $\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^{I}$. Then the TSFHA operator reduced to TSFOWA operator.

Proof: Straightforward.

5.4. T-Spherical Fuzzy Geometric Aggregation Operators

In this section, the WGA operators for TSFSs are developed including TSFWG operator, TSFOWG operator and TSFHG operator. We investigated the basic properties of these operators like monotonicity, idempotency and boundedness etc. The fitness of these AOs is validated using mathematical induction.

5.4.1. Definition

For some TSFNs I_j, the TSFWG operator is a mapping defined as:

$$TSFWG(I_1, I_2, I_3 \dots I_m) = \prod_{j=1}^{m} I_j^{w_j}$$

5.4.2. Theorem

The aggregated value of some TSFNs I_j using TSFWG operator is a TSFN and is given by:

$$TSFWG(I_1, I_2, I_3 \ldots I_m) = \left(\prod_{j=1}^{m} (\varsigma_j)^{w_j}, \prod_{j=1}^{m} (i_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - d_j'^{n})^{w_j}} \right)$$

Proof: Using mathematical induction:

For $m = 2$

$w_1 I_1 = \left((\varsigma_1)^{w_1}, (i_1)^{w_1}, \sqrt[n]{1 - (1 - d_1'^{n})^{w_1}} \right)$ and

$w_2 I_2 = \left((\varsigma_2)^{w_2}, (i_2)^{w_2}, \sqrt[n]{1 - (1 - d_2'^{n})^{w_2}} \right)$

$w_1 I_1 \otimes w_2 I_2$

$$= \left((\varsigma_1)^{w_1}, (i_1)^{w_1}, \sqrt[n]{1 - (1 - d_1'^{n})^{w_1}} \right) \otimes \left((\varsigma_2)^{w_2}, (i_2)^{w_2}, \sqrt[n]{1 - (1 - d_2'^{n})^{w_2}} \right)$$

$$= \left(\begin{array}{c} (\varsigma_1)^{w_1}.(\varsigma_2)^{w_2}, (i_1)^{w_1}.(i_2)^{w_2}, \\ \sqrt[n]{\left(\sqrt[n]{1 - (1 - d_1'^{n})^{w_1}}\right)^n + \left(\sqrt[n]{1 - (1 - d_2'^{n})^{w_2}}\right)^n - \sqrt[n]{1 - (1 - d_1'^{n})^{w_1}}.\sqrt[n]{1 - (1 - d_2'^{n})^{w_2}}} \end{array} \right)$$

$$= \left((\varsigma_1)^{w_1}.(\varsigma_2)^{w_2}, (i_1)^{w_1}.(i_2)^{w_2}, \sqrt[n]{1 - (1 - d_1'^{n})^{w_1}.(1 - d_2'^{n})^{w_2}} \right)$$

$$= \left(\prod_{j=1}^{2} (\varsigma_j)^{w_j}, \prod_{j=1}^{2} (i_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{2} (1 - d_j'^{n})^{w_j}} \right)$$

True for $m = 2$. Assume that result holds for $m = k$ i.e.

$$TSFWG(I_1, I_2, I_3 \ldots I_k) = \left(\prod_{j=1}^{k} (\varsigma_j)^{w_j}, \prod_{j=1}^{k} (i_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{k} (1 - d_j'^{n})^{w_j}} \right)$$

To prove for $m = k + 1$. Consider

$$TSFWG(I_1, I_2, I_3 \dots I_k, I_{k+1}) = \prod_{j=1}^{k+1} I_j^{w_j} = \prod_{j=1}^{k} I_j^{w_j} \otimes I_{k+1}^{w_{k+1}}$$

$$= \left(\prod_{j=1}^{k} (\varsigma_j)^{w_j}, \prod_{j=1}^{k} (i_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{k} \left(1 - \text{d'}_j^{n}\right)^{w_j}} \right) \oplus \left((\varsigma_2)^{w_{k+1}}, (i_2)^{w_{k+1}}, \left(\sqrt[n]{1 - (1 - \text{d'}_2^{n})^{w_{k+1}}} \right) \right)$$

Finally,

$$TSFWG(I_1, I_2, I_3 \dots I_k, I_{k+1}) = \left(\prod_{j=1}^{k+1} (\varsigma_j)^{w_j}, \prod_{j=1}^{k+1} (i_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{k+1} \left(1 - \text{d'}_j^{n}\right)^{w_j}} \right)$$

Hence the result holds for $m = k + 1$.

The following Theorem is based on the basic properties of aggregation.

5.4.3. Theorem

1. (Idempotency)

If for all $j = 1, 2, 3, \dots m$, $I_j = I = (\varsigma, i, \text{d'})$. Then $TSFWG(I_1, I_2, I_3 \dots I_m) = I$

2. (Boundedness)

If $I^- = \left(\min_j \varsigma_j, \max_j i_j, \max_j \text{d'}_j \right)$ and $I^+ = \left(\max_j \varsigma_j, \min_j i_j, \min_j \text{d'}_j \right)$. Then

$$I^- \leq TSFWG(I_1, I_2, I_3 \dots I_m) \leq I^+$$

3. (Monotonicity)

Let $I_j = \left(\varsigma_{I_j}, i_{I_j}, \text{d'}_{I_j} \right)$ and $P_j = \left(\varsigma_{P_j}, i_{P_j}, \text{d'}_{P_j} \right)$ be two TSFNs such that $I_j \leq P_j \ \forall\, j$.

Then

$$TSFWG(I_1, I_2, I_3 \dots I_m) \leq TSFWG(P_1, P_2, P_3 \dots P_m)$$

5.4.4. Example

Consider for $n = 2$, we have three TSFNs $I_1 = (0.67, 0.34, 0.58)$, $I_2 = (0.43, 0.59, 0.31)$ and $I_3 = (0.78, 0.63, 0.48)$ and $w = (0.5, 0.3, 0.2)^t$ be the weight vector. Then

$TSFWG(I_1, I_2, I_3)$

$$= \begin{pmatrix} (0.58^2)^{0.5}(0.43^2)^{0.3}(0.78^2)^{0.2}, (0.34^2)^{0.5}(0.59^2)^{0.3}(0.63^2)^{0.2}, \\ \sqrt[2]{1 - ((1 - 0.58^2)^{0.5})((1 - 0.31^2)^{0.3})((1 - 0.48^2)^{0.2})} \end{pmatrix}$$

$$TSFWG(I_1, I_2, I_3) = (0.60464, 0.453799, 0.385256)$$

In TSFWG operators, the weight is applied to TSFNs. Sometimes in MADM problems, when we need to weight the ordered position of the TSFN, then we need to develop TSFOWG operator and when we need to weight the TSFNs as well as its ordered position we use the concept of TSFHG operator. Therefore, we developed TSFOWG operators and TSFHG operators and explained them with the help of numerical examples.

5.4.5. Definition

For some TSFNs I_j, the TSFOWG operator is a mapping defined as:

$$TSFOWG(I_1, I_2, I_3 \dots I_m) = \prod_{j=1}^{m} I_{\sigma(j)}^{w_j}$$

Where $I_{\sigma(j)}$ is the j^{th} largest of TSFNs I_j.

5.4.6. Theorem

The aggregated value of some TSFNs I_j using TSFOWG operator is a TSFN and is given by:

$$TSFOWG(I_1, I_2, I_3 \dots I_m) = \left(\prod_{j=1}^{m} (\text{ş}_{\sigma(j)})^{w_j}, \prod_{j=1}^{m} (i_{\sigma(j)})^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - \text{d'}^{n}_{\sigma(j)})^{w_j}} \right)$$

Proof: Similar.

5.4.7. Example

We solve Example 5.4.4 using TSFOWG operators. To do that, first we need to calculate the score values of TSFNs to arrange them in ordered positions and then simply aggregate them using TSFOWG operator. The score values are: $\check{S}(I_1) = 0.3045$, $\check{S}(I_2) = 0.0888$, $\check{S}(I_3) = 0.37$. Based on score values, the new ordered position of TSFNSs are:

$I_{\sigma(1)} = I_3 = (0.78, 0.63, 0.48)$

$I_{\sigma(2)} = I_1 = (0.67, 0.34, 0.58)$

$I_{\sigma(3)} = I_2 = (0.43, 0.59, 0.31)$

$TSFOWG(I_3, I_1, I_2) = (0.856729, 0.516755, 0.322408)$

5.4.8. Definition

For some TSFNs I_j, the TSFHG operator is a mapping defined as:

$$ISFHG(I_1, I_2, I_3 \dots I_m) = \prod_{j=1}^{m} \dot{I}^{w_j}_{\sigma(j)}$$

Where $\dot{I}_{\sigma(j)}$ is the j^{th} largest of TSFNs $\dot{I}_j$ and $\dot{I}_j = I_j^{m\omega_j}$ such that m is the number of TSFNs and $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^I$ is the weight vector of I_j.

While applying TSFHG operator, we first determine $\dot{I}_j = I_j^{m\omega_j}$ using the weight vector $\omega = (\omega_1, \omega_2, \omega_3 \dots \omega_n)^I$. Then the weighted TSFNs $\dot{I}_j$ are rearranged where $\dot{I}_{\sigma(j)}$ is the j^{th} largest of TSFNs $\dot{I}_j$. Finally, the TSFHG operator is used to aggregate the TSFNs $\dot{I}_j$. Using the basic operations of TSFNs, the following theorem is proposed.

5.4.9. Theorem

The aggregated value of some TSFNs I_j using TSFHG operator is a TSFN and is given by:

$$TSFHG(I_1, I_2, I_3 \dots I_m) = \left(\prod_{j=1}^{m} \left(\dot{s}_{\sigma(j)}\right)^{w_j}, \prod_{j=1}^{m} \left(\dot{i}_{\sigma(j)}\right)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \dot{d}_{\sigma(j)}^{n}\right)^{w_j}} \right)$$

Proof: Similar.

5.4.10. Example

Consider for $n = 2$, we have three TSFNs $I_1 = (0.67, 0.34, 0.58)$, $I_2 = (0.43, 0.59, 0.31)$ and $I_3 = (0.78, 0.63, 0.48)$ and $\omega = (0.4, 0.35, 0.25)^t$ be the weight vector of given TSFNs while $w = (0.5, 0.3, 0.2)^t$ is the aggregated associated weighted vector. Then

$\dot{I}_1 = I_1^{3\omega_1} = I_1^{3\times 0.4}$

$= \left((0.67^2)^{3\times0.4}, (0.34^2)^{3\times0.4}, \sqrt[2]{1 - (1 - 0.58^2)^{3\times0.4}}\right)$

$= (0.618429, 0.274015, 0.623421)$

Similarly, $\dot{I}_2 = (0.412232, 0.574638, 0.317261)$ and $\dot{I}_3 = (0.82997, 0.707, 0.42229)$.

Now we use score function to find the ordered position of $\dot{I}_j$ as follows:

$\check{S}(\dot{I}_1) = 0.006199$, $\check{S}(\dot{I}_2) = -0.06928$, $\check{S}(\dot{I}_3) = -0.51055$. Based on score values, the new ordered position of TSFNSs are:

$\dot{I}_{\sigma(1)} = \dot{I}_1 = (0.618429, 0.274015, 0.623421)$

$\dot{I}_{\sigma(2)} = \dot{I}_3 = (0.829986, 0.70714, 0.422288)$

$\dot{I}_{\sigma(3)} = \dot{I}_2 = (0.412232, 0.574638, 0.317261)$

$TSFWG(\dot{I}_1, \dot{I}_3, \dot{I}_2) = (0.483282, 0.41902, 0.683548)$

5.4.11. Theorem

If we assume the weight vector w to be $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$. Then the TSFHG operator reduced to TSFWG operator.

Proof: As $\dot{I}_j = I_j^{m\omega_j}$ and $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$. So $\dot{I}_j^{w_j} = I_j^{\omega_j}$ and

$$TSFHG(I_1, I_2, I_3 \dots I_m) = \prod_{j=1}^{m} \dot{I}_{\sigma(j)}^{w_j} = \prod_{j=1}^{m} I_j^{w_j} = TSFWG(I_1, I_2, I_3 \dots I_m)$$

5.4.12. Theorem

If we assume the weight vector ω to be $\omega = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$. Then the TSFHG operator reduces to TSFOWG operator.

Proof: Straightforward.

5.5. Application in Multi-Attribute Decision Making

In this section, we aim to use the TSFWA and TSFWG AOs in MADM problem. In MADM process, we rank the set of alternatives $\mathcal{A}_j$ based on some attributes K_j having weights in the form of weight vector w. In this phenomenon, a panel of decision makers evaluated the alternatives $\mathcal{A}_j$ and provided their information in the form of TSFNs i.e. in a decision matrix containing TSFNs. Then the TSFWA and TSFWG AOs are utilized to aggregate the information for the evaluation of best alternative. The steps of algorithm of the MADM process are described below:

5.5.1. Algorithm:

Step 1: The information in the form of TSFNs under some attributes about the alternatives are gathered and a decision matrix is formed.

Step 2: In step 2, the data provided by the decision makers is aggregated using AOs of TSFNs.

Step 3: In step 3, the scores of the aggregated data is computed.

Step 4: The final step involves the ranking of score values and most suitable alternative is obtained.

To demonstrate the MADM algorithm, we present the following numerical example.

5.5.2. Example

A multinational company is designing its financial policy for the upcoming year about where to invest to get a potential profit. For this, the research department of the company came with four plans about where to invest after some initial screening. These four alternatives are, $\mathcal{A}_1$: *Asian Markets*, $\mathcal{A}_2$: *Local Markets*, $\mathcal{A}_3$: *European Markets* and $\mathcal{A}_4$: *African Markets*. The evaluation of suitable market to invest in is based on four attributes which are, P_1: The growth perspective, P_2: risk perspective, P_3: political and social perspective and P_4: environmental perspective. The WV is $w = (0.2, 0.1, 0.3, 0.4)^{\mathcal{T}}$. The stepwise demonstration of MADM process is as follows:

Step 1: The formation of decision matrix in Table 20. Note that all the data provided in Table 20 are purely TSFNs for $n = 3$.

	P_1	P_2	P_3	P_4
$\mathcal{A}_1$	(0.53, 0.33, 0.38)	(0.65,0.24,0.74)	(0.61, 0.39, 0.45)	(0.55, 0.88, 0.29)
$\mathcal{A}_2$	(0.40, 0.71, 0.15)	(0.48, 0.46,0.67)	(0.69, 0.46, 0.29)	(0.61, 0.73, 0.43)
$\mathcal{A}_3$	(0.33, 0.53, 0.79)	(0.71, 0.49, 0.16)	(0.53, 0.39, 0.84)	(0.50,0.90, 0.01)
$\mathcal{A}_4$	(0.64, 0.38,0.73)	(0.33,0.64,0.76)	(0.27,0.89,0.07)	(0.74, 0.36,0.19)

Table 20 (Decision Matrix)

Step 2: The data provided by the decision makers in Table 20 is aggregated using TSFWA operators in this step. The steps involved are already explained, here we just provided the results which are:

$$\mathcal{A}_1 = TSFWA(I_{11}, I_{12}, I_{13}, I_{14}) = (0.548023, 0.49755, 0.383533)$$

$$\mathcal{A}_2 = TSFWA(I_{21}, I_{22}, I_{23}, I_{24}) = (0.602926, 0.603509, 0.323543)$$

$$\mathcal{A}_3 = TSFWA(I_{31}, I_{32}, I_{33}, I_{34}) = (0.521966, 0.592777, 0.11946)$$

$$\mathcal{A}_4 = TSFWA(I_{41}, I_{42}, I_{43}, I_{44}) = (0.623935, 0.505723, 0.211727)$$

Step 3: Now we compute the score values of the date obtained in Step 2.

$SC(I_1) = (0.548023)^3 - (0.056417)^3 = 0.10817$

$SC(I_2) = (0.602926)^3 - (0.033869)^3 = 0.185307$

$SC(I_3) = (0.521966)^3 - (0.001705)^3 = 0.140504$

$SC(I_4) = (0.623935)^3 - (0.009491)^3 = 0.233403$

Step 4: Step 4 involves the comparison of score values obtained in Step 3. The comparison is as follows:

$$SC(I_4) > SC(I_2) > SC(I_3) > SC(I_1)$$

Clearly, the score of I_4 is greater among all so the firm needed to go with policy 4 i.e. to invest in *African markets* according to the evaluation of the data using TSFWA operators. Such type of decision making could be very helpful in management sciences, economics and problems of engineering and computer sciences where one need to choose among some alternatives based on expert's opinion.

5.6. Comparative Study and Advantages

In this section we are about to establish a comparative study of proposed study and existing work which will demonstrate the diverse nature of the AOs of TSFSs along with the limitations of the AOs of IFSs, PyFSs and PFSs. Here we are interested in showing that the proposed AOs are the generalizations of the AOs of IFSs [29, 30], PyFSs [44, 45], q-ROFSs [85] and PFSs [50, 83]. The following remark will explain how the AOs of IFSs, PyFSs, q-ROFSs and PFSs becomes particular cases of T-spherical fuzzy AOs.

5.6.1. Remark

Consider the TSFWA and TSFWG operator as follows:

$$TSFWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt[n]{1-\prod_{j=1}^{m}(1-ş_j^n)^{w_j}}, \prod_{j=1}^{m}(i_j)^{w_j}, \prod_{j=1}^{m}(d'_j)^{w_j}\right)$$

$$TSFWG(I_1, I_2, I_3 \dots I_m) = \left(\prod_{j=1}^{m}(ş_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \sqrt[n]{1-\prod_{j=1}^{m}(1-d'^n_j)^{w_j}}\right)$$

- If we take the $n = 2$, we obtained WAA and WGA of SFSs given as:

$$SFWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt{1-\prod_{j=1}^{m}(1-ş_j^2)^{w_j}}, \prod_{j=1}^{m}(i_j)^{w_j}, \prod_{j=1}^{m}(d'_j)^{w_j}\right)$$

$$SFWG(I_1, I_2, I_3 \dots I_m) = \left(\prod_{j=1}^{m}(ş_j)^{w_j}, \prod_{j=1}^{m}(i_j)^{w_j}, \sqrt{1-\prod_{j=1}^{m}(1-d'^2_j)^{w_j}}\right)$$

- If we take the $n = 1$, we obtained WAA operator and WGA operator of PFSs proposed in [50, 83].
- If we take the $i_j = 0$, we obtained WAA operator and WGA operator of q-ROFSs proposed in [85].
- If we take the $n = 2$ and $i_j = 0$, we obtained WAA operator and WGA operator of PyFSs proposed in [44, 45].
- If we take the $n = 1$ and $i_j = 0$, we obtained WAA operator and WGA operator of PyFSs proposed in [29, 30].

Hence, it is proved that the WAA operators and WGA operators proposed in this manuscript are the generalizations of the AOs of IFSs, PyFSs and PFSs and can handle the data which the existing tools could not.

As described in Section 5.1, that the theory of IFSs, PyFSs and q-ROFSs can only deal with situations where we face two types of opinion i.e. yes and no. These types of structures failed to describe the voting phenomena or situations where opinion might be described by MG, NG, AG and RG. In view of all these facts, it is claimed that the aggregation theory proposed in this manuscript is better than the theory that is already developed and can model human opinion more effectively.

Further, we show that the proposed AOs of TSFSs can be applied to solve the problems lying in the environment of IFSs, PyFSs, q-ROFSs and PFSs etc. For this purpose, we consider our Example 5.5.2 and dropped the AG from each triplet of Table 20. The new decision matrix obtained is provided in Table 21.

	P_1	P_2	P_3	P_4
$\mathcal{A}_1$	$(0.53, 0.38)$	$(0.65, 0.74)$	$(0.61, 0.45)$	$(0.55, 0.29)$
$\mathcal{A}_2$	$(0.40, 0.15)$	$(0.48, 0.67)$	$(0.69, 0.29)$	$(0.61, 0.43)$
$\mathcal{A}_3$	$(0.33, 0.79)$	$(0.71, 0.16)$	$(0.53, 0.84)$	$(0.50, 0.01)$
$\mathcal{A}_4$	$(0.64, 0.73)$	$(0.33, 0.76)$	$(0.27, 0.07)$	$(0.74, 0.19)$

Table 21 (Decision Matrix after dropping the abstinence grade)

The information obtained after dropping the abstinence grade are now PyFNs as for all the duplets, the sum of square of the MG and BG is less than 1. Therefore, we use the following special case of the Remark 5.6.1 to aggregate the information obtained in Table 21.

$$PyFWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt{1 - \prod_{j=1}^{m} (1 - ş_j^2)^{w_j}}, \prod_{j=1}^{m} (d'_j)^{w_j} \right)$$

The aggregated results are

$\mathcal{A}_1 = (0.548023, 0.383533)$

$\mathcal{A}_2 = (0.602926, 0.323543)$

$\mathcal{A}_3 = (0.521966, 0.11946)$

$\mathcal{A}_4 = (0.623935, 0.211727)$

To find the most suitable alternative, we use the score function and the score values are given by:

$SC(\mathcal{A}_1) = (0.548023)^3 - (0.383533)^3 = 0.10817$

$SC(\mathcal{A}_2) = (0.602926)^3 - (0.323543)^3 = 0.185307$

$SC(\mathcal{A}_3) = (0.521966)^3 - (0.11946)^3 = 0.140504$

$SC(\mathcal{A}_4) = (0.623935)^3 - (0.11946)^3 = 0.233403$

Based on the score values, we obtained the following ranking pattern

$$SC(\mathcal{A}_4) > SC(\mathcal{A}_2) > SC(\mathcal{A}_3) > SC(\mathcal{A}_1)$$

The result is consistent, and it shows that the proposed TSFWA operators are applicable in existing fuzzy environments and can solve any problem without any limitations. Here this point must be noted that the result is similar to that of Example 5.5.2 after dropping AG. It is not necessary that the results may always get change. However, it is evident that the accuracy of the results must change as in in TSF environment, accuracy rate is more because of less information loss.

Note that later at some stages, the work proposed in this chapter was further improved by Garg et al. [94] where he points out some cases where TSFWA and TSFWG AOs failed. Further, Liu et al. [95] also pointed out the weakness of the AOs of TSFSs proposed in Chapter 2 and of this chapter. One may refer to [94, 95] for better understanding of the new modified AOs.

Chapter 6

Hamacher Aggregation Operators of T-Spherical Fuzzy Sets [6]

In this chapter, it is observed that the Hamacher aggregation (HA) operators of IFSs, PyFSs, q-ROFSs and PFSs have some limitations in their applicability. To serve the purpose, some HA operators based on T-spherical fuzzy numbers are introduced. The concepts of TSF Hamacher weighted averaging (TSFHWA) and TSF Hamacher weighted geometric (TSFHWG) aggregation operators are proposed which described four aspects of human opinion including yes, no, abstinence and refusal degree with no limitations. Such type of AOs efficiently described the cases that left unsolved by the existing AOs. The validity of the proposed AOs is examined, and some basic properties are discussed. The proposed new HA operators are used to analyze the performance of search and rescue robots using a MADM approach as their performance in an emergency is eminent. The proposed HA operators have two variable parameters namely n and γ which effects the decision-making process and their sensitivity towards decision-making results is also analyzed. A comparative analysis of the results obtained using proposed HA operators in view of the variable parameters n and γ is established to discuss any advantages or disadvantages.

[6] The work in this chapter has been published in the following paper:
Ullah K., Mahmood T. and Garg, H. Evaluation of the Performance of Search and Rescue Robots Using T-spherical Fuzzy Hamacher Aggregation Operators. International Journal of Fuzzy Systems, 2020. https://doi.org/10.1007/s40815-020-00803-2

6.1.2. Remark

The Hamacher product and the Hamacher sum defined above reduces to algebraic product and algebraic sum [58] for $\gamma = 1$ and it reduces to Einstein sum and Einstein product [58] for $\gamma = 2$ respectively.

Now we investigate several HA operators so far developed and their limitations in order to develop the ground for new HA operators. Recently, Darko and Liang [93] proposed HA operators in q-rung orthopair fuzzy environment. Such HA operators are capable of handling the imprecise information of Pythagorean fuzzy settings and intuitionistic fuzzy settings as well.

6.1.3. Definition [93]

The averaging and geometric HA operators for some q-ROFNs are defined as

$$q - ROFHWA(I_1, I_2, I_3, \ldots I_m)$$

$$= \left(\sqrt[n]{\frac{\prod_{j=1}^{m}\left(1 + (\gamma - 1)\text{ş}_j^n\right)^{w_j} - \prod_{j=1}^{m}\left(1 - \text{ş}_j^n\right)^{w_j}}{\prod_{j=1}^{m}\left(1 + (\gamma - 1)\text{ş}_j^n\right)^{w_j} + (\gamma - 1)\prod_{j=1}^{m}\left(1 - \text{ş}_j^n\right)^{w_j}}}, \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m}\mathrm{d'}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1 + (\gamma - 1)\left(1 - \mathrm{d'}_j^n\right)\right)^{w_j} + (\gamma - 1)\prod_{j=1}^{m}\left(\mathrm{d'}_j^n\right)^{2w_j}}} \right)$$

$q - ROFHWG(I_{1,}I_2, I_3, \ldots I_m)$

$$= \left(\frac{\sqrt[n]{\gamma} \prod_{j=1}^{m} \varsigma_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{m} \left(1 + (\gamma - 1)(1 - \varsigma_j^n)\right)^{w_j} + (\gamma - 1) \prod_{j=1}^{m} (\varsigma_j^n)^{2w_j}}}, \sqrt[n]{\frac{\prod_{j=1}^{m} (1 + (\gamma - 1)\mathrm{d'}_j^n)^{w_j} - \prod_{j=1}^{m} (1 - \mathrm{d'}_j^n)^{w_j}}{\prod_{j=1}^{m} (1 + (\gamma - 1)\mathrm{d'}_j^n)^{w_j} + (\gamma - 1) \prod_{j=1}^{m} (1 - \mathrm{d'}_j^n)^{w_j}}} \right)$$

The HA operators for PyFSs and IFSs were proposed by Wu and Wei [54] and Huang [53] respectively which are the special cases of the HA operators of q-ROFSs proposed by Darko and Liang [93]. The intuitionistic fuzzy HA operators proposed by Huang [53] by placing $q = 1$ in Definition 6.1.3 while the HA operators of PyFSs proposed by Wu and Wei [54] can be obtained by placing $q = 2$ in Definition 6.1.3.

However, the HA operators of IFNs, PyFNs and q-ROFNs deal with only those circumstances where human opinion is described by the MG and the NG only which leads to information loss as a human opinion has abstinence and refusal degree too. In situations where human opinion has some abstinence as well as refusal degree involved, these concepts fail to be applied as suggested by Cuong [9]. In such situations, the framework of PFNs is a better option. Keeping this in view, Wei [58] developed HA operators in picture fuzzy environment. The concept of picture fuzzy HA operators is described as follows:

6.1.4. Definition [58]

The averaging and geometric HA operators for a some PFNs are defined as:

$PFHWA(I_{1,}I_2, I_3, \ldots I_m)$

$$= \begin{pmatrix} \frac{\prod_{j=1}^{m}(1+(\gamma-1)ş_j)^{w_j} - \prod_{j=1}^{m}(1-ş_j)^{w_j}}{\prod_{j=1}^{m}(1+(\gamma-1)ş_j)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(1-ş_j)^{w_j}}, \\ \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m} i_j^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)(1-i_j)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(i_j)^{2w_j}}, \\ \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m} \text{d'}_j^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)(1-\text{d'}_j)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(\text{d'}_j)^{2w_j}} \end{pmatrix}$$

$PFHWG(I_1, I_2, I_3, \ldots I_m)$

$$= \begin{pmatrix} \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m} ş_j^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)(1-ş_j)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(ş_j)^{2w_j}}, \\ \frac{\prod_{j=1}^{m}(1+(\gamma-1)i_j)^{w_j} - \prod_{j=1}^{m}(1-i_j)^{w_j}}{\prod_{j=1}^{m}(1+(\gamma-1)i_j)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(1-i_j)^{w_j}}, \\ \frac{\prod_{j=1}^{m}(1+(\gamma-1)i_j)^{w_j} - \prod_{j=1}^{m}(1-i_j)^{w_j}}{\prod_{j=1}^{m}(1+(\gamma-1)i_j)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(1-i_j)^{w_j}} \end{pmatrix}$$

HA operators of PFSs proposed by Wei [58] have been successfully utilized in MADM problems. Obviously, HA operators proposed by Wei [58] deals with imprecise information involving human opinion, are better than HA operators proposed by Wu and Wei [54], Huang [53] and Darko and Liang [93] as the chance of information loss is reduced sufficiently due to the ability of taking into account the AG and RG alongside of the MG and the NG.

There are some situations where the concept of picture fuzzy information aggregation could not be applied especially when the triplets under observation i.e. $(ş, i, \text{d'})$ does not satisfy the condition $0 \leq şum(ş, i, \text{d'}) \leq 1$ e.g. the triplet $(0.71, 0.52, 0.91)$ is not a PFN because the sum of three components exceeds 1. To deal with such situations, we proposed the framework of SFSs and consequently TSFSs in Chapter 2. In view of the Definition of

TSFS, every triplet of the form (ς, i, d') can be regarded as a TSFN for some $n \in \mathbb{Z}^+$. Therefore in this chapter, we propose the idea of HA operators for TSFSs which can process any kind of data which the existing HA operators fail to process.

6.2. T-Spherical fuzzy Hamacher operations

The aim of this section is to propose the Hamacher operations in TSF environment and show their superiority over the Hamacher operations of IFNs, PyFNs and PFNs. Using Hamacher t-norm and Hamacher t-conorm we propose the following HA operations.

6.2.1. Definition

For two TSFNs $I_1 = (\varsigma_1, i_1, d'_1)$ and $I_2 = (\varsigma_2, i_2, d'_2)$ and for $\lambda, \gamma > 0$. The T-spherical fuzzy Hamacher operations are:

1. $$I_1 \oplus I_2 = \left(\begin{array}{c} \sqrt[n]{\dfrac{\varsigma_1^n + \varsigma_2^n - \varsigma_1^n \varsigma_2^n - (1-\gamma)\varsigma_1^n \varsigma_2^n}{1-(1-\gamma)\varsigma_1^n \varsigma_2^n}}, \\ \dfrac{i_1 i_2}{\sqrt[n]{\gamma + (1-\gamma)(i_1^n + i_2^n - i_1^n i_2^n)}}, \dfrac{d'_1 d'_2}{\sqrt[n]{\gamma + (1-\gamma)(d'^n_1 + d'^n_2 - d'^n_1 d'^n_2)}} \end{array} \right)$$

2. $$I_1 \otimes I_2 = \left(\begin{array}{c} \dfrac{\varsigma_1 \varsigma_2}{\sqrt[n]{\gamma + (1-\gamma)(\varsigma_1^n + \varsigma_2^n - \varsigma_1^n \varsigma_2^n)}}, \\ \sqrt[n]{\dfrac{i_1^n + i_2^n - i_1^n i_2^n - (1-\gamma) i_1^n i_2^n}{1-(1-\gamma) i_1^n i_2^n}}, \sqrt[n]{\dfrac{d'^n_1 + d'^n_2 - d'^n_1 d'^n_2 - (1-\gamma) d'^n_1 d'^n_2}{1-(1-\gamma) d'^n_1 d'^n_2}} \end{array} \right)$$

3. $$\lambda I_1 = \left(\begin{array}{c} \sqrt[n]{\dfrac{(1+(\gamma-1)\varsigma_1^n)^\lambda - (1-\varsigma_1^n)^\lambda}{(1+(\gamma-1)\varsigma_1^n)^\lambda + (\gamma-1)(1-\varsigma_1^n)^\lambda}}, \\ \dfrac{\sqrt[n]{\gamma}\, i_1^\lambda}{\sqrt[n]{\left(1+(\gamma-1)(1-i_1^n)\right)^\lambda + (\gamma-1)(i_1^n)^{2\lambda}}}, \dfrac{\sqrt[n]{\gamma}\, d'^\lambda_1}{\sqrt[n]{\left(1+(\gamma-1)(1-d'^n_1)\right)^\lambda + (\gamma-1)(d'^n_1)^{2\lambda}}} \end{array} \right)$$

4. $$I_1^\lambda = \left(\begin{array}{c} \dfrac{\sqrt[n]{\gamma}\, \varsigma_1^\lambda}{\sqrt[n]{\left(1+(\gamma-1)(1-\varsigma_1^n)\right)^\lambda + (\gamma-1)(\varsigma_1^n)^{2\lambda}}}, \\ \sqrt[n]{\dfrac{(1+(\gamma-1)i_1^n)^\lambda - (1-i_1^n)^\lambda}{(1+(\gamma-1)i_1^n)^\lambda + (\gamma-1)(1-i_1^n)^\lambda}}, \sqrt[n]{\dfrac{(1+(\gamma-1)d'^n_1)^\lambda - (1-d'^n_1)^\lambda}{(1+(\gamma-1)d'^n_1)^\lambda + (\gamma-1)(1-d'^n_1)^\lambda}} \end{array} \right)$$

The Hamacher operations proposed in Definition 6.2.1 are more generalized than the existing Hamacher operations of IFSs, PyFSs, q-ROFSs and PFSs. Unlike Hamacher operations of IFSs, PyFSs and PFSs, the T-spherical fuzzy Hamacher operations described the MG, the AG, the NG and the RG with no limitations. The reason is that for every triplet (ş, i, d'), there is a $n \in \mathbb{Z}^+$ that make the triplet a TSFN.

There are some conditions under which the proposed Hamacher operations of TSFNs reduced to the environment of SFSs, PFSs, q-ROFSs, PyFSs and IFSs respectively. Using T-spherical fuzzy Hamacher operations proposed in Definition 6.2.1, we proposed the Hamacher operations for SFNs, PFNs, q-ROFNs. PyFNs and IFNs in the following remark as its special cases.

6.2.2. Remark

The T-spherical fuzzy Hamacher operations reduced to Hamacher operations of

- SFNs; if we take $n = 2$.
- PFNs; if we take $n = 1$.
- Q-ROFNs; if we take the AG i.e. i as zero.
- PyFNs; if we take $n = 2$ and the AG i.e. i as zero.
- IFNs; if we take $n = 1$ and the AG i.e. i as zero.

6.3. T-Spherical fuzzy Hamacher Averaging Operators

This section is based on the averaging AOs based on Hamacher operations. Using Hamacher operation proposed in Definition 6.2.1, we proposed TSFHWA operator. The induction method is used for the validation of the proposed TSFHWA operator and some properties are also investigated.

6.3.1. Definition

Let $I_i = (\c{s}, i, \mathrm{d'})$ be a collection of TSFNs. Then TSFHWA operator is a map $T^n \to T$ such that

$$TSFHWA(I_1, I_2, I_3, \dots I_m) = \underset{j=1}{\overset{m}{\oplus}} w_j I_j$$

Now, using Definition 6.2.1 and Definition 6.3.1, we propose the following result.

6.3.2. Theorem

Let $I_j = (\c{s}, i, \mathrm{d'})$ be a collection of TSFNs. Then TSFHWA operator is a TSFN and is having the form

$TSFHWA(I_1, I_2, I_3, \dots I_m)$

$$= \left(\begin{array}{c} \sqrt[n]{\dfrac{\prod_{j=1}^{m}(1+(\gamma-1)\c{s}_j^n)^{w_j} - \prod_{j=1}^{m}(1-\c{s}_j^n)^{w_j}}{\prod_{j=1}^{m}(1+(\gamma-1)\c{s}_j^n)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(1-\c{s}_j^n)^{w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} i_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)(1-i_j^n)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(i_j^n)^{2w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} \mathrm{d'}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)(1-\mathrm{d'}_j^n)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(\mathrm{d'}_j^n)^{2w_j}}} \end{array} \right)$$

Proof: Using mathematical induction,

For $m = 2$

$w_1 I_1 \oplus w_2 I_2$

$$= \left(\begin{pmatrix} \sqrt[n]{\frac{(1+(\gamma-1)ş_1^n)^{w_1}-(1-ş_1^n)^{w_1}}{(1+(\gamma-1)ş_1^n)^{w_1}+(\gamma-1)(1-ş_1^n)^{w_1}}}, \\ \frac{\sqrt[n]{\gamma}\, i_1^{w_1}}{\sqrt[n]{\left(1+(\gamma-1)(1-i_1^n)\right)^{w_1}+(\gamma-1)(i_1^n)^{2w_1}}}, \\ \frac{\sqrt[n]{\gamma}\, \mathrm{d'}_1^{w_1}}{\sqrt[n]{\left(1+(\gamma-1)(1-\mathrm{d'}_1^n)\right)^{w_1}+(\gamma-1)(\mathrm{d'}_1^n)^{2w_1}}} \end{pmatrix} \oplus \begin{pmatrix} \sqrt[n]{\frac{(1+(\gamma-1)ş_2^n)^{w_2}-(1-ş_2^n)^{w_2}}{(1+(\gamma-1)ş_2^n)^{w_2}-(\gamma-1)(1-ş_2^n)^{w_2}}}, \\ \frac{\sqrt[n]{\gamma}\, i_2^{w_2}}{\sqrt[n]{\left(1+(\gamma-1)(1-i_2^n)\right)^{w_2}+(\gamma-1)(i_2^n)^{2w_2}}}, \\ \frac{\sqrt[n]{\gamma}\, \mathrm{d'}_2^{w_2}}{\sqrt[n]{\left(1+(\gamma-1)(1-\mathrm{d'}_2^n)\right)^{w_2}+(\gamma-1)(\mathrm{d'}_2^n)^{2w_2}}} \end{pmatrix} \right)$$

$$w_1 I_1 \oplus w_2 I_2 = \begin{pmatrix} \sqrt[n]{\frac{\prod_{j=1}^{2}\left(1+(\gamma-1)ş_j^n\right)^{w_j}-\prod_{j=1}^{2}\left(1-ş_j^n\right)^{w_j}}{\prod_{j=1}^{2}\left(1+(\gamma-1)ş_j^n\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(1-ş_j^n\right)^{w_j}}}, \\ \frac{\sqrt[n]{\gamma}\prod_{j=1}^{2} i_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{2}\left(1+(\gamma-1)\left(1-i_j^n\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(i_j^n\right)^{2w_j}}}, \\ \frac{\sqrt[n]{\gamma}\prod_{j=1}^{2} \mathrm{d'}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{2}\left(1+(\gamma-1)\left(1-\mathrm{d'}_j^n\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}\left(\mathrm{d'}_j^n\right)^{2w_j}}} \end{pmatrix}$$

The result holds true for $m = 2$.

Now, suppose the result is true for $m = k$ and

$TSFHWA(I_1, I_2, I_3, \dots I_k) =$

$$\left(\begin{array}{c} \sqrt[n]{\dfrac{\prod_{j=1}^{k}\left(1+(\gamma-1)ş_j^n\right)^{w_j}-\prod_{j=1}^{k}\left(1-ş_j^n\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)ş_j^n\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-ş_j^n\right)^{w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{k} i_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-i_j^n)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(i_j^n)^{2w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{k} \mathrm{d'}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-\mathrm{d'}_j^n)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(\mathrm{d'}_j^n)^{2w_j}}} \end{array} \right)$$

To prove for $m = k + 1$

$$TSFHWA(I_1, I_2, I_3, \dots I_k, I_{k+1}) = TSFHWA(I_1, I_2, I_3, \dots I_k) \oplus I_{k+1}$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \sqrt[n]{\dfrac{\prod_{j=1}^{k}\left(1+(\gamma-1)ş_j^n\right)^{w_j}-\prod_{j=1}^{k}\left(1-ş_j^n\right)^{w_j}}{\prod_{j=1}^{k}\left(1+(\gamma-1)ş_j^n\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}\left(1-ş_j^n\right)^{w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{k} i_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-i_j^n)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(i_j^n)^{2w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{k} \mathrm{d'}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-\mathrm{d'}_j^n)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(\mathrm{d'}_j^n)^{2w_j}}} \end{array} \right) \oplus \\ \left(\begin{array}{c} \sqrt[n]{\dfrac{(1+(\gamma-1)ş_{k+1}^n)^{w_{k+1}}-(1-ş_{k+1}^n)^{w_{k+1}}}{(1+(\gamma-1)ş_{k+1}^n)^{w_{k+1}}-(\gamma-1)(1-ş_{k+1}^n)^{w_{k+1}}}}, \\ \dfrac{\sqrt[n]{\gamma}\, i_{k+1}^{w_{k+1}}}{\sqrt[n]{\left(1+(\gamma-1)(1-i_{k+1}^n)\right)^{w_{k+1}}+(\gamma-1)(i_{k+1}^n)^{2w_{k+1}}}}, \\ \dfrac{\sqrt[n]{\gamma}\, \mathrm{d'}_{k+1}^{w_{k+1}}}{\sqrt[n]{\left(1+(\gamma-1)(1-\mathrm{d'}_{k+1}^n)\right)^{w_{k+1}}+(\gamma-1)(\mathrm{d'}_{k+1}^n)^{2w_{k+1}}}} \end{array} \right) \end{array} \right)$$

$TSFHWA(I_1, I_2, I_3, \dots I_{k+1})$

$$= \left(\begin{array}{c} \sqrt[n]{\dfrac{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\varsigma_j^n\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-\varsigma_j^n\right)^{w_j}}{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\varsigma_j^n\right)^{w_j} + (\gamma-1)\prod_{j=1}^{k+1}\left(1-\varsigma_j^n\right)^{w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{k+1} i_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\left(1-i_j^n\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{k+1}\left(i_j^n\right)^{2w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{k+1} \text{d}'^{w_j}_j}{\sqrt[n]{\prod_{j=1}^{k+1}\left(1+(\gamma-1)\left(1-\text{d}'^n_j\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{k+1}\left(\text{d}'^n_j\right)^{2w_j}}} \end{array} \right)$$

Hence the result holds for $m = k + 1$ and therefore holds true for all values of m.

Now, we discussed some basic properties of the proposed TSFHWA operator as follows:

6.3.3. Theorem

The HA operator of TSFNs satisfies the following properties:

1. (Idempotency)

If $I_j = I = (\varsigma, i, \text{d}')\ \forall\ j = 1, 2, 3, \dots m$. Then $TSFHWA(I_1, I_2, I_3 \dots I_m) = I$

2. (Boundedness)

If $I^- = \left(\min_j \varsigma_j, \max_j i_j, \max_j \text{d}'_j\right)$ and $I^+ = \left(\max_j \varsigma_j, \min_j i_j, \min_j \text{d}'_j\right)$. Then

$$I^- \leq TSFHWA(I_1, I_2, I_3 \dots I_m) \leq I^+$$

3. (Monotonicity)

Let I_j and P_j be two TSFNs such that $I_j \leq P_j\ \forall\ j$. Then

$$TSFHWA(I_1, I_2, I_3 \dots I_m) \leq TSFHWA(P_1, P_2, P_3 \dots P_m)$$

Proof: Similar

The TSFHWA AO weighs the TSFN only. In MADM problem, there are circumstances when the ordered position of the TSFN matters. For, those situations, the concept of ordered weighted averaging operators play a significant role and TSFHOWA operator is proposed as follows.

6.3.4. Definition

Let $I_i = (\text{ş}, i, \text{d'})$ be a collection of TSFNs. Then TSFHOWA operator is a map $T^n \rightarrow T$ such that

$$TSFHOWA(I_{1,}I_2, I_3, \dots I_m) = \bigoplus_{j=1}^{m} w_j I_{\sigma(j)}$$

Where $\sigma(j)$ is such that $I_{\sigma(j-1)} \geq I_{\sigma(j)}\ \forall\ j$.

6.3.5. Theorem

Let $I_i = (\text{ş}, i, \text{d'})$ be a collection of TSFNs. Then TSFHOWA operator is a TSFN having the form

$TSFHOWA(I_{1,}I_2, I_3, \dots I_m)$

$$= \left(\sqrt[n]{\frac{\prod_{j=1}^{m}\left(1+(\gamma-1)\S_{\sigma(j)}^{n}\right)^{w_j}-\prod_{j=1}^{m}\left(1-\S_{\sigma(j)}^{n}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)\S_{\sigma(j)}^{n}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{m}\left(1-\S_{\sigma(j)}^{n}\right)^{w_j}}}, \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m} i_{\sigma(j)}^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-i_{\sigma(j)}^{n}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{m}\left(i_{\sigma(j)}^{n}\right)^{2w_j}}}, \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m} \mathrm{d'}_{\sigma(j)}^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-\mathrm{d'}_{\sigma(j)}^{n}\right)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{m}\left(\mathrm{d'}_{\sigma(j)}^{n}\right)^{2w_j}}} \right)$$

6.3.6. Remark

The TSFHOWA operator satisfies the properties of idempotency, monotonicity and boundedness stated in Theorem 6.3.3.

The TSFHWA operator and TSFHOWA operator discussed two different scenarios. The first one only weighs the T-spherical fuzzy argument while the later weighs the ordered position of the TSF argument. We need to develop such operator that weighs both, the ordered position as well as the argument itself. Therefore, we propose the concept of hybrid operator that takes into account the weight of both, the argument and its ordered position.

6.3.7. Definition

Let $I_i = (\S, i, \mathrm{d'})$ be a collection of TSFNs. Then TSFHHA operator is a map $T^n \to T$ such that

$$TSFHHA(I_1, I_2, I_3, \dots I_m) = \bigoplus_{j=1}^{m} w_j \dot{I}_{\sigma(j)}$$

where $\dot{I}_{\sigma(j)}$ is the j^{th} largest of the TSFN $\dot{I}_j = m\text{ω}_j I_j$ with ω_j as the weight vector of T-spherical fuzzy arguments I_j such that $\text{ω}_j \in [0,1]$ and $\sum_1^{\text{d}'} \text{ω}_j = 1$ and m is the balancing coefficient.

6.3.8. Theorem

Let $I_i = (\text{ş}, i, \text{d}')$ be a collection of TSFNs. Then TSFHHA operator is a TSFN having the form

$$TSFHHA(I_1, I_2, I_3, \dots I_m)$$

$$= \left(\sqrt[n]{\frac{\prod_{j=1}^{m}\left(1+(\gamma-1)\dot{\text{ş}}_{\sigma(j)}^{n}\right)^{w_j} - \prod_{j=1}^{m}\left(1-\dot{\text{ş}}_{\sigma(j)}^{n}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)\dot{\text{ş}}_{\sigma(j)}^{n}\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(1-\dot{\text{ş}}_{\sigma(j)}^{n}\right)^{w_j}}}, \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m}\dot{i}_{\sigma(j)}^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-\dot{i}_{\sigma(j)}^{n}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(\dot{i}_{\sigma(j)}^{n}\right)^{2w_j}}}, \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m}\dot{\text{d}}'^{w_j}_{\sigma(j)}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-\dot{\text{d}}'^{n}_{\sigma(j)}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(\dot{\text{d}}'^{n}_{\sigma(j)}\right)^{2w_j}}} \right)$$

6.3.9. Remark

The TSFHHA operator reduced to TSFHWA operator if we take $w_j = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m} \dots \frac{1}{m}\right)^T$ while it reduced to TSFHOWA operator if we take $\text{ω}_j = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m} \dots \frac{1}{m}\right)^T$.

6.3.10. Remark

The TSFHHA operator satisfies the properties of idempotency, monotonicity and boundedness stated in Theorem 6.3.3.

6.4. T-Spherical fuzzy Hamacher Geometric Operators

This section is based on the geometric aggregation operators of TSFNs. We propose TSFHWG operator and the induction method is used for the validation of the proposed HA operator. Some other properties of the TSFHWG operator are also investigated.

6.4.1. Definition

Let $I_i = (\text{ş}, i, \text{d'})$ be a collection of TSFNs. Then TSFHWG operator is a map $T^n \to T$ such that

$$TSFHWG(I_1, I_2, I_3, \dots I_m) = \bigotimes_{j=1}^{m} w_j I_j$$

Now, using Definition 6.2.1, we propose the following result.

6.4.2. Theorem

Let $I_i = (\text{ş}, i, \text{d'})$ be a collection of TSFNs. Then TSFHWG operator is a TSFN having the form

$$TSFHWG(I_1, I_2, I_3, \dots I_m) = \bigotimes_{j=1}^{m} I_j^{w_j}$$

$$= \left(\begin{array}{c} \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} \text{ş}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)(1-\text{ş}_j^n)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(\text{ş}_j^n)^{2w_j}}}, \\ \sqrt[n]{\dfrac{\prod_{j=1}^{m}(1+(\gamma-1)i_j^n)^{w_j} - \prod_{j=1}^{m}(1-i_j^n)^{w_j}}{\prod_{j=1}^{m}(1+(\gamma-1)i_j^n)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(1-i_j^n)^{w_j}}}, \\ \sqrt[n]{\dfrac{\prod_{j=1}^{m}(1+(\gamma-1)\text{d'}_j^n)^{w_j} - \prod_{j=1}^{m}(1-\text{d'}_j^n)^{w_j}}{\prod_{j=1}^{m}(1+(\gamma-1)\text{d'}_j^n)^{w_j} + (\gamma-1)\prod_{j=1}^{m}(1-\text{d'}_j^n)^{w_j}}} \end{array} \right)$$

Proof: Using mathematical induction,

For $m = 2$

$w_1 I_1 \otimes w_2 I_2$

$$= \left(\begin{pmatrix} \frac{\sqrt[n]{\gamma}\,\varsigma_1^{w_1}}{\sqrt[n]{\left(1+(\gamma-1)(1-\varsigma_1^n)\right)^{w_1}+(\gamma-1)(\varsigma_1^n)^{2w_1}}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)i_1^n)^{w_1}-(1-i_1^n)^{w_1}}{(1+(\gamma-1)i_1^n)^{w_1}+(\gamma-1)(1-i_1^n)^{w_1}}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)d_1'^n)^{w_1}-(1-d_1'^n)^{w_1}}{(1+(\gamma-1)d_1'^n)^{w_1}+(\gamma-1)(1-d_1'^n)^{w_1}}} \end{pmatrix} \otimes \begin{pmatrix} \frac{\sqrt[n]{\gamma}\,\varsigma_2^{w_2}}{\sqrt[n]{\left(1+(\gamma-1)(1-\varsigma_2^n)\right)^{w_2}-(\gamma-1)(\varsigma_2^n)^{2w_2}}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)i_2^n)^{w_2}-(1-i_2^n)^{w_2}}{(1+(\gamma-1)i_2^n)^{w_2}+(\gamma-1)(1-i_2^n)^{w_2}}}, \\ \sqrt[n]{\frac{(1+(\gamma-1)d_2'^n)^{w_2}-(1-d_2'^n)^{w_2}}{(1+(\gamma-1)d_2'^n)^{w_2}+(\gamma-1)(1-d_2'^n)^{w_2}}} \end{pmatrix} \right)$$

$$w_1 I_1 \otimes w_2 I_2 = \begin{pmatrix} \frac{\sqrt[n]{\gamma}\prod_{j=1}^{2}\varsigma_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{2}\left(1+(\gamma-1)(1-\varsigma_j^n)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{2}(\varsigma_j^n)^{2w_j}}}, \\ \sqrt[n]{\frac{\prod_{j=1}^{2}(1+(\gamma-1)i_j^n)^{w_j}-\prod_{j=1}^{2}(1-i_j^n)^{w_j}}{\prod_{j=1}^{2}(1+(\gamma-1)i_j^n)^{w_j}+(\gamma-1)\prod_{j=1}^{2}(1-i_j^n)^{w_j}}}, \\ \sqrt[n]{\frac{\prod_{j=1}^{2}(1+(\gamma-1)d_j'^n)^{w_j}-\prod_{j=1}^{2}(1-d_j'^n)^{w_j}}{\prod_{j=1}^{2}(1+(\gamma-1)d_j'^n)^{w_j}+(\gamma-1)\prod_{j=1}^{2}(1-d_j'^n)^{w_j}}} \end{pmatrix}$$

The result holds true for $m = 2$.

Suppose the result is true for $m = k$ and

$TSFHWG(I_1, I_2, I_3, \dots I_k) =$

$$\left(\frac{\sqrt[n]{\gamma}\prod_{j=1}^{k} \text{ş}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-\text{ş}_j^n)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(\text{ş}_j^n)^{2w_j}}}, \sqrt[n]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)i_j^n)^{w_j}-\prod_{j=1}^{k}(1-i_j^n)^{w_j}}{\prod_{j=1}^{k}(1+(\gamma-1)i_j^n)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(1-i_j^n)^{w_j}}}, \sqrt[n]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)\text{d'}_j^n)^{w_j}-\prod_{j=1}^{k}(1-\text{d'}_j^n)^{w_j}}{\prod_{j=1}^{k}(1+(\gamma-1)\text{d'}_j^n)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(1-\text{d'}_j^n)^{w_j}}}\right)$$

To prove for $m = k + 1$,

$TSFHWG(I_1, I_2, I_3, \dots I_k, I_{k+1}) = TSFHWG(I_1, I_2, I_3, \dots I_k) \otimes I_{k+1}$

$$= \left(\left(\frac{\sqrt[n]{\gamma}\prod_{j=1}^{k} \text{ş}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k}\left(1+(\gamma-1)(1-\text{ş}_j^n)\right)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(\text{ş}_j^n)^{2w_j}}}, \sqrt[n]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)i_j^n)^{w_j}-\prod_{j=1}^{k}(1-i_j^n)^{w_j}}{\prod_{j=1}^{k}(1+(\gamma-1)i_j^n)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(1-i_j^n)^{w_j}}}, \sqrt[n]{\frac{\prod_{j=1}^{k}(1+(\gamma-1)\text{d'}_j^n)^{w_j}-\prod_{j=1}^{k}(1-\text{d'}_j^n)^{w_j}}{\prod_{j=1}^{k}(1+(\gamma-1)\text{d'}_j^n)^{w_j}+(\gamma-1)\prod_{j=1}^{k}(1-\text{d'}_j^n)^{w_j}}}\right) \otimes \left(\frac{\sqrt[n]{\gamma}\,\text{ş}_{k+1}^{w_{k+1}}}{\sqrt[n]{\left(1+(\gamma-1)(1-\text{ş}_{k+1}^n)\right)^{w_{k+1}}+(\gamma-1)(\text{ş}_{k+1}^n)^{2w_{k+1}}}}, \sqrt[n]{\frac{(1+(\gamma-1)i_{k+1}^n)^{w_{k+1}}-(1-i_{k+1}^n)^{w_{k+1}}}{(1+(\gamma-1)i_{k+1}^n)^{w_{k+1}}+(\gamma-1)(1-i_{k+1}^n)^{w_{k+1}}}}, \sqrt[n]{\frac{(1+(\gamma-1)\text{d'}_{k+1}^n)^{w_{k+1}}-(1-\text{d'}_{k+1}^n)^{w_{k+1}}}{(1+(\gamma-1)\text{d'}_{k+1}^n)^{w_{k+1}}+(\gamma-1)(1-\text{d'}_{k+1}^n)^{w_{k+1}}}}\right)\right)$$

$$TSFHWG(I_1, I_2, I_3, \dots I_{k+1})$$

$$= \left(\frac{\sqrt[n]{\gamma} \prod_{j=1}^{k+1} \text{ş}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{k+1}\left(1+(\gamma-1)(1-\text{ş}_j^n)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{k+1}(\text{ş}_j^n)^{2w_j}}}, \sqrt[n]{\frac{\prod_{j=1}^{k+1}(1+(\gamma-1)i_j^n)^{w_j} - \prod_{j=1}^{k+1}(1-i_j^n)^{w_j}}{\prod_{j=1}^{k+1}(1+(\gamma-1)i_j^n)^{w_j} + (\gamma-1)\prod_{j=1}^{k+1}(1-i_j^n)^{w_j}}}, \sqrt[n]{\frac{\prod_{j=1}^{k+1}(1+(\gamma-1)\text{d'}_j^n)^{w_j} - \prod_{j=1}^{k+1}(1-\text{d'}_j^n)^{w_j}}{\prod_{j=1}^{k+1}(1+(\gamma-1)\text{d'}_j^n)^{w_j} + (\gamma-1)\prod_{j=1}^{k+1}(1-\text{d'}_j^n)^{w_j}}} \right)$$

Hence the result holds for $m = k + 1$ and therefore holds true for all values of m.

Now, we discussed some basic properties of the proposed TSFHWG operator as follows:

6.4.3. Theorem

The TSFHWG operator satisfies the following properties:

1. (Idempotency)

If $I_j = I = (\text{ş}, i, \text{d'})\ \forall\ j = 1, 2, 3, \dots m$. Then $TSFHWG(I_1, I_2, I_3 \dots I_m) = I$

2. (Boundedness)

If $I^- = \left(\min_j \text{ş}_j, \max_j i_j, \max_j \text{d'}_j\right)$ and $I^+ = \left(\max_j \text{ş}_j, \min_j i_j, \min_j \text{d'}_j\right)$. Then

$$I^- \leq TSFHWG(I_1, I_2, I_3 \dots I_m) \leq I^+$$

3. (Monotonicity)

Let I_j and P_j be two TSFNs such that $I_j \leq P_j\ \forall\ j$. Then

$$TSFHWG(I_1, I_2, I_3 \dots I_m) \leq TSFHWG(P_1, P_2, P_3 \dots P_m)$$

The TSFHWG aggregation operator weighs the TSFN only. In MADM problem, there are circumstances when the ordered position of the TSFN matters. For those situations, the concept of ordered weighted averaging operators play a significant role and TSFHOWG operator is proposed as follows.

6.4.4. Definition

Let $I_i = (\text{ş}, i, \text{d'})$ be a collection of TSFNs. Then TSFHOWG operator is a map $T^n \to T$ such that

$$TSFHOWG(I_1, I_2, I_3, \dots I_m) = \bigotimes_{j=1}^{m} w_j I_{\sigma(j)}$$

Where $\sigma(j)$ is such that $I_{\sigma(j-1)} \geq I_{\sigma(j)} \ \forall\, j$.

6.4.5. Theorem

Let $I_i = (\text{ş}, i, \text{d'})$ be a collection of TSFNs. Then TSFHOWG operator is a TSFN having the form

$$TSFHOWG(I_1, I_2, I_3, \dots I_m) = \bigotimes_{j=1}^{m} w_j I_{\sigma(j)}$$

$$= \left(\begin{array}{c} \sqrt[n]{\dfrac{\prod_{j=1}^{m}\left(1+(\gamma-1)\text{ş}_{\sigma(j)}^{n}\right)^{w_j} - \prod_{j=1}^{m}\left(1-\text{ş}_{\sigma(j)}^{n}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)\text{ş}_{\sigma(j)}^{n}\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(1-\text{ş}_{\sigma(j)}^{n}\right)^{w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} i_{\sigma(j)}^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-i_{\sigma(j)}^{n}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(i_{\sigma(j)}^{n}\right)^{2w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} \text{d'}_{\sigma(j)}^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-\text{d'}_{\sigma(j)}^{n}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(\text{d'}_{\sigma(j)}^{n}\right)^{2w_j}}} \end{array} \right)$$

6.4.6. Remark

The TSFHOWG operator satisfies the properties of idempotency, monotonicity and boundedness stated in Theorem 6.4.3.

The TSFHWG operator and TSFHOWG operator discussed two different scenarios. The first one only weighs the TSF argument while the later weighs the ordered position of the TSF argument. We propose the concept of Hamacher hybrid geometric operator that weighs both argument and its ordered position.

6.4.7. Definition

Let $I_i = (\text{ş}, i, \text{d}')$ be a collection of TSFNs. Then TSFHHG operator is a map $T^n \to T$ such that

$$TSFHHG(I_1, I_2, I_3, \dots I_m) = \bigotimes_{j=1}^{m} w_j \dot{I}_{\sigma(j)}$$

where $\dot{I}_{\sigma(j)}$ is the jth largest of the TSFN $\dot{I}_j = I_j^{m\mathrm{w}_j}$ with w_j as the weight vector of T-spherical fuzzy arguments I_j such that $\mathrm{w}_j \in [0, 1]$ and $\sum_1^{\text{d}'} \mathrm{w}_j = 1$ and m is the balancing coefficient.

6.4.8. Theorem

Let $I_i = (\text{ş}, i, \text{d}')$ be a collection of TSFNs. Then TSFHHG operator is a TSFN having the form

$TSFHHG(I_1, I_2, I_3, \dots I_m)$

$$
= \left(\sqrt[n]{\frac{\prod_{j=1}^{m}\left(1+(\gamma-1)\dot{s}_{\sigma(j)}^{n}\right)^{w_j} - \prod_{j=1}^{m}\left(1-\dot{s}_{\sigma(j)}^{n}\right)^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)\dot{s}_{\sigma(j)}^{n}\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(1-\dot{s}_{\sigma(j)}^{n}\right)^{w_j}}}, \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m} i_{\sigma(j)}^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-i_{\sigma(j)}^{n}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(i_{\sigma(j)}^{n}\right)^{2w_j}}}, \frac{\sqrt[n]{\gamma}\prod_{j=1}^{m} \dot{d}_{\sigma(j)}^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-\dot{d}_{\sigma(j)}^{n}\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(\dot{d}_{\sigma(j)}^{n}\right)^{2w_j}}} \right)
$$

6.4.9. Remark

The TSFHHG operator reduced to TSFHWG operator if we take $w_j = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m} \dots \frac{1}{m}\right)^T$ while it reduced to TSFHOWG operator if we take $\omega_j = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m} \dots \frac{1}{m}\right)^T$.

6.4.10. Remark

The TSFHHG operator satisfy the properties of idempotency, monotonicity and boundedness stated in Theorem 6.4.3.

6.5. Some Special Cases

The goal of this section is to prove the generalization of TSFHWA and TSFHWG operators over previous HA operators discussed in Section 6.1. We state some restrictions upon which the proposed HA operators reduced to previous HA operators. In other words, the existing HA operators become special cases of the proposed TSFHWA and TSFHWG operators.

6.5.1. Remark

Consider the TSFHWA and TSFHWG aggregation operators

$TSFHWA(I_1, I_2, I_3, \dots I_m)$

$$= \left(\begin{array}{c} \sqrt[n]{\dfrac{\prod_{j=1}^{m}\left(1+(\gamma-1)\text{ş}_j^n\right)^{w_j} - \prod_{j=1}^{m}\left(1-\text{ş}_j^n\right)^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)\text{ş}_j^n\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(1-\text{ş}_j^n\right)^{w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} i_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-i_j^n\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(i_j^n\right)^{2w_j}}}, \\ \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} d_j'^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-d_j'^n\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(d_j'^n\right)^{2w_j}}} \end{array} \right)$$

$TSFHWG(I_1, I_2, I_3, \dots I_m)$

$$= \left(\begin{array}{c} \dfrac{\sqrt[n]{\gamma}\prod_{j=1}^{m} \text{ş}_j^{w_j}}{\sqrt[n]{\prod_{j=1}^{m}\left(1+(\gamma-1)\left(1-\text{ş}_j^n\right)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(\text{ş}_j^n\right)^{2w_j}}}, \\ \sqrt[n]{\dfrac{\prod_{j=1}^{m}\left(1+(\gamma-1)i_j^n\right)^{w_j} - \prod_{j=1}^{m}\left(1-i_j^n\right)^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)i_j^n\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(1-i_j^n\right)^{w_j}}}, \\ \sqrt[n]{\dfrac{\prod_{j=1}^{m}\left(1+(\gamma-1)d_j'^n\right)^{w_j} - \prod_{j=1}^{m}\left(1-d_j'^n\right)^{w_j}}{\prod_{j=1}^{m}\left(1+(\gamma-1)d_j'^n\right)^{w_j} + (\gamma-1)\prod_{j=1}^{m}\left(1-d_j'^n\right)^{w_j}}} \end{array} \right)$$

The TSFHWA and TSFHWG operators reduced to the HA operators of

- SFNs (this paper); if we take $n = 2$.
- PFNs [58]; if we take $n = 1$.
- Q-ROFNs [93]; if we take the AG i.e. i as zero.
- PyFNs [54]; if we take $n = 2$ and the AG i.e. i as zero.
- IFNs [53]; if we take $n = 1$ and the AG i.e. i as zero.

All this analysis shows the generalization of TSFHWA and TSFHWG operators over the HA operations of IFSs, PyFSs, q-ROFSs, PFSs and SFSs. The HA operators of q-ROFSs, PyFSs and IFSs used two grades (namely MG and NG) to quantify fuzziness while that of PFSs, SFSs and TSFSs use three grades (MG, AG and NG) to quantify fuzziness. Obviously, the results obtained using PFSs, SFSs and TSFSs would be more significant than using duplets as the chances of information loss reduced significantly. Further, as described in Chapter 2 that PFS and SFS have their limitation in range (shown in Figure 6, Figure 7, Figure 8 and Figure 9, in Chapter 0.1 hence proving that TSFS is the only fuzzy framework that gives us better results conveniently with no information loss comparative to other fuzzy frameworks.

6.6. Multi-Attribute Decision Making for Evaluating the Performance of Search and Rescue Robots

This section aims to utilize the proposed HA operators of TSFNs in MADM problem. We utilize both TSFHWA and TSFHWG operators in MADM algorithm followed by numerical demonstration and show the significance of the parameter n in decision making process.

MADM is a process of selecting a best alternative among a list of finite alternatives using AOs and similarity or distance measures etc. Here, the interesting fact is that information about the alternative is in the form of TSFNs which not only discuss the MG and NG but also the AG and RG in uncertain environment. Let us assume the collection of alternatives be denoted by $\mathcal{A}_k$ (k is finite) and the attributes based on which the alternatives are assessed be denoted by $\mathcal{G}_j$ (j is finite) under the weight vector w_j. Let

$\mathcal{D}_{k\times j} = (\mathrm{T})_{k\times j} = (m, i, n)$ be the decision matrix containing the information about the alternatives in the form of TSFNs. The MADM algorithm containing proposed HA operators is demonstrated as follows.

6.6.1. Algorithm

The steps of MADM algorithm using HA operators are described below followed by a comprehensive flowchart in Figure 12.

Step 1: This step involves information gathering form decision makers about the given alternatives. The decision makers provide their evaluation about alternatives in the form of TSFNs keeping the view the attributes.

Step 2: The aim here is to compute the value of n which makes every value in the decision matrix a TSFN. We take the least value of n for which every triplet of the decision matrix is a TSFN.

Step 3: This step is based on normalization of decision matrix provided in Step 1 to ensure that every attribute is of benefit type. If an attribute is of cost type, then we use the definition of complement of TSFN proposed in Definition 2.2.1 to make all values are of benefit type.

Step 4: This step involves the aggregation of normalized data obtained in Step 3 using TSFHWA and TSFHWG operators proposed in Section 6.3 and 6.4 respectively.

Step 5: In Step 5, the following score function of TSFNs proposed in Section 2.3 in order to rank the alternatives based on greater score values.

$$S(I) = \c{s}^n - \mathrm{d}^n . r^n$$

Step 6: In step 6, best alternative is chosen based on the score values obtained in Step 5.

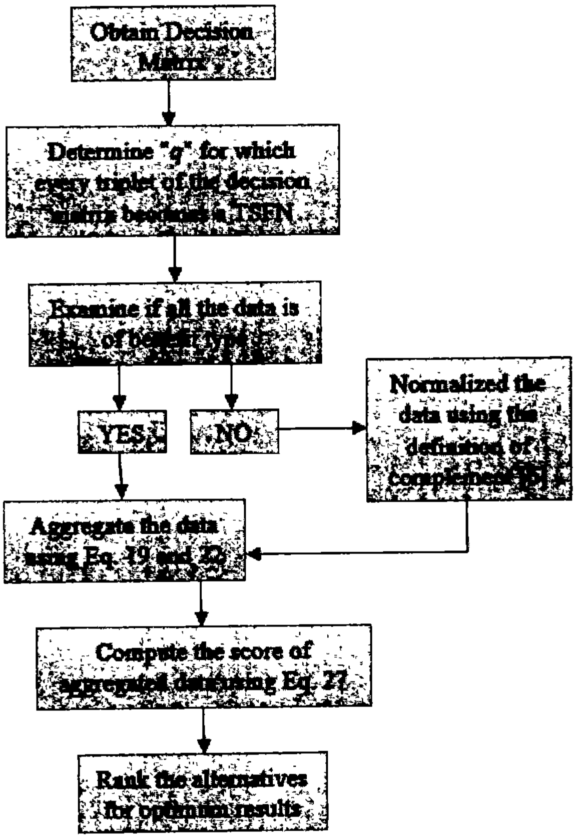

Figure 12 (flow chart of the proposed MADM algorithm)

In this section, we used the MADM algorithm to evaluate the performance of search and rescue robots.

The performance of search and rescue robots [96] is eminent in situations of emergencies. Rescue robots took place of the responders in case of emergency as they can take photos of the scenes and could record some live videos that can help a lot in understanding the intensity of the situation. These search and rescue robots can be helpful in caves, tunnels and wilderness for finding victims or a potential hazard. The motivation to use these search and rescue robots is their aped and completeness of the task without increasing risks to victims or rescuer.

6.6.2. Example

In this example, we consider the problem of evaluating the performance of search and rescue robots. This example is adapted from [97] where the performance of search and rescue robots was examined using intuitionistic fuzzy normalized weighted Bonferroni hormonic mean-based operators. However, in intuitionistic fuzzy environment, only two aspects of human opinion were studied which leads to information loss as the abstinence and refusal degree of human opinion are neglected. Based on existing literature [97], the attributes that have an essential role in the evaluation of search and rescue robots include G_1; *viability*, G_2; *athletic ability*, G_3; *working ability* and G_4; *communication and control capability*. Let the number of search and rescue robots that needs to be assessed be 4, denoted by $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$. The weight vector for the four attributes be $w = (0.32, 0.23, 0.18, 0.27)^T$. The evaluation here involves the opinion of the experts which they provide in the form of a decision matrix keeping in view the four attributes. The opinion of the experts is expressed in the form of TSFNs describing the MG, AG, NG and RG of alternatives based on attributes. The experts from the respective field gave their opinion and a decision matrix is provided in Table 22 and the stepwise computations are provided as:

Step 1: The initial evaluation about the search and rescue robots by the experts in the form of TSFNs is given in Table 22. The three components involved in the triplets respectively represents MG, the AG and the NG.

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	(0.64, 0.34, 0.71)	(0.72,0.45,0.63)	(0.51, 0.59, 0.68)	(0.56, 0.88, 0.71)
$\mathcal{A}_2$	(0.74, 0.69, 0.56)	(0.65, 0.59,0.68)	(0.91, 0.50, 0.63)	(0.87, 0.61, 0.79)
$\mathcal{A}_3$	(0.54, 0.33, 0.39)	(0.75, 0.54, 0.67)	(0.84, 0.66, 0.70)	(0.72,0.88, 0.35)
$\mathcal{A}_4$	(0.77, 0.63,0.45)	(0.88,0.60,0.44)	(0.67,0.77,0.55)	(0.70, 0.40,0.50)

Table 22 (Decision matrix containing opinion of experts about robots)

Step 2: In this step, we compute the value of n for which every triplet given in Table 22 becomes a TSFN. To do so, first we analyze the sum all three components of the triplets i.e. $\varsigma um\ (\varsigma^n, i^n, d'^n)$ for $n = 1, 2, 3, 4, 5, 6 \ldots$ unless we found an n for which $0 \leq \varsigma um\ (\varsigma^n, i^n, d'^n) \leq 1$. After some iterations, we found that all the triplets of Table 22 are TSFNs for $n = 5$ and the sum i.e. $\varsigma um\ (\varsigma^5, i^5, d'^5)$ for each entry of Table 22 is given in Table 23.

0.292341	**0.311188**	**0.251388**	**0.763228**
0.433377	0.332915	0.754526	0.890586
0.058852	0.418234	0.711515	0.726476
0.388375	0.621984	0.456019	0.20956

Table 23 ($m^n + i^n + n^n$ for $n = 5$)

Step 4: Assuming that all the attributes in this problem are of benefit type, we proceed to the aggregation of the decision matrix proposed in Table 22. We use the TSFHWA operator and TSFHWG operator to aggregate the decision matrix and the results are listed in Table 24 below.

	TSFHWA operator	*TSFHWG operator*
$\mathcal{A}_1$	(0.63136459,0.527244, 0.694886)	(0.61476783, 0.70588141, 0.89609987)
$\mathcal{A}_2$	(0.81580694, 0.613081, 0.665947)	(0.80208765, 0.62469032, 0.89583012)
$\mathcal{A}_3$	(0.72527687, 0.556551, 0.479825)	(0.69448866, 0.71784277, 0.88174917)
$\mathcal{A}_4$	(0.77847638, 0.577191, 0.478821)	(0.772873, 0.63365776, 0.87523554)

Table 24 (Aggregated information)

Utilizing the score function, we computed the scores of the aggregated information in Table 25 below.

Scores	*TSFHWA operator*	*TSFHWG operator*
$\mathcal{A}_1$	−0.07829324	−0.87437565
$\mathcal{A}_2$	0.24920943	−0.44116969
$\mathcal{A}_3$	0.17835228	−0.67101258
$\mathcal{A}_4$	0.26568982	−0.41245345

Table 25 (Scores of aggregated information)

Step 5: By comparing the score values obtained in Table 25, the arrangement of alternatives is given in Table 26 followed by their geometrical comparison in Figure 13.

	Ranking
TSFHWA operator	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
TSFHWG operator	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$

Table 26 (Ranking of alternatives)

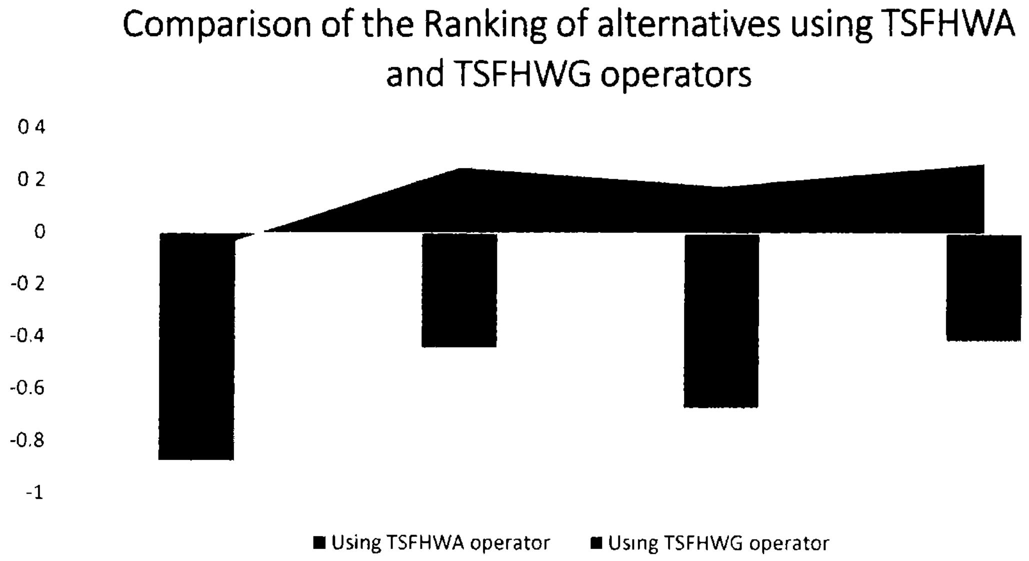

Figure 13 (Comparison of the ranking of the alternatives using TSFHWA and TSFHWG operators)

The ranking analysis shows that the search and rescue robot number $\mathcal{A}_4$ is the most reliable one for dealing with emergency situations. Further, in this case the rankings are the same whether we use TSFHWA operator or TSFHWG operator which is not compulsory always as the result may differ depending upon the data provided by the decision makers.

6.6.3. Effect of "γ" on ranking of alternatives

In the Definition Hamacher operations, the condition on constant γ is that $\gamma > 0$. Therefore, we aim to analyze the effect of variation in γ on the ranking results. For this purpose, we solved the Example 6.6.2 for various values of γ and the ranking results are given in Table 27.

γ	*TSFHWA operator*	*TSFHWG operator*
5	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
10	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
50	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
100	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
105	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$
500	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$

Table 27 (Ranking results for various values of γ)

The analysis from Table 27 clearly indicates that varying γ have no effect on ranking in case of TSFHWA operator from $\gamma = 2$ to $\gamma = 500$. On the other hand, it is observed that the ranking results get changed in case of TSFHWG operator for $\gamma = 105$ which is shown in highlighted text in Table 27. However, from $\gamma = 105$ to $\gamma = 500$, the results remain the same. This shows the significance of variation in γ in decision analysis. At this stage, as for $\gamma \geq 105$, the results are stable so the it must be taken as $\gamma = 105$ for consistent results.

6.6.4. Effect of variation in "n" on ranking results

In TSFHWA and TSFHWG operators, γ is not the only variable constant but the value of "n" also plays an important role. Here, we analyze the impact of "n" on ranking results. In the Example 6.6.2, we have chosen "$n = 5$" because it was the least value of "n" that made each triplet provided by the decision maker a TSFN and the problem could not be solved for any lesser value of "n". Now we analyze the impact of larger values of "n" on ranking results. The ranking results for various values of "n" are shown in Table 28.

n	**TSFHWA operator**	**TSFHWG operator**
5	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
6	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
11	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$
50	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$
100	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$
200	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$	$\mathcal{A}_2 > \mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1$

Table 28 (Ranking results for various values of $\gamma = 2$ and various values of "n")

The ranking results seems consistent after analyzing Table 28 as the ranking pattern gets changed at $n = 6$ in case of TSFHWA operator while the ranking pattern get changed at $n = 11$ in case of TSFHWG operators. However, the ranking pattern after $n = 11$ remains the same in both cases no matter how high we raise the value of "n". This shows a stability in ranking results at $n = 11$. The ranking pattern for below and above the stability points in case of TSFHWA and TSFHWG operator is geometrically shown in Figure 14 and Figure 15 respectively.

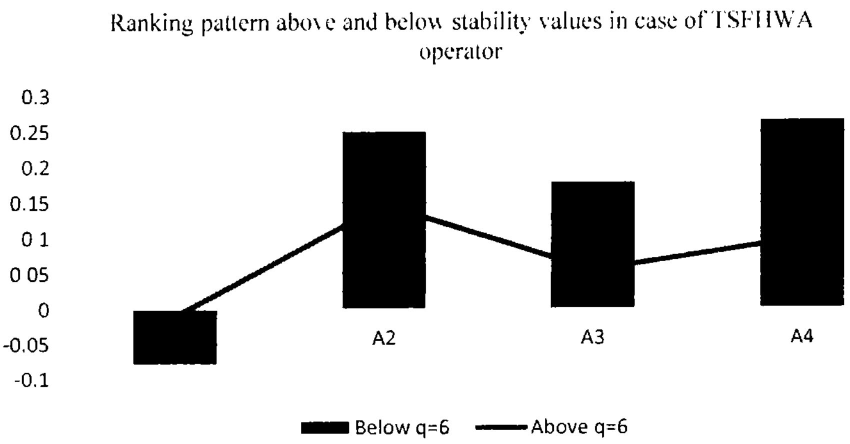

Figure 14 (Ranking pattern above and below stability values in case of TSFHWA operator)

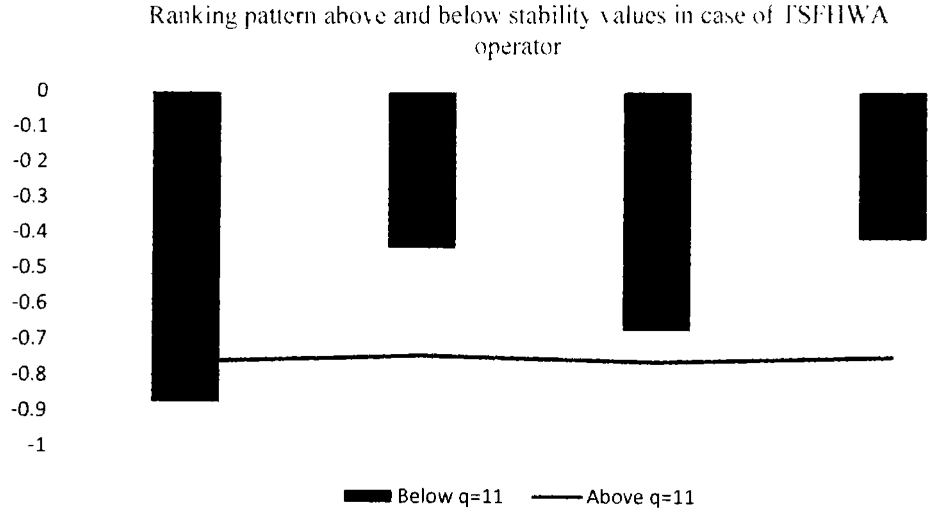

Figure 15 (Ranking pattern above and below stability values in case of TSFHWG operator)

6.7. Comparative Study

The goal of this section is to establish a comparison of the proposed work with that of existing literature to demonstrate the advantages of the new proposed work over existing theory. We establish two types of comparison here.

In our first approach, we take the numerical Example 6.6.2 and dropped the AG from each triplet of the decision matrix. Doing so will reduce the decision matrix given in Table 22 to the environment of q-ROFS. Hence using a special case of proposed HA operators discussed in Section 6.5, we solved the given problem and analyze the results.

In second approach, we apply the AOs proposed in Chapter 2, Chapter 5, AOs proposed by Garg et al. [94] and Liu et al. [95] to the decision matrix provided in Table 22 and analyze the results.

6.7.1. Example

Consider the Example 6.6.2. If we drop the AG from the decision matrix provided in Table 22. We obtained the following decision matrix in Table 29. The rest of the information like weight vector is the same as taken in Example 6.6.2.

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$
$\mathcal{A}_1$	(0.64,0.71)	(0.72,0.63)	(0.51, 0.68)	(0.56, 0.71)
$\mathcal{A}_2$	(0.74, 0.56)	(0.65, 0.68)	(0.91, 0.63)	(0.87, 0.79)
$\mathcal{A}_3$	(0.54, 0.39)	(0.75, 0.67)	(0.84,, 0.70)	(0.72, 0.35)
$\mathcal{A}_4$	(0.77, 0.45)	(0.88,0.44)	(0.67, 0.55)	(0.70, 0.50)

Table 29 (Decision matrix containing opinion of experts about robots after dropping the abstinence grade)

After examining the duplets obtained in Table 29, all the duplets are q-ROFNs for $n = 4$. Now, this type of information can be processed by using the averaging and geometric HA operators of q-ROFSs proposed by Darko and Liang [93]. The aggregated information using q-ROFHWA and q-ROFHWG operators is given in Table 30.

	$q - ROFHWA$ operator	$q - ROFHWG$ operator
$\mathcal{A}_1$	(0.62759479, 0.701401)	(0.60888807, 0.9862327)
$\mathcal{A}_2$	(0.81273797, 0.672338)	(0.77880727, 0.98623275)
$\mathcal{A}_3$	(0.71992669, 0.482812)	(0.68152261, 0.9862327)
$\mathcal{A}_4$	(0.77591318, 0.480725)	(0.75470923, 0.9862327)

Table 30 Aggregated information using q-ROFHWA and q-ROFHWG operators)

The scores of aggregated information obtained in Table 30 are given in Table 31 as follows.

Scores	$q-ROFHWA\ operator$	$q-ROFHWG\ operator$
$\mathcal{A}_1$	−0.10792025	−0.74974108
$\mathcal{A}_2$	0.27938052	−0.2865601
$\mathcal{A}_3$	0.22593439	−0.47844049
$\mathcal{A}_4$	0.32555303	−0.23847328

Table 31 (Score values of alternatives)

The ranking of alternatives based on score values of Table 31 are given in Table 32 as follows.

	$Ranking$
$q-ROFHWA\ operator$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
$q-ROFHWG\ operator$	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$

Table 32 (Ranking of alternatives)

This ranking is different than the one obtained in Table 26 using TSFHWA and TSFHWG operators. This shows that by dropping the AG, some information loss accrued which causes the change in ranking pattern. Therefore, the use of TSFHWA and TSFHWG operators is eminent for modeling of human opinion without any loss of information. AG is always been a part of human opinion and the frameworks of IFS, PyFS and q-ROFS lack such component. This make the T-spherical fuzzy environment and proposed T-spherical fuzzy HA operators more reliable than existing fuzzy frameworks.

6.7.2. Example

In this example, the information of the decision makers about the alternatives in Example 6.6.2 is processed by several existing aggregation tools to assess the validity of the ranking results obtained in Table 26. Here it is important to note that the existing HA operators of

IFSs [53, 57], PyFSs [54, 55], q-ROFSs [93] and PFSs [58] cannot solve the data and information given in Example 6.6.2. Now we use the AOs proposed in Chapter 2, Chapter 4, by Garg et al. [94], Liu et al. [95] on the data provided in Table 22 to compare the results. The ranking results obtained by using the stated existing AOs are listed in Table 33.

Aggregation operator	By	*Ranking*
TSFHWA operator	This Chapter	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
TSFHWG operator	This Chapter	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
TSFWG operator	Chapter 2	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
TSFWA operator	Chapter 4	$\mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_3 > \mathcal{A}_1$
TSFWG operator	Chapter 4	$\mathcal{A}_4 > \mathcal{A}_2 > \mathcal{A}_3 > \mathcal{A}_1$
TSFWGIA operator	Garg et al. [94]	$\mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1 > \mathcal{A}_2$
TSFPMM operator	Liu et al. [95]	$\mathcal{A}_4 > \mathcal{A}_3 > \mathcal{A}_1 > \mathcal{A}_2$

Table 33 (Ranking of alternatives using existing and proposed)

Analyzing the ranking of alternative given in Table 33, it is evident that the results obtained using proposed TSFHWA and TSFHWG operators are consistent as these are much similar to the ranking results obtained by Garg et al. [94] and Liu et al. [95]. This must be noted that the AOs proposed by Garg et al. [94] is an improved version of the AOs proposed in Chapter 2 and Chapter 4 in this thesis.

6.8. Discussion and Advantages of the Proposed Work

Working in T-spherical fuzzy environment allows you to discuss four aspects of human opinion namely MG, AG, NG and RG unlike intuitionistic, Pythagorean and q-rung ortho pair fuzzy environments. Hence, the chance of losing information in T-spherical fuzzy environment will always be negligible than other fuzzy environments. One such example is discussed in Example 6.7.1 where the AG of the T-spherical fuzzy information is dropped, and we observed the consequences in the results. Here, it is worth mentioning that the framework of PFS also allows you to describe the four aspects of human opinion but in a limited range (Discussed in Chapter 1). Therefore, the more convenient option is to use T-spherical fuzzy framework. Some key advantages of the proposed work are stated below:

1. T-spherical fuzzy HA operators generalizes existing HA operators describing the abstinence and refusal degree of human opinion with unlimited range while the existing HA operators either leads to the loss of information or have limited range.
2. The existing HA operators becomes special cases of the proposed T-spherical fuzzy HA operators.
3. T-spherical fuzzy HA operators can solve the problems studied in existing literature while the existing HA operators of IFSs, PyFSs, q-ROFSs and PFSs cannot take over the problems discussed in T-spherical fuzzy environment.

The results obtained using T-spherical fuzzy HA operators are stable than those obtained using existing literature. This is because the proposed HA operators have two variable parameters which are responsible for changing the ranking of alternatives but provides some stable results at a certain stage as discussed in Example 6.6.2.

Chapter 7

Interval Valued T-Spherical Fuzzy Set and its Applications [7]

Expressing the measure of uncertainty, in terms of an interval instead of a crisp number, provides improved results in fuzzy mathematics. Several such concepts are established, including the IVFS, the IVIFS, and the IVPFS. The goal of this chapter is to enhance the T-spherical fuzzy set (TSFS) by introducing the concept of interval-valued TSFS (IVTSFS), which describes the uncertainty measure in terms of the MG, AG, NG, and the RG. The novelty of the IVTSFS over the pre-existing fuzzy structures is analyzed. The basic operations are proposed for IVTSFSs and their properties are investigated. Two AOs for IVTSFSs are developed, including WAA and WGA operators, and their validity is examined using the induction method. Several consequences of new operators, along with their comparative studies, are elaborated. A multi-attribute decision-making method in the context of IVTSFSs is developed, followed by a brief numerical example where the selection of the best policy, among a list of investment policies of a multinational company, is to be evaluated. The advantages of using the framework of IVTSFSs are described theoretically and numerically, hence showing the limitations of pre-existing concepts.

So far, there are several fuzzy frameworks developed. Some of them use crisp values from $[0, 1]$ for expressing the MG, AG, NG such as IFS, PyFS, q-ROFS, PFS, SFS, SVNS and TSFS. There exist some fuzzy frameworks that takes closed subintervals of $[0, 1]$ to

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[7] The work in this chapter is taken from the following published article:
Ullah K., Hassan N., Mahmood T. Jan N. and Hassan M. Evaluation of Investment Policy Based on Multi-Attribute Decision-Making Using Interval Valued T-Spherical Fuzzy Aggregation Operators. Symmetry, 2019, 11 (3), 357-380. https://doi.org/10.3390/sym11030357

represent the MG, AG and NG of the uncertain event such as IVFS, IVIFS, IVPyFS, IVq-ROFS, IVPFS, IVNS etc. The question might arise that how the quantification of MGs using intervals produce better results than by quantifying them using crisp values. To answer the question, in the next section, we analyzed the results of a problem using two approaches and establish a comparison.

7.1. The Significance of Interval-Valued Fuzzy Structures

Sometimes, the impression of an event cannot be characterized by a crisp number. For example, suppose if the MG of an object is $[0.4, 0.8]$ and one chose the MG to 0.7 or 0.5 from the interval. It means some information is totally ignored about the object. The concepts of IFFS, IVIFS and other fuzzy frameworks are introduced to reduce the information loss in such cases. The interval-valued frameworks for the IFS, PyFS, q-ROPFS, PFS and SVNS are developed in [14, 15, 16, 10, 17]. Now, we discussed the significance of using interval-valued frameworks instead of the ordinary fuzzy framework with crisp representations numerically. Note that in our study, $w = (w_1, w_2, w_3, \dots w_m)^T$ shall denote the weight vector, such that $w_j \in [0,1]$ for $j = 1, 2, 3 \dots m$ and $\sum_{j=1}^{m} w_j = 1$.

Consider the following example in which the selection of the best candidate is to be carried out among four candidates $(C_i, i = 1, 2, 3, 4)$ based on four attributes $(G_j, j = 1, 2, 3, 4)$ under the weight vector $w = (0.22, 0.34, 0.27, 0.17)^T$. In this case, the evaluation information of the candidates is in the form of IVIFNs, which are given in Table 34.

[redacted]				
[redacted]	$\begin{pmatrix}[0.2, 0.3],\\ [0.0, 0.5]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.6],\\ [0.1, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.4, 0.5],\\ [0.2, 0.4]\end{pmatrix}$	$\begin{pmatrix}[0.7, 0.8],\\ [0.1, 0.2]\end{pmatrix}$
[redacted]	$\begin{pmatrix}[0.6, 0.7],\\ [0.2, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.6],\\ [0.0, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6, 0.7],\\ [0.2, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.6, 0.7],\\ [0.1, 0.2]\end{pmatrix}$
[redacted]	$\begin{pmatrix}[0.4, 0.5],\\ [0.3, 0.4]\end{pmatrix}$	$\begin{pmatrix}[0.3, 0.6],\\ [0.1, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.6],\\ [0.3, 0.4]\end{pmatrix}$	$\begin{pmatrix}[0.6, 0.7],\\ [0.1, 0.3]\end{pmatrix}$
[redacted]	$\begin{pmatrix}[0.6, 0.7],\\ [0.2, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.5, 0.7],\\ [0.1, 0.3]\end{pmatrix}$	$\begin{pmatrix}[0.7, 0.8],\\ [0.1, 0.2]\end{pmatrix}$	$\begin{pmatrix}[0.3, 0.4],\\ [0.1, 0.2]\end{pmatrix}$

Table 34 The decision matrix where $(C_j, j = 1,2,3,4)$ denotes alternatives and $(G_k, k = 1,2,3,4)$ denotes the attributes.

To aggregate the data provided in Table 34 based on IVIFNS denoted by I_j, Xu and Cai [37] developed the following aggregation tool:

$$IVIFWA(I_1, I_2, I_3 \dots I_m)$$

$$= \left(\left[1 - \prod_{j=1}^{m}\left(1 - (\text{ş}_j^l)\right)^{w_j}, 1 - \prod_{j=1}^{m}\left(1 - (\text{ş}_j^u)\right)^{w_j}\right], \left[\prod_{j=1}^{m}(\text{d'}_j^l)^{w_j}, \prod_{j=1}^{m}(\text{d'}_j^u)^{w_j}\right]\right)$$

The aggregated results are:

$$IVIFWA(I_{11}, I_{12}, I_{13}, I_{14}) = ([0.6235, 0.7569], [0.0000, 0.3386])$$

$$IVIFWA(I_{21}, I_{22}, I_{23}, I_{24}) = ([0.6777, 0.8180], [0.0000, 0.2800])$$

$$IVIFWA(I_{31}, I_{32}, I_{33}, I_{34}) = ([0.5730, 0.7745], [0.1713, 0.3454])$$

$$IVIFWA(I_{41}, I_{42}, I_{43}, I_{44}) = ([0.6493, 0.8352], [0.1165, 0.2510])$$

The following score of IVIFN, $I = ([\text{ş}^l, \text{ş}^u], [\text{d'}^l, \text{d'}^u])$ is used to get the score values of the aggregated information.

$$SC(I) = \frac{1}{2}(ş^{l} - d'^{l} + ş^{u} - d'^{u})$$

Using this score's functions, we have:

$SC(C_1) = 0.5209, SC(C_2) = 0.6079, SC(C_3) = 0.4154, SC(C_4) = 0.5585$

Clearly,

$$SC(C_2) > SC(C_4) > SC(C_1) > SC(C_3)$$

Hence, C_2 is the suitable candidate, according to the MADM method based on the IVIFWA operators. Now our target is to convert the information provided in Table 34 to intuitionistic fuzzy environment. By doing so, some information loss will occur.

Assigning a crisp value to the MG and NG of the above problem reduces the information in Table 34 in the context of IFSs. Such information can be aggregated using the aggregation operators proposed by Xu [29]. The decision matrix for the problem discussed above in the context of IFSs is given in Table 2.

	G_1	G_2	G_3	G_4
C_1	$(0.3, 0.5)$	$(0.6, 0.3)$	$(0.5, 0.4)$	$(0.8, 0.2)$
C_2	$(0.7, 0.3)$	$(0.6, 0.3)$	$(0.7, 0.3)$	$(0.7, 0.2)$
C_3	$(0.5, 0.4)$	$(0.6, 0.3)$	$(0.6, 0.4)$	$(0.7, 0.3)$
C_4	$(0.6, 0.2)$	$(0.5, 0.1)$	$(0.7, 0.1)$	$(0.3, 0.1)$

Table 35 The decision matrix where $(C_i, i = 1,2,3,4)$ denotes alternatives and $(G_j, j = 1,2,3,4)$ denotes the attributes.

The WAA operator of the IFSs, proposed by Xu [29], is given by:

$$IFWA(I_1, I_2, I_3 \dots I_m) = \left(1 - \prod_{j=1}^{m}\left(1 - (ş_j)\right)^{w_j}, \prod_{j=1}^{m}(d'_j)^{w_j}\right)$$

$IFWA(I_{11}, I_{12}, I_{13}, I_{14}) = (0.7569, 0.3386)$

$IFWA(I_{21}, I_{22}, I_{23}, I_{24}) = (0.8180, 0.2800)$

$IFWA(I_{31}, I_{32}, I_{33}, I_{34}) = (0.7745, 0.3454)$

$IFWA(I_{41}, I_{42}, I_{43}, I_{44}) = (0.7489, 0.1165)$

For the ranking of IFN I, the score function developed in [29] is given by:

$$SC(I) = \text{ş} - \text{d}'$$

Using this score function, we have:

$SC(C_1) = 0.4183, SC(C_2) = 0.5380, SC(C_3) = 0.4291, SC(C_4) = 0.6324$

Clearly, $SC(c_4) > SC(c_2) > SC(c_3) > SC(c_1)$

Thus, C_4 is the suitable candidate, according to the MADM method based on the IFWA operators. It can be clearly seen that the results are drastically changed, and this is only because of the information loss due to converting the data from interval valued intuitionistic fuzzy environment to intuitionistic fuzzy environment. This shows the importance of using interval-valued fuzzy frameworks over crisp-valued fuzzy frameworks.

The concept of the SFS and TSFS developed in Chapter 2 of this thesis is a generalization of the q-ROFS, PyFS, PFS, IFS, and FS. Keeping the importance of interval valued fuzzy frameworks in view of above discussed example, we aim to introduce the concept of interval valued TSFS (IVTSFS). The concept of the IVTSFS will provide flexibility in modeling the MG, AG, NG and RG in terms of closed subintervals rather than by a crisp number. So far, we have studied the applications of TSFSs in medical diagnosis, pattern recognition, MADM and clustering and it is evident that the framework of IVTSFS will provide more better results as the MG, AG, NG and RG are expressed in terms of closed subintervals instead of crisp numbers.

In the next section, the definition of the IVTSFS is proposed and its superiority over the existing fuzzy concepts is shown, with the help of some useful remarks.

7.2. Interval Valued T-Spherical Fuzzy Set

In this section, we aim to define the concept of IVTSFS and show its generalization and superiority over the previously defined fuzzy frameworks. We also aim to develop the basic set theoretic operations of IVTSFSs and compare it with previously defined operations. For ranking purpose, the score function of IVTSFS is also introduced and its significance is discussed. Note that $IVTSFS(X)$ shall denote the set of all the IVTSFSs.

7.2.1. Definition

An IVTSFS has the shape $I = \{(ş(\varkappa), i(\varkappa), \text{d}'(\varkappa)) : \varkappa \in X\}$, where $ş(\varkappa)$, $i(\varkappa)$, and $\text{d}'(\varkappa)$ are mappings from X to a closed subinterval of $[0, 1]$, such that $ş(\varkappa) = [ş^l(\varkappa), ş^u(\varkappa)]$, $i(\varkappa) = [i^l(\varkappa), i^u(\varkappa)]$, and $\text{d}'(\varkappa) = [\text{d}'^l(\varkappa), \text{d}'^u(\varkappa)]$ with a restriction that $0 \leq şum\ ((ş^u)^n, (i^u)^n, (\text{d}'^u)^n) \leq 1$ for some $n \in \mathbb{Z}^+$. The term RG is defined by

$$r(\varkappa) = ([r^l(\varkappa), r^u(\varkappa)]) = \left(\begin{bmatrix} (1 - (ş^u)^n(\varkappa) - (i^u)^n(\varkappa) - (\text{d}'^u)^n)^{\frac{1}{n}}, \\ (1 - (ş^l)^n(\varkappa) - (i^l)^n(\varkappa) - (\text{d}'^l)^n)^{\frac{1}{n}} \end{bmatrix}\right)$$

and the pair $(ş(\varkappa), i(\varkappa), \text{d}'(\varkappa))$ is considered to be the interval valued TSFN (IVTSFN).

The existing frameworks become the special cases of the IVTSFS. Consider the following theorem:

7.2.2. Theorem

An IVTSFS reduces to:

1. TSFS: if we consider $ş^l = ş^u, i^l = i^u$ and $d'^l = d'^u$. [This thesis]
2. IVSFS: if we consider $n = 2$. [This thesis]
3. SFS: if we consider $n = 2$ and $ş^l = ş^u, i^l = i^u$, and $d'^l = d'^u$. [This thesis]
4. IVPFS: if we consider $n = 1$. [10]
5. PFS: if we consider $n = 1$ and $ş^l = ş^u, i^l = i^u$ and $d'^l = d'^u$. [10]
6. IVq-ROPFS: if we consider $i^l = i^u = 0$. [16]
7. q-ROPFS: if we consider $ş^l = ş^u, i^l = i^u$ and $d'^l = d'^u$. [8]
8. IVPyFS: if we consider $n = 2$ and $i^l = i^u = 0$. [15]
9. PyFS: if we consider $n = 2$ and $ş^l = ş^u, i^l = i^u = 0$ and $d'^l = d'^u$. [7]
10. IVIFS: if we consider $n = 1$ and $i^l = i^u = 0$.[14]
11. IFS: if we consider $n = 1$ and $ş^l = ş^u, i^l = i^u = 0$ and $d'^l = d'^u$. [5]
12. IVFS: if we consider $n = 1$ and $i^l = i^u = d'^l = d'^u = 0$. [13]
13. FS: if we consider $n = 1$ and $ş^l = ş^u, i^l = i^u = 0 = d'^l = d'^u$. [1]

Next, the operations of the sum, product, scalar multiplications, and power operation for IVTSFNs will be introduced. With the help of a remark, it is shown that these operations are the generalizations of the operations of existing concepts under some restrictions.

7.2.3. Definition

For $I_1, I_2 \in IVTSFS(X)$, let $I_1 = \left([\c{s}_1^l, \c{s}_1^u], [i_1^l, i_1^u], [d_1'^l, d_1'^u]\right)$ and $I_2 = \left([\c{s}_2^l, \c{s}_2^u], [i_2^l, i_2^u], [d_2'^l, d_2'^u]\right)$ be two IVTSFNs and $\lambda > 0$. Then, the operations of the sum, product, scalar multiplications, and power are defined as:

1. $$I_1 \oplus I_2 = \left\{\left(\left[\begin{array}{c}\sqrt[n]{\c{s}_1^{l^n} + \c{s}_2^{l^n} - \c{s}_1^{l^n}.\c{s}_2^{l^n}},\\ \sqrt[n]{\c{s}_1^{u^n} + \c{s}_2^{u^n} - \c{s}_1^{u^n}.\c{s}_2^{u^n}}\end{array}\right], [\, i_1^l.\, i_2^l, i_1^u.\, i_2^u], [d_1'^l.\, d_2'^l, d_1'^u.\, d_2'^u]\right)\right\}$$

2. $$I_1 \otimes I_2 = \left\{\left([\c{s}_1^l.\c{s}_2^l, \c{s}_1^u.\c{s}_2^u], [\, i_1^l.\, i_2^l, i_1^u.\, i_2^u], \left[\begin{array}{c}\sqrt[n]{d_1'^{l^n} + d_2'^{l^n} - d_1'^{l^n}.d_2'^{l^n}},\\ \sqrt[n]{d_1'^{u^n} + d_2'^{u^n} - d_1'^{u^n}.d_2'^{u^n}}\end{array}\right]\right)\right\}$$

3. $$\lambda I_1 = \left(\left[\sqrt[n]{1 - (1 - \c{s}_1^{l^n})^{\lambda}}, \sqrt[n]{1 - (1 - \c{s}_1^{u^n})^{\lambda}}\right], \left[(i_1^l)^{\lambda}, (i_1^u)^{\lambda}\right], \left[(d_1'^l)^{\lambda}, (d_1'^u)^{\lambda}\right]\right)$$

4. $$I_1^{\lambda} = \left(\left[(\c{s}_1^l)^{\lambda}, (\c{s}_1^u)^{\lambda}\right], \left[(i_1^l)^{\lambda}, (i_1^u)^{\lambda}\right], \left[\sqrt[n]{1 - (1 - d_1'^{l^n})^{\lambda}}, \sqrt[n]{1 - (1 - d_2'^{u^n})^{\lambda}}\right]\right)$$

7.2.4. Remark

The operations in Definition 7.2.3 become valid for:

- TSFSs: if we consider $\c{s}^l = \c{s}^u, i^l = i^u$, and $d'^l = d'^u$. [This thesis]
- IVSFSs: if we consider $n = 2$. [This thesis]
- SFSs: if we consider $n = 2$ and $\c{s}^l = \c{s}^u, i^l = i^u$ and $d'^l = d'^u$. [This thesis]
- IVPFSs: if we consider $n = 1$. [This thesis]
- PFSs: if we consider $n = 1$ and $\c{s}^l = \c{s}^u, i^l = i^u$ and $d'^l = d'^u$. [This thesis]
- IVq-ROPFSs: if we consider $i^l = i^u = 0$. [This thesis]

- q-ROPFSs: if we consider $ş^l = ş^u, i^l = i^u$ and $d'^l = d'^u$. [85]
- IVPyFSs: if we consider $n = 2$ and $i^l = i^u = 0$. [47]
- PyFSs: if we consider $n = 2$ and $ş^l = ş^u, i^l = i^u = 0$ and $d'^l = d'^u$. [45, 46]
- IVIFSs: if we consider $n = 1$ and $i^l = i^u = 0$. [37]
- IFSs: if we consider $n = 1$ and $ş^l = ş^u, i^l = i^u = 0$ and $d'^l = d'^u$. [29, 30]
- IVFSs: if we consider $n = 1$ and $i^l = i^u = d'^l = d'^u = 0$.
- FSs: if we consider $n = 1$ and $ş^l = ş^u, i^l = i^u = 0 = d'^l = d'^u$.

7.2.5. Theorem

The following properties hold true for $Ą, \mathcal{B} \in IVTSFS(X)$, where $\lambda, \lambda_1, \lambda_3 > 0$.

1. $I_1 \oplus I_2 = I_2 \oplus I_1$
2. $I_1 \otimes I_2 = I_2 \otimes I_1$
3. $\lambda(I_1 \oplus I_2) = \lambda I_1 \oplus \lambda I_2$
4. $(I_1 \otimes I_2)^\lambda = {I_1}^\lambda \otimes {I_2}^\lambda$
5. $\lambda_1 I_1 \oplus \lambda_2 I_1 = (\lambda_1 + \lambda_2) I_1$
6. ${I_1}^{\lambda_1} \otimes {I_1}^{\lambda_2} = {I_1}^{\lambda_1 + \lambda_2}$
7. $({I_1}^c)^\lambda = (\lambda I_1)^c$
8. $\lambda({I_1}^c) = \left({I_1}^\lambda\right)^c$
9. ${I_1}^c \oplus {I_2}^c = (I_1 \otimes I_2)^c$
10. ${I_1}^c \otimes {I_2}^c = (I_1 \oplus I_2)^c$

Proof. Trivial.

The ranking of two numbers in fuzzy algebraic structures has certain importance, especially in MADM, medical diagnosis and in pattern recognition. There are several rules defined for the ranking of two numbers of a certain fuzzy framework studied by a number of scientists. Here, we follow the idea of Joshi and Kumar [43], to define a score function for an IVTSFN.

7.2.6. Definition

For an IVTSFN $I = ([\c{s}^l, \c{s}^u], [i^l, i^u], [d'^l, d'^u])$, the score function is defined as:

$$SC(I) = \frac{(\c{s}^l)^n\left(1-(i^l)^n-(d'^l)^n\right)+(\c{s}^u)^n\left((1-(i^u)^n-(d'^u)^n\right))}{3} \text{ and } SC(I) \in [0,1]$$

7.2.7. Remark

Using the restrictions stated in Theorem 1, the score value for the TSFS, IVSFS, SFS, IVPFS, and PFS, as well as other fuzzy algebraic structures, can be obtained analogously.

7.2.8. Example

Let $I_1 = ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8])$ and $I_2 = ([0.2, 0.4], [0.2, 0.3], [0.3, 0.8])$ be two IVTSFNs for $n = 4$. Then, using the score function, we have:

$$SC(I_1) = 0.0227 \text{ and } SC(I_2) = 0.0066$$

Clearly, $SC(I_1) > SC(I_2)$. So $I_1 > I_2$.

7.3. Averaging Aggregation Operators for Interval-Valued T-Spherical Fuzzy Sets

In this section, some AOs for IVTSFSs are proposed. These operators include the interval valued T-spherical fuzzy weighted averaging (IVTSFWA) operator, interval valued T-spherical fuzzy ordered weighted averaging (IVTSFOWA) operator and interval valued T-spherical fuzzy hybrid averaging (IVTSFHA) operator. The validation and basic properties of these aggregation tools are studied and supported by numerical examples.

7.3.1. Definition

The IVTSFWA operator for IVTSFNs I_j $(j = 1, 2, 3 \ldots m)$ is of the form:

$$IVTSFWA(I_1, I_2, I_3 \ldots I_m) = \sum_{j=1}^{m} w_j I_j$$

7.3.2. Theorem

The aggregated value of IVTSFNs I_j, using the IVTSFWA operator, is an IVTSFN which is given as:

$$IVTSFWA(I_1, I_2, I_3 \ldots I_m) = \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (s_j^l)^n\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (s_j^u)^n\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{m} (i_j^l)^{w_j}, \prod_{j=1}^{m} (i_j^u)^{w_j} \right], \left[\prod_{j=1}^{m} (d_j^l)^{w_j}, \prod_{j=1}^{m} (d_j^u)^{w_j} \right] \end{array} \right)$$

Proof: Using mathematical induction.

For $m = 2$.

$$w_1 I_1 = \left(\left[\sqrt[n]{1-\left(1-\text{ş}_{I_1}^{l\ n}\right)^{w_1}}, \sqrt[n]{1-\left(1-\text{ş}_{I_1}^{u\ n}\right)^{w_1}}\right], \left[\left(i_{I_1}^{l}\right)^{w_1}, \left(i_{I_1}^{u}\right)^{w_1}\right], \left[\left(\text{d'}_{I_1}^{l}\right)^{w_1}, \left(\text{d'}_{I_1}^{u}\right)^{w_1}\right]\right)$$

And

$$w_2 I_2 = \left(\left[\sqrt[n]{1-\left(1-\text{ş}_{I_2}^{l\ n}\right)^{w_2}}, \sqrt[n]{1-\left(1-\text{ş}_{I_2}^{u\ n}\right)^{w_2}}\right], \left[\left(i_{I_2}^{l}\right)^{w_2}, \left(i_{I_2}^{u}\right)^{w_2}\right], \left[\left(\text{d'}_{I_2}^{l}\right)^{w_2}, \left(\text{d'}_{I_2}^{u}\right)^{w_2}\right]\right)$$

$w_1 I_1 \oplus w_2 I_2$

$$= \left(\begin{array}{c} \left(\left[\sqrt[n]{1-\left(1-\text{ş}_{I_1}^{l\ n}\right)^{w_1}}, \sqrt[n]{1-\left(1-\text{ş}_{I_1}^{u\ n}\right)^{w_1}}\right], \left[\left(i_{I_1}^{l}\right)^{w_1}, \left(i_{I_1}^{u}\right)^{w_1}\right], \left[\left(\text{d'}_{I_1}^{l}\right)^{w_1}, \left(\text{d'}_{I_1}^{u}\right)^{w_1}\right]\right) \oplus \\ \left(\left[\sqrt[n]{1-\left(1-\text{ş}_{I_2}^{l\ n}\right)^{w_2}}, \sqrt[n]{1-\left(1-\text{ş}_{I_2}^{u\ n}\right)^{w_2}}\right], \left[\left(i_{I_2}^{l}\right)^{w_2}, \left(i_{I_2}^{u}\right)^{w_2}\right], \left[\left(\text{d'}_{I_2}^{l}\right)^{w_2}, \left(\text{d'}_{I_2}^{u}\right)^{w_2}\right]\right) \end{array}\right)$$

$$= \left(\begin{array}{c} \left[\begin{array}{c} \sqrt[n]{\left(\sqrt[n]{1-\left(1-\text{ş}_{I_1}^{l\ n}\right)^{w_1}}\right)^n + \left(\sqrt[n]{1-\left(1-\text{ş}_{I_2}^{l\ n}\right)^{w_2}}\right)^n - \sqrt[n]{1-\left(1-\text{ş}_{I_1}^{l\ n}\right)^{w_1}} . \sqrt[n]{1-\left(1-\text{ş}_{I_2}^{l\ n}\right)^{w_2}}}, \\ \sqrt[n]{\left(\sqrt[n]{1-\left(1-\text{ş}_{I_1}^{u\ n}\right)^{w_1}}\right)^n + \left(\sqrt[n]{1-\left(1-\text{ş}_{I_2}^{u\ n}\right)^{w_2}}\right)^n - \sqrt[n]{1-\left(1-\text{ş}_{I_1}^{u\ n}\right)^{w_1}} . \sqrt[n]{1-\left(1-\text{ş}_{I_2}^{u\ n}\right)^{w_2}}} \end{array}\right] \\ \left[\left(i_{I_1}^{l}\right)^{w_1} . \left(i_{I_2}^{l}\right)^{w_2}, \left(i_{I_1}^{u}\right)^{w_1} . \left(i_{I_2}^{u}\right)^{w_2}\right] \\ \left[\left(\text{d'}_{I_1}^{l}\right)^{w_1} . \left(\text{d'}_{I_2}^{l}\right)^{w_2}, \left(\text{d'}_{I_1}^{u}\right)^{w_1} . \left(\text{d'}_{I_2}^{u}\right)^{w_2}\right] \end{array}\right)$$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1-\left(1-\text{ş}_{I_1}^{l\ n}\right)^{w_1} . \left(1-\text{ş}_{I_2}^{l\ n}\right)^{w_2}}, \sqrt[n]{1-\left(1-\text{ş}_{I_1}^{u\ n}\right)^{w_1} . \left(1-\text{ş}_{I_2}^{u\ n}\right)^{w_2}}\right] \\ \left[\left(i_{I_1}^{l}\right)^{w_1} . \left(i_{I_2}^{l}\right)^{w_2}, \left(i_{I_1}^{u}\right)^{w_1} . \left(i_{I_2}^{u}\right)^{w_2}\right] \\ \left[\left(\text{d'}_{I_1}^{l}\right)^{w_1} . \left(\text{d'}_{I_2}^{l}\right)^{w_2}, \left(\text{d'}_{I_1}^{u}\right)^{w_1} . \left(\text{d'}_{I_2}^{u}\right)^{w_2}\right] \end{array}\right)$$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1-\prod_{j=1}^{2}\left(1-\text{ş}_{I_j}^{l\ n}\right)^{w_j}}, \sqrt[n]{1-\prod_{j=1}^{2}\left(1-\text{ş}_{I_j}^{u\ n}\right)^{w_j}}\right], \\ \left[\prod_{j=1}^{2}\left(i_{I_j}^{l\ n}\right)^{w_j}, \prod_{j=1}^{2}\left(i_{I_j}^{u\ n}\right)^{w_j}\right], \left[\prod_{j=1}^{2}\left(\text{d'}_{I_j}^{l\ n}\right)^{w_j}, \prod_{j=1}^{2}\left(\text{d'}_{I_j}^{u\ n}\right)^{w_j}\right] \end{array}\right)$$

Hence, for $m = 2$, the result is true.

Supposedly, we claim that the result is valid for $m = k$, i.e.,

$IVTSFWA(I_1, I_2, I_3 \dots I_k)$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{k} \left(1 - \S_j^{l\,n}\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{k} \left(1 - \S_j^{u\,n}\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{k} \left(i_{I_j}^{l\,n}\right)^{w_j}, \prod_{j=1}^{k} \left(i_{I_j}^{u\,n}\right)^{w_j} \right], \left[\prod_{j=1}^{k} \left(d'^{l\,n}_{I_j}\right)^{w_j}, \prod_{j=1}^{k} \left(d'^{u\,n}_{I_j}\right)^{w_j} \right] \end{array} \right)$$

To prove the result for $m = k + 1$, consider:

$$IVTSFWA(I_1, I_2, I_3 \dots I_k, I_{k+1}) = \sum_{j=1}^{k+1} w_j I_j = \sum_{j=1}^{k} w_j I_j \oplus w_{k+1} I_{k+1}$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{k} \left(1 - \S_j^{l\,n}\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{k} \left(1 - \S_j^{u\,n}\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{k} \left(i_{I_j}^{l\,n}\right)^{w_j}, \prod_{j=1}^{k} \left(i_{I_j}^{u\,n}\right)^{w_j} \right], \left[\prod_{j=1}^{k} \left(d'^{l\,n}_{I_j}\right)^{w_j}, \prod_{j=1}^{k} \left(d'^{u\,n}_{I_j}\right)^{w_j} \right] \end{array} \right) \oplus \\ \left(\begin{array}{c} \left[\sqrt[n]{1 - \left(1 - \S_{k+1}^{l\,n}\right)^{w_{k+1}}}, \sqrt[n]{1 - \left(1 - \S_{k+1}^{u\,n}\right)^{w_{k+1}}} \right], \\ \left[\left(i_{I_{k+1}}^{l\,n}\right)^{w_{k+1}}, \left(i_{I_{k+1}}^{u\,n}\right)^{w_{k+1}} \right], \left[\left(d'^{l\,n}_{I_{k+1}}\right)^{w_{k+1}}, \left(d'^{u\,n}_{I_{k+1}}\right)^{w_{k+1}} \right] \end{array} \right) \end{array} \right)$$

Finally, we get:

$IVTSFWA(I_1, I_2, I_3 \dots I_k, I_{k+1})$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{k+1} \left(1 - \S_j^{l\,n}\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{k+1} \left(1 - \S_j^{u\,n}\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{k+1} \left(i_{I_j}^{l\,n}\right)^{w_j}, \prod_{j=1}^{k+1} \left(i_{I_j}^{u\,n}\right)^{w_j} \right], \left[\prod_{j=1}^{k+1} \left(d'^{l\,n}_{I_j}\right)^{w_j}, \prod_{j=1}^{k+1} \left(d'^{u\,n}_{I_j}\right)^{w_j} \right] \end{array} \right)$$

Hence, the result is valid for $m = k + 1$. So, using the induction method, it is proven that the result is valid for all $m \in \mathbb{Z}^+$. In the following theorem, we state that the IVTSFWA operator satisfies the basic characteristics of an AOs.

7.3.3. Theorem

For the IVTSFWA operator, the following properties hold true:

1. Idempotency:

If $I_j = I$ for all $j = 1, 2, 3, \dots m$, then $IVTSFWA(I_1, I_2, I_3 \dots I_m) = I$

Proof: Let $I_j = I = ([\c{s}^l, \c{s}^u], [i^l, i^u], [d'^l, d'^u])$ for all j, then:

$IVTSFWA(I, I, I \dots I)$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{m} (1 - (\c{s}^l)^n)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} (1 - (\c{s}^u)^n)^{w_j}} \right], \\ \left[\prod_{j=1}^{m} (i^l)^{w_j}, \prod_{j=1}^{m} (i^u)^{w_j} \right], \left[\prod_{j=1}^{m} (d'^l)^{w_j}, \prod_{j=1}^{m} (d'^u)^{w_j} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1 - (1 - (\c{s}^l)^n)^{\sum_{i=1}^{m} w_i}}, \sqrt[n]{1 - (1 - (\c{s}^u)^n)^{\sum_{i=1}^{m} w_i}} \right], \\ \left[(i^l)^{\sum_{i=1}^{m} w_i}, (i^u)^{\sum_{i=1}^{m} w_i} \right], \left[(d'^l)^{\sum_{i=1}^{m} w_i}, (d'^u)^{\sum_{i=1}^{m} w_i} \right] \end{array} \right)$$

$$= ([\c{s}^l, \c{s}^u], [i^l, i^u], [d'^l, d'^u]) = I$$

2. Boundedness:

If $I^- = \min_j(\c{s}_j, i_j\, d'_j)$ and $I^+ = \max_j(\c{s}_j, i_j\, d'_j)$ are the least and greatest IVTSFNs, then:

$$I^- \leq IVTSFWA(I_1, I_2, I_3 \dots I_m) \leq I^+$$

Proof: Straightforward.

3. Monotonicity:

For the two collections of IVTSFNs, I_j and P_j. If $I_j \leq P_j \; \forall \, j$, then:

$$IVTSFWA(I_1, I_2, I_3 \dots I_m) \leq IVTSFWA(P_1, P_2, P_3 \dots P_m)$$

Proof: As

$$IVTSFWA(I_1, I_2, I_3 \dots I_m) = w_1 I_1 + w_2 I_2 + w_3 I_3 + \cdots w_m I_m$$

and

$$IVTSFWA(P_1, P_2, P_3 \dots P_m) = w_1 P_1 + w_2 P_2 + w_3 P_3 + \cdots w_m P_m$$

As $I_j \leq P_j \; \forall \, j$. So,

$$IVTSFWA(I_1, I_2, I_3 \dots I_m) \leq IVTSFWA(P_1, P_2, P_3 \dots P_m)$$

In MADM problems, sometimes the ordered position of the information has importance and needs to be weighted. For this reason, the concept of the IVTSFOWA operator is proposed.

7.3.4. Definition

The IVTSFOWA operator for IVTSFNs I_j $(j = 1, 2, 3 \dots m)$ is of the form:

$$IVTSFOWA(I_1, I_2, I_3 \dots I_m) = \sum_{j=1}^{m} w_j I_{\alpha(j)}$$

where $I_{\alpha(j)}$ denotes the j^{th} largest value of I_j.

7.3.5. Theorem

The aggregated value of the IVTSFNs, I_j, using an IVTSFOWA operator, is an IVTSFN and is given as:

$$IVTSFOWA(I_1, I_2, I_3 \dots I_m)$$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (s_{\alpha(j)}^{l})^{n}\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (s_{\alpha(j)}^{u})^{n}\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{m} \left(i_{\alpha(j)}^{l}\right)^{w_j}, \prod_{j=1}^{m} \left(i_{\alpha(j)}^{l}\right)^{w_j} \right], \left[\prod_{j=1}^{m} \left(d_{\alpha(j)}'^{l}\right)^{w_j}, \prod_{j=1}^{m} \left(d_{\alpha(j)}'^{l}\right)^{w_j} \right] \end{array} \right)$$

Proof: This result can analogously be proved.

7.3.6. Example

Let $I_1 = ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8])$, $I_2 = ([0.2, 0.4], [0.2, 0.6], [0.3, 0.8])$, $I_3 = ([0.3, 0.3], [0.4, 0.5], [0.7, 0.8])$, and $I_4 = ([0.1, 0.5], [0.2, 0.2], [0.3, 0.5])$ be the four IVTSFNs for $n = 4$ and let $w = (0.22, 0.34, 0.27, 0.17)^T$ be the weight vector. Then, using the score function defined in Definition 7.2.6, we have $SC(I_1) = 0.0227$, $SC(I_2) = 0.0055$, $SC(I_3) = 0.0089$, and $SC(I_4) = 0.0066$. So, the IVTSFNs are ranked as: $I_1 > I_3 > I_4 > I_2$ and we get:

$$I_{\alpha(1)} = ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8])$$

$$I_{\alpha(2)} = ([0.3, 0.3], [0.4, 0.5], [0.7, 0.8])$$

$$I_{\alpha(3)} = ([0.1, 0.5], [0.2, 0.2], [0.3, 0.5])$$

$$I_{\alpha(4)} = ([0.2, 0.4], [0.2, 0.6], [0.3, 0.8])$$

Now, the aggregated value of I_1, I_2, I_3, and I_4, using the IVTSFOWA operator, is given by:

$IVTSFOWA(I_1, I_3, I_4, I_2) = ([0.2691, 0.4573], [0.2532, 0.4485], [0.3994, 0.8000])$

As discussed, whenever the ordered position of IVTSFNs is necessary, we use the IVTSFOWA operator. If along with this, the argument also needs to be weighted, then we develop the IVTSFHA operator.

7.3.7. Definition

The IVTSFHA operator for the IVTSFN I_j $(j = 1, 2, 3 \dots m)$ is of the form:

$$IVTSFHA(I_1, I_2, I_3 \dots I_m) = \sum_{j=1}^{m} w_j \dot{I}_{\alpha(j)}$$

where $\dot{I}_j$ can be computed as $\dot{I}_j = m\mathfrak{w} I_j$, $\dot{I}_{\alpha(j)}$ denotes the j^{th} largest value of $\dot{I}_j$, and further $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3 \dots \mathfrak{w}_n)^T$ is the weight vector of $\mathrm{I_j}$.

7.3.8. Theorem

The aggregated value of the IVTSFNs I_j, using the IVTSFHA operator, is an IVTSFN and is given as:

$$IVTSFHA(I_1, I_2, I_3 \dots I_m)$$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (\dot{\mathrm{s}}^{l}_{\alpha(j)})^n\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (\dot{\mathrm{s}}^{u}_{\alpha(j)})^n\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{m} (\dot{i}^{l}_{\alpha(j)})^{w_j}, \prod_{j=1}^{m} (\dot{i}^{l}_{\alpha(j)})^{w_j} \right], \left[\prod_{j=1}^{m} (\dot{\mathrm{d}}'^{l}_{\alpha(j)})^{w_j}, \prod_{j=1}^{m} (\dot{\mathrm{d}}'^{l}_{\alpha(j)})^{w_j} \right] \end{array} \right)$$

Proof: This result can analogously be proved.

7.3.9. Example

Let $I_1 = ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8])$, $I_2 = ([0.2, 0.4], [0.2, 0.6], [0.3, 0.8])$, $I_3 = ([0.3, 0.3], [0.4, 0.5], [0.7, 0.8])$, and $I_4 = ([0.1, 0.5], [0.2, 0.2], [0.3, 0.5])$ be the four IVTSFNs for $n = 4$ and let $\omega = (0.22, 0.34, 0.27, 0.17)^T$ be the weight vector of the IVTSFNs. Furthermore, let $w = (0.2, 0.1, 0.3, 0.4)^T$ be the aggregated associated weight vector. Then, using $\dot{I}_j = m\mathfrak{w}I_j$, we have:

$$\dot{I}_1 = m\mathfrak{w}I_1$$

$$= \begin{pmatrix} \left[\sqrt[4]{1 - (1-(0.3)^4)^{4\times0.22}}, \sqrt[4]{1 - (1-(0.6)^4)^{4\times0.22}}\right], \\ [(0.2)^{4\times0.22}, (0.7)^{4\times0.22}], [(0.5)^{4\times0.22}, (0.8)^{4\times0.22}] \end{pmatrix}$$

$$= ([0.3466, 0.6379], [0.0004, 0.0528], [0.0138, 0.0901\,])$$

Similarly,

$$\dot{I}_2 = m\mathfrak{w}I_2 = ([0.1120, 0.2876], [0.0005, 0.0441], [0.0028, 0.1393])$$

$$\dot{I}_3 = m\mathfrak{w}I_3 = ([0.2725, 0.2725], [0.0069, 0.0069], [0.0169, 0.1106])$$

$$\dot{I}_4 = m\mathfrak{w}I_4 = ([0.3347, 0.5363], [0.0003, 0.0014], [0.0014, 0.0696])$$

Now, using the score function defined in Definition 6, we have $SC(\dot{I}_1) = 0.5240$, $SC(\dot{I}_2) = 0.1900$, $SC(\dot{I}_3) = 0.3461$, and $SC(\dot{I}_4) = 0.5003$. So, the IVTSFNs $\dot{I}_j$ are ranked as:

$$I_1 > I_4 > I_3 > I_2$$

And we get:

$$\dot{I}_{\alpha(1)} = ([0.3466, 0.6379], [0.0004, 0.0528], [0.0138, 0.0901])$$

$$\dot{I}_{\alpha(2)} = ([0.3347, 0.5363], [0.0003, 0.0014], [0.0014, 0.0696])$$

$$\dot{I}_{\alpha(3)} = ([0.2725, 0.2725], [0.0069, 0.0069], [0.0169, 0.1106])$$

$$\dot{I}_{\alpha(4)} = ([0.1120, 0.2876], [0.0005, 0.0441], [0.0028, 0.1393])$$

Now, the aggregated value of $\dot{I}_1, \dot{I}_2, \dot{I}_3$, and $\dot{I}_4$, using the IVTSFHA operator, is given by:

$$IVTSFHA(\dot{I}_1, \dot{I}_4, \dot{I}_3, \dot{I}_1) = ([0.3129, 0.5153], [0.0008, 0.0066], [0.0050, 0.0903])$$

7.3.10. Theorem

If $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$, then the IVTSFHA operator becomes the IVTSFWA operator.

Proof: As $\dot{I}_j = m\varpi_j I_j$ and $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$. So $w_j \dot{I}_j = \varpi_j I_j$ and

$$IVTSFHA(I_1, I_2, I_3 \dots I_m) = \sum_{j=1}^{m} w_j \dot{I}_{\sigma(j)} = \sum_{j=1}^{m} \varpi_j I_j = IVTSFWA(I_1, I_2, I_3 \dots I_m)$$

7.4. Geometric Aggregation Operators for Interval-Valued T-Spherical Fuzzy Sets:

In this section, some geometric AOs for IVTSFSs are proposed. These operators include the interval valued T-spherical fuzzy weighted geometric (IVTSFWG) operator, interval valued T-spherical fuzzy ordered weighted geometric (IVTSFOWG) operator and interval valued T-spherical fuzzy hybrid geometric (IVTSFHG) operator. The validation and basic properties of these aggregation tools are studied and supported by numerical examples.

7.4.1. Definition

The IVTSFWG operator for the IVTSFNs I_j $(j = 1, 2, 3 \dots m)$ is of the form:

$$IVTSFWG(I_1, I_2, I_3 \dots I_m) = \prod_{j=1}^{m} I_j^{w_j}$$

7.4.2. Theorem

The aggregated value of the IVTSFNs I_j, using the IVTSFWG operator, is an IVTSFN and is given as:

$$IVTSFWG(I_1, I_2, I_3 \dots I_m)$$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{m} (\mathfrak{s}_j^l)^{w_j}, \prod_{j=1}^{m} (\mathfrak{s}_j^u)^{w_j} \right], \left[\prod_{j=1}^{m} (i_j^l)^{w_j}, \prod_{j=1}^{m} (i_j^u)^{w_j} \right], \\ \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (\mathrm{d'}_j^l)^n\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (\mathrm{d'}_j^u)^n\right)^{w_j}} \right] \end{array} \right)$$

Proof: Using mathematical induction.

For $m = 2$.

$$I_1^{w_1} = \left([(\mathfrak{s}_{I_1}^l)^{w_1}, (\mathfrak{s}_{I_1}^u)^{w_1}], [(i_{I_1}^l)^{w_1}, (i_{I_1}^u)^{w_1}], \left[\sqrt[n]{1 - (1 - \mathrm{d'}_{I_1}^{l\,n})^{w_1}}, \sqrt[n]{1 - (1 - \mathrm{d'}_{I_1}^{u\,n})^{w_1}} \right] \right)$$

And

$$I_2^{w_2} = \left([(\mathfrak{s}_{I_2}^l)^{w_2}, (\mathfrak{s}_{I_2}^u)^{w_2}], [(i_{I_2}^l)^{w_2}, (i_{I_2}^u)^{w_2}], \left[\sqrt[n]{1 - (1 - \mathrm{d'}_{I_2}^{l\,n})^{w_2}}, \sqrt[n]{1 - (1 - \mathrm{d'}_{I_2}^{u\,n})^{w_2}} \right] \right)$$

$$I_1^{w_1} \otimes I_2^{w_2}$$

$$= \left(\begin{array}{c} \left([(\S_{I_1}^l)^{w_1}, (\S_{I_1}^u)^{w_1}], [(i_{I_1}^l)^{w_1}, (i_{I_1}^u)^{w_1}], \left[\sqrt[n]{1-(1-{d'}_{I_1}^{l\,n})^{w_1}}, \sqrt[n]{1-(1-{d'}_{I_1}^{u\,n})^{w_1}} \right] \right) \otimes \\ \left([(\S_{I_2}^l)^{w_2}, (\S_{I_2}^u)^{w_2}], [(i_{I_2}^l)^{w_2}, (i_{I_2}^u)^{w_2}], \left[\sqrt[n]{1-(1-{d'}_{I_2}^{l\,n})^{w_2}}, \sqrt[n]{1-(1-{d'}_{I_2}^{u\,n})^{w_2}} \right] \right) \end{array} \right)$$

$$= \left(\begin{array}{c} [(\S_{I_1}^l)^{w_1}.(\S_{I_2}^l)^{w_2}, (\S_{I_1}^u)^{w_1}.(\S_{I_2}^u)^{w_2}], [(i_{I_1}^l)^{w_1}.(i_{I_2}^l)^{w_2}, (i_{I_1}^u)^{w_1}.(i_{I_2}^u)^{w_2}], \\ \left[\begin{array}{c} \sqrt[n]{\left(\sqrt[n]{1-(1-{d'}_{I_1}^{l\,n})^{w_1}}\right)^n + \left(\sqrt[n]{1-(1-{d'}_{I_2}^{l\,n})^{w_2}}\right)^n - \sqrt[n]{1-(1-{d'}_{I_1}^{l\,n})^{w_1}}.\sqrt[n]{1-(1-{d'}_{I_2}^{l\,n})^{w_2}}}, \\ \sqrt[n]{\left(\sqrt[n]{1-(1-{d'}_{I_1}^{u\,n})^{w_1}}\right)^n + \left(\sqrt[n]{1-(1-{d'}_{I_2}^{u\,n})^{w_2}}\right)^n - \sqrt[n]{1-(1-{d'}_{I_1}^{u\,n})^{w_1}}.\sqrt[n]{1-(1-{d'}_{I_2}^{u\,n})^{w_2}}} \end{array} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} [(\S_{I_1}^l)^{w_1}.(\S_{I_2}^l)^{w_2}, (\S_{I_1}^u)^{w_1}.(\S_{I_2}^u)^{w_2}], [(i_{I_1}^l)^{w_1}.(i_{I_2}^l)^{w_2}, (i_{I_1}^u)^{w_1}.(i_{I_2}^u)^{w_2}], \\ \left[\sqrt[n]{1-(1-{d'}_{I_1}^{l\,n})^{w_1}.(1-{d'}_{I_2}^{l\,n})^{w_2}}, \sqrt[n]{1-(1-{d'}_{I_1}^{u\,n})^{w_1}.(1-{d'}_{I_2}^{u\,n})^{w_2}} \right] \end{array} \right)$$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{2} (\S_j^l)^{w_j}, \prod_{j=1}^{2} (\S_j^u)^{w_j} \right], \left[\prod_{j=1}^{2} (i_j^l)^{w_j}, \prod_{j=1}^{2} (i_j^u)^{w_j} \right], \\ \left[\sqrt[n]{1-\prod_{j=1}^{2}(1-({d'}_j^l)^n)^{w_j}}, \sqrt[n]{1-\prod_{j=1}^{2}(1-({d'}_j^u)^n)^{w_j}} \right] \end{array} \right)$$

Hence, for $m = 2$, the result is true.

Based on this assumption, we claim that the result is valid for $m = k$ i.e.,

$IVTSFWG(I_1, I_2, I_3 \dots I_k)$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{k}(\text{ş}_j^l)^{w_j}, \prod_{j=1}^{k}(\text{ş}_j^u)^{w_j}\right], \left[\prod_{j=1}^{k}(i_j^l)^{w_j}, \prod_{j=1}^{k}(i_j^u)^{w_j}\right], \\ \left[\sqrt[n]{1-\prod_{j=1}^{k}\left(1-(\text{d}'^l_j)^n\right)^{w_j}}, \sqrt[n]{1-\prod_{j=1}^{k}\left(1-(\text{d}'^u_j)^n\right)^{w_j}}\right] \end{array} \right)$$

To prove the result for $m = k + 1$, consider:

$$IVTSFWG(I_1, I_2, I_3 \dots I_k, I_{k+1}) = \prod_{j=1}^{k+1} I_j^{w_j} = \prod_{j=1}^{k} I_j^{w_j} \otimes I_{k+1}^{w_{k+1}}$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[\prod_{j=1}^{k}(\text{ş}_j^l)^{w_j}, \prod_{j=1}^{k}(\text{ş}_j^u)^{w_j}\right], \left[\prod_{j=1}^{k}(i_j^l)^{w_j}, \prod_{j=1}^{k}(i_j^u)^{w_j}\right], \\ \left[\sqrt[n]{1-\prod_{j=1}^{k}\left(1-(\text{d}'^l_j)^n\right)^{w_j}}, \sqrt[n]{1-\prod_{j=1}^{k}\left(1-(\text{d}'^u_j)^n\right)^{w_j}}\right] \end{array} \right) \otimes \\ \left(\begin{array}{c} \left[(\text{ş}^l_{I_{k+1}})^{w_{k+1}}, (\text{ş}^u_{I_{k+1}})^{w_{k+1}}\right], \left[(i^l_{I_{k+1}})^{w_{k+1}}, (i^u_{I_{k+1}})^{w_{k+1}}\right], \\ \left[\sqrt[n]{1-(1-\text{d}'^l_{I_{k+1}}{}^n)^{w_{k+1}}}, \sqrt[n]{1-(1-\text{d}'^u_{I_{k+1}}{}^n)^{w_{k+1}}}\right] \end{array} \right) \end{array} \right)$$

Finally, we get:

$IVTSFWG(I_1, I_2, I_3 \dots I_k, I_{k+1})$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{k+1}(\text{ş}_j^l)^{w_j}, \prod_{j=1}^{k+1}(\text{ş}_j^u)^{w_j}\right], \left[\prod_{j=1}^{k+1}(i_j^l)^{w_j}, \prod_{j=1}^{k+1}(i_j^u)^{w_j}\right], \\ \left[\sqrt[n]{1-\prod_{j=1}^{k+1}\left(1-(\text{d}'^l_j)^n\right)^{w_j}}, \sqrt[n]{1-\prod_{j=1}^{k+1}\left(1-(\text{d}'^u_j)^n\right)^{w_j}}\right] \end{array} \right)$$

Hence, the result is valid for $m = k + 1$. Consequently, using the induction method, it is proven that the result is valid for all $m \in \mathbb{Z}^+$.

In the following theorem, we state that the IVTSFWG operator satisfies the basic characteristics of aggregation operators.

7.4.3. Theorem

For an IVTSFWG operator, the following properties hold true.

1. Idempotency:

   If $I_j = I$ for all $j = 1, 2, 3, \dots m$, then $IVTSFWG(I_1, I_2, I_3 \dots I_m) = I$

2. Boundedness:

   If $I^- = \min_j(\varsigma_j, \iota_j\, d'_j)$ and $I^+ = \max_j(\varsigma_j, \iota_j\, d'_j)$ are the least and greatest IVTSFNs,

then

$$I^- \leq IVTSFWG(I_1, I_2, I_3 \dots I_m) \leq I^+$$

3. Monotonicity:

   For the two collections of IVTSFNs, I_j and P_j, if $I_j \leq P_j\ \forall\, j$ then

$$IVTSFWA(I_1, I_2, I_3 \dots I_m) \leq IVTSFWA(P_1, P_2, P_3 \dots P_m)$$

Proof: Similar

To handle situations where the ordered position of the information is important, the concept of the IVTSFOWG operator is proposed.

7.4.4. Definition

The IVTSFOWG operator for IVTSFNs I_j $(j = 1, 2, 3 \dots m)$ is of the form:

$$IVTSFOWG(I_1, I_2, I_3 \dots I_m) = \prod_{j=1}^{m} I_{\alpha(j)}^{w_j}$$

where $I_{\alpha(j)}$ denotes the j^{th} largest value of I_j.

7.4.5. Theorem

The aggregated value of the IVTSFNs I_j, using the IVTSFOWG operator, is an IVTSFN and is given as:

$IVTSFOWG(I_1, I_2, I_3 \dots I_m)$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{m} \left(\varsigma^{l}_{\alpha(j)} \right)^{w_j}, \prod_{j=1}^{m} \left(\varsigma^{u}_{\alpha(j)} \right)^{w_j} \right], \left[\prod_{j=1}^{m} \left(i^{l}_{\alpha(j)} \right)^{w_j}, \prod_{j=1}^{m} \left(i^{u}_{\alpha(j)} \right)^{w_j} \right], \\ \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \left(d'^{l}_{\alpha(j)} \right)^{n} \right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \left(d'^{u}_{\alpha(j)} \right)^{n} \right)^{w_j}} \right] \end{array} \right)$$

7.4.6. Example

Let $I_1 = ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8])$, $I_2 = ([0.2, 0.4], [0.2, 0.6], [0.3, 0.8])$, $I_3 = ([0.3, 0.3], [0.4, 0.5], [0.7, 0.8])$, and $I_4 = ([0.1, 0.5], [0.2, 0.2], [0.3, 0.5])$ be the four IVTSFNs for $n = 4$ and let $w = (0.22, 0.34, 0.27, 0.17)^t$ be the weight vector. Then, using the score function, we have $SC(I_1) = 0.0227$, $SC(I_2) = 0.0055$, $SC(I_3) = 0.0089$, and $SC(I_4) = 0.0066$. So, the IVTSFNs are ranked as:

$$I_1 > I_3 > I_4 > I_2$$

And we get:

$$I_{\alpha(1)} = ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8])$$

$$I_{\alpha(2)} = ([0.3, 0.3], [0.4, 0.5], [0.7, 0.8])$$

$$I_{\alpha(3)} = ([0.1, 0.5], [0.2, 0.2], [0.3, 0.5])$$

$$I_{\alpha(4)} = ([0.2, 0.4], [0.2, 0.6], [0.3, 0.8])$$

Now, the aggregated value of $I_1, I_2, I_3,$ and I_4, using the IVTSFOWA operator, is given by:

$IVTSFOWG(I_1, I_3, I_4, I_2) = ([0.2081, 0.4212], [0.2532, 0.4336], [0.5844, 0.7626])$

Next, the notion of the IVTSFHG operator is proposed.

7.4.7. Definition

The IVTSFHG operator for IVTSFNs I_j $(j = 1, 2, 3 \ldots m)$ is of the form:

$$IVTSFHG(I_1, I_2, I_3 \ldots I_m) = \prod_{j=1}^{m} \dot{I}_{\alpha(j)}^{w_j}$$

where $\dot{I}_j$ can be computed as $\dot{I}_j = I_j^{m\mathfrak{w}}$, $\dot{I}_{\alpha(j)}$ denotes the j^{th} largest value of $\dot{I}_j$, and $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2, \mathfrak{w}_3 \ldots \mathfrak{w}_n)^t$ is the weight vector of I_j.

7.4.8. Theorem

The aggregated value of IVTSFNs I_j, using the IVTSFHG operator, is an IVTSFN and is given as:

$IVTSFHG(I_1, I_2, I_3 \ldots I_m)$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{m} \left(\dot{\$}_{\alpha(j)}^{l}\right)^{w_j}, \prod_{j=1}^{m} \left(\dot{\$}_{\alpha(j)}^{u}\right)^{w_j} \right], \left[\prod_{j=1}^{m} \left(\dot{i}_{\alpha(j)}^{l}\right)^{w_j}, \prod_{j=1}^{m} \left(\dot{i}_{\alpha(j)}^{u}\right)^{w_j} \right], \\ \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \left(\dot{d}_{\alpha(j)}^{l}\right)^{n}\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - \left(\dot{d}_{\alpha(j)}^{u}\right)^{n}\right)^{w_j}} \right] \end{array} \right)$$

Proof: This result can analogously be proved.

7.4.9. Example

Let $I_1 = ([0.3, 0.6], [0.2, 0.7], [0.5, 0.8])$, $I_2 = ([0.2, 0.4], [0.2, 0.6], [0.3, 0.8])$, $I_3 = ([0.3, 0.3], [0.4, 0.5], [0.7, 0.8])$, and $I_4 = ([0.1, 0.5], [0.2, 0.2], [0.3, 0.5])$ be the four IVTSFNs for $n = 4$ and let $\mathfrak{w} = (0.22, 0.34, 0.27, 0.17)^T$be the weight vector of IVTSFNs. Furthermore, let $w = (0.2, 0.1, 0.3, 0.4)^T$ be the aggregated associated weight vector. Then, using $\dot{I}_j = I_j^{m\mathfrak{w}}$, we have:

$$\dot{I}_1 = I_1^{m\mathfrak{w}}$$

$$= \begin{pmatrix} [(0.3)^{4\times 0.22}, (0.6)^{4\times 0.22}], [(0.2)^{4\times 0.22}, (0.7)^{4\times 0.22}], \\ \left[\sqrt[4]{1-(1-(0.5)^4)^{4\times 0.22}}, \sqrt[4]{1-(1-(0.8)^4)^{4\times 0.22}}\right] \end{pmatrix}$$

$$= ([0.0018, 0.0285], [0.0004, 0.0528], [0.5434, 0.8217])$$

Similarly,

$$\dot{I}_2 = I_2^{m\mathfrak{w}} = ([0.0005, 0.0087], [0.0005, 0.0441], [0.1945, 0.7382])$$

$$\dot{I}_3 = I_3^{m\mathfrak{w}} = ([0.0022, 0.0022], [0.0069, 0.0069], [0.4730, 0.7858])$$

$$\dot{I}_4 = I_4^{m\mathfrak{w}} = ([0.0003, 0.0044], [0.0003, 0.0014], [0.4410, 0.8592])$$

Now, using the score function of IVTSFNs, we have $SC(\dot{I}_1) = 0.0020$, $SC(\dot{I}_2) = 0.0011$, $SC(\dot{I}_3) = 0.0013$, and $SC(\dot{I}_4) = 0.0004$. So, the IVTSFNs are ranked as:

$$\dot{I}_1 > \dot{I}_3 > \dot{I}_2 > \dot{I}_4$$

Now, the aggregated value of $\dot{I}_1, \dot{I}_2, \dot{I}_3$, and $\dot{I}_4$, using the IVTSFHG operator, is given by:

$$IVTSFHG(I_1, I_3, I_2, I_4) = ([0.0008, 0.0055], [0.0008, 0.0066], [0.4681, 0.8235])$$

7.4.10. Theorem

If $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$, then the IVTSFHG operator becomes the IVTSFWG operator.

Proof: As $\dot{I}_j = I_j^{m\varpi_j}$ and $w = \left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}\right)^t$. So $\dot{I}_j^{w_j} = I_j^{\varpi_j}$ and

$$IVTSFHG(I_1, I_2, I_3 \dots I_m) = \prod_{j=1}^{m} \dot{I}_{\alpha(j)}^{w_j} = \prod_{j=1}^{m} I_j^{\varpi_j} = IVTSFWG(I_1, I_2, I_3 \dots I_m)$$

7.5. Multi-Attribute Decision-Making Investment Planning

MADM is one of the most suitable areas where the aggregation of the tools of fuzzy structures is applied to get useful results. A detailed literature survey of MADM problems has been discussed in Chapter 0.1. In this section, our aim is to develop the MADM method in the context of IVTSFSs.

In a MADM problem, the selection of the best or most suitable option is carried out using a list of options. Consider a list of m alternatives, denoted by A_i, being observed by decision makers with k attributes, denoted by H_k. The decision makers evaluated the m number of alternatives with k attributes and provided their information in the form of IVTSFNs. Furthermore, $w = (w_1, w_2, w_3, \dots w_k)^t$ denoted the weight vector of the attributes. The detailed steps of the MADM process are given as:

Step 1: The decision makers assess the alternatives, given the attributes, and provide their information in the form of a decision matrix.

Step 2: Apply the proposed aggregation tools to the decision matrix obtained in Step 1.

Step 3: Compute the scores of the IVTSFNs obtained in Step 2.

Step 4: Analyze the score values of the alternatives and rank them to obtain the best alternative.

The following example demonstrates the proposed MADM algorithm in detail.

7.5.1. Example

A multinational company needed to announce its policy on its investment for the upcoming financial year. Keeping in mind its previous performance, the company needed to evaluate its financial policies to launch the best investment. It had four policies to be evaluated after some initial screening. These four policies included four possible places (countries) for investments i.e., A_1: *To invest in Pakistan,* A_2: *To invest in Iran,* A_3; *To invest in the UAE, and* A_4: *To invest in Bangladesh.* Assuming that all of the attributes are of benefit, the four investment plans were evaluated based on the following attributes: H_1: *Comfort zone,* H_2: *Government regulations,* H_3: *Interest of the people, and* H_4: *Competition in the markets.* The attribute H_1 shows that evaluation policies are based on the law and considers the situation of the country for the investment. The attribute H_2 allows the decision makers to evaluate the policy, keeping in mind the relevant government policies that directly affect the business policies. The attribute H_3 allows the decision makers to evaluate the policy based on the cultural values and overall lifestyle of the people of the country, as any investment policy is directly linked to the consumers and distributors. The fourth attribute, H_4, provides a chance for the decision makers to evaluate the policy by considering the competition in the market. If the competition in the market is low, then the success of the policy may be assured. All of the four factors had a great impact on the success of the policy and the decision makers provided their evaluations with all four attributes in mind. Let $w = (0.22, 0.34, 0.27, 0.17)^t$ be the weight vector of the attributes. The policy makers gave their opinions in the form of IVTSFNs, as follows.

The description of imprecise information, in terms of intervals instead of crisp numbers, provides consistent and improved results numerically, as described in Section 7.1. Furthermore, the description of imprecision using the MG, AG, NG, and the RG, improves the accuracy of the human opinion about an uncertain event. Therefore, the evaluation of the investment policy, in this case, is studied in the environment of IVTSFSs, using the WA and WG aggregation operators. The stepwise demonstration of the MADM process is as follows:

Step 1: The evaluation of the alternatives, A_i, by the decision makers in the form of a decision matrix.

$$D_{4\times 4} = \begin{pmatrix} & H_1 & H_2 & H_3 & H_4 \\ A_1 & \begin{pmatrix}[0.3,0.5],\\ [0.4,0.6],\\ [0.2,0.6]\end{pmatrix} & \begin{pmatrix}[0.1,0.5],\\ [0.3,0.6],\\ [0.4,0.7]\end{pmatrix} & \begin{pmatrix}[0.2,0.7],\\ [0.4,0.4],\\ [0.3,0.5]\end{pmatrix} & \begin{pmatrix}[0.4,0.9],\\ [0.5,0.6],\\ [0.3,0.4]\end{pmatrix} \\ A_2 & \begin{pmatrix}[0.4,0.5],\\ [0.2,0.7],\\ [0.6,0.7]\end{pmatrix} & \begin{pmatrix}[0.4,0.7],\\ [0.3,0.6],\\ [0.5,0.8]\end{pmatrix} & \begin{pmatrix}[0.4,0.8],\\ [0.2,0.4],\\ [0.1,0.6]\end{pmatrix} & \begin{pmatrix}[0.3,0.8],\\ [0.4,0.6],\\ [0.2,0.2]\end{pmatrix} \\ A_3 & \begin{pmatrix}[0.8,0.8],\\ [0.0,0.3],\\ [0.5,0.6]\end{pmatrix} & \begin{pmatrix}[0.2,0.7],\\ [0.1,0.2],\\ [0.4,0.8]\end{pmatrix} & \begin{pmatrix}[0.7,0.9],\\ [0.3,0.3],\\ [0.4,0.8]\end{pmatrix} & \begin{pmatrix}[0.3,0.4],\\ [0.4,0.8],\\ [0.4,0.4]\end{pmatrix} \\ A_4 & \begin{pmatrix}[0.2,0.2],\\ [0.5,0.5],\\ [0.2,0.6]\end{pmatrix} & \begin{pmatrix}[0.5,0.7],\\ [0.3,0.6],\\ [0.3,0.3]\end{pmatrix} & \begin{pmatrix}[0.2,0.8],\\ [0.3,0.4],\\ [0.1,0.3]\end{pmatrix} & \begin{pmatrix}[0.1,0.9],\\ [0.3,0.7],\\ [0.5,0.6]\end{pmatrix} \end{pmatrix}$$

The selection of the best location for investment is carried out using two approaches. Firstly, by using the IVTSFWA operators and secondly, by using IVTSFWG operators. It can be seen that all of the values in the decision matrix are IVTSFNs for $n = 5$. For a lesser n, some values are not IVTSFNs. Therefore, we take $n = 5$.

Step 2: By applying the IVTSFWA operators to the decision matrix provided in Step 1, we get:

$$A_1 = IVTSFWA(A_{11}, A_{12}, A_{13}, A_{14})$$
$$= ([0.2986, 0.7225], [0.3768, 0.5378], [0.3026, 0.5618])$$

$$A_2 = IVTSFWA(A_{21}, A_{22}, A_{23}, A_{24})$$
$$= ([0.3891,0.7365], [0.2583, 0.5563], [0.2884, 0.5679])$$

$$A_3 = IVTSFWA(A_{31}, A_{32}, A_{33}, A_{34})$$
$$= ([0.6634, 0.7986], [0.0000,0.3088], [0.4201,0.6675])$$

$$A_4 = IVTSFWA(A_{41}, A_{42}, A_{43}, A_{44})$$
$$= ([0.4050, 0.7725], [0.3357, 0.5304], [0.2225, 0.3931])$$

Step 3: This step involves the computation of the score values as such:

$SC(A_1) = 0.1083, SC(A_2) = 0.1423, SC(A_3) = 0.3846$, and $SC(A_4) = 0.1846$

Step 4: In this step, the score values obtained in Step 3 are analyzed. Based on the score values, the ranking of the alternatives is given:

$$SC(A_3) > SC(A_4) > SC(A_2) > SC(A_1)$$

The ranking analysis shows that A_3 has the greatest value, hence the policy of making investments in the UAE is the best, according to the proposed decision-making method using the IVTSFWA operators.

Now, the same data is analyzed to find the best policy of investment using IVTSFWG operators.

Step 2: By applying the IVTSFWG operators to the decision matrix provided in Step 1, we get:

$$A_1 = IVTSFWG(A_{11}, A_{12}, A_{13}, A_{14})$$
$$= ([0.1944, 0.6051], [0.3768, 0.5378], [0.3413, 0.6132])$$

$$A_2 = IVTSFWG(A_{21}, A_{22}, A_{23}, A_{24})$$
$$= ([0.3809, 0.6894], [0.2583, 0.5563], [0.4900, 0.7090])$$

$$A_3 = IVTSFWG(A_{31}, A_{32}, A_{33}, A_{34})$$
$$= ([0.4077, 0.7015], [0.0000, 0.3088], [0.4312, 0.7455])$$

$$A_4 = IVTSFWG(A_{41}, A_{42}, A_{43}, A_{44})$$
$$= ([0.2427, 0.5749], [0.3357, 0.5304], [0.3627, 0.5040])$$

Step 3: This step involves the computation of the score values as:

$$SC(A_1) = 0.0520, C(A_2) = 0.0993, SC(A_3) = 0.1263, \text{ and } SC(A_4) = 0.0589$$

Step 4: In this step, the score values obtained in Step 3 are analyzed. Based on the score values, the ranking of alternatives is given as:

$$SC(A_3) > SC(A_2) > SC(A_4) > SC(A_1)$$

The ranking analysis shows that A_3 has the greatest value, thus the policy of making *investments in the UAE* is the best option, according to the proposed decision-making method using IVTSFWG operators. Hence, in this case, we obtain the same results by using IVTAFWA operators as we do by using IVTSFWG operators. However, we will not necessarily always obtain the same results using different aggregation tools, i.e., the results obtained using IVTSFWA and IVTSFWG operators may differ. Thus, this type of information cannot be processed by using the pre-existing aggregation tools because of their limited structures. Our proposed aggregation operators, on the other hand, can aggregate such types of information. The other interesting fact is that the information provided in this problem is similar to human opinion, as it describes the MG, AG, NG, and the RG. Furthermore, if we used only a single crisp value for each membership grade instead of intervals, then the result would drastically change, as illustrated in Section 7.1.

7.6. Consequences of the Proposed Work and a Comparative Study

The aim of this section is to analyze some consequences of our proposed work. We prove that the previous AOs of IFSs, IVIFSs, PyFSs, IVPyFSs, q-ROFSs, IVq-ROFSs, PFSs, IVPFSs, SFSs and TSFSs becomes the special cases of the proposed AOs.

Consider the IVTSFWA and IVTSFWG operators, respectively.

$$IVTSFWA(I_1, I_2, I_3 \ldots I_m)$$

$$= \left(\begin{array}{c} \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (ş_j^l)^n\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (ş_j^u)^n\right)^{w_j}} \right], \\ \left[\prod_{j=1}^{m} (i_j^l)^{w_j}, \prod_{j=1}^{m} (i_j^u)^{w_j} \right], \left[\prod_{j=1}^{m} (d'^l_j)^{w_j}, \prod_{j=1}^{m} (d'^u_j)^{w_j} \right] \end{array} \right)$$

$$IVTSFWG(I_1, I_2, I_3 \ldots I_m)$$

$$= \left(\begin{array}{c} \left[\prod_{j=1}^{m} (ş_j^l)^{w_j}, \prod_{j=1}^{m} (ş_j^u)^{w_j} \right], \left[\prod_{j=1}^{m} (i_j^l)^{w_j}, \prod_{j=1}^{m} (i_j^u)^{w_j} \right], \\ \left[\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (d'^l_j)^n\right)^{w_j}}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (d'^u_j)^n\right)^{w_j}} \right] \end{array} \right)$$

1. If we use $ş^l = ş^u = ş$, $i^l = i^u = i$, and $d'^l = d'^u = d'$, then the WAA and the WGA operators of TSFSs are obtained and are given as:

$$TSFWA(I_1, I_2, I_3 \ldots I_m) = \left(\sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (ş_j)^n\right)^{w_j}}, \prod_{j=1}^{m} (i_j)^{w_j}, \prod_{j=1}^{m} (d'_j)^{w_j} \right)$$

$$TSFWG(I_1, I_2, I_3 \ldots I_m) \left(\prod_{j=1}^{m} (ş_j)^{w_j}, \prod_{j=1}^{m} (i_j)^{w_j}, \sqrt[n]{1 - \prod_{j=1}^{m} \left(1 - (d'_j)^n\right)^{w_j}} \right)$$

2. If we use $n = 2$, then the WAA and the WGA operators of IVSFSs are obtained and are given as:

$IVSFWA(I_1, I_2, I_3 \dots I_m)$

$$= \left(\left[\sqrt{1 - \prod_{j=1}^{m} \left(1 - (\text{ş}_j^{l})^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{m} \left(1 - (\text{ş}_j^{u})^2\right)^{w_j}} \right], \left[\prod_{j=1}^{m} (i_j^{l})^{w_j}, \prod_{j=1}^{m} (i_j^{u})^{w_j} \right], \left[\prod_{j=1}^{m} (\text{d}'^{l}_j)^{w_j}, \prod_{j=1}^{m} (\text{d}'^{u}_j)^{w_j} \right] \right)$$

$IVSFWG(I_1, I_2, I_3 \dots I_m)$

$$= \left(\left[\prod_{j=1}^{m} (\text{ş}_j^{l})^{w_j}, \prod_{j=1}^{m} (\text{ş}_j^{u})^{w_j} \right], \left[\prod_{j=1}^{m} (i_j^{l})^{w_j}, \prod_{j=1}^{m} (i_j^{u})^{w_j} \right], \left[\sqrt{1 - \prod_{j=1}^{m} \left(1 - (\text{d}'^{l}_j)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{m} \left(1 - (\text{d}'^{u}_j)^2\right)^{w_j}} \right] \right)$$

3. If we use $n = 2$, $\text{ş}^{l} = \text{ş}^{u} = \text{ş}$, $i^{l} = i^{u} = i$, and $\text{d}'^{l} = \text{d}'^{u} = \text{d}'$, then the WAA and the WGA operators of SFSs are obtained and are given as:

$$SFWA(I_1, I_2, I_3 \dots I_m) = \left(\sqrt{1 - \prod_{j=1}^{m} \left(1 - (\text{ş}_j)^2\right)^{w_j}}, \prod_{j=1}^{m} (i_j)^{w_j}, \prod_{j=1}^{m} (\text{d}'_j)^{w_j} \right)$$

$$SFWG(I_1, I_2, I_3 \dots I_m) \left(\prod_{j=1}^{m} (\text{ş}_j)^{w_j}, \prod_{j=1}^{m} (i_j)^{w_j}, \sqrt{1 - \prod_{j=1}^{m} \left(1 - (\text{d}'_j)^2\right)^{w_j}} \right)$$

4. If we use $n = 1$, then the WAA and the WGA operators of IVPFSs are obtained and are given as:

$IVPFWA(I_1, I_2, I_3 \dots I_m)$

$$
=\left(\begin{array}{c}\left[1-\prod_{j=1}^{m}\left(1-(\text{ş}_j^l)\right)^{w_j},1-\prod_{j=1}^{m}\left(1-(\text{ş}_j^u)\right)^{w_j}\right],\\ \left[\prod_{j=1}^{m}(i_j^l)^{w_j},\prod_{j=1}^{m}(i_j^u)^{w_j}\right],\left[\prod_{j=1}^{m}(\text{d}'^l_j)^{w_j},\prod_{j=1}^{m}(\text{d}'^u_j)^{w_j}\right]\end{array}\right)
$$

$IVPFWG(I_1, I_2, I_3 \dots I_m)$

$$
=\left(\begin{array}{c}\left[\prod_{j=1}^{m}(\text{ş}_j^l)^{w_j},\prod_{j=1}^{m}(\text{ş}_j^u)^{w_j}\right],\left[\prod_{j=1}^{m}(i_j^l)^{w_j},\prod_{j=1}^{m}(i_j^u)^{w_j}\right],\\ \left[1-\prod_{j=1}^{m}\left(1-(\text{d}'^l_j)^2\right)^{w_j},1-\prod_{j=1}^{m}\left(1-(\text{d}'^u_j)^2\right)^{w_j}\right]\end{array}\right)
$$

5. If we use $n = 1$, $\text{ş}^l = \text{ş}^u = \text{ş}$, $i^l = i^u = i$, and $\text{d}'^l = \text{d}'^u = \text{d}'$, then the WAA and the WGA operators of PFSs are obtained. Note that these PFWA operator and PFWG operators are the same as the one proposed by Garg [50] and Wei [83].
6. If we use $i^l = i^u = i = 0$, then the WAA and the WGA operators of the IVq-ROPFSs are obtained and are given as:

$IVq-ROPFWA(I_1, I_2, I_3 \dots I_m)$

$$
=\left(\left[\sqrt[n]{1-\prod_{j=1}^{m}\left(1-(\text{ş}_j^l)^n\right)^{w_j}},\sqrt[n]{1-\prod_{j=1}^{m}\left(1-(\text{ş}_j^u)^n\right)^{w_j}}\right],\left[\prod_{j=1}^{m}(\text{d}'^l_j)^{w_j},\prod_{j=1}^{m}(\text{d}'^u_j)^{w_j}\right]\right)
$$

$IVq-ROPFWG(I_1, I_2, I_3 \dots I_m)$

$$
=\left(\left[\prod_{j=1}^{m}(\text{ş}_j^l)^{w_j},\prod_{j=1}^{m}(\text{ş}_j^u)^{w_j}\right],\left[\sqrt[n]{1-\prod_{j=1}^{m}\left(1-(\text{d}'^l_j)^n\right)^{w_j}},\sqrt[n]{1-\prod_{j=1}^{m}\left(1-(\text{d}'^u_j)^n\right)^{w_j}}\right]\right)
$$

7. If we use $ş^l = ş^u = ş$, $i^l = i^u = i = 0$, and $d'^l = d'^u = d'$, then the WAA and the WGA operators of the newly developed q-ROPFSs are obtained as proposed by Liu and Wang [85].
8. If we use $n = 2$ and $i^l = i^u = i = 0$, then the WAA and the WGA operators of the IVPyFSs are obtained. Note that these IVPyFWA and IWPyFWG operators are the same as those proposed by Rehman et al. [46, 47].
9. If we use $n = 2$, $ş^l = ş^u = ş$, $i^l = i^u = i = 0$, and $d'^l = d'^u = d'$, then the WAA and the WGA operators of the PyFSs are obtained. Note that these PyFWA and PyFWG operators are the same as those proposed by Rahman et al. [44] and Peng and Yang [45].

(8) If we use $n = 1$ and $i^l = i^u = i = 0$, then the WAA and the WGA operators of the IVIFSs are obtained. Note that these IVIFWA and IVIFWG operators are the same as those proposed by in [35, 37].

(9) If we use $n = 1$, $ş^l = ş^u = ş$, $i^l = i^u = i = 0$, and $d'^l = d'^u = d'$, then the WAA and the WGA operators of the IFSs are obtained. Note that these IFWA and IFW operators are the same as those proposed by Xu [29] and Xu and Yager [30].

Hence, using the proposed definitions of the IVTSFWA and IVTSFWG operators and with the help of some restrictions, we successfully proposed WAA and WGA operators for IVq-ROPFSs, q-ROPFSs, TSFSs, IVSFSs, SFSs, IVPFSs, and PFSs. Furthermore, it is also shown that the proposed operators are the generalizations of the aggregation tools of PFSs, PyFSs, IVPyFSs, IFSs, and IVIFs.

Now, we discuss the viability of the proposed IVTSF aggregation operators in the problems of the existing frameworks.

Firstly, consider the MADM problem discussed in Chapter 2, Example 2.6.2.2, with three alternatives and three attributes. The decision matrix is shown in Table 36.

	H_1	H_2	H_3
A_1	$(0.8, 0.5, 0.4)$	$(0.7, 0.4, 0.4)$	$(0.3, 0.5, 0.4)$
A_2	$(0.9, 0.2, 0.4)$	$(0.6, 0.3, 0.2)$	$(0.4, 0.1, 0.7)$
A_3	$(0.5, 0.5, 0.5)$	$(0.8, 0.2, 0.3)$	$(0.6, 0.4, 0.3)$

Table 36 (The decision matrix)

This type of MADM problem can easily be solved by placing $ş^l = ş^u = ş, i^l = i^u = i$ and $d'^l = d'^u = d'$ into the proposed IVTSFWA and IVTSFWG operators.

Now, consider the MADM problem studied by Garg [50], with four alternatives and four attributes. The decision matrix is shown in Table 37.

	H_1	H_2	H_3	H_3
A_1	$(0.2, 0.1, 0.6)$	$(0.5,0.3, 0.1)$	$(0.5, 0.1, 0.3)$	$(0.4, 0.3, 0.2)$
A_2	$(0.1, 0.4,0.4)$	$(0.6, 0.3,0.1)$	$(0.5,0.2, 0.2)$	$(0.2, 0.1,0.7)$
A_3	$(0.3,0.2, 0.2)$	$(0.6,0.2,0.1)$	$(0.4, 0.1,0.3)$	$(0.3, 0.3, 0.4)$
A_4	$(0.3, 0.1,0.6)$	$(0.1,0.2,0.6)$	$(0.1, 0.3, 0.5)$	$(0.2, 0.3, 0.2)$

Table 37 (The decision matrix by Garg [50])

This type of MADM problem can easily be solved by placing $n = 1, ş^l = ş^u = ş, i^l = i^u = i$, and $d'^l = d'^u = d'$ into the proposed IVTSFWA operator.

Similarly, the MADM problems solved by [29-37, 44-5138] can also be solved using the proposed AOs.

7.7. Advantages of Proposed Work

In this section, the advantages of the proposed new aggregation tools over the pre-existing aggregation tools are discussed and further demonstrated with a numerical example. If we consider MADM problem in Example 7.5, it becomes quite clear that none of the existing aggregation tools of the previous fuzzy frameworks could solve the data involved in that problem, thus showing the limitations of the existing AOs.

On the other hand, if we consider an example in the context of the existing fuzzy frameworks, then such a problem can easily be solved by using the proposed aggregation tools of the IVTSFSs. For instance, consider a decision matrix of four alternatives and three attributes in the MADM problem discussed by Garg [98].

$$D_{4\times 3} = \begin{pmatrix} & H_1 & H_2 & H_3 \\ A_1 & \begin{pmatrix} [0.4, 0.5], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.4, 0.6], \\ [0.2, 0.4] \end{pmatrix} & \begin{pmatrix} [0.1, 0.3], \\ [0.5, 0.6] \end{pmatrix} \\ A_2 & \begin{pmatrix} [0.6, 0.7], \\ [0.2, 0.3] \end{pmatrix} & \begin{pmatrix} [0.6, 0.7], \\ [0.2, 0.3] \end{pmatrix} & \begin{pmatrix} [0.4, 0.7], \\ [0.1, 0.2] \end{pmatrix} \\ A_3 & \begin{pmatrix} [0.3, 0.6], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.5, 0.6], \\ [0.3, 0.4] \end{pmatrix} & \begin{pmatrix} [0.5, 0.6], \\ [0.1, 0.3] \end{pmatrix} \\ A_4 & \begin{pmatrix} [0.7, 0.8], \\ [0.1, 0.2] \end{pmatrix} & \begin{pmatrix} [0.6, 0.7], \\ [0.1, 0.3] \end{pmatrix} & \begin{pmatrix} [0.3, 0.4], \\ [0.1, 0.2] \end{pmatrix} \end{pmatrix}$$

Here, the four alternatives denote four possible policies of investment and our aim is to find the best policy. Now, this problem can easily be solved using the aggregation operators of the IVTSFNs by placing $n = 2$ and $i^l = i^u = i = 0$. The IVTSFWA operator for $n = 2$ and $i^l = i^u = i = 0$, becomes:

$$IVTSFWA(I_1, I_2, I_3 \dots I_m)$$

$$= \left(\left[\sqrt{1 - \prod_{j=1}^{m} \left(1 - (s_j^l)^2\right)^{w_j}}, \sqrt{1 - \prod_{j=1}^{m} \left(1 - (s_j^u)^2\right)^{w_j}} \right], \left[\prod_{j=1}^{m} (d'^l_j)^{w_j}, \prod_{j=1}^{m} (d'^u_j)^{w_j} \right] \right)$$

The aggregation results are:

$$A_1 = ([0.2692, 0.4079], [0.3955, 0.5206])$$

$$A_2 = ([0.4586, 0.6408], [0.1995, 0.3153])$$

$$A_3 = ([0.3972, 0.5612], [0.2197, 0.3946])$$

$$A_4 = ([0.4910, 0.6024], [0.1565, 0.2736])$$

Using the score function, we get:

$$SC(A_2) > SC(A_4) > SC(A_3) > SC(A_1)$$

Hence, the score values show that A_2 is the best policy to be implemented. This result is well in accordance with the results obtained by Garg [98]. Therefore, the claim that the IVTSFWA and IVTSFWG operators can deal with data in the form of pre-existing fuzzy frameworks is proven to be true.

Chapter 8

A Comparative Study of Several Aspects of Single Valued Neutrosophic Sets and T-Spherical Fuzzy Sets

In this thesis, we have established several AOs, SMs and CCs and studied their viability in MADM, pattern recognition, medical diagnosis and clustering. We also established the comparison of the proposed and previous study and numerically examined it. TSFS is a generalization of IFS and other fuzzy frameworks so is the SVNS. Keeping in mind, the broad idea reflected by the thesis title, the goal of this chapter is to establish a comparative study of TSFSs and SVNSs as both are two widely studied generalizations of IFSs. Our focus is to examine the frameworks of TSFSs and SVNSs and point out their limitations (if any) and advantages. We also aim to apply some AOs and SMs to some problems analyze the results comparatively. Based on the results, we aim to draw some conclusions about these two fuzzy frameworks.

For convenience, we aim to recall the definitions of TSFS and SVNS and their basic operations for better understanding.

8.1. Comparison of the Basic Notions

The aim of this section is to discuss the basic definitions of SVNSs and TSFSs and related terms. We aim to examine both generalizations of the IFSs critically and point out the weak and strong points.

The concept of SVNS is developed by Wang et al. [11] after Smarandache [12] proposed the concept of NS theory. The reason to introduce the notion of SVNS is due to the limitations of IFSs proposed by Atanassov [5]. The concept of IFS is defined as:

8.1.1. Definition

On a set X, an IFS is of the shape $I = \left\{\left(\varkappa, \left(ş(\varkappa), \text{d}'(\varkappa)\right)\right) : 0 \leq şum(ş, \text{d}') \leq 1\right\}$. Further, $r(\varkappa) = 1 - şum(ş, \text{d}')$ represents the hesitancy degree of $\varkappa \in X$ and the pair $(ş, \text{d}')$ is termed as a IFN.

So far, several generalizations of IFSs are proposed known as PyFS, q-ROFS, SVNS, PFS, SFS and TSFS. There are two main drawbacks in IFS; one, it restricts assigning values to MG and NG in a certain range as depicted in Figure 1. Two, it described only two aspects of human opinion i.e. MG and NG and there is no discussion about the AG and RG. However, it described the hesitancy degree but that is dependent on MG and NG. To enhance the first drawback of IFS, Yager developed the concept of PyFS [6] and q-ROFS [8]. To deal with second drawback, Smarandache [12] proposed neutrosophic logic and NS theory which were further extended to the concept of SVNS proposed by Wang et al. [11]. The concept of SVNS proposed by Wang et al. [11] is defined as:

8.1.2. Definition [17]

On a set X, a SVNS is of the shape $I = \left\{\left(\varkappa, \left(ş(\varkappa), i(\varkappa), \text{d}'(\varkappa)\right)\right) : 0 \leq şum(ş, i, \text{d}') \leq 3\right\}$. Further, the triplet $(ş, i, \text{d}')$ is termed as a SVNN.

If we observe the definition, Wang et al. [11] introduced the neutral membership degree denoted by i left undescribed by Atanassov IFS. Further, in the definition, Wang et al. [11]

introduced the restriction on MG, neutral MG and NG as $0 \leq \text{ş}um(\text{ş}, i, d') \leq 3$. This condition, without any doubts, allows us to assign any value to $\text{ş}, i$ and d' from $[0, 1]$ interval. This concept of SVNS indeed generalizes the framework of IFS improving both the drawbacks that exists in IFSs.

However, there arises some questions:

1. What is the significance of using the restriction "$0 \leq \text{ş}um(\text{ş}, i, d') \leq 3$"? As in almost all the fuzzy frameworks the lower and upper limits are 0 and 1.
2. What would be the meaning of a SVNN described as $(1, 1, 1)$? Because generally if MG increases the other two decreases in any real-life event. For example, if we take the phenomenon of voting. There is a fixed number of voters, so if the number of "votes in favor" of someone increases. The number of "votes against" hence decreased in general but in SVNS, it does not seem so.
3. How could we describe the SVNN expressed as $(1, 1, 0)$. Though it is according to definition but its ambiguous.
4. In real life situations, yes, neutral MG exists so is refusal grade. The NS theory provide no information about discussing the refusal grade of human opinion. However, it described the RG in along with AG combined.

In parallel but after few years of SVNSs, Cuong [9] introduced the idea of PFS as another generalization of IFS. Cuong's PFS is based on a MG, AG, NG and a RG and is defined as:

8.1.3. Definition [9]

On a set X, a PFS is of the shape $I = \{(\varkappa, (\c{s}(\varkappa), i(\varkappa), \mathrm{d}'(\varkappa))) : 0 \leq \c{s}um(\c{s}, i, \mathrm{d}') \leq 1\}$. Further, $r(\varkappa) = 1 - \c{s}um(\c{s}, i, \mathrm{d}')$ represents the RG of $\varkappa \in X$ and the triplet $(\c{s}, i, \mathrm{d}')$ is termed as a PFN.

Cuong's concept of PFS also described the abstinence and refusal aspect of human opinion left undescribed by Atanassov's IFS with a condition $0 \leq \c{s}um(\c{s}, i, \mathrm{d}') \leq 1$ and with a RG defined as $r(\varkappa) = 1 - \c{s}um(\c{s}, i, \mathrm{d}')$. This concept is also the generalization of IFS and described the RG as well along with AG. Because, when referring to real-life voting, Cuong suggested that one may vote in favor, or remain abstain, or vote against but there are some who do not come to vote at all. Hence, PFS described all the four aspects of human opinion. Further, in PFS if MG increases, the AG, NG and RG decreases keeping the condition i.e. $0 \leq \c{s}um(\c{s}, i, \mathrm{d}') \leq 1$ in observation. In other words, we can say that MG, AG, NG and RG are not independent.

Though, Cuong's PFS proved to be a more realistic approach but it has a restriction, $0 \leq \c{s}um(\c{s}, i, \mathrm{d}') \leq 1$, that keeps us in a certain range for assigning the values to MG, AG and NG. For example, if a situation is considered where the triplet $(\c{s}, i, \mathrm{d}')$ has values from $[0, 1]$ such as $\c{s} = 0.99, i = 0.57$ & $\mathrm{d}' = 0.49$. In such case, the sum of components i.e. $\c{s}um(\c{s}, i, \mathrm{d}')$ exceeds $[0, 1]$. However, if the power on constraints is raised to n where $n \in \mathbb{Z}^+$ then we can assign any value of our choice to $\c{s}, i, \mathrm{d}'$ in the interval $[0, 1]$. In the current example if n is taken as 6. Then $\c{s}um(\c{s}^6, i^6, \mathrm{d}'^6) = 0.986556 \in [0, 1]$. The choice of n is up to decision makers and it may affect the results in aggregation process. We recalled the definition of TSFS as:

8.1.4. Definition

For any universal set X, a TSFS is of the form $I = \{(\varkappa, (\c{s}, , \text{d}')) \ \forall \varkappa \in X\}$. Here $\c{s}, i$ and d' are mappings from $X \rightarrow [0, 1]$ denoting the MG, AG and NG respectively provided that for some least $n \in \mathbb{Z}^+$, $0 \leq \c{s}um\left(\c{s}^n(\varkappa), i^n(\varkappa), \text{d}'^n(\varkappa)\right) \leq 1$ and $r(\varkappa) = \sqrt[n]{1 - \c{s}um\left(\c{s}^n(\varkappa), i^n(\varkappa), \text{d}'^n(\varkappa)\right)}$ is known as the RD of $\varkappa$ in I. The triplet $(\c{s}, i, \text{d}')$ is considered as a T-spherical fuzzy number (TSFN).

A TSFS have all the features that a PFS has and the plus point is TSFS has no limitation as for every triplet $(\c{s}, i, \text{d}')$ we have an n such that $0 \leq \c{s}um\left(\c{s}^n(\varkappa), i^n(\varkappa), \text{d}'^n(\varkappa)\right) \leq 1$. This means that a TSFS allows the MG, AG and NG to be independent (except the absolute cases i.e. $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$). However, the RG of TSFS is still dependent on other grades.

Now we establish a comparison between TSFS and SVNS by discussing their similarities and differences as follows.

8.1.5. Similarities

The similarities between SVNS and TSFS are as follows:

1. Both TSFS and SVNS discussed the abstinence of real-life events with an AG.
2. Both TSFS and SVNS generalizes the concept of IFS and can handle the problem which IFS fails to handle.
3. In both frameworks, we the MG, AG, NG are independent of each other and can have any value from $[0, 1]$. However, in TSFSs, this independency is not possible in absolute cases.

8.1.6. Differences

The differences of SVNS and TSFS are as follows:

1. SVNS discussed MG, AG and NG but not RG as it combined RG with AG while TSFS discussed the RG separately.
2. The definition of SVNS leads to some SVNNs which has no meaning i.e. it allows the SVNNs like $(1,1,1)$ and $(1,1,0)$ while TSFS has no such cases.
3. The restriction of SVNS i.e. "$0 \leq şum(ş, i, d') \leq 3$" allows the MG, AG and NG independent by enlarging the restriction from 1 to 3. However TSFS also ensure independency of MG, AG, NG by introducing a parameter n keeping the range of restriction to 1.

The following are two remarks show the relationship of TSFNs and SVNSs.

8.1.7. Remark

In general, every SVNN can be considered as a TSFN.

Proof: This is true as for every SVNN $(ş, i, d')$ we can have an n such that $0 \leq şum(ş^n, i^n, d'^n) \leq 1$.

Some absolute cases such as (1,1,1), $(1,1,0)$, $(0,1,1)$, $(1,0,1)$ cannot be considered as TSFNs.

8.1.8. Remark

Every TSFN can be considered as a SVNN.

In the next section, we apply the AOs and SMs of both TSFS and SVNS to a common problem to analyze the results.

8.2. Comparative Study in Multi-Attribute Decision Making

The aim of this section is to investigate a MADM problem using AOs of TSFSs and SVNSs to analyze the results. For this purpose, we take a decision matrix from Example 5.5.2 with 4 alternatives say A_1, A_2, A_3 and A_4 and 4 attributes say H_1, H_2, H_3 and H_4. The weight vector for this problem is $w = (0.2, 0.1, 0.3, 0.4)^t$.

We already solved this problem using TSFWA aggregation operator in Section 5.5 and the ranking results are provided in Table 38.

Score values using TSFWA operator	
$SC(A_1)$	0.10817
$SC(A_2)$	0.185307
$SC(A_3)$	0.140504
$SC(A_4)$	0.233403

Table 38 (Score values using TSFWA operator)

Now we solve this problem using WAA operators of SVNS proposed by Sahin [089]. For convenience, the decision matrix discussed in Example 5.5.2 is provided again in Table 39.

	H_1	H_2	H_3	H_4
A_1	$(0.53, 0.33, 0.38)$	$(0.65,0.24,0.74)$	$(0.61, 0.39, 0.45)$	$(0.55, 0.88, 0.29)$
A_2	$(0.40, 0.71, 0.15)$	$(0.48, 0.46,0.67)$	$(0.69, 0.46, 0.29)$	$(0.61, 0.73, 0.43)$
A_3	$(0.33, 0.53, 0.79)$	$(0.71, 0.49, 0.16)$	$(0.53, 0.39, 0.84)$	$(0.50,0.90, 0.01)$
A_4	$(0.64, 0.38,0.73)$	$(0.33,0.64,0.76)$	$(0.27,0.89,0.07)$	$(0.74, 0.36,0.19)$

Table 39 (Decision Matrix)

We applied the following WAA operator of SVNSs to the decision matrix given in Table 39 and the aggregated results for each alternative are listed as follows:

$$A_1 = (0.548023, 0.49755, 0.383533)$$

$$A_2 = (0.602926, 0.603509, 0.323543)$$

$$A_3 = (0.521966, 0.592777, 0.11946)$$

$$A_4 = (0.623935, 0.505723, 0.211727)$$

The score values of alternatives based on score value of SVNNs proposed by Sahin [089] are given in Table 40:

Score values using TSFWA operator	
$SC(A_1)$	0.083624
$SC(A_2)$	−0.0097
$SC(A_3)$	0.077252
$SC(A_4)$	0.056143

Table 40 (Score values using SVNWA operator)

If we compare the score values of alternatives using TSFWA operators and SVNWA operator, it is observed that the results are quite different. The reason behind this is that in

SVNS, the RG is totally ignored while computing the scores. In fact, there is no concept of RG in SVNSs. That is the reason that the AOs of both TSFSs and SVNSs produce different results from same data. The geometrical comparison of the ranking results of the alternatives is given in Figure 16.

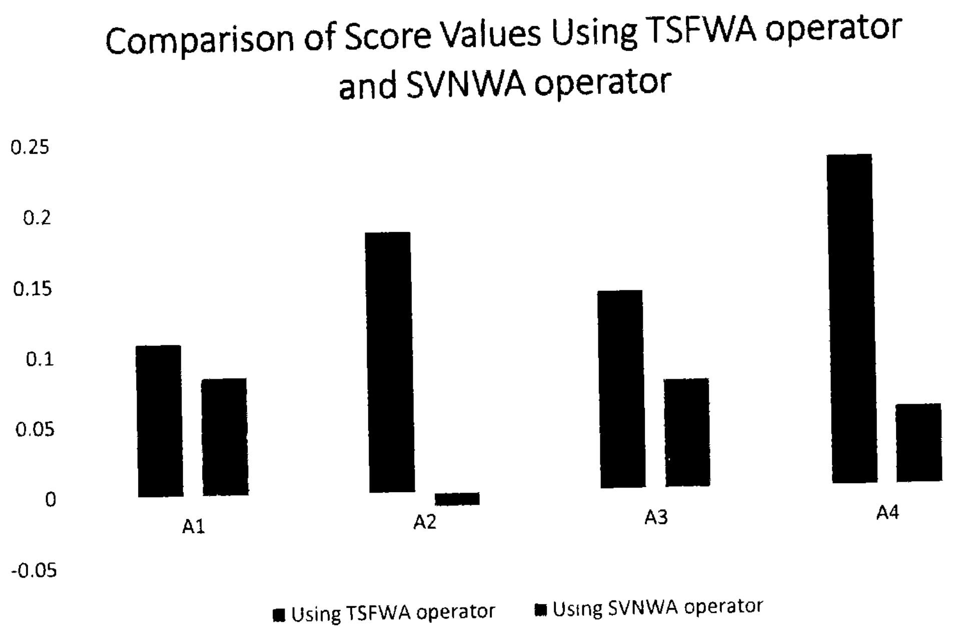

Figure 16 (Comparison of score values)

Figure 16 clearly indicate that TSF AOs yields A_4 as best alternative while AOs of SVNNs yields A_1 as best alternative. This huge difference is only because TSFS allows you to consider RG while SVNS does not.

8.3. Comparative Study in Pattern Recognition

The aim of this section is to investigate a pattern recognition problem using CCs of TSFSs and SVNSs. To serve the purpose, we consider Example 3.5.1.1 from Section 3.5.1 and solve it using the CCs of SVNSs proposed by Ye [99]. For convenience, we consider the information about four building material and an unknown material from Example 3.5.1.1 in Table 41.

	P_1	P_2	P_3	P_4	P
x_1	(0.56, 0.47, 0.22)	(0.81,0.3,0.37)	(0.43, 0.43, 0.55)	(0.57, 0.51, 0.39)	(0.34,0.56,0.78)
x_2	(0.11, 0.11, 0.11)	(0.59, 0.66,0.66)	(0.91, 0.34, 0.68)	(0.56,0.76,0.31)	(0.47, 0.38,0.84)
x_3	(0.35, 0.45, 0.61)	(0.42,0.56, 0.71)	(0.81, 0.41, 0.35)	(0.27,0.59, 0.72)	(0.55,0.44,0.65)
x_4	(0.33, 0.54,0.31)	(0.59,0.45,0.9)	(0.44,0.55,0.77)	(0.46, 0.46,0.45)	(0.76,0.46,0.85)
x_5	(0.35, 0.2, 0.64)	(0.16,0.33,0.42)	(0.55, 0.44, 0.29)	(0.57,0.66,0.91)	(0.13, 0.35,0.57)
x_6	(0.47, 0.37, 0.68)	(0.68, 0.46, 0.88)	(0.47, 0.66, 0.75)	(0.41,0.73, 0.41)	(0.24, 0.54, 0.45)
x_7	(0.78, 0.55,0.03)	(0.49, 0.54, 0.39)	(0.58,0.34, 0.23)	(0.21, 0.43, 0.13)	(0.82,0.46, 0.69)

Table 41 (Data on building materials)

Now, we apply CCs of SVNSs proposed by Ye [99] and CC of TSFSs proposed in Definition 4.2.2 on the information provided in Table 41, we get the following results in Table 42.

Similarity Measures	(P_1, P)	(P_2, P)	(P_3, P)	(P_4, P)
CC_{SVNS} [99]	0.381181	0.544776	0.520383	0.469287
CC_{TSFS} [4.2.8]	0.650146	0.730458	0.684701	0.444453

Table 42 (Correlation coefficient of P_i with P in neutrosophic environment)

The geometrical demonstration of the CCs obtained in Table 42 is provided in Figure 17.

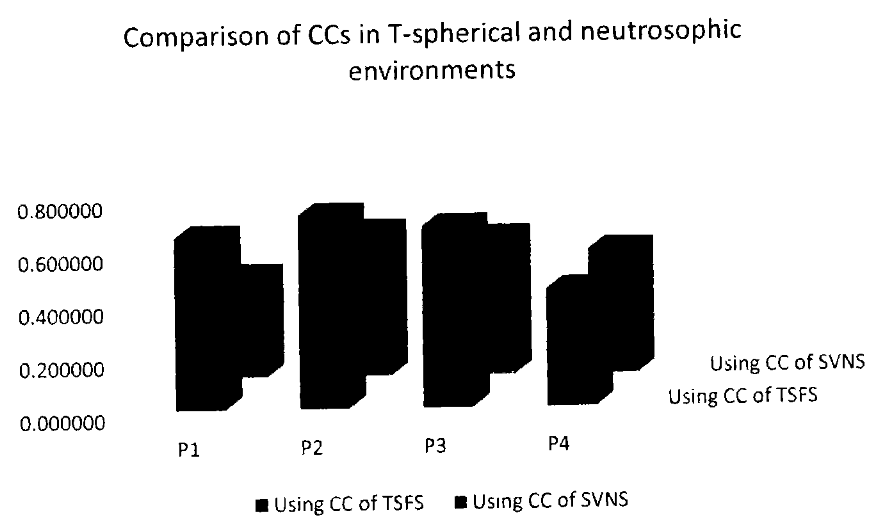

Figure 17 (Comparison of CCs in T-spherical and neutrosophic environments)

Upon analyzing information obtained in Table 42 and Figure 17, it can be seen that the unknown building material P has a greater CC with P_2 using CC of TSFSs or SVNSs. However, it is eminent from the data that the CC of P and P_2 in TSF environment is 0.73 which much higher than the CC of P and P_2 in neutrosophic environment which is 0.54. This difference in behavior is prompted only because of the involvement of RG in the information which is considered in TSF environment but not in neutrosophic environment.

Based on the critical analysis of TSFSs and SVNSs in Section 8.1, 8.2 and 8.3, we conclude our comparative analysis in the following Section 8.4.

8.4. Conclusion of Comparative Study of TSFS and SVNSs

The aim of this section is to draw some conclusive points from the comparative study established in this chapter. We established a comparison in basic operations of TSFSs and SVNSs and then discussed two problems in these environments to signify their importance. Some key points of conclusion are as follows:

1. Every SVNN is a TSFS in general.
2. Every TSFN can also be considered as a SVNN but in that case, the RG is ignored.
3. There are some triplets which are SVNNs but having no meaning. In case of TSFSs, this does not happen.
4. The concept of RG is not defined in neutrosophic environment however it is defined in TSF environment and plays a significant role in real-life problems.
5. In SVNSs, increment or decrement in the values of the MG, AG and NG does not affect each other at all and provides independency for MG, AG and NG. However in TSFSs, the MG, AG and NGs are independent but the RG is dependent on these three grades for some $n \in \mathbb{Z}^+$.

Keeping in mind all these conclusive points, it is evident that both, TSFS and SVNS are handful in dealing with real life problems than other fuzzy frameworks such as IFS, PyFS, q-ROFS, PFS and SFS.

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