Some Contributions to Orthopair Fuzzy Sets and Their Applications

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A THESIS SUBMITTED IN PRACTICAL FULLFILLMENT OF THE REQUIRNMENT FOR THE DEGREE OF DOCTOR OF PHILISOPHY (PhD) IN MATHEMATICS AT THE DEPARTMENT OF MATHEMATICS & STATISTICS, FACULTY OF BASIC AND APPLIED SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY ISLAMABAD

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2020

Dedicated

To

My

Loving Mother and Father (late)

My family

And respectful teachers

Whose affection is reason on every success in my life.

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First and foremost, all praise to **'ALLAH'**, Lord of the world, the Almighty, and I find no words to thank **'ALLAH'**, who created me and all the universe around us. Who is the sole creator of all the things tinier than an electron to huge galaxies. Special praises and thanks for his last messenger, the holy Prophet Muhammad (SAW). The holy Prophet said that I am the light, whoever follows me, will never be in darkness. Special praises for Ima Athar (as) whose teachings are a blessing for the whole mankind. May Allah guide us and the whole humanity to the right path.

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0.0. Research Profile

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5. **A. Hussain**, T. Mahmood and M.I. Ali. Dombi Aggregation operators for q-Rung Orthopair Fuzzy Soft Sets with Their Application in Multicriteria Decision Making. (submitted)
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0.1. Introduction

Nowadays the process of decision making ($\mathcal{DM}$) is a complex issue involving professionals of different genre. Every organization has to take decision at one or another point as a part of managerial process. Therefore, every organization extensively needs a team of professional experts to make every kind of complex decision. Multi-criteria decision making ($\mathcal{MCDM}$) has a high potential and disciplined process to improve and evaluate multiple conflicting criteria in all areas of the $\mathcal{DM}$. In this competitive environment, an enterprise needs the most accurate and rapid response to change the customer needs. So, $\mathcal{MCDM}$ has the ability to handle successfully the evaluation process of multiple contradictory criteria. For an intelligent decision, the experts analyze each and every character of an alternative and then they take the decision. But remember, that an individual alone cannot come out with final decision because $\mathcal{DM}$ problems consist of cumulative and consultative process. $\mathcal{DM}$ plays a significant role and is the key component to determine both organizational and managerial activities. Since intellectual minds are engaged in this process, so it needs solid scientific knowledge coupled with experience and skills in addition to mental maturity. For an intelligent and successful decision, the experts require a careful preparation and analysis of each and every character for an alternative and they can take a good decision if they are armed with all the data and information they need. To handle this complexity Zadeh [1] originated a dominant and pioneer concept of fuzzy set (FS). For each domain in FS, a value is assigned from unit closed interval and called membership grade ($\mathcal{MG}$). From the inception of FS it has been generalized in different directions from which one of the most significant concept is intuitionistic fuzzy (IF) set (IFS). Atanassov [2] initiated this dominant concept of IFS which is characterized by two mappings called $\mathcal{MG}$ and nonmembership grade ($\mathcal{NMG}$). IFS is defined on the bases of restriction which means that the sum of $\mathcal{MG}$ and $\mathcal{NMG}$ must not exceed the unit interval $[0, 1]$. Similarly, in many situations of real life problems, usually people irresolute or hesitant to assign $\mathcal{MG}$ to an alternative according to its corresponding attribute that make complication for experts for final decision. Therefore, Torra [3] initiated the notion of hesitant fuzzy set that allows a single alternative of the reference set which has several possible values as a $\mathcal{MG}$. The notion of IFS appears as a hot research area after its origination. Since the inception of IFS' researches proposed various sources in different directions from which one of the key concept is of the aggregation operators. Aggregation operators have the ability to reduce the set of finite values in $\mathcal{DM}$ process into a single value which was a major issue for decision makers that how to get the unique result from the collected information taken from different sources. Xu [4] investigated the series of aggregation operators such as IF weighted averaging (IFWA), IF ordered weighted averaging (IFOWA) and IF hybrid averaging (IFHA) operators under IF environment. The series of geometric operators namely IF weighted geometric (IFWG), IF ordered weighted geometric (IFOWG) and IF hybrid geometric (IFHG) operators were presented by Xu and Yager [5]. Zhao et al. [6] initiated the idea of generalized IFWA, generalized IFOWA and generalized IFHA operators by utilizing the IF information. Wang and Liu [7, 8] presented the notion of IF Einstein weighted averaging and geometric (IFEWA/G) aggregation operators by using Einstein operations. Zhao and Wei [9] investigated the idea of the IF Einstein hybrid averaging and the IF Einstein hybrid geometric operators, and then presented their application in decision making. He et al. [10] initiated the idea of IF interactive aggregation operators. Garg [11, 12] proposed the generalized concept of IF interactive operators and present novel IF operational laws. Ye [13] investigated hybrid arithmetic and geometric operators and initiated their applications in $\mathcal{DM}$ by using IF environment. Liu et al. [14] presented the study of prioritized aggregation operator for hesitant IFS. In literature, different techniques were used to handle the ranking with score or accuracy functions but all these techniques had some drawbacks. So, Ali et al. [15] initiated a graphical technique for ranking the IF values. From the inception and appearance of the dominant concept of IFS a lot of research was done by different scholars in several directions. However, there exists some deficiency in this prominent notion due to which it fails to handle the situation and it is not always possible for the professional experts to provide their choices in the range of IFS.

To cover this shortcoming, Yager [16] investigated the powerful paradigm of Pythagorean fuzzy set (PyFS) in which the square sum of $\mathcal{MG}$ and $\mathcal{NMG}$ must lie between the real numbers $0 \; and \; 1$. PyFS relaxes and widens the boundary range by providing additional space to the decision makers. Yager [17] originated the PyF weighted averaging and weighted geometric (PyFWA/ PyFWG) aggregation operators under the PyF environment. Peng and Yang [18] initiated the concept of subtraction and division operators, and proved some of its basic properties. Peng and Yang [19] investigated the notion of PyF Choquet integral average and PyF Choquet integral geometric operators. Garg [20, 21] proposed some PyF Einstein averaging and PyF Einstein geometric operators and presented their basic characteristics. Garg [22] investigated confidence PyF weighted and ordered weighted averaging operators with their basic properties. The idea of symmetric PyFWA (SPyFWA) and symmetric PyFWG (SPyFWG) operators are initiated by Ma and Xu [23]. Wei and Lu [24] proposed the concept of PyF power averaging and power geometric operators and presented their desirable characteristics of these investigated operators. Wei [25] presented some interaction averaging and geometric operators by using PyF information. The concept of Hamacher operations for PyF averaging and geometric operators was presented by Wu and Wei [26]. However, PyFS also has some shortcomings and decision makers are restricted to their boundary limitation and they cannot provide their preferred values freely.

In many scenarios of real life's problems professional experts have diverse opinions to handle the $\mathcal{DM}$ problems in which some energetic perspective are in support or against of some plans, entities or projects. For example, in a certain country government lunched a mega project to portray its achievement and performance. The ruling party leaders and members highly appreciate and rate their project by assigning $\mathcal{MG}$ about 0.92, whereas the opposition leaders depreciate the same project and have strong opposite point of views about it and try to defame it by providing $\mathcal{NMG}$ may be 0.85. So in this case $(0.92)^2 + (0.85)^2 > 1$ but $(0.92)^q + (0.85)^q < 1$ for $q \geq 6$. Due to restrictions, this decisive information cannot be effectively handled by IFS and PyFS.

Recently, some improvements have been made in the dominant notion of FS as, Yager [27] investigated the generalized concept of FS, IFS and PyFS and called it q-rung orthopair fuzzy (q-ROF) set (q-ROFS). It is observed that the parameter q is the most useful characteristic of this concept which has the capability to cover the boundary range that can be required. The input range of q-ROFS is more flexible, wider and suitable because when the rung increases, the orthopair provides additional space to the boundary constraint. In q-ROFS, the sum of qth power of $\mathcal{MG}$ and qth power of $\mathcal{NMG}$ must be confined to the unit interval $[0,\ 1]$ for $q \geq 1$. Thus, the concept of q-ROFS is more powerful and stronger than IFS and PyFS because these are the special cases of q-ROFS. The basic properties of q-ROFS are proposed by Yager and Alajlan [28] and have been utilized in knowledge representation. Ali [29] presented another view of q-ROFS by using the concept of orbits. The concepts of q-ROF weighted averaging (q-ROFWA) and q-ROF weighted geometric (q-ROFWG) were proposed by Liu and Wang [30]. Liu and Liu [31] presented the combined study of Bonferroni mean (BM) operators and q-ROFS to investigate the q-ROF BM operators and also studied q-ROF geometric BM operators with their desirable properties. Wang et al. [32] investigated the combine concept of Muirhead means (MM) operators with q-ROFS to get the new aggregation operators that are q-ROF MM operators and give their applications in decision making. Joshi and Gegov [33] incorporated the confidence level of experts to the original information under q-ROF environment to propose some aggregation operators such as confidence q-ROFWA (Cq-ROFWA) and confidence q-ROFWG (Cq-ROFWG), Cq-ROFOWA, Cq-ROFOWG operators. Yang et al. [34] presented the concept of q-RO normal fuzzy sets and defined the operational laws and score function for it. They also initiated some aggregation operators for the same concept that are q-RONFWA, q-RONFOWG. Furthermore, Hussain et al. [35] proposed hesitant q-ROFWA and hesitant q-ROFWG operators and discussed their desirable properties.

The dominant concept of rough set (RS) was first proposed by Pawlak [36] who generalized the classical set theory to cope with the imprecise, vague and uncertain information. By the definition of Pawlak's RS, a universal set is characterized by two approximation sets known as upper and lower approximations. The lower approximation consists of those alternatives which contain a subset and the upper approximation consists of those alternatives having nonempty intersection with a subset. Further equivalence relation plays a key role in Pawlak's RS for approximations but this condition too restricts the practical and theoretical aspects of RS. So, researchers used generalized structure by using nonequivalence structure for detail [37, 38, 39, 40, 41, 42]. From the inception researchers used the hybrid study of rough set theory (RST) with different concepts. The hybrid study of RS and IFS was proposed by Chakrabarty [43] to obtain the notion of IF rough set (IFRS) and IFRS became the hot and progressive research area for the researchers, for detail see [44, 45, 46, 47]. Zhou and Wu [48] proposed the combined study of RS and IFS by using crisp and fuzzy approximation space. Zhou and Wu [49] initiated the constructive and axiomatic approaches under the IF rough environment. Hussain et al. [50] initiated the concept of rough PyF ideals by using the algebraic structure of semigroups. Zhang et al. [51]

proposed multigranulation rough set technique over two universes by using the PyF environment. Hussain et al. [52] presented the concept of covering based q-ROF rough set model by utilizing fuzzy β-covering and fuzzy β-covering neighborhood.

Molodtsov [53] investigated the pioneer concept of soft set (S_{ft}S) which generalized the classical set theory and is free from inherit complexity which the contemporary theories faced. It is observed that S_{ft}S has very close relation with fuzzy set and rough set. The S_{ft}S theory regarded as an effective mathematical tool for handling the uncertain, ambiguous and imprecise data. Maji et al. [54] proposed the hybrid notion of S_{ft}S with fuzzy set to obtain fuzzy S_{ft}S (FS_{ft}S) which plays a bridge role between these two theories. Ali et al. [55] improved some existing definition and operations in S_{ft}S theory. Maji et al. [56] investigated the hybrid notion of S_{ft}S and IFS to achieve IF soft (IFS_{ft}) set (IFS_{ft}S) which play a key role for the scholars. The concept of generalized IFS_tS was proposed by Agarwal et al. [57]. Arora and Garg [58] presented the concept of IFS_{ft} weighted averaging (IFS_{ft}WA) and IFS_{ft} weighted geometric (IFS_{ft}WG) operators. Garg and Arora [59] proposed the notion of some power averaging and geometric aggregation operators by utilizing generalized IFS_{ft}S. Arora [60] initiated the notion of IFS_{ft}WA and IFS_{ft}WG by using the Einstein operations. Feng et al. [61] improved some existing literature related to generalized IFS_{ft}S and proposed some new operations for the developed concept. Hussain et al. [62] presented the combined study S_{ft}S, rough set and PyFS to achieve the new concept of soft rough PyFS (S_{ft}RPyFS) and PyF soft RS (PyFS_{ft}RS). Riaz and Hashmi [63] presented the hybrid study of S_{ft}S, rough set, PyFS and m-polar fuzzy set to get the new notion of Pythagorean m-polar fuzzy soft rough set. Hussain et al. [64] presented the hybrid structure of S_{ft}S with q-ROFS to get the prominent concept of q-ROF soft (q-ROFS_{ft}) set (q-ROFS_{ft}S) and proposed some aggregation operators such as q-ROFS_{ft} weighted averaging (q-ROFS_{ft}WA), q-ROFS_{ft} ordered weighted averaging (q-ROFS_{ft}OWA) and q-ROFS_{ft} hybrid averaging (q-ROFS_{ft}HA).

Dombi [65] initiated the new concept of Dombi t-norm and Dombi t-conorm operators having good precedence with general operational parameter which possess the resilience of variability. The behaviour of operational parameter is very important to express the experts' attitude in decision making. Seikh et al. [66] presented IF Dombi weighted averaging (IFDWA) and IF Dombi weighted geometric (IFDWG) operators based on Dombi t-norm and t-conorm. Liu et al. [67] initiated the idea of IF Dombi bonferroni mean operators and proposed their application for decision making. Akram et al. [68] and Jana et al. [69] presented the concept of PyF Dombi weighted averaging (PyFDWA) and PyF Dombi weighted geometric (PyFDWG) aggregation operators. Also Khan et al. [70] initiated the Dombi operations in PyF environment. Jana et al. [71] gave the idea of q-ROF Dombi weighted averaging and Dombi weighted geometric (q-ROFDWA and q-ROFDWG) aggregation operators with their fundamental desirable characteristics, Zhong et al. [72] investigated the concept of power partitioned Heronian mean operators based on Dombi operation law for q-ROF environment. From the analysis of existing literature, it is observed that aggregation operators have a great importance in decision making to aggregate the collective evaluated information of different sources into a single value. According to the best of our knowledge up-till now no application of the aggregation operators with the hybridization of q-ROFS with soft set and rough set is reported in q-ROF environment. Therefore, this motivates the thesis towards the combined study of S_{ft}S, RS, PyFS and q-ROFS to get the new concepts of PyFS_{ft}RS, q-ROFRS, q-ROFS_{ft}S, q-ROFS_{ft}S by using Dombi operations, q-ROFS_{ft} rough set (q-ROFS_{ft}RS) and further we investigated aggregation operators based on soft information and soft rough information in this thesis. Moreover, the basic desirable properties of these aggregation operators are investigated in detail. Different techniques for $\mathcal{MCDM}$ and step wise algorithms for $\mathcal{DM}$ are demonstrated by utilizing the proposed approaches. Numerical examples for developed approaches are presented and comparative study of the investigated models with some existing methods are done in detail in chapter wise which show that the proposed models are more effective and applicable than existing approaches.

0.2. Chapter wise study

Chapter 1

Chapter one is devoted for the basic and rudimentary concepts and definitions concerning FS, IFS, PyFS, q-ROFS, RS and S_{ft}S which will be helpful for our subsequent chapters. Furthermore, some fundamental operational laws on these concepts are presented.

Chapter 2

In this Chapter we are going to present the hybrid study of S_{ft}Ss, RSs and PyFSs to get the new concepts of soft rough Pythagorean fuzzy sets (S_{ft}RPyFS) and Pythagorean fuzzy soft rough sets (PyFS_{ft}RS). The aim of this chapter is to originate the two new notions that are S_{ft}RPyFS and PyFS_{ft}RS, and to investigate some important properties of S_{ft}RPyFS and PyFS_{ft}RS in detail. Furthermore, classical representations of PyFS_{ft}R approximation operators are presented. Then the proposed operators are applied to $\mathcal{DM}$ problem in which the experts provide their preferences in PyFS_{ft}R environment. Finally through an illustrative example, it is shown that how the proposed operators work in decision making problems.

Chapter 3

In this chapter a comprehensive model is originated to handle the $\mathcal{DM}$ problems in which experts have quite different opinions in favor or against of some plans, entities or projects. Therefore, a new technique is adopted to investigate the hybrid notion of RS with q-ROFSs by using the concept of fuzzy β-covering and fuzzy β-covering neighborhoods to get the new notion of covering based q-ROF rough set (CBq-ROFRS). Furthermore, by applying the developed concept of CBq-ROFRS to TOPSIS and its application to multi-attribute decision making ($\mathcal{MADM}$) are discussed in detail. In real scenario CBq-ROFRS model is an important tools to discuss the complex and uncertain information. This method has stronger capacity than IFS and PyFS to cope with uncertainty. From the analysis, it is clear that CBq-ROFRS degenerate into covering based IF rough set (CBIFRS) if the rung $q = 1$ and degenerate into covering based PyF rough set (CBq-PyFRS) if the rung $q = 2$. Thus the proposed concept is generalization of CBIFRS and CBPyFRS. Moreover, an illustrative example is presented to demonstrate how the developed model helps us in $\mathcal{DM}$ problems and a comparative study of the proposed model with some existing methods are presented whish show that the developed approach is more capable and superior than existing methods.

Chapter 4

In 1999, Molodtsov investigated the pioneer notion of S_{ft}Ss which provides a general framework for mathematical problems by affix parameterization tools during the analysis as compared to FSs and q-ROFSs. From the analysis of existing literature and best of our knowledge, there has been no research on the hybrid model of S_{ft}Ss and q-ROFSs that is q-rung orthopair fuzzy soft set (q-ROFS_{ft}S). Therefore, for the scope of future motive, the proposed concept has enough space for the new research. The aim of this chapter is to investigate the notion of q-ROFS_{ft}S, which plays a bridge role between these two concepts. Therefore, our main contribution in this chapter is to investigate the q-ROFS_{ft} weighted averaging (q-ROFS_{ft}WA), q-ROFS_{ft} ordered weighted averaging (q-ROFS_{ft}OWA) and q-ROFS_{ft} hybrid averaging (q-ROFS_{ft}HA) operators under q-ROFS_{ft} environment. Further, the fundamental properties of these aggregation operators are studied. On the bases of developed approach an algorithm for $\mathcal{MCDM}$ is being presented. An application of medical diagnosis problems is solved on the proposed algorithm under the q-ROFS_{ft} environment. Finally, a comparison of the developed operators with some existing operators are being presented showing the superiority and efficiency of the developed approach than the existing literature.

Chapter 5

This chapter consists of the combined study of the pioneer paradigm of S_{ft}S and q-ROFS that is the notion of q-ROFS_{ft}S. The notion of q-ROFS_{ft}S is free from that inherited complexities which are associated to the contemporary theories. In this chapter our main contribution is to originate the concept of q-ROFS_{ft} weighted geometric (q-ROFS_{ft}WG), q-ROFS_{ft} ordered weighted geometric (q-ROFS_{ft}OWG) and q-ROFS_{ft} hybrid geometric (q-ROFS_{ft}HG) operators in q-ROFS_{ft} environment.

Moreover, some dominant properties such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity of these developed operators are studied in detail. Based on these proposed approaches, a model is build up for $\mathcal{MCDM}$ and its step wise algorithm is being presented. Finally, utilizing the developed approach an illustrative example is solved under q-ROFS_{ft} environment. Further, a comparative analysis of the investigated models with existing methods is presented with detail which shows the competence and ability of the developed models.

Chapter 6

Recently, some improvement has been made in the dominant notion of fuzzy set that is Yager investigated the generalized concept of FS, IFS and PyFS which he called it q-ROFS. It is observed that the rung q is the most useful characteristic of this concept which has the capability to cover the boundary range that can be required. The input range of q-ROFS is more flexible, wider and suitable because when the rung q increase, the orthopair provides additional space to the boundary constraint. The aim of this chapter is to present the Dombi aggregation operators using q-ROFS_{ft} environments. Since Dombi operational parameter possess natural flexibility with resilience of variability. The behaviour of Dombi operational parameter is very important to express the experts' attitude in decision making. In this chapter, we present q-ROFS_{ft} Dombi average (q-ROFS_{ft}DA) aggregation operators including q-ROFS_{ft} Dombi weighted average (q-ROFS_{ft}DWA), q-ROFS_{ft} Dombi ordered weighted average (q-ROFS_{ft}DOWA) and q-ROFS_{ft} Dombi hybrid average (q-ROFS_{ft}DHA) operators. The basic properties of these operators are presented in detail such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity. By applying these developed approaches, this chapter contains the technique and algorithm for $\mathcal{MCDM}$. Further a numerical example is developed to illustrate the flexibility and applicability of the developed operators.

Chapter 7

The aim of this chapter is to present the notion of q-ROFS_{ft}S based on the Dombi operations. Since Dombi operational parameter possess natural flexibility with resilience of variability. The behaviour of Dombi operational parameter is very important to express the experts' attitude in decision making. In this chapter, we investigate q-ROFS_{ft} Dombi geometric (q-ROFS_{ft}DG) aggregation operators including q-ROFS_{ft} Dombi weighted geometric (q-ROFS_{ft}DWG), q-ROFS_{ft} Dombi ordered weighted geometric (q-ROFS_{ft}DOWG) and q-ROFS_{ft} Dombi hybrid geometric (q-ROFS_{ft}DHG) operators. The basic properties of these operators are presented in detail such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity. A $\mathcal{MCDM}$ technique and algorithm is developed based on above mentioned approach.

Chapter 8

The aim of this chapter is to investigate the hybrid concept of S_{ft}S and RS with the notion of q-ROFS to obtain the new notion of q-ROF soft rough set (q-ROFS_{ft}RS). In addition, some averaging aggregation operators such as q-ROFS_{ft}R weighted averaging (q-ROFS_{ft}RWA), q-ROFS_{ft}R ordered weighted averaging (q-ROFS_{ft}ROWA) and q-ROFS_{ft}R hybrid averaging (q-ROFS_{ft}RHA) operators are presented. Then basic desirable properties of these investigated averaging operators are discussed in detail. Moreover, we investigated the geometric aggregation operators such as q-ROFS_{ft}R weighted geometric (q-ROFS_{ft}RWG), q-ROFS_{ft}R ordered weighted geometric (q-ROFS_{ft}ROWG) and q-ROFS_{ft}R hybrid geometric (q-ROFS_{ft}RHG) operators, and proposed the basic desirable characteristics of investigated geometric operators. The technique for $\mathcal{MCDM}$ and step wise algorithm for $\mathcal{DM}$ by utilizing the proposed approaches are demonstrated clearly. Finally, a numerical example for the developed approach is presented and a comparative study of the investigated models with some existing methods is brought to light in detail which shows that the proposed models are more effective and applicable than existing approaches.

Chapter 1

Preliminaries

This chapter is devoted for the basic and rudimentary concepts and definitions concerning FS, IFS, PyFS, q-ROFS, RS and S_{ft}S which will be helpful for our subsequent chapters. Further, some fundamental operational laws on these concepts are presented.

1.1. Fuzzy sets

Zaheh [1] initiated the pioneer and dominant concept of FS in 1965, which brought revolution not only in the field of mathematics and logic but also in different fields of science and technology. This concept nicely handles the uncertainty by assigning the $\mathcal{MG}$ from the unit interval [0,1] and is defined as.

1.1.1. Definition [1]

Let T be a universal set. A FS $\mathcal{F}$ on T is of the form

$$\mathcal{F} = \{< \hbar, \mu_{\mathcal{F}}(\hbar) > | \hbar \in T\},$$

where $\mu_{\mathcal{F}}(\hbar)$ denotes the $\mathcal{MG}$, that is $\mu_{\mathcal{F}}: T \to [0,1]$ of an element $\hbar \in T$ such that $0 < \mu_{\mathcal{F}}(\hbar) < 1$. The collection of all FSs on the set T is represented by FS^T.

Let $\mathcal{F} = \{< \hbar, \mu_{\mathcal{F}}(\hbar) > | \hbar \in T\}$ and $\mathcal{F}_1 = \left\{< \hbar, \mu_{\mathcal{F}_1}(\hbar) > \middle| \hbar \in T\right\}$ be any two fuzzy sets. Then the basic operations on them are given as:

i. $\mathcal{F} \subseteq \mathcal{F}_1$ iff $\mu_{\mathcal{F}}(\hbar) \leq \mu_{\mathcal{F}_1}(\hbar)$ for all $\hbar \in T$;

ii. $\mathcal{F} \cup \mathcal{F}_1 = \left(\hbar, \max\left(\mu_{\mathcal{F}}(\hbar), \mu_{\mathcal{F}_1}(\hbar)\right)\right)$ for $\hbar \in T$;

iii. $\mathcal{F} \cap \mathcal{F}_1 = \left(\hbar, \min\left(\mu_{\mathcal{F}}(\hbar), \mu_{\mathcal{F}_1}(\hbar)\right)\right)$ for $\hbar \in T$;

iv. $\mathcal{F} = \mathcal{F}_1$ iff $\mathcal{F} \subseteq \mathcal{F}_1$ and $\mathcal{F}_1 \subseteq \mathcal{F}$;

v. $\mathcal{F}^c = \left(\hbar, 1 - \mu_{\mathcal{F}}(\hbar)\right)$ for $\hbar \in T$, where $\mathcal{F}^c$ is the complement of $\mathcal{F}$.

1.2. Intuitionistic fuzzy sets

From the above definition it is clear that FS deals only $\mathcal{MG}$, but in many scenario of real life not only the grade of $\mathcal{MG}$, the $\mathcal{NMG}$ also required. To cope on this shortcoming Atanassove [2] investigated the prominent concept of IFS, which covered the deficiency of FS. The IFS consists of $\mathcal{MG}$ and $\mathcal{NMG}$ and their sum belongs to [0,1] and is defined as:

1.2.1. Definition [2]

Let T be a universal of discourse. An IFS $\mathcal{J}$ on T is of the form

$$\mathcal{J} = \{< \hbar, \mu_{\mathcal{J}}(\hbar), \psi_{\mathcal{J}}(\hbar) > | \hbar \in T\},$$

where $\mu_{\mathcal{J}}(\hbar)\ and\ \psi_{\mathcal{J}}(\hbar)$ denotes the $\mathcal{MG}$ and $\mathcal{NMG}$ that is $\mu_{\mathcal{J}}: T \to [0,1]\ and\ \psi_{\mathcal{J}}: T \to [0,1]$ of an element $\hbar \in T$ with the constraint $0 \leq \mu_{\mathcal{J}}(\hbar) + \psi_{\mathcal{J}}(\hbar) \leq 1$. Moreover, the degree of hesitancy is given as $\pi_{\mathcal{J}} = 1 - \left(\mu_{\mathcal{J}}(\hbar) + \psi_{\mathcal{J}}(\hbar)\right)$. If there is no confusion then $\mathcal{J} =< \hbar, \mu_{\mathcal{J}}(\hbar), \psi_{\mathcal{J}}(\hbar) >$, is represented by $\mathcal{J} = \left(\mu_{\mathcal{J}}, \psi_{\mathcal{J}}\right)$ and is known to be IF value (IFV). The collection of all IFSs on the set T is represented by IFS^T. The graphical interpretation of IFS is given in Fig. 1.

Let $\mathcal{J}, \mathcal{J}_1 \in IFS^T$ be any two IFSs. Then the basic operations on them are given as:

i. $\mathcal{J} \subseteq \mathcal{J}_1$ iff $\mu_{\mathcal{J}}(\hbar) \leq \mu_{\mathcal{J}_1}(\hbar)\ and\ \psi_{\mathcal{J}}(\hbar) \geq \psi_{\mathcal{J}_1}(\hbar)$ for all $\hbar \in T$;

ii. $\mathcal{J} \cup \mathcal{J}_1 = \left(\hbar, \max\left(\mu_{\mathcal{J}}(\hbar), \mu_{\mathcal{J}_1}(\hbar)\right), \min\left(\psi_{\mathcal{J}}(\hbar), \psi_{\mathcal{J}_1}(\hbar)\right)\right)$ for $\hbar \in T$;

iii. $\mathcal{J} \cap \mathcal{J}_1 = \left(\hbar, \min\left(\mu_{\mathcal{J}}(\hbar), \mu_{\mathcal{J}_1}(\hbar)\right), \max\left(\psi_{\mathcal{J}}(\hbar), \psi_{\mathcal{J}_1}(\hbar)\right)\right)$ for $\hbar \in T$;

iv. $\mathcal{J} = \mathcal{J}_1$ iff $\mathcal{J} \subseteq \mathcal{J}_1$ and $\mathcal{J}_1 \subseteq \mathcal{J}$;

v. $\mathcal{J}^c = \left(\hbar, \psi_{\mathcal{J}}(\hbar), \mu_{\mathcal{J}}(\hbar)\right)$ for $\hbar \in T$, where $\mathcal{J}^c$ is the complement of $\mathcal{J}$.

1.2.2. Definition [4, 5]

Let $\mathcal{J} = \left(\mu_{\mathcal{J}}(\hbar), \psi_{\mathcal{J}}(\hbar)\right)$ and $\mathcal{J}_1 = \left(\mu_{\mathcal{J}_1}(\hbar), \psi_{\mathcal{J}_1}(\hbar)\right)$ be any two IFVs and $\lambda > 0$. Then the basic operations on them are given as:

i. Ring sum operation:

$$\mathcal{J} \oplus \mathcal{J}_2 = \left\{\left(\hbar, \mu_{\mathcal{J}}(\hbar) + \mu_1(\hbar) - \mu_{\mathcal{J}}(\hbar)\mu_{\mathcal{J}_1}(\hbar), \psi_{\mathcal{J}}(\hbar)\psi_{\mathcal{J}_1}(\hbar)\right) | \hbar \in T\right\};$$

ii. Ring product operation:

$$\mathcal{J} \otimes \mathcal{J}_1 = \left\{\left(\hbar, \mu_{\mathcal{J}}(\hbar)\mu_{\mathcal{J}_1}(\hbar), \psi_{\mathcal{J}}(\hbar) + \psi_{\mathcal{J}_1}(\hbar) - \psi_{\mathcal{J}}(\hbar)\psi_{\mathcal{J}_1}(\hbar)\right) | \hbar \in T\right\};$$

iii. $\lambda\, \mathcal{J} = \left(1 - \left(1 - \mu_{\mathcal{J}}(\hbar)\right)^{\lambda}, \psi_{\mathcal{J}}{}^{\lambda}(\hbar)\right);$

iv. $\mathcal{J}^{\lambda} = \left(\mu_{\mathcal{J}}^{\lambda}(\mathscr{k}), 1 - \left(1 - \psi_{\mathcal{J}}(\mathscr{k})\right)^{\lambda}\right).$

To make comparison between two or more IFVs various authors used several methods for ranking IFVs. First of all Chen and Tan [73] presented the concept of score function for ranking IFVs. Greater the score value better that IFV is and is defined as.

1.2.3. Definition [72]

Let $\mathcal{J} = (\mu_{\mathcal{J}}, \psi_{\mathcal{J}})$ be an IFV. Then a score function of $\mathcal{J}$ can be given as below:

$$\mathcal{S}c(\mathcal{J}) = \mu_{\mathcal{J}} - \psi_{\mathcal{J}}; \qquad \mathcal{S}c(\mathcal{J}) \in [-1,1].$$

Now comparing any two IFVs $\mathcal{J}_i = (\mu_{\mathcal{J}_i}, \psi_{\mathcal{J}_i})$ $(i = 1,2)$, then

i. $\mathcal{J}_1 \succ \mathcal{J}_2$ if $\mathcal{S}c(\mathcal{J}_1) > \mathcal{S}c(\mathcal{J}_2)$;
ii. $\mathcal{J}_1 \prec \mathcal{J}_2$ if $\mathcal{S}c(\mathcal{J}_1) < \mathcal{S}c(\mathcal{J}_2)$;

In case when $\mathcal{S}c(\mathcal{J}_1) = \mathcal{S}c(\mathcal{J}_2)$, then two IFVs can be compared by using accuracy function, which is defined as:

1.2.4. Definition [74]

Let $\mathcal{J} = (\mu_{\mathcal{J}}, \psi_{\mathcal{J}})$ be an IFV. Then an accuracy function of $\mathcal{J}$ can be denoted and defined as below:

$$Ac(\mathcal{J}) = \mu_{\mathcal{J}} + \psi_{\mathcal{J}}; \qquad Ac(\mathcal{J}) \in [0,1].$$

Now comparing any two IFVs $\mathcal{J}_i = (\mu_{\mathcal{J}_i}, \psi_{\mathcal{J}_i})$ $(i = 1,2)$, if $\mathcal{S}c(\mathcal{J}_1) = \mathcal{S}c(\mathcal{J}_2)$, then we have

i. $\mathcal{J}_1 \succ \mathcal{J}_2$ if $Ac(\mathcal{J}_1) > Ac(\mathcal{J}_2)$;
ii. $\mathcal{J}_1 \prec \mathcal{J}_2$ if $Ac(\mathcal{J}_1) < Ac(\mathcal{J}_2)$;
iii. $\mathcal{J}_1 = \mathcal{J}_2$ if $Ac(\mathcal{J}_1) = Ac(\mathcal{J}_2)$.

1.2.5. Theorem [4, 5]

Let $\mathcal{J} = (\mu_{\mathcal{J}}, \psi_{\mathcal{J}})$ and $\mathcal{J}_1 = (\mu_{\mathcal{J}_1}, \psi_{\mathcal{J}_1})$ be any two IFVs and $\lambda, \lambda_1, \lambda_2 > 0$. Then the following are holds:

i. $\mathcal{J} \oplus \mathcal{J}_1 = \mathcal{J}_1 \oplus \mathcal{J}$;
ii. $\lambda(\mathcal{J} \oplus \mathcal{J}_1) = \lambda\mathcal{J} \oplus \lambda\mathcal{J}_1$;
iii. $(\lambda_1 + \lambda_2)\mathcal{J} = \lambda_1\mathcal{J} + \lambda_2\mathcal{J}$;

iv. $\mathcal{J} \otimes \mathcal{J}_1 = \mathcal{J}_1 \otimes \mathcal{J}$;

v. $(\mathcal{J} \otimes \mathcal{J}_1)^{\lambda} = \mathcal{J}^{\lambda} \otimes {\mathcal{J}_1}^{\lambda}$;

vi. $\mathcal{J}^{(\lambda_1+\lambda_2)} = \mathcal{J}^{\lambda_1} \otimes \mathcal{J}^{\lambda_2}$.

1.3. Pythagorean fuzzy sets

From the above analysis, it is clear that the prominent concept of IFS [2] deals with both $\mathcal{MG}$ and $\mathcal{NMG}$ with the constraint that the sum of $\mathcal{MG}$ and $\mathcal{NMG}$ must belongs to $[0,1]$. However, in some scenario of real life problems the values assigned to $\mathcal{MG}$ and $\mathcal{NMG}$ from $[0,1]$ but their sum exceeds 1. To cope on this situation Yager [16] investigated the dominant notion of PyFS characterized by $\mathcal{MG}$ and $\mathcal{NMG}$ which provided the more space to decision makers as compared to IFS. PyFS satisfies the constraint that the square sum of $\mathcal{MG}$ and $\mathcal{NMG}$ must not exceed the real numbers 0 *and* 1. For example if the decision maker assigns the values to $\mathcal{MG} = \frac{\sqrt{3}}{2}$ and $\mathcal{NMG} = \frac{1}{2}$, then their sum is bigger than 1, so IFS cannot handle it. PyFS is capable to handle this situation, that is $\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \leq 1$. Therefore, PyFS provides more space and freedom for the experts to assign values as compared to IFS, which is defined as:

1.3.1. Definition [16]

Let T be a universal of discourse. A PyFS $\aleph$ on T is of the form

$$\aleph = \{< \mathcal{k}, \mu_{\aleph}(\mathcal{k}), \psi_{\aleph}(\mathcal{k}) > | \mathcal{k} \in T\},$$

where $\mu_{\aleph}(\mathcal{k})$ *and* $\psi_{\aleph}(\mathcal{k})$ denotes the $\mathcal{MG}$ and $\mathcal{NMG}$ that is $\mu_{\aleph}: T \to [0,1]$ *and* $\psi_{\aleph}: T \to [0,1]$ of an element $\mathcal{k} \in T$ with the constraint $0 \leq \mu_{\aleph}(\mathcal{k})^2 + \psi_{\aleph}(\mathcal{k})^2 \leq 1$. Moreover, the degree of hesitancy is given as $\pi_{\aleph} = \sqrt{1 - (\mu_{\aleph}(\mathcal{k})^2 + \psi_{\aleph}(\mathcal{k})^2)}$. If there is no confusion then $\aleph =< \mathcal{k}, \mu_{\aleph}(\mathcal{k}), \psi_{\aleph}(\mathcal{k}) >$, is represented by $\aleph = (\mu_{\aleph}, \psi_{\aleph})$ and is known to be PyF value (PyFV). The collection of all PyFS on the set T is represented by $PyFS^T$. The graphical interpretation of PyFS is given in Fig. 1.

1.3.2. Definition [17, 75]

Let $\aleph, \aleph_1 \in PyFS^T$ be two PyFSs. Then the basic operations on them are given as:

i. $\aleph \subseteq \aleph_1$ iff $\mu_{\aleph}(\mathcal{k}) \leq \mu_{\aleph_1}(\mathcal{k})$ *and* $\psi_{\aleph}(\mathcal{k}) \geq \psi_{\aleph_1}(\mathcal{k})$ for all $\mathcal{k} \in T$;

ii. $\aleph \cup \aleph_2 = \left(\hbar, \max\left(\mu_{\aleph}(\hbar), \mu_{\aleph_1}(\hbar)\right), \min\left(\psi_{\aleph}(\hbar), \psi_{\aleph_1}(\hbar)\right)\right)$ for $\hbar \in T$;

iii. $\aleph \cap \aleph_2 = \left(\hbar, \min\left(\mu_{\aleph}(\hbar), \mu_{\aleph_1}(\hbar)\right), \max\left(\psi_{\aleph}(\hbar), \psi_{\aleph_1}(\hbar)\right)\right)$ for $\hbar \in T$;

iv. $\aleph_1 = \aleph_2$ iff $\aleph \subseteq \aleph_1$ and $\aleph_1 \subseteq \aleph$;

v. $\aleph^c = \left(\hbar, \psi_{\aleph}(\hbar), \mu_{\aleph}(\hbar)\right)$ for $\hbar \in T$, where $\aleph^c$ is the complement of $\aleph$.

1.3.3. Definition [76]

Let $\aleph = (\mu_{\aleph}, \psi_{\aleph})$ and $\aleph_1 = \left(\mu_{\aleph_1}, \psi_{\aleph_1}\right)$ be two PyFVs and $\lambda > 0$. Then the basic operations on them are given as:

i. Ring sum operation:

$$\aleph \oplus \aleph_2 = \left\{\left(\hbar, \sqrt{\mu_{\aleph}^2(\hbar) + \mu_{\aleph_1}^2(\hbar) - \mu_{\aleph}^2(\hbar)\mu_{\aleph_1}^2(\hbar)}, \psi_{\aleph}(\hbar)\psi_{\aleph_1}(\hbar)\right) | \hbar \in T\right\};$$

ii. Ring product operation:

$$\aleph \otimes \aleph_1 = \left\{\left(\hbar, \mu_{\aleph}(\hbar)\mu_{\aleph_1}(\hbar), \sqrt{\psi_{\aleph}^2(\hbar) + \psi_{\aleph_1}^2(\hbar) - \psi_{\aleph}^2(\hbar)\psi_{\aleph_1}^2(\hbar)}\right) | \hbar \in T\right\};$$

iii. $$\lambda \aleph = \left(\sqrt{1 - \left(1 - \mu_{\aleph}^2(\hbar)\right)^{\lambda}}, \psi_{\aleph}^{\lambda}(\hbar)\right);$$

iv. $$\aleph^{\lambda} = \left(\mu_{\aleph}^{\lambda}(\hbar), \sqrt{1 - \left(1 - \psi_{\aleph}^2(\hbar)\right)^{\lambda}}\right).$$

1.3.4. Definition [76]

Let $\aleph = (\mu_{\aleph}, \psi_{\aleph})$ be a PyFV. Then the score function of $\aleph$ can be defined as follows:

$$\mathcal{S}c(\aleph) = (\mu_{\aleph})^2 - (\psi_{\aleph})^2; \qquad \mathcal{S}c(\aleph) \in [-1,1].$$

Now comparing any two PyFVs $\aleph_i = (\mu_{\aleph_i}, \psi_{\aleph_i})$ $(i = 1,2)$, then

i. $\aleph_1 \succ \aleph_2$ if $\mathcal{S}c(\aleph_1) > \mathcal{S}c(\aleph_2)$;

ii. $\aleph_1 \prec \aleph_2$ if $\mathcal{S}c(\aleph_1) < \mathcal{S}c(\aleph_2)$;

In case when $\mathcal{S}c(\aleph_1) = \mathcal{S}c(\aleph_2)$ then $\aleph_1 \sim \aleph_2$. Further to cope on this issue Peng and Yang [18] initiated the accuracy function, which is defined as:

1.3.5. Definition [18]

Let $\aleph = (\mu_{\aleph}, \psi_{\aleph})$ be a PyFV. Then an accuracy function of $\aleph$ is given as below:

$$Ac(\aleph) = (\mu_{\aleph})^2 + (\psi_{\aleph})^2; \qquad Ac(\aleph) \in [0,1].$$

Now comparing any two PyFVs $\aleph_i = \left(\mu_{\aleph_i}, \psi_{\aleph_i}\right)(i = 1,2)$, then

i. If $\mathcal{S}c(\aleph_1) = \mathcal{S}c(\aleph_2)$, then we have
ii. $\aleph_1 \succ \aleph_2$ if $Ac(\aleph_1) > Ac(\aleph_2)$;
iii. $\aleph_1 \prec \aleph_2$ if $Ac(\aleph_1) < Ac(\aleph_2)$;
iv. $\aleph_1 \sim \aleph_2$ if $Ac(\aleph_1) = Ac(\aleph_2)$.

1.3.6. Theorem [76]

Let $\aleph = (\mu_\aleph, \psi_\aleph)$ and $\aleph_1 = \left(\mu_{\aleph_1}, \psi_{\aleph_1}\right)$ be any two PyFVs and $\lambda, \lambda_1, \lambda_2 > 0$. Then the following results are holds:

i. $\aleph \oplus \aleph_1 = \aleph_1 \oplus \aleph$;
ii. $\lambda(\aleph + \aleph_1) = \lambda\aleph \oplus \lambda\aleph_1$;
iii. $(\lambda_1 + \lambda_2)\aleph = \lambda_1\aleph \oplus \lambda_2\aleph$
iv. $\aleph \otimes \aleph_1 = \aleph_1 \otimes \aleph$;
v. $(\aleph \otimes \aleph_1)^\lambda = \aleph^\lambda \otimes {\aleph_1}^\lambda$;
vi. $\aleph^{(\lambda_1+\lambda_2)} = \aleph^{\lambda_1} \otimes \aleph^{\lambda_2}$.

1.4. q-Rung orthopair fuzzy sets

From the Definitions of IFS and PyFS, it is clear that in some practical problems both these notions fail to cope the scenario. So researchers face difficulties to handle these shortcomings. Yager [27] originated the generalized concept of both IFS and PyFS, which is known as q-ROFS. In this concept the sum qth power of $\mathcal{MG}$ and qth power of $\mathcal{NMG}$ must not exceed the real numbers $0 \; and \; 1$ for $q \geq 1$. Hence q-ROFS freely allows the alternative to their corresponding criteria provided by $\mathcal{DM}$ just by adjusting the value of rung q, which is defined as follows:

1.4.1. Definition [27]

Let T be a universal set. A q-ROFS $\mathfrak{J}$ on the set T is of the form

$$\mathfrak{J} = \left\{< \hbar, \mu_{\mathfrak{J}}(\hbar), \psi_{\mathfrak{J}}(\hbar) >_q \middle| \hbar \in T \;\; for \; q \geq 1\right\},$$

where $\mu_{\mathfrak{J}}(\hbar) \; and \; \psi_{\mathfrak{J}}(\hbar)$ denotes the $\mathcal{MG}$ and $\mathcal{NMG}$ that is $\mu_{\mathfrak{J}}: T \to [0,1] \; and \; \psi_{\mathfrak{J}}: T \to [0,1]$ of an element $\hbar \in T$ with the constraint $0 \leq \mu_{\mathfrak{J}}(\hbar)^q + \psi_{\mathfrak{J}}(\hbar)^q \leq 1$ for $q \geq 1$. Moreover, the degree of hesitancy is given as $\pi_{\mathfrak{J}} = \sqrt[q]{1 - (\mu_{\mathfrak{J}}(\hbar)^q + \psi_{\mathfrak{J}}(\hbar)^q)}$. If there is no confusion then $\mathfrak{J} =< \hbar, \mu_{\mathfrak{J}}(\hbar), \psi_{\mathfrak{J}}(\hbar) >_q$, is represented by $\mathfrak{J} = (\mu_{\mathfrak{J}}, \psi_{\mathfrak{J}})$ and is known to be q-ROF value (q-ROFV). The collection of all q-ROFSs is on the set T is represented by $\mathrm{q-ROF}S^{T}$. The graphical interpretation of q-ROFS is given in Fig. 1.

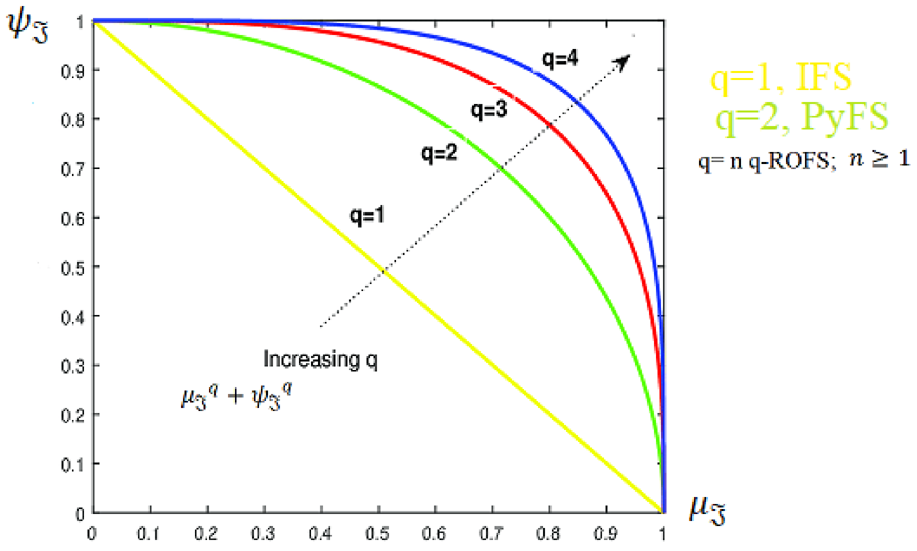

Fig. 1. Graphical Interpretation of q-ROFS

1.4.2. **Definition** [27]

Let $\mathfrak{J}, \mathfrak{J}_1 \in \mathrm{q-ROF}S^{T}$ be any two q-ROFSs. Then the basic operations on them are given as:

i. $\mathfrak{J} \subseteq \mathfrak{J}_1$ iff $\mu_{\mathfrak{J}}(k) \leq \mu_{\mathfrak{J}_1}(k)$ *and* $\psi_{\mathfrak{J}}(k) \geq \psi_{\mathfrak{J}_1}(k)$ for all $k \in T$;

ii. $\mathfrak{J} \cup \mathfrak{J}_1 = \left(k, \max\left(\mu_{\mathfrak{J}}(k), \mu_{\mathfrak{J}_1}(k)\right), \min\left(\psi_{\mathfrak{J}}(k), \psi_{\mathfrak{J}_1}(k)\right)\right)$ for $k \in T$;

iii. $\mathfrak{J} \cap \mathfrak{J}_1 = \left(k, \min\left(\mu_{\mathfrak{J}}(k), \mu_{\mathfrak{J}_1}(k)\right), \max\left(\psi_{\mathfrak{J}}(k), \psi_{\mathfrak{J}_1}(k)\right)\right)$ for $k \in T$;

iv. $\mathfrak{J} = \mathfrak{J}_1$ iff $\mathfrak{J} \subseteq \mathfrak{J}_1$ and $\mathfrak{J}_1 \subseteq \mathfrak{J}$;

v. $\mathfrak{J}^c = \left(k, \psi_{\mathfrak{J}}(k), \mu_{\mathfrak{J}}(k)\right)$ for $k \in T$, where $\mathfrak{J}^c$ is the complement of $\mathfrak{J}$.

1.4.3. **Definition** [27]

Let $\mathfrak{J} = (\mu_{\mathfrak{J}}, \psi_{\mathfrak{J}})$ and $\mathfrak{J}_1 = (\mu_{\mathfrak{J}_1}, \psi_{\mathfrak{J}_1})$ be any two q-ROFVs. Then the basic operations on them are given as:

i. Ring sum operation:

$$\mathfrak{I}\oplus\mathfrak{I}_1 = \left\{\left(\hbar, \sqrt[q]{\mu_{\mathfrak{I}}^q(\hbar)+\mu_{\mathfrak{I}_1}^q(\hbar)-\mu_{\mathfrak{I}}^q(\hbar)\mu_{\mathfrak{I}_1}^q(\hbar)}, \psi_{\mathfrak{I}}(\hbar)\psi_{\mathfrak{I}_1}(\hbar)\right) \mid \hbar \in T\right\};$$

ii. Ring product operation:

$$\mathfrak{I}\otimes\mathfrak{I}_1 = \left\{\left(\hbar, \mu_{\mathfrak{I}}(\hbar)\mu_{\mathfrak{I}_1}(\hbar), \sqrt[q]{\psi_{\mathfrak{I}}^q(\hbar)+\psi_{\mathfrak{I}_1}^q(\hbar)-\psi_{\mathfrak{I}}^q(\hbar)\psi_{\mathfrak{I}_1}^q(\hbar)}\right) \mid \hbar \in T\right\};$$

iii. $$\lambda\mathfrak{I} = \left(\sqrt[q]{1-\left(1-\mu_{\mathfrak{I}}^q(\hbar)\right)^{\lambda}}, \psi_{\mathfrak{I}}^{\lambda}(\hbar)\right);$$

iv. $$\mathfrak{I}^{\lambda} = \left(\mu_{\mathfrak{I}}^{\lambda}(\hbar), \sqrt[q]{1-\left(1-\psi_{\mathfrak{I}}^q(\hbar)\right)^{\lambda}}\right).$$

To make comparison between two or more q-ROFVs, a score function is used to differentiate two or more q-ROFVs. In many cases of real problems if score function fails to differentiate the q-ROFVs. To solve this issue an accuracy function is investigated, which are defined below.

1.4.4. Definition [30]

Let $\mathfrak{I} = (\mu_{\mathfrak{I}}, \psi_{\mathfrak{I}})$ be a q-ROFV, then the score function of $\mathfrak{I}$ can be given as follows:

$$\mathcal{S}c(\mathfrak{I}) = {\mu_{\mathfrak{I}}}^q - {\psi_{\mathfrak{I}}}^q; \qquad \mathcal{S}c(\mathfrak{I}) \in [-1,1].$$

Now comparing any two q-ROFVs $\mathfrak{I}_i = \left(\mu_{\mathfrak{I}_i}, \psi_{\mathfrak{I}_i}\right)(i = 1,2)$, then

i. $\mathfrak{I}_1 \succ \mathfrak{I}_2$ if $\mathcal{S}c(\mathfrak{I}_1) > \mathcal{S}c(\mathfrak{I}_2)$;
ii. $\mathfrak{I}_1 \prec \mathfrak{I}_2$ if $\mathcal{S}c(\mathfrak{I}_1) < \mathcal{S}c(\mathfrak{I}_2)$;

In case when $\mathcal{S}c(\mathfrak{I}_1) = \mathcal{S}c(\mathfrak{I}_2)$ then $\mathfrak{I}_1 \sim \mathfrak{I}_2$. Further two q-ROFVs can be compared by using accuracy function, which is defined as:

1.4.5. Definition [30]

Let $\mathfrak{I} = (\mu_{\mathfrak{I}}, \psi_{\mathfrak{I}})$ be a q-ROFV. Then the accuracy function of $\mathfrak{I}$ is given as:

$$Ac(\mathfrak{I}) = {\mu_{\mathfrak{I}}}^q + {\psi_{\mathfrak{I}}}^q; \qquad Ac(\mathfrak{I}) \in [0,1].$$

Now comparing any two q-ROFVs $\mathfrak{J}_i = (\mu_{\mathfrak{J}_i}, \psi_{\mathfrak{J}_i})$ $(i = 1,2)$, if $\mathcal{S}c(\mathfrak{J}_1) = \mathcal{S}c(\mathfrak{J}_2)$, then we have

i. $\mathfrak{J}_1 \succ \mathfrak{J}_2$ if $Ac(\mathfrak{J}_1) > Ac(\mathfrak{J}_2)$;
ii. $\mathfrak{J}_1 \prec \mathfrak{J}_2$ if $Ac(\mathfrak{J}_1) < Ac(\mathfrak{J}_2)$;
iii. $\mathfrak{J}_1 \sim \mathfrak{J}_2$ if $Ac(\mathfrak{J}_1) = Ac(\mathfrak{J}_2)$.

1.4.6. Theorem [30]

Let $\mathfrak{J} = (\mu_{\mathfrak{J}}, \psi_{\mathfrak{J}})$ and $\mathfrak{J}_1 = (\mu_{\mathfrak{J}_1}, \psi_{\mathfrak{J}_1})$ be any two q-ROFVs and $\lambda, \lambda_1, \lambda_2 > 0$. Then the following results are holds:

(i) $\mathfrak{J} \oplus \mathfrak{J}_1 = \mathfrak{J}_1 \oplus \mathfrak{J}$;
(ii) $\lambda(\mathfrak{J} + \mathfrak{J}_1) = \lambda\mathfrak{J} \oplus \lambda\mathfrak{J}_1$;
(iii)$(\lambda_1 + \lambda_2)\mathfrak{J} = \lambda_1\mathfrak{J} \oplus \lambda_2\mathfrak{J}$;
(iv) $\mathfrak{J} \otimes \mathfrak{J}_1 = \mathfrak{J}_1 \otimes \mathfrak{J}$;
(v) $(\mathfrak{J} \otimes \mathfrak{J}_1)^{\lambda} = \mathfrak{J}^{\lambda} \otimes {\mathfrak{J}_1}^{\lambda}$;
(vi) $\mathfrak{J}^{(\lambda_1+\lambda_2)} = \mathfrak{J}^{\lambda_1} \otimes \mathfrak{J}^{\lambda_2}$.

1.5. q-Rung orthopair fuzzy weighted averaging aggregation operator

Here we will define a brief concept of q-ROFWA and q-ROFWG aggregation operators.

1.5.1. Definition [30]

Let $\mathfrak{J}_i = (\mu_i, \psi_i)$ $(i = 1,2 \dots, n)$ be q-ROFVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ be the weight vectors such that $\sum_{i=1}^{n} \bar{\bar{w}} = 1$ with $\bar{\bar{w}}_i \in [0,1]$. Then the mapping for q-ROFWA aggregation operator is defined as: $\mathrm{q-ROF}WA: \mathcal{H}^n \to \mathcal{H}$ (where $\mathcal{H}^n$ is the collection of q-ROFVs)

$$\mathrm{q-ROF}WA(\mathfrak{J}_1, \mathfrak{J}_2, \dots \mathfrak{J}_n) = \oplus_{i=1}^{n} \bar{\bar{w}}_i \mathfrak{J}_i$$
$$= \left(\sqrt[q]{1 - \prod_{i=1}^{n} \left(1 - \mu_i^q\right)^{\bar{\bar{w}}}}, \prod_{i=1}^{n} \psi_i^{\bar{\bar{w}}} \right).$$

1.5.2. Definition [30]

Let $\mathfrak{J}_i = (\mu_i, \psi_i)$ $(i = 1,2 \dots, n)$ be q-ROFVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ be the weight vectors such that $\sum_{i=1}^{n} \bar{\bar{w}} = 1$ with $\bar{\bar{w}}_i \in [0,1]$ respectively. Then the mapping for q-

ROFWG aggregation operator is defined as: $\mathrm{q}-\mathrm{ROF}WG: \mathcal{H}^n \rightarrow \mathcal{H}$ (where $\mathcal{H}^n$ is the collection of q-ROFVs).

$$\mathrm{q}-\mathrm{ROF}WG(\mathfrak{J}_1, \mathfrak{J}_2, \ldots \mathfrak{J}_n) = \otimes_{i=1}^{n} \mathfrak{J}_i^{\overline{\overline{w}}_i}$$

$$= \left(\prod_{i=1}^{n} \mu_i^{\overline{\overline{w}}}, \sqrt[q]{1 - \prod_{i=1}^{\overline{\overline{n}}} \left(1 - \psi_i^q\right)^{\overline{\overline{w}}}} \right).$$

1.5.3. Remarks

(i) If $q = 1$, then $\mathrm{q}-\mathrm{ROF}WA$ reduced to $\mathrm{IF}WA$ by Xu [4] and $\mathrm{q}-\mathrm{ROF}WG$ reduced to $\mathrm{IF}WG$ by Xu and Yager [5].

$$\mathrm{IF}WA(\mathcal{J}_1, \mathcal{J}_2, \ldots \mathcal{J}_n) = \oplus_{i=1}^{n} \overline{\overline{w}}_i \mathcal{J}_i$$

$$= \left(1 - \prod_{i=1}^{n} (1 - \mu_i)^{\overline{\overline{w}}}, \prod_{i=1}^{n} \psi_i^{\overline{\overline{w}}} \right)$$

and

$$\mathrm{IF}WG(\mathcal{J}_1, \mathcal{J}_2, \ldots \mathcal{J}_n) = \oplus_{i=1}^{n} \overline{\overline{w}}_i \mathcal{J}_i$$

$$= \left(\prod_{i=1}^{n} \mu_i^{\overline{\overline{w}}}, 1 - \prod_{i=1}^{n} (1 - \psi_i)^{\overline{\overline{w}}} \right).$$

(ii) If $q = 2$, then $\mathrm{q}-\mathrm{ROF}WA$ reduced to $\mathrm{PyF}WA$ and $\mathrm{q}-\mathrm{ROF}WG$ reduced to $\mathrm{PyF}WG$ given in [17, 75].

$$\mathrm{PyF}WA(\aleph_1, \aleph_2, \ldots \aleph_n) = \oplus_{i=1}^{n} \overline{\overline{w}}_i \aleph_i$$

$$= \left(\sqrt[2]{1 - \prod_{i=1}^{n} (1 - \mu_i^2)^{\overline{\overline{w}}}}, \prod_{i=1}^{n} \psi_i^{\overline{\overline{w}}} \right)$$

and

$$\mathrm{PyF}WG(\aleph_1, \aleph_2, \ldots \aleph_n) = \oplus_{i=1}^{n} \overline{\overline{w}}_i \aleph_i$$

$$= \left(\prod_{i=1}^{n} \mu_i^{\overline{\overline{w}}}, \sqrt[2]{1 - \prod_{i=1}^{n} (1 - \psi_i^2)^{\overline{\overline{w}}}} \right).$$

1.6. Rough set theory

In 1982, the pioneer concept of rough set (RS) was first proposed by Pawlak [36] who generalized the classical set theory to cope imprecise, vague and uncertain information by an easy way. By definition, Pawlak's RS of a universal set is characterized by two approximation sets known as lower and upper approximations. The lower approximation consists of those elements which contain in the set and the upper approximation consists of those objects having nonempty intersection with the set. Further equivalence relation plays a key role in Pawlak's RS for approximations and RS theory has been extensively used in various directions of theoretical as well as in practical applications.

Let T_1, T_2 be two nonempty set and consider a Cartesian product $T_1 \times T_2 = \{(k_1, k_2) | k_1 \in T_1, k_2 \in T_2\}$. A subset of a Cartesian product is called binary relation and it is denoted by $\mathcal{R}$. The statements (k_1, k_2) in $\mathcal{R}$ is read as "k_1 is $\mathcal{R}$-related to k_2" and is represented as $k_1 \mathcal{R} k_2$ or $(k_1, k_2) \in \mathcal{R}$.

Suppose T be a universal set and $\mathcal{R}$ be a binary relation over $T \times T$ having the following properties:

(i) $\mathcal{R}$ is reflexive, i.e. $\forall\, k \in T, k\mathcal{R}k$,

(ii) $\mathcal{R}$ is symmetric, i.e. for any $k_1, k_2 \in T,\ if\ k_1 \mathcal{R} k_2, then\ k_2 \mathcal{R} k_1$,

(iii) $\mathcal{R}$ is transitive, i.e. for any $k_1, k_2, k_3 \in T$ if $k_1 \mathcal{R} k_2$ and $k_2 \mathcal{R} k_3$, then $k_1 \mathcal{R} k_3$.

Then $\mathcal{R}$ is called an equivalence relation. The set of those objects of T which are related to $k \in T$, is said to be equivalence class of k and is represented as $[k]_{\mathcal{R}} = \{u \in T | k\mathcal{R}u\}$. The pair $(T, \mathcal{R})$ is known to be an approximation space. Consider a nonempty subset $\mathcal{K}$ of T, then $\mathcal{K}$ is definable if it can be written in the union of some equivalence classes of T, otherwise $\mathcal{K}$ is undefinable. So in this case set $\mathcal{K}$ can be approximated in the form of definable subsets called lower and upper approximation which are given as:

$$\underline{\mathcal{R}}(\mathcal{K}) = \{k \in T |\ [k]_{\mathcal{R}} \subseteq \mathcal{K}\}$$

$$\overline{\mathcal{R}}(\mathcal{K}) = \{k \in T |\ [k]_{\mathcal{R}} \cap \mathcal{K} \neq \emptyset\}$$

$\underline{\mathcal{R}}(\mathcal{K})$ is the greatest definable set in T contained in $\mathcal{K}$ and $\overline{\mathcal{R}}(\mathcal{K})$ is the least set in T containing $\mathcal{K}$. The set $\mathcal{R}(\mathcal{K}) = (\underline{\mathcal{R}}(\mathcal{K}), \overline{\mathcal{R}}(\mathcal{K}))$ is said to RS if $\underline{\mathcal{R}}(\mathcal{K}) \neq \overline{\mathcal{R}}(\mathcal{K})$. The geometrical interpretation of RS is shown in Fig. 2.

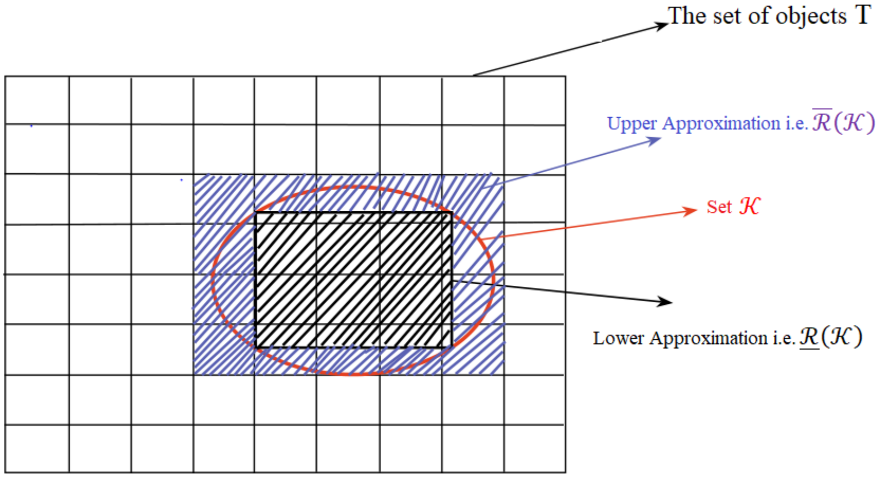

Fig. 2, Geometrical Interpretation of Rough Set

1.6.1. Example

Let $T = \{\hslash_1, \hslash_2, \hslash_3, \hslash_4, \hslash_5, \hslash_6\}$ and

$$\mathcal{R} = \left\{\begin{matrix}(\hslash_1,\hslash_1),(\hslash_2,\hslash_2),(\hslash_3,\hslash_3),(\hslash_4,\hslash_4),(\hslash_5,\hslash_5),(\hslash_6,\hslash_6),(\hslash_1,\hslash_2),(\hslash_2,\hslash_1),\\ (\hslash_3,\hslash_5),(\hslash_5,\hslash_3)\end{matrix}\right\}.$$

Then $\mathcal{R}$ is an equivalence relation on T. Now the equivalence classes of $\mathcal{R}$ are $[\hslash_1]_{\mathcal{R}} = [\hslash_2]_{\mathcal{R}} = \{\hslash_1, \hslash_2\}, [\hslash_3]_{\mathcal{R}} = [\hslash_5]_{\mathcal{R}} = \{\hslash_3, \hslash_5\}, [\hslash_4]_{\mathcal{R}} = \{\hslash_4\}$ *and* $[\hslash_6]_{\mathcal{R}} = \{\hslash_6\}$. Let us consider $\mathcal{K} = \{\hslash_1, \hslash_3, \hslash_4, \hslash_6\} \subseteq T$. Then $\underline{\mathcal{R}}(\mathcal{K}) = \{\hslash_4, \hslash_6\}$, $\overline{\mathcal{R}}(\mathcal{K}) = \{\hslash_1, \hslash_2, \hslash_3, \hslash_4, \hslash_5, \hslash_6\}$. As $\underline{\mathcal{R}}(\mathcal{K}) \neq \overline{\mathcal{R}}(\mathcal{K})$. Hence $\mathcal{R}(\mathcal{K}) = (\underline{\mathcal{R}}(\mathcal{K}), \overline{\mathcal{R}}(\mathcal{K}))$ is a rough set.

In next definition Yao [77] presented the rough set based on set valued mapping (SVM) means using nonequivalence structure.

1.6.2. Definition [77]

Consider a universal set T. Suppose $\mathcal{R} \subseteq T \times T$ be the crisp relation. Suppose $\mathcal{R}^*$ be the set valued mapping (SVM) i.e. $\mathcal{R}^* = T \to \mathcal{P}^*(T)$ defined by $\mathcal{R}^*(\hslash) = \{u \in T | (\hslash, u) \in \mathcal{R} \text{ and } \hslash \in T\}$. Then the pair $(T, \mathcal{R})$ is known to be an approximation space. Suppose $\emptyset \neq \mathfrak{J} \subseteq T$, then the lower and upper approximation of $\mathfrak{J}$ w.r.t approximation space $(T, \mathcal{R})$ is denoted and defined as:

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{\hslash \in T | \mathcal{R}^*(\hslash) \subseteq \mathfrak{J}\}$$

$$and \qquad \bar{\mathcal{R}}(\mathfrak{J}) = \{\hslash \in T | \mathcal{R}^*(\hslash) \cap \mathfrak{J} \neq \emptyset\}.$$

The pair $\left(\underline{\mathcal{R}}(\mathfrak{J}), \bar{\mathcal{R}}(\mathfrak{J})\right)$ is known to be a crisp rough set, where $\underline{\mathcal{R}}(\mathfrak{J}) \neq \bar{\mathcal{R}}(\mathfrak{J})$. Hence $\underline{\mathcal{R}}(\mathfrak{J}), \bar{\mathcal{R}}(\mathfrak{J}): \mathcal{P}^*(T) \to \mathcal{P}^*(T)$ is called crisp lower and upper approximation operators w.r.t $(T, \mathcal{R})$, where $\mathcal{P}^*(T)$ is the collection of power set of T.

1.6.3. Theorem

Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are any two equivalence relations on set T and $\mathcal{K}_1, \mathcal{K}_2$ are the non-empty subsets of T. Then the following are hold:

i. $\underline{\mathcal{R}}(\mathcal{K}_1) \subseteq \mathcal{K}_1 \subseteq \overline{\mathcal{R}}(\mathcal{K}_1)$,

ii. $\overline{\mathcal{R}}(\mathcal{K}_1 \cup \mathcal{K}_2) = \overline{\mathcal{R}}(\mathcal{K}_1) \cup \overline{\mathcal{R}}(\mathcal{K}_2)$,

iii. $\underline{\mathcal{R}}(\mathcal{K}_1 \cap \mathcal{K}_2) = \underline{\mathcal{R}}(\mathcal{K}_1) \cap \underline{\mathcal{R}}(\mathcal{K}_2)$,

iv. $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \overline{\mathcal{R}}(\mathcal{K}_1) \subseteq \overline{\mathcal{R}}(\mathcal{K}_2)$,

v. $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \underline{\mathcal{R}}(\mathcal{K}_1) \subseteq \underline{\mathcal{R}}(\mathcal{K}_2)$,

vi. $\underline{\mathcal{R}}(\mathcal{K}_1 \cup \mathcal{K}_2) \supseteq \underline{\mathcal{R}}(\mathcal{K}_1) \cup \underline{\mathcal{R}}(\mathcal{K}_2)$,

vii. $\overline{\mathcal{R}}(\mathcal{K}_1 \cap \mathcal{K}_2) \subseteq \overline{\mathcal{R}}(\mathcal{K}_1) \cap \overline{\mathcal{R}}(\mathcal{K}_2)$.

1.7. Covering based intuitionistic fuzzy rough set

Here in this subsection we are going to present a brief structure of CBIFRS and its related structure.

1.7.1. Definition [78]

Let T be a universal set. The set $\mathcal{K} = \{\mathcal{C} \neq \phi : \mathcal{C} \subseteq T\}$ is called cover of T, if $\cup\, \mathcal{C} = T$. So in this case the pair $(T, \mathcal{K})$ is said to be covering approximation space (CAS).

1.7.2. Definition [78]

Let $(T, \mathcal{K})$ be a CAS. Then $\mathcal{N}_{\mathcal{K}}(\mathscr{k}) = \cap \{\mathcal{C} : \mathcal{C} \in \mathcal{K}\ and\ \mathscr{k} \in \mathcal{C}\}$is known as the neighborhood of $\mathscr{k} \in T$ w.r.t $(T, \mathcal{K})$.

1.7.3. Definition [79]

Let T be a universal set. Let $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m\}$ with $\mathcal{F}_i \in FS^T (i = 1, \ldots, m)$. If $\bigvee_{\mathcal{F}_i \in \mathcal{F}} \mathcal{F}_i(\mathscr{k}) = 1$ for each $\mathscr{k} \in T$, so in this case $\mathcal{F}$ is called a fuzzy covering of T. The pair $(T, \mathcal{F})$ is called a fuzzy CAS.

1.7.4. Definition [80]

Suppose T is a universal of discourse. For any $\beta \in (0,1]$ and $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m\}$ with $\mathcal{F}_m \in FS^T$ $(i = 1, \ldots,\ m)$, then $\mathcal{F}$ is called a fuzzy β-covering of T, if $\cup\, \mathcal{F}_i(\mathscr{k}) \geqslant \beta$ for each $\mathscr{k} \in T$. The pair $(T, \mathcal{F})$ is said to be fuzzy β-covering approximation space.

1.7.5. Definition [80]

Consider $(T, \mathcal{F})$ be a fuzzy CAS and $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m\}$ be a fuzzy β-covering of T, for some $\beta \in (0,1]$. Then fuzzy β-neighborhood is defined as $\mathcal{N}_{\mathscr{k}}^{\beta} = \cap\, \{\mathcal{F}_m \in \mathcal{F} : \mathcal{F}_i(\mathscr{k}) \geqslant \beta\ (i = 1, 2, \ldots, m)\}$ for $\mathscr{k} \in T$.

1.7.6. Definition [80]

Let $(T, \mathcal{F})$ be a fuzzy CAS and $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m\}$ be a fuzzy β-covering of T, for some $\beta \in (0,1]$. Then β-neighborhood of y is denoted and defined as $(\mathcal{N}_y^{\beta})^* = \{\mathscr{k} \in T : \mathcal{N}_y^{\beta}(\mathscr{k}) \geqslant \beta\}$ for each $y \in T$.

1.7.7. Definition [81]

Let T be any set and $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_m\}$, where $\mathcal{J}_i \in IFS^T$ and $i = 1,2, \ldots, m$. For any IFV $\beta = (\mu_\beta, \psi_\beta)$, $\mathcal{J}$ is called intuitionistic fuzzy β-covering (IF β-covering) of T, if $(\bigcup_{i=1}^{m} \mathcal{J}_i)(\mathscr{k}) \geqslant \beta$ for all $\mathscr{k} \in T$. Here $(T, \mathcal{J})$ is called a IF covering approximation space (IFCAS).

Suppose that $(T, \mathcal{J})$ is a IFCAS and $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_m\}$ be a IF β-covering of T for some $\beta = (\mu_\beta, \psi_\beta)$. Then $\mathcal{N}_{\mathcal{J}(\mathscr{k})}^{\beta} = \cap\, \{\mathcal{J}_j \in \mathcal{J} : \mathcal{J}_j \geqslant \beta,\ \mathrm{j} = 1,2, \ldots, \mathrm{m}\}$ is the IFβ-neighborhood of $\mathscr{k}$ in T.

An IFβ-neighborhood system is denoted and defined as $\mathcal{N}_{\mathcal{J}}^{\beta} = \{\mathcal{N}_{\mathcal{J}(\mathscr{k})}^{\beta} : \mathscr{k} \in T\}$ which is induced by IF β-covering $\mathcal{J}$. By using IF matrix to represent a IFβ-neighborhood system as follows:

$$\mathbb{M}_{\mathcal{J}}^{\beta} = \left[\mathcal{N}_{\mathcal{J}(\mathscr{k}_i)}^{\beta}(\mathscr{k}_j)\right]_{\mathscr{k}_i \times \mathscr{k}_j \in T \times T}$$

1.7.8. Remarks [81]

(i) If $\beta = (1,0)$, then in this case IF β-covering reduced to a crisp covering and similarly if $\beta = (1,0)$, then IF β-neighborhood reduced to a crisp neighborhood.

(ii) If $\beta = (\hbar, 0)$, such that $0 < \hbar < 1$, then in this case IF β-covering reduced to a fuzzy covering and similarly if If $\beta = (\hbar, 0)$, then IF β-neighborhood reduced to a fuzzy β-neighborhood respectively.

1.7.9. Definition [81]

Consider a IFCAS (T, C), where $\mathcal{J} = \{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \dots, \mathcal{J}_{\mathrm{m}}\}$ is the set of IF β-covering of T for some $\beta = (\mu_\beta, \psi_\beta)$ and $T = \{\hbar_1, \hbar_2, \dots, \hbar_n\}$. Consider that the neighborhood system $\mathcal{N}_{\mathcal{J}}^{\beta} = \left\{\mathcal{N}_{\mathcal{J}(\hbar)}^{\beta} : \hbar \in T\right\}$ induced by IF β-covering of $\mathcal{J}$ such that

$$\mathcal{N}_{\mathcal{J}(\hbar_i)}^{\beta} = \left\{< \hbar_j, \mu_{\mathcal{N}_{\mathcal{J}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j), \psi_{\mathcal{N}_{\mathcal{J}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) >_q \,|\text{for all } i = 1, \dots, n \text{ and } j = 1, \dots m\right\}$$

Now for any $\mathfrak{J} \in \mathrm{IFS}^T$, where $\mathfrak{J} = \left\{< \hbar_j, \mu_{\mathfrak{J}}(\hbar_j), \psi_{\mathfrak{J}}(\hbar_j)\right) >_q |j = 1, \dots, m\}$, the lower and upper approximations of $\mathfrak{J}$ w.r.t $\mathcal{N}_{\mathcal{J}(\hbar)}^{\beta}$ is represented and defined by

$$\mathcal{N}_{\mathcal{J}}^{\beta}(\mathfrak{J}) = \left(\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}), \overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})\right),$$

where

$$\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}) = \left\{< \hbar_i, \mu_{\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\hbar_i), \psi_{\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\hbar_i) >_q |i = 1, \dots, n\right\}$$

and

$$\overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}) = \left\{< \hbar_i, \mu_{\overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\hbar_i), \psi_{\overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\hbar_i) >_q |i = 1, \dots, n\right\}$$

such that

$$\mu_{\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\hbar_i) = \bigwedge_{j=1}^{m}\left\{\mu_{\mathcal{N}_{\mathcal{J}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \vee \mu_{\mathfrak{J}}(\hbar_j)\right\}$$

$$\psi_{\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\hbar_i) = \bigvee_{j=1}^{m}\left\{\psi_{\mathcal{N}_{\mathcal{J}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \wedge \psi_{\mathfrak{J}}(\hbar_j)\right\}$$

$$\mu_{\overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) = \bigvee_{j=1}^{m} \left\{ \mu_{\mathcal{N}_{\mathcal{J}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \wedge \mu_{\mathfrak{J}}(\mathcal{k}_j) \right\}$$

$$\psi_{\overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) = \bigwedge_{j=1}^{m} \left\{ \psi_{\mathcal{N}_{\mathcal{J}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \vee \psi_{\mathfrak{J}}(\mathcal{k}_j) \right\}$$

So the operators $\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}), \overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}) : IFS^T \to IFS^T$ are said to be lower and upper IF rough approximation operators w.r.t $\mathcal{N}_{\mathcal{J}}^{\beta}$.

Therefore, the covering based IF rough set is the pair $\mathcal{N}_{\mathcal{J}}^{\beta}(\mathfrak{J}) = (\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}), \overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}))$, when ever $\underline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J}) \neq \overline{\mathcal{N}_{\mathcal{J}}^{\beta}}(\mathfrak{J})$.

1.8. Soft sets

In 1999, Molodtsov [53] investigated the pioneer concept of soft set (S_{ft}S) which is defended on parameterizations tool to cope the uncertainty and vague data. Various traditional concepts such as fuzzy sets by Zadeh [1], rough sets theory by Pawlak [36], IFS by Atanassov [2] etc. are generally used by scholars to handle the complexity during analysis but grossly all these notions have the lack of parameterizations tool. Therefore, these concepts cannot be impressively applied to real life problems. So, Molodtsov investigated the novel concept of S_{ft}S which is defined as:

1.8.1. Definition [53]

Consider a fix set T called universal and $\mathbb{E}$ represents the set of parameters with $\mathcal{K} \subseteq \mathbb{E}$. The pair $(\mathcal{H}, \mathcal{K})$ is known to be a S_{ft}S over T, where $\mathcal{H}$ is a function given by $\mathcal{H}: \mathcal{K} \to P(T)$. $P(T)$ denotes the power set of T.

1.8.2. Definition [82]

Let $(\mathcal{H}_1, \mathcal{K}_1)$ and $(\mathcal{H}_2, \mathcal{K}_2)$ be two S_{ft}Ss over a common universe T. Then $(\mathcal{H}_2, \mathcal{K}_2)$ is known to be a soft subset of $(\mathcal{H}_1, \mathcal{K}_1)$ if $\mathcal{K}_2 \subseteq \mathcal{K}_1$ and $\mathcal{H}_2(s) \subseteq \mathcal{H}_1(s)$, for all $s \in \mathcal{K}_2$.

1.8.3. Definition [82]

Two S_{ft}Ss $(\mathcal{H}_1, \mathcal{K}_1)$ and $(\mathcal{H}_2, \mathcal{K}_2)$ over a common universe T are called soft equal if $(\mathcal{H}_1, \mathcal{K}_1)$ is a soft subset of $(\mathcal{H}_2, \mathcal{K}_2)$ and $(\mathcal{H}_2, \mathcal{K}_2)$ is a soft subset of $(\mathcal{H}_1, \mathcal{K}_1)$.

1.8.4. Definition [82]

Let $(\mathcal{H}_1, \mathcal{K}_1)$ and $(\mathcal{H}_2, \mathcal{K}_2)$ be two S_{ft}Ss over a common universe T. Then their union is a S_{ft}S (E, C), where $C = \mathcal{K}_1 \cup \mathcal{K}_2$ and for all $s \in C$,

$$E(s) = \begin{cases} \mathcal{H}_1(s) & if\ s \in \mathcal{K}_1 \backslash \mathcal{K}_2 \\ \mathcal{H}_2(s) & if\ s \in \mathcal{K}_2 \backslash \mathcal{K}_1 \\ \mathcal{H}_1(s) \cup \mathcal{H}_2(s) & if\ s \in \mathcal{K}_2 \cap \mathcal{K}_1 \end{cases}$$

This relation is represented as $(\mathcal{H}_1, \mathcal{K}_1)\ \widetilde{\cup}\ (\mathcal{H}_2, \mathcal{K}_2) = (E, C)$.

1.8.5. Definition [55]

Let $(\mathcal{H}_1, \mathcal{K}_1)$ and $(\mathcal{H}_2, \mathcal{K}_2)$ be two S_{ft}Ss over a same universe T. Then their extended intersection is a S_{ft}S (E, C), where $C = \mathcal{K}_1 \cup \mathcal{K}_2$ and for all $s \in C$,

$$E(s) = \begin{cases} \mathcal{H}_1(s) & if\ s \in \mathcal{K}_1 \backslash \mathcal{K}_2 \\ \mathcal{H}_2(s) & if\ s \in \mathcal{K}_2 \backslash \mathcal{K}_1 \\ \mathcal{H}_1(s) \cap \mathcal{H}_2(s) & if\ s \in \mathcal{K}_2 \cap \mathcal{K}_1 \end{cases}$$

This relation is represented as $(\mathcal{H}_1, \mathcal{K}_1)\ \widetilde{\cap}\ (\mathcal{H}_2, \mathcal{K}_2) = (E, C)$.

1.8.6. Definition [54]

Consider a S_{ft}S $(\mathcal{H}, \mathbb{E})$ and $\mathcal{K} \subseteq \mathbb{E}$. A pair $(\widetilde{\mathcal{H}}, \mathcal{K})$ is said to be a fuzzy S_{ft}S (FS_{ft}S) over T, where $\widetilde{\mathcal{H}}$ is a function given by $\widetilde{\mathcal{H}}: \mathcal{K} \to \mathcal{H}^{(T)}$; $\mathcal{H}^{(T)}$ represents the collection of all fuzzy subsets of T, and is given as

$$\widetilde{\mathcal{H}}_{s_j} = \{\prec \mathscr{k}_i, \mu_j(\mathscr{k}_i) \succ |\mathscr{k}_i \in T\}$$

If s_j is any parameter and $\widetilde{\mathcal{H}}_{s_j}$ represents a crisp subset of a universal set X, then FS_{ft}S reduces to S_{ft}S.

1.9. Intuitionistic fuzzy soft set

In this subsection we will present the concept of IFS_{ft}S and their basic operational laws.

1.9.1. Definition [56]

Let $(\mathcal{H}, \mathbb{E})$ be a S_{ft}S over T. A pair $(\mathcal{J}, \mathcal{K})$ over T is said to be an IFS_{ft}S such that $\mathcal{J}$ is function given by $\mathcal{J}: \mathcal{K} \to IFS^T$. Then IF$S_{ft}$S denoted and defined as:

$$\mathcal{J}_{s_j}(\mathscr{k}_i) = \left\{\left(\mathscr{k}_i, \mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i)\right) |\mathscr{k}_i \in T \text{ and } s_j \in \mathbb{E}\right\},$$

where $\mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i)$ represents the $\mathcal{MG}$ and $\mathcal{NMG}$ an object $\mathscr{k}_i \in T$ to the set $\mathcal{J}_{s_j}$ with $0 \leq \mu_j(\mathscr{k}_i) + \psi_j(\mathscr{k}_i) \leq 1$. For the sake of simplicity $\mathcal{J}_{s_j}(\mathscr{k}_i) = \left(\mathscr{k}_i, \mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i)\right)$ is denoted as $\mathcal{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, which represents IF$S_{ft}$ value (IF$S_{ft}V$). Further, $\pi_{\mathcal{J}_{s_{ij}}} = 1 - (\mu_{ij} + \psi_{ij})$ is known as hesitancy degree.

1.9.2. Definition [58, 60]

Let $\mathcal{J}_{s_{i1}} = (\mu_1(\mathscr{k}_i), \psi_1(\mathscr{k}_i))$ *and* $\mathcal{J}_{s_{i2}} = (\mu_2(\mathscr{k}_i), \psi_2(\mathscr{k}_i)), (i = 1,2, \dots, m)$ be two IF$S_{ft}Vs$. Then the fundamental operations are given as:

(i) $\mathcal{J}_{s_{i1}} \cup \mathcal{J}_{s_{i2}} = \left(\mathscr{k}_i, \max\left(\mu_1(\mathscr{k}_i), \mu_2(\mathscr{k}_i)\right), \min\left(\psi_1(\mathscr{k}_i), \psi_2(\mathscr{k}_i)\right)\right)$ for $\mathscr{k}_i \in T$;

(ii) $\mathcal{J}_{s_{i1}} \cap \mathcal{J}_{s_{i2}} = \left(\mathscr{k}_i, \min\left(\mu_1(\mathscr{k}_i), \mu_2(\mathscr{k}_i)\right), \max\left(\psi_1(\mathscr{k}_i), \psi_2(\mathscr{k}_i)\right)\right)$ for $\mathscr{k}_i \in T$;

(iii) $\mathcal{J}^c_{s_{i1}} = \left(\mathscr{k}_i, \psi_1(\mathscr{k}_i), \mu_1(\mathscr{k}_i)\right)$ for $\mathscr{k}_i \in T$, where $\mathcal{J}^c_{s_{i1}}$ is complement of $\mathcal{J}_{s_{i1}}$.

1.9.3. Definition [58, 60]

Let $\mathcal{J}_{s_{11}} = (\mu_1(\mathscr{k}_1), \psi_1(\mathscr{k}_1))$ *and* $\mathcal{J}_{s_{12}} = (\mu_2(\mathscr{k}_1), \psi_2(\mathscr{k}_1))$ be two IF$S_{ft}Vs$. Then the fundamental operations are given as:

i. $$\mathcal{J}_{s_{11}} \oplus \mathcal{J}_{s_{12}} = \left(\sqrt[q]{\mu_1^q(\mathscr{k}_1) + \mu_2^q(\mathscr{k}_1) - \mu_1^q(\mathscr{k}_1)\mu_2^q(\mathscr{k}_1)}, \psi_1(\mathscr{k}_1)\psi_2(\mathscr{k}_1)\right);$$

ii. $$\mathcal{J}_{s_{11}} \otimes \mathcal{J}_{s_{12}} = \left(\mu_1(\mathscr{k}_1)\mu_2(\mathscr{k}_1), \sqrt[q]{\psi_1^q(\mathscr{k}_1) + \psi_2^q(\mathscr{k}_1) - \psi_1^q(\mathscr{k}_1)\psi_2^q(\mathscr{k}_1)}\right);$$

iii. $$\lambda \mathcal{J}_{s_{11}} = \left(\sqrt[q]{1 - \left(1 - \mu_1^q(\mathscr{k}_1)\right)^\lambda}, \psi_1^\lambda(\mathscr{k}_1)\right) \text{ for } \lambda > 0;$$

iv. $$\mathcal{J}^\lambda_{s_{11}} = \left(\mu_1^\lambda(\mathscr{k}_1), \sqrt[q]{1 - \left(1 - \psi_1^q(\mathscr{k}_1)\right)^\lambda}\right) \text{ for } \lambda > 0.$$

1.9.4. Definition [58]

Let $\mathcal{J}_{s_{11}} = (\mu_{11}, \psi_{11})$ be any IFS_{ft}V. Then the score function of $\mathcal{J}_{s_{11}}$ can be given as:

$$\mathcal{S}c(\mathcal{J}_{s_{11}}) = \mu_{11} - \psi_{11}; \qquad \mathcal{S}c(\mathcal{J}_{s_{11}}) \in [-1,1].$$

Greater the score value, greater the IFS_{ft}V is.

1.10. Intuitionistic fuzzy soft weighted averaging and geometric aggregation operators

Here we will discuss the detail of IFS_{ft}WA and IFS_{ft}WG aggregation operators.

1.10.1. Definition [58]

Let $\mathcal{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(i = 1,2 \dots, n, and\ j = 1,2 \dots, m)$ be IF$S_{ft}Vs$. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ be the weight vectors for experts $\mathcal{k}_i$ and parameters s_j such that $\sum_{i=1}^{n} \bar{\bar{w}} = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}} = 1$ with $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0, 1]$ respectively. Then the mapping for IF$S_{ft}WA$ aggregation operator is defined as: IF$S_{ft}WA: \mathcal{H}^n \to \mathcal{H}$ (where $\mathcal{H}^n$ is the collection of IFS_{ft}Vs)

$$\text{IF}S_{ft}WA(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots \mathcal{J}_{s_{nm}}) = \oplus_{j=1}^{m} \bar{\bar{u}}_j \left(\oplus_{i=1}^{n} \bar{\bar{w}}_i \mathcal{J}_{s_{ij}} \right)$$

$$= \left(1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \mu_{ij})^{\bar{\bar{w}}} \right)^{\bar{\bar{u}}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}} \right)^{\bar{\bar{u}}} \right).$$

1.10.2. Definition [58]

Let $\mathcal{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(i = 1,2 \dots, n, and\ j = 1,2 \dots, m)$ be IF$S_t Vs$. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ be the weight vectors for experts $\mathcal{k}_i$ and parameters s_j such that $\sum_{i=1}^{n} \bar{\bar{w}} = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}} = 1$ with $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0, 1]$ respectively. Then the mapping for IF$S_{ft}WG$ aggregation operator is defined as: IF$S_{ft}WG: \mathcal{H}^n \to \mathcal{H}$ (where $\mathcal{H}^n$ is the collection of IFS_{ft}Vs)

$$\text{IF}S_{ft}WG(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots \mathcal{J}_{s_{nm}}) = \otimes_{j=1}^{m} \left(\otimes_{i=1}^{n} \mathcal{J}_{s_{ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}$$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \mu_{ij}^{\bar{\bar{w}}} \right)^{\bar{\bar{u}}}, 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \psi_{ij})^{\bar{\bar{w}}} \right)^{\bar{\bar{u}}} \right).$$

1.10.3. Theorem [58]

Let $\mathcal{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(i = 1,2 \dots, n, and\ j = 1,2 \dots, m)$ be IF$S_t Vs$. Suppose $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ be the weight vectors for experts $\mathcal{k}_i$ and parameters s_j such that $\sum_{i=1}^{n} \bar{\bar{w}} = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}} = 1$ with $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0, 1]$ respectively. Then the IF$S_{ft}WA$ aggregation operator has the following properties.

(i) (Idempotency): If $\mathcal{J}_{s_{ij}} = \mathfrak{T}_s$, where $\mathfrak{T}_s = (\mu_s, \psi_s)$, then

$$\text{IF}S_{ft}WA(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}) = \mathfrak{T}_s.$$

(ii) (Boundedness): If $\mathcal{J}^-_{s_{ij}} = \left(\min_j \min_i\{\mu_{ij}\}, \max_j \max_i\{\psi_{ij}\}\right)$ and

$\mathcal{J}^+_{s_{ij}} = \left(\max_j \max_i\{\mu_{ij}\}, \min_j \min_i\{\psi_{ij}\}\right)$, then

$$\mathcal{J}^-_{s_{ij}} \leq \text{IF}S_{ft}WA(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}) \leq \mathcal{J}^+_{s_{ij}}.$$

(iii) (Shift-invariance): If $\mathfrak{T}_s = (\mu_s, \psi_s)$ is another IFS_tV, then

$$\text{IF}S_{ft}WA(\mathcal{J}_{s_{11}} \oplus \mathfrak{T}_s, \mathcal{J}_{s_{12}} \oplus \mathfrak{T}_s, \dots, \mathcal{J}_{s_{nm}} \oplus \mathfrak{T}_s) = \text{IF}S_{ft}WA(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}) \oplus \mathfrak{T}_s.$$

(iv) (Homogeneity): For $\lambda > 0$, then we have

$$\text{IF}S_{ft}WA(\lambda\mathcal{J}_{s_{11}}, \lambda\mathcal{J}_{s_{12}}, \dots, \lambda\mathcal{J}_{s_{nm}}) = \lambda\, \text{IF}S_{ft}WA(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}).$$

1.10.4. Theorem [58]

Let $\mathcal{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(i = 1,2 \dots, n, and\ j = 1,2 \dots, m)$ be IFS_{ft}Vs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ be the weight vectors for experts $\mathscr{k}_i$ and parameters s_j such that $\sum_{i=1}^{n} \bar{\bar{w}} = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}} = 1$ with $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0, 1]$ respectively. Then the IFS_{ft}WG aggregation operator has the properties.

(i) (Idempotency): If $\mathcal{J}_{s_{ij}} = \mathfrak{T}_s$, where $\mathfrak{T}_s = (\mu_s, \psi_s)$, then

$$\text{IF}S_{ft}WG(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}) = \mathfrak{T}_s.$$

(ii) (Boundedness): If $\mathcal{J}^-_{s_{ij}} = \left(\min_j \min_i\{\mu_{ij}\}, \max_j \max_i\{\psi_{ij}\}\right)$ and

$\mathcal{J}^+_{s_{ij}} = \left(\max_j \max_i\{\mu_{ij}\}, \min_j \min_i\{\psi_{ij}\}\right)$, then

$$\mathcal{J}^-_{s_{ij}} \leq \text{IF}S_{ft}WG(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}) \leq \mathcal{J}^+_{s_{ij}}.$$

(iii) (Shift-invariance): If $\mathfrak{T}_s = (\mu_s, \psi_s)$ is another IFS_{ft}V, then

$$\text{IF}S_{ft}WG(\mathcal{J}_{s_{11}} \oplus \mathfrak{T}_s, \mathcal{J}_{s_{12}} \oplus \mathfrak{T}_s, \dots, \mathcal{J}_{s_{nm}} \oplus \mathfrak{T}_s) = \text{IF}S_{ft}WG(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}) \oplus \mathfrak{T}_s.$$

(iv) (Homogeneity): For $\lambda > 0$, then we have

$$\text{IF}S_{ft}WG(\lambda\mathcal{J}_{s_{11}}, \lambda\mathcal{J}_{s_{12}}, \dots, \lambda\mathcal{J}_{s_{nm}}) = \lambda\, \text{IF}S_{ft}WG(\mathcal{J}_{s_{11}}, \mathcal{J}_{s_{12}}, \dots, \mathcal{J}_{s_{nm}}).$$

1.11. Soft rough set

In this section, the combine study of S_{ft}S and RS are presented to get the new notion of S_{ft}RS by based on crisp S_{ft}S relation form universe T to set of parameter $\mathbb{E}$.

1.11.1. Definition [83]

Suppose a S_{ft}S $(\mathcal{H}, \mathbb{E})$ over a universal set T. The relation $\mathcal{R}$ from T with $\mathbb{E}$ is said to be a crisp soft relation, whish is denoted and defined as:

$$\mathcal{R} = \{< (\mathscr{k}, s), \mu_{\mathcal{R}}(\mathscr{k}, s) > |(\mathscr{k}, s) \in T \times \mathbb{E}\},$$

where $\mu_{\mathcal{R}}: T \times \mathbb{E} \to \{0, 1\}$ with $\mu_{\mathcal{R}}(\mathscr{k}, s) = \begin{cases} 1 & (\mathscr{k}, s) \in \mathcal{R} \\ 0 & (\mathscr{k}, s) \notin \mathcal{R} \end{cases}$

1.11.2. Definition [84]

Consider a universal set T and $\mathbb{E}$ be the fixed set of parameter. Suppose $\mathcal{R} \subseteq T \times \mathbb{E}$ be the crisp soft relation over $T \times \mathbb{E}$. Suppose $\mathcal{R}^*$ be the set valued mapping (SVM) i.e. $\mathcal{R}^* = T \to \mathcal{P}^*(\mathbb{E})$ defined by $\mathcal{R}^*(\mathscr{k}) = \{s \in \mathbb{E} | (\mathscr{k}, s) \in \mathcal{R} \ and \ \mathscr{k} \in T\}$, where $\mathcal{P}^*(\mathbb{E})$ is the collection of power set of $\mathbb{E}$.

The relation $\mathcal{R}^*$ is known to be serial if $\forall \ \mathscr{k} \in T, \mathcal{R}^*(\mathscr{k}) \neq \emptyset$. Then the pair $(T, \mathbb{E}, \mathcal{R})$ is known to be a crisp soft approximation (S_{ft}A) space. Suppose $\emptyset \neq \mathcal{K} \subseteq \mathbb{E}$, then the lower and upper soft approximation of $\mathcal{K}$ w.r.t S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ is denoted and defined as:

$$\underline{\mathcal{R}}(\mathcal{K}) = \{\mathscr{k} \in T | \mathcal{R}^*(\mathscr{k}) \subseteq \mathcal{K}\} \qquad (1)$$

$$and \quad \bar{\mathcal{R}}(\mathcal{K}) = \{\mathscr{k} \in T | \mathcal{R}^*(\mathscr{k}) \cap \mathcal{K} \neq \emptyset\} \qquad (2)$$

The pair $\left(\underline{\mathcal{R}}(\mathcal{K}), \bar{\mathcal{R}}(\mathcal{K})\right)$ is known to be a crisp soft rough set, where $\underline{\mathcal{R}}(\mathcal{K}) \neq \bar{\mathcal{R}}(\mathcal{K})$. Hence $\underline{\mathcal{R}}(\mathcal{K}), \bar{\mathcal{R}}(\mathcal{K}): \mathcal{P}^*(\mathbb{E}) \to \mathcal{P}^*(T)$ is called lower and upper crisp soft rough approximation (S_{ft}RA) operators w.r.t $(T, \mathbb{E}, \mathcal{R})$.

1.11.3. Example

Consider a universal set $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_4, \mathscr{k}_5\}$ and let $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ be the set of parameters. Then the S_{ft}S $(\mathcal{H}, \mathbb{E})$ over T are defined as:

$$\mathcal{H}(s_1) = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3\}, \quad \mathcal{H}(s_2) = \emptyset, \quad \mathcal{H}(s_3) = \{\mathscr{k}_2, \mathscr{k}_4\}, \quad \mathcal{H}(s_4) = T,$$

Now consider a crisp soft relation $\mathcal{R}$ over $T \times \mathbb{E}$ is given as:

$$\mathcal{R} = \left\{ \begin{matrix} (\hbar_1, s_1), (\hbar_2, s_1), (\hbar_3, s_1), (\hbar_2, s_3), (\hbar_4, s_3), (\hbar_1, s_4), (\hbar_2, s_4), (\hbar_3, s_4), \\ (\hbar_4, s_4), (\hbar_5, s_4) \end{matrix} \right\}$$

Now from the definition of SVM $\mathcal{R}^*$, are given as

$$\mathcal{R}^*(\hbar_1) = \{s_1, s_4\}, \quad \mathcal{R}^*(\hbar_2) = \{s_1, s_2, s_4\}, \quad \mathcal{R}^*(\hbar_3) = \{s_1, s_4\}, \quad \mathcal{R}^*(\hbar_4) = \{s_3, s_4\}, \quad \mathcal{R}^*(\hbar_5) = \{s_4\}$$

If $\mathcal{K} = \{s_2, s_3, s_4\} \subseteq \mathbb{E}$ be the set of parameter then by Eqs. (1) and (2), $\underline{\mathcal{R}}(\mathcal{K}), \bar{\mathcal{R}}(\mathcal{K})$ are given as:

$$\underline{\mathcal{R}}(\mathcal{K}) = \{\hbar_4, \hbar_5\}, \qquad \bar{\mathcal{R}}(\mathcal{K}) = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\} = T.$$

1.11.4. Theorem [84]

Suppose $(T, \mathbb{E}, \mathcal{R})$ be a crisp S_{ft}RA space. Consider $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{P}^*(\mathbb{E})$, then lower and upper approximation satisfied the following properties:

i. $\underline{\mathcal{R}}(\mathcal{K}_1) = \sim\left(\overline{\mathcal{R}}(\sim\mathcal{K}_1)\right)$, where $\sim\mathcal{K}_1$ is complement of $\mathcal{K}_1$;

ii. $\overline{\mathcal{R}}(\mathcal{K}_1) = \sim\left(\underline{\mathcal{R}}(\sim\mathcal{K}_1)\right)$;

iii. $\overline{\mathcal{R}}(\mathcal{K}_1 \cup \mathcal{K}_2) = \overline{\mathcal{R}}(\mathcal{K}_1) \cup \overline{\mathcal{R}}(\mathcal{K}_2)$;

iv. $\underline{\mathcal{R}}(\mathcal{K}_1 \cap \mathcal{K}_2) = \underline{\mathcal{R}}(\mathcal{K}_1) \cap \underline{\mathcal{R}}(\mathcal{K}_2)$;

v. $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \overline{\mathcal{R}}(\mathcal{K}_1) \subseteq \overline{\mathcal{R}}(\mathcal{K}_2)$;

vi. $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \underline{\mathcal{R}}(\mathcal{K}_1) \subseteq \underline{\mathcal{R}}(\mathcal{K}_2)$;

vii. $\underline{\mathcal{R}}(\mathcal{K}_1 \cup \mathcal{K}_2) \supseteq \underline{\mathcal{R}}(\mathcal{K}_1) \cup \underline{\mathcal{R}}(\mathcal{K}_2)$;

viii. $\overline{\mathcal{R}}(\mathcal{K}_1 \cap \mathcal{K}_2) \subseteq \overline{\mathcal{R}}(\mathcal{K}_1) \cap \overline{\mathcal{R}}(\mathcal{K}_2)$.

1.12. Intuitionistic fuzzy soft rough set

Zhang [84] et al. originated the hybrid notion of S_{ft}S and RS with IFS to initiate the new concept of IFS_{ft}RS. They have presented some desirable properties of the IFS_{ft}RS.

1.12.1. Definition [83]

Consider a S_{ft}S $(\mathcal{H}, \mathcal{K})$ over a universal set T. Then a relation $\mathcal{R}$ is known to be a fuzzy soft relation (FS_{ft}R) from $T \times \mathbb{E}$ and denoted as:

$$\mathcal{R} = \{< (\hbar, s), \mu_{\mathcal{R}}(\hbar, s) > |(\hbar, s) \in T \times \mathbb{E}\},$$

where $\mu_{\mathcal{R}}: T \times \mathbb{E} \to [0,1]$. If $T = \{\hbar_1, \hbar_2, \dots, \hbar_m\}$ and $\mathbb{E} = \{s_1, s_2, \dots, s_n\}$, then the F$S_{ft}$R $\mathcal{R}$ from $T\ to\ \mathbb{E}$ is given in the following Table 1.1

Table 1.1, Tabular representation of FS_{ft}R

	s_1	s_2	$\cdots$	s_n
$\hbar_1$	$\mu_{\mathcal{R}}(\hbar_1, s_1)$	$\mu_{\mathcal{R}}(\hbar_1, s_2)$	$\cdots$	$\mu_{\mathcal{R}}(\hbar_1, s_n)$
$\hbar_2$	$\mu_{\mathcal{R}}(\hbar_2, s_1)$	$\mu_{\mathcal{R}}(\hbar_2, s_2)$	$\cdots$	$\mu_{\mathcal{R}}(\hbar_2, s_n)$
	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\hbar_m$	$\mu_{\mathcal{R}}(\hbar_m, s_1)$	$\mu_{\mathcal{R}}(\hbar_m, s_2)$	$\cdots$	$\mu_{\mathcal{R}}(\hbar_m, s_n)$

1.12.2. Definition [84]

Consider an S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$. Now for any $\mathcal{J} = \{< \hbar, \mu_{\mathcal{J}}(\hbar), \psi_{\mathcal{J}}(\hbar) > | \hbar \in T\}$, where $\mathcal{J} \in IFS^T$. Then the lower and upper approximation $\mathcal{J}$ w.r.t $(T, \mathbb{E}, \mathcal{R})$ are denoted by $\underline{\mathcal{R}}(\mathcal{J})\ and\ \overline{\mathcal{R}}(\mathcal{J})$ and are defined as:

$$\underline{\mathcal{R}}(\mathcal{J}) = \{< \hbar, \mu_{\underline{\mathcal{R}}(\mathcal{J})}(\hbar), \psi_{\underline{\mathcal{R}}(\mathcal{J})}(\hbar) > | \hbar \in T\}$$

$$\overline{\mathcal{R}}(\mathcal{J}) = \{< \hbar, \mu_{\overline{\mathcal{R}}(\mathcal{J})}(\hbar), \psi_{\overline{\mathcal{R}}(\mathcal{J})}(\hbar) > | \hbar \in T\},$$

where

$$\mu_{\underline{\mathcal{R}}(\mathcal{J})}(\hbar) = \bigwedge\nolimits_{\hbar \in \mathbb{E}} \{(1 - \mu_{\mathcal{R}}(\hbar, s)) \vee \mu_{\mathcal{J}}(\hbar)\}$$

$$\psi_{\underline{\mathcal{R}}(\mathcal{J})}(\hbar) = \bigvee\nolimits_{\hbar \in \mathbb{E}} \{\mu_{\mathcal{R}}(\hbar, s) \wedge \psi_{\mathcal{J}}(\hbar)\}$$

$$\mu_{\overline{\mathcal{R}}(\mathcal{J})}(\hbar) = \bigvee\nolimits_{\hbar \in \mathbb{E}} \{\mu_{\mathcal{R}}(\hbar, s) \wedge \mu_{\mathcal{J}}(\hbar)\}$$

$$\psi_{\overline{\mathcal{R}}(\mathcal{J})}(\hbar) = \bigwedge\nolimits_{\hbar \in T} \{(1 - \mu_{\mathcal{R}}(\hbar, s)) \vee \psi_{\mathcal{J}}(\hbar)\}$$

The pair $\left(\underline{\mathcal{R}}(\mathcal{J}), \bar{\mathcal{R}}(\mathcal{J})\right)$ is called an IFS_{ft}RS, where $\underline{\mathcal{R}}(\mathcal{J}) \neq \bar{\mathcal{R}}(\mathcal{J})$. Hence $\underline{\mathcal{R}}(\mathcal{J}), \bar{\mathcal{R}}(\mathcal{J}): IFS^{\mathbb{E}} \to IFS^T$ is called lower and upper IFS_tR approximation operators w.r.t $(T, \mathbb{E}, \mathcal{R})$.

The upper and lower approximation that is $\underline{\mathcal{R}}(\mathcal{J})$ *and* $\bar{\mathcal{R}}(\mathcal{J})$ is again IFN. Therefore, $\underline{\mathcal{R}}(\mathcal{J}), \bar{\mathcal{R}}(\mathcal{J}): IFS(\mathbb{E}) \rightarrow IFS(T)$ is called lower and upper IFS_tR approximation operators w.r.t $(T, \mathbb{E}, \mathcal{R})$.

1.12.3. Example

Consider a universal set $T = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\}$ and let $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ be the set of parameters. Suppose $\mathcal{R}$ be the FS_{ft}R over $T \times \mathbb{E}$ as given in Table 1.2.

Now to define $\mathcal{J} \in IFS^{\mathbb{E}}$ that is,

$$\mathcal{J} = \{(s_1, 0.7, 0.2), (s_2, 0.6, 0.3), (s_3, 0.4, 0.5), (s_4, 0.8, 0.1)\}.$$

Now to calculate $\underline{\mathcal{R}}(\mathcal{J}), \bar{\mathcal{R}}(\mathcal{J})$ are given as:

$$\underline{\mathcal{R}}(\mathcal{J}) = \{(\hbar_1, 0.4,0.5), (\hbar_2, 0.4,0.5), (\hbar_3, 0.6,0.3), (\hbar_4, 0.4,0.5), (\hbar_5, 0.4,0.5)\},$$

$$\bar{\mathcal{R}}(\mathcal{J}) = \{(\hbar_1, 0.7,0.2), (\hbar_2, 0.6,0.3), (\hbar_3, 0.6,0.4), (\hbar_4, 0.8,0.2), (\hbar_5, 0.6,0.3)\}.$$

Table 1.2, Tabular representation of FS_{ft}R

	s_1	s_2	s_3	s_4
$\hbar_1$	0.9	0.4	0.7	0.3
$\hbar_2$	0.6	0.8	0.9	0.5
$\hbar_3$	0.3	0.6	0.2	0.1
$\hbar_4$	0.7	0.3	0.6	0.8
$\hbar_5$	0.5	0.9	0.8	0.4

1.12.4. Theorem [84]

Suppose $(T, \mathbb{E}, \mathcal{R})$ be the fuzzy S_{ft}RA space. Consider $\mathcal{J}_1, \mathcal{J}_2 \in IFS^{\mathbb{E}}$, then lower and upper approximation operators $\underline{\mathcal{R}}(\mathcal{J}_1)$ and $\bar{\mathcal{R}}(\mathcal{J}_2)$ satisfied the following properties:

i. $\underline{\mathcal{R}}(\mathcal{J}_1) = \sim\left(\overline{\mathcal{R}}(\sim\mathcal{J}_1)\right)$, where $\sim\mathcal{J}_1$ is complement of $\mathcal{J}_1$,

ii. $\overline{\mathcal{R}}(\mathcal{J}_1) = \sim\left(\underline{\mathcal{R}}(\sim\mathcal{J}_1)\right)$,

iii. $\overline{\mathcal{R}}(\mathcal{J}_1 \cup \mathcal{J}_2) = \overline{\mathcal{R}}(\mathcal{J}_1) \cup \overline{\mathcal{R}}(\mathcal{J}_2)$,

iv. $\underline{\mathcal{R}}(\mathcal{J}_1 \cap \mathcal{J}_2) = \underline{\mathcal{R}}(\mathcal{J}_1) \cap \underline{\mathcal{R}}(\mathcal{J}_2)$,

v. $\mathcal{J}_1 \subseteq \mathcal{J}_2 \Rightarrow \overline{\mathcal{R}}(\mathcal{J}_1) \subseteq \overline{\mathcal{R}}(\mathcal{J}_2)$,

vi. $\mathcal{J}_1 \subseteq \mathcal{J}_2 \Rightarrow \underline{\mathcal{R}}(\mathcal{J}_1) \subseteq \underline{\mathcal{R}}(\mathcal{J}_2)$,

vii. $\underline{\mathcal{R}}(\mathcal{J}_1 \cup \mathcal{J}_2) \supseteq \underline{\mathcal{R}}(\mathcal{J}_1) \cup \underline{\mathcal{R}}(\mathcal{J}_2)$,

viii. $\overline{\mathcal{R}}(\mathcal{J}_1 \cap \mathcal{J}_2) \subseteq \overline{\mathcal{R}}(\mathcal{J}_1) \cap \overline{\mathcal{R}}(\mathcal{J}_2)$.

1.13. TOPSIS method with Pythagorean fuzzy information

Let $T = \{\mathscr{k}_1, \mathscr{k}_2, \ldots, \mathscr{k}_n\}$ be any set of n feasible alternatives, $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ be the set of m attributes and consider the weight vector $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \ldots, \overline{\overline{w}}_m)^T$ for all attributes such that $0 \leq \overline{\overline{w}}_i \leq 1$ and $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$. Decision makers $\mathcal{D}_{mem}$ and $\mathcal{D}_{non-mem}$ express their preference evaluation for alternatives $\mathscr{k}_i (i = 1, \ldots n)$ corresponding to the set of attribute $\mathcal{P}_j (j = 1, \ldots, m)$ by $\mu_{ij}\ and\ \psi_{ij}$ respectively. So combining these two values as a PyFVs, we have PyF decision matrix $\mathcal{P}_j(\mathscr{k}_i) = (\mu_{ij}, \psi_{ij})$. This means that the decision maker $\mathcal{D}_{mem}$ provides $\mathcal{MG}\ \mu_{ij}$ to an object $\mathscr{k}_i$ against to the attribute $\mathcal{P}_j$. Whereas the expert $\mathcal{D}_{non-mem}$ provides $\mathcal{NMG}\ \psi_{ij}$ to an object $\mathscr{k}_i$ against to the attribute $\mathcal{P}_j$ and their decision matrix is given as:

$$\mathcal{P}_j(\mathscr{k}_j) = \begin{pmatrix} (\mu_{11}, \psi_{11}) & (\mu_{12}, \psi_{12}) & \cdots & (\mu_{1j}, \psi_{1j}) \\ (\mu_{21}, \psi_{21}) & (\mu_{22}, \psi_{22}) & \cdots & (\mu_{2j}, \psi_{2j}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{i1}, \psi_{i1}) & (\mu_{i2}, \psi_{i2}) & \cdots & (\mu_{ij}, \psi_{ij}) \end{pmatrix}$$

By using the PyF TOPSIS approach, we will present PyF positive ideal solution and PyF negative ideal solution through the score function by Definition 1.3.4, which is given as:

$$P^+ = \{\mathcal{P}_j, max\{s(\mathcal{P}_j(\mathscr{k}_i))\}/j = 1, \ldots, m\}$$

$$= \{< \mathcal{P}_1, \mu_1^+, \psi_1^+ >, < \mathcal{P}_2, \mu_2^+, \psi_2^+ >, \ldots < \mathcal{P}_2, \mu_m^+, \psi_m^+ >\}$$

$$P^- = \{\mathcal{P}_j, min\{s(\mathcal{P}_j(\mathscr{k}_i))\}/j = 1, \ldots, m\}$$

$$= \{< \mathcal{P}_1, \mu_1^-, \psi_1^- >, < \mathcal{P}_2, \mu_2^-, \psi_2^- >, \ldots < \mathcal{P}_2, \mu_m^-, \psi_m^- >\}$$

Further to calculate the weighted distances $\mathcal{D}^+$ and $\mathcal{D}^-$ for an object $\mathscr{k}_i$ and PyF-PIS P^+ and PyF-NIS P^- are defined as the following:

$$\mathcal{D}^+(\mathscr{k}_i, P^+) = \sum_{j=1}^{m} \overline{\overline{w}}_j\ d\left(\mathcal{P}_j(\mathscr{k}_i), \mathcal{P}_j(P^+)\right)$$

$$= \left\{ \frac{1}{2} \sum_{j=1}^{m} \overline{\overline{w}}_j \left(\left| \mu_{ij}^2 - \left(\mu_j^+\right)^2 \right| + \left| \psi_{ij}^2 - \left(\psi_j^+\right)^2 \right| + \left| \pi_{ij}^2 - \left(\pi_j^+\right)^2 \right| \right) \right\} \; for\ (i = 1, \ldots, n)$$

Usually inferior the value of $\mathcal{D}^+$, better the alternative $\mathcal{k}_i$ and let,

$$\mathcal{D}^+{}_{min}(\mathcal{k}_i, P^+) = \min_{1 \le i \le n} \mathcal{D}^+(\mathcal{k}_i, P^+)$$

and

$$\mathcal{D}^-(\mathcal{k}_i, P^-) = \sum_{j=1}^{m} \overline{\overline{w}}_j \, d\left(\mathcal{P}_j(\mathcal{k}_i), \mathcal{P}_j(P^-) \right)$$

$$= \left\{ \frac{1}{2} \sum_{j=1}^{m} \overline{\overline{w}}_j \left(\left| \mu_{ij}^2 - \left(\mu_j^+\right)^2 \right| + \left| \psi_{ij}^2 - \left(\psi_j^+\right)^2 \right| + \left| \pi_{ij}^2 - \left(\pi_j^+\right)^2 \right| \right) \right\} \; for\ (i = 1, \ldots, n)$$

Greater the value of $\mathcal{D}^-$, better the alternative $\mathcal{k}_i$ and let,

$$\mathcal{D}^-{}_{\max}(\mathcal{k}_i, P^-) = \max_{1 \le i \le n} \mathcal{D}^-(\mathcal{k}_i, P^-)$$

In TOPSIS method to get the ranking of alternative $\mathcal{k}_i$, we use the revised closeness formula which is defined as:

$$\xi(\mathcal{k}_i) = \frac{\mathcal{D}^-(\mathcal{k}_i, P^-)}{\mathcal{D}^-{}_{\max}(\mathcal{k}_i, P^-)} - \frac{\mathcal{D}^+(\mathcal{k}_i, P^+)}{\mathcal{D}^+{}_{min}(\mathcal{k}_i, P^+)}$$

Larger the value of $\xi(\mathcal{k}_i)$, better that alternative is.

Chapter 2

Pythagorean fuzzy soft rough sets

In this chapter we are going to present the hybrid study of S_{ft}Ss, RSs and PyFSs to get the new concepts of soft rough Pythagorean fuzzy sets (S_{ft}RPyFS) and Pythagorean fuzzy soft rough sets (PyFS_{ft}RS). The aim of this chapter is to originate the two new notions that are S_{ft}RPyFS and PyFS_{ft}RS, and to investigate some important properties of S_{ft}RPyFS and PyFS_{ft}RS in detail. Furthermore, classical representations of PyFS_{ft}R approximation operators are presented. Then the proposed operators are applied to $\mathcal{DM}$ problem in which the experts provide their preferences in PyFS_{ft}R environment. Finally through an illustrative example, it is shown that how the proposed operators work in $\mathcal{DM}$ problems.

2.1.1. Definition

Suppose $T^* = \{(\ell_1, \ell_2): (\ell_1, \ell_2) \in [0,1] \times [0,1] \; such\; that \; \ell_1^2 + \ell_2^2 \leq 1\}$ with the ordered relation $\preccurlyeq$ represented as:

$$(\ell_1, \ell_2) \preccurlyeq (g_1, g_2) \Leftrightarrow \ell_1 \leq g_1 \;\; and \;\; g_2 \leq \ell_2 \;\; \forall\, (\ell_1, \ell_2), (g_1, g_2) \in T^* \qquad (2.1)$$

For an arbitrary $(\ell_1, \ell_2), (g_1, g_2)$ are known to be incomparable if the Eq. (2.1) is not satisfied.

2.1.2. Lemma

The ordered set T^* is a complete lattice w.r.t ordere d relation $\preccurlyeq$.

Next $\forall\, (\ell_1, \ell_2), (g_1, g_2) \in T^*$, the operation $\wedge \; and \; \vee$ on $(T^*, \preccurlyeq_{T^*})$ are given as:

$$(\ell_1, \ell_2) \wedge (g_1, g_2) = \{\min(\ell_1, g_1), \max(\ell_2, g_2)\}$$

$$(\ell_1, \ell_2) \vee (g_1, g_2) = \{\max(\ell_1, g_1), \min(\ell_2, g_2)\}.$$

2.2. Soft rough Pythagorean fuzzy set

In this section, we will present the concept of S_{ft}RPyFS by combining the crisp soft relation from T to $\mathbb{E}$ with the rough PyFS. Furthermore, the concept of S_{ft}RPyF approximation operators are investigated, and some basic properties of proposed operators are also discussed.

2.2.1. Definition

Consider a crisp S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$. Now for any $\mathfrak{J} = \{< s, \mu_{\mathfrak{J}}(s), \psi_{\mathfrak{J}}(s) > | s \in \mathbb{E}\}$, where $\mathfrak{J} \in PyFS^{\mathbb{E}}$. Then the lower and upper soft approximations of $\mathfrak{J}$ w.r.t S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ are represented by $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ and are defined as:

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{< \hbar, \mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar), \psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar) > | \hbar \in T\}$$

$$\overline{\mathcal{R}}(\mathfrak{J}) = \{< \hbar, \mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\hbar), \psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\hbar) > | \hbar \in T\},$$

where

$$\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar) = \bigwedge_{s \in \mathcal{R}^*(\hbar)} \mu_{\mathfrak{J}}(s) \ and \ \psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar) = \bigvee_{s \in \mathcal{R}^*(\hbar)} \psi_{\mathfrak{J}}(s)$$

with $0 \leq \left(\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2 + \left(\psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2 \leq 1$ and

$$\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\hbar) = \bigvee_{s \in \mathcal{R}^*(\hbar)} \mu_{\mathfrak{J}}(s) \ and \ \psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\hbar) = \bigwedge_{s \in \mathcal{R}^*(\hbar)} \psi_{\mathfrak{J}}(s)$$

with $0 \leq \left(\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2 + \left(\psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2 \leq 1$

The pair $\left(\underline{\mathcal{R}}(\mathfrak{J}), \bar{\mathcal{R}}(\mathfrak{J})\right)$ is called a S_{ft}RPyFS of $\mathfrak{J}$ w.r.t $(T, \mathbb{E}, \mathcal{R})$, where $\underline{\mathcal{R}}(\mathfrak{J}) \neq \bar{\mathcal{R}}(\mathfrak{J})$. Hence $\underline{\mathcal{R}}(\mathfrak{J}), \bar{\mathcal{R}}(\mathfrak{J}): PFS^{\mathbb{E}} \to PyFS^{T}$ are called lower and upper S_{ft}RPyF approximation operators w.r.t $(T, \mathbb{E}, \mathcal{R})$.

2.2.2. Remark

Consider a crisp S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and let for any $\mathcal{F} = \{< s, \mu_{\mathcal{F}}(s) > | s \in \mathbb{E}\}$, where $\mathcal{F} \in FS^{\mathbb{E}}$. Then the defined lower and upper S_{ft}RPyF operators $\underline{\mathcal{R}}(\mathcal{F})$ and $\bar{\mathcal{R}}(\mathcal{F})$ reduce to soft rough fuzzy operators, that is

$$\underline{\mathcal{R}}(\mathcal{F}) = \{< \hbar, \mu_{\underline{\mathcal{R}}(\mathcal{F})}(\hbar) > | \hbar \in T\} \quad and \quad \overline{\mathcal{R}}(\mathcal{F}) = \{< \hbar, \mu_{\overline{\mathcal{R}}(\mathcal{F})}(\hbar), > | \hbar \in T\},$$

where

$$\mu_{\underline{\mathcal{R}}(\mathcal{F})}(\hbar) = \bigwedge_{\hbar_1 \in \mathcal{R}^*(\hbar)} \mu_{\mathcal{F}}(\hbar_1) \quad and \quad \mu_{\overline{\mathcal{R}}(\mathcal{F})}(\hbar) = \bigvee_{\hbar_1 \in \mathcal{R}^*(\hbar)} \mu_{\mathcal{F}}(\hbar_1)$$

with $0 \le \mu_{\underline{\mathcal{R}}(\mathcal{F})}(\mathscr{k}) \le 1$ and $0 \le \mu_{\overline{\mathcal{R}}(\mathcal{F})}(\mathscr{k}) \le 1$.

The pair $\left(\underline{\mathcal{R}}(\mathcal{F}), \bar{\mathcal{R}}(\mathcal{F})\right)$ is called a soft rough FS of $\mathcal{F}$ w.r.t S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$, where $\underline{\mathcal{R}}(\mathcal{F}) \neq \bar{\mathcal{R}}(\mathcal{F})$.

2.2.3. Remark

Consider a crisp S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and let $\mathfrak{J} \in P(\mathbb{E})$, where $P(\mathbb{E})$ is a power set of $\mathbb{E}$. Then the defined lower and upper S_{ft}RPyF operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ reduce to crisp S_{ft}RA operators. Hence, it is observed that S_{ft}RPyFS is the generalization of S_{ft}RS.

2.2.4. Example

Consider a universal set $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_4, \mathscr{k}_5\}$ let $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ be the set of parameters. Then a S_{ft}S $(\mathcal{H}, \mathbb{E})$ over T is given as:

$$\mathcal{H}(s_1) = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3\}, \ \ \mathcal{H}(s_2) = \emptyset, \ \ \mathcal{H}(s_3) = \{\mathscr{k}_2, \mathscr{k}_4\}, \quad \mathcal{H}(s_4) = T.$$

Let $\mathcal{R}$ be the crisp soft relation from T to $\mathbb{E}$, given by

$$\mathcal{R} = \left\{ \begin{matrix} (\mathscr{k}_1, s_1), (\mathscr{k}_2, s_1), (\mathscr{k}_3, s_1), (\mathscr{k}_2, s_3), (\mathscr{k}_4, s_3), (\mathscr{k}_1, s_4), (\mathscr{k}_2, s_4), (\mathscr{k}_3, s_4), \\ (\mathscr{k}_4, s_4), (\mathscr{k}_5, s_4) \end{matrix} \right\}.$$

Now from the above relation the SVM $\mathcal{R}^*$ is given as:

$$\mathcal{R}^*(\mathscr{k}_1) = \{s_1, s_4\}, \quad \mathcal{R}^*(\mathscr{k}_2) = \{s_1, s_3, s_4\}, \quad \mathcal{R}^*(\mathscr{k}_3) = \{s_1, s_4\}, \quad \mathcal{R}^*(\mathscr{k}_4) = \{s_3, s_4\}, \quad \mathcal{R}^*(\mathscr{k}_5) = \{s_4\}.$$

Next to define an arbitrary PyFS $\mathfrak{J}$ that is $\mathfrak{J} \in PyFS^{\mathbb{E}}$ as given below:

$$\mathfrak{J} = \{< s_1, 0.9, 0.3 >, < s_2, 0.8, 0.5 >, < s_3, 0.4, 0.7 >, < s_4, 0.7, 0.5 >\}$$

Now to determine the S_{ft}RPyF lower and upper approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ that are

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{< \mathscr{k}_1, 0.7, 0.5 >, < \mathscr{k}_2, 0.4, 0.7 >, < \mathscr{k}_3, 0.7, 0.5 >, < \mathscr{k}_4, 0.4, 0.7 >, < \mathscr{k}_5, 0.7, 0.5 >\}$$

$$\bar{\mathcal{R}}(\mathfrak{J}) = \{< \mathscr{k}_1, 0.9, 0.3 >, < \mathscr{k}_2, 0.9, 0.3 >, < \mathscr{k}_3, 0.9, 0.3 >, < \mathscr{k}_4, 0.7, 0.5 >, < \mathscr{k}_5, 0.7, 0.5 >\}.$$

2.2.5. Theorem

Suppose a crisp $S_{ft}A$ space $(T, \mathbb{E}, \mathcal{R})$. Then for any $\mathfrak{J}, \mathfrak{J}_1, \mathfrak{J}_2 \in PFS^{\mathbb{E}}$, the S_{ft}RPyF approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ hold the following characteristics:

i. $\underline{\mathcal{R}}(\mathfrak{J}) = \sim\left(\overline{\mathcal{R}}(\sim\mathfrak{J})\right)$, where $\sim\mathfrak{J}$ is complement of $\mathfrak{J}$,

ii. $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$,

iii. $\mathfrak{J}_1 \subseteq \mathfrak{J}_2 \Rightarrow \underline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \underline{\mathcal{R}}(\mathfrak{J}_2)$,

iv. $\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) \supseteq \underline{\mathcal{R}}(\mathfrak{J}_1) \cup \underline{\mathcal{R}}(\mathfrak{J}_2)$,

v. $\overline{\mathcal{R}}(\mathfrak{J}) = \sim\left(\underline{\mathcal{R}}(\sim\mathfrak{J})\right)$,

vi. $\overline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) = \overline{\mathcal{R}}(\mathfrak{J}_1) \cup \overline{\mathcal{R}}(\mathfrak{J}_2)$,

vii. $\mathfrak{J}_1 \subseteq \mathfrak{J}_2 \Rightarrow \overline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \overline{\mathcal{R}}(\mathfrak{J}_2)$,

viii. $\overline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) \subseteq \overline{\mathcal{R}}(\mathfrak{J}_1) \cap \overline{\mathcal{R}}(\mathfrak{J}_2)$.

Proofs: i. Now by using the definition of S_{ft}RPyFS, we have

$$\sim\left(\overline{\mathcal{R}}(\sim\mathfrak{J})\right) = \{< \hbar, \psi_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(\hbar), \mu_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(\hbar) > | \hbar \in T\}$$

$$= \left\{< \hbar, \bigwedge_{s\in\mathcal{R}^*(\hbar)} \psi_{(\sim\mathfrak{J})}(s), \bigvee_{s\in\mathcal{R}^*(\hbar)} \mu_{(\sim\mathfrak{J})}(s) > | \hbar \in T\right\}$$

$$= \left\{< \hbar, \bigwedge_{s\in\mathcal{R}^*(\hbar)} \mu_{(\mathfrak{J})}(s), \bigvee_{s\in\mathcal{R}^*(\hbar)} \psi_{(\mathfrak{J})}(s) > | \hbar \in T\right\}$$

$$= \{< \hbar, \mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar), \psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar) > | \hbar \in T\}$$

this implies that $\sim\left(\overline{\mathcal{R}}(\sim\mathfrak{J})\right) = \underline{\mathcal{R}}(\mathfrak{J})$.

ii. Now to show that $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$

$$\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \{< \hbar, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1\cap\mathfrak{J}_2)}(\hbar), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1\cap\mathfrak{J}_2)}(\hbar) > | \hbar \in T\}$$

$$= \left\{< \hbar, \bigwedge_{s\in\mathcal{R}^*(\hbar)} \mu_{(\mathfrak{J}_1\cap\mathfrak{J}_2)}(s), \bigvee_{s\in\mathcal{R}^*(\hbar)} \psi_{(\mathfrak{J}_1\cap\mathfrak{J}_2)}(s) > | \hbar \in T\right\}$$

$$= \left\{< \hbar, \bigwedge_{s\in\mathcal{R}^*(\hbar)} \{\mu_{(\mathfrak{J}_1)}(s) \wedge \mu_{(\mathfrak{J}_2)}(s)\}, \bigvee_{s\in\mathcal{R}^*(\hbar)} \{\psi_{(\mathfrak{J}_1)}(s) \vee \psi_{(\mathfrak{J}_2)}(s)\} > | \hbar \in T\right\}$$

$$= \left\{< \hbar, \left\{\bigwedge_{s\in\mathcal{R}^*(\hbar)} \mu_{(\mathfrak{J}_1)}(s) \wedge \bigwedge_{s\in\mathcal{R}^*(\hbar)} \mu_{(\mathfrak{J}_2)}(s)\right\}, \left\{\bigvee_{s\in\mathcal{R}^*(\hbar)} \psi_{(\mathfrak{J}_1)}(s) \vee \bigvee_{s\in\mathcal{R}^*(\hbar)} \psi_{(\mathfrak{J}_2)}(s)\right\} > | \hbar \in T\right\}$$

$$= \left\{< \mathscr{k}, \left\{\mu_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) \wedge \mu_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k})\right\}, \left\{\psi_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) \vee \psi_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k})\right\} > | \mathscr{k} \in T\right\}$$

this implies $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$.

iii. Now to prove that if $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$ $then$ $\underline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \underline{\mathcal{R}}(\mathfrak{J}_2)$,

$$\underline{\mathcal{R}}(\mathfrak{J}_1) = \left\{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) > | \mathscr{k} \in T\right\}$$

$$= \left\{< \mathscr{k}, \bigwedge_{s \in \mathcal{R}^*(\mathscr{k})} \mu_{(\mathfrak{J}_1)}(s), \bigvee_{s \in \mathcal{R}^*(\mathscr{k})} \psi_{(\mathfrak{J}_1)}(s) > | \mathscr{k} \in T\right\}$$

$$\leq \left\{< \mathscr{k}, \bigwedge_{s \in \mathcal{R}^*(\mathscr{k})} \mu_{(\mathfrak{J}_2)}(s), \bigvee_{s \in \mathcal{R}^*(\mathscr{k})} \psi_{(\mathfrak{J}_2)}(s) > | \mathscr{k} \in T\right\}$$

$$= \left\{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\right\}$$

$$= \underline{\mathcal{R}}(\mathfrak{J}_2)$$

this implies $\underline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \underline{\mathcal{R}}(\mathfrak{J}_2)$.

iv. To prove that $\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) \supseteq \underline{\mathcal{R}}(\mathfrak{J}_1) \cup \underline{\mathcal{R}}(\mathfrak{J}_2)$

$$\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) = \left\{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\right\}$$

$$= \left\{< \mathscr{k}, \bigwedge_{s \in \mathcal{R}^*(\mathscr{k})} \mu_{(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(s), \bigvee_{s \in \mathcal{R}^*(\mathscr{k})} \psi_{(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(s) > | \mathscr{k} \in T\right\}$$

$$= \left\{< \mathscr{k}, \bigwedge_{s \in \mathcal{R}^*(\mathscr{k})} \left\{\mu_{(\mathfrak{J}_1)}(s) \vee \mu_{(\mathfrak{J}_2)}(s)\right\}, \bigvee_{s \in \mathcal{R}^*(\mathscr{k})} \left\{\psi_{(\mathfrak{J}_1)}(s) \wedge \psi_{(\mathfrak{J}_2)}(s)\right\} > | \mathscr{k} \in T\right\}$$

$$\geq \left\{< \mathscr{k}, \left\{\bigwedge_{s \in \mathcal{R}^*(\mathscr{k})} \mu_{(\mathfrak{J}_1)}(s) \vee \bigwedge_{s \in \mathcal{R}^*(\mathscr{k})} \mu_{(\mathfrak{J}_2)}(s)\right\}, \left\{\bigvee_{s \in \mathcal{R}^*(\mathscr{k})} \psi_{(\mathfrak{J}_1)}(s) \wedge \bigvee_{s \in \mathcal{R}^*(\mathscr{k})} \psi_{(\mathfrak{J}_2)}(s)\right\} > | \mathscr{k} \in T\right\}$$

$$= \left\{< \mathscr{k}, \left\{\mu_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) \vee \mu_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k})\right\}, \left\{\psi_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) \wedge \psi_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k})\right\} > | \mathscr{k} \in T\right\}$$

this implies $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) \supseteq \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$.

The proofs of v. to viii. are easy and follow the above results.

Furthermore, by counter example it is observe that the equality does not hold in part iv. and viii.

2.2.6. Example

Suppose the crisp soft relation $\mathcal{R}$ on $T \times \mathbb{E}$ as given in Example 2.2.4, i.e.

$$\mathcal{R} = \left\{ \begin{matrix} (\hbar_1, s_1), (\hbar_2, s_1), (\hbar_3, s_1), (\hbar_2, s_3), (\hbar_4, s_3), (\hbar_1, s_4), (\hbar_2, s_4), (\hbar_3, s_4), \\ (\hbar_4, s_4), (\hbar_5, s_4) \end{matrix} \right\}$$

Now from the above relation the SVM $\mathcal{R}^*$ is given as:

$$\mathcal{R}^*(\hbar_1) = \{s_1, s_4\}, \quad \mathcal{R}^*(\hbar_2) = \{s_1, s_3, s_4\}, \quad \mathcal{R}^*(\hbar_3) = \{s_1, s_4\}, \quad \mathcal{R}^*(\hbar_4) = \{s_3, s_4\}, \quad \mathcal{R}^*(\hbar_5) = \{s_4\}$$

Next to define an arbitrary PyFSs $\mathfrak{J}_1, \mathfrak{J}_2$ that is $\mathfrak{J}_1, \mathfrak{J}_2 \in PFS^{\mathbb{E}}$ as given below:

$$\mathfrak{J}_1 = \{< s_1, 0.9,0.3 >, < s_2, 0.8,0.5 >, < s_3, 0.4,0.7 >, < s_4, 0.7,0.5 >\}$$

$$\mathfrak{J}_2 = \{< s_1, 0.85,0.5 >, < s_2, 0.9,0.4 >, < s_3, 0.5,0.6 >, < s_4, 0.4,0.3 >\}$$

Consider

$$\mathfrak{J}_1 \cup \mathfrak{J}_2 = \{< s_1, 0.9,0.3 >, < s_2, 0.9,0.4 >, < s_3, 0.5,0.6 >, < s_4, 0.7,0.3 >\}$$

Now to determine the S_{ft}RPyF lower and upper approximation operators $\underline{\mathcal{R}}(\mathfrak{J}_1)$ and $\underline{\mathcal{R}}(\mathfrak{J}_2)$ that are

$$\underline{\mathcal{R}}(\mathfrak{J}_1) = \{< \hbar_1, 0.7,0.5 >, < \hbar_2, 0.4,0.7 >, < \hbar_3, 0.7,0.5 >, < \hbar_4, 0.4,0.7 >, < \hbar_5, 0.7,0.5 >\}$$

$$\underline{\mathcal{R}}(\mathfrak{J}_2) = \{< \hbar_1, 0.4,0.5 >, < \hbar_2, 0.4,0.6 >, < \hbar_3, 0.4,0.5 >, < \hbar_4, 0.4,0.6 >, < \hbar_5, 0.4,0.3 >\}$$

Further

$$\underline{\mathcal{R}}(\mathfrak{J}_1) \cup \underline{\mathcal{R}}(\mathfrak{J}_2) = \left\{ \begin{matrix} < \hbar_1, 0.7,0.5 >, < \hbar_2, 0.4,0.6 >, < \hbar_3, 0.7,0.5 >, \\ < \hbar_4, 0.4,0.7 >, < \hbar_5, 0.7,0.3 > \end{matrix} \right\}$$

and

$$\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) = \{< \hbar_1, 0.7,0.3 >, < \hbar_2, 0.5,0.6 >, < \hbar_3, 0.7,0.3 >, < \hbar_4, 0.5,0.6 >, < \hbar_5, 0.7,0.3 >\}$$

From the above analysis it is observed that $\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) \nsubseteq \underline{\mathcal{R}}(\mathfrak{J}_1) \cup \underline{\mathcal{R}}(\mathfrak{J}_2)$ because

$$\left\{< \mathcal{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{I}_1 \cup \mathfrak{I}_2)}(\mathcal{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{I}_1 \cup \mathfrak{I}_2)}(\mathcal{k}) > | \mathcal{k} \in T\right\} \nleq \left\{< \mathcal{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{I}_1) \cup \underline{\mathcal{R}}(\mathfrak{I}_2)}(\mathcal{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{I}_1) \cup \underline{\mathcal{R}}(\mathfrak{I}_2)}(\mathcal{k}) > | \mathcal{k} \in T\right\}$$

This implies $< \mathcal{k}_1, 0.7, 0.3 > \nleq < \mathcal{k}_1, 0.7, 0.5 > \Rightarrow 0.7 \leq 0.7 \;\; but\; 0.3 \ngeq 0.5$.

Similarly, we can show that

$$\overline{\mathcal{R}}(\mathfrak{I}_1) \cup \overline{\mathcal{R}}(\mathfrak{I}_2) \nsubseteq \overline{\mathcal{R}}(\mathfrak{I}_1 \cup \mathfrak{I}_2).$$

2.3. Pythagorean fuzzy soft rough set

In this section, motivated from the concept of IF S_{ft}RS [84] we investigate the novel notion of PyFS_{ft}RS. Furthermore, the basic properties of PyFS_{ft}R approximation operators are also investigated in detail.

2.3.1. Definition

Consider the set of parameters $\mathbb{E}$ and let T be the universe of discourse. Then the pair $(\mathfrak{I}, \mathbb{E})$ is called Pythagorean fuzzy S_{ft}S (PyFS_{ft}S) over a universe of discourse T, where $\mathfrak{I}$ is a mapping given by $\mathfrak{I}: \mathbb{E} \to PyFS^T$ such that $\forall\, s \in \mathbb{E}, \mathfrak{I}(s) = \left\{< \mathcal{k}, \mu_{\mathfrak{I}(s)}(\mathcal{k}), \psi_{\mathfrak{I}(s)}(\mathcal{k}) > | \mathcal{k} \in T\right\} \in PyFS^T$ and $\mu_{\mathfrak{I}(s)}, \psi_{\mathfrak{I}(s)}: T \to [0, 1]$ denotes the $\mathcal{MG}$ and $\mathcal{NMG}$ of an alternative $\mathcal{k} \in T$ to the set $\mathfrak{I}(s)$, with the constraint $0 \leq \left(\mu_{\mathfrak{I}(s)}(\mathcal{k})\right)^2 + \left(\psi_{\mathfrak{I}(s)}(\mathcal{k})\right)^2 \leq 1$.

2.3.2. Definition

Consider the set of parameters $\mathbb{E}$ and let T be the universe of discourse. Let $\mathcal{R}$ be an arbitrary FS_{ft}R over $T \times \mathbb{E}$. Then the triplet $(T, \mathbb{E}, \mathcal{R})$ is called fuzzy S_{ft}A space. Consider for any $\mathfrak{I} = \left\{< s, \mu_{\mathfrak{I}}(s), \psi_{\mathfrak{I}}(s) > | s \in \mathbb{E}\right\} \in PyFS^{\mathbb{E}}$, then the lower and upper approximation $\underline{\mathcal{R}}(\mathfrak{I})$ and $\overline{\mathcal{R}}(\mathfrak{I})$ of $\mathfrak{I}$ w.r.t fuzzy S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ is denoted and defined as:

$$\underline{\mathcal{R}}(\mathfrak{I}) = \left\{< \mathcal{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k}) > | \mathcal{k} \in T\right\}$$
$$\overline{\mathcal{R}}(\mathfrak{I}) = \left\{< \mathcal{k}, \mu_{\overline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k}), \psi_{\overline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k}) > | \mathcal{k} \in T\right\},$$

where

$$\mu_{\underline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k}) = \bigwedge_{s \in \mathbb{E}} \left\{\left(1 - \mu_{\mathcal{R}}(\mathcal{k}, s)\right) \vee \mu_{\mathfrak{I}}(s)\right\} \; and \;\; \psi_{\underline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k}) = \bigvee_{s \in \mathbb{E}} \left\{\mu_{\mathcal{R}}(\mathcal{k}, s) \wedge \psi_{\mathfrak{I}}(s)\right\}$$

with $0 \leq \left(\mu_{\underline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k})\right)^2 + \left(\psi_{\underline{\mathcal{R}}(\mathfrak{I})}(\mathcal{k})\right)^2 \leq 1$ and

$$\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigvee_{s\in\mathbb{E}}\{\mu_{\mathcal{R}}(\mathscr{k},s)\wedge\mu_{\mathfrak{J}}(s)\}\ and\ \psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigwedge_{s\in\mathbb{E}}\{(1-\mu_{\mathcal{R}}(\mathscr{k},s))\vee\psi_{\mathfrak{J}}(s)\}$$

with $0 \leq \left(\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})\right)^2 + \left(\psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})\right)^2 \leq 1$.

The pair $\left(\underline{\mathcal{R}}(\mathfrak{J}), \bar{\mathcal{R}}(\mathfrak{J})\right)$ is called PyFS_{ft}RS of $\mathfrak{J}$ w.r.t fuzzy S_{ft}A space $(T,\mathbb{E},\mathcal{R})$, where $\underline{\mathcal{R}}(\mathfrak{J}) \neq \bar{\mathcal{R}}(\mathfrak{J})$. Hence $\underline{\mathcal{R}}(\mathfrak{J}),\ \bar{\mathcal{R}}(\mathfrak{J}): PyFS^{\mathbb{E}} \to PyFS^{T}$ are called lower and upper PyFS_{ft}R approximation operators w.r.t $(T,\mathbb{E},\mathcal{R})$.

From the above analysis, it is clear that the lower and upper PyFS_{ft}R approximation operators $\underline{\mathcal{R}}(\mathfrak{J}),\ \bar{\mathcal{R}}(\mathfrak{J})$ are again the PyFVs implies $\underline{\mathcal{R}}(\mathfrak{J}),\ \bar{\mathcal{R}}(\mathfrak{J}) \in PyFS^{T}$ that is:

$$\begin{aligned}
&\left(\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})\right)^2 + \left(\psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})\right)^2 \\
&= \bigwedge_{s\in\mathbb{E}}\left[\left(1-(\mu_{\mathcal{R}}(\mathscr{k},s))^2\right)\vee\left(\mu_{\mathfrak{J}}(s)\right)^2\right] + \bigvee_{s\in\mathbb{E}}\left[(\mu_{\mathcal{R}}(\mathscr{k},s))^2\wedge\left(\psi_{\mathfrak{J}}(s)\right)^2\right] \\
&= 1-\bigvee_{s\in\mathbb{E}}\left[(\mu_{\mathcal{R}}(\mathscr{k},s))^2\wedge\left\{1-\left(\mu_{\mathfrak{J}}(s)\right)^2\right\}\right] + \bigvee_{s\in\mathbb{E}}\left[(\mu_{\mathcal{R}}(\mathscr{k},s))^2\wedge\left(\psi_{\mathfrak{J}}(s)\right)^2\right] \\
&\leq 1-\bigvee_{s\in\mathbb{E}}\left[(\mu_{\mathcal{R}}(\mathscr{k},s))^2\wedge\left\{1-\left(\mu_{\mathfrak{J}}(s)\right)^2\right\}\right] + \bigvee_{s\in\mathbb{E}}\left[(\mu_{\mathcal{R}}(\mathscr{k},s))^2\wedge\left\{1-\left(\mu_{\mathfrak{J}}(s)\right)^2\right\}\right] \\
&= 1
\end{aligned}$$

This implies $\left(\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})\right)^2 + \left(\psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})\right)^2 \leq 1$.

Hence it is clear that $\underline{\mathcal{R}}(\mathfrak{J}) \in PFS^{T}$. Similarly $\bar{\mathcal{R}}(\mathfrak{J}) \in PFS^{T}$. Therefore, the lower and upper PyFS_{ft}R approximation operators w.r.t $(T,\mathbb{E},\mathcal{R})$ is again a PyFV.

2.3.3. Remarks

Consider the crisp S_{ft}A space $(T,\mathbb{E},\mathcal{R})$ and $\mathfrak{J} \in PFS^{\mathbb{E}}$, then lower and upper PyFS_{ft}R approximation operator's $\underline{\mathcal{R}}(\mathfrak{J})\ and\ \bar{\mathcal{R}}(\mathfrak{J})$ reduced into the following form.

$$\underline{\mathcal{R}}(\mathfrak{J}) = \left\{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) > |\mathscr{k} \in T\right\}$$

$$\overline{\mathcal{R}}(\mathfrak{J}) = \left\{< \mathscr{k}, \mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}), \psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) > |\mathscr{k} \in T\right\},$$

where

$$\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigwedge_{s\in\mathcal{R}^*(\mathscr{k})}\mu_{\mathfrak{J}}(s)\ and\ \psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigvee_{s\in\mathcal{R}^*(\mathscr{k})}\psi_{\mathfrak{J}}(s)$$

and

$$\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigvee_{s\in\mathcal{R}^*(\mathscr{k})} \mu_{\mathfrak{J}}(s) \ and \ \psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigwedge_{s\in\mathcal{R}^*(\mathscr{k})} \psi_{\mathfrak{J}}(s)$$

Thus in this case the lower and upper PyFS_{ft}R approximation operators reduced to S_{ft}RPyF approximation operators. Hence it is observed that PyFS_{ft}RS is the generalization of S_{ft}RPyFS.

2.3.4. Remarks

Consider a FS_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and $\mathfrak{J} \in FS^{\mathbb{E}}$, then lower and upper PyFS_{ft}R approximation operator's $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ reduced into the following form.

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) > | \mathscr{k} \in T\}$$

$$\overline{\mathcal{R}}(\mathfrak{J}) = \{< \mathscr{k}, \mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) > | \mathscr{k} \in T\},$$

where

$$\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigwedge_{s\in\mathcal{R}^*(\mathscr{k})} \mu_{\mathfrak{J}}(s) \ and \ \psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigvee_{s\in\mathcal{R}^*(\mathscr{k})} \psi_{\mathfrak{J}}(s)$$

and

$$\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigvee_{s\in\mathcal{R}^*(\mathscr{k})} \mu_{\mathfrak{J}}(s) \ and \ \psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigwedge_{s\in\mathcal{R}^*(\mathscr{k})} \psi_{\mathfrak{J}}(s)$$

Thus in this case the lower and upper PyFS_{ft}R approximation operators reduced to S_{ft}RF approximation operators. Hence it is observed that PyFS_{ft}RS is the generalization of S_{ft}RFS.

2.3.5. Example

Suppose $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_4, \mathscr{k}_5\}$ be a universal of discourse and $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ be the set of parameter. Consider $(\mathcal{H}, \mathbb{E})$ be a FS_{ft}S over T. Let $\mathcal{R}$ be the FS_{ft}R over $T \times \mathbb{E}$ which is given in Table 2.1.

Next to define a PyFS $\mathfrak{J} \in PyFS^{\mathbb{E}}$ as follows:

$$\mathfrak{J} = \{< s_1, 0.8, 0.4 >, < s_2, 0.9, 0.2 >, < s_3, 0.7, 0.6 >, < s_4, 0.6, 0.3 >\}$$

Now to determine the PyFS_{ft}R lower and upper approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\underline{\mathcal{R}}(\mathfrak{J})$ that are

Table 2.1. $\mathrm{F}S_{ft}\mathrm{R}$ $\mathcal{R}$ over $T \times \mathbb{E}$

$\mathcal{R}$	s_1	s_2	s_3	s_4
k_1	0.8	0.3	0.5	0.7
k_2	0.9	0.6	0.4	0.6
k_3	0.7	0.1	0.8	0.3
k_4	0.2	0.9	0.3	0.6
k_5	0.8	0.3	0.9	0.5

$$\underline{\mathcal{R}}(\mathfrak{J}_1) = \{< k_1, 0.6, 0.5 >, < k_2, 0.6, 0.4 >, < k_3, 0.7, 0.6 >, < k_4, 0.6, 0.3 >, < k_5, 0.6, 0.6 >\}$$

$$\bar{\mathcal{R}}(\mathfrak{J}_2) = \{< k_1, 0.8, 0.3 >, < k_2, 0.8, 0.4 >, < k_3, 0.7, 0.4 >, < k_4, 0.9, 0.2 >, < k_5, 0.8, 0.4 >\}$$

2.3.6. Theorem

Suppose a fuzzy $S_{ft}A$ space $(T, \mathbb{E}, \mathcal{R})$. Then for any $\mathfrak{J}, \mathfrak{J}_1, \mathfrak{J}_2 \in PFS^{\mathbb{E}}$, the $\mathrm{PyF}S_{ft}\mathrm{R}$ approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ hold the following characteristics.

i. $\underline{\mathcal{R}}(\mathfrak{J}) = \sim\left(\overline{\mathcal{R}}(\sim\mathfrak{J})\right)$, where $\sim\mathfrak{J}$ is complement of $\mathfrak{J}$,

ii. $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$,

iii. $\mathfrak{J}_1 \subseteq \mathfrak{J}_2 \Rightarrow \underline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \underline{\mathcal{R}}(\mathfrak{J}_2)$,

iv. $\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) \supseteq \underline{\mathcal{R}}(\mathfrak{J}_1) \cup \underline{\mathcal{R}}(\mathfrak{J}_2)$,

v. $\overline{\mathcal{R}}(\mathfrak{J}) = \sim\left(\underline{\mathcal{R}}(\sim\mathfrak{J})\right)$,

vi. $\overline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) = \overline{\mathcal{R}}(\mathfrak{J}_1) \cup \overline{\mathcal{R}}(\mathfrak{J}_2)$,

vii. $\mathfrak{J}_1 \subseteq \mathfrak{J}_2 \Rightarrow \overline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \overline{\mathcal{R}}(\mathfrak{J}_2)$,

viii. $\overline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) \subseteq \overline{\mathcal{R}}(\mathfrak{J}_1) \cap \overline{\mathcal{R}}(\mathfrak{J}_2)$.

Proof: i. By applying definition of $\mathrm{PyF}S_{ft}\mathrm{R}$ approximation operators, we have

$$\sim\left(\overline{\mathcal{R}}(\sim\mathfrak{J})\right) = \{< k, \psi_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(k), \mu_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(k) > | k \in T\}$$

$$= \left\{< k, \bigwedge_{s\in\mathbb{E}}\{(1-\mu_{\mathcal{R}}(k,s)) \vee \psi_{(\sim\mathfrak{J})}(s)\}, \bigvee_{s\in\mathbb{E}}\{\mu_{\mathcal{R}}(k,s) \wedge \mu_{(\sim\mathfrak{J})}(s)\} > | k \in T\right\}$$

$$= \left\{< \mathscr{k}, \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(\mathscr{k},s)) \vee \mu_{\mathfrak{J}}(s)\}, \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(\mathscr{k},s) \wedge \psi_{\mathfrak{J}}(s)\} > | \mathscr{k} \in T\right\}$$

$$= \{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) > | \mathscr{k} \in T\}$$

this implies $\sim\left(\overline{\mathcal{R}}(\sim\mathfrak{J})\right) = \underline{\mathcal{R}}(\mathfrak{J})$.

ii. Now to prove that $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$, we have

$$\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1\cap\mathfrak{J}_2)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1\cap\mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\}$$

$$= \left\{< \mathscr{k}, \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(\mathscr{k},s)) \vee \mu_{(\mathfrak{J}_1\cap\mathfrak{J}_2)}(s)\}, \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(\mathscr{k},s) \wedge \psi_{(\mathfrak{J}_1\cap\mathfrak{J}_2)}(s)\} > | \mathscr{k} \in T\right\}$$

$$= \left\{< \mathscr{k}, \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(\mathscr{k},s)) \vee [\mu_{\mathfrak{J}_1}(s) \wedge \mu_{\mathfrak{J}_2}(s)]\}, \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(\mathscr{k},s) \wedge [\psi_{\mathfrak{J}_1}(s) \vee \psi_{\mathfrak{J}_2}(s)]\} > | \mathscr{k} \in T\right\}$$

$$= \left\{< \mathscr{k}, \left[\bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(\mathscr{k},s)) \wedge \mu_{\mathfrak{J}_1}(s)\} \vee \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(\mathscr{k},s)) \wedge \mu_{\mathfrak{J}_2}(s)\}\right], \left[\bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(\mathscr{k},s) \wedge \psi_{\mathfrak{J}_1}(s)\} \vee \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(\mathscr{k},s) \wedge \psi_{\mathfrak{J}_2}(s)\}\right] > | \mathscr{k} \in T\right\}$$

$$= \{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) \wedge \mu_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) \vee \psi_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\}$$

this implies $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$.

iii. If $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$ then we have to prove that $\underline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \underline{\mathcal{R}}(\mathfrak{J}_2)$

$$\underline{\mathcal{R}}(\mathfrak{J}_1) = \{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(\mathscr{k}) > | \mathscr{k} \in T\}$$

$$= \left\{< \mathscr{k}, \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(\mathscr{k},s)) \vee \mu_{(\mathfrak{J}_1)}(s)\}, \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(\mathscr{k},s) \wedge \psi_{(\mathfrak{J}_1)}(s)\} > | \mathscr{k} \in T\right\}$$

$$\leq \left\{< \mathscr{k}, \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(\mathscr{k},s)) \vee \mu_{(\mathfrak{J}_2)}(s)\}, \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(\mathscr{k},s) \wedge \psi_{(\mathfrak{J}_2)}(s)\} > | \mathscr{k} \in T\right\}$$

$$= \{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\}$$

This implies $\underline{\mathcal{R}}(\mathfrak{J}_1) \subseteq \underline{\mathcal{R}}(\mathfrak{J}_2)$.

iv. To prove that $\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) \supseteq \underline{\mathcal{R}}(\mathfrak{J}_1) \cup \underline{\mathcal{R}}(\mathfrak{J}_2)$

$$\underline{\mathcal{R}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) = \{< \mathscr{k}, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1\cup\mathfrak{J}_2)}(\mathscr{k}), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1\cup\mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\}$$

$$= \left\{ < k, \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(k,s)) \vee \mu_{(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(s)\}, \right.$$

$$\left. \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(k,s) \wedge \psi_{(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(s)\} > | k \in T \right\}$$

$$= \left\{ < k, \bigwedge_{s\in\mathbb{E}} \{(1-\mu_{\mathcal{R}}(k,s)) \vee [\mu_{\mathfrak{J}_1}(s) \vee \mu_{\mathfrak{J}_2}(s)]\}, \right.$$

$$\left. \bigvee_{s\in\mathbb{E}} \{\mu_{\mathcal{R}}(k,s) \wedge [\psi_{\mathfrak{J}_1}(s) \wedge \psi_{\mathfrak{J}_2}(s)]\} > | k \in T \right\}$$

$$\geq \left\{ < \left[\bigwedge_{s\in\mathbb{E}} (1-\mu_{\mathcal{R}}(k,s)) \vee \mu_{\mathfrak{J}_1}(s)\right] \vee \left[\bigwedge_{s\in\mathbb{E}} (1-\mu_{\mathcal{R}}(k,s)) \vee \mu_{\mathfrak{J}_2}(s)\right], \right.$$

$$\left. \left[\bigvee_{s\in\mathbb{E}} \mu_{\mathcal{R}}(k,s) \wedge \psi_{\mathfrak{J}_1}(s)\right] \wedge \left[\bigvee_{s\in\mathbb{E}} \mu_{\mathcal{R}}(k,s) \wedge \psi_{\mathfrak{J}_2}(s)\right] > \right\}$$

$$= \{ < k, \mu_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(k) \vee \mu_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(k), \psi_{\underline{\mathcal{R}}(\mathfrak{J}_1)}(k) \wedge \psi_{\underline{\mathcal{R}}(\mathfrak{J}_2)}(k) > | k \in T \}$$

this implies $\underline{\mathcal{R}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{R}}(\mathfrak{J}_1) \cap \underline{\mathcal{R}}(\mathfrak{J}_2)$.

The proofs of v. to viii. are easy and follow the above results.

2.3.7. Definition

Suppose $\mathfrak{J} = \{ < k, \mu_{\mathfrak{J}}(k), \psi_{\mathfrak{J}}(k) > | k \in T\} \in PFS^T$ be any PyFS and $\alpha, \beta \in [0,1]$ with $\alpha^2 + \beta^2 \leq 1$.Then the-$(\alpha, \beta)$ level cut set on $\mathfrak{J}$ is defined and denoted as:

$$\mathfrak{J}_\alpha^\beta = \{ k \in T | \mu_{\mathfrak{J}}(k) \geq \alpha, \psi_{\mathfrak{J}}(k) \leq \beta\}$$

Then the set $\mathfrak{J}_\alpha = \{k \in T | \mu_{\mathfrak{J}}(k) \geq \alpha\}$ is known as membership set of α-level cut which is generated as $\mathfrak{J}$. Similarly,

$\mathfrak{J}_{\alpha^+} = \{k \in T | \mu_{\mathfrak{J}}(k) > \alpha\}$ is known as membership set of strong α-level cut which is generated as $\mathfrak{J}$. The set $\mathfrak{J}^\beta = \{k \in T | \psi_{\mathfrak{J}}(k) \leq \beta\}$ is known as membership set of β-level cut which is generated as $\mathfrak{J}$. Similarly,

$\mathfrak{J}^{\beta^+} = \{k \in T | \psi_{\mathfrak{J}}(k) < \beta\}$ is known as membership set of strong β-level cut which is generated as $\mathfrak{J}$.

On the same way the other level cut sets of PyFS $\mathfrak{J}$ are denoted and defined as:

$\mathfrak{J}_{\alpha^+}^\beta = \{k \in T | \mu_{\mathfrak{J}}(k) > \alpha, \psi_{\mathfrak{J}}(k) \leq \beta\}$ is known as (α^+, β)-level cut set on $\mathfrak{J}$,

$\mathfrak{J}_\alpha^{\beta^+} = \{k \in T | \mu_{\mathfrak{J}}(k) \geq \alpha, \psi_{\mathfrak{J}}(k) < \beta\}$ is known as (α, β^+)-level cut set on $\mathfrak{J}$,

$\mathfrak{J}_{\alpha^+}^{\beta^+} = \{k \in T | \mu_{\mathfrak{J}}(k) > \alpha, \psi_{\mathfrak{J}}(k) < \beta\}$ is known as (α^+, β^+)-level cut set on $\mathfrak{J}$.

2.3.8. Theorem

Let $\mathfrak{J}, \mathfrak{J}_1, \mathfrak{J}_2$ be any PyFSs. Then the (α, β)-level cut set for PyFS satisfied the following, for $\alpha, \beta \in [0,1]$ with $\alpha^2 + \beta^2 \leq 1$.

(i) $\mathfrak{I}_{\alpha}^{\beta} = \mathfrak{I}_{\alpha} \cap \mathfrak{I}^{\beta}$;

(ii) $(\sim\mathfrak{I})_{\alpha} = \sim\mathfrak{I}^{\alpha^{+}}$, $(\sim\mathfrak{I})^{\beta} = \sim\mathfrak{I}_{\beta^{+}}$;

(iii) $\mathfrak{I}_1 \subseteq \mathfrak{I}_2 \Rightarrow (\mathfrak{I}_1)_{\alpha}^{\beta} \subseteq (\mathfrak{I}_2)_{\alpha}^{\beta}$;

(iv) $(\mathfrak{I}_1 \cap \mathfrak{I}_2)_{\alpha} = (\mathfrak{I}_1)_{\alpha} \cap (\mathfrak{I}_2)_{\alpha}$; $(\mathfrak{I}_1 \cap \mathfrak{I}_2)^{\beta} = (\mathfrak{I}_1)^{\beta} \cap (\mathfrak{I}_2)^{\beta}$; $(\mathfrak{I}_1 \cap \mathfrak{I}_2)_{\alpha}^{\beta} = (\mathfrak{I}_1)_{\alpha}^{\beta} \cap (\mathfrak{I}_2)_{\alpha}^{\beta}$;

(v) $(\mathfrak{I}_1 \cup \mathfrak{I}_2)_{\alpha} = (\mathfrak{I}_1)_{\alpha} \cup (\mathfrak{I}_2)_{\alpha}$; $(\mathfrak{I}_1 \cup \mathfrak{I}_2)^{\beta} = (\mathfrak{I}_1)^{\beta} \cup (\mathfrak{I}_2)^{\beta}$; $(\mathfrak{I}_1 \cup \mathfrak{I}_2)_{\alpha}^{\beta} \supseteq (\mathfrak{I}_1)_{\alpha}^{\beta} \cup (\mathfrak{I}_2)_{\alpha}^{\beta}$;

(vi) $\alpha_1 \geq \alpha_2 \ and \ \beta_1 \leq \beta_2 \Rightarrow \mathfrak{I}_{\alpha_1} \subseteq \mathfrak{I}_{\alpha_2}$; $\mathfrak{I}_{\beta_1} \subseteq \mathfrak{I}_{\beta_2}$; $\mathfrak{I}_{\alpha_1}^{\beta_1} \subseteq \mathfrak{I}_{\alpha_2}^{\beta_2}$;

Proof: The proof of (i) and (iii) are easy and follows from Definition 2.3.7.

(ii). To show that $(\sim\mathfrak{I})_{\alpha} = \sim\mathfrak{I}^{\alpha^{+}}$

If

$$\mathfrak{I} = \left\{< \mathcal{k}, \mu_{\mathfrak{I}}(\mathcal{k}), \psi_{\mathfrak{I}}(\mathcal{k}) > | \mathcal{k} \in T\right\}, \text{then } \sim\mathfrak{I} = \left\{< \mathcal{k}, \psi_{\mathfrak{I}}(\mathcal{k}), \mu_{\mathfrak{I}}(\mathcal{k}) > | \mathcal{k} \in T\right\}$$

Now

$$(\sim\mathfrak{I})_{\alpha} = \left\{\mathcal{k} \in T | \psi_{\mathfrak{I}}(\mathcal{k}) \geq \alpha\right\} \qquad (2.2)$$

Nest

$$\mathfrak{I}^{\alpha^{+}} = \left\{\left\{\mathcal{k} \in T | \psi_{\mathfrak{I}}(\mathcal{k}) < \alpha\right\}\right\}$$

This implies

$$\sim\mathfrak{I}^{\alpha^{+}} = \left\{\left\{\mathcal{k} \in T | \psi_{\mathfrak{I}}(\mathcal{k}) \geq \alpha\right\}\right\} \qquad (2.3)$$

Thus from Eqs. (2.2) and (2.3), we have

$$(\sim\mathfrak{I})_{\alpha} = \sim\mathfrak{I}^{\alpha^{+}}.$$

Similarly, it can be proved that

$$(\sim\mathfrak{I})^{\beta} = \sim\mathfrak{I}_{\beta^{+}}$$

(iv). To prove that $(\mathfrak{I}_1 \cap \mathfrak{I}_2)_{\alpha} = (\mathfrak{I}_1)_{\alpha} \cap (\mathfrak{I}_2)_{\alpha}$

As

$$\mathfrak{I}_1 \cap \mathfrak{I}_2 = \left\{< \mathcal{k}, min\left[\mu_{\mathfrak{I}_1}(\mathcal{k}), \mu_{\mathfrak{I}_2}(\mathcal{k})\right], max\left[\psi_{\mathfrak{I}_1}(\mathcal{k}), \psi_{\mathfrak{I}_2}(\mathcal{k})\right] > | \mathcal{k} \in T\right\}$$

Now

$$(\mathfrak{I}_1 \cap \mathfrak{I}_2)_{\alpha} = \left\{< \mathcal{k} \in T | min\left[\mu_{\mathfrak{I}_1}(\mathcal{k}), \mu_{\mathfrak{I}_2}(\mathcal{k})\right] \geq \alpha\right\}$$
$$= \left\{< \mathcal{k} \in T | \mu_{\mathfrak{I}_1}(\mathcal{k}) \geq \alpha\right\} \cap \left\{< \mathcal{k} \in T | \mu_{\mathfrak{I}_2}(\mathcal{k}) \geq \alpha\right\}$$
$$= (\mathfrak{I}_1)_{\alpha} \cap (\mathfrak{I}_2)_{\alpha}$$

Next

$$(\mathfrak{I}_1 \cap \mathfrak{I}_2)^\beta = \{\mathscr{k} \in T | max[\psi_{\mathfrak{I}_1}(\mathscr{k}), \psi_{\mathfrak{I}_2}(\mathscr{k})] \leq \beta\}$$
$$= \{\mathscr{k} \in T | \psi_{\mathfrak{I}_1}(\mathscr{k}) \leq \beta\} \cap \{\mathscr{k} \in T | \psi_{\mathfrak{I}_2}(\mathscr{k}) \leq \beta\}$$
$$= (\mathfrak{I}_1)^\beta \cap (\mathfrak{I}_2)^\beta$$

So, by using (i) we have

$$(\mathfrak{I}_1 \cap \mathfrak{I}_2)_\alpha^\beta = (\mathfrak{I}_1 \cap \mathfrak{I}_2)_\alpha \cap (\mathfrak{I}_1 \cap \mathfrak{I}_2)^\beta = (\mathfrak{I}_1)_\alpha \cap (\mathfrak{I}_2)_\alpha \cap (\mathfrak{I}_1)^\beta \cap (\mathfrak{I}_2)^\beta$$
$$= \{(\mathfrak{I}_1)_\alpha \cap (\mathfrak{I}_1)^\beta\} \cap \{(\mathfrak{I}_2)_\alpha \cap (\mathfrak{I}_2)^\beta\} = (\mathfrak{I}_1)_\alpha^\beta \cap (\mathfrak{I}_2)_\alpha^\beta$$

this implies $(\mathfrak{I}_1 \cap \mathfrak{I}_2)_\alpha^\beta = (\mathfrak{I}_1)_\alpha^\beta \cap (\mathfrak{I}_2)_\alpha^\beta$.

(v). Consider

$$\mathfrak{I}_1 \cup \mathfrak{I}_2 = \{< \mathscr{k}, max[\mu_{\mathfrak{I}_1}(\mathscr{k}), \mu_{\mathfrak{I}_2}(\mathscr{k})], min[\psi_{\mathfrak{I}_1}(\mathscr{k}), \psi_{\mathfrak{I}_2}(\mathscr{k})] > | \mathscr{k} \in T\}$$

Now

$$(\mathfrak{I}_1 \cup \mathfrak{I}_2)_\alpha = \{< \mathscr{k} \in T | max[\mu_{\mathfrak{I}_1}(\mathscr{k}), \mu_{\mathfrak{I}_2}(\mathscr{k})] \geq \alpha\}$$
$$= \{< \mathscr{k} \in T | \mu_{\mathfrak{I}_1}(\mathscr{k}) \geq \alpha\} \cup \{< \mathscr{k} \in T | \mu_{\mathfrak{I}_2}(\mathscr{k}) \geq \alpha\}$$
$$= (\mathfrak{I}_1)_\alpha \cup (\mathfrak{I}_2)_\alpha$$

Next

$$(\mathfrak{I}_1 \cap \mathfrak{I}_2)^\beta = \{\mathscr{k} \in T | min[\psi_{\mathfrak{I}_1}(\mathscr{k}), \psi_{\mathfrak{I}_2}(\mathscr{k})] \leq \beta\}$$
$$= \{\mathscr{k} \in T | \psi_{\mathfrak{I}_1}(\mathscr{k}) \leq \beta\} \cup \{\mathscr{k} \in T | \psi_{\mathfrak{I}_2}(\mathscr{k}) \leq \beta\}$$
$$= (\mathfrak{I}_1)^\beta \cup (\mathfrak{I}_2)^\beta$$

As we know that $\mathfrak{I}_1 \subseteq \mathfrak{I}_1 \cup \mathfrak{I}_2$ *and* $\mathfrak{I}_2 \subseteq \mathfrak{I}_1 \cup \mathfrak{I}_2$

So, by using (iii) we have

$$(\mathfrak{I}_1)_\alpha^\beta \subseteq (\mathfrak{I}_1 \cup \mathfrak{I}_2)_\alpha^\beta \quad and \quad (\mathfrak{I}_1)_\alpha^\beta \subseteq (\mathfrak{I}_1 \cup \mathfrak{I}_2)_\alpha^\beta$$

this implies $(\mathfrak{I}_1 \cap \mathfrak{I}_2)_\alpha^\beta = (\mathfrak{I}_1)_\alpha^\beta \cap (\mathfrak{I}_2)_\alpha^\beta$.

(vi). Consider for any $\mathscr{k} \in (\mathfrak{I}_1)_{\alpha_1}$, then by Definition 2.3.7, we get $\mu_{\mathfrak{I}_1}(\mathscr{k}) \geq \alpha_1 \geq \alpha_2 \Rightarrow \mu_{\mathfrak{I}_1}(\mathscr{k}) \geq \alpha_2 \Rightarrow \mathscr{k} \in (\mathfrak{I}_1)_{\alpha_2}$, therefore we have $(\mathfrak{I}_1)_{\alpha_1} \subseteq (\mathfrak{I}_1)_{\alpha_2}$.

Similarly, we get $(\mathfrak{I}_1)^{\beta_1} \subseteq (\mathfrak{I}_2)^{\beta_2}$. Consequently $(\mathfrak{I}_1)_{\alpha_1} \cap (\mathfrak{I}_2)^{\beta_1} \subseteq (\mathfrak{I}_1)_{\alpha_2} \cap (\mathfrak{I}_2)^{\beta_2}$, then by using (i) we have $(\mathfrak{I}_1)_{\alpha_1}^{\beta_1} \subseteq (\mathfrak{I}_2)_{\alpha_2}^{\beta_2}$.

Suppose a FS_{ft}R $\mathcal{R}$ from T to $\mathbb{E}$, denoted by

$$\mathcal{R}_\alpha = \{(\mathscr{k}, s) \in T \times \mathbb{E} | \mu_\mathcal{R}(\mathscr{k}, s) \geq \alpha\}$$
$$\mathcal{R}_\alpha(\mathscr{k}) = \{s \in \mathbb{E} | \mu_\mathcal{R}(\mathscr{k}, s) \geq \alpha\} \quad for\ \alpha \in [0,1]$$
$$\mathcal{R}_{\alpha^+} = \{(\mathscr{k}, s) \in T \times \mathbb{E} | \mu_\mathcal{R}(\mathscr{k}, s) > \alpha\}$$
$$\mathcal{R}_{\alpha^+}(\mathscr{k}) = \{s \in \mathbb{E} | \mu_\mathcal{R}(\mathscr{k}, s) > \alpha\} \quad for\ \alpha \in [0,1]$$
$$\mathcal{R}^\alpha = \{(\mathscr{k}, s) \in T \times \mathbb{E} | \psi(\mathscr{k}, s) \leq \alpha\}$$

$$\mathcal{R}^{\alpha}(\mathcal{k}) = \{s \in \mathbb{E} | \psi(\mathcal{k}, s) \leq \alpha\} \qquad for\ \alpha \in [0,1]$$

$$\mathcal{R}^{\alpha^+} = \{(\mathcal{k}, s) \in T \times \mathbb{E} | \psi(\mathcal{k}, s) < \alpha\}$$

$$\mathcal{R}^{\alpha^+}(\mathcal{k}) = \{s \in \mathbb{E} | \psi(\mathcal{k}, s) < \alpha\} \qquad for\ \alpha \in [0,1]$$

Then $\mathcal{R}_{\alpha}, \mathcal{R}_{\alpha^+}, \mathcal{R}^{\alpha}, \mathcal{R}^{\alpha^+}$ are crisp S_{ft}R on $T \times \mathbb{E}$.

In Theorems 2.3.9 and 2.3.10, it is shown that PyFS_{ft}R approximation operators can be represented by crisp S_{ft}RA operators.

2.3.9. Theorem

Suppose a FS_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and $\mathfrak{J} \in PFS^{\mathbb{E}}$. Then the upper PyF$S_{ft}$R approximation operator can be shown as follows, for all $\mathcal{k} \in T$.

(i).

$$\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) = \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{\mathcal{R}_{\alpha}}(\mathfrak{J}_{\alpha})(\mathcal{k})\right]$$

$$= \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{\mathcal{R}_{\alpha}}(\mathfrak{J}_{\alpha^+})(\mathcal{k})\right]$$

$$= \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha})(\mathcal{k})\right]$$

$$= \bigvee_{\alpha \in [0,1]} \left[\alpha \wedge \overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha^+})(\mathcal{k})\right]$$

(ii).

$$\psi_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) = \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee \left(1 - \overline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}^{\alpha})(\mathcal{k})\right)\right]$$

$$= \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee \left(1 - \overline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}^{\alpha^+})(\mathcal{k})\right)\right]$$

$$= \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee \left(1 - \overline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}^{\alpha})(\mathcal{k})\right)\right]$$

$$= \bigwedge_{\alpha \in [0,1]} \left[\alpha \vee \left(1 - \overline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}^{\alpha^+})(\mathcal{k})\right)\right]$$

(iii).

$$\left[\overline{\mathcal{R}}(\mathfrak{J})\right]_{\alpha^+} \subseteq \overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha^+}) \subseteq \overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha}) \subseteq \overline{\mathcal{R}_{\alpha}}(\mathfrak{J}_{\alpha}) \subseteq \left[\overline{\mathcal{R}}(\mathfrak{J})\right]_{\alpha}$$

(iv).

$$\left[\overline{\mathcal{R}}(\mathfrak{J})\right]^{\alpha^+} \subseteq \overline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}^{\alpha^+}) \subseteq \overline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}^{\alpha}) \subseteq \overline{\mathcal{R}_{(1-\alpha)}}(\mathfrak{J}^{\alpha}) \subseteq \left[\overline{\mathcal{R}}(\mathfrak{J})\right]^{\alpha}.$$

Proof: (i). For any $\mathcal{k} \in T$, we have

$$\bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{\mathcal{R}_{(\alpha)}}\mathfrak{J}_{\alpha}(\mathscr{k})\right]=Sup\{\alpha\in[0,1]|\mathscr{k}\in\overline{\mathcal{R}_{(\alpha)}}\mathfrak{J}_{\alpha}\}$$

$$=Sup\{\alpha\in[0,1]|\mathcal{R}_{(\alpha)}(\mathscr{k})\cap\mathfrak{J}_{\alpha}\neq\emptyset\}$$

$$=Sup\{\alpha\in[0,1]|\exists s\in\mathbb{E}[s\in\mathcal{R}_{(\alpha)}(\mathscr{k}),s\in\mathfrak{J}_{\alpha}]\}$$

$$=Sup\{\alpha\in[0,1]|\exists s\in\mathbb{E}[\mu_{\mathcal{R}}(\mathscr{k},s)\geq\alpha,\mu_{\mathfrak{J}}(s)\geq\alpha]\}$$

$$=\bigvee_{s\in\mathbb{E}}[\mu_{\mathcal{R}}(\mathscr{k},s)\wedge\mu_{\mathfrak{J}}(s)]$$

$$=\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})$$

On the same way we can prove that

$$\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k})=\bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{\mathcal{R}_{\alpha}}(\mathfrak{J}_{\alpha^+})(\mathscr{k})\right]$$

$$=\bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha})(\mathscr{k})\right]$$

$$=\bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha^+})(\mathscr{k})\right]$$

(ii). The upper crisp S_{ft}RA operator according to definition S_{ft}RS, we have

$$\bigwedge_{\alpha\in[0,1]}\left[\alpha\vee\left(1-\overline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}^{\alpha})(\mathscr{k})\right)\right]=\inf\{\alpha\in[0,1]|\mathscr{k}\in\overline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}^{\alpha})\}$$

$$=\inf\{\alpha\in[0,1]|\overline{\mathcal{R}_{1-\alpha}}(\mathscr{k})\cap\mathfrak{J}^{\alpha}\neq\emptyset\}$$

$$=\inf\{\alpha\in[0,1]|\exists s\in\mathbb{E}[s\in\overline{\mathcal{R}_{1-\alpha}},s\in\mathfrak{J}^{\alpha}]\}$$

$$=\inf\{\alpha\in[0,1]|\exists s\in\mathbb{E}[\mu_{\mathcal{R}}(\mathscr{k},s)\geq 1-\alpha,\psi_{\mathfrak{J}}(s)\leq\alpha]\}$$

$$=\bigwedge_{s\in\mathbb{E}}\{(1-\mu_{\mathcal{R}}(\mathscr{k},s))\vee\psi_{\mathfrak{J}}(s)\}$$

$$=\psi_{\overline{\mathcal{R}}(\mathfrak{J})}(s)$$

On the same way we can prove that

$$\psi_{\overline{\mathcal{R}}(\mathfrak{J})}(s)=\bigwedge_{\alpha\in[0,1]}\left[\alpha\vee\left(1-\overline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}^{\alpha^+})(\mathscr{k})\right)\right]$$

$$=\bigwedge_{\alpha\in[0,1]}\left[\alpha\vee\left(1-\overline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}^{\alpha})(\mathscr{k})\right)\right]$$

$$=\bigwedge_{\alpha\in[0,1]}\left[\alpha\vee\left(1-\overline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}^{\alpha^+})(\mathscr{k})\right)\right]$$

(iii). It is easily verified that $\overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha^+})\subseteq\overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha})\subseteq\overline{\mathcal{R}_{\alpha}}(\mathfrak{J}_{\alpha})$. We have just to prove that $[\overline{\mathcal{R}}(\mathfrak{J})]_{\alpha^+}\subseteq\overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha^+})$ and $\overline{\mathcal{R}_{\alpha}}(\mathfrak{J}_{\alpha})\subseteq[\overline{\mathcal{R}}(\mathfrak{J})]_{\alpha}$

Let for all $\mathcal{k} \in \left[\overline{\mathcal{R}}(\mathfrak{J})\right]_{\alpha^+}$ implies $\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) > \alpha$. Now according to definition of PyFS_{ft}R approximation operators

$\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) = \vee_{s\in\mathbb{E}}\left[\mu_{\mathcal{R}}(\mathcal{k}, s) \wedge \mu_{\mathfrak{J}}(s)\right] > \alpha$ holds. So there exist $s_0 \in \mathbb{E}$ such that $\mu_{\mathcal{R}}(\mathcal{k}, s_0) \wedge \mu_{\mathfrak{J}}(s_0) > \alpha$, this implies that $\mu_{\mathcal{R}}(\mathcal{k}, s_0) > \alpha$ and $\mu_{\mathfrak{J}}(s_0) > \alpha$. So, $s_0 \in \mathcal{R}_{\alpha^+}(\mathcal{k})$ and $s_0 \in \mathfrak{J}_{\alpha^+}$, thus as a result we get $\mathcal{R}_{\alpha^+}(\mathcal{k}) \cap \mathfrak{J}_{\alpha^+} \neq \emptyset$.Therefore, by definition of crisp S_{ft}RS, we have $s_0 \in \overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha^+})$. Hence $\left[\overline{\mathcal{R}}(\mathfrak{J})\right]_{\alpha^+} \subseteq \overline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}_{\alpha^+})$.

Next for any $\mathcal{k} \in \overline{\mathcal{R}_\alpha}(\mathfrak{J}_\alpha)$, we have $\overline{\mathcal{R}_\alpha}(\mathfrak{J}_\alpha)(\mathcal{k}) = 1$. Since $\mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) = \vee_{\beta\in[0,1]}[\beta \wedge \overline{\mathcal{R}_\beta}(\mathfrak{J}_\beta)(\mathcal{k})] \geq \alpha \wedge \overline{\mathcal{R}_\alpha}(\mathfrak{J}_\alpha)(\mathcal{k}) = \alpha \Rightarrow \mu_{\overline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) \geq \alpha$, thus $\mathcal{k} \in \left[\overline{\mathcal{R}}(\mathfrak{J})\right]_\alpha$. Therefore $\overline{\mathcal{R}_\alpha}(\mathfrak{J}_\alpha) \subseteq \left[\overline{\mathcal{R}}(\mathfrak{J})\right]_\alpha$.

2.3.10. Theorem

Suppose a FS_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and $\mathfrak{J} \in PyFS^{\mathbb{E}}$. Then the lower PyF$S_{ft}$A operator can be shown as: for all $\mathcal{k} \in T$

(i)

$$\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) = \bigwedge_{\alpha\in[0,1]}\left[\alpha \vee \underline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}_{\alpha^+})(\mathcal{k})\right]$$

$$= \bigwedge_{\alpha\in[0,1]}\left[\alpha \vee \underline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}_{\alpha})(\mathcal{k})\right]$$

$$= \bigwedge_{\alpha\in[0,1]}\left[\alpha \vee \underline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}_{\alpha^+})(\mathcal{k})\right]$$

$$= \bigwedge_{\alpha\in[0,1]}\left[\alpha \vee \underline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}_{\alpha})(\mathcal{k})\right]$$

(ii)

$$\psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathcal{k}) = \bigvee_{\alpha\in[0,1]}\left[\alpha \wedge \left(1 - \underline{\mathcal{R}_{\alpha}}(\mathfrak{J}^{\alpha})(\mathcal{k})\right)\right]$$

$$= \bigvee_{\alpha\in[0,1]}\left[\alpha \wedge \left(1 - \underline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}^{\alpha})(\mathcal{k})\right)\right]$$

$$= \bigvee_{\alpha\in[0,1]}\left[\alpha \wedge \left(1 - \underline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}^{\alpha^+})(\mathcal{k})\right)\right]$$

$$= \bigvee_{\alpha\in[0,1]}\left[\alpha \wedge \left(1 - \underline{\mathcal{R}_{\alpha}}(\mathfrak{J}^{\alpha^+})(\mathcal{k})\right)\right]$$

(iii)

$$\left[\underline{\mathcal{R}}(\mathfrak{J})\right]_{\alpha^+} \subseteq \underline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}_{\alpha^+}) \subseteq \underline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}_{\alpha^+}) \subseteq \underline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}_{\alpha}) \subseteq \left[\underline{\mathcal{R}}(\mathfrak{J})\right]_{\alpha}$$

(iv)

$$\left[\underline{\mathcal{R}}(\mathfrak{J})\right]^{\alpha^+} \subseteq \underline{\mathcal{R}_{(1-\alpha)^+}}\left(\mathfrak{J}^{\alpha^+}\right) \subseteq \underline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}^{\alpha}) \subseteq \underline{\mathcal{R}_{(1-\alpha)}}(\mathfrak{J}^{\alpha}) \subseteq \left[\underline{\mathcal{R}}(\mathfrak{J})\right]^{\alpha}$$

Proof: The proof of (i) and (iii) according to Theorem 2.3.8 and 2.3.9. Now for any $\mathscr{k} \in T$, consider

$$\begin{aligned}
\mu_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(\mathscr{k}) &= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{\mathcal{R}_{\alpha}}(\sim\mathfrak{J}_{\alpha})(\mathscr{k})\right] \\
&= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \overline{\mathcal{R}_{\alpha}}\left(\sim\mathfrak{J}^{\alpha^+}\right)(\mathscr{k})\right] \\
&= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(\sim\underline{\mathcal{R}_{\alpha}}\left(\mathfrak{J}^{\alpha^+}\right)(\mathscr{k})\right)\right] \\
&= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(1-\underline{\mathcal{R}_{\alpha}}\left(\mathfrak{J}^{\alpha^+}\right)(\mathscr{k})\right)\right]
\end{aligned}$$

$$\begin{aligned}
\psi_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(\mathscr{k}) &= \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\left(1-\overline{\mathcal{R}_{1-\alpha}}(\sim\mathfrak{J}^{\alpha})(\mathscr{k})\right)\right] \\
&= \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\left(1-\overline{\mathcal{R}_{1-\alpha}}(\sim\mathfrak{J}_{\alpha^+})(\mathscr{k})\right)\right] \\
&= \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\left\{1-\left(\sim\overline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}_{\alpha^+})(\mathscr{k})\right)\right\}\right] \\
&= \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\underline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}_{\alpha^+})(\mathscr{k})\right]
\end{aligned}$$

Therefore, by the duality of upper and lower PyFS_{ft}R approximation operators (see Theorem 2.3.6, we can get

$$\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \psi_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(\mathscr{k}) = \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\underline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}_{\alpha^+})(\mathscr{k})\right]$$

$$\psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \mu_{\overline{\mathcal{R}}(\sim\mathfrak{J})}(\mathscr{k}) = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(1-\underline{\mathcal{R}_{\alpha}}\left(\mathfrak{J}^{\alpha^+}\right)(\mathscr{k})\right)\right]$$

Similarly, the proof of above result, we can get

$$\begin{aligned}
\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) &= \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\underline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}_{\alpha})(\mathscr{k})\right] \\
&= \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\underline{\mathcal{R}_{(1-\alpha)^+}}(\mathfrak{J}_{\alpha^+})(\mathscr{k})\right] \\
&= \bigwedge_{\alpha\in[0,1]} \left[\alpha\vee\underline{\mathcal{R}_{1-\alpha}}(\mathfrak{J}_{\alpha})(\mathscr{k})\right]
\end{aligned}$$

And

$$\psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\mathscr{k}) = \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(1 - \underline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}^{\alpha})(\mathscr{k})\right)\right]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(1 - \underline{\mathcal{R}_{\alpha^+}}(\mathfrak{J}^{\alpha^+})(\mathscr{k})\right)\right]$$
$$= \bigvee_{\alpha\in[0,1]} \left[\alpha \wedge \left(1 - \underline{\mathcal{R}_{\alpha}}(\mathfrak{J}^{\alpha})(\mathscr{k})\right)\right]$$

The proofs of (iii) and (iv) are easy and follows form Theorem 2.3.9.

2.4. Application of Pythagorean fuzzy soft rough set in decision making

Here in this section, the technique for $\mathcal{DM}$ process is constructed on the basis of proposed approaches. For this, we have used the ring sum and ring product operations on PyFSs. By the operation the basic concept of this method and approach to $\mathcal{DM}$ is given, which is based on PyFS_{ft}RS approach.

2.4.1. Ring sum and ring product

2.4.1.1. Definition [76]

Let $\mathfrak{J}_1 = \{< \mathscr{k}, \mu_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k}), \psi_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k}) > | \mathscr{k} \in T\}$ and $\mathfrak{J}_2 = \{< \mathscr{k}, \mu_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k}), \psi_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\} \in PyFS^T$. Then the ring sum for $\mathfrak{J}_1$ and $\mathfrak{J}_2$ can be defined as:

$$\mathfrak{J}_1 \oplus \mathfrak{J}_2 = \left\{\sqrt{\left(\mu_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k})\right)^2 + \left(\mu_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k})\right)^2 - \left(\mu_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k})\right)^2 \left(\mu_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k})\right)^2},\right.$$
$$\left.\psi_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k})\psi_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k}) | \mathscr{k} \in T\right\}.$$

2.4.1.2. Definition [76]

Let $\mathfrak{J}_1 = \{< \mathscr{k}, \mu_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k}), \psi_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k}) > | \mathscr{k} \in T\}$ and $\mathfrak{J}_2 = \{< \mathscr{k}, \mu_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k}), \psi_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k}) > | \mathscr{k} \in T\} \in PyFS^T$. Then the ring product for $\mathfrak{J}_1$ and $\mathfrak{J}_2$ can be defined as:

$$\mathfrak{J}_1 \otimes \mathfrak{J}_2 = \left\{\mu_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k})\mu_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k}), \sqrt{\begin{matrix}\left(\psi_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k})\right)^2 + \left(\psi_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k})\right)^2 \\ - \left(\psi_{\mathcal{R}(\mathfrak{J}_1)}(\mathscr{k})\right)^2 \left(\psi_{\mathcal{R}(\mathfrak{J}_2)}(\mathscr{k})\right)^2\end{matrix}} | \mathscr{k} \in T\right\}.$$

Consider a fuzzy soft approximation space$(T, \mathbb{E}, \mathcal{R})$ in which T is the universal set, $\mathbb{E}$ be the initial set of parameters and $\mathcal{R}$ be the FS_{ft}R on $T \times \mathbb{E}$. We will initiate the general steps and $\mathcal{DM}$ algorithms of the proposed concepts as follows:

2.4.2. Algorithm

This subsection is devoted for the step wise algorithm of the proposed model and consists of the following steps:

Step (i): First to find the FS_{ft}R $\mathcal{R}$ from $T \times \mathbb{E}$ or fuzzy SS_{ft} $(\mathcal{H}, \mathbb{E})$ over T, accordance to the interests of decision maker.

Step (ii): For the evaluation of certain decision input each person has different point of views on the attribute of the same parameter, so to find the optimum normal decision object $\mathfrak{J}$ in accordance with the demand of expert/decision maker.

Step (iii): From Definition PyFS_{ft}RS, calculate the PyFS_{ft}R approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$.

Step (iv): By ring sum or ring product operation calculate the choice set.

$$\xi = \{\bar{\mathcal{R}}(\mathfrak{J}) \oplus \underline{\mathcal{R}}(\mathfrak{J})\}$$

$$= \left\{\sqrt{\left(\mu_{\bar{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2 + \left(\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2 - \left(\mu_{\bar{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2 \left(\mu_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar)\right)^2}, \right.$$

$$\left. \psi_{\bar{\mathcal{R}}(\mathfrak{J})}(\hbar)\psi_{\underline{\mathcal{R}}(\mathfrak{J})}(\hbar) | \hbar \in T \right\}$$

$$\xi = \{< \hbar, \mu_\xi(\hbar), \psi_\xi(\hbar) > | \hbar \in T\}.$$

Step (v): Compute the top level threshold value $\lambda = (\mu, \psi)$ such that $\mu = \max_{1\le i\le n} \mu_\xi(\hbar_i)$ and $\psi = \min_{1\le i\le n} \psi_\xi(\hbar_i)$. It is clear that in choice set ξ the PyFV, λ is the maximum choice value. If $\mu_\xi(\hbar_j) \ge_{T^*} \mu$ and $\psi_\xi(\hbar_j) \le_{T^*} \psi$ then the optimum decision value is $\hbar_j$.

The final decision is only one, one may go back to the second step and change the optimum decision object in the final step of the given algorithm, when there exist too many "optimal choices" to be chosen.

The concept of the proposed algorithm is illustrated with the help of the following example.

2.5. Illustrative example

For a certain senior position of a doctor in Pakistan Institute of Medical Sciences (PIMS) Cardiac Centre, the appointment of new faculty has to face a very complex evaluation and $\mathcal{DM}$ process. The skill and ability of a candidate may be judged with respect to various attributes like "physical and surgical productivity" "managerial skill" "ability to work under pressure" "research productivity" etc. In order to take the right decision about the candidate the professional experts opinions are needed for these criteria.

Consider that $T = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\}$ be set of five candidates who fulfil the requirements for the senior faculty position in PIMS. In order to appoint the most qualified and suitable candidates for the required position, a team of experts is organized and chaired by Prof. Z as a director. The team of experts will judge the candidates upon the criteria in the parameter set $\mathbb{E} = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, where

$s_1 = physical\ and\ surgical\ productivity,$ $s_2 = managerial\ skill$

$s_3 = experience\ and\ research\ productivity,$

$s_4 = abality\ to\ work\ under\ pressure$

$s_5 = academic\ leadership\ quality,$ $s_6 = contribution\ to\ PIMS$

According to the background and experience, the team of experts wants to appoint the candidate which qualifies with the parameters of $\mathbb{E}$ who deserves extremely from candidate in T.

Ring sum for PyFS_{ft}RS

Step (i). Consider that the experts explain the gorgeous and attractiveness of the candidates by calculating a FS_{ft}R $\mathcal{R}$ from $T \times \mathbb{E}$ which is given in the following Table 2.2.

Table 2.2. FS_{ft}R $\mathcal{R}$ from $T \times \mathbb{E}$

$\mathcal{R}$	s_1	s_2	s_3	s_4	s_5	s_6
$\hbar_1$	0.6	0.5	0.8	0.7	0.9	0.4
$\hbar_2$	0.8	0.4	0.7	0.6	0.5	0.3
$\hbar_3$	0.6	0.3	0.4	0.5	0.1	0.6
$\hbar_4$	0.4	0.5	0.1	0.3	0.8	0.2
$\hbar_5$	0.5	0.4	0.8	0.7	0.3	0.9

Step (ii): Suppose a committee of professionals present the optimum normal decision object $\mathfrak{J}$ which is a PyF subset over the set of parameters $\mathbb{E}$, which is given as:

$$\mathfrak{J} = \{< s_1, 0.5,0.3 >, < s_2, 0.6,0.4 >, < s_3, 0.7,0.2 >, < s_4, 0.4,0.5 >, < s_1, 0.5,0.4 >, < s_1, 0.8,0.2 >\}.$$

Therefore, the characteristics of the candidates upon the Criteria of the given parameters can be described by the PyFS. For example, the standard of the candidate under criteria/parameter s_1 is $(0.5,0.3)$. The value 0.5 is the degree of membership and the value 0.3 is the degree of non-membership of candidate under criteria s_1 respectively. In other words, candidate is qualified for the $\mathcal{MG}$ is 0.5 and disqualified for the $\mathcal{NMG}$ that is 0.3.

Step (iii): From Definition PyFS_{ft}RS, calculate the PyFS_{ft}R approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$.

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{< \mathscr{k}_1, 0.4,0.5 >, < \mathscr{k}_2, 0.5,0.4 >, < \mathscr{k}_3, 0.5,0.5 >, < \mathscr{k}_4, 0.2,0.5 >, < \mathscr{k}_5, 0.5,0.4 >\}$$

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{< \mathscr{k}_1, 0.7,0.2 >, < \mathscr{k}_2, 0.7,0.3 >, < \mathscr{k}_3, 0.6,0.4 >, < \mathscr{k}_4, 0.5,0.4 >, < \mathscr{k}_5, 0.8,0.2 >\}$$

Step (iv): By ring sum operation calculate the choice set.

$$\xi = \{\bar{\mathcal{R}}(\mathfrak{J}) \oplus \underline{\mathcal{R}}(\mathfrak{J})\}$$
$$= \{< \mathscr{k}_1, 0.75601,0.1 >, < \mathscr{k}_2, 0.78581,0.12 >, < \mathscr{k}_3, 0.72111,0.2 >, < \mathscr{k}_4, 0.52915,0.2 >, < \mathscr{k}_5, 0.8544,0.08 >\}$$

Step (v): Compute the top level threshold value $\lambda = (\mu, \psi)$ such that $\mu = \max_{1 \le i \le n} \mu_\xi(\mathscr{k}_i)$ and $\psi = \min_{1 \le i \le n} \psi_\xi(\mathscr{k}_i)$. It is clear that in choice set ξ the PyF value λ is the maximum choice value. If $\mu_\xi(\mathscr{k}_j) \ge_{T^*} \mu$ and $\psi_\xi(\mathscr{k}_j) \le_{T^*} \psi$, then the optimum decision value is $\mathscr{k}_j$. Hence the optimum decision is $\lambda = \mathscr{k}_5 = (0.8544,0.08)$.

Ring product for PyFS_{ft}RS

Now to calculate the optimal decision through ring product operator, we have

$$\xi = \{\bar{\mathcal{R}}(\mathfrak{J}) \oplus \underline{\mathcal{R}}(\mathfrak{J})\}$$

$$= \{< \mathscr{k}_1, 0.28,0.52915 >, < \mathscr{k}_2, 0.35,0.48539 >, < \mathscr{k}_3, 0.3,0.60828 >,$$
$$< \mathscr{k}_4, 0.1,0.60828 >, < \mathscr{k}_5, 0.4,0.44 >\}.$$

Hence, the optimal decision is $\lambda = \mathscr{k}_5 = (0.4,0.44)$. Therefore, the most qualified and suitable candidate for the required position is $\mathscr{k}_5$.

Ring sum for S_{ft}RPyFS

Step (i). Consider that the experts explain the gorgeous and attractiveness of the candidates through the proposed model of S_{ft}RPyFS. Consider a S_{ft}S $(\mathcal{H}, \mathbb{E})$ over T defined as follows:

$$\mathcal{H}(s_1) = \{\mathscr{k}_1, \mathscr{k}_6\}, \quad \mathcal{H}(s_2) = \emptyset, \quad \mathcal{H}(s_3) = \{\mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_4, \mathscr{k}_5\}, \quad \mathcal{H}(s_4) = \{\mathscr{k}_1, \mathscr{k}_3, \mathscr{k}_4\},$$
$$\mathcal{H}(s_5) = \{\mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_5\}, \quad \mathcal{H}(s_6) = \{\mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_5\}$$

Now to define crisp S_{ft}R $\mathcal{R}$ from $T \times \mathbb{E}$, that is

$$\mathcal{R} = \left\{ \begin{matrix} (\mathscr{k}_1, s_1), (\mathscr{k}_6, s_1), (\mathscr{k}_2, s_3), (\mathscr{k}_3, s_3), (\mathscr{k}_4, s_3), (\mathscr{k}_5, s_3), (\mathscr{k}_1, s_4), (\mathscr{k}_3, s_4), (\mathscr{k}_4, s_4), \\ (\mathscr{k}_2, s_5), (\mathscr{k}_3, s_5), (\mathscr{k}_5, s_5), (\mathscr{k}_2, s_6), (\mathscr{k}_3, s_6), (\mathscr{k}_5, s_6), \end{matrix} \right\}$$

Furthermore, from Definition of S_{ft}RS to obtain the SVM $\mathcal{R}^*$, that is

$$\mathcal{R}^*(\mathscr{k}_1) = \{s_1, s_4, s_6\}, \ \mathcal{R}^*(\mathscr{k}_2) = \{s_3, s_5\}, \ \mathcal{R}^*(\mathscr{k}_3) = \{s_3, s_4, s_5, s_6\}, \ \mathcal{R}^*(\mathscr{k}_4) = \{s_3, s_4\}, \ \mathcal{R}^*(\mathscr{k}_5) = \{s_3, s_5, s_6\},$$

Step (ii). Now consider the team of experts present the optimum normal decision object $\mathfrak{J}$ which is a PyF subset over the set $\mathbb{E}$ as follows:

$$\mathfrak{J} = \{< s_1, 0.5,0.3 >, < s_2, 0.6,0.4 >, < s_3, 0.7,0.2 >, < s_4, 0.4,0.5 >, < s_5, 0.5,0.4 >, < s_1, 0.8,0.2 >\}.$$

Therefore, the characteristics of the candidates upon the criteria of the given parameters can be described by the PyFSs. For example, the standard of the candidate under criteria/parameter s_1 is $(0.5,0.3)$. The value 0.5 is $\mathcal{MG}$ and the value 0.3 is $\mathcal{NMG}$ of candidate under criteria s_1 respectively. In other words, candidates is qualified for the $\mathcal{MG}$ is 0.5 and disqualified for the $\mathcal{NMG}$ is 0.3.

Step (iii): From Definition S_{ft}RPyFS, calculate the S_{ft}RPyF approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$.

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{< \mathscr{k}_1, 0.4,0.5 >, < \mathscr{k}_2, 0.5,0.4 >, < \mathscr{k}_3, 0.4,0.5 >, < \mathscr{k}_4, 0.4,0.5 >,$$
$$< \mathscr{k}_5, 0.5,0.4 >\}$$

$$\underline{\mathcal{R}}(\mathfrak{J}) = \{< \mathcal{k}_1, 0.8, 0.2 >, < \mathcal{k}_2, 0.7, 0.2 >, < \mathcal{k}_3, 0.8, 0.2 >, < \mathcal{k}_4, 0.7, 0.2 >, < \mathcal{k}_5, 0.8, 0.2 >\}$$

Step (iv): By ring sum operation calculate the choice set.

$$\xi = \{\bar{\mathcal{R}}(\mathfrak{J}) \oplus \underline{\mathcal{R}}(\mathfrak{J})\}$$
$$= \{< \mathcal{k}_1, 0.83522, 0.1 >, < \mathcal{k}_2, 0.78581, 0.08 >, < \mathcal{k}_3, 0.83522, 0.1 >, < \mathcal{k}_4, 0.75604, 0.1 >, < \mathcal{k}_5, 0.8544, 0.08 >\}$$

Step (v): Compute the top level threshold value $\lambda = (\mu, \psi) \in T^*$ such that $\mu = \max_{1 \leq i \leq n} \mu_\xi(\mathcal{k}_i)$ and $\psi = \min_{1 \leq i \leq n} \psi_\xi(\mathcal{k}_i)$. It is clear that in choice set ξ the PyFV, λ is the maximum choice value. If $\mu_\xi(\mathcal{k}_j) \geq_{T^*} \mu$ and $\psi_\xi(\mathcal{k}_j) \leq_{T^*} \psi$, then the optimum decision value is $\mathcal{k}_j$. Hence the optimum decision is $\lambda = \mathcal{k}_5 = (0.8544, 0.08)$.

Ring product for S_{ft}RPyFS

Now to calculate the optimal decision through ring product operator, we have

$$\xi = \{\bar{\mathcal{R}}(\mathfrak{J}) \oplus \underline{\mathcal{R}}(\mathfrak{J})\}$$
$$= \{< \mathcal{k}_1, 0.32, 0.52915 >, < \mathcal{k}_2, 0.35, 0.44 >, < \mathcal{k}_3, 0.32, 0.52915 >, < \mathcal{k}_4, 0.28, 0.52915 >, < \mathcal{k}_5, 0.4, 0.44 >\}.$$

Hence, the optimal decision is $\lambda = \mathcal{k}_5 = (0.4, 0.44)$. Therefore, the most qualified and suitable candidate for the required position is $\mathcal{k}_5$.

2.5.1. Comparative study

From the above analysis it is clear, that the proposed approach is better than intuitionistic fuzzy rough set (IFRS) [49], soft rough intuitionistic fuzzy set (S_{ft}RIFS) and intuitionistic fuzzy soft rough set (IFS_{ft}RS) [84]. The advantages of the proposed method with existing literature are given below.

2.5.1.1. Advantages

(a) Consider a crisp S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and let $\mathfrak{J} = \{< s, \mu_{\mathfrak{J}}(s) > | s \in T\} \in FS^{\mathbb{E}}$. Then the defined S_{ft}RPyF approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ degenerate into the S_{ft}R fuzzy set.

(b) Suppose a crisp S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and for a crisp set $\mathfrak{J} \in P^{\mathbb{E}}$ of $\mathbb{E}$. Then the defined S_{ft}RPyF approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ degenerate into S_{ft}RA operators as defined in Definition of S_{ft}RS.

(c) By taking crisp S_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and let $\mathfrak{J} \in PFS^{\mathbb{E}}$. Then the PyF$S_{ft}$R approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ in definition of PyFS_{ft}RS, degenerate into S_{ft}RPyF approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ in definition S_{ft}RPyFS.

(d) By taking FS_{ft}A space $(T, \mathbb{E}, \mathcal{R})$ and let $\mathfrak{J} \in FS^{\mathbb{E}}$. Then the PyF$S_{ft}$R approximation operators $\underline{\mathcal{R}}(\mathfrak{J})$ and $\bar{\mathcal{R}}(\mathfrak{J})$ in definition PyFS_{ft}RS, degenerate into soft fuzzy rough approximation operators defined by Sun and Ma [85].

Now to verify the effectiveness of the developed approach with some existing methods are presented in Table 2.3. by considering the above Illustrative Example. IFRS [49] having no information about parameterizations tools, so due to lake of this information the method developed in [86] failed to handle the proposed example. On the other hand, if the sum of PyF value $\left(\mu_{\mathfrak{J}}(\mathscr{k}), \psi_{\mathfrak{J}}(\mathscr{k})\right)$ is greater than 1, that is $\mu_{\mathfrak{J}}(\mathscr{k}) + \psi_{\mathfrak{J}}(\mathscr{k}) > 1$ in optimum normal decision object $\mathfrak{J}$ of Step (ii). So in this case the method presented in [84] failed to tackle the situation. Thus from the comparative study it is clear that the proposed method is more superior and provides more freedom to the decision makers for the selection of $\mathcal{MG}$ and $\mathcal{NMG}$ as compare to existing literature.

Table 2.3. Comparative study of the proposed method with some existing literature

Methods	Ranking
IFRS [49]	Failed to handle
S_{ft}RIFS [84]	$\mathscr{k}_5 > \mathscr{k}_2 > \mathscr{k}_1 \simeq \mathscr{k}_3 > \mathscr{k}_4$
IFS_{ft}RS [84]	$\mathscr{k}_5 > \mathscr{k}_2 > \mathscr{k}_1 \simeq \mathscr{k}_3 > \mathscr{k}_4$
S_{ft}RPyFS (proposed)	$\mathscr{k}_5 > \mathscr{k}_2 > \mathscr{k}_1 \simeq \mathscr{k}_3 > \mathscr{k}_4$
PyFS_{ft}RS (proposed)	$\mathscr{k}_5 > \mathscr{k}_2 > \mathscr{k}_1 \simeq \mathscr{k}_3 > \mathscr{k}_4$

2.5.2. Conclusion

The theories of RS, S_{ft}S, IFS and PyFS all are important mathematical tools for dealing with uncertainties. In this manuscript, we have presented two new concepts: S_{ft}RPyFS and PyFS_{ft}RS, which can be seen as two new generalization of S_{ft}RS models. Then we have investigated some important properties of S_{ft}RPyFS and PyFS_{ft}RS with detail.

Moreover, ?the classical representations of PyFS_{ft}R approximation operators are presented. In addition, the validity and effectiveness of the proposed operators are checked by applying them to the problems of $\mathcal{DM}$ in which the experts provide their preferences in PyFS_{ft}R environment. Finally, through a numerical example it is demonstrated that how the proposed operators work in $\mathcal{DM}$ problems. By comparative analysis, we find that it is more effective to deal with $\mathcal{DM}$ problem with the evaluation of PyF information based on S_{ft}RPyFS and PyFS_{ft}RS models than $\mathcal{DM}$ problem with the evaluation of S_{ft}RIFS and IFS_{ft}RS models.

Chapter 3

Covering based orthopair fuzzy rough set model hybrid with TOPSIS

In this chapter a comprehensive model is originated to handle the $\mathcal{DM}$ problems in which the experts have quite different opinions in favor or against some plans, entities or projects. Therefore, a new technique is applied to investigate the hybrid notions of RS with q-ROFS by using the concept of fuzzy β-covering and fuzzy β-covering neighborhoods to get the new notion of covering based q-ROF rough set (CBq-ROFRS). Furthermore, by applying the developed concept of CBq-ROFRS on TOPSIS and present its application to the $\mathcal{MADM}$. In real scenario CBq-ROFRS model is an important tool to discuss the complex and uncertain information. This method has stronger capacity than IFS and PyFS to cope the uncertainty. From the analysis, it is clear that CBq-ROFRS degenerates into covering based IF rough set (CBIFRS) if the rung $q = 1$ and degenerate into covering based PyF rough set (CBPyFRS) if the rung $q = 2$. Thus the proposed concept is the generalization of both CBIFRS and CBPyFRS. Moreover, an illustrative example is presented to show how the developed model will be helpful in $\mathcal{DM}$ problems and a comparative study of the developed method with some other methods is presented which show that the developed approach is more capable and superior than the existing methods.

3.1. Covering based q-rung orthopair fuzzy rough set

Here in this section we are going to investigate the hybrid structure of q-ROFSs, fuzzy CAS and fuzzy RSs to get the generalized structure of CBq-ROFRS. First we define the PyF covering approximation space (PyFCAS).

3.1.1. Definition

Let T be any set and $\aleph = \{\aleph_1, \aleph_2, \ldots, \aleph_m\}$, where $\aleph_i \in PFS^T$ and $i = 1,2,\ldots,m$. For any PyFV $\beta = (\mu_\beta, \psi_\beta)$, $\aleph$ is called Pythagorean fuzzy β-covering (PyF β-covering) of T, if $(\bigcup_{i=1}^{m} \aleph_i)(\hbar) \geqslant \beta$ for all $\hbar \in T$. Here $(T, \aleph)$ is called a PyFCAS.

Suppose that $(T, \aleph)$ be a PyFCAS and $\aleph = \{\aleph_1, \aleph_2, \ldots, \aleph_m\}$ be a PyF β-covering of T for some $\beta = (\mu_\beta, \psi_\beta)$. Then $\mathcal{N}_{\aleph(\hbar)}^{\beta} = \cap \{\aleph_j \in \aleph : \aleph_j \geqslant \beta,\ \mathrm{j} = 1,2,\ldots,\mathrm{m}\}$ PyF β neighbor-hood of $\hbar$ in T.

A PyF β-neighborhood system is denoted and defined as $\mathcal{N}_{\aleph}^{\beta} = \{\mathcal{N}_{\aleph(\hbar)}^{\beta} : \hbar \in T\}$ which is induced by PyF β-covering $\aleph$. By using PyF matrix to represent a PyF β-neighborhood system follows the structure given below:

$$\mathbb{M}_{\aleph}^{\beta} = \left[\mathcal{N}_{\aleph(\hbar_i)}^{\beta}(\hbar_j)\right]_{\hbar_i \times \hbar_j \in T \times T}$$

3.1.2. Remarks

(i) If $\beta = (1,0)$, then in this case PyF β-covering reduced to a crisp covering and if $\beta = (1,0)$, then PyF β-neighborhood reduced to a crisp neighborhood.

(ii) If $\beta = (\hbar, 0)$ such that $0 < \hbar < 1$, then in this case PyF β-covering reduced to a fuzzy covering and if $\beta = (\hbar, 0)$, then PyF β-neighborhood reduced to a fuzzy β-neighborhood respectively.

3.1.3. Definition

Let T be any set and $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m\}$, where $\mathcal{P}_i \in q - ROF^T$ and $i = 1,2,\dots,m$. For any q-ROFV $\beta = (\mu_\beta, \psi_\beta)$, $\mathcal{P}$ is said to be q-ROF β-covering (q-ROFβ-covering) of T, if $(\bigcup_{i=1}^{m} \mathcal{P}_i)(\hbar) \succcurlyeq \beta$ for all $\hbar \in T$. Here $(T, \mathcal{P})$ is called a q-ROF covering approximation space $(q - ROFCAS)$.

Suppose that $(T, \mathcal{P})$ be a $q - ROFCAS$ and $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m\}$ be a q-ROFβ-covering of T for some $\beta = (\mu_\beta, \psi_\beta)$.Then $\mathcal{N}_{\mathcal{P}(\hbar)}^{\beta} = \cap \{\mathcal{P}_j \in \mathcal{P} : \mathcal{P}_j(\hbar) \succcurlyeq \beta, \mathrm{j} = 1, 2, \dots, \mathrm{m}\}$ is called q-ROFβ-neighborhood of T.

A q-ROFβ-neighborhood system is denoted and defined as $\mathcal{N}_{\mathcal{P}}^{\beta} = \{ \mathcal{N}_{\mathcal{P}(\hbar)}^{\beta} : \hbar \in T\}$ which is induced by q-ROFβ-covering $\mathcal{P}$. By using q-ROF matrix to represent a q-ROFβ-neighborhood system as follows:

$$\mathbb{M}_{\mathcal{P}}^{\beta} = \left[\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}(\hbar_j)\right]_{\hbar_i \times \hbar_j \in T \times T}$$

3.1.4. Remark

(i) If $\beta = (1,0)$, then q-ROFβ-covering reduced to a crisp covering and if $\beta = (1,0)$, then in this case q-ROFβ-neighborhood reduced to a crisp neighborhood.

(ii) If $\beta = (\mathscr{k}, 0)$, such that $0 < \mathscr{k} < 1$, then q-ROFβ-covering reduced to a fuzzy covering and if $\beta = (\mathscr{k}, 0)$, then q-ROFβ-neighborhood reduced to a fuzzy β-neighborhood respectively.

Proof. (i) Let $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m\}$ q-ROF β-covering. Then by definition $(\bigcup_{i=1}^{m} \mathcal{P}_i)(\mathscr{k}) \succcurlyeq \beta$ for all $\mathscr{k} \in T$. If $\beta = (1,0)$, then there exist at least a q-ROFV $\alpha = (\mu_\alpha, \psi_\alpha) = (1,0)$ such that $(1,0) = \mathcal{P}_j(\mathscr{k})$, (for some $j = 1,2,\dots,m$) for $\mathscr{k} \in T$. Thus $\bigcup_{\mathcal{P}_i \in \mathcal{P}} \mathcal{P}_i = T$. Hence, if $\beta = (1,0)$, then in this case q-ROF β-covering reduceds into crisp cover.

Next consider $\mathcal{N}_{\mathcal{P}(\mathscr{k})}^{\beta} = \cap \{\mathcal{P}_j \in \mathcal{P}: \mathcal{P}_j(\mathscr{k}) \succcurlyeq \beta, j = 1,2,\dots,m\}$ be a q-ROF β-covering neighborhood of T. If $\beta = (1,0)$, then there exists at least a q-ROFV $\alpha = (\mu_\alpha, \psi_\alpha) = (1,0) = \mathcal{P}_i(\mathscr{k})$ such that $\alpha = \mathcal{P}_i(\mathscr{k}) \succcurlyeq \beta$, for $\mathscr{k} \in T$. Then each $\mathcal{N}_{\mathcal{P}(\mathscr{k})}^{\beta}$ contain at least a q-ROFV $\alpha = (\mu_\alpha, \psi_\alpha) = (1,0)$ for $\mathscr{k} \in T$. Thus $\mathcal{N}_{\mathscr{k}}^{\beta} = \cap \{\mathcal{P}_j : \mathcal{P}_j \in \mathcal{P}$ and $\mathscr{k} \in \mathcal{P}_j, j = 1,2, \dots, m\}$. Hence, if $\beta = (1,0)$, then in this case q-ROF β-covering neighborhood reduced into crisp neighborhood.

Similarly we can prove the (ii).

3.1.5. Definition [87]

Suppose $\mathfrak{J} = (\mu_{\mathfrak{J}}, \psi_{\mathfrak{J}})$ be a q-ROFV, then score function of $\mathfrak{J}$ is given as

$$\mathcal{S}c(\mathfrak{J}) = \frac{1}{2}(1 + \mu_{\mathfrak{J}}^{q} - \psi_{\mathfrak{J}}^{q}), \ \ \mathcal{S}c(\mathfrak{J}) \in [0,1], for\ q \geq 1.$$

Greater the score value of $\mathcal{S}c(\mathfrak{J})$, then superior the orthopair is.

3.1.6. Example

Suppose that $(T, \mathcal{P})$ be a q-ROFCAS and $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5\}$ be the set of q-ROFSs of T such that $T = \{\mathscr{k}_1, \mathscr{k}_2, \dots, \mathscr{k}_6\}$ with $\beta = (0.8,0.7)$ as given in Table 3.1.

Hence $\mathcal{P}$ is a q-ROF β-covering of T. Then

$\mathcal{N}_{\mathcal{P}(\mathscr{k}_1)}^{(0.8,0.7)} = \mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_5$, $\mathcal{N}_{\mathcal{P}(\mathscr{k}_2)}^{(0.8,0.7)} = \mathcal{P}_1 \cap \mathcal{P}_2$, $\mathcal{N}_{\mathcal{P}(\mathscr{k}_3)}^{(0.8,0.7)} = \mathcal{P}_1 \cap \mathcal{P}_3$,

$\mathcal{N}_{\mathcal{P}(\mathscr{k}_4)}^{(0.8,0.7)} = \mathcal{P}_1 \cap \mathcal{P}_4 \cap \mathcal{P}_5$, $\mathcal{N}_{\mathcal{P}(\mathscr{k}_5)}^{(0.8,0.7)} = \mathcal{P}_2 \cap \mathcal{P}_4$, $\mathcal{N}_{\mathcal{P}(\mathscr{k}_4)}^{(0.8,0.7)} = \mathcal{P}_1 \cap \mathcal{P}_3$

Table 3.1. Tabular representation of q-ROF β-covering

$T/\mathcal{P}$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$
$\mathscr{k}_1$	(0.9,0.5)	(0.85,0.65)	(0.7,0.8)	(0.5,0.9)	(0.9,0.3)
$\mathscr{k}_2$	(0.89,0.7)	(0.93,0.45)	(0.79,0.65)	(0.69,0.8)	(0.65,0.95)
$\mathscr{k}_3$	(0.95,0.6)	(0.69,0.85)	(0.98,0.43)	(0.7,0.4)	(0.5,0.9)
$\mathscr{k}_4$	(0.85,0.7)	(0.6,0.9)	(0.55,0.85)	(0.97,0.3)	(0.89,0.4)
$\mathscr{k}_5$	(0.6,0.87)	(0.9,0.45)	(0.69,0.85)	(0.92,0.6)	(0.8,0.75)
$\mathscr{k}_6$	(0.88,0.55)	(0.6,0.9)	(0.9,0.63)	(0.8,0.75)	(0.5,0.89)

From $\mathcal{N}_{\mathcal{P}}^{\beta} = \left\{\mathcal{N}_{\mathcal{P}(\mathscr{k})}^{\beta} : \mathscr{k} \in T\right\}$ is obtained the Table 3.2 given below,

Table 3.2. Tabular representation of $\mathcal{N}_{\mathcal{P}}^{(0.8,0.7)}$

	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$	$\mathscr{k}_4$	$\mathscr{k}_5$	$\mathscr{k}_6$
$\mathscr{k}_1$	(0.85,0.65)	(0.65,0.95)	(0.5,0.9)	(0.6,0.9)	(0.6,0.87)	(0.5,0.9)
$\mathscr{k}_2$	(0.85,0.65)	(0.89,0.7)	(0.69,0.85)	(0.6,0.9)	(0.6,0.87)	(0.6,0.9)
$\mathscr{k}_3$	(0.7,0.8)	(0.79,0.7)	(0.95,0.6)	(0.55,0.85)	(0.6,0.87)	(0.88,0.63)
$\mathscr{k}_4$	(0.5,0.9)	(0.65,0.95)	(0.5,0.9)	(0.85,0.7)	(0.6,0.87)	(0.5,0.89)
$\mathscr{k}_5$	(0.5,0.9)	(0.69,0.8)	(0.69,0.85)	(0.92,0.6)	(0.6,0.9)	(0.6,0.9)
$\mathscr{k}_6$	(0.7,0.8)	(0.79,0.7)	(0.95,0.6)	(0.55,0.85)	(0.6,0.8)	(0.88,0.63)

Therefore,

$$\mathbb{M}_{\mathcal{P}}^{(0.8,0.7)} = \begin{bmatrix} (0.85,0.65) & (0.65,0.95) & (0.5,0.9) & (0.6,0.9) & (0.6,0.87) & (0.5,0.9) \\ (0.85,0.65) & (0.89,0.7) & (0.69,0.85) & (0.6,0.9) & (0.6,0.87) & (0.6,0.9) \\ (0.7,0.8) & (0.79,0.7) & (0.95,0.6) & (0.55,0.85) & (0.6,0.87) & (0.88,0.63) \\ (0.5,0.9) & (0.65,0.95) & (0.5,0.9) & (0.85,0.7) & (0.6,0.87) & (0.5,0.89) \\ (0.5,0.9) & (0.69,0.8) & (0.69,0.85) & (0.6,0.9) & (0.9,0.6) & (0.6,0.9) \\ (0.7,0.8) & (0.79,0.7) & (0.95,0.6) & (0.55,0.85) & (0.6,0.87) & (0.88,0.63) \end{bmatrix}$$

3.1.7. Definition

Consider a q-ROFCAS (T, C), where $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \dots, \mathcal{P}_{\mathrm{m}}\}$ is the set of q-ROF β-covering of T for some $\beta = (\mu_\beta, \psi_\beta)$ and $T = \{\mathscr{k}_1, \mathscr{k}_2, \dots, \mathscr{k}_n\}$. Consider that the neighborhood system $\mathcal{N}_{\mathcal{P}}^{\beta} = \left\{\mathcal{N}_{\mathcal{P}(\mathcal{k})}^{\beta} : \mathcal{k} \in T\right\}$ induced by q-ROF β-covering of $\mathcal{P}$ such that

$$\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta} = \left\{< \mathcal{k}_j, \mu_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j), \psi_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) >_q \mid \text{for all } i = 1, \dots, n \text{ and } j = 1, \dots m\right\}$$

Now for any $\mathfrak{J} \in \mathrm{q-ROFS}^{T}$, where $\mathfrak{J} = \left\{< \mu_{\mathfrak{J}}(\mathcal{k}_j), \psi_{\mathfrak{J}}(\mathcal{k}_j)\right) >_q \mid j = 1, \dots, m\}$, the lower and upper approximations of $\mathfrak{J}$ w.r.t $\mathcal{N}_{\mathcal{P}(\mathcal{k})}^{\beta}$ is represented and defined by

$$\mathcal{N}_{\mathcal{P}}^{\beta}(\mathfrak{J}) = \left(\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}), \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})\right),$$

where

$$\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}) = \left\{< \mathcal{k}_i, \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i), \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) >_q \mid i = 1, \dots, n\right\}$$

and

$$\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}) = \left\{< \mathcal{k}_i, \mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i), \psi_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) >_q \mid i = 1, \dots, n\right\}$$

such that

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) = \bigwedge_{j=1}^{m} \left\{\mu_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \wedge \mu_{\mathfrak{J}}(\mathcal{k}_j)\right\}$$

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) = \bigvee_{j=1}^{m} \left\{\psi_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \vee \psi_{\mathfrak{J}}(\mathcal{k}_j)\right\}$$

$$\mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) = \bigvee_{j=1}^{m} \left\{\mu_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \vee \mu_{\mathfrak{J}}(\mathcal{k}_j)\right\}$$

$$\psi_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})}(\mathcal{k}_i) = \bigwedge_{j=1}^{m} \left\{\psi_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \wedge \psi_{\mathfrak{J}}(\mathcal{k}_j)\right\}$$

So the operators $\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}), \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})$:q-ROF$(T) \rightarrow$ q-ROF(T) are said to be lower and upper q-ROF rough (q-ROFR) approximation operators with respect to $\mathcal{N}_{\mathcal{P}}^{\beta}$.

Therefore, the CBq-ROFRSs is the pair $\mathcal{N}_{\mathcal{P}}^{\beta}(\mathfrak{J}) = (\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}), \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}))$, when ever $\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}) \neq \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})$.

3.1.8. Remark

i. If the value of $q = 1$, then the notion of CBq-ROFRS is reduced to CBIFRS.
ii. If the value of $q = 2$, then the notion of CBq-ROFRS is reduced to CBPyFRS.
iii. The notion of CBq-ROFRS is the generalization of CBIFRS and CBPyFRS models.

3.1.9. Example

Consider that $\mathfrak{J} \in \text{q} - \text{ROFS}^{T}$, that is

$$\mathfrak{J} = \begin{Bmatrix} (\hbar_1, 0.91, 0.62), (\hbar_2, 0.58, 0.83), (\hbar_3, 0.8, 0.75), (\hbar_4, 0.95, 0.35), \\ (\hbar_5, 0.8, 0.7), (\hbar_6, 0.98, 0.37) \end{Bmatrix}$$

and if we consider $\mathbb{M}_{\mathcal{P}}^{\beta} = \left[\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}(\hbar_j)\right]_{\hbar_i \times \hbar_j \in T \times T}$ as given in Example 3.1.6, where $\beta = (0.8,0.7)$. Then

$$\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}) = \begin{Bmatrix} (\hbar_1, 0.5, 0.95), (\hbar_2, 0.58, 0.9), (\hbar_3, 0.55, 0.87), (\hbar_4, 0.5, 0.95), \\ (\hbar_5, 0.5, 0.9), (\hbar_6, 0.55, 0.87) \end{Bmatrix}$$

$$\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}) = \begin{Bmatrix} (\hbar_1, 0.98, 0.35), (\hbar_2, 0.98, 0.35), (\hbar_3, 0.98, 0.35), (\hbar_4, 0.98, 0.35), \\ (\hbar_5, 0.98, 0.35), (\hbar_6, 0.98, 0.35) \end{Bmatrix}$$

3.1.10. Definition [88]

Let us consider that $\mathfrak{J}_1 = (\mu_{\mathfrak{J}_1}, \psi_{\mathfrak{J}_1})$ and $\mathfrak{J}_2 = (\mu_{\mathfrak{J}_2}, \psi_{\mathfrak{J}_2})$ be two q-ROFSs. Then the distance between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ is defined as follows:

$$D(\mathfrak{J}_1, \mathfrak{J}_2) = \left\{ \frac{1}{2n} \sum_{\hbar \in T} \left|\mu_{(\mathfrak{J}_1)}(\hbar) - \mu_{(\mathfrak{J}_2)}(\hbar)\right|^p + \frac{1}{2n} \sum_{\hbar \in T} \left|\psi_{(\mathfrak{J}_1)}(\hbar) - \psi_{(\mathfrak{J}_2)}(\hbar)\right|^p \right\}^{\frac{1}{p}}, \text{where } p \geq 1.$$

3.1.11. Theorem

Let $(T, \mathcal{P})$ be a q-ROFCAS and $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ be a q-ROFβ-covering of T for some $\beta = (\mu_\beta, \psi_\beta)$. Consider that the neighborhood system $\mathcal{N}_{\mathcal{P}}^{\beta} = \{\mathcal{N}_{\mathcal{P}(\mathscr{k})}^{\beta} | \mathscr{k} \in T\}$ induced by q-ROFβ-covering $\mathcal{P}$. Now for any $\mathfrak{J}_1, \mathfrak{J}_2 \in \text{q} - \text{ROFS}^T$, following are holds:

i. $\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}) \subseteq \mathfrak{J} \subseteq \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J})$;

ii. If $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$, then $\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \subseteq \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)$ and $\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \subseteq \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)$;

iii. $\sim \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) = \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\sim \mathfrak{J}_1)$ and $\sim \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) = \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\sim \mathfrak{J}_1)$;

iv. $\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \cap \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)$;

v. $\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) \supseteq \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \cup \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)$;

vi. $\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) = \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \cup \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)$;

vii. $\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) \subseteq \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \cap \overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)$.

Proof: Proof of i. to iii. are easy and follows the definition of CBq-ROFRS.

iv: As we know that

$$\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \left\{< \mathscr{k}_i, \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cap \mathfrak{J}_2)}(\mathscr{k}_i), \quad \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}\mathfrak{J}_1 \cap \mathfrak{J}_2}(\mathscr{k}_i) >_q | i = 1, \ldots, n\right\}$$

As

$$\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) = \left\{< \mathscr{k}_i, \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1)}(\mathscr{k}_i), \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}\mathfrak{J}_1}(\mathscr{k}_i) >_q | i = 1,2, \ldots n\right\}$$

In order to show $\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cap \mathfrak{J}_2) = \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \cap \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1)$, we have to prove

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cap \mathfrak{J}_2)}(\mathscr{k}_i) = \left\{\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1)}(\mathscr{k}_i) \cap \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)}(\mathscr{k}_i)\right\}$$

and

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cap \mathfrak{J}_2)}(\mathscr{k}_i) = \left\{\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1)}(\mathscr{k}_i) \vee \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)}(\mathscr{k}_i)\right\}$$

Now consider

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1\cap\mathfrak{I}_2)}(\mathscr{k}_i) = \bigwedge_{j=1}^{m}\left\{\mu_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\wedge\mu_{\mathfrak{I}_1\cap\mathfrak{I}_2}(\mathscr{k}_j)\right\}$$

$$= \bigwedge_{j=1}^{m}\left\{\mu_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\wedge\{\mu_{\mathfrak{I}_1}(\mathscr{k}_j)\cap\mu_{\mathfrak{I}_2}(\mathscr{k}_j)\}\right\}$$

$$= \bigwedge_{j=1}^{m}\left\{\mu_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\wedge\mu_{\mathfrak{I}_1}(\mathscr{k}_j)\right\}\wedge\bigwedge_{j=1}^{m}\left\{\mu_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\wedge\mu_{\mathfrak{I}_2}(\mathscr{k}_j)\right\}$$

$$= \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1)}(\mathscr{k}_i)\wedge\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_2)}(\mathscr{k}_i)$$

Next

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1\cap\mathfrak{I}_2)}(\mathscr{k}_i) = \bigvee_{j=1}^{m}\left\{\psi_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\vee\psi_{\mathfrak{I}_1\cap\mathfrak{I}_2}(\mathscr{k}_j)\right\}$$

$$= \bigvee_{j=1}^{m}\left\{\psi_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\vee\{\psi_{\mathfrak{I}_1}(\mathscr{k}_j)\cap\psi_{\mathfrak{I}_2}(\mathscr{k}_j)\}\right\}$$

$$= \bigvee_{j=1}^{m}\left\{\psi_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\vee\psi_{\mathfrak{I}_2}(\mathscr{k}_j)\right\}\vee\bigvee_{j=1}^{m}\left\{\psi_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i,\mathscr{k}_j)\vee\psi_{\mathfrak{I}_2}(\mathscr{k}_j)\right\}$$

$$= \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1)}(\mathscr{k}_i)\vee\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_2)}(\mathscr{k}_i)$$

Therefore

$$\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1\cap\mathfrak{I}_2) = \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1)\cap\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1).$$

v: Next to prove

$$\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1\cup\mathfrak{I}_2) \supseteq \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1)\cup\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1)$$

we have to show $\mathscr{k}_i \in T$

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1\cup\mathfrak{I}_2)}(\mathscr{k}_i) \geq \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1)}(\mathscr{k}_i)\vee\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_2)}(\mathscr{k}_i)$$

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1\cup\mathfrak{I}_2)}(\mathscr{k}_i) \leq \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_1)}(\mathscr{k}_i)\wedge\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{I}_2)}(\mathscr{k}_i)$$

Now consider

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\hbar_i) = \bigwedge_{j=1}^{m} \left\{ \mu_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \wedge \mu_{(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\hbar_j) \right\}$$

$$= \bigwedge_{j=1}^{m} \left\{ \mu_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \wedge \{\mu_{\mathfrak{J}_1}(\hbar_j) \vee \mu_{\mathfrak{J}_2}(\hbar_j)\} \right\}$$

$$\geq \bigwedge_{j=1}^{m} \left\{ \mu_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \wedge \mu_{\mathfrak{J}_1}(\hbar_j) \right\} \vee \bigwedge_{j=1}^{m} \left\{ \mu_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \wedge \mu_{\mathfrak{J}_2}(\hbar_j) \right\}$$

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\hbar_i) \geq \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1)}(\hbar_i) \vee \mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)}(\hbar_i)$$

Furthermore,

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\hbar_i) = \bigvee_{j=1}^{m} \left\{ \psi_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \vee \psi_{(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\hbar_i) \right\}$$

$$= \bigvee_{j=1}^{m} \left\{ \psi_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \vee \{\psi_{(\mathfrak{J}_1)}(\hbar_i) \wedge \psi_{(\mathfrak{J}_2)}(\hbar_i)\} \right\}$$

$$\leq \bigvee_{j=1}^{m} \left\{ \psi_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \vee \psi_{(\mathfrak{J}_1)}(\hbar_i) \right\} \wedge \bigvee_{j=1}^{m} \left\{ \psi_{\mathcal{N}_{\mathcal{P}(\hbar_i)}^{\beta}}(\hbar_i, \hbar_j) \vee \psi_{(\mathfrak{J}_2)}(\hbar_i)\} \right\}$$

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cup \mathfrak{J}_2)}(\hbar_i) \leq \left\{ \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1)}(\hbar_i) \wedge \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)}(\hbar_i) \right\}$$

Therefore,

$$\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1 \cup \mathfrak{J}_2) \supseteq \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_1) \cup \underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathfrak{J}_2)$$

Proofs of vi: and vii: are directly follows from the above proofs of iv: and v.

3.2. Multi-attribute decision making model by utilizing q-ROFRS hybrid with TOPSIS

$\mathcal{MADM}$ has a high potential and disciplined process to improve and evaluate multiple conflicting criteria in all areas of $\mathcal{DM}$. In this competitive environment an enterprise needs the more accurate and more repaid response to change the customer needs. So, $\mathcal{MADM}$ has the ability to handle successfully the evaluation process of multiple contradictory attribute. For an intelligent decision the experts analyze each and every characteristic of an alternative and then they take the decision. Further, we will present the model for $\mathcal{MADM}$ and their basic steps of construction by utilizing the proposed aggregation operators under q-ROFR information.

Let $T = \{\mathcal{k}_1, \mathcal{k}_2, \dots, \mathcal{k}_n\}$ be any set of n feasible alternatives, $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_m\}$ be the set of m attributes and consider the weight vector $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ for all attributes such that $0 \leq \overline{\overline{w}}_i \leq 1$ and $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$. Decision makers $\mathcal{D}_{mem}$ and $\mathcal{D}_{non-mem}$ express their preference evaluation for alternatives $\mathcal{k}_i (i = 1, \dots n)$ corresponding to the set of attribute $\mathcal{P}_j (j = 1, \dots, m)$ by μ_{ij} *and* ψ_{ij} respectively. So combining these two values as a q-ROFV we have q-ROF decision matrix $\mathcal{P}_j(\mathcal{k}_i) = (\mu_{ij}, \psi_{ij})$. This means that the decision maker $\mathcal{D}_{mem}$ provides $\mathcal{MG}$ μ_{ij} to an object $\mathcal{k}_i$ against to the attribute $\mathcal{P}_j$. Whereas the expert $\mathcal{D}_{non-mem}$ provides $\mathcal{NMG}$ ψ_{ij} to an object $\mathcal{k}_i$ against to the attribute $\mathcal{P}_j$ and their decision matrix is given as:

$$\mathcal{P}_j(\mathcal{k}_j) = \begin{pmatrix} (\mu_{11}, \psi_{11}) & (\mu_{12}, \psi_{12}) & \cdots & (\mu_{1j}, \psi_{1j}) \\ (\mu_{21}, \psi_{21}) & (\mu_{22}, \psi_{22}) & \cdots & (\mu_{2j}, \psi_{2j}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{i1}, \psi_{i1}) & (\mu_{i2}, \psi_{i2}) & \cdots & (\mu_{ij}, \psi_{ij}) \end{pmatrix}$$

In order to tackle a $\mathcal{MADM}$ problem with the help of the proposed model, first we will discuss the technique for proposed model and steps wise algorithm for $\mathcal{MADM}$ problem for the proposed CBq-ROFRS model, which mainly consists of three steps. In the first step decision makers $\mathcal{D}_{mem}$ and $\mathcal{D}_{non-mem}$ provide their input to find a q-ROFSs as explained above. By using the q-ROF TOPSIS (q-ROF-TOPSIS) approach, we will present q-ROF positive ideal solution (q-ROF-PIS) $P^+ = \{\mathcal{P}_j, max\{\mathcal{Sc}(\mathcal{P}_j(\mathcal{k}_i))\}/j = 1, \dots, m\}$ and q-ROF negative ideal solution (q-ROF-NIS) $P^- = \{\mathcal{P}_j, min\{\mathcal{Sc}(\mathcal{P}_j(\mathcal{k}_i))\}/j = 1, \dots, m\}$, through the score function by Definition 3.1.5. With the help of Definition 3.1.10, the distance $\mathcal{D}^+$ and $\mathcal{D}^-$ are determined among alternatives $\mathcal{k}_i$ and q-ROF-PIS P^+ and q-ROF-NIS P^-. Therefore, the new q-ROFS $\mathcal{D} = \{< \mathcal{k}, \mu_{\mathcal{D}}(\mathcal{k}), \psi_{\mathcal{D}}(\mathcal{k}) >_q | \mathcal{k} \in T\} = (\mu_{\mathcal{D}}, \psi_{\mathcal{D}})_q = (\mathcal{D}^+, \mathcal{D}^-)_q$ can be constructed. Therefore, a multi-attribute q-rung orthopair fuzzy decision making information system (MAq-ROFDMIS) $(T, \mathcal{P}, P, \mathcal{D})$ has been obtained. Then to find the optimal object or ranking among all the objects, they are arranged according to the preference evaluation. In second phase the lower and upper approximations of the q-ROFSs are calculated with the precision parameter $\beta (0 < \beta \leq 1)$ (where β the precision parameter, which is used on the CBq-ROFRS model to explain the consistency consensus threshold by the decision maker). Finally to find the optimal object via ranking or $\mathcal{DM}$ process among all the alternatives and then arranged according the preference evaluation.

Here the detail of first step is presented and first suggesting the q-ROF-TOPSIS method. In this method the optimal alternative should have the shortest distance (that is the alternative should have higher score value) from q-ROF-PIS P^+ and the farthest distance (that is the alternative should have least score value) from the q-ROF-NIS P^-. By use of Definition 3.1.5, to identify q-ROF-PIS P^+ and q-ROF-NIS P^- obtains the following structure.

$$P^+ = \{\mathcal{P}_j, max\{s(\mathcal{P}_j(\mathcal{k}_i))\}/j = 1,\ldots,m\}$$

$$= \{< \mathcal{P}_1, \mu_1^+, \psi_1^+ >, < \mathcal{P}_2, \mu_2^+, \psi_2^+ >, \ldots < \mathcal{P}_2, \mu_m^+, \psi_m^+ >\}$$

and

$$P^- = \{\mathcal{P}_j, min\{s(\mathcal{P}_j(\mathcal{k}_i))\}/j = 1,\ldots,m\}$$

$$= \{< \mathcal{P}_1, \mu_1^-, \psi_1^- >, < \mathcal{P}_2, \mu_2^-, \psi_2^- >, \ldots < \mathcal{P}_2, \mu_m^-, \psi_m^- >\}$$

Further with the help of Definition 3.1.10, to calculate the weighted distances $\mathcal{D}^+$ and $\mathcal{D}^-$ for an object $\mathcal{k}_i$ and q-ROF-PIS P^+ and q-ROF-NIS P^- is defined as the following:

$$\mathcal{D}^+ = \sum_{j=1}^{m} \bar{\bar{w}}_j \; d\left(\mathcal{P}_j(\mathcal{k}_i), \mathcal{P}_j(P^+)\right)$$

$$= \left\{\frac{1}{2n}\sum_{j=1}^{m} \bar{\bar{w}}_j \left|\mu_{ij}(\mathcal{k}) - \mu_j(\mathcal{k})^+\right|^p + \frac{1}{2n}\sum_{j=1}^{m} \bar{\bar{w}}_j \left|\psi_{ij}(\mathcal{k}) - \psi_j(\mathcal{k})^+\right|^p\right\}^{\frac{1}{p}} \; for \; (i = 1,\ldots,n)$$

and

$$\mathcal{D}^- = \sum_{j=1}^{m} \bar{\bar{w}}_j \, d\left(\mathcal{P}_j(\mathcal{k}_i), \mathcal{P}_j(P^-)\right)$$

$$= \left\{\frac{1}{2n}\sum_{j=1}^{m} \bar{\bar{w}}_j \left|\mu_{ij}(\mathcal{k}) - \mu_j(\mathcal{k})^-\right|^p + \frac{1}{2n}\sum_{j=1}^{m} \bar{\bar{w}}_j \left|\psi_{ij}(\mathcal{k}) - \psi_j(\mathcal{k})^-\right|^p\right\}^{\frac{1}{p}} \; for \; (i = 1,\ldots,n)$$

Therefore we put together the new q-ROFS $\mathcal{D} = (\mu_{\mathcal{D}}, \psi_{\mathcal{D}}) = (\mathcal{D}^+, \mathcal{D}^-)$.

3.2.1. Definition

A q-ROF triangular norm (in short q-ROF t-norm) is a function $T: [0,1] \times [0,1] \to [0,1]$, having the following characteristic:

(i) Commutative

(ii) Associative

(iii) Increasing

(iv) $T(\mathcal{k}, 1) = \mathcal{k}\ \forall \mathcal{k} \in [0,1]$.

Similarly a q-ROF triangular t-conorm (in short q-ROF t-conorm) is a function $T: [0,1] \times [0,1] \to [0,1]$, having the following characteristic:

(v) Commutative

(vi) Associative

(vii) Increasing

(viii) $T(\mathcal{k}, 0) = \mathcal{k}\ \forall \mathcal{k} \in [0,1]$.

Here the q-ROF t-norm and q-ROF t-conorm are used for $\mathcal{MADM}$ problem.

$$T_{\mathfrak{J}}(\mathcal{k}^1, \mathcal{k}^2) = \frac{\mathcal{k}^1 \mathcal{k}^2}{\sqrt[q]{1 + (1 - \mathcal{k}_1^q)(1 - \mathcal{k}_2^q)}} \quad and \quad T_{\mathfrak{J}}(\mathcal{k}^1, \mathcal{k}^2) = \sqrt[q]{\frac{\mathcal{k}_1^q + \mathcal{k}_2^q}{1 + \mathcal{k}_1^q \mathcal{k}_2^q}}$$

Further by use a definition of CBq-ROFRS, the lower and upper approximations of best and worst q-ROFDM alternatives are found based on consistency consensus threshold $\beta (0 < \beta \leq 1)$ as given below.

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathcal{k}_i) = \bigwedge_{j=1}^{m} \left\{ \mu_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \wedge \mu_{\mathcal{D}}(\mathcal{k}_j) \right\}$$

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathcal{k}_i) = \bigvee_{j=1}^{m} \left\{ \psi_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \vee \psi_{\mathcal{D}}(\mathcal{k}_j) \right\}$$

and

$$\mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathcal{k}_i) = \bigvee_{j=1}^{m} \left\{ \mu_{\mathcal{N}_{\mathcal{P}(\mathcal{k}_i)}^{\beta}}(\mathcal{k}_i, \mathcal{k}_j) \vee \mu_{\mathcal{D}}(\mathcal{k}_j) \right\}$$

$$\psi_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i) = \bigwedge_{j=1}^{m} \left\{ \psi_{\mathcal{N}_{\mathcal{P}(\mathscr{k}_i)}^{\beta}}(\mathscr{k}_i, \mathscr{k}_j) \wedge \psi_{\mathcal{D}}(\mathscr{k}_j) \right\}$$

Finally based on Definition 3.2.1, to find the rank of all the alternatives and then arrange them according to the preference evaluation based on consistency consensus threshold $\beta(0 < \beta \leq 1)$.

3.2.2. Definition

Suppose that MAq-ROFDMIS is $(T, \mathcal{P}, P, \mathcal{D})$. For the q-ROFDM object $\mathcal{D} = (\mathcal{D}^+, \mathcal{D}^-) \in \mathrm{q-ROFS}^T$ represented by the preference information of decision maker $\mathcal{D}$ and risk preference threshold $\alpha(0 < \alpha \leq 1)$.

Now define the ranking function of alternative $\mathscr{k}_i$ $(i = 1, \ldots, n)$ as:

$$T_{\mathfrak{J}}(\mathscr{k}_i) = \alpha T_{\mathfrak{J}}\left(\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i), \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i)\right) + (1-\alpha) T_{\mathfrak{J}}\left(\mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i), \psi_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i)\right)$$

The ranking function shows that $0 \leq T_{\mathfrak{J}}(\mathscr{k}_I) \leq 1$.

3.2.3. Algorithm

By utilizing the above interpretation, the step wise decision algorithm for the developed model based on CBq-ROFRS is summarized as follows:

input MAq-ROFDMIS (U, $\mathcal{P}$,P, $\mathcal{D}$);

output The sort ordering for all alternatives;

Step (i): Determine the q-ROF-NIS P^+ and q-ROF-NIS P^-,

Step (ii): Determine the $\mathcal{D}^+ = \mu_{\mathcal{D}}$ and $\mathcal{D}^- = \psi_{\mathcal{D}}$ between the alternatives and the q-ROF-PIS P^+ and q-ROF-NIS P^-,

Step (iii): Next find the lower and upper approximations

$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i), \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i), \mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i)$ and $\eta_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\mathscr{k}_i)$,

Step (iv): Determine the ranking function $T_{\mathfrak{J}}(\mathcal{k}_i)$,

Step (v): Finally ranking of all object in a specific ordered to get best optimum option of professional experts.

The flow for TOPSIS method under q-ROFRS is given in Fig. 3.1.

3.3. Numerical example

In this section we will initiate an illustrative example to prove the quality and Excellency of the developed model based on CBq-ROFS that relates the evaluation and rank of appointment of new faculty position in Universities. Then q-ROF-TOPSIS provides the desired ranking.

For a certain senior position in Universities, the appointment of a new faculty has to face a very complex evaluation and decision making process. The skill and ability of a candidate may be judged with respect to various attributes like as "managerial skills" "ability to work under pressure" "research productivity" etc. In order to take the right decision about the candidate the opinions of professional experts are needed for these criteria.

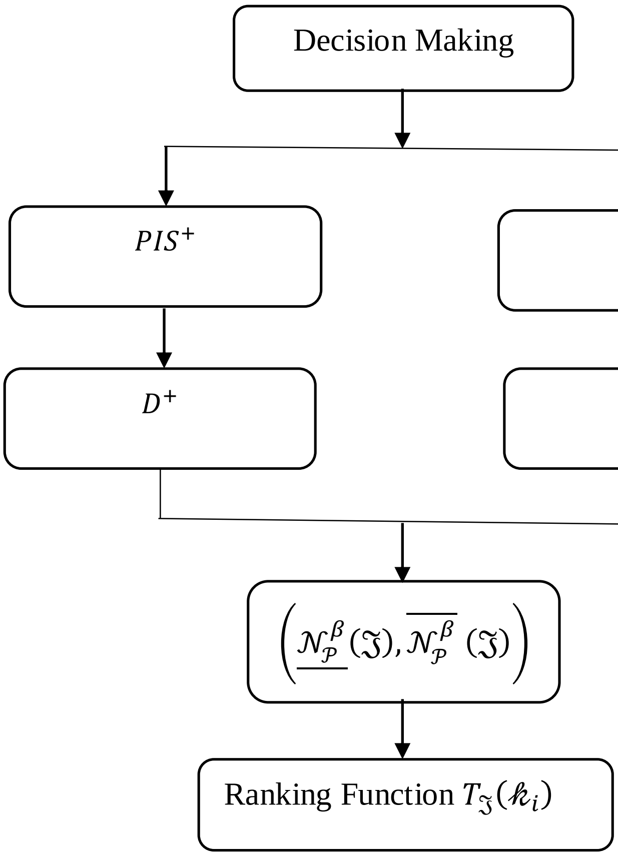

Fig. 3.1. Flow chart for TOPSIS method under q-ROFRS

Consider that $T = \{\mathcal{k}_1, \mathcal{k}_2, \mathcal{k}_3, \mathcal{k}_4, \mathcal{k}_5\}$ be the set of five candidates who fulfil the requirements for the senior faculty position in Y University. In order to appoint the most qualified and suitable person for the required position, a team of experts is organized and chaired by Prof. Z as a director. The team of experts will judge the candidates upon the criteria in the set of attribute $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6\}$, where

$\mathcal{P}_1 =$ Research productivity, $\mathcal{P}_2 =$ Managerial skill,

$\mathcal{P}_3 =$ Impact on research community $\mathcal{P}_4 =$ Ability to work under pressure,

$\mathcal{P}_5 =$ Academic leadership qualities, $\mathcal{P}_6 =$ Contribution to Y University

According to the background and expertise, the team of experts wants to appoint the candidate who qualifies with the criteria of $\mathcal{P}$ to the utmost extent from candidate in T. Suppose that the evaluation values of each alternative with respect to each attribute provided by the decision makers $\mathcal{D}_{mem}$ and $\mathcal{D}_{non-mem}$ are presented in the decision matrix given in Table 3.3, and the weights of all the attributes set is given as below:

$\bar{\bar{w}}_1 = 0.2,\ \bar{\bar{w}}_2 = 0.18,\ \bar{\bar{w}}_3 = 0.22,\ \bar{\bar{w}}_4 = 0.12,\ \bar{\bar{w}}_5 = 0.15,\ \bar{\bar{w}}_6 = 0.13$

Table 3.3, Tabular representation of q-ROFSs for $\mathcal{P}$

$T/\mathcal{P}$	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	$\mathcal{P}_6$
$\mathcal{k}_1$	(0.98,0.3)	(0.7,0.4)	(0.8,0.2)	(0.9,0.1)	(0.7,0.6)	(0.4,0.3)
$\mathcal{k}_2$	(0.9,0.4)	(0.7,0.8)	(0.7,0.5)	(0.6,0.3)	(0.65,0.87)	(0.6,0.2)
$\mathcal{k}_3$	(0.8,0.7)	(0.7,0.2)	(0.95,0.4)	(0.8,0.4)	(0.5,0.2)	(0.8,0.3)
$\mathcal{k}_4$	(0.8,0.3)	(0.6,0.5)	(0.7,0.4)	(0.9,0.2)	(0.8,0.65)	(0.4,0.2)
$\mathcal{k}_5$	(0.5,0.2)	(0.95,0.4)	(0.8,0.3)	(0.7,0.1)	(0.6,0.3)	(0.94,0.38)

For example, the characteristics of a candidate $\mathcal{k}_1$ under attribute $\mathcal{P}_1$ is (0.98,0.3), the value 0.98 is the $\mathcal{MG}$ and the value 0.3 is the $\mathcal{NMG}$ of candidate $\mathcal{k}_1$ under criterion $\mathcal{P}_1$ respectively. In other words, candidate $\mathcal{k}_1$ is qualified and suitable on $\mathcal{MG}$ 0.98 and disqualified on $\mathcal{NMG}$ of 0.3.

Further the step wise algorithm for the proposed $\mathcal{MADM}$ approach based on CBq-ROFRS consists of the following steps:

Step (i): Now to compute the q-ROF-PIS P^+ and q-ROF-NIS P^- by use of Definition 3.1.5, when $q = 3$ as follows.

$$P^+ = \left\{\begin{matrix}(\mathcal{P}_1, 0.98,0.3), (\mathcal{P}_2, 0.95,0.4), (\mathcal{P}_3, 0.95,0.4), (\mathcal{P}_4, 0.9,0.1), (\mathcal{P}_5, 0.8,0.65), \\ (\mathcal{P}_6, 0.94,0.38)\end{matrix}\right\}$$

$$P^- = \left\{\begin{matrix}(\mathcal{P}_1, 0.5,0.2), (\mathcal{P}_2, 0.7,0.8), (\mathcal{P}_3, 0.7,0.5), (\mathcal{P}_4, 0.6,0.3), (\mathcal{P}_5, 0.65,0.87), \\ (\mathcal{P}_6, 0.4,0.3)\end{matrix}\right\}$$

Step (ii): Furthermore, to compute the distance $\mathcal{D}^+ = \mu_{\mathcal{D}}$ and $\mathcal{D}^+ = \psi_{\mathcal{D}}$, between the alternatives and the q-ROF-PIS P^+ and q-ROF-NIS P^- when $p = 3$;

$$\mathcal{D}^+ = \mu_{\mathcal{D}} = \frac{0.12943}{\hbar_1}, \frac{0.13642}{\hbar_2}, \frac{0.14903}{\hbar_3}, \frac{0.14129}{\hbar_4}, \frac{0.13849}{\hbar_5}$$

$$\mathcal{D}^- = \psi_{\mathcal{D}} = \frac{0.15782}{\hbar_1}, \frac{0.10907}{\hbar_2}, \frac{0.2201}{\hbar_3}, \frac{0.11109}{\hbar_4}, \frac{0.17627}{\hbar_5}$$

Step (iii): Next, to determine the lower and upper approximation that is $\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i), \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i), \mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i)$ and $\eta_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i)$,

First to compute q-ROFβ-neighborhood for each $\hbar_i \in T$ $(i = 1,2,\dots,5)$, and let the consistency threshold q-ROF $\beta = (0.8,0.4)$. Then

$$\mathcal{N}_{\mathcal{P}(\hbar_1)}^{(0.8,0.4)} = \mathcal{P}_1 \cap \mathcal{P}_3 \cap \mathcal{P}_4, \quad \mathcal{N}_{\mathcal{P}(\hbar_2)}^{(0.8,0.4)} = \mathcal{P}_1, \quad \mathcal{N}_{\mathcal{P}(\hbar_3)}^{(0.8,0.4)} = \mathcal{P}_3 \cap \mathcal{P}_4 \cap \mathcal{P}_6,$$

$$\mathcal{N}_{\mathcal{P}(\hbar_4)}^{(0.8,0.4)} = \mathcal{P}_1 \cap \mathcal{P}_4, \quad \mathcal{N}_{\mathcal{P}(\hbar_5)}^{(0.8,0.4)} = \mathcal{P}_2 \cap \mathcal{P}_3 \cap \mathcal{P}_6$$

Further Table 3.4 for $\mathcal{N}_{\mathcal{P}}^{(0.8,0.4)}$

Now to compute

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i), \quad \psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i), \quad \mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i) \ and \ \eta_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})}(\hbar_i)$$

$$\mu_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})} = \frac{0.12943}{\hbar_1}, \frac{0.12943}{\hbar_2}, \frac{0.12943}{\hbar_3}, \frac{0.12943}{\hbar_4}, \frac{0.12943}{\hbar_5}$$

$$\psi_{\underline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})} = \frac{0.7}{\hbar_1}, \frac{0.8}{\hbar_2}, \frac{0.5}{\hbar_3}, \frac{0.7}{\hbar_4}, \frac{0.8}{\hbar_5}$$

Table 3.4, Tabular representation of $\mathcal{N}_{\mathcal{P}}^{(0.8,0.4)}$

$\mathcal{N}_{\mathcal{P}}^{(0.8,0.4)}$	k_1	k_2	k_3	k_4	k_5
k_1	(0.8,0.3)	(0.6,0.5)	(0.8,0.7)	(0.7,0.4)	(0.5,0.3)
k_2	(0.7,0.4)	(0.7,0.8)	(0.7,0.2)	(0.6,0.5)	(0.59,0.4)
k_3	(0.4,0.3)	(0.6,0.5)	(0.8,0.4)	(0.4,0.4)	(0.7,0.38)
k_4	(0.9,0.3)	(0.6,0.4)	(0.8,0.7)	(0.8,0.3)	(0.5,0.2)
k_5	(0.4,0.4)	(0.6,0.8)	(0.7,0.4)	(0.4,0.5)	(0.8,0.4)

$$\mu_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})} = \frac{0.7}{k_1}, \frac{0.8}{k_2}, \frac{0.8}{k_3}, \frac{0.9}{k_4}, \frac{0.8}{k_5}$$

$$\eta_{\overline{\mathcal{N}_{\mathcal{P}}^{\beta}}(\mathcal{D})} = \frac{0.10907}{k_1}, \frac{0.10907}{k_2}, \frac{0.10907}{k_3}, \frac{0.10907}{k_4}, \frac{0.10907}{k_5}$$

Step (iv): Now to compute the ranking function $T_{\mathfrak{I}}(k_i)$, and for this let the risk preference threshold $\alpha = 0.75$, where $(0 < \alpha \leq 1)$ as follows:

$$T_{\mathfrak{I}} = \frac{0.076550}{k_1}, \frac{0.084171}{k_2}, \frac{0.058484}{k_3}, \frac{0.080097}{k_4}, \frac{0.087149}{k_5}$$

Step (v): Finally, rank the obtained results and arranged them in a specific ordered to get the most desirable option from k_i.

$$k_5 > k_2 > k_4 > k_1 > k_3$$

Hence through the process of decision making finally we get most desirable applicant for the required position by the utilizing CBq-ROFRS model based on $\mathcal{MADM}$ method. Hence, from the illustrative example it is observed that the $5th$ candidate is the most desirable and perfect applicant for the required position.

3.3.1. Comparative analysis

Yager [16], developed the concept to PyFSs and presented an important model based on PyFWA operator to solve $\mathcal{MCDM}$ problems. On the same concept Zhang and Xu [76], presented TOPSIS to solve $\mathcal{MCDM}$ with PyF information. These methods fail to handle situations when the $\mathcal{MG}$ is 0.9 and $\mathcal{NMG}$ is 0.8. In this case $(0.9)^2 + (0.8)^2 > 1$ and the methods proposed in [16] and [76] fail to tackle the situation. The proposed

Table 3.5, Comparative analysis of different methods

Methods	*Score values*					**Ranking**
	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$	$\mathscr{k}_4$	$\mathscr{k}_5$	
CFRSs [80]			*Fail to handle*			×
CFRSs [89]			*Fail to handle*			×
CIFRSs [90]			*Fail to handle*			×
PyFSs [16]			*Fail to handle*			×
PyFSs [76]			*Fail to handle*			×
CBPyFRS(proposed)			*Fail to handle*			×
CBq-ROFRS (proposed)	0.076550	0.084171	0.058484	0.080097	0.087	$\mathscr{k}_5 > \mathscr{k}_2 > \mathscr{k}_4 > \mathscr{k}_1 > \mathscr{k}_3$

method handle such situations very easily, for example $(0.9)^q + (0.8)^q < 1\ for\ q \geq 5$. So from the analysis it is clear that the presented model is more suitable to meet a variety of situations by adjusting the values of q. Therefore the proposed method is the more superior than the methods proposed in [16] and [76] because the input range of developed model is more flexible, wider and suitable because when the rung increases, the orthopair provides additional space to the boundary constraint. Therefore the proposed method is more suitable because it provides more space to the decision maker in decision making problems.

3.3.2. Conclusion

$\mathcal{MADM}$ has the high potential and discipline process to improve and evaluate multiple conflicting criteria in all areas of the decision making. A comprehensive model is originated to handle the $\mathcal{DM}$ problems in which some energetic perspective are in support and against of some plans, entities or projects. The investigated concept is interesting in that case, where the professionals have contradictions in their decision about some proposal or plan. Therefore, a new technique is developed to investigate the hybrid notions of RS with q-ROFS by using the concept of fuzzy β-covering and fuzzy β- neighborhoods to get the new notion of CBq-ROFRS. Furthermore, by applying the developed concept of CBq-ROFRS on TOPSIS and presenting its application for $\mathcal{MADM}$. In real scenario CBq-ROFRS model is an important tools to discuss the complex and uncertain information. This method has stronger capacity than IFS and PyFS to cope the uncertainty. From the analysis, it is clear that CBq-ROFRS degenerates into CBIFRS if the rung $q = 1$ and degenerate into CBPyFRS if the rung $q = 2$. Thus the proposed concept is generalization of CBIFRS and CBPyFyRS. Moreover, an illustrative example is presented to describe how the developed model helps us in $\mathcal{DM}$ problems and a comparative study of the proposed model with some existing methods is presented which shows that the developed approach is more capable and superior than the existing methods. The comparative analysis of the developed model with existing methods is given in Table 3.5 by considering the above Illustrative Example. From Table 3.5 it is clear that the methods proposed in [80] and [89] are failed to handle situation because only handle the fuzzy $\mathcal{MG}$ and having no information about $\mathcal{NMG}$. Similarly the CIFRSs method proposed in [90] also failed to handle it due to the limitation on $\mathcal{MG}$ and $\mathcal{NMG}$ that their sum is less than or equal to 1. Analogously the methods proposed in [16, 76] and CPyFRSs are also failed to handle the situation due to the limitation on $\mathcal{MG}$ and $\mathcal{NMG}$ that their square sum is less than or equal to 1. The main advantages of the proposed method has the ability to cope these situations and provides a huge space and freedom to the decision makers to assign values freely by adjusting the value of q and hence the method proposed in this paper is superior than existing methods.

Chapter 4

Orthopair fuzzy soft average aggregation operators

In 1999, Molodtsov investigated the pioneer notion of S_{ft}S which provides a general framework for mathematical problems by affix parameterization tools during the analysis as compared to fuzzy set and q-ROFS. From the analysis of existing literature and best of our knowledge, there has been no research on the hybrid model of S_{ft}S and q-ROFS that is q-rung orthopair fuzzy soft set (q-ROFS_{ft}S). Therefore, for the scope of future motive, the proposed concept has enough space for the new research. The aim of this chapter is to investigate the notion of q-ROFS_{ft}S, which plays a bridge role between these two notions. Therefore, our main contribution in this chapter is to investigate the q-ROFS_{ft} weighted averaging (q-ROFS_{ft}WA), q-ROFS_{ft} ordered weighted averaging (q-ROFS_{ft}OWA) and q-ROFS_{ft} hybrid averaging (q-ROFS_{ft}HA) operators in q-ROFS_{ft} environment. Further, the fundamental properties of these aggregation operators are studied. On the base of developed approach an algorithm for $\mathcal{MCDM}$ is being presented. An application of medical diagnosis problems is solved on the proposed algorithm under the q-ROFS_{ft} environment. Finally, comparison between the developed operators with some existing operators are being presented showing the superiority and efficiency of the developed approach than the existing literature.

4.1. Pythagorean fuzzy soft set

Yager [16] investigated the dominant concept of PyFS, in which the square sum of $\mathcal{MG}$ and $\mathcal{NMG}$ belongs to $[0,1]$. The input range of PyFS is more flexible and provides additional space to the experts for selecting their decision choice. Here we will present the hybrid model of S_{ft}S and PyFS that is Pythagorean fuzzy soft set (PyFS_{ft}S) which is defined as:

4.1.1. Definition

Consider a soft set $(\mathcal{H}, \mathbb{E})$ over a universe of discourse T. A pair $(\mathcal{T}, \mathbb{E})$ is known to be a PyFS_{ft}S over T, where $\mathcal{T}$ is a function given by $\mathcal{T}: \mathbb{E} \rightarrow PyFS^{(T)}$, which is given as

$$\mathcal{T}_{s_j}(\hbar_i) = \{\prec \hbar_i, \mu_j(\hbar_i), \psi_j(\hbar_i) \succ | \hbar_i \in T \text{ and } s_j \in \mathbb{E}\},$$

where $\mu_j(\hbar_i), \psi_j(\hbar_i)$ denotes the $\mathcal{MG}$ and $\mathcal{NMG}$ of an object $\hbar_i \in T$ to the set $\mathcal{T}_{s_j}$ respectively, and satisfying the condition that $0 \leq \left(\mu_j(\hbar_i)\right)^2 + \left(\psi_j(\hbar_i)\right)^2 \leq 1$. For simplicity $\mathcal{T}_{s_j}(\hbar_i) = \prec \hbar_i, \mu_j(\hbar_i), \psi_j(\hbar_i) \succ$ is denoted by $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ is known as PyF soft value (PyFS_{ft}V).

4.1.2. Definition

Let $\mathfrak{I}_{s_{i1}} = (\mu_{i1}, \psi_{i1}), \mathfrak{I}_{s_{i2}} = (\mu_{i2}, \psi_{i2})$ $(i = 1, 2, \dots, m)$ be any two PyFS_{ft}Vs. Then the basic operations on them are given as follows:

(i) $\mathfrak{I}_{s_{i1}} \cup \mathfrak{I}_{s_{i2}} = \left(\mathscr{k}_i, max\left(\mu_1(\mathscr{k}_i), \mu_2(\mathscr{k}_i)\right), min\left(\psi_1(\mathscr{k}_i), \psi_2(\mathscr{k}_i)\right)\right)$ for $\mathscr{k}_i \in T$;

(ii) $\mathfrak{I}_{s_{11}} \cap \mathfrak{I}_{s_{12}} = \left(\mathscr{k}_i, min\left(\mu_1(\mathscr{k}_i), \mu_2(\mathscr{k}_i)\right), max\left(\psi_1(\mathscr{k}_i), \psi_2(\mathscr{k}_i)\right)\right)$ for $\mathscr{k}_i \in T$;

(iii) ${\mathfrak{I}_{s_{i1}}}^c = \left(\mathscr{k}_i, \psi_1(\mathscr{k}_i), \mu_1(\mathscr{k}_i)\right)$, where ${\mathfrak{I}_{s_{i1}}}^c$ denotes the complement of $\mathfrak{I}_{s_{i1}}$;

(iv) $\mathfrak{I}_{s_{i1}} \subseteq \mathfrak{I}_{s_{i2}}\ if\ \mu_{i1}(\mathscr{k}_i) \le \mu_{i2}(\mathscr{k}_i)\ and\ \psi_{i1}(\mathscr{k}_i) \ge \psi_{i2}(\mathscr{k}_i)$ for all $\mathscr{k}_i \in T$;

4.1.3. Definition

Let $\mathfrak{I}_{s_{11}} = \left(\mu_1(\mathscr{k}_1), \psi_1(\mathscr{k}_1)\right), \mathfrak{I}_{s_{12}} = \left(\mu_2(\mathscr{k}_1), \psi_2(\mathscr{k}_1)\right)$ be any two PyFS_{ft}Vs and $\lambda > 0$. Then some basic operations are given below:

(i) $\mathfrak{I}_{s_{11}} \oplus \mathfrak{I}_{s_{12}} = \left(\sqrt{\mu_1^2(\mathscr{k}_1) + \mu_2^2(\mathscr{k}_1) - \mu_1^2(\mathscr{k}_1)\mu_2^2(\mathscr{k}_1)}\ , \psi_1(\mathscr{k}_1)\psi_2(\mathscr{k}_1)\right)$;

(ii) $\mathfrak{I}_{s_{11}} \otimes \mathfrak{I}_{s_{12}} = \left(\mu_1(\mathscr{k}_1)\mu_2(\mathscr{k}_1)\ , \sqrt{\psi_1^2(\mathscr{k}_1) + \psi_2^2(\mathscr{k}_1) - \psi_1^2(\mathscr{k}_1)\psi_2^2(\mathscr{k}_1)}\right)$;

(iii) $\lambda\mathfrak{I}_{s_{11}} = \left(\sqrt{1 - [1 - \mu_1^2(\mathscr{k}_1)]^\lambda}, \psi_1^\lambda(\mathscr{k}_1)\right)$;

(iv) $\mathfrak{I}_{s_{11}}^\lambda = \left(\mu_1^\lambda(\mathscr{k}_1), \sqrt{1 - [1 - \psi_1^2(\mathscr{k}_1)]^\lambda}\right)$.

4.2. q-Rung orthopair fuzzy soft set

Recently in 2017 Yager [27], investigated the prominent concepts of q-ROFS in which the sum of q^{th} power of $\mathcal{MG}$ and q^{th} power of $\mathcal{NMG}$ belongs to $[0,1]$. In this section, we will investigate the hybrid model of S_{ft}S and q-ROFS that is q-ROFS_{ft}S and their desirable properties are discussed in detail.

4.2.1. Definition

Consider a S_{ft}S $(\mathcal{H}, \mathbb{E})$ over a universal set T and a pair $(\mathcal{T}, \mathcal{K})$ is known to be a q-ROFS_{ft}S over T, where $\mathcal{T}$ is a function given by $\mathcal{T}: \mathbb{E} \to q - ROFS^{(T)}$, which is defined as:

$$\mathcal{T}_{s_j}(\mathscr{k}_i) = \left\{\prec \mathscr{k}_i, \mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i) \succ_q | \mathscr{k}_i \in T, s_j \in \mathbb{E}\ and\ q \ge 1\right\},$$

where $\mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i)$ denotes the $\mathcal{MG}$ and $\mathcal{NMG}$ of an object $\mathscr{k}_i \in T$ to the set $\mathcal{T}_{s_j}$ respectively, and satisfying the condition that $0 \le \left(\mu_j(\mathscr{k}_i)\right)^q + \left(\psi_j(\mathscr{k}_i)\right)^q \le 1$ and $q \ge 1$. For the simplicity $\mathcal{T}_{s_j}(\mathscr{k}_i) = \prec \mathscr{k}_i, \mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i) \succ_q$, is denoted by $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ which represents a q-ROFS_{ft} value (q-ROFS_{ft}V). Moreover, the degree of hesitancy for q-ROFS_{ft}V is defined as $\pi_{\mathfrak{I}_{s_{ij}}} = \sqrt[q]{1 - \left(\left(\mu_{ij}\right)^q + \left(\psi_{ij}\right)^q\right)}$. The set of all q-ROF$S_{ft}$S on the set T is represented by $q - ROFS_{ft}S^{(T)}$.

Let $\Im_{s_{i1}} = (\mu_{i1}, \psi_{i1}), \Im_{s_{i2}} = (\mu_{i2}, \psi_{i2})$ $(i = 1, 2, \dots, m)$ be any two q-ROFS_{ft}Vs and $\lambda > 0$. Then some basic operations on q-ROFS_{ft}Vs are given as follows:

i. $\Im_{s_{i1}} \cup \Im_{s_{i2}} = \left(\ell_i, \left(max\left(\mu_1(\ell_i), \mu_2(\ell_i)\right), min\left(\psi_1(\ell_i), \psi_2(\ell_i)\right)\right)\right)$ for $\ell_i \in T$;

ii. $\Im_{s_{i1}} \cap \Im_{s_{i2}} = \left(\ell_i, \left(min\left(\mu_1(\ell_i), \mu_2(\ell_i)\right), max\left(\psi_1(\ell_i), \psi_2(\ell_i)\right)\right)\right)$ for $\ell_i \in T$;

iii. $\Im^c_{s_{i1}} = \left(\ell_i, \left(\psi_1(\ell_i), \mu_1(\ell_i)\right)\right)$, where $\Im^c_{s_{i1}}$ denotes the complement of $\Im_{s_{i1}}$;

iv. $\Im_{s_{i1}} \preccurlyeq \Im_{s_{i2}}\ if\ \mu_1(\ell_i) \leq \mu_2(\ell_i),\ \psi_1(\ell_i) \geq \psi_2(\ell_i)$ for all $\ell_i \in T$;

v. $\Im_{s_{i1}} \oplus \Im_{s_{i2}} = \left(\sqrt[q]{\mu_1^q(\ell_i) + \mu_2^q(\ell_i) - \mu_1^q(\ell_i)\mu_2^q(\ell_i)}\ , \psi_1^q(\ell_i)\psi_2^q(\ell_i)\right)$;

vi. $\Im_{s_{i1}} \otimes \Im_{s_{i2}} = \left(\mu_1(\ell_i)\mu_2(\ell_i)\ , \sqrt[q]{\psi_1^q(\ell_i) + \psi_2^q(\ell_i) - \psi_1^q(\ell_i)\psi_2^q(\ell_i)}\right)$;

vii. $\lambda\Im_{s_{i1}} = \left(\sqrt[q]{1 - \left[1 - \mu_{i1}^q(\ell_i)\right]^\lambda}, \psi_{i1}^\lambda(\ell_i)\right)$;

viii. $\Im_{i1}^\lambda = \left(\mu_{i1}^\lambda(\ell_i)\ , \sqrt[q]{1 - \left[1 - \psi_{i1}^q(\ell_i)\right]^\lambda}\right)$.

4.2.2. Example

Consider that a person wants to buy a cellphone form the set under consideration of five possible alternatives that is $T = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$. Let $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ be the corresponding set of parameters, where $s_1 =$ high quality audio, video and voice call, $s_2 =$ impressive design with high resolution camera, $s_3 =$ high battery timing, $s_4 =$ reasonable price. On the basis of above criteria a decision maker evaluate the alternatives with rating values and described the result in the form of q-ROFS_{ft}Vs as given in Table 4.1;

Table 4.1. tabular representation of q-ROFS_{ft}S $(\Im, \mathcal{K})$; for $q \geq 3$

T	s_1	s_2	s_3	s_4
ℓ_1	$(0.9, 0.5)$	$(0.8, 0.4)$	$(0.6, 0.3)$	$(0.95, 0.4)$
ℓ_2	$(0.8, 0.2)$	$(0.5, 0.1)$	$(0.7, 0.6)$	$(0.8, 0.3)$
ℓ_3	$(0.93, 0.2)$	$(0.7, 0.3)$	$(0.5, 0.4)$	$(0.75, 0.4)$
ℓ_4	$(0.7, 0.4)$	$(0.8, 0.6)$	$(0.93, 0.2)$	$(0.7, 0.2)$
ℓ_5	$(0.82, 0.6)$	$(0.9, 0.4)$	$(0.7, 0.1)$	$(0.92, 0.45)$

4.2.3. Definition

Consider $\Im_{s_{11}} = (\mu_{11}, \psi_{11})$ be a q-ROFS_{ft}V. Then the score function for $\Im_{s_{11}}$ can be given as,

$$\mathcal{S}c(\Im_{s_{11}}) = \mu_{11}^q - \psi_{11}^q + \left(\frac{e^{\mu_{11}^q - \psi_{11}^q}}{e^{\mu_{11}^q - \psi_{11}^q + 1}} - \frac{1}{2}\right)\pi_{\Im_{s_{11}}}^q \text{ for } q \geq 1 \text{ and } S(\Im_{s_{11}}) \in [-1,1].$$

Let $\Im_{s_{11}} = (\mu_{11}, \psi_{11})$ *and* $\Im_{s_{12}} = (\mu_{12}, \psi_{12})$ be two q-ROFS_{ft}Vs. Then

(i) If $\mathcal{S}c(\Im_{s_{11}}) > \mathcal{S}c(\Im_{s_{12}})$, then $\Im_{s_{11}} \succcurlyeq \Im_{s_{12}}$;
(ii) If $\mathcal{S}c(\Im_{s_{11}}) < \mathcal{S}c(\Im_{s_{12}})$, then $\Im_{s_{11}} \preccurlyeq \Im_{s_{12}}$;
(iii) If $\mathcal{S}c(\Im_{s_{11}}) = \mathcal{S}c(\Im_{s_{12}})$, then
(a) If $\pi_{\Im_{s_{11}}}^q > \pi_{\Im_{s_{12}}}^q$ then $\Im_{s_{11}} \prec \Im_{s_{12}}$;
(b) If $\pi_{\Im_{s_{11}}}^q = \pi_{\Im_{s_{12}}}^q$ then $\Im_{s_{11}} = \Im_{s_{12}}$.

4.2.4. Theorem

Let $\Im_{s_{11}} = (\mu_{11}, \psi_{11}), \Im_{s_{12}} = (\mu_{12}, \psi_{12})$ be any two q-ROFS_{ft}Vs and $\lambda, \lambda_1, \lambda_2 > 0$. Then the following properties are holds:

(i) $\Im_{s_{11}} \oplus \Im_{s_{12}} = \Im_{s_{12}} \oplus \Im_{s_{11}}$;
(ii) $\Im_{s_{11}} \otimes \Im_{s_{12}} = \Im_{s_{12}} \otimes \Im_{s_{11}}$;
(iii) $\lambda(\Im_{s_{11}} \oplus \Im_{s_{12}}) = \lambda\Im_{s_{11}} \oplus \lambda\Im_{s_{12}}$;
(iv) $(\lambda_1 + \lambda_2)\Im_{s_{11}} = \lambda_1\Im_{s_{11}} \oplus \lambda_2\Im_{s_{11}}$;
(v) $\Im_{s_{11}}^{(\lambda_1+\lambda_2)} = \Im_{s_{11}}^{\lambda_1} \otimes \Im_{s_{11}}^{\lambda_2}$;
(vi) $\Im_{s_{11}}^{\lambda} \otimes \Im_{s_{12}}^{\lambda} = (\Im_{s_{11}} \otimes \Im_{s_{11}})^{\lambda}$.

Proof. Proofs are straightforward.

4.3. q-Rung orthopair fuzzy soft average aggregation operator

In this section, we present the detail study of q-ROFS_{ft}WA, q-ROFS_{ft}OWA and q-ROFS_{ft}HA operators and also discuss some of their related properties in detail.

4.3.1. q-Rung orthopair fuzzy soft weighted averaging operators

In this subsection, we investigate q-ROFS_{ft}WA operator and some of their basic properties.

4.3.1.1. Definition

Let $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ *for* $i = 1,2, \dots, n$ *and* $j = 1,2, \dots, m$, be the collection of q-ROFS_{ft}Vs, and consider the weight vectors $\bar{\bar{w}} = \{\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n\}$ *and* $\bar{\bar{u}} = \{\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m\}$ for the experts $\mathscr{k}_i$ and for the parameters $s_j{}'s$ respectively; and having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^n \bar{\bar{w}}_i = 1$ *and* $\sum_{j=1}^m \bar{\bar{u}}_j = 1$. Then the mapping for q-ROFS_{ft}WA operator is defined as: $\text{q} - \text{ROF}S_{ft}\text{WA}: \text{X}^n \rightarrow X$, (where X is the collections of all q-ROFS_{ft}Vs)

$$q-\mathrm{ROF}S_{ft}\mathrm{WA}(\Im_{s_{11}},\Im_{s_{12}},\ldots,\Im_{s_{nm}})=\oplus_{j=1}^{m}\bar{\bar{u}}_j\left(\oplus_{i=1}^{n}\bar{\bar{w}}_i\Im_{s_{ij}}\right).$$

The aggregated result for q-ROFS_{ft}WA operator is described in the following Theorem 4.3.1.2.

4.3.1.2. Theorem

Let $\Im_{s_{ij}}=(\mu_{ij},\psi_{ij})$ $for\ i=1,2,\ldots,n\ and\ j=1,2,\ldots,m,$ be the collections of q-ROFS_{ft}Vs. Then the aggregated result for q-ROFS_{ft}WA operator is given as:

$$q-\mathrm{ROF}S_{ft}\mathrm{WA}(\Im_{s_{11}},\Im_{s_{12}},\ldots,\Im_{s_{nm}})=\oplus_{j=1}^{m}\bar{\bar{u}}_j\left(\oplus_{i=1}^{n}\bar{\bar{w}}_i\Im_{s_{ij}}\right)$$
$$=\left(\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mu_{ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right), \tag{4.1}$$

where $\bar{\bar{w}}=\{\bar{\bar{w}}_1,\bar{\bar{w}}_2,\ldots,\bar{\bar{w}}_n\}$ be the weight vector for the experts $\hbar_i$ and $\bar{\bar{u}}=\{\bar{\bar{u}}_1,\bar{\bar{u}}_2,\ldots,\bar{\bar{u}}_m\}$ be the weight vector for the parameters $s_j's$ respectively; and having the conditions that $\bar{\bar{w}}_i,\bar{\bar{u}}_j\in[0,1]$ with $\sum_{i=1}^{n}\bar{\bar{w}}_i=1\ and\ \sum_{j=1}^{m}\bar{\bar{u}}_j=1.$

Proof. Consider mathematical induction to prove the given result.

As we know by operation laws, that

$$\Im_{s_{11}}\oplus\Im_{s_{12}}=\left(\sqrt[q]{(\mu_{11})^q+(\mu_{12})^q-(\mu_{11})^q(\mu_{12})^q}\ ,\psi_{11}\psi_{12}\right)\ and$$
$$\lambda\Im=\left(\sqrt[q]{1-[1-\mu^q]^{\lambda}},\psi^{\lambda}\right)\quad for\ \lambda\geq 1$$

First we will show that the $Eq.$ (4.1) is true for $n=2\ and\ m=2$, so we have

$$q-\mathrm{ROF}S_{ft}\mathrm{WA}(\Im_{s_{11}},\Im_{s_{12}})=\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{2}\bar{\bar{w}}_i\Im_{s_{ij}}\right)$$
$$=\bar{\bar{u}}_1(\oplus_{i=1}^{2}\bar{\bar{w}}_i\Im_{s_{i1}})\oplus\bar{\bar{u}}_2(\oplus_{i=1}^{2}\bar{\bar{w}}_i\Im_{s_{i2}})$$
$$=\bar{\bar{u}}_1(\bar{\bar{w}}_1\Im_{s_{11}}\oplus\bar{\bar{w}}_2\Im_{s_{21}})\oplus\bar{\bar{u}}_2(\bar{\bar{w}}_1\Im_{s_{12}}\oplus\bar{\bar{w}}_2\Im_{s_{22}})$$
$$=\bar{\bar{u}}_1\left\{\left(\sqrt[q]{1-(1-\mu_{11}^q)^{\bar{\bar{w}}_1}},\psi_{11}^{\bar{\bar{w}}_1}\right)\oplus\left(\sqrt[q]{1-(1-\mu_{21}^q)^{\bar{\bar{w}}_2}},\psi_{21}^{\bar{\bar{w}}_2}\right)\right\}\oplus$$
$$\bar{\bar{u}}_2\left\{\left(\sqrt[q]{1-(1-\mu_{12}^q)^{\bar{\bar{w}}_1}},\psi_{12}^{\bar{\bar{w}}_1}\right)\oplus\left(\sqrt[q]{1-(1-\mu_{22}^q)^{\bar{\bar{w}}_2}},\psi_{22}^{\bar{\bar{w}}_2}\right)\right\}$$
$$=\bar{\bar{u}}_1\left(\sqrt[q]{1-\prod_{i=1}^{2}(1-\mu_{i1}^q)^{\bar{\bar{w}}_i}},\prod_{i=1}^{2}\psi_{i1}^{\bar{\bar{w}}_i}\right)\oplus\upsilon_2\left(\sqrt[q]{1-\prod_{i=1}^{2}(1-\mu_{i2}^q)^{\bar{\bar{w}}_i}},\prod_{i=1}^{2}\psi_{i2}^{\bar{\bar{w}}_i}\right)$$

$$= \left(\sqrt[q]{1 - \left(\prod_{i=1}^{2} \left(1 - \mu_{i1}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_1}}, \left(\prod_{i=1}^{2} \psi_{i1}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_1} \right) \oplus \left(\begin{array}{c} \sqrt[q]{1 - \left(\prod_{i=1}^{2} \left(1 - \mu_{i2}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_2}}, \\ \left(\prod_{i=1}^{2} \psi_{i2}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_2} \end{array} \right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{2} \left(\prod_{i=1}^{2} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{2} \left(\prod_{i=1}^{2} \psi_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)$$

Hence the result is true for $n = 2 \ and \ m = 2$,

Next suppose that $Eq.$ (4.1) is true for $n = k_1 \ and \ m = k_2$

$$\text{q} - \text{ROF}S_{ft}\text{WA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{k_1 k_2}}\right) = \oplus_{j=1}^{k_2} \bar{\bar{u}}_j \left(\oplus_{i=1}^{k_1} \bar{\bar{w}}_i \mathfrak{J}_{s_{ij}} \right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{k_2} \left(\prod_{i=1}^{k_1} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{k_2} \left(\prod_{i=1}^{k_1} \psi_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)$$

We show that $Eq.$ (4.1) is true for $n = k_1 + 1 \ and \ m = k_2 + 1$

$$\text{q} - \text{ROF}S_{ft}\text{WA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{k_1 k_2}}, \mathfrak{J}_{s_{(k_1+1)(k_2+1)}}\right) =$$

$$\left\{ \oplus_{j=1}^{k_2} \bar{\bar{u}}_j \left(\oplus_{i=1}^{k_1} \bar{\bar{w}}_i \mathfrak{J}_{s_{ij}} \right) \right\} \oplus \bar{\bar{u}}_{(k_1+1)} \left(\bar{\bar{w}}_{(k_2+1)} \mathfrak{J}_{s_{(k_1+1)(k_2+1)}} \right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{k_2} \left(\prod_{i=1}^{k_1} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{k_2} \left(\prod_{i=1}^{k_1} \psi_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right) \oplus \bar{\bar{u}}_{(k_1+1)} \left(\bar{\bar{w}}_{(k_2+1)} \mathfrak{J}_{s_{(k_1+1)(k_2+1)}} \right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{(k_2+1)} \left(\prod_{i=1}^{(k_1+1)} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{(k_2+1)} \left(\prod_{i=1}^{(k_1+1)} \psi_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)$$

Hence $Eq.$ (4.1) is true for $n = k_1 + 1 \ and \ m = k_2 + 1$. Therefore, by mathematical induction the $Eq.$ (4.1) is true for all $m, n \geq 1$.

Moreover, to show that the aggregated result achieved from q-ROFS_{ft}WA operator is also a q-ROFS_{ft}V. Now for any $\mathfrak{J}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$, $(i = 1, 2, \dots, n) \ and \ (j = 1, 2, \dots, m)$, where $0 \leq \mu_{ij}, \ \psi_{ij} \leq 1$, satisfying that $0 \leq \mu_{ij}^{q} + \psi_{ij}^{q} \leq 1$, with weight vectors $\bar{\bar{w}} = \{\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n\}$ and $\bar{\bar{u}} = \{\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m\}$ for the experts $\mathscr{k}_i$ and for the parameters $s_j{}'s$ respectively; and having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1 \ and \ \sum_{j=1}^{m} \bar{\bar{u}}_j = 1$.

As $0 \leq \mu_{ij} \leq 1 \Rightarrow 0 \leq 1 - \mu_{ij} \leq 1 \Rightarrow 0 \leq \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i} \leq 1$

$$\Rightarrow 0 \leq \prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i} \leq 1 \Rightarrow 0 \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq 1$$

$$\Rightarrow 0 \leq \sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \leq 1$$

Similarly,

$$0 \leq \psi_{ij} \leq 1 \Rightarrow 0 \leq \prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i} \leq 1 \Rightarrow 0 \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq 1$$

As

$$\mu_{ij}^{q} + \psi_{ij}^{q} \leq 1 \Rightarrow \psi_{ij}^{q} \leq 1 - \mu_{ij}^{q} \Rightarrow \prod_{i=1}^{n} \left(\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i} \leq \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)$$

$$\Rightarrow \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Rightarrow \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)^{q} \leq \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \tag{4.2}$$

Now we have

$$0 \leq \left\{\sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\}^{q} + \left\{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}^{q}$$

by $Eq. (4.2)$, we have

$$\leq 1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} + \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} = 1$$

Therefore,

$$0 \leq \left\{\sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\}^{q} + \left\{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}^{q} \leq 1$$

Hence, it is proved that the aggregated result achieved from q-ROFS_{ft}WA operator is also a q-ROFS_{ft}V.

4.3.1.3. Remark

(a) If the value of q is fixed, that is $q = 1$, then the proposed q-ROFS_{ft}WA operator reduces to IFS_{ft}WA operator.
(b) If the value of q is fixed, that is $q = 2$, then the proposed q-ROFS_{ft}WA operator reduces to PyFS_{ft}WA operator.
(c) If there is only one parameter, that is s_1 (mean $m = 1$), then the proposed q-ROFS_{ft}WA operator reduces to q-ROFWA operator.

Hence from Remark 4.3.1.3, it is clear that the developed q-ROFS_{ft}WA operators is the generalized case of IFWA, IFS_{ft}WA and PyFS_{ft}WA operators.

4.3.1.4. Example

Suppose Mr. X wants to purchase a house from the set of five houses in the domain set $T = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\}$ and let $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ be the set of criterion (parameters), i.e. s_i $(i = 1,2,3,4)$ stands for $s_1 =$ beautiful, $s_2 =$ in green surrounding, $s_3 =$ expenxive, $s_4 =$ safety respectively. Suppose the weight vectors $\bar{\bar{w}} = (0.24,0.23,0.22,0.15,0.16)^T$ and $\bar{\bar{u}} = (0.28,0.19,0.3,0.23)^T$ for the experts $\hbar_i$ and for the parameters $s_j{}'s$ respectively. The experts provide their evaluation for each house to their corresponding criterion (parameters) in the form of q-ROFS_{ft}Vs, which is presented in Table 4.2.

By using $Eq.$ (4.1), we have

$$\mathrm{q-ROF}S_{ft}\mathrm{WA}(\Im_{s_{11}}, \Im_{s_{12}}, \ldots, \Im_{s_{54}}) = \left(\sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \mu_{ij}^{q})^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)$$

$$= \left(\sqrt[3]{\begin{array}{c} 1 - \left\{ \begin{array}{c} (1-0.88^3)^{0.24}(1-0.96^3)^{0.23}(1-0.91^3)^{0.22} \\ (1-0.75^3)^{0.15}(1-0.82^3)^{0.16} \end{array} \right\}^{0.28} \\ \left\{ \begin{array}{c} (1-0.8^3)^{0.24}(1-0.75^3)^{0.23}(1-0.9^3)^{0.22}(1-0.85^3)^{0.15} \\ (1-0.9^3)^{0.16} \end{array} \right\}^{0.19} \\ \left\{ \begin{array}{c} (1-0.65^3)^{0.24}(1-0.77^3)^{0.23}(1-0.86^3)^{0.22}(1-0.93^3)^{0.15} \\ (1-0.78^3)^{0.16} \end{array} \right\}^{0.3}, \\ \left\{ \begin{array}{c} (1-0.93^3)^{0.24}(1-0.87^3)^{0.23}(1-0.7^3)^{0.22}(1-0.9^3)^{0.15} \\ (1-0.94^3)^{0.16} \end{array} \right\}^{0.23} \end{array}} \right.$$

$$\left. \begin{array}{c} \left\{ \begin{array}{c} (0.4^{0.24})(0.2^{0.23})(0.4^{0.22})(0.6^{0.15}) \\ (0.5^{0.16}) \end{array} \right\}^{0.28} \left\{ \begin{array}{c} (0.4^{0.24})(0.3^{0.23})(0.3^{0.22}) \\ (0.45^{0.15})(0.33^{0.16}) \end{array} \right\}^{0.19} \\ \left\{ \begin{array}{c} (0.25^{0.24})(0.4^{0.23})(0.34^{0.22})(0.2^{0.15}) \\ (0.3^{0.16}) \end{array} \right\}^{0.3} \left\{ \begin{array}{c} (0.32^{0.24})(0.25^{0.23})(0.5^{0.22}) \\ (0.2^{0.15})(0.45^{0.16}) \end{array} \right\}^{0.23} \end{array} \right)$$

$$= (0.866891, 0.334196).$$

In the following, in view of Theorem 4.3.1.2 some properties of the developed q-ROFS_{ft}WA operator for the collections of q-ROFS_{ft}Vs $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, is being presented.

Table 4.2, Tabular representation of q-ROFS_{ft}S $(\Im, \mathcal{K})$ for q ≥ 3

T	s_1	s_2	s_3	s_4
k_1	(0.88, 0.4)	(0.8, 0.4)	(0.65, 0.25)	(0.93, 0.32)
k_2	(0.96, 0.2)	(0.75, 0.3)	(0.77, 0.4)	(0.87, 0.25)
k_3	(0.91, 0.4)	(0.9, 0.3)	(0.86, 0.34)	(0.7, 0.5)
k_4	(0.7, 0.6)	(0.85, 0.45)	(0.93, 0.2)	(0.9, 0.2)
k_5	(0.82, 0.5)	(0.9, 0.33)	(0.78, 0.3)	(0.94, 0.45)

4.3.1.5. Theorem

Suppose the collections of q-ROFS_{ft}Vs $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, with weight vectors $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ for the experts k_i and for the parameters $s_j's$ respectively, such that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the q-ROF$S_{ft}$WA operator holds the following properties:

$\boldsymbol{i}$: ($\boldsymbol{Idempotency}$): If $\Im_{s_{ij}} = \mathcal{E}_s$, $(\forall\, i = 1,2,\dots,n$ and $j = 1,2,\dots,m)$, where $\mathcal{E}_s = (b, d)$, then

$$\text{q} - \text{ROF}S_{ft}\text{WA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) = \mathcal{E}_s.$$

$\boldsymbol{ii}$: ($\boldsymbol{Boundedness}$): If $\Im_{s_{ij}}^{-} = \left(\min_j \min_i \{\mu_{ij}\}, \max_j \max_i \{\psi_{ij}\}\right)$ and

$\Im_{s_{ij}}^{+} = \left(\max_j \max_i \{\mu_{ij}\},\ \min_j \min_i \{\psi_{ij}\}\right)$, then

$$\Im_{s_{ij}}^{-} \leq \text{ q} - \text{ROF}S_{ft}\text{WA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \leq \Im_{s_{ij}}^{+}.$$

$\boldsymbol{iii}$: ($\boldsymbol{Monotonicity}$): If $\mathcal{E}_{s_{ij}} = (b_{ij}, d_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, be the another collection of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij}$ and $\psi_{ij} \geq d_{ij}$, then

$$\text{q} - \text{ROF}S_{ft}\text{WA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \leq \text{q} - \text{ROF}S_{ft}\text{WA}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{nm}}).$$

$\boldsymbol{iv}$: ($\boldsymbol{Shift\ Invariance}$): If $\tilde{\mathcal{E}}_s = (b, d)$, is another q-ROF$S_{ft}$V, then

$$\text{q} - \text{ROF}S_{ft}\text{WA}(\Im_{s_{11}} \oplus \mathcal{E}_s, \Im_{s_{12}} \oplus \mathcal{E}_s, \dots, \Im_{s_{nm}} \oplus \mathcal{E}_s) \\ = \text{q} - \text{ROF}S_{ft}\text{WA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \oplus \mathcal{E}_s.$$

$\boldsymbol{iv}$: ($\boldsymbol{Homogeneity}$): If λ is any real number such that $\lambda > 0$, then

$$\text{q} - \text{ROF}S_{ft}\text{WA}(\lambda\Im_{s_{11}}, \lambda\Im_{s_{12}}, \dots, \lambda\Im_{s_{nm}}) = \lambda \text{q} - \text{ROF}S_{ft}\text{WA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}).$$

Proof.i: (***Idempotency***) As it is given that if for all $\Im_{s_{ij}} = \mathcal{E}_s = (b, d)$ ($\forall\, i = 1,2, \dots, n$ *and* $j = 1,2, \dots, m$), then from Eq. 4.1, we have

$$\begin{aligned}
&\mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}\right) \\
&= \left(\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mu_{ij}^{q}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\psi_{ij}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right) \\
&= \left(\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-b^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}d^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right) \\
&= \left(\sqrt[q]{1-\left((1-b^{q})^{\sum_{i=1}^{n}\overline{\overline{w}}_i}\right)^{\sum_{j=1}^{m}\overline{\overline{u}}_j}}, \left(d^{\sum_{i=1}^{n}\overline{\overline{w}}_i}\right)^{\sum_{j=1}^{m}\overline{\overline{u}}_j}\right) \\
&= \left(\sqrt[q]{1-(1-b^{q})}, d\right) = (b, d) = \tilde{\mathcal{E}}_s
\end{aligned}$$

Therefore, $\mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}\right) = \mathcal{E}_s$.

ii: (***Boundedness***) As $\Im_{s_{ij}}^{-} = \left(\min_j \min_i \{\mu_{ij}\}, \max_j \max_i \{\psi_{ij}\}\right)$ and $\Im_{s_{ij}}^{+} = \left(\max_j \max_i \{\mu_{ij}\}, \min_j \min_i \{\psi_{ij}\}\right)$. To prove that $\Im_{s_{ij}}^{-} \leq \mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}\right) \leq \Im_{s_{ij}}^{+}$,

Now for each $i = 1,2, \dots, n$ *and* $j = 1,2, \dots, m$, we have

$$\begin{aligned}
&\min_j \min_i \{\mu_{ij}\} \leq \mu_{ij} \leq \max_j \max_i \{\mu_{ij}\} \Leftrightarrow 1-\max_j \max_i \{\mu_{ij}^{q}\} \leq 1-\mu_{ij}^{q} \\
&\qquad \leq 1-\min_j \min_i \{\mu_{ij}^{q}\} \\
&\Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\max_j \max_i \{\mu_{ij}^{q}\}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mu_{ij}^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \\
&\qquad \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\min_j \min_i \{\mu_{ij}^{q}\}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \\
&\Leftrightarrow \left(\left(1-\max_j \max_i \{\mu_{ij}^{q}\}\right)^{\sum_{i=1}^{n}\overline{\overline{w}}_i}\right)^{\sum_{j=1}^{m}\overline{\overline{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mu_{ij}^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \\
&\qquad \leq \left(\left(1-\min_j \min_i \{\mu_{ij}^{q}\}\right)^{\sum_{i=1}^{n}\overline{\overline{w}}_i}\right)^{\sum_{j=1}^{m}\overline{\overline{u}}_j} \\
&\Leftrightarrow \left(1-\max_j \max_i \{\mu_{ij}^{q}\}\right) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mu_{ij}^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \leq \left(1-\min_j \min_i \{\mu_{ij}^{q}\}\right)
\end{aligned}$$

$$\Leftrightarrow 1-\left(1-\min_j \min_i\{\mu_{ij}^q\}\right) \leq 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mu_{ij}^q\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$
$$\leq 1-\left(1-\max_j \max_i\{\mu_{ij}^q\}\right)$$

Hence

$$\min_j \min_i\{\mu_{ij}\} \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mu_{ij}^q\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}$$
$$\leq \max_j \max_i\{\mu_{ij}\} \tag{4.3}$$

Next for each $i = 1,2,\dots,n \; and \; j = 1,2,\dots,m$, we have

$$\min_j \min_i\{\psi_{ij}\} \leq \psi_{ij} \leq \max_j \max_i\{\psi_{ij}\}$$
$$\Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\min_j \min_i\{\psi_{ij}\}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\psi_{ij}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$
$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\max_j \max_i\{\psi_{ij}\}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$
$$\Leftrightarrow \left(\left(\min_j \min_i\{\psi_{ij}\}\right)^{\sum_{i=1}^{n}\overline{\overline{w}}_i}\right)^{\sum_{j=1}^{m}\overline{\overline{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\psi_{ij}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$
$$\leq \left(\left(\max_j \max_i\{\psi_{ij}\}\right)^{\sum_{i=1}^{n}\overline{\overline{w}}_i}\right)^{\sum_{j=1}^{m}\overline{\overline{u}}_j}$$

this implies that

$$\min_j \min_i\{\psi_{ij}\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\psi_{ij}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \leq \max_j \max_i\{\psi_{ij}\} \tag{4.4}$$

Therefore, from $Eqs.$ (4.3) and (4.4), we have

$$\min_j \min_i\{\mu_{ij}\} \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mu_{ij}^q\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}} \leq \max_j \max_i\{\mu_{ij}\}$$

and

$$\min_j \min_i\{\psi_{ij}\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\psi_{ij}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \leq \max_j \max_i\{\psi_{ij}\}$$

Let $\delta = \mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) = (\mu_\delta, \psi_\delta)$, then by score function given in Definition 4.2.4, we have

$$\mathcal{S}c(\delta)=\mu_\delta^q-\psi_\delta^q+\left(\frac{e^{\mu_\delta^q-\psi_\delta^q}}{e^{\mu_\delta^q-\psi_\delta^q}+1}-\frac{1}{2}\right)\pi_\delta^q$$
$$\leq\left(\max_j\max_i\{\mu_{ij}\}\right)^q-\left(\min_j\min_i\{\psi_{ij}\}\right)^q+$$
$$+\left(\frac{e^{\left(\max_j\max_i\{\mu_{ij}\}\right)^q-\left(\min_j\min_i\{\psi_{ij}\}\right)^q}}{e^{\left(\max_j\max_i\{\mu_{ij}\}\right)^q-\left(\min_j\min_i\{\psi_{ij}\}\right)^q}+1}-\frac{1}{2}\right)\pi^q_{\mathfrak{I}^+_{s_{ij}}}=\mathcal{S}c\left(\mathfrak{I}^+_{s_{ij}}\right)$$
this implies
$$\mathcal{S}c(\delta)\leq\mathcal{S}c\left(\mathfrak{I}^+_{s_{ij}}\right)$$
and $\mathcal{S}c(\delta)=\mu_\delta^q-\psi_\delta^q+\left(\frac{e^{\mu_\delta^q-\psi_\delta^q}}{e^{\mu_\delta^q-\psi_\delta^q+1}}-\frac{1}{2}\right)\pi_\delta^q\geq\left(\min_j\min_i\{\mu_{ij}\}\right)^q-\left(\max_j\max_i\{\psi_{ij}\}\right)^q+$
$$+\left(\frac{e^{\left(\min_j\min_i\{\mu_{ij}\}\right)^q-\left(\max_j\max_i\{\psi_{ij}\}\right)^q}}{e^{\left(\min_j\min_i\{\mu_{ij}\}\right)^q-\left(\max_j\max_i\{\psi_{ij}\}\right)^q}+1}-\frac{1}{2}\right)\pi^q_{\mathfrak{I}^-_{s_{ij}}}=\mathcal{S}c\left(\mathfrak{I}^-_{s_{ij}}\right)$$
this implies $\mathcal{S}c(\delta)\geq\mathcal{S}c\left(\mathfrak{I}^-_{s_{ij}}\right)$.

In view of that direction, consider the following cases,

Case i: If $\mathcal{S}c(\delta)<\mathcal{S}c\left(\mathfrak{I}^+_{s_{ij}}\right)$ *and* $\mathcal{S}c(\delta)>\mathcal{S}c\left(\mathfrak{I}^-_{s_{ij}}\right)$, by the comparison of two q-ROFS_{ft}Vs, we get
$$\mathfrak{I}^-_{s_{ij}}<\text{q}-\text{ROF}S_{ft}\text{WA}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\ldots,\mathfrak{I}_{s_{nm}}\right)<\mathfrak{I}^+_{s_{ij}}\,.$$
Case ii: If $\mathcal{S}c(\delta)=\mathcal{S}c\left(\mathfrak{I}^+_{s_{ij}}\right)$, that is
$$\mu_\delta^q-\psi_\delta^q+\left(\frac{e^{\mu_\delta^q-\psi_\delta^q}}{e^{\mu_\delta^q-\psi_\delta^q}+1}-\frac{1}{2}\right)\pi_\delta^q=\left(\max_j\max_i\{\mu_{ij}\}\right)^q-\left(\min_j\min_i\{\psi_{ij}\}\right)^q+$$
$$+\left(\frac{e^{\left(\max_j\max_i\{\mu_{ij}\}\right)^q-\left(\min_j\min_i\{\psi_{ij}\}\right)^q}}{e^{\left(\max_j\max_i\{\mu_{ij}\}\right)^q-\left(\min_j\min_i\{\psi_{ij}\}\right)^q}+1}-\frac{1}{2}\right)\pi^q_{\mathfrak{I}^+_{s_{ij}}},$$
then by using the above inequalities, we get

$\mu_\delta=\max_j\max_i\{\mu_{ij}\}$ *and* $\psi_\delta=\min_j\min_i\{\psi_{ij}\}$. Thus $\pi_\delta^q=\pi^q_{\mathfrak{I}^+_{s_{ij}}}$,

Hence by comparison of two q-ROFS_{ft}Vs, we have
$$\text{q}-\text{ROF}S_{ft}\text{WA}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\ldots,\mathfrak{I}_{s_{nm}}\right)=\mathfrak{I}^+_{s_{ij}}\,.$$
Case iii: If $\mathcal{S}c(\delta)=\mathcal{S}c\left(\mathfrak{I}^-_{s_{ij}}\right)$, that is

$$\mu_\delta^q-\psi_\delta^q+\left(\frac{e^{\mu_\delta^q-\psi_\delta^q}}{e^{\mu_\delta^q-\psi_\delta^q}+1}-\frac{1}{2}\right)\pi_\delta^q=\left(\min_j\min_i\{\mu_{ij}\}\right)^q-\left(\max_j\max_i\{\psi_{ij}\}\right)^q+$$

$$+\left(\frac{e^{\left(\min_j\min_i\{\mu_{ij}\}\right)^q-\left(\max_j\max_i\{\psi_{ij}\}\right)^q}}{e^{\left(\min_j\min_i\{\mu_{ij}\}\right)^q-\left(\max_j\max_i\{\psi_{ij}\}\right)^q}+1}-\frac{1}{2}\right)\pi_{\mathfrak{I}_{s_{ij}}^-}^q,$$

then by using the above inequalities, we get

$$\mu_\delta=\min_j\min_i\{\mu_{ij}\}\ and\ \psi_\delta=\max_j\max_i\{\psi_{ij}\}$$

Thus

$$\pi_\delta^q=\pi_{\mathfrak{I}_{s_{ij}}^-}^q$$

this implies

$$\mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\dots,\mathfrak{I}_{s_{nm}}\right)=\mathfrak{I}_{s_{ij}}^-.$$

Therefore, it is proved that

$$\mathfrak{I}_{s_{ij}}^-\le\mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\dots,\mathfrak{I}_{s_{nm}}\right)\le\mathfrak{I}_{s_{ij}}^+.$$

$\boldsymbol{iii}$: $(\boldsymbol{Monotonicity})$ Since $\mu_{ij}\le b_{ij}\ and\ \psi_{ij}\ge d_{ij},\ (i=1,2,\dots,n)\ and\ (j=1,2,\dots,m)$, then this implies that

$$\mu_{ij}\le b_{ij}\Rightarrow 1-b_{ij}\le 1-\mu_{ij}\Rightarrow 1-b_{ij}^q\le 1-\mu_{ij}^q$$

$$\Rightarrow\prod_{j=1}^m\left(\prod_{i=1}^n(1-b_{ij}^q)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\le\prod_{j=1}^m\left(\prod_{i=1}^n(1-\mu_{ij}^q)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Rightarrow 1-\prod_{j=1}^m\left(\prod_{i=1}^n(1-\mu_{ij}^q)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\le 1-\prod_{j=1}^m\left(\prod_{i=1}^n(1-b_{ij}^q)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\sqrt[q]{1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-\mu_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\le\sqrt[q]{1-\prod_{j=1}^m\left(\prod_{i=1}^n\left(1-b_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\qquad(4.5)$$

Furthermore

$$\psi_{ij}\ge d_{ij}\Rightarrow\left(\prod_{i=1}^n(\psi_{ij})^{\bar{\bar{w}}_i}\right)\ge\prod_{i=1}^n(d_{ij})^{\bar{\bar{w}}_i}$$

$$\Rightarrow\prod_{j=1}^m\left(\prod_{i=1}^n(\psi_{ij})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\ge\prod_{j=1}^m\left(\prod_{i=1}^n(d_{ij})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\qquad(4.6)$$

Let $\delta_{\mathfrak{I}}=\mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\dots,\mathfrak{I}_{s_{nm}}\right)=(\mu_{\delta_{\mathfrak{I}}},\psi_{\delta_{\mathfrak{I}}})$ and

$$\delta_{\mathcal{E}}=\mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\mathcal{E}_{s_{11}},\mathcal{E}_{s_{12}},\dots,\mathcal{E}_{s_{nm}}\right)=(b_{\delta_{\mathcal{E}}},d_{\delta_{\mathcal{E}}})$$

From $Eqs.$ (4.5) and (4.6), we have

$$\mu_{\delta_{\mathfrak{I}}}\le b_{\delta_{\mathcal{E}}}\ and\ \psi_{\delta_{\mathfrak{I}}}\ge d_{\delta_{\mathcal{E}}}$$

then by score function given in Definition 4.2.4, we have

$$\mathcal{S}c(\delta_{\mathfrak{I}}) \leq \mathcal{S}c(\delta_{\mathcal{E}})$$

In view of that direction, consider the following cases,

Case i: If $\mathcal{S}c(\delta_{\mathfrak{I}}) < \mathcal{S}c(\delta_{\mathcal{E}})$, by the comparison of two q-ROFS_{ft}Vs, we get

$$\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{nm}}\right) < \mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{WA}\left(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \ldots, \mathcal{E}_{s_{nm}}\right).$$

Case ii: If $\mathcal{S}c(\delta_{\mathfrak{I}}) = \mathcal{S}c(\delta_{\mathcal{E}})$, that is

$$\mathcal{S}c(\delta_{\mathfrak{I}}) = \mu_{\delta_{\mathfrak{I}}}^{q} - \psi_{\delta_{\mathfrak{I}}}^{q} + \left(\frac{e^{\mu_{\delta_{\mathfrak{I}}}^{q} - \psi_{\delta_{\mathfrak{I}}}^{q}}}{e^{\mu_{\delta_{\mathfrak{I}}}^{q} - \psi_{\delta_{\mathfrak{I}}}^{q}} + 1} - \frac{1}{2}\right)\pi_{\delta_{\mathfrak{I}}}^{q}$$

$$= \mu_{\delta_{\mathcal{E}}}^{q} - \psi_{\delta_{\mathcal{E}}}^{q} + \left(\frac{e^{\mu_{\delta_{\mathcal{E}}}^{q} - \psi_{\delta_{\mathcal{E}}}^{q}}}{e^{\mu_{\delta_{\mathcal{E}}}^{q} - \psi_{\delta_{\mathcal{E}}}^{q}} + 1} - \frac{1}{2}\right)\pi_{\delta_{\mathcal{E}}}^{q} = \mathcal{S}c(\delta_{\mathcal{E}}),$$

then by above inequality, we have

$$\mu_{\delta_{\mathfrak{I}}} = b_{\delta_{\mathcal{E}}} \; and \; \psi_{\delta_{\mathfrak{I}}} = d_{\delta_{\mathcal{E}}}$$

Hence $\pi_{\delta_{\mathfrak{I}}}^{q} = \pi_{\delta_{\mathcal{E}}}^{q} \Rightarrow \left(\mu_{\delta_{\mathfrak{I}}}, \psi_{\delta_{\mathfrak{I}}}\right) = \left(b_{\delta_{\mathcal{E}}}, d_{\delta_{\mathcal{E}}}\right)$

Therefore, it is proved that

$$\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{nm}}\right) \leq \mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{WA}\left(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \ldots, \mathcal{E}_{s_{nm}}\right).$$

iv: (***Shift Invariance***) Since $\mathcal{E}_s = (b, d)$ and $\mathfrak{I}_{s_{ij}} = \left(\mu_{s_{ij}}, \psi_{s_{ij}}\right)$ are the q-ROFS_{ft}Vs, so

$$\mathfrak{I}_{s_{11}} \oplus \mathcal{E}_s = \left(\sqrt[q]{1 - (1 - \mu_{11}^{q})(1 - b^{q})}\ , \psi_{11} d\right)$$

Therefore,

$$\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}} \oplus \mathcal{E}_s, \mathfrak{I}_{s_{12}} \oplus \mathcal{E}_s, \ldots, \mathfrak{I}_{s_{nm}} \oplus \mathcal{E}_s\right) = \oplus_{j=1}^{m} \bar{\bar{u}}_j \left(\oplus_{i=1}^{n} \bar{\bar{w}}_i \left(\mathfrak{I}_{s_{nm}} \oplus \mathcal{E}_s\right)\right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}(1 - b^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\ , \prod_{j=1}^{m}\left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i} d^{\bar{\bar{u}}_i}\right)^{\bar{\bar{u}}_j}\right)$$

$$= \left(\sqrt[q]{1 - (1 - b^{q})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\ , d\prod_{j=1}^{m}\left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\ , \prod_{j=1}^{m}\left(\prod_{i=1}^{n} \psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right) \oplus (b, d)$$

$$= \mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{WA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{nm}}\right) \oplus \mathcal{E}_s$$

Hence the required result is proved.

iv: (***Homogeneity***) Consider $\lambda > 0$ be any real number and $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ be a q-ROFS_{ft}V, then

$$\lambda \Im_{s_{ij}} = \left(\sqrt[q]{1 - \left(1 - \mu_{ij}^{q}\right)^{\lambda}}, \psi_{ij}^{\lambda} \right)$$

Now

$$\begin{aligned}
&\mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\lambda\Im_{s_{11}}, \lambda\Im_{s_{12}}, \dots, \lambda\Im_{s_{nm}}\right) \\
&\quad = \left(\sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mu_{ij}^{q}\right)^{\lambda \bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\psi_{ij}^{\lambda \bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \right) \\
&= \left(\sqrt[q]{1 - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{w}}_j}\right)^{\lambda}}, \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\psi_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)^{\lambda} \right) \\
&= \lambda \mathrm{q-ROF}S_{ft}\mathrm{WA}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}\right)
\end{aligned}$$

Therefore, the required property is proved.

4.3.2. q-Rung orthopair fuzzy soft ordered weighted averaging operators

From the above analyses of q-ROFS_{ft}WA operator, it is clear that q-ROFS_{ft}WA operator just weighed the values of q-ROFS_{ft}N, while q-ROFS_{ft}OWA operator weigh the ordered positions via scoring the q-ROFS_{ft} values rather than weighting the q-ROFS_{ft} values themselves. So, here we will present the detailed study of q-ROFS_{ft}OWA operator and also studied their related properties.

4.3.2.1. Definition

Let $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(for\ i = 1,2,\dots,n\ and\ j = 1,2,\dots,m)$, be the collections of q-ROFS_{ft}Vs, and consider the weight vectors $\bar{\bar{w}} = \{\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n\}\ and\ \bar{\bar{u}} = \{\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m\}$ for the experts $\mathscr{k}_i$ and for the parameters $s_j's$ respectively, and having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n}\bar{\bar{w}}_i = 1\ and\ \sum_{j=1}^{m}\bar{\bar{u}}_j = 1$. Then the mapping for q-ROFS_{ft}OWA operator is defined as: $\mathrm{q-ROF}S_{ft}\mathrm{OWA}: \mathrm{X}^n \to X$, (where X is the collections of all q-ROFS_{ft}Vs)

$$\mathrm{q-ROF}S_{ft}\mathrm{OWA}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}\right) = \oplus_{j=1}^{m}\bar{\bar{u}}_j \left(\oplus_{i=1}^{n}\bar{\bar{w}}_i \Im_{\sigma s_{ij}}\right).$$

The aggregated result for q-ROFS_{ft}OWA operator is described in the following Theorem 4.3.2.2.

4.3.2.2. Theorem

Let $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(for\ i = 1,2,\dots,n\ and\ j = 1,2,\dots,m)$, be the collections of q-ROFS_{ft}Vs. Then the aggregated result for q-ROFS_{ft}OWA operator is given as:

$$\text{q}-\text{ROF}S_{ft}\text{OWA}(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\dots,\mathfrak{I}_{s_{nm}}) = \oplus_{j=1}^{m}\bar{\bar{u}}_j\left(\oplus_{i=1}^{n}\bar{\bar{w}}_i\mathfrak{I}_{\sigma s_{ij}}\right)$$

$$= \left(\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mu_{\sigma ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\psi_{\sigma ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right), \quad (4.7)$$

where $\mathfrak{I}_{\sigma s_{ij}} = (\mu_{\sigma ij},\psi_{\sigma ij})$, represents the permutations of i^{th} *and* j^{th} largest object of the collections of $i\times j$ q-ROFS_{ft}Vs $\mathfrak{I}_{s_{ij}} = (\mu_{ij},\psi_{ij})$.

Proof. Proof is similar to Theorem 4.3.1.2.

4.3.2.3. Remark

a. If the value of q is fixed, that is $q = 1$, then the proposed q-ROFS_{ft}OWA operator reduces to IFS_{ft}OWA operator.
b. If the value of q is fixed, that is $q = 2$, then the proposed q-ROFS_{ft}OWA operator reduces to PyFS_{ft}OWA operator.
c. If there is only one parameter, that is s_1 (means $m = 1$), then the proposed q-ROFS_{ft}OWA operator reduces to q-ROFOWA operator.

Hence from Remark 4.3.2.3, it is clear that IFS_{ft}OWA, PyFS_{ft}OWA and q-ROFOWA operators are the special cases of the proposed q-ROFS_{ft}OWA operator.

4.3.2.4. Example

Consider the collections q-ROFS_{ft}Vs $\mathfrak{I}_{s_{ij}} = (\mu_{ij},\psi_{ij})$ as given in Table 4.2, of Example 4.3.1.4, then by using score function from Definition 4.2.4, the tabular representations of $\mathfrak{I}_{\sigma s_{ij}} = (\mu_{\sigma ij},\psi_{\sigma ij})$ is given in Table 4.3.

Now by using $Eq.(4.7)$, to find the q-ROFS_{ft}OWA operator, we have

$$\text{q}-\text{ROF}S_{ft}\text{OWA}(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\dots,\mathfrak{I}_{s_{nm}}) = \oplus_{j=1}^{m}\bar{\bar{u}}_j\left(\oplus_{i=1}^{n}\bar{\bar{w}}_i\mathfrak{I}_{\sigma s_{ij}}\right)$$

$$= \left(\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\mu_{\sigma ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\psi_{\sigma ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)$$

Table 4.3, Tabular representation of q-ROFS_{ft}S $\mathfrak{I}_{\sigma s_{ij}} = (\mu_{\sigma ij},\psi_{\sigma ij})$ for q ≥ 3

T	s_1	s_2	s_3	s_4
k_1	(0.96, 0.2)	(0.9, 0.3)	(0.93, 0.2)	(0.93, 0.32)
k_2	(0.91, 0.4)	(0.9, 0.33)	(0.86, 0.34)	(0.9, 0.2)
k_3	(0.88, 0.4)	(0.85, 0.45)	(0.78, 0.3)	(0.94, 0.45)
k_4	(0.82, 0.5)	(0.8, 0.4)	(0.77, 0.4)	(0.87, 0.25)
k_5	(0.75, 0.6)	(0.75, 0.3)	(0.65, 0.25)	(0.7, 0.5)

$$= \left(\sqrt[3]{\begin{matrix} 1 - \left\{ \begin{matrix} (1-0.96^3)^{0.24}(1-0.91^3)^{0.23}(1-0.88^3)^{0.22}(1-0.82^3)^{0.15} \\ (1-0.75^3)^{0.16} \end{matrix} \right\}^{0.28} \\ \left\{ \begin{matrix} (1-0.9^3)^{0.24}(1-0.9^3)^{0.23}(1-0.85^3)^{0.22}(1-0.8^3)^{0.15} \\ (1-0.75^3)^{0.16} \end{matrix} \right\}^{0.19} \\ \left\{ \begin{matrix} (1-0.93^3)^{0.24}(1-0.86^3)^{0.23}(1-0.78^3)^{0.22}(1-0.77^3)^{0.15} \\ (1-0.65^3)^{0.16} \end{matrix} \right\}^{0.3} \\ \left\{ \begin{matrix} (1-0.93^3)^{0.24}(1-0.9^3)^{0.23}(1-0.94^3)^{0.22}(1-0.87^3)^{0.15} \\ (1-0.7^3)^{0.16} \end{matrix} \right\}^{0.23} \end{matrix}}, \right.$$

$$\left. \begin{matrix} \left\{ \begin{matrix} (0.2^{0.24})(0.4^{0.23})(0.4^{0.22})(0.5^{0.15}) \\ (0.6^{0.16}) \end{matrix} \right\}^{0.28} \left\{ \begin{matrix} (0.3^{0.24})(0.33^{0.23})(0.45^{0.22}) \\ (0.4^{0.15})(0.3^{0.16}) \end{matrix} \right\}^{0.19} \\ \left\{ \begin{matrix} (0.2^{0.24})(0.34^{0.23})(0.3^{0.22})(0.4^{0.15}) \\ (0.25^{0.16}) \end{matrix} \right\}^{0.3} \left\{ \begin{matrix} (0.32^{0.24})(0.2^{0.23})(0.45^{0.22}) \\ (0.25^{0.15})(0.5^{0.16}) \end{matrix} \right\}^{0.23} \end{matrix} \right)$$

$$= (0.878279, 0.32812)$$

In the following, in view of Theorem 4.3.2.2, some properties of the proposed q-ROFS_{ft}OWA operator for the collections of q-ROFS_{ft}Vs $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij}),\ (i = 1, 2, \dots, n)\ and\ (j = 1, 2, \dots, m)$, is being presented.

4.3.2.5. Theorem

Suppose the collections of q-ROFS_{ft}Vs $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij}),\ (i = 1, 2, \dots, n)\ and\ (j = 1, 2, \dots, m)$, with weight vectors $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ for the experts $\mathscr{k}_i$ and for the parameters $s_j's$ respectively, such that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1\ and\ \sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the q-ROF$S_{ft}$OWA operator has the following properties:

i: (***Idempotency***): If $\Im_{s_{ij}} = \mathcal{E}_{\sigma s},\ (\forall\ i = 1,2,\dots,n\ and\ j = 1,2,\dots,m)$, where $\mathcal{E}_{\sigma s} = (\boldsymbol{b}, d)$, then

$$\text{q} - \text{ROF}S_{ft}\text{OWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) = \mathcal{E}_{\sigma s}.$$

ii: (***Boundedness***): If $\Im_{\sigma s_{ij}}^{-} = \left(\min_j \min_i \{\mu_{\sigma ij}\}, \max_j \max_i \{\psi_{\sigma ij}\}\right)$ and $\Im_{\sigma s_{ij}}^{+} = \left(\max_j \max_i \{\mu_{\sigma ij}\}, \min_j \min_i \{\psi_{\sigma ij}\}\right)$, then

$$\Im_{\sigma s_{ij}}^{-} \leq \text{q} - \text{ROF}S_{ft}\text{OWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \leq \Im_{\sigma s_{ij}}^{+}.$$

iii: (***Monotonicity***): If $\mathcal{E}_{s_{ij}} = (b_{ij}, d_{ij}),\ (i = 1, 2, \dots, n)\ and\ (j = 1, 2, \dots, m)$, be the another collection of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij}\ and\ \psi_{ij} \geq d_{ij}$, then

$$\text{q} - \text{ROF}S_{ft}\text{OWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \leq \text{q} - \text{ROF}S_{ft}\text{OWA}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{nm}}).$$

iv: (***Shift Invariance***): If $\mathcal{E}_s = (b, d)$, is another q-ROFS_{ft}V, then

$$\text{q} - \text{ROF}S_{ft}\text{OWA}(\Im_{s_{11}} \oplus \mathcal{E}_s, \Im_{s_{12}} \oplus \mathcal{E}_s, \dots, \Im_{s_{nm}} \oplus \mathcal{E}_s) = \text{q} - \text{ROF}S_{ft}\text{OWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \oplus \mathcal{E}_s.$$

iv: (***Homogeneity***): If λ is any real number such that $\lambda \geq 0$, then

$$\text{q} - \text{ROF}S_{ft}\text{OWA}(\lambda\Im_{s_{11}}, \lambda\Im_{s_{12}}, \dots, \lambda\Im_{s_{nm}}) = \lambda\text{q} - \text{ROF}S_{ft}\text{OWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}).$$

Proof. Proofs are straightforward and follows from $Theorem$ 4.3.1.5.

4.3.3. q-Rung orthopair fuzzy soft hybrid averaging operators

From the above analyses of q-ROFS_{ft}WA and q-ROFS_{ft}OWA operators, it is clear that q-ROFS_{ft}WA operator just weighed the values of q-ROFS_{ft}V, while q-ROFS_{ft}OWA operator weight the ordered positions via score function of the q-ROFS_{ft} values rather than weighting the q-ROFS_{ft} values themselves. So, it is clear that weights denotes distinct attributes in both q-ROFS_{ft}WA and q-ROFS_{ft}OWA operators. However, at the same time both the operators weigh only one of them. Therefore, here we will present the detail study of q-ROFS_{ft}HA operator which measure both q-ROFS_{ft} values and its order position at the same time and also studied their related properties in detail.

4.3.3.1. Definition

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(for\ i = 1,2,\ldots,n\ and\ j = 1,2,\ldots,m)$, be the collections of q-ROFS_{ft}Vs, and consider the weight vectors $\bar{\bar{w}} = \{\bar{\bar{w}}_1, \bar{\bar{w}}_2, \ldots, \bar{\bar{w}}_n\}\ and\ \bar{\bar{u}} = \{\bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots, \bar{\bar{u}}_m\}$ for the experts $\mathscr{k}_i$ and for the parameters $s_j's$ respectively; and having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1\ and\ \sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the mapping for q-ROFS_{ft}HA operator is defined as; $\mathrm{q-ROF}S_{ft}\mathrm{HA}: \mathrm{X}^n \to X$, (where X is the collections of all q-ROFS_{ft}Vs):

$$\mathrm{q-ROF}S_{ft}\mathrm{HA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{nm}}) = \oplus_{j=1}^{m} \bar{\bar{u}}_j \left(\oplus_{i=1}^{n} \bar{\bar{w}}_i \tilde{\mathfrak{J}}_{s_{ij}}\right).$$

The aggregated result for q-ROFS_{ft}HA operator is described in the following Theorem 4.3.3.2.

4.3.3.2. Theorem

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $(for\ i = 1,2,\ldots,n\ and\ j = 1,2,\ldots,m)$, with $\bar{\bar{v}} = (v_1, v_2, \ldots, v_n)^T$ and $\mathfrak{r} = (\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_m)^T$ are the weight vectors of $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, such that $v_i, \mathfrak{r}_j \in [0,1]$ with $\sum_{i=1}^{n} v_i = 1\ and\ \sum_{j=1}^{m} \mathfrak{r}_j = 1$ and n is the known as balancing coefficient represents the number of elements in $i^{th}\ row\ and\ j^{th}\ column$ with aggregation associated vectors $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \ldots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots, \bar{\bar{u}}_m)^T$ for the experts $\mathscr{k}_i$ and for the parameters $s_j's$ respectively, such that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1\ and\ \sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the aggregated result for q-ROFS_{ft}HA operator is given as:

$$\mathrm{q-ROF}S_{ft}\mathrm{HA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{nm}}) = \oplus_{j=1}^{m} \bar{\bar{u}}_j \left(\oplus_{i=1}^{n} \bar{\bar{w}}_i \tilde{\mathfrak{J}}_{s_{ij}}\right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \tilde{\mu}_{ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \tilde{\psi}_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right), \tag{4.8}$$

where $\tilde{\mathfrak{J}}_{s_{ij}} = n v_i \mathfrak{r}_j \mathfrak{J}_{s_{ij}}$, represents the permutation of $i^{th}\ and\ j^{th}$ largest object of the collections of $i \times j$ q-ROFS_{ft}Vs $\tilde{\mathfrak{J}}_{s_{ij}} = (\tilde{\mu}_{ij}, \tilde{\psi}_{ij})$.

Proof: Proof is similar to Theorem 4.3.1.2.

4.3.3.3. Remark

a. If the value of q is fixed, that is $q = 1$, then the developed q-ROFS_{ft}HA operator reduces to IFS_{ft}HA operator.
b. If the value of q is fixed, that is $q = 2$, then the proposed q-ROFS_{ft}HA operator reduces to PyFS_{ft}HA operator.
c. If there is only one parameter, that is s_1 (means $m = 1$), then the proposed q-ROFS_{ft}HA operator reduces to q-ROFHA operator.
d. If $v\mathfrak{r} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then the proposed q-ROFS_{ft}HA operator reduces to q-ROFS_{ft}WA operator.
e. If $\bar{\bar{w}}\bar{\bar{u}} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then the proposed q-ROFS_{ft}HA operator reduces to q-ROFS_{ft}OWA operator.

Hence from Remark 4.3.3.3, it is clear that IFS_{ft}HA, PyFS_{ft}HA, q-ROFHA, q-ROFS_{ft}WA and q-ROFS_{ft}OWA operators are the special cases of the proposed q-ROFS_{ft}HA operator.

4.3.3.4. Example

Consider the collections q-ROFS_{ft}Vs $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ as given in Table 4.2, of Example 4.3.1.4, with $v = (0.26,0.22,0.1,0.27,0.15)^T$ and $\mathfrak{r} = (0.23,0.28,0.2,0.29)^T$ be the weight vectors of them, and having associated aggregate vectors $\bar{\bar{w}} = (0.27,0.18,0.1,0.18,0.27)^T$ and $\bar{\bar{u}} = (0.26,0.24,0.24,0.26)^T$. Then by using operation laws as given in $Eq.(4.9)$ and their score results are given in Table 4.4. The corresponding q-ROFS_{ft}Vs $\tilde{\mathfrak{I}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{I}_{s_{ij}}$, of the permutation of i^{th} and j^{th} largest object of the collections of $i \times j$ q-ROFS_fVs $\tilde{\mathfrak{I}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{I}_{s_{ij}}$, is given in Table 4.5. Since

$$\tilde{\mathfrak{I}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{I}_{s_{ij}} = \left(\sqrt[3]{1 - \left(1 - \mu_{ij}^3\right)^{nv_i\mathfrak{r}_j}}, \psi^{nv_i\mathfrak{r}_j}\right) \tag{4.9}$$

Table 4.4, Tabular representation of score values of q-ROFS_{ft}Vs $\tilde{\mathfrak{I}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{I}_{s_{ij}}$ for q ≥ 3

	s_1	s_2	s_3	s_4
k_1	-0.16007	-0.15175	-0.29021	0.195882
k_2	0.13525	-0.19738	-0.45464	0.027663
k_3	-0.59745	-0.46069	-0.65469	-0.71362
k_4	-0.49017	-0.10964	0.092527	0.276974
k_5	-0.59292	-0.27435	-0.52888	0.048367

Table 4.5, Tabular representation of q-ROFS_{ft}Vs $\widetilde{\mathfrak{I}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{I}_{s_{ij}}$ for q ≥ 3

	s_1	s_2	s_3	s_4
k_1	(0.7495,0.6655)	(0.6711, 0.7395)	(0.7089, 0.6476)	(0.7369, 0.5325)
k_2	(0.6617, 0.7604)	(0.6125, 0.7164)	(0.4311, 0.6934)	(0.7716, 0.6508)
k_3	(0.5389, 0.8533)	(0.5375, 0.6912)	(0.5007, 0.8174)	(0.8642, 0.8406)
k_4	(0.5059, 0.8873)	(0.6213, 0.7923)	(0.4515, 0.8348)	(0.6621, 0.6426)
k_5	(0.5299, 0.9)	(0.5508, 0.8449)	(0.4581, 0.8977)	(0.3895, 0.9044)

Now by using $Eq. 4.8$, of $Theorem$ 4.3.3.2,

$$\text{q} - \text{ROF}S_{ft}\text{HA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) = \oplus_{j=1}^{m} \bar{\bar{u}}_j \left(\oplus_{i=1}^{n} \bar{\bar{w}}_i \widetilde{\mathfrak{I}}_{s_{ij}}\right)$$

$$= \left(\sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \tilde{\mu}_{ij}^{\,q})^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \tilde{\psi}_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right) = (0.635733, 0.753867)$$

In the following, in view of Theorem 6, some properties of the developed q-ROFS_{ft}HA operator for the collections of q-ROFS_{ft}Vs $\mathfrak{I}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, is being presented.

4.3.3.5. Theorem

Suppose the collections of q-ROFS_{ft}Vs $\mathfrak{I}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, with $v = (v_1, v_2, \dots, v_n)^T$ and $\mathfrak{r} = (\mathfrak{r}_1, \mathfrak{r}_2, \dots, \mathfrak{r}_m)^T$ be the weight vectors of them, such that $v_i, \mathfrak{r}_j \in [0,1]$ with $\sum_{i=1}^{n} v_i = 1$ and $\sum_{j=1}^{m} \mathfrak{r}_j = 1$ and n is the known as balancing coefficient represent the number of elements in i^{th} row and j^{th} $column$. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ be the aggregate associated weight vectors for the experts k_i and for the parameters $s_j's$ respectively, such that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the q-ROF$S_{ft}$HA operator satisfy the following properties:

i: (***Idempotency***): If $\mathfrak{I}_{s_{ij}} = \tilde{\mathcal{E}}_s$, $(\forall\, i = 1,2, \dots, n$ and $j = 1,2, \dots, m)$, where $\tilde{\mathcal{E}}_s = nv_i\mathfrak{r}_j\mathcal{E}_s$, then

$$\text{q} - \text{ROF}S_{ft}\text{HA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) = \tilde{\mathcal{E}}_s.$$

ii: (***Boundedness***): If $\widetilde{\mathfrak{I}}_{s_{ij}}^{-} = \left(\min_j \min_i\{\tilde{\mu}_{ij}\}, \max_j \max_i\{\tilde{\psi}_{ij}\}\right)$ and

$\widetilde{\mathfrak{I}}_{s_{ij}}^{+} = \left(\max_j \max_i\{\tilde{\mu}_{ij}\}, \min_j \min_i\{\tilde{\psi}_{ij}\}\right)$, then

$$\widetilde{\mathfrak{I}}_{s_{ij}}^{-} \leq \text{q} - \text{ROF}S_{ft}\text{HA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) \leq \widetilde{\mathfrak{I}}_{s_{ij}}^{+}.$$

iii: (***Monotonicity***): If $\mathcal{E}_{s_{ij}} = (b_{ij}, d_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, be the another collection of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij}$ and $\psi_{ij} \geq d_{ij}$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HA}(\Im_{s_{11}},\Im_{s_{12}},\dots,\Im_{s_{nm}}) \leq \mathrm{q-ROF}S_{ft}\mathrm{HA}(\mathcal{E}_{s_{11}},\mathcal{E}_{s_{12}},\dots,\mathcal{E}_{s_{nm}}).$$

$\boldsymbol{iv}$: ($\boldsymbol{Shift\ Invariance}$): If $\mathcal{E}_s = (b, d)$, is another q-ROFS_{ft}V, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HA}(\Im_{s_{11}}\oplus\mathcal{E}_s,\Im_{s_{12}}\oplus\mathcal{E}_s,\dots,\Im_{s_{nm}}\oplus\mathcal{E}_s) = \mathrm{q-ROF}S_{ft}\mathrm{HA}(\Im_{s_{11}},\Im_{s_{12}},\dots,\Im_{s_{nm}})\oplus\mathcal{E}_s.$$

$\boldsymbol{iv}$: ($\boldsymbol{Homogeneity}$): If λ is any real number such that $\lambda \geq 0$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HA}(\lambda\Im_{s_{11}},\lambda\Im_{s_{12}},\dots,\lambda\Im_{s_{nm}}) = \lambda\mathrm{q-ROF}S_{ft}\mathrm{HA}(\Im_{s_{11}},\Im_{s_{12}},\dots,\Im_{s_{nm}}).$$

Proof. Proofs are straightforward and follows from $Theorem$ 4.3.1.5.

As aggregation operators are used to create a framework for $\mathcal{MCDM}$ problems. So, in coming section we will present the application for the proposed aggregation operators.

4.4. Model for $\mathcal{MCDM}$ under q-rung orthopair fuzzy soft information

Decision making is a pre-planned process of selecting the logical choice among several objects. $\mathcal{DM}$ plays an important role in real life situation. A good decision can change the course of our lives. An intelligent decision maker judges the limitations, advantages and characteristics of each alternatives and then he could reaches to the final decision. Here we will present the mathematical description of the proposed model for $\mathcal{MCDM}$ under q-ROFS_{ft} environment. The general concept and step wise algorithm for the given approach is as follows:

Let $T = \{\hbar_1, \hbar_2, \dots, \hbar_l\}$ be the set of l different alternatives, which is assessed by n senior experts $\mathfrak{D}_1, \mathfrak{D}_2, \dots, \mathfrak{D}_n$ and let $\mathbb{E} = \{s_1, s_2, \dots, s_m\}$ be the corresponding set m parameters. A team of n senior experts has been constituted to evaluate each alternative $\hbar_e$ $(e = 1,2,\dots,l)$ according to their corresponding parameter s_j $(j = 1,2,\dots,m)$. Assume that the committee of experts provide their assessment in terms of q-ROFS_{ft}Vs $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ with weight vectors $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ for the experts $\hbar_i$ and for the parameters s_j respectively, such that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n}\bar{\bar{w}}_i = 1\ and\ \sum_{j=1}^{m}\bar{\bar{u}}_j = 1$. So the collective information are expressed in a decision matrix $\mathbb{M} = [\Im_{\hbar_{ij}}]_{n\times\boldsymbol{m}}$. In ordered to use the assessments of senior experts, the aggregated q-ROFS_{ft}V ξ_e for alternative $\hbar_e$ $(e = 1,2,\dots,l)$ is given as $\xi_e = (\mu_e, \psi_e)$ by applying the proposed aggregation operators. Finally determine the score function for overall aggregated q-ROFS_{ft}Vs $\Im_{\hbar_e}$ $(e = 1,2,\dots,l)$ for the alternatives and rank them in a specific order to get the best optimal solution.

4.4.1. Algorithm

In the following, the step wise algorithm for solving $\mathcal{MCDM}$ problems with the help of proposed operators consists of the following steps.

Step 1. Collect the expert's assessment information for each alternative to their corresponding parameters and then construct a decision matrix $\mathbb{M} = [\Im_{\hbar_{ij}}]_{n\times\boldsymbol{m}}$ as:

$$\mathbb{M} = \begin{bmatrix} (\mu_{11}, \psi_{11}) & (\mu_{12}, \psi_{12}) & \cdots & (\mu_{1m}, \psi_{1m}) \\ (\mu_{21}, \psi_{21}) & (\mu_{22}, \psi_{22}) & \cdots & (\mu_{2m}, \psi_{2m}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{1m}, \psi_{1m}) & (\mu_{2m}, \psi_{2m}) & \cdots & (\mu_{nm}, \psi_{nm}) \end{bmatrix}$$

Step 2. Normalize the q-ROFS_{ft} decision matrix $\mathbb{M} = [\Im_{\hbar_{ij}}]_{n\times \boldsymbol{m}}$ by changing assessment value of cost parameter into benefit parameter if there is any, by using the formula from [91] that is,

$$p_{ij} = \begin{cases} \Im^{c}_{\hbar_{ij}} \ ; \ for\ cost\ type\ parameter \\ \Im_{\hbar_{ij}} \ ; \ for\ benefit\ type\ parameter' \end{cases}$$

where $\Im^{c}_{\hbar_{ij}} = (\psi_{ij}, \mu_{ij})$ represents the complement of $\Im_{\hbar_{ij}} = (\mu_{ij}, \psi_{ij})$.

Step 3. By applying the proposed aggregation operators aggregate the q-ROFS_{ft}Vs $\Im_{\hbar_{ij}} = (\mu_{ij}, \psi_{ij})$ for each alternative $\hbar_e$ $(e = 1,2, \dots, l)$ into collective decision matrix ξ_e.

Step 4. Calculate the score value for ξ_e by using Definition 9, of the overall alternative $\hbar_e$ $(e = 1,2, \dots, l)$.

Step 5. Arrange the ranking result in a specific order for alternative $\hbar_e$ $(e = 1,2, \dots, l)$ and chose the best optimal result.

The flow chart of above algorithm for q-ROFS_{ft}WA is given in Fig. 4.1.

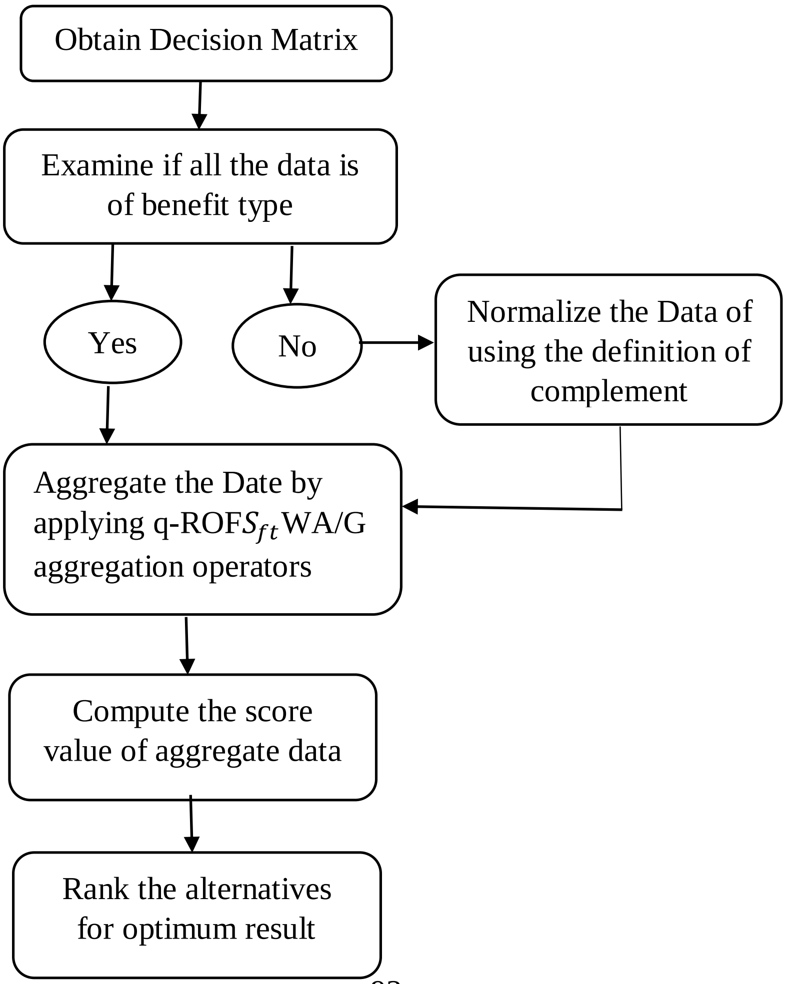

Fig. 4.1. flow chart for q-ROFS_{ft}WA and q-ROFS_{ft}WG

4.5. A Numerical example of the proposed model to $\mathcal{MCDM}$

This section is devoted for the presentation of an illustrative example to demonstrate the effectiveness and validity of the developed model with q-ROF soft information.

Consider a team of experts consist of five senior doctors $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3, \mathfrak{D}_4$ and $\mathfrak{D}_5$, whose weight vectors $\overline{\overline{w}} = (0.18,0.24,0.21,0.15,0.22)^T$, will present their evaluation for four different patients $\mathcal{k}_1, \mathcal{k}_2, \mathcal{k}_3$ and $\mathcal{k}_4$ based on the constraint of parameters set $\mathbb{E} = \{s_1 = chest\,pain, s_2 = fever, s_3 = cough, s_4 = fatigue, s_5 = vomit\}$ having weight vector $\overline{\overline{u}} = (0.26,0.22,0.1,0.27,0.15)^T$. The doctors present their evaluation for each alternative to their corresponding symptom in the form of q-ROFS_{ft}Vs. Now we apply the step wise algorithm of the proposed model to diagnose the illness of desirable patients.

$$\boldsymbol{By\ using\ q - ROFS_{ft}WA\ operator}$$

Step 1. The doctors present their evaluation for the illness of each alternative (patient) to their corresponding symptoms (parameters) in the form of q-ROFS_{ft}Vs, which is given in Tables $4.6 - 4.9$ respectively.

Step 2. There is no need to normalize the given q-ROF soft matrix because all the parameters of the same type.

Step 3. The experts/doctors evaluation for each patient $\mathcal{k}_i$ $(i = 1,2,3,4)$ is aggregated by applying the $Eq.4.1$, $for\ q = 3$, so we have

$$\xi_1 = (0.715197, 0.188439), \quad \xi_2 = (0.745295,0.189273),$$
$$\xi_3 = (0.775728,0.164921), \quad \xi_4 = (0.754479,0.158639)$$

Step 4. Calculate the score value by using Definition 4.2.4, for each aggregated value ξ_i $(i = 1,2,3,4)$ in **Step 3**, that is

$$\mathcal{S}c(\xi_1) = 0.414877, \quad \mathcal{S}c(\xi_2) = 0.46537, \quad \mathcal{S}c(\xi_3) = 0.522354,$$
$$\mathcal{S}c(\xi_4) = 0.484856$$

Step 5. Finally rank the results in descending order to get the best optimal result. Hence from the score values, we get the ranking result as:

$$\mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_4) > \mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_1)$$

Therefore, form overall analysis of the experts, it is observed that patient $\mathcal{k}_3$ has more serious illness than the others patients.

$$\boldsymbol{For\ q - ROFS_{ft}OWA\ operator}$$

Step 1. Same as above.

Step 2. Same as above.

Step 3. The experts/doctors evaluation for each patient $\mathcal{k}_i$ $(i = 1,2,3,4)$ is aggregated by applying the $Eq.4.7$, so we have

$$\xi_1 = (0.722349, 0.186069), \quad \xi_2 = (0.750879,0.185528),$$
$$\xi_3 = (0.775689,0.168362), \quad \xi_4 = (0.75392,0.161782)$$

Step 4. Calculate the score value by using Definition 4.2.4, for each aggregated value ξ_i $(i = 1,2,3,4)$ in **Step 3**, that is

$$\mathcal{S}c(\xi_1) = 0.426939, \quad \mathcal{S}c(\xi_2) = 0.475573, \quad \mathcal{S}c(\xi_3) = 0.521928, \quad \mathcal{S}c(\xi_4) = 0.483572$$

Step 5. Finally rank the results in descending order to get the best optimal result. Hence from the score values, we get the ranking result as:

$$\mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_4) > \mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_1)$$

Therefore, form overall analysis of the experts, it is observed that for q-ROFS_{ft}OWA operator the best optimal solution is again same as q-ROFS_{ft}WA. Hence patient $\mathcal{k}_3$ has more serious illness than the others patients.

$$\boldsymbol{For\ q-ROFS_{ft}HA\ operator}$$

Step 1. Same as above.

Step 2. Same as above.

Step 3. The experts/doctors evaluation for each patient $\mathcal{k}_i$ $(i = 1,2,3,4)$ is aggregated by applying the $Eq. 4.8$, with $\mathfrak{u} = (0.15,0.2,0.17,0.3,0.18)^T$ and $\mathfrak{r} = (0.16,0.21,0.13,0.26,\ 0.24)^T$ be the weight vectors of $\Im_{\mathcal{k}_{ij}} = (\mu_{ij}, \psi_{ij})$, and n is the balancing coefficient represents the number of elements in i^{th} $row\ and$ j^{th} $column.$ Let $\bar{\bar{w}} = (0.18,0.24,0.21,\ 0.15,0.22)^T$ and $\bar{\bar{u}} = (0.26,0.22,0.1,\ 0.27,0.15)^T$ be the aggregate associated weight vectors for the experts $\mathcal{k}_i$ and for the parameters $s_j{}'s$ respectively, so we have

$$\xi_1 = (0.457099, 0.709743), \quad \xi_2 = (0.478573,0.708384),$$
$$\xi_3 = (0.489985,0.696883), \quad \xi_4 = (0.474105,0.692815)$$

Step 4. Calculate the score value by using Definition 4.2.4, for each aggregated value ξ_i $(i = 1,2,3,4)$ in **Step 3**, that is

$$\mathcal{S}c(\xi_1) = -0.29764,\ \mathcal{S}c(\xi_2) = -0.27858,\ \mathcal{S}c(\xi_3) = -0.2507,\ \mathcal{S}c(\xi_4) = -0.25753$$

Step 5. Finally rank the results in descending order to get the best optimal result. Hence from the score values, we get the ranking result as:

$$\mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_4) > \mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_1)$$

Therefore, form overall analysis of the experts, it is observed that for q-ROFS_{ft}HA operator the best optimal solution is again same like q-ROFS_{ft}WA and q-ROFS_{ft}OWA. Hence patient $\mathcal{k}_3$ has more serious illness than the others patients.

Table 4.6, q-ROFS_{ft} matrix for patient $\mathcal{k}_1$

	s_1 = Chest pain	s_2 = Fever	s_3 = Cough	s_4 = Fatigue	s_5 = Vomit
$\mathfrak{D}_1$	$(0.7,0.25)$	$(0.7,0.22)$	$(0.88,0.1)$	$(0.9,0.1)$	$(0.73,0.2)$
$\mathfrak{D}_2$	$(0.6, 0.1)$	$(0.6,0.13)$	$(0.85,0.12)$	$(0.65,0.25)$	$(0.81,0.18)$
$\mathfrak{D}_3$	$(0.54,0.15)$	$(0.7,0.2)$	$(0.75,0.24)$	$(0.68,0.25)$	$(0.6,0.26)$
$\mathfrak{D}_4$	$(0.65,0.2)$	$(0.8,0.18)$	$(0.85,0.13)$	$(0.8,0.18)$	$(0.7,0.28)$
$\mathfrak{D}_5$	$(0.6,0.3)$	$(0.75,0.18)$	$(0.67,0.25)$	$(0.6,0.3)$	$(0.45,0.15)$

Table 4.7, q-ROFS_{ft} matrix for patient $\mathcal{k}_2$

	s_1 = Chest pain	s_2 = Fever	s_3 = Cough	s_4 = Fatigue	s_5 = Vomit
$\mathfrak{D}_1$	$(0.8,0.15)$	$(0.75,0.22)$	$(0.76,0.1)$	$(0.8,0.19)$	$(0.7,0.25)$
$\mathfrak{D}_2$	$(0.75, 0.18)$	$(0.8,0.15)$	$(0.8,0.18)$	$(0.5,0.25)$	$(0.8,0.16)$
$\mathfrak{D}_3$	$(0.78,0.13)$	$(0.7,0.2)$	$(0.7,0.25)$	$(0.76,0.21)$	$(0.76,0.23)$
$\mathfrak{D}_4$	$(0.9,0.1)$	$(0.65,0.33)$	$(0.76,0.15)$	$(0.87,0.12)$	$(0.65,0.18)$
$\mathfrak{D}_5$	$(0.65,0.3)$	$(0.55,0.2)$	$(0.6,0.3)$	$(0.7,0.23)$	$(0.55,0.15)$

Table 4.8, q-ROFS_{ft} matrix for patient $\mathcal{k}_3$

	s_1 = Chest pain	s_2 = Fever	s_3 = Cough	s_4 = Fatigue	s_5 = Vomit
$\mathfrak{D}_1$	$(0.71,0.25)$	$(0.78,0.1)$	$(0.88,0.11)$	$(0.81,0.18)$	$(0.78,0.2)$
$\mathfrak{D}_2$	$(0.8,0.15)$	$(0.85,0.12)$	$(0.9,0.1)$	$(0.65,0.25)$	$(0.74,0.23)$
$\mathfrak{D}_3$	$(0.76,0.1)$	$(0.88,0.11)$	$(0.84,0.12)$	$(0.86,0.1)$	$(0.79,0.2)$
$\mathfrak{D}_4$	$(0.78,0.22)$	$(0.75,0.25)$	$(0.74,0.2)$	$(0.75,0.25)$	$(0.65,0.16)$
$\mathfrak{D}_5$	$(0.6,0.25)$	$(0.8,0.19)$	$(0.75,0.16)$	$(0.6,0.2)$	$(0.5,0.1)$

Table 4.9, q-ROFS_{ft} matrix for patient $\mathscr{k}_4$

	$s_1 =$ Chest pain	$s_2 =$ Fever	$s_3 =$ Cough	$s_4 =$ Fatigue	$s_5 =$Vomit
$\mathfrak{D}_1$	(0.76,0.22)	(0.75,0.22)	(0.85,0.14)	(0.78,0.2)	(0.65,0.26)
$\mathfrak{D}_2$	(0.72,0.12)	(0.79,0.18)	(0.6,0.12)	(0.73,0.15)	(0.8,0.14)
$\mathfrak{D}_3$	(0.82,0.16)	(0.83,0.1)	(0.84,0.13)	(0.82,0.12)	(0.77,0.2)
$\mathfrak{D}_4$	(0.6,0.27)	(0.6,0.3)	(0.7,0.2)	(0.83,0.13)	(0.6,0.25)
$\mathfrak{D}_5$	(0.55,0.1)	(0.81,0.12)	(0.8,0.15)	(0.72,0.17)	(0.5,0.15)

4.5.1. Comparative study

To show the superiority and influence of proposed model, a comparative study has been presented of the proposed model with some existing literature, based on different aggregation operators (see [4, 30, 58]). If we assign the value to $\mathcal{MG}$ 0.9 and $\mathcal{NMG}$ 0.5, then their sum $0.9 + 0.5 > 1$. So in this case the methods presented in [4, 58] will fail to cope the situation. Similarly, if we consider Tables 4.6 to 4.9, then the methods developed in [4, 30] will also fail to tackle the situation, and the developed approach cope all these situations. For this, different parameters of q-ROF soft numbers are aggregated by applying weighted averaging operator having weight vectors $\bar{\bar{u}} = (0.26,0.22,0.1,0.27,0.15)^T$, and got the q-ROF soft matrix for different candidates $\mathscr{k}_i$ $(i = 1,2,3,4)$ as summarized in Table 4.10. Then based on this evaluated matrix a comparative analysis of different aggregation operators has been presented, and their corresponding results for each candidates are given in Table 4.11. From this table, it is clear that patient $\mathscr{k}_3$ has more illness diagnosed by the expert doctors. The characteristic analysis of the developed approach with some existing studies is given in Table 4.12. So, from Table 4.12, it is clear that the methods given in [4, 30] have no information about parameter study. The advantages of the developed concept is that they have the ability to solve the real life problems by using their parameterizations properties. Hence, the developed concept can be utilized for solving the $\mathcal{DM}$ problems rather than other existing operators in q-ROFS_{ft} environment.

Table 4.10, Aggregated values of q-ROFS_{ft} matrix for patients

	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$	$\mathscr{k}_4$
$\mathfrak{D}_1$	(0.5137,0.1660)	(0.4413,0.1751)	(0.5180,0.1531)	(0.4527,0.2046)
$\mathfrak{D}_2$	(0.4224,0.1445)	(0.4508,0.1764)	(0.5481,0.1459)	(0.4114,0.1415)
$\mathfrak{D}_3$	(0.3308,0.2162)	(0.4047,0.2015)	(0.5803,0.1238)	(0.5465,0.1377)
$\mathfrak{D}_4$	(0.4768,0.1888)	(0.4859,0.1696)	(0.3991,0.2113)	(0.3101,0.2291)
$\mathfrak{D}_5$	(0.2881,0.2193)	(0.2256,0.2246)	(0.3276,0.1685)	(0.3784,0.1347)

Table 4.11. Comparative Studies of different methods

Methods	*Score values of patients*				Ranking
	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$	$\mathscr{k}_4$	
IFWA [4]	0.258398,	0.241765,	0.31298,	0.232224	$\xi_3 > \xi_1 > \xi_2 > \xi_4$
IFOWA [4]	0.23115,	0.23056,	0.329916,	0.277871	$\xi_3 > \xi_4 > \xi_1 > \xi_2$
IFHA [4]	0.26103,	0.273152,	0.337673,	0.246599	$\xi_3 > \xi_2 > \xi_1 > \xi_4$
IFS_{ft}WA [58]	0.516859,	0.548324,	0.604673,	0.590214	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFS_{ft}WA (proposed)	0.522097,	0.565965,	0.621904,	0.593809	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFS_{ft}OWA (proposed)	0.532526,	0.575719,	0.621094,	0.591066	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFS_{ft}HA (proposed)	-0.39452,	-0.37378,	-0.34634,	-0.34975	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFWA [30]	0.083494,	0.076265,	0.111612,	0.067345	$\xi_3 > \xi_1 > \xi_2 > \xi_4$
q-ROFS_{ft}WA (proposed)	0.414877,	0.46537,	0.522354,	0.484856	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}OWA(proposed)	0.426939,	0.475573,	0.521928,	0.483572	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}HA (proposed)	-0.29764,	-0.27858,	-0.2507,	-0.25753	$\xi_3 > \xi_4 > \xi_2 > \xi_1$

Table 4.12. Characteristic analysis of different models

	Fuzzy information	*Aggragate parameter informatiom*
IFWA [4]	*Yes*	*No*
IFOWA [4]	*Yes*	*No*
IFHA [4]	*Yes*	*No*
IFS_{ft}WA [58]	*Yes*	*Yes*
q-ROFWA [30]	*Yes*	*No*
Proposed Operators	*Yes*	*Yes*

4.5.2. Conclusion

Decision making is a pre-planned process of selecting the logical choice among several objects. Therefore, $\mathcal{DM}$ plays a significant role in real life situation. In this manuscript we have presented the hybrid study of S_{ft}S and q-ROFS to get new concept of q-ROFS_{ft}S, which provides a general framework for mathematical problems by affix parameterization tools during the analysis as compared to other method. Based on this concept we have established the aggregation operators that are q-ROFS_{ft}WA, q-ROFS_{ft}OWA and q-ROFS_{ft}HA and also studied their corresponding operational laws in q-ROFS_{ft} environment. Furthermore, we have investigated their desirable properties in detail. Based on proposed model a medical $\mathcal{DM}$ problem has been presented under the q-ROFS_{ft} environment. Then we have shown the justification of the developed approach with some existing methods and a characteristic analysis showing the influence and superiority of the developed method than the existing literature. The advantages of the developed concepts are that they have the ability to solve the real life problems by using their parameterizations properties. Hence, the developed concept can be utilized for solving the $\mathcal{DM}$ problems rather than other existing operators in q-ROFS_{ft} environment.

Chapter 5

Orthopair fuzzy soft geometric aggregation operators

Hussain et al. [64] presented the combined study of S_{ft}S and q-ROFS to get the new notion called q-ROFS_{ft}S. The notion of q-ROFS_{ft}S is free from that inherited complexities which suffered the contemporary theories because parameterization tool is the most significant character of q-ROFS_{ft}S. In this chapter our main contribution is to originate the concept of q-ROFS_{ft}WG, q-ROFS_{ft}OWG and q-ROFS_{ft}HG operators in q-ROFS_{ft} environment. Moreover, some dominant properties such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity of these developed operators are studied in detail. Based on these proposed approaches, a model is built up for $\mathcal{MCDM}$ and its algorithm has being presented. Finally, utilizing the developed approach an illustrative example is solved under q-ROFS_{ft} environment. Further a comparative analysis of the investigated models with existing methods are presented in detail which shows the competence and ability of the developed models.

5.1. q-Rung orthopair fuzzy soft set

In this section, a detail study of the hybrid model of S_{ft}S and q-ROFS that is q-ROFS_{ft}S is being presented and their desirable operations are discussed in detail.

5.1.1. **Definition** [64]

Suppose a soft set $(\mathcal{H}, \mathbb{E})$ over a universal set T and a pair $(\mathcal{T}, \mathcal{S})$ is said to be a q-ROFS_{ft}S over T, where $\mathcal{T}$ is a mapping denoted by $\mathcal{T}: \mathbb{E} \to q - ROFS^{(S)}$, which is given as:

$$\mathcal{T}_{s_j}(\mathscr{k}_i) = \{\prec \mathscr{k}_i, \mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i) \succ_q \mid \mathscr{k}_i \in T, s_j \in \mathbb{E} \; and \; q \geq 1\},$$

where $\mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i)$ denote the $\mathcal{MG}$ and $\mathcal{NMG}$ of an alternative $\mathscr{k}_i \in T$ to the set $\mathcal{T}_{s_j}$ respectively, and hold the restriction that $0 \leq \left(\mu_j(\mathscr{k}_i)\right)^q + \left(\psi_j(\mathscr{k}_i)\right)^q \leq 1$ and $q \geq 1$. For the simplicity $\mathcal{T}_{s_j}(\mathscr{k}_i) = \prec \mathscr{k}_i, \mu_j(\mathscr{k}_i), \psi_j(\mathscr{k}_i) \succ_q$, is denoted by $\mathfrak{I}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ representing a q-ROFS_{ft} number (q-ROFS_{ft}V). Further, the hesitancy degree for q-ROFS_{ft}V is given as $\pi_{\mathfrak{I}_{s_{ij}}} = \sqrt[q]{1 - \left((\mu_{ij})^q + (\psi_{ij})^q\right)}$. The set of all q-ROFS_{ft}S is denoted by q-ROF$S_{ft}S^{(T)}$.

For detail study of q-ROFS_{ft}S and its basic operations and relation see Chapter 4, Section 4.2.

On the analysis of q-ROFS_{ft}V we presented the score function which estimates the ranking between two or more alternatives satisfy the desirable choice of experts.

5.1.2. Definition [64]

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ be a q-ROFS_{ft}V. Then score function for $\mathfrak{J}_{s_{ij}}$ is given as

$$\mathcal{S}c\left(\mathfrak{J}_{s_{ij}}\right) = \mu_{ij}^q - \psi_{ij}^q + \left(\frac{e^{\mu_{ij}^q - \psi_{ij}^q}}{e^{\mu_{ij}^q - \psi_{ij}^q} + 1} - \frac{1}{2}\right)\pi_{\mathfrak{J}_{s_{ij}}}^q \quad for\ q \geq 1 \quad and \quad \mathcal{S}c\left(\mathfrak{J}_{s_{ij}}\right) \in [-1,1]$$

Let $\mathfrak{J}_{s_{11}} = (\mu_{11}, \psi_{11})\ and\ \ \mathfrak{J}_{s_{12}} = (\mu_{12}, \psi_{12})$ be two q-ROFS_{ft}Ns. Then

(i) If $\mathcal{S}c(\mathfrak{J}_{s_{11}}) > \mathcal{S}c(\mathfrak{J}_{s_{12}})$, then $\mathfrak{J}_{s_{11}} \succcurlyeq \mathfrak{J}_{s_{12}}$;

(ii) If $\mathcal{S}c(\mathfrak{J}_{s_{11}}) < \mathcal{S}c(\mathfrak{J}_{s_{12}})$, then $\mathfrak{J}_{s_{11}} \preccurlyeq \mathfrak{J}_{s_{12}}$;

(iii) If $\mathcal{S}c(\mathfrak{J}_{s_{11}}) = \mathcal{S}c(\mathfrak{J}_{s_{12}})$, then

(a) If $\pi_{\mathfrak{J}_{s_{11}}} > \pi_{\mathfrak{J}_{s_{12}}}$ then $\mathfrak{J}_{s_{11}} \prec \mathfrak{J}_{s_{12}}$;

(b) If $\pi_{\mathfrak{J}_{s_{11}}} = \pi_{\mathfrak{J}_{s_{12}}}$ then$\mathfrak{J}_{s_{11}} = \mathfrak{J}_{s_{12}}$.

5.2. q-Rung orthopair fuzzy soft geometric aggregation operator

This section is allotted to the detailed study of q-ROFS_{ft}WG, q-ROFS_{ft}OWG and q-ROFS_{ft}HG operators and their basic properties have been provided in detail.

5.2.1. q-Rung orthopair fuzzy soft weighted geometric operators

This subsection, consists of the detail study of q-ROFS_{ft}WG operator and discussed their fundamental properties.

5.2.1.1. Definition

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ i = 1,2, \dots, n\ and\ j = 1,2, \dots, m,$ be the collection of q-ROFS_{ft}Vs, and suppose the weight vectors $\overline{\overline{w}} = \{\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_n\}$ for the decision makers $\mathcal{k}_i$ and $\overline{\overline{u}} = \{\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_m\}$ and for the parameters s_j' respectively; and satisfying the restrictions that $\overline{\overline{w}}_i, \overline{\overline{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \overline{\overline{w}}_i = 1\ and\ \sum_{j=1}^{m} \overline{\overline{u}}_j = 1$. Then q-ROF$S_{ft}$WG operator is a mapping denoted and defined as: $\mathrm{q-ROF}S_{ft}\mathrm{WG}: \mathrm{X}^n \rightarrow X$, (where X contains the collection of q-ROFS_{ft}Vs)

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n} \mathfrak{J}_{s_{ij}}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$

The following Theorem 5.3.1.2, describes the aggregation result for q-ROFS_{ft}WG operator.

5.2.1.2. Theorem

Suppose the collections $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ i = 1,2, \dots, n\ and\ j = 1,2, \dots, m,$ of q-ROFS_{ft}Vs. Then the aggregation result for q-ROFS_{ft}WG operator is defined as:

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n} \mathfrak{J}_{s_{ij}}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \mu_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}, \quad \sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \psi_{ij}^{q})^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}} \right), \tag{5.1}$$

where $\bar{\bar{w}} = \{\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n\}$ and be weight vector for decision makers $\bar{\bar{u}} = \{\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m\}$ be the for the parameters s_j respectively; which satisfying that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1 \ and \ \sum_{j=1}^{m} \bar{\bar{u}}_j = 1$.

Proof. By utilizing mathematical induction to prove the aggregation result of Eq. 5.1.

Consider the operation laws of q-ROFS_{ft}S, that is

$$\mathfrak{J}_{s_{11}} \otimes \mathfrak{J}_{s_{12}} = \left(\mu_{11}\mu_{12}, \sqrt[q]{(\psi_{11})^q + (\psi_{12})^q - (\psi_{11})^q(\psi_{12})^q} \right) \quad and$$
$$\mathfrak{J}^{\alpha} = \left(, \mu^{\alpha}, \sqrt[q]{1 - [1 - \psi^q]^{\alpha}} \right) \quad for\ \alpha \geq 1$$

First we will show that the $Eq.5.1$ is true for $n = 2 \ and \ m = 2$, so we have

$$\text{q} - \text{ROF}S_{ft}\text{WG}\left(\mathfrak{J}_{s_{ij}}, \mathfrak{J}_{s_{ij}}\right) = \otimes_{j=1}^{2} \left(\otimes_{i=1}^{2} \mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}$$
$$= \left(\otimes_{i=1}^{2} \mathfrak{J}_{s_{i1}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_1} \otimes \left(\otimes_{i=1}^{2} \mathfrak{J}_{s_{i2}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_2}$$
$$= \left(\mathfrak{J}_{s_{11}}^{\bar{\bar{w}}_1} \otimes \mathfrak{J}_{s_{21}}^{\bar{\bar{w}}_2} \right)^{\bar{\bar{u}}_1} \otimes \left(\mathfrak{J}_{s_{12}}^{\bar{\bar{w}}_1} \otimes \mathfrak{J}_{s_{22}}^{\bar{\bar{w}}_2} \right)^{\bar{\bar{u}}_2}$$
$$= \left\{ \begin{array}{c} \left(\mu_{11}^{\bar{\bar{w}}_1}, \sqrt[q]{1 - \left(1 - \psi_{11}^{q}\right)^{\bar{\bar{w}}_1}} \right) \otimes \\ \left(\mu_{21}^{\bar{\bar{w}}_2}, \sqrt[q]{1 - \left(1 - \psi_{21}^{q}\right)^{\bar{\bar{w}}_2}} \right) \end{array} \right\}^{\bar{\bar{u}}_1} \otimes \left\{ \begin{array}{c} \left(\mu_{12}^{\bar{\bar{w}}_1}, \sqrt[q]{1 - \left(1 - \psi_{12}^{q}\right)^{\bar{\bar{w}}_1}} \right) \otimes \\ \left(\mu_{22}^{\bar{\bar{w}}_2}, \sqrt[q]{1 - \left(1 - \psi_{22}^{q}\right)^{\bar{\bar{w}}_2}} \right) \end{array} \right\}^{\bar{\bar{u}}_2}$$
$$= \left\{ \left(\prod_{i=1}^{2} \mu_{i1}^{\bar{\bar{w}}_i}, \sqrt[q]{1 - \prod_{i=1}^{2} \left(1 - \psi_{i1}^{q}\right)^{\bar{\bar{w}}_i}} \right) \right\}^{\bar{\bar{u}}_1} \otimes \left\{ \left(\prod_{i=1}^{2} \mu_{i2}^{\bar{\bar{w}}_i}, \sqrt[q]{1 - \prod_{i=1}^{2} \left(1 - \psi_{i2}^{q}\right)^{\bar{\bar{w}}_i}} \right) \right\}^{\bar{\bar{u}}_2}$$
$$= \left(\left(\prod_{i=1}^{2} \mu_{i1}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_1}, \sqrt[q]{1 - \left(\prod_{i=1}^{2} \left(1 - \psi_{i1}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_1}} \right) \otimes$$
$$\left(\left(\prod_{i=1}^{2} \mu_{i1}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_2}, \sqrt[q]{1 - \left(\prod_{i=1}^{2} \left(1 - \psi_{i1}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_2}} \right)$$
$$= \left(\prod_{j=1}^{2} \left(\prod_{i=1}^{2} \mu_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{2} \left(\prod_{i=1}^{2} \left(1 - \psi_{ij}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}} \right)$$

Hence the result is true for $n = 2 \ and \ m = 2$, Next suppose that $Eq. 5.1$ is true for $n = k_1 \ and \ m = k_2$

$$\text{q}-\text{ROF}S_{ft}\text{WG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{k_1k_2}}\right) = \otimes_{j=1}^{k_2}\left(\otimes_{i=1}^{k_1}\mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$= \left(\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[q]{1-\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right)$$

We show that $Eq. 5.1$ is true for $n = k_1 + 1 \ and \ m = k_2 + 1$

$$\text{q}-\text{ROF}S_{ft}\text{WG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{k_1k_2}}, \mathfrak{J}_{s_{(k_1+1)(k_2+1)}}\right)$$

$$= \text{q}-\text{ROF}S_{ft}\text{WG}\left(\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{k_1k_2}}\right), \mathfrak{J}_{s_{(k_1+1)(k_2+1)}}\right)$$

$$= \left\{\otimes_{j=1}^{k_2}\left(\otimes_{i=1}^{k_1}\mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\} \otimes \left(\mathfrak{J}_{s_{(k_1+1)(k_2+1)}}^{\bar{\bar{w}}_{(k_2+1)}}\right)^{\bar{\bar{u}}_{(k_1+1)}}$$

$$= \left(\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[q]{1-\prod_{j=1}^{k_2}\left(\prod_{i=1}^{k_1}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right) \otimes \left(\mathfrak{J}_{s_{(k_1+1)(k_2+1)}}^{\bar{\bar{w}}_{(k_2+1)}}\right)^{\bar{\bar{u}}_{(k_1+1)}}$$

$$= \left(\prod_{j=1}^{(k_2+1)}\left(\prod_{i=1}^{(k_1+1)}\mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[q]{1-\prod_{j=1}^{(k_2+1)}\left(\prod_{i=1}^{(k_1+1)}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right)$$

Hence $Eq. 5.1$ is true for $n = k_1 + 1 \ and \ m = k_2 + 1$. Therefore, by induction process $Eq. 5.1$ is true for all values of $m, n \geq 1$.

Moreover, to prove the aggregated result achieved from q-ROFS_{ft}WG operator is again a q-ROFS_{ft}V. Now for any $\mathfrak{J}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$, $(i = 1, 2, \dots, n) \ and \ (j = 1, 2, \dots, m)$, where $0 \leq \mu_{ij}, \ \psi_{ij} \leq 1$, satisfying that $0 \leq \mu_{ij}^q + \psi_{ij}^q \leq 1$, with weight vectors $\bar{\bar{w}} = \{\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n\}$ and $\bar{\bar{u}} = \{\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m\}$ for the decision maker $\mathscr{k}_i$ and for the parameters s_j respectively; which satisfying that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n}\bar{\bar{w}}_i = 1 \ and \ \sum_{j=1}^{m}\bar{\bar{u}}_j = 1$.

As,

$$0 \leq \mu_{ij} \leq 1 \Rightarrow 0 \leq \prod_{i=1}^{n}\mu_{ij}^{\bar{\bar{w}}_i} \leq 1 \Rightarrow 0 \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq 1$$

Similarly, $0 \leq \psi_{ij} \leq 1 \Rightarrow 0 \leq 1 - \psi_{ij} \leq 1 \Rightarrow 0 \leq \left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i} \leq 1$

$$\Rightarrow 0 \leq \prod_{i=1}^{n}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i} \leq 1 \Rightarrow 0 \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq 1$$

$$\Rightarrow 0 \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \leq 1$$

As

$$\mu_{ij}^{q}+\psi_{ij}^{q} \leq 1 \Rightarrow \mu_{ij}^{q} \leq 1-\psi_{ij}^{q} \Rightarrow \prod_{i=1}^{n}\left(\mu_{ij}^{q}\right)^{\bar{\bar{w}}_i} \leq \left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)$$

$$\Rightarrow \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Rightarrow \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)^{q} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \tag{5.2}$$

Now we have

$$0 \leq \left\{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}^{q} + \left\{\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\}^{q}$$

by $Eq. 5.2$, wehave

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} + 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} = 1$$

Therefore,

$$0 \leq \left\{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}^{q} + \left\{\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\}^{q} \leq 1$$

Therefore, from the above analysis we observed that the aggregation result obtained from q-ROFS_{ft}WG operator is again a q-ROFS_{ft}V.

5.2.1.3. Remark

a. When rung $q = 1$, then in this case the developed q-ROFS_{ft}WG operator degenerates into IFS_{ft}WG operator.
b. When rung $q = 2$, then in this case the investigated q-ROFS_{ft}WG operator degenerates into PyFS_{ft}WG operator.
c. If the parameter set contain just one element, i.e. s_1(mean $m = 1$), then in this case the developed q-ROFS_{ft}WG operator degenerates to q-ROFWG operator.

It is clear from Remark 5.3.1.3, that IFWG, IFS_{ft}WG, PyFS_{ft}WG and q-ROFWG operators are the special cases of the developed operator.

5.2.1.4. Example

Consider a decision maker Mr. Z purchase a house in the domain set $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_4, \mathscr{k}_5\}$ and let $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ be the criterion (parameters) set, i.e. s_i $(i = 1,2,3,4)$ stands for $s_1 =$ beautiful, $s_2 =$ in green surrounding, $s_3 =$ expenxive, $s_4 =$ safety respectively. Suppose $\overline{\overline{w}} = \{0.26,0.12,0.23,0.2,0.19\}$ be the weight vectors for expert $\mathscr{k}_i$ and $\overline{\overline{u}} = \{0.26,0.21,0.29,0.24\}$ be the weight vector for parameters s_j respectively. The decision maker gives their assessment for each alternative against his parameters in the form of q-ROFS_{ft}Vs, which is given in Table 5.2.

By using $Eq. 5.1$, we have

$$\text{q} - \text{ROF}S_{ft}\text{WG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{54}}\right) = \left(\prod_{j=1}^{4}\left(\prod_{i=1}^{5}\mu_{ij}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}, \sqrt[3]{1 - \prod_{j=1}^{4}\left(\prod_{i=1}^{6}(1-\psi_{ij}^{3})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}\right)$$

Table 5.2, Tabular notation of q-ROFS_{ft}S $(T, \mathcal{S})$ for q = 3

$\mathcal{S}$	s_1	s_2	s_3	s_4
$\mathscr{k}_1$	(0.78, 0.34)	(0.86, 0.42)	(0.72, 0.26)	(0.93, 0.4)
$\mathscr{k}_2$	(0.93, 0.25)	(0.76, 0.36)	(0.87, 0.41)	(0.87, 0.5)
$\mathscr{k}_3$	(0.91, 0.24)	(0.92, 0.35)	(0.86, 0.42)	(0.77, 0.25)
$\mathscr{k}_4$	(0.75, 0.26)	(0.85, 0.34)	(0.93, 0.25)	(0.94, 0.28)
$\mathscr{k}_5$	(0.85, 0.35)	(0.94, 0.35)	(0.78, 0.3)	(0.92, 0.46)

$$= \left(\begin{matrix}\left\{\begin{matrix}(0.78^{0.26})(0.93^{0.12})(0.91^{0.23}) \\ (0.75^{0.2})(0.85^{0.19})\end{matrix}\right\}^{0.26} \left\{\begin{matrix}(0.86^{0.26})(0.76^{0.12})(0.92^{0.23}) \\ (0.85^{0.2})(0.94^{0.19})\end{matrix}\right\}^{0.21} \\ \left\{\begin{matrix}(0.72^{0.26})(0.87^{0.12})(0.86^{0.23}) \\ (0.93^{0.2})(0.78^{0.19})\end{matrix}\right\}^{0.29} \left\{\begin{matrix}(0.93^{0.26})(0.87^{0.12})(0.77^{0.23}) \\ (0.94^{0.2})(0.92^{0.19})\end{matrix}\right\}^{0.24}\end{matrix}, \right.$$

$$\left.\sqrt[3]{\begin{matrix}1 - \left\{\begin{matrix}(1-0.34^3)^{0.26}(1-0.25^3)^{0.12}(1-0.24^3)^{0.23} \\ (1-0.26^3)^{0.2}(1-0.35^3)^{0.19}\end{matrix}\right\}^{0.26} \\ \left\{\begin{matrix}(1-0.42^3)^{0.26}(1-0.36^3)^{0.12}(1-0.35^3)^{0.23} \\ (1-0.34^3)^{0.2}(1-0.35^3)^{0.19}\end{matrix}\right\}^{0.21} \\ \left\{\begin{matrix}(1-0.26^3)^{0.26}(1-0.41^3)^{0.12}(1-0.42^3)^{0.23} \\ (1-0.25^3)^{0.2}(1-0.3^3)^{0.19}\end{matrix}\right\}^{0.29} \\ \left\{\begin{matrix}(1-0.4^3)^{0.26}(1-0.5^3)^{0.12}(1-0.25^3)^{0.23} \\ (1-0.28^3)^{0.2}(1-0.46^3)^{0.19}\end{matrix}\right\}^{0.24}\end{matrix}}\right)$$

$$= (0.849189, 0.350549).$$

From the analysis of Theorem 5.3.1.2, the q-ROFS_{ft}WG operator fulfill the following properties for the collection q-ROFS_{ft}Vs $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, is being presented.

5.2.1.5. Theorem

Let $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, the collections of q-ROFS_{ft}Vs with weight vectors $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ for the decision makers $\mathscr{k}_i$ and for the parameters s_j respectively, such that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the q-ROF$S_{ft}$WG operator satisfying the following properties;

$\boldsymbol{i}$: $(\boldsymbol{Idempotency})$: If $\Im_{s_{ij}} = \mathcal{E}_s$, $(\forall\, i = 1,2, \dots, n$ and $j = 1,2, \dots, m)$, where $\mathcal{E}_s = (b, d)$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) = \mathcal{E}_s.$$

$\boldsymbol{ii}$: $(\boldsymbol{Boundedness})$: If $\Im_{s_{ij}}^{-} = \left(\min_j \min_i \{\mu_{ij}\}, \max_j \max_i \{\psi_{ij}\}\right)$ and

$\Im_{s_{ij}}^{+} = \left(\max_j \max_i \{\mu_{ij}\},\ \min_j \min_i \{\psi_{ij}\}\right)$, then

$$\Im_{s_{ij}}^{-} \leq \mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \leq \Im_{s_{ij}}^{+}.$$

$\boldsymbol{iii}$: $(\boldsymbol{Monotonicity})$: If $\mathcal{E}_{s_{ij}} = (b_{ij}, d_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, be the another collection of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij}$ and $\psi_{ij} \geq d_{ij}$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \leq \mathrm{q-ROF}S_{ft}\mathrm{WG}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{nm}}).$$

$\boldsymbol{iv}$: $(\boldsymbol{Shift\ Invariance})$: If $\tilde{\mathcal{E}}_s = (b, d)$, is another q-ROF$S_{ft}$V, then

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}} \otimes \mathcal{E}_s, \Im_{s_{12}} \otimes \mathcal{E}_s, \dots, \Im_{s_{nm}} \otimes \mathcal{E}_s) = \mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \otimes \mathcal{E}_s.$$

$\boldsymbol{iv}$: $(\boldsymbol{Homogeneity})$: For a real number $\lambda > 0$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}(\lambda\Im_{s_{11}}, \lambda\Im_{s_{12}}, \dots, \lambda\Im_{s_{nm}}) = \lambda\mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}).$$

$\boldsymbol{Proof.i}$: $(\boldsymbol{Idempotency})$: As it is given that if for all $\Im_{s_{ij}} = \mathcal{E}_s = (b, d)(\forall\, i = 1,2, \dots,\ n$ and $j = 1,2, \dots, m)$, then from Theorem 5.3.1.2, we have

$$\begin{aligned}
&\mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \\
&= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n} \mu_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \psi_{ij}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right) \\
&= \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n} b^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - d^q)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right) \\
&= \left(\left(b^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j}, \sqrt[q]{1 - \left((1 - d^q)^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j}}\right) \\
&= \left(b, \sqrt[q]{1 - (1 - d^q)}\right) = (b\ , d) = \tilde{\mathcal{E}}_s
\end{aligned}$$

Therefore, $\mathrm{q-ROF}S_{ft}\mathrm{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) = \mathcal{E}_s$.

ii: (***Boundedness***): As $\Im_{s_{ij}}^{-} = \left(\min_j \min_i\{\mu_{ij}\}, \max_j \max_i\{\psi_{ij}\}\right)$ and

$\Im_{s_{ij}}^{+} = \left(\max_j \max_i\{\mu_{ij}\}, \min_j \min_i\{\psi_{ij}\}\right)$. To prove that

$$\Im_{s_{ij}}^{-} \leq \text{ q} - \text{ROF}S_{ft}\text{WG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \leq \Im_{s_{ij}}^{+},$$

Now for every $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$, we have

$$\min_j \min_i\{\mu_{ij}\} \leq \mu_{ij} \leq \max_j \max_i\{\mu_{ij}\}$$

$$\Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\min_j \min_i\{\mu_{ij}\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mu_{ij})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\max_j \max_i\{\mu_{ij}\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Leftrightarrow \left(\left(\min_j \min_i\{\mu_{ij}\}\right)^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mu_{ij})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \left(\left(\max_j \max_i\{\mu_{ij}\}\right)^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j}$$

this implies that

$$\min_j \min_i\{\mu_{ij}\} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mu_{ij})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \max_j \max_i\{\mu_{ij}\} \tag{5.3}$$

Next for each $i = 1,2, \dots, n$ and $j = 1,2, \dots, m$, we have

$$\min_j \min_i\{\psi_{ij}\} \leq \psi_{ij} \leq \max_j \max_i\{\psi_{ij}\} \Leftrightarrow 1 - \max_j \max_i\{\psi_{ij}^{q}\} \leq 1 - \psi_{ij}^{q}$$
$$\leq 1 - \min_j \min_i\{\psi_{ij}^{q}\}$$

$$\Leftrightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \max_j \max_i\{\psi_{ij}^{q}\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - \psi_{ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \min_j \min_i\{\psi_{ij}^{q}\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Leftrightarrow \left(\left(1 - \max_j \max_i\{\psi_{ij}^{q}\}\right)^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - \psi_{ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \left(\left(1 - \min_j \min_i\{\psi_{ij}^{q}\}\right)^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j}$$

$$\Leftrightarrow \left(1 - \max_j \max_i\{\psi_{ij}^{q}\}\right) \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - \psi_{ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \left(1 - \min_j \min_i\{\psi_{ij}^{q}\}\right)$$

$$\Leftrightarrow 1-\left(1-\min_j \min_i\{\psi_{ij}^q\}\right) \le 1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$
$$\le 1-\left(1-\max_j \max_i\{\psi_{ij}^q\}\right)$$

Hence

$$\min_j \min_i\{\psi_{ij}\} \le \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}$$
$$\le \max_j \max_i\{\psi_{ij}\} \qquad (5.4)$$

Therefore, from $Eqs.\,(5.3)\ and\ (5.4)$, we have

$$\min_j \min_i\{\mu_{ij}\} \le \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mu_{ij})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \le \max_j \max_i\{\mu_{ij}\}$$

and

$$\min_j \min_i\{\psi_{ij}\} \le \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \le \max_j \max_i\{\psi_{ij}\}$$

Let $\delta = \mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) = (\mu_\delta, \psi_\delta)$, then by using score function, we have

$$\mathcal{S}c(\delta) = \mu_\delta^q - \psi_\delta^q + \left(\frac{e^{\mu_\delta^q-\psi_\delta^q}}{e^{\mu_\delta^q-\psi_\delta^q}+1} - \frac{1}{2}\right)\pi_\delta^q$$
$$\le \left(\max_j \max_i\{\mu_{ij}\}\right)^q - \left(\min_j \min_i\{\psi_{ij}\}\right)^q +$$
$$+\left(\frac{e^{\left(\max_j \max_i\{\mu_{ij}\}\right)^q-\left(\min_j \min_i\{\psi_{ij}\}\right)^q}}{e^{\left(\max_j \max_i\{\mu_{ij}\}\right)^q-\left(\min_j \min_i\{\psi_{ij}\}\right)^q}+1} - \frac{1}{2}\right)\pi_{\mathfrak{I}_{s_{ij}}^+}^q = \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^+\right) \Rightarrow \mathcal{S}c(\delta)$$
$$\le \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^+\right)$$

and

$$\mathcal{S}c(\delta) = \mu_\delta^q - \psi_\delta^q + \left(\frac{e^{\mu_\delta^q-\psi_\delta^q}}{e^{\mu_\delta^q-\psi_\delta^q}+1} - \frac{1}{2}\right)\pi_\delta^q$$
$$\ge \left(\min_j \min_i\{\mu_{ij}\}\right)^q - \left(\max_j \max_i\{\psi_{ij}\}\right)^q +$$
$$+\left(\frac{e^{\left(\min_j \min_i\{\mu_{ij}\}\right)^q-\left(\max_j \max_i\{\psi_{ij}\}\right)^q}}{e^{\left(\min_j \min_i\{\mu_{ij}\}\right)^q-\left(\max_j \max_i\{\psi_{ij}\}\right)^q}+1} - \frac{1}{2}\right)\pi_{\mathfrak{I}_{s_{ij}}^-}^q = \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^-\right) \Rightarrow \mathcal{S}c(\delta)$$
$$\ge \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^-\right).$$

From the above study, the following cases arises,

$\boldsymbol{Case\ i}$: If $\mathcal{S}c(\delta) < \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^{+}\right)$ and $\mathcal{S}c(\delta) > \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^{-}\right)$, by comparing two q-ROF$S_{ft}$Vs, we get

$$\mathfrak{I}_{s_{ij}}^{-} < \mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) < \mathfrak{I}_{s_{ij}}^{+}.$$

$\boldsymbol{Case\ ii}$: If $\mathcal{S}c(\delta) = \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^{+}\right)$, that is

$$\mu_{\delta}^{q} - \psi_{\delta}^{q} + \left(\frac{e^{\mu_{\delta}^{q}-\psi_{\delta}^{q}}}{e^{\mu_{\delta}^{q}-\psi_{\delta}^{q}}+1} - \frac{1}{2}\right)\pi_{\delta}^{q} = \left(\max_{j}\max_{i}\{\mu_{ij}\}\right)^{q} - \left(\min_{j}\min_{i}\{\psi_{ij}\}\right)^{q} +$$
$$+\left(\frac{e^{\left(\max_{j}\max_{i}\{\mu_{ij}\}\right)^{q}-\left(\min_{j}\min_{i}\{\psi_{ij}\}\right)^{q}}}{e^{\left(\max_{j}\max_{i}\{\mu_{ij}\}\right)^{q}-\left(\min_{j}\min_{i}\{\psi_{ij}\}\right)^{q}}+1} - \frac{1}{2}\right)\pi_{\mathfrak{I}_{s_{ij}}^{+}}^{q},$$

then by using the above inequalities, we get

$\mu_{\delta} = \max_{j}\max_{i}\{\mu_{ij}\}$ and $\psi_{\delta} = \min_{j}\min_{i}\{\psi_{ij}\}$. Thus $\pi_{\delta}^{q} = \pi_{\mathfrak{I}_{s_{ij}}^{+}}^{q}$,

Thus from the comparison of two q-ROFS_{ft}Vs, we have

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) = \mathfrak{I}_{s_{ij}}^{+}.$$

$\boldsymbol{Case\ iii}$: If $\mathcal{S}c(\delta) = \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}^{-}\right)$, that is

$$\mu_{\delta}^{q} - \psi_{\delta}^{q} + \left(\frac{e^{\mu_{\delta}^{q}-\psi_{\delta}^{q}}}{e^{\mu_{\delta}^{q}-\psi_{\delta}^{q}}+1} - \frac{1}{2}\right)\pi_{\delta}^{q} = \left(\min_{j}\min_{i}\{\mu_{ij}\}\right)^{q} - \left(\max_{j}\max_{i}\{\psi_{ij}\}\right)^{q} +$$
$$+\left(\frac{e^{\left(\min_{j}\min_{i}\{\mu_{ij}\}\right)^{q}-\left(\max_{j}\max_{i}\{\psi_{ij}\}\right)^{q}}}{e^{\left(\min_{j}\min_{i}\{\mu_{ij}\}\right)^{q}-\left(\max_{j}\max_{i}\{\psi_{ij}\}\right)^{q}}+1} - \frac{1}{2}\right)\pi_{\mathfrak{I}_{s_{ij}}^{-}}^{q},$$

then by using the above inequalities, we get

$$\mu_{\delta} = \min_{j}\min_{i}\{\mu_{ij}\}\ and\ \psi_{\delta} = \max_{j}\max_{i}\{\psi_{ij}\}$$

Thus

$$\pi_{\delta}^{q} = \pi_{\mathfrak{I}_{s_{ij}}^{-}}^{q}$$

this implies $\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) = \mathfrak{I}_{s_{ij}}^{-}$.

Therefore, it is proved that

$$\mathfrak{I}_{s_{ij}}^{-} \leq \mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{nm}}\right) \leq \mathfrak{I}_{s_{ij}}^{+}.$$

$\boldsymbol{iii}$: ($\boldsymbol{Monotonicity}$): Since $\mu_{ij} \leq b_{ij}$ and $\psi_{ij} \geq d_{ij}$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, then this implies that

$$\mu_{ij} \leq b_{ij} \Rightarrow \left(\prod_{i=1}^{n}\left(\mu_{ij}\right)^{\bar{\bar{w}}_{i}}\right) \leq \prod_{i=1}^{n}\left(b_{ij}\right)^{\bar{\bar{w}}_{i}}$$

$$\Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(\mu_{ij})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(b_{ij})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \quad (5.5)$$

Furthermore,

$$\psi_{ij} \geq d_{ij} \Rightarrow 1 - d_{ij} \geq 1 - \psi_{ij} \Rightarrow 1 - d_{ij}^{q} \geq 1 - \psi_{ij}^{q}$$

$$\Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - d_{ij}^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \geq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - \psi_{ij}^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$

$$\Rightarrow 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - \psi_{ij}^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j} \geq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1 - d_{ij}^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$

$$\sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \psi_{ij}^{q}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}} \geq \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - d_{ij}^{q}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}} \quad (5.6)$$

Let $\delta_{\mathfrak{J}} = \text{q} - \text{ROF}\mathcal{S}_{ft}\text{WG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) = (\mu_{\delta_{\mathfrak{J}}}, \psi_{\delta_{\mathfrak{J}}})$ and

$$\delta_{\mathcal{E}} = \text{q} - \text{ROF}\mathcal{S}_{ft}\text{WG}(\mathcal{E}_{s_{11}}, \quad \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{nm}}) = (b_{\delta_{\mathcal{E}}}, d_{\delta_{\mathcal{E}}})$$

From $Eqs.$ (5.5) and (5.6), we have

$$\mu_{\delta_{\mathfrak{J}}} \leq b_{\delta_{\mathcal{E}}} \; and \; \psi_{\delta_{\mathfrak{J}}} \geq d_{\delta_{\mathcal{E}}}$$

then from a score function, we have

$$\mathcal{S}c(\delta_{\mathfrak{J}}) \leq \mathcal{S}c(\delta_{\mathcal{E}})$$

In view of above condition, the following cases arises,

Case i: If $\mathcal{S}c(\delta_{\mathfrak{J}}) < \mathcal{S}c(\delta_{\mathcal{E}})$, by comparing two q-ROF$\mathcal{S}_{ft}$Vs, we get

$$\text{q} - \text{ROF}\mathcal{S}_{ft}\text{WG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) < \text{q} - \text{ROF}\mathcal{S}_{ft}\text{WG}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{nm}}).$$

Case ii: If $\mathcal{S}c(\delta_{\mathfrak{J}}) = \mathcal{S}c(\delta_{\mathcal{E}})$, that is

$$\mathcal{S}c(\delta_{\mathfrak{J}}) = \mu_{\delta_{\mathfrak{J}}}^{q} - \psi_{\delta_{\mathfrak{J}}}^{q} + \left(\frac{e^{\mu_{\delta_{\mathfrak{J}}}^{q} - \psi_{\delta_{\mathfrak{J}}}^{q}}}{e^{\mu_{\delta_{\mathfrak{J}}}^{q} - \psi_{\delta_{\mathfrak{J}}}^{q}} + 1} - \frac{1}{2}\right)\pi_{\delta_{\mathfrak{J}}}^{q} = \mu_{\delta_{\mathcal{E}}}^{q} - \psi_{\delta_{\mathcal{E}}}^{q} + \left(\frac{e^{\mu_{\delta_{\mathcal{E}}}^{q} - \psi_{\delta_{\mathcal{E}}}^{q}}}{e^{\mu_{\delta_{\mathcal{E}}}^{q} - \psi_{\delta_{\mathcal{E}}}^{q}} + 1} - \frac{1}{2}\right)\pi_{\delta_{\mathcal{E}}}^{q} = \mathcal{S}c(\delta_{\mathcal{E}}),$$

then by above inequality, we have

$$\mu_{\delta_{\mathfrak{J}}} = b_{\delta_{\mathcal{E}}} \; and \; \psi_{\delta_{\mathfrak{J}}} = d_{\delta_{\mathcal{E}}}$$

Hence

$$\pi_{\delta_{\mathfrak{J}}}^{q}=\pi_{\delta_{\mathcal{E}}}^{q}\Longrightarrow\left(\mu_{\delta_{\mathfrak{J}}},\psi_{\delta_{\mathfrak{J}}}\right)=\left(b_{\delta_{\mathcal{E}}},d_{\delta_{\mathcal{E}}}\right)$$

Therefore, it is proved that

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\dots,\mathfrak{J}_{s_{nm}}\right)\leq\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathcal{E}_{s_{11}},\mathcal{E}_{s_{12}},\dots,\mathcal{E}_{s_{nm}}\right).$$

$\boldsymbol{iv}$: $(\boldsymbol{Shift\ Invariance})$ Since $\mathcal{E}_s=(b,d)$ and $\mathfrak{J}_{s_{ij}}=\left(\mu_{s_{ij}},\psi_{s_{ij}}\right)$ are the q-ROFS_{ft}Vs, so

$$\mathfrak{J}_{s_{11}}\otimes\mathcal{E}_s=\left(\mu_{11}b,\sqrt[q]{1-(1-\psi_{11}^{q})(1-d^{q})}\right)$$

Therefore,

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{J}_{s_{11}}\otimes\mathcal{E}_s,\mathfrak{J}_{s_{12}}\otimes\mathcal{E}_s,\dots,\mathfrak{J}_{s_{nm}}\otimes\mathcal{E}_s\right)=\otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\left(\mathfrak{J}_{s_{nm}}\otimes\mathcal{E}_s\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$
$$=\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\overline{\overline{w}}_i}\,b^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j},\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\overline{\overline{w}}_i}(1-d^{q})^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}\right)$$
$$=\left(b\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j},\sqrt[q]{1-(1-d^{q})\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}\right)$$
$$=\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j},\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}\right)\otimes(b,d)$$
$$=\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\dots,\mathfrak{J}_{s_{nm}}\right)\otimes\mathcal{E}_s$$

Thus we get required proof.

$\boldsymbol{iv}$: $(\boldsymbol{Homogeneity})$ Consider for real number $\lambda>0$ and $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ be a q-ROFS_{ft}V, then

$$\mathfrak{J}_{s_{ij}}^{\lambda}=\left(\mu_{ij}^{\lambda},\sqrt[q]{1-\left(1-\psi_{ij}^{q}\right)^{\lambda}}\right)$$

Now

$$\mathrm{q-ROF}S_{ft}\mathrm{WG}\left(\lambda\mathfrak{J}_{s_{11}},\lambda\mathfrak{J}_{s_{12}},\dots,\lambda\mathfrak{J}_{s_{nm}}\right)$$
$$=\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\mu_{ij}^{\lambda\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j},\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\psi_{ij}^{q}\right)^{\lambda\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}}\right)$$

$$= \left(\left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \mu_{ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)^{\lambda}, \sqrt[q]{1 - \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \psi_{ij}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)^{\lambda}} \right)$$

$$= \lambda \mathrm{q} - \mathrm{ROF}S_{ft}\mathrm{WG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}\right)$$

Hence, the proof is completed.

5.2.2. q-Rung orthopair fuzzy soft ordered weighted geometric operators

From the analysis of q-ROFS_{ft}WG operator, it is observed that q-ROFS_{ft}WG operator only weights the values of q-ROFS_{ft}V, while q-ROFS_{ft}OWG operator weights the ordered positions of q-ROFS_{ft}V through scoring instead of weighting the q-ROFS_{ft} values itself. So, in this subsection we will investigate the detailed study of q-ROFS_{ft}OWG operator and its related properties.

5.2.2.1. Definition

Let $\mathfrak{J}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ $(for\ i = 1,2, \dots, n\ and\ j = 1,2, \dots, m)$, be the collections of q-ROFS_{ft}Vs, and suppose the weight vectors $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ for the decision makers $\mathcal{k}_i$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ for the parameters s_j respectively, and satisfying the restrictions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1\ and\ \sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then q-ROF$S_{ft}$OWG operator is a mapping denoted and defined as: $\mathrm{q} - \mathrm{ROF}S_{ft}\mathrm{OWG}: \mathrm{X}^n \to X$, (where X contains the collections of q-ROFS_{ft}Vs)

$$\mathrm{q} - \mathrm{ROF}S_{ft}\mathrm{OWG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}\right) = \otimes_{j=1}^{m} \left(\otimes_{i=1}^{n} \mathfrak{J}_{\sigma s_{ij}}^{u_i} \right)^{z_j}.$$

The following Theorem 5.3.2.2, described the aggregation result for q-ROFS_{ft}OWA operator.

5.2.2.2. Theorem

Consider the collections $\mathfrak{J}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ $(for\ i = 1,2, \dots, n\ and\ j = 1,2, \dots, m)$, of q-ROF$S_{ft}$Vs. Then the aggregation result using q-ROFS_{ft}OWG operator is defined as:

$$\mathrm{q} - \mathrm{ROF}S_{ft}\mathrm{OWG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}\right) = \otimes_{j=1}^{m} \left(\otimes_{i=1}^{n} \mathfrak{J}_{\sigma s_{ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}$$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \mu_{\sigma ij}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} (1 - \psi_{\sigma ij}^{q})^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}} \right), \tag{5.7}$$

where $\Im_{\sigma s_{ij}} = (\mu_{\sigma ij}, \psi_{\sigma ij})$, denotes the permutations of i^{th} and j^{th} largest value of an alternative of the collections of $ith\ row\ and\ jth\ column$ of q-ROFS_{ft}Vs $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$.

Proof. Proof is easy and directly follows from Theorem 5.3.1.2.

5.2.2.3. Remark

a. When rung $q = 1$, then the investigated q-ROFS_{ft}OWG operator degenerate into IFS_{ft}OWG operator.
b. When rung $q = 2$, then the investigated q-ROFS_{ft}OWG operator degenerate into PyFS_{ft}OWG operator.
c. If the parameter set contains just one parameter that is s_1 (means $m = 1$), then the developed q-ROFS_{ft}OWG operator in this manuscript reduces to q-ROFOWG operator.

Thus from the analysis of Remark 5.3.2.3, we observed that IFS_{ft}OWG, PyFS_{ft}OWG and q-ROFOWG operators are the specially derived from the developed q-ROFS_{ft}OWG operator.

5.2.2.4. Example

Suppose that $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ be the collection q-ROFS_{ft}Vs. Take the values of q-ROFS_{ft}Vs from Table 5.2 of Example 5.3.1.4, then by utilizing score function, the tabular notations of $\Im_{\sigma s_{ij}} = (\mu_{\sigma ij}, \psi_{\sigma ij})$ is given in Table 5.3. Now by $Eq.$ (5.7), we have

$$\text{q} - \text{ROF}S_{ft}\text{OWG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) = \otimes_{j=1}^{m} \left(\otimes_{i=1}^{n} \Im_{\sigma s_{ij}}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}$$

$$= \left(\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \mu_{\sigma ij}^{\overline{\overline{w}}_i} \right)^{\overline{\overline{u}}_j}, \quad \sqrt[q]{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \psi_{\sigma ij}^{q}\right)^{\overline{\overline{w}}_i} \right)^{\overline{\overline{u}}_j}} \right)$$

Table 5.3, Tabular notation of q-ROFS_{ft}S $\Im_{\sigma s_{ij}} = (\mu_{\sigma ij}, \psi_{\sigma ij})$ for q = 3

S	s_1	s_2	s_3	s_4
k_1	(0.93, 0.25)	(0.94, 0.35)	(0.93, 0.25)	(0.94, 0.28)
k_2	(0.91, 0.24)	(0.92, 0.35)	(0.87, 0.41)	(0.93, 0.4)
k_3	(0.85, 0.35)	(0.85, 0.34)	(0.86, 0.42)	(0.92, 0.46)
k_4	(0.78, 0.34)	(0.86, 0.42)	(0.78, 0.3)	(0.87, 0.5)
k_5	(0.75, 0.26)	(0.76, 0.36)	(0.72, 0.26)	(0.77, 0.25)

$$= \begin{pmatrix} \left\{ \begin{matrix} (0.93^{0.26})(0.91^{0.12})(0.85^{0.23}) \\ (0.78^{0.2})(0.75^{0.19}) \end{matrix} \right\}^{0.26} \left\{ \begin{matrix} (0.94^{0.26})(0.92^{0.12})(0.85^{0.23}) \\ (0.86^{0.2})(0.76^{0.19}) \end{matrix} \right\}^{0.21} \\ \left\{ \begin{matrix} (0.93^{0.26})(0.87^{0.12})(0.86^{0.23}) \\ (0.78^{0.2})(0.72^{0.19}) \end{matrix} \right\}^{0.29} \left\{ \begin{matrix} (0.94^{0.26})(0.93^{0.12})(0.92^{0.23}) \\ (0.87^{0.2})(0.77^{0.19}) \end{matrix} \right\}^{0.24}, \\ \sqrt[3]{\begin{matrix} 1 - \left\{ \begin{matrix} (1-0.25^3)^{0.26}(1-0.24^3)^{0.12}(1-0.35^3)^{0.23} \\ (1-0.34^3)^{0.2}(1-0.26^3)^{0.19} \end{matrix} \right\}^{0.26} \\ \left\{ \begin{matrix} (1-0.35^3)^{0.26}(1-0.35^3)^{0.12}(1-0.34^3)^{0.23} \\ (1-0.42^3)^{0.2}(1-0.36^3)^{0.19} \end{matrix} \right\}^{0.21} \\ \left\{ \begin{matrix} (1-0.25^3)^{0.26}(1-0.41^3)^{0.12}(1-0.42^3)^{0.23} \\ (1-0.3^3)^{0.2}(1-0.26^3)^{0.19} \end{matrix} \right\}^{0.29} \\ \left\{ \begin{matrix} (1-0.28^3)^{0.26}(1-0.4^3)^{0.12}(1-0.46^3)^{0.23} \\ (1-0.5^3)^{0.2}(1-0.25^3)^{0.19} \end{matrix} \right\}^{0.24} \end{matrix}} \end{pmatrix}$$

$$= (0.854398, 0.353285)$$

From the analysis of Theorem 5.3.2.2, the q-ROFS_{ft}OWG operator fulfill the following properties for the collection q-ROFS_{ft}Vs $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, have been initiated.

5.2.2.5. Theorem

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$ be the collection of q-ROFS_{ft}Vs with weight vector $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ for experts $\mathscr{k}_i$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ be weight vector the parameters s_j respectively, such that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the q-ROF$S_{ft}$OWG operator satisfied the following:

$\boldsymbol{i}$: $(\boldsymbol{Idempotency})$: If $\mathfrak{J}_{s_{ij}} = \mathcal{E}_{\sigma s}$, $(\forall\, i = 1,2, \dots, n$ and $j = 1,2, \dots, m)$, where $\mathcal{E}_{\sigma s} = (b, d)$, then

$$\text{q} - \text{ROF}S_{ft}\text{OWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) = \mathcal{E}_{\sigma s}.$$

$\boldsymbol{ii}$: $(\boldsymbol{Boundedness})$: If $\mathfrak{J}^{-}_{\sigma s_{ij}} = \left(\min_j \min_i \{\mu_{\sigma ij}\}, \max_j \max_i \{\psi_{\sigma ij}\}\right)$ and

$\mathfrak{J}^{+}_{\sigma s_{ij}} = \left(\max_j \max_i \{\mu_{\sigma ij}\}, \min_j \min_i \{\psi_{\sigma ij}\}\right)$, then

$$\mathfrak{J}^{-}_{\sigma s_{ij}} \leq \text{q} - \text{ROF}S_{ft}\text{OWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) \leq \mathfrak{J}^{+}_{\sigma s_{ij}}.$$

$\boldsymbol{iii}$: $(\boldsymbol{Monotonicity})$: If $\mathcal{E}_{s_{ij}} = (b_{ij}, d_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, be the another collection of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij}$ and $\psi_{ij} \geq d_{ij}$, then

$$\text{q} - \text{ROF}S_{ft}\text{OWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) \leq \text{q} - \text{ROF}S_{ft}\text{OWG}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{nm}}).$$

$\boldsymbol{iv}$: $(\boldsymbol{Shift\ Invariance})$: If $\mathcal{E}_s = (b, d)$, is another q-ROFS_{ft}V, then

$$\text{q} - \text{ROF}S_{ft}\text{OWG}(\mathfrak{J}_{s_{11}} \otimes \mathcal{E}_s, \mathfrak{J}_{s_{12}} \otimes \mathcal{E}_s, \dots, \mathfrak{J}_{s_{nm}} \otimes \mathcal{E}_s) \\ = \text{q} - \text{ROF}S_{ft}\text{OWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) \otimes \mathcal{E}_s.$$

$\boldsymbol{iv}$: $(\boldsymbol{Homogeneity})$: If any $\lambda > 0$, then

$$\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{OWG}\left(\lambda\mathfrak{I}_{s_{11}},\lambda\mathfrak{I}_{s_{12}},\ldots,\lambda\mathfrak{I}_{s_{nm}}\right)=\lambda\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{OWG}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\ldots,\mathfrak{I}_{s_{nm}}\right).$$

Proof. Proofs are straightforward like Theorem 5.3.1.5.

5.2.3. q-Rung orthopair fuzzy soft hybrid geometric operators

In this subsection, we will initiate the detailed study of q-ROFS_{ft}HG operator and it is observe that q-ROFS_{ft}HG operator weights q-ROFS_{ft}Vs and its order position as well. Here we will discuss its fundamental properties of q-ROFS_{ft}HG operators such as Idempotency, Boundedness, Monotonicity, etc. with detail.

5.2.3.1. Definition

Let$\mathfrak{I}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ $(for\ i=1,2,\ldots,n\ and\ j=1,2,\ldots,m)$, be the collections of q-ROFS_{ft}Vs, and consider the weight vectors $\bar{\bar{w}}=\{\bar{\bar{w}}_1,\bar{\bar{w}}_2,\ldots,\bar{\bar{w}}_n\}\ and\ \bar{\bar{u}}=\{\bar{\bar{u}}_1,\bar{\bar{u}}_2,\ldots,\bar{\bar{u}}_m\}$ for the professional experts $\hbar_i$ and for the parameters $s_j's$ respectively; and satisfying that $\bar{\bar{w}}_i,\bar{\bar{u}}_j\in[0,1]$ with $\sum_{i=1}^{n}\bar{\bar{w}}_i=1\ and\ \sum_{j=1}^{m}\bar{\bar{u}}_j=1$. Then q-ROF$S_{ft}$HG operator is a mapping denoted and defined as; $\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{HG}:\mathrm{X}^n\rightarrow X$, (where X contains the collections of all q-ROFS_{ft}Vs)

$$\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{HG}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\ldots,\mathfrak{I}_{s_{nm}}\right)=\otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\tilde{\tilde{\mathfrak{I}}}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}.$$

Based on Definition 5.3.3.1, the following Theorem 5.3.3.2, described the aggregation result for q-ROFS_{ft}HG operator.

5.2.3.2. Theorem

Suppose the collection $\mathfrak{I}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ $(for\ i=1,2,\ldots,n\ and\ j=1,2,\ldots,m)$ of q-ROFVs, with $v=(v_1,v_2,\ldots,v_n)^T\ and\ \mathfrak{r}=(\mathfrak{r}_1,\mathfrak{r}_2,\ldots,\mathfrak{r}_m)^T$ are the weight vectors of $\mathfrak{I}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$, such that $v_i,\mathfrak{r}_j\in[0,1]$ with $\sum_{i=1}^{n}v_i=1\ and\ \sum_{j=1}^{m}\mathfrak{r}_j=1$ and n denotes the number of elements and is called the balancing coefficient in $i^{th}\ row\ and\ j^{th}\ column$ with aggregation associated vectors $\bar{\bar{w}}=(\bar{\bar{w}}_1,\bar{\bar{w}}_2,\ldots,\bar{\bar{w}}_n)^T$ and $\bar{\bar{u}}=(\bar{\bar{u}}_1,\bar{\bar{u}}_2,\ldots,\bar{\bar{u}}_m)^T$ for the decision makers $\hbar_i$ and for the parameters $s_j's$ respectively, with $\bar{\bar{w}}_i,\bar{\bar{u}}_j\in[0,1]$ such that $\sum_{i=1}^{n}\bar{\bar{w}}_i=1\ and\ \sum_{j=1}^{m}\bar{\bar{u}}_j=1$.Then the aggregated result for q-ROFS_{ft}HG operator is given as:

$$\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{HG}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\ldots,\mathfrak{I}_{s_{nm}}\right)=\otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\tilde{\tilde{\mathfrak{I}}}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$=\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\tilde{\mu}_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j},\sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}(1-\tilde{\psi}_{ij}^{q})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right),\tag{5.8}$$

where $\tilde{\tilde{\mathfrak{I}}}_{s_{ij}}=\left(\mathfrak{I}_{s_{ij}}\right)^{nv_i\mathfrak{r}_j}$, represents the largest alternative of permutation of $i^{th}\ and\ j^{th}$ of the collections of $i\times j$ q-ROFS_{ft}Ns $\tilde{\tilde{\mathfrak{I}}}_{s_{ij}}=\left(\tilde{\mu}_{ij},\tilde{\psi}_{ij}\right)$.

Proof: Proof is straightforward like Theorem 5.3.1.2.

5.2.3.3. Remark

(a) When $q = 1$, then the investigated q-ROFS_{ft}HG operator degenerates into IFS_{ft}HG operator.

(b) When $q = 2$, then in this case the investigated q-ROFS_{ft}HG operator degenerates into PyFS_{ft}HG operator.

(c) When the parameter set contains just one alternative that is s_1 (means $m = 1$), then the investigated q-ROFS_{ft}HG operator degenerates to q-ROFHG operator.

(d) When $v\mathfrak{r} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then the investigated q-ROFS_{ft}HG operator degenerates into q-ROFS_{ft}WG operator.

(e) When $\bar{\bar{w}}\bar{\bar{u}} = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then in this case the investigated q-ROFS_{ft}HG operator degenerates into q-ROFS_{ft}OWG operator.

Thus from the analysis of Remark 5.3.3.3, we analyzed that IFS_{ft}HG, PyFS_{ft}HG, q-ROFHG, q-ROFS_{ft}WG and q-ROFS_{ft}OWG operators are the special derived cases of the developed q-ROFS_{ft}HG operator.

5.2.3.4. Example

Suppose that $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ be the collection of q-ROFS_{ft}Vs as described in Table 5.2, of Example 5.3.1.4, with $v = (0.26,0.22,0.1,0.27,0.15)^T$ be the weight vectors of experts $\mathfrak{r} = (0.23,0.28,0.2,0.29)^T$ be the weight vector for parameter. Let the associated aggregate vectors $\bar{\bar{w}} = (0.27,0.18,0.1,0.18,0.27)^T$and $\bar{\bar{u}} = (0.26,0.24,0.24,0.26)^T$.By applying $Eq.(5.9)$ and their score values are express in Table 5.4. The permutation of largest values of the collection q-ROFS_{ft}Vs $\tilde{\mathfrak{J}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{J}_{s_{ij}}$, of i^{th} row and j^{th} column are expressed in Table 5.5. Since

$$\tilde{\mathfrak{J}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{J}_{s_{ij}} = \left(\sqrt[3]{1-\left(1-\mu_{ij}^3\right)^{nv_i\mathfrak{r}_j}}, \psi^{nv_i\mathfrak{r}_j}\right) \tag{5.9}$$

Table 5.4, The score values of q-ROFS_{ft}Vs $\tilde{\mathfrak{J}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{J}_{s_{ij}}$ for q ≥ 3

	s_1	s_2	s_3	s_4
k_1	-0.22765	-0.08604	-0.26668	0.109494
k_2	-0.01185	-0.25119	-0.36466	-0.23585
k_3	-0.48968	-0.47161	-0.69634	-0.50436
k_4	-0.14662	0.008862	0.03341	0.295625
k_5	-0.45766	-0.21373	-0.52886	-0.33226

Table 5.5, Tabular description of q-ROFS_{ft}Vs $\widetilde{\mathfrak{J}}_{s_{ij}} = nv_i\mathfrak{r}_j\mathfrak{J}_{s_{ij}}$ for q ≥ 3

	s_1	s_2	s_3	s_4
$\hbar_1$	$(0.6967, 0.7042)$	$(0.6711, 0.6651)$	$(0.7089, 0.6878)$	$(0.7942, 0.6075)$
$\hbar_2$	$(0.5389, 0.6582)$	$(0.6752, 0.7292)$	$(0.4854, 0.7045)$	$(0.7716, 0.7079)$
$\hbar_3$	$(0.5594, 0.7243)$	$(0.6777, 0.8022)$	$(0.5949, 0.8219)$	$(0.6621, 0.8016)$
$\hbar_4$	$(0.5331, 0.8344)$	$(0.5463, 0.7301)$	$(0.4515, 0.8348)$	$(0.654, 0.8446)$
$\hbar_5$	$(0.5299, 0.8486)$	$(0.5752, 0.8633)$	$(0.4581, 0.9167)$	$(0.439, 0.8179)$

Now by using $Eq.$ 5.8, of $Theorem$ 5.3.3.2,

$$\mathrm{q-ROF}S_{ft}\mathrm{HG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\widetilde{\mathfrak{J}}_{\sigma s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$
$$= \left(\prod_{j=1}^{4}\left(\prod_{i=1}^{5}\tilde{\mu}_{ij}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[3]{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{5}(1-\tilde{\psi}_{ij}^{3})^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right)$$
$$= (0.595792, 0.630295)$$

Based on Theorem 5.3.3.2, the investigated q-ROFS_{ft}HG operator satisfied some basic properties.

5.2.3.5. Theorem

Suppose $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$, $(i = 1, 2, \dots, n)$ and $(j = 1, 2, \dots, m)$, be the collection of q-ROFS_{ft}Vs with $v = (v_1, v_2, \dots, v_n)^T$ be the weight vectors of $\hbar_i$ and $\mathfrak{r} = (\mathfrak{r}_1, \mathfrak{r}_2, \dots, \mathfrak{r}_m)^T$ be the weight vectors of s_j, with $v_i, \mathfrak{r}_j \in [0,1]$ such that $\sum_{i=1}^{n} v_i = 1$ and $\sum_{j=1}^{m} \mathfrak{r}_j = 1$. Here n represent the number of alternatives in i^{th} row and j^{th} $column$ and is called balancing coefficient. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_n)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_m)^T$ be the aggregate associated weight vectors for the experts $\hbar_i$ and for the parameters $s_j's$ respectively, with $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ such that $\sum_{i=1}^{n} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{m} \bar{\bar{u}}_j = 1$. Then the following properties are held for q-ROFS_{ft}HG operator:

$\boldsymbol{i}$: ($\boldsymbol{Idempotency}$): If $\mathfrak{J}_{s_{ij}} = \mathcal{E}_s$, $(\forall\, i = 1,2,\dots,n$ and $j = 1,2,\dots,m)$, where $\mathcal{E}_s = (\mu_s, \psi_s)$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) = \mathcal{E}_s.$$

$\boldsymbol{ii}$: ($\boldsymbol{Boundedness}$): If $\mathfrak{J}_{s_{ij}}^{-} = \left(\min_j \min_i\{\mu_{ij}\}, \max_j \max_i\{\psi_{ij}\}\right)$ and

$\mathfrak{J}_{s_{ij}}^{+} = \left(\max_j \max_i\{\mu_{ij}\}, \min_j \min_i\{\psi_{ij}\}\right)$, then

$$\mathfrak{J}_{s_{ij}}^{-} \le \mathrm{q-ROF}S_{ft}\mathrm{HG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{nm}}) \le \mathfrak{J}_{s_{ij}}^{+}.$$

iii: (***Monotonicity***): If $\mathcal{E}_{s_{ij}} = (b_{ij}, d_{ij})$, $(i = 1,2, \dots, n)$ *and* $(j = 1,2, \dots, m)$, be the another collection of q-ROFS_{ft}Vs such that $\mu_{ij} \le b_{ij}$ *and* $\psi_{ij} \ge d_{ij}$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \le \mathrm{q-ROF}S_{ft}\mathrm{HG}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{nm}}).$$

iv: (***Shift Invariance***): If $\mathcal{E}_s = (b, d)$, is another q-ROFS_{ft}V, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HG}(\Im_{s_{11}} \otimes \mathcal{E}_s, \Im_{s_{12}} \otimes \mathcal{E}_s, \dots, \Im_{s_{nm}} \otimes \mathcal{E}_s) = \mathrm{q-ROF}S_{ft}\mathrm{HG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}) \otimes \mathcal{E}_s.$$

iv: (***Homogeneity***): For any $\lambda > 0$, then

$$\mathrm{q-ROF}S_{ft}\mathrm{HG}(\lambda\Im_{s_{11}}, \lambda\Im_{s_{12}}, \dots, \lambda\Im_{s_{nm}}) = \lambda\mathrm{q-ROF}S_{ft}\mathrm{HG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{nm}}).$$

Proof. Proofs are easy and follow the Theorem 5.3.1.5.

5.3. An approach for $\mathcal{MCDM}$ under q-rung orthopair fuzzy soft information

This section is allotted for the $\mathcal{DM}$ process for the developed aggregation operators. In $\mathcal{DM}$ aggregation operators plays an important role because it aggregates the several evaluation values of experts into a single value. $\mathcal{DM}$ is a pre-planned process of identifying and selecting the best choice out of many alternatives. $\mathcal{DM}$ is a hard process because it can vary so obviously from one scenario to the next. Therefore, it is very important to judge the characteristics and limitations of alternative. Also $\mathcal{DM}$ is a better approach to increase the chance of selecting most appropriate alternative of the choice. It is essential to know that how much truly background information is required for decision maker and the best effective strategy in $\mathcal{DM}$ is to keep an eye and focus on your goal.

Suppose the set $T = \{\mathscr{k}_1, \mathscr{k}_2, \dots, \mathscr{k}_l\}$ of different objects and consider $\mathbb{E} = \{s_1, s_2, \dots, s_n\}$ be set of parameter against alternatives $\mathscr{k}_e$ $(e = 1,2, \dots, l)$. The team of m professional experts $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_m$ is going to evaluate each object $\mathscr{k}_s$ against their given parameter s_j. The group of professional experts describe its evaluation in the form of $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ with weight vector $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ for senior experts $\mathcal{D}_i$ and let $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vector for the parameters s_j with $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ such that $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ *and* $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. The collective evaluation of professional experts are described in a decision matrix $\mathbb{M} = [\Im_{\mathscr{k}_{ij}}]_{m\times n}$. By applying the developed model on evaluated decision matrix $\mathbb{M} = [\Im_{\mathscr{k}_{ij}}]_{m\times n}$ we will get an aggregated q-ROFS_{ft}V $\xi_e = (\mu_e, \psi_e)$ for every object to against parameters. Finally by applying the score function on each aggregated q-ROFS_{ft}V $\xi_e = (\mu_e, \psi_e)$ for each object $\mathscr{k}_e$ and rank them in a specific ordered to the most desirable option out of total.

5.3.1. Algorithm

Based on above analysis, the algorithm for the proposed model for solving MCDM application is given below.

Step 1. Collect the evaluation information of professional experts for every object to corresponding parameters and then established the decision matrix $\mathbb{M} = [\Im_{\mathscr{k}_{ij}}]_{m\times n}$ as:

$$\mathbb{M} = \begin{bmatrix} (\mu_{11}, \psi_{11}) & (\mu_{12}, \psi_{12}) & \cdots & (\mu_{1n}, \psi_{1n}) \\ (\mu_{21}, \psi_{21}) & (\mu_{22}, \psi_{22}) & \cdots & (\mu_{2n}, \psi_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \psi_{m1}) & (\mu_{m2}, \psi_{m2}) & \cdots & (\mu_{mn}, \psi_{mn}) \end{bmatrix}$$

Step 2. Normalize the decision matrix by interchanging the cost and benefit parameters if there is any by applying the formula from [91] that is,

$$p_{ij} = \begin{cases} \mathfrak{J}^{c}_{\mathscr{k}_{ij}} & ; for\ cost\ type\ parameter \\ \mathfrak{J}_{\mathscr{k}_{ij}} & ; for\ benefit\ type\ parameter' \end{cases}$$

where $\mathfrak{J}^{c}_{\mathscr{k}_{ij}} = (\psi_{ij}, \mu_{ij})$ represents the complement of $\mathfrak{J}_{\mathscr{k}_{ij}} = (\mu_{ij}, \psi_{ij})$.

Step 3. By applying the developed model on evaluated decision matrix $\mathbb{M} = [\mathfrak{J}_{\mathscr{k}_{ij}}]_{m\times \boldsymbol{n}}$ we will get an aggregated q-ROFS_{ft}V $\xi_e = (\mu_e, \psi_e)$ for each alternative $\mathscr{k}_e (e = 1,2, \dots, l)$ to their corresponding parameters.

Step 4. Determine the score value on each aggregated q-ROFS_{ft}V $\xi_e = (\mu_e, \psi_e)$ for each alternative $\mathscr{k}_e$.

Step 5. Finally rank the score value in a specific ordered to get best choice out of total.

The flow chart of above algorithm for q-ROFS_{ft}WG operator is given in Fig. 4.1.

5.4. An Illustrative example for the proposed model to $\mathcal{MCDM}$

In this subsection through an illustrative example we will present the medical diagnose problem by applying the developed model to determine the applicability and superiority of the developed methods based on q-ROF soft information adopted from [64].

Suppose a team of five professional Doctors $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4\ and\ \mathcal{D}_5$ are going to describe their assessment report for four different under medical treatment patients $\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3\ and\ \mathscr{k}_4$ having weight vector $\bar{\bar{w}} = (0.18,0.24,0.21,0.15,0.22)^T$. Let $\mathbb{E} = \{s_1 = chest\ pain, s_2 = fever, s_3 = cough, s_4 = fatigue, s_5 = vomit\}$ be the set of parameters having weight vector $\bar{\bar{u}} = (0.26,0.22,0.1,0.27,0.15)^T$. The experts mean professional Doctors present their assessment report for each under medical treatment patient against their symptom in the form of q-ROFS_{ft} decision matrix. Based on above analysis, to diagnose the most illness patient via the algorithm for the proposed model is given below.

By using $\mathbf{q-ROF}S_{ft}\mathbf{WG}$ operator

Step 1. The collective evaluation information of professional experts for each patient to oppose parameters (symptoms) and their established the decision matrix $\mathbb{M} = [\mathfrak{J}_{\mathscr{k}_{ij}}]_{m\times \boldsymbol{n}}$ are given in Tables $5.6 - 5.9$ respectively:

Step 2. All the parameters are the same type so no need to normalize the assessment information in decision matrix.

Step 3. By applying the developed model on each evaluated decision matrix $\mathbb{M} = [\mathfrak{J}_{\mathscr{k}_{ij}}]_{m\times \boldsymbol{n}}$ for each patient $\mathscr{k}_i$ by using $Eq. 5.1,\ for\ q = 3$, and the aggregated result is given below:

$$\xi_1 = (0.676098, 0.217227), \quad \xi_2 = (0.711392, 0.213948),$$
$$\xi_3 = (0.745244, 0.192632), \quad \xi_4 = (0.726185, 0.183019)$$

Step 4. Determine the score function on each aggregated q-ROFV $\xi_e = (\mu_e, \psi_e)$ for each alternative $\mathcal{k}_e$ in **Step 3**, that is

$$S(\xi_1) = 0.349273, \quad S(\xi_2) = 0.404847, \quad S(\xi_3) = 0.464826,$$
$$S(\xi_4) = 0.4337$$

Step 5. In final step rank the score value in a specific ordered to get best choice out of total.

$$S(\xi_3) > S(\xi_4) > S(\xi_2) > S(\xi_1)$$

Hence, form the analysis of above calculation it is clear that under medical treatment patient $\mathcal{k}_3$ has diagnose more illness in list.

By $\mathbf{q-ROF}\boldsymbol{S_{ft}}\mathbf{OWG}$ operator

Step 1. Similar as above.

Step 2. Similar as above.

Step 3. By applying the developed model on each evaluated decision matrix $\mathbb{M} = [\Im_{\mathcal{k}_{ij}}]_{m\times n}$ for each patient $\mathcal{k}_i$ by using $Eq. 5.7,\ for\ q = 3$, and the aggregated result is given below:

$$\xi_1 = (0.682695, 0.212683), \quad \xi_2 = (0.716155, 0.210147),$$
$$\xi_3 = (0.747071, 0.194893), \quad \xi_4 = (0.726911, 0.188214)$$

Step 4. Determine the score value on each aggregated q-ROFV $\xi_e = (\mu_e, \psi_e)$ for each alternative $\mathcal{k}_e$ in **Step 3**, that

$$S(\xi_1) = 0.360011, \quad S(\xi_2) = 0.41323, \quad S(\xi_3) = 0.467677,$$
$$S(\xi_4) = 0.434245$$

Step 5. In final step rank the score value in a specific ordered to get best choice out of total.

$$S(\xi_3) > S(\xi_4) > S(\xi_2) > S(\xi_1)$$

Hence, form the analysis of above calculation it is clear that under medical treatment patient $\mathcal{k}_3$ has diagnose more illness in list.

For $\mathbf{q-ROF}\boldsymbol{S_{ft}}\mathbf{HG}$ operator

Step 1. Similar as above.

Step 2. Similar as above.

Step 3. By applying the developed model on each evaluated decision matrix $\mathbb{M} = [\Im_{\mathcal{k}_{ij}}]_{m\times n}$ for each patient $\mathcal{k}_i$ by using $Eq. 5.8,\ for\ q = 3$, with $v = (0.15, 0.2, 0.17, 0.3, 0.18)^T\ and\ \mathfrak{r} = (0.16, 0.21, 0.13, 0.26, 0.24)^T$ be the weight vectors of $\Im_{\mathcal{k}_{ij}} = (\mu_{ij}, \psi_{ij})$, and n represent the number of alternatives in $i^{th}\ row\ and\ j^{th}\ column$ and is called balancing coefficient. Let $\bar{\bar{w}} =$

$(0.18,0.24,0.21,0.15,0.22)^T$ and $\bar{\bar{u}} = (0.26,0.22,0.1,0.27,0.15)^T$ be the aggregate associated weight vectors for professional Doctor $\mathcal{D}_i$ and for the parameters $s_j{}'s$ respectively, the aggregated result is given below:

$$\xi_1 = (0.418696, 0.732854), \quad \xi_2 = (0.444599, 0.735594),$$
$$\xi_3 = (0.469079, 0.715699), \quad \xi_4 = (0.450482, 0.71448)$$

Step 4. Determine the score value on each aggregated q-ROFV $\xi_e = (\mu_e, \psi_e)$ for each alternative $\hbar_e$ in **Step 3**, that

$$S(\xi_1) = -0.3625,\ S(\xi_2) = -0.34969,\ S(\xi_3) = -0.2981,\ S(\xi_4) = -0.31024$$

Table 5.6. q-ROFS_{ft} matrix for patient $\hbar_1$

	s_1 = Chest pain	s_2 = Fever	s_3 = Cough	s_4 =Fatigue	s_5 = Vomit
$\mathcal{D}_1$	$(0.7,0.25)$	$(0.7,0.22)$	$(0.88,0.1)$	$(0.9,0.1)$	$(0.73,0.2)$
$\mathcal{D}_2$	$(0.6, 0.1)$	$(0.6,0.13)$	$(0.85,0.12)$	$(0.65,0.25)$	$(0.81,0.18)$
$\mathcal{D}_3$	$(0.54,0.15)$	$(0.7,0.2)$	$(0.75,0.24)$	$(0.68,0.25)$	$(0.6,0.26)$
$\mathcal{D}_4$	$(0.65,0.2)$	$(0.8,0.18)$	$(0.85,0.13)$	$(0.8,0.15)$	$(0.7,0.28)$
$\mathcal{D}_5$	$(0.6,0.3)$	$(0.75,0.18)$	$(0.67,0.25)$	$(0.6,0.3)$	$(0.45,0.15)$

Table 5.7. q-ROFS_{ft} matrix for patient $\hbar_2$

	s_1 =Chest pain	s_2 =Fever	s_3 =Cough	s_4 =Fatigue	s_5 =Vomit
$\mathcal{D}_1$	$(0.8,0.15)$	$(0.75,0.22)$	$(0.76,0.1)$	$(0.8,0.19)$	$(0.7,0.25)$
$\mathcal{D}_2$	$(0.75, 0.18)$	$(0.8,0.15)$	$(0.8,0.18)$	$(0.5,0.25)$	$(0.8,0.16)$
$\mathcal{D}_3$	$(0.78,0.13)$	$(0.7,0.2)$	$(0.7,0.25)$	$(0.76,0.21)$	$(0.76,0.23)$
$\mathcal{D}_4$	$(0.9,0.1)$	$(0.65,0.33)$	$(0.76,0.15)$	$(0.87,0.12)$	$(0.65,0.18)$
$\mathcal{D}_5$	$(0.65,0.3)$	$(0.55,0.2)$	$(0.6,0.3)$	$(0.7,0.23)$	$(0.55,0.15)$

Table 5.8. q-ROFS_{ft} matrix for patient $\hbar_3$

	s_1 =Chest pain	s_2 =Fever	s_3 =Cough	s_4 =Fatigue	s_5 =Vomit
$\mathcal{D}_1$	$(0.71,0.25)$	$(0.78,0.1)$	$(0.88,0.11)$	$(0.81,0.18)$	$(0.78,0.2)$
$\mathcal{D}_2$	$(0.8,0.15)$	$(0.85,0.12)$	$(0.9,0.1)$	$(0.65,0.25)$	$(0.74,0.23)$
$\mathcal{D}_3$	$(0.76,0.1)$	$(0.88,0.11)$	$(0.84,0.12)$	$(0.86,0.1)$	$(0.79,0.2)$
$\mathcal{D}_4$	$(0.78,0.22)$	$(0.75,0.25)$	$(0.74,0.2)$	$(0.75,0.25)$	$(0.65,0.16)$
$\mathcal{D}_5$	$(0.6,0.25)$	$(0.8,0.19)$	$(0.75,0.16)$	$(0.6,0.2)$	$(0.5,0.1)$

Step 5. In final step rank the score value in a specific ordered to get best choice out of total:

$$S(\xi_3) > S(\xi_4) > S(\xi_2) > S(\xi_1)$$

Hence, form the analysis of above calculation it is clear that under medical treatment patient $\mathscr{k}_3$ has diagnose more illness in list.

5.4.1. Comparative analysis

To present the applicability and superiority of the investigated aggregation models, a comparative study is being given in the following (see [5, 7, 21, 30, 58]). If we consider PyFVs, so in this case the methods investigated in [5, 7, 21, 58] are failed to handle the decision makers prefer choice. Similarly, if we acknowledge Tables 5.6 to 5.9, then the methods initiated in [5, 7, 21, 30] are failed to handle the experts prefer evaluations and the methods investigated in this manuscript still handle

Table 5.9. q-ROFS_{ft} matrix for patient $\mathscr{k}_4$

	s_1 =Chest pain	s_2 =Fever	s_3 =Cough	s_4 =Fatigue	s_5 =Vomit
$\mathcal{D}_1$	$(0.76,0.22)$	$(0.75,0.22)$	$(0.85,0.14)$	$(0.78,0.2)$	$(0.65,0.26)$
$\mathcal{D}_2$	$(0.72,0.12)$	$(0.79,0.18)$	$(0.6,0.12)$	$(0.73,0.15)$	$(0.8,0.14)$
$\mathcal{D}_3$	$(0.82,0.16)$	$(0.83,0.1)$	$(0.84,0.13)$	$(0.82,0.12)$	$(0.77,0.2)$
$\mathcal{D}_4$	$(0.6,0.27)$	$(0.6,0.3)$	$(0.7,0.2)$	$(0.83,0.13)$	$(0.6,0.25)$
$\mathcal{D}_5$	$(0.55,0.1)$	$(0.81,0.12)$	$(0.8,0.15)$	$(0.72,0.17)$	$(0.55,0.15)$

all these scenarios. By applying proposed weighted geometric operators on Tables 5.6 to 5.9 to aggregate the different parameters of q-ROFS_{ft}Ns with weigh vector $\bar{\bar{u}} = (0.26,0.22,0.1,0.27,0.15)^T$ to achieve the decision matrix as summarized in Table 5.10 for different patients $\mathscr{k}_i$ $(i = 1,2,3,4)$. Based on Table 5.10, a comparative study of the different existing models have been presented and their summarized results for each patient $\mathscr{k}_i$ are given Table 5.11. Hence, form the above calculation of Table 5.11, it is clear that under medical treatment patient b_3 has diagnose more illness in list. The Characteristic summery of proposed models with some existing literatures are presented in Table 5.12. Thus from the analysis of Table 5.12, it is observed that existing models give in [5, 7, 21, 30] having no information about parameterization tools. The main advantage of the investigated model is the capability to solve real problems by utilizing parameterization properties. Therefore, the proposed approach is more capable and superior than existing methods under q-ROFS_{ft} environment.

5.4.2. Conclusion

Decision making is a pre-plan process of identifying and choosing the logical choice out of several alternatives. $\mathcal{DM}$ is a hard process because it can vary so obviously from one scenario to the next. Therefore, it is very important to judge the characteristics and limitations of alternative. Also $\mathcal{DM}$ is a batter approach to increase the chance of selecting most appropriate alternative of the choice. It is essential to know that how much truly background information is required for decision maker and the best effective strategy in $\mathcal{DM}$ is to keep an eye and focus on your goal. The pioneer paradigm of S_{ft}S was investigated by Molodtsov by affixing parameterization tools in ordinary sets. S_{ft}S theory is free from inherit complexity and a nice mathematical tool to cope uncertainties in parametric manner. The aim of this manuscript is to initiate the combine study of S_{ft}S and q-ROFS to get the new notion called q-ROFS_{ft}S. The notion of q-ROFS_{ft}S is free from those complexities which suffering the ordinary theories because parameterization tool is the most significant character of q-ROFS_{ft}S. In this manuscript our main contribution to originate the concept of q-ROFS_{ft}WG, q-ROFS_{ft}OWG and q-ROFS_{ft}HG operators in q-ROFS_{ft}S environment. Moreover, some dominant properties of these developed operators are studied with detail. Based on these proposed approach, a model is build up for $\mathcal{MCDM}$ and their step wise algorithm is being presented. Finally, utilizing the developed approach an illustrative example is solved under q-ROFS_{ft} environment. Further a comparative analysis of the investigated models with existing methods are presented in detail which shows the competence and ability of the developed models. The main advantage of the investigated model is the capability to solve real problems by utilizing parameterization properties. Therefore, the proposed approach is more capable and superior than existing methods under q-ROFS_{ft} environment.

Table 5.10, Aggregated values of q-ROFS_{ft} matrix for patients

	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$	$\mathscr{k}_4$
$\mathcal{D}_1$	$(0.7713, 0.1999)$	$(0.7691,0.1960)$	$(0.7783,0.1932)$	$(0.7538,0.2168)$
$\mathcal{D}_2$	$(0.6641, 0.1820)$	$(0.6929,0.1974)$	$(0.7666,0.1951)$	$(0.7358,0.1478)$
$\mathcal{D}_3$	$(0.6388, 0.2267)$	$(0.7453,0.2019)$	$(0.8245,0.1301)$	$(0.8164,0.1470)$
$\mathcal{D}_4$	$(0.7474, 0.1984)$	$(0.7774,0.2151)$	$(0.7406,0.2280)$	$(0.6651,0.2457)$
$\mathcal{D}_5$	$(0.6103, 0.2607)$	$(0.6184,0.2483)$	$(0.6360,0.2015)$	$(0.6687,0.1411)$

Table 5.11. Comparative Studies of different methods

Methods	$Score\ values\ of\ patients$				Ranking
	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	
IFWG [5]	0.46119,	0.49917,	0.5572,	0.55418	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFOWG [5]	0.47123,	0.50500,	0.55447,	0.54260	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFHG [5]	0.44665,	0.48839,	0.53872,	0.52907	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFS_fWG [58]	0.47536,	0.51160,	0.56806,	0.55816	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
IFEWG [7]	0.44290,	0.50071,	0.55935,	0.55612	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFWG [21]	0.41024,	0.46069,	0.51919,	0.49886	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFEWG [21]	0.16317,	0.21196,	0.25247,	0.25114	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFS_{ft}WG (proposed)	0.46427,	0.51474,	0.57343,	0.55374	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFS_{ft}OWG (proposed)	0.47513,	0.52336,	0.57502,	0.54971	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
PyFS_{ft}HG (proposed)	-0.44065,	-0.435,	-0.38704,	-0.39273	
q-ROFWG [30]	0.29865,	0.35021,	0.40678,	0.38045	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}WG (proposed)	0.349273,	0.404847,	0.464826,	0.4337	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}OWG (proposed)	0.360011,	0.41323,	0.467677,	0.434245	$\xi_3 > \xi_4 > \xi_2 > \xi_1$
q-ROFS_{ft}HG (proposed)	-0.3625,	-0.34969,	-0.2981,	-0.31024	$\xi_3 > \xi_4 > \xi_2 > \xi_1$

Table 5.12. Characteristic analysis of different models

	$Fuzzy\ information$	$Aggragate\ parameter\ informatiom$
IFWG [5]	Yes	No
IFOWG [5]	Yes	No
IFHG [5]	Yes	No
IFEWG [7]	Yes	No
IFS_{ft}WG [58]	Yes	Yes
PyFWG [21]	Yes	No
PyFEWG [21]	Yes	No
q-ROFWG [30]	Yes	No
Proposed Operators	Yes	Yes

Chapter 6

Orthopair fuzzy soft Dombi averaging aggregation operators

Recently, some improvement has been suggested in the dominant notion of fuzzy set, by Yager. He investigated the generalized concept of FS, IFS and PyFS and called it q-ROFS. It is observed that the rung q is the most useful characteristic of this concept which has the capability to cover the boundary range that can be required. The input range of q-ROFS is more flexible, wider and suitable because when the rung q increase, the orthopair provides additional space to the boundary constraint. The aim of this chapter is to present the Dombi aggregation operators using q-ROFS_{ft} environments. Since Dombi operational parameter possess natural flexibility with resilience of variability. The behaviour of Dombi operational parameter is very important to express the experts' attitude in $\mathcal{DM}$. In this chapter, we present q-ROFS_{ft}DA aggregation operators including q-ROFS_{ft}DWA, q-ROFS_{ft}DOWA and q-ROFS_{ft}DHA operators. The basic properties of these operators are presented in detail such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity. By applying developed approach, this chapter contains the technique and algorithm for $\mathcal{MCDM}$. Further a numerical example is given to illustrative the flexibility and applicability of the developed operators.

6.1. q-Rung orthopair fuzzy soft set

This section is devoted for the detail and hybrid study of the prominent paradigm of S_{ft}S and the recent developed pioneer notion of q-ROFS to obtain the new concept of q-ROFS_{ft}S. For a detail study of q-ROFS_{ft}S and its basic operations and relations see Chapter 4, Section 4.2.

To rank two or more q-ROFS_{ft}Vs score function plays an important role which estimates the ranking values of alternatives satisfy the desirable choice of experts

6.1.1. Definition

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ be a q-ROFS_{ft}V. Then the score function of $\mathfrak{J}_{s_{ij}}$ is denoted and defined as:

$$\mathcal{S}c\left(\mathfrak{I}_{s_{ij}}\right) = \frac{1}{2}\left(1 + \mu_{ij}^{q} - \psi_{ij}^{q}\right), \quad \mathcal{S}c\left(\mathfrak{I}_{s_{ij}}\right) \in [0,1].$$

Greater the score value superior that orthopair is.

6.1.2. Definition

Let $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ be a q-ROF$\mathcal{S}_{ft}$V. Then the accuracy function of $\mathfrak{I}_{s_{ij}}$ is denoted and defined as:

$$Ac\left(\mathfrak{I}_{s_{ij}}\right) = \mu_{ij}^{q} + \psi_{ij}^{q}, \quad Ac\left(\mathfrak{I}_{s_{ij}}\right) \in [0,1],$$

Let $\mathfrak{I}_{s_{1j}} = \left(\mu_{1j}, \psi_{1j}\right)\ for\ (j = 1,2)$ be two q-ROF$\mathcal{S}_{ft}$V and $\mathcal{S}c\left(\mathfrak{I}_{s_{11}}\right), \mathcal{S}c\left(\mathfrak{I}_{s_{12}}\right)$ be the score functions of $\mathfrak{I}_{s_{11}}$ and $\mathfrak{I}_{s_{12}}$, and $Ac\left(\mathfrak{I}_{s_{11}}\right), Ac\left(\mathfrak{I}_{s_{11}}\right)$ be the accuracy functions of $\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}$ respectively. Then

(i) If $\mathcal{S}c\left(\mathfrak{I}_{s_{11}}\right) > \mathcal{S}c\left(\mathfrak{I}_{s_{12}}\right)$, then $\mathfrak{I}_{s_{11}} \succ \mathfrak{I}_{s_{12}}$,

(ii) If $\mathcal{S}c\left(\mathfrak{I}_{s_{11}}\right) = \mathcal{S}c\left(\mathfrak{I}_{s_{12}}\right)$, then

a) If $Ac\left(\mathfrak{I}_{s_{11}}\right) > Ac\left(\mathfrak{I}_{s_{11}}\right)$, then $\mathfrak{I}_{s_{11}} \succ \mathfrak{I}_{s_{12}}$,

b) If $Ac\left(\mathfrak{I}_{s_{11}}\right) = Ac\left(\mathfrak{I}_{s_{11}}\right)$, then $\mathfrak{I}_{s_{11}} = \mathfrak{I}_{s_{12}}$.

6.2. Dombi operations on q-rung orthopair fuzzy soft set

In 1982 Dombi [65] initiated the new type of sum and product operators which is known as Dombi t-norm and Dombi t-conorm and is given below:

6.2.1. Definition [65]

Let $f\ and\ g$ be any two real numbers and $\beta \geq 1$. Then the Dombi norms for them are defined in the subsequent expression:

$$T_D(f,g) = \frac{1}{1 + \left\{\left(\frac{1-f}{f}\right)^{\beta} + \left(\frac{1-g}{g}\right)^{\beta}\right\}^{\frac{1}{\beta}}},$$

$$T_D^{*}(f,g) = 1 - \frac{1}{1 + \left\{\left(\frac{f}{1-f}\right)^{\beta} + \left(\frac{g}{1-g}\right)^{\beta}\right\}^{\frac{1}{\beta}}}.$$

In view of above definition we can establish the new operation laws for q-ROFS_{ft}Ns as follows:

6.2.2. Definition

Consider $\mathfrak{J}_{s_{11}} = (\mu_{11}, \psi_{11})$ and $\mathfrak{J}_{s_{12}} = (\mu_{12}, \psi_{12})$ be any two q-ROFS_{ft}Vs, $\beta \geq 1$ and $\rho > 0$. Then the Dombi operations of t-norm and t-conorm for q-ROFS_{ft}Vs are defined as follows:

(i) $$\mathfrak{J}_{s_{11}} \oplus \mathfrak{J}_{s_{12}} = \left(\sqrt[q]{1 - \frac{1}{1+\left\{\left(\frac{\mu_{11}{}^q}{1-\mu_{11}{}^q}\right)^\beta + \left(\frac{\mu_{12}{}^q}{1-\mu_{12}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\left(\frac{1-\psi_{11}}{\psi_{11}}\right)^\beta + \left(\frac{1-\psi_{12}}{\psi_{12}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right);$$

(ii) $$\mathfrak{J}_{s_{11}} \otimes \mathfrak{J}_{s_{12}} = \left(\frac{1}{1+\left\{\left(\frac{1-\mu_{11}}{\mu_{11}}\right)^\beta + \left(\frac{1-\mu_{12}}{\mu_{12}}\right)^\beta\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\left(\frac{\psi_{11}{}^q}{1-\psi_{11}{}^q}\right)^\beta + \left(\frac{\psi_{12}{}^q}{1-\psi_{12}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}} \right);$$

(iii) $$\rho\mathfrak{J}_{s_{11}} = \left(\sqrt[q]{1 - \frac{1}{1+\left\{\rho\left(\frac{\mu_{11}{}^q}{1-\mu_{11}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\rho\left(\frac{1-\psi_{11}}{\psi_{11}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right);$$

(iv) $$\mathfrak{J}_{s_{11}}^{\rho} = \left(\frac{1}{1+\left\{\rho\left(\frac{1-\mu_{11}}{\mu_{11}}\right)^\beta\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\rho\left(\frac{\psi_{11}{}^q}{1-\psi_{11}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}} \right).$$

By using the above operation laws we can easily obtain the following results for $\rho, \rho_1, \rho_2 > 0$:

(i) $\mathfrak{J}_{s_{11}} \oplus \mathfrak{J}_{s_{12}} = \mathfrak{J}_{s_{12}} \oplus \mathfrak{J}_{s_{11}}$;

(ii) $\mathfrak{J}_{s_{11}} \otimes \mathfrak{J}_{s_{12}} = \mathfrak{J}_{s_{12}} \oplus \mathfrak{J}_{s_{11}}$;

(iii) $\rho(\mathfrak{J}_{s_{11}} \oplus \mathfrak{J}_{s_{12}}) = \rho\mathfrak{J}_{s_{12}} \oplus \rho\mathfrak{J}_{s_{11}}$;

(iv) $\rho_1\mathfrak{J}_{s_{11}} \oplus \rho_1\mathfrak{J}_{s_{11}} = (\rho_1 + \rho_2)\mathfrak{J}_{s_{11}}$;

(v) $\mathfrak{J}_{s_{11}}^{\rho} \otimes \mathfrak{J}_{s_{12}}^{\rho} = \left(\mathfrak{J}_{s_{11}} \otimes \mathfrak{J}_{s_{12}}\right)^{\rho}$;

(vi) $\mathfrak{J}_{s_{11}}^{\rho_1} \otimes \mathfrak{J}_{s_{11}}^{\rho_2} = \mathfrak{J}_{s_{11}}^{(\rho_1+\rho_2)}$.

Proofs are easy and straightforward.

6.3. q-Rung orthopair fuzzy soft Dombi average aggregation operators

In this section, in view of defined Dombi operation laws we will extend Dombi operators to q-ROFS_{ft} environment such as q-ROFS_{ft}WA, q-ROFS_{ft}DOWA and q-ROFS_{ft}DHA operators and investigate their fundamental properties with details.

6.3.1. q- Rung orthopair fuzzy soft Dombi weighted averaging operators

This subsection is devoted for the study of q-ROFS_{ft}DWA operator and discuss their basic properties in details.

6.3.1.1. **Definition**

Let $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)\, for\ (i = 1,2, \dots, \text{m and } j = 1, 2, \dots, \text{n})$ be the collection of q-ROFS_{ft}Vs. Suppose $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ be the weight vectors for expert $\mathscr{k}_i$ and $\overline{\overline{u}} = (\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\overline{\overline{w}}_i, \overline{\overline{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$ and $\sum_{j=1}^{n} \overline{\overline{u}}_j = 1$. Then q-ROF$S_{ft}$DWA operator is a mapping denoted and define as: $q-\text{ROF}S_t\text{DWA}: X^n \to X$, (where X represents the collection of q-ROFS_{ft}Vs) such that

$$q-\text{ROF}S_{ft}\text{DWA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{mn}}\right) = \oplus_{j=1}^{n} \overline{\overline{u}}_j (\oplus_{i=1}^{m} \overline{\overline{w}}_i \mathfrak{I}_{s_{ij}}) \tag{6.1}$$

Based on Eq. (6.1) we can obtain the aggregated result for q-ROFS_{ft}DWA operator as described in Theorem 6.3.1.2.

6.3.1.2. **Theorem**

Suppose the collection $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)\, for\ (i = 1,2, \dots, m\ and\ j = 1, 2, \dots, n)$ of q-ROFS_{ft}Vs. Consider $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ be the weight vectors for experts $\mathscr{k}_i$ and $\overline{\overline{u}} = (\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\overline{\overline{w}}_i, \overline{\overline{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$ and $\sum_{j=1}^{n} \overline{\overline{u}}_j = 1$. Then the aggregated result for q-ROFS_{ft}DWA operator is stated as:

$$q-\text{ROF}S_{ft}\text{DWA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{mn}}\right) = \oplus_{j=1}^{n} \overline{\overline{u}}_j (\oplus_{i=1}^{m} \overline{\overline{w}}_i \mathfrak{I}_{s_{ij}})$$

$$= \left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^\beta\right)\right\}^{\frac{1}{\beta}}} \right) \quad (6.2)$$

Proof: The required proof can be obtained by using mathematical induction.

From Dombi operational laws, we have

$$\Im_{s_{11}} \oplus \Im_{s_{12}} = \left(\sqrt[q]{1 - \frac{1}{1+\left\{\left(\frac{\mu_{11}{}^q}{1-\mu_{11}{}^q}\right)^\beta + \left(\frac{\mu_{12}{}^q}{1-\mu_{12}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\left(\frac{1-\psi_{11}}{\psi_{11}}\right)^\beta + \left(\frac{1-\psi_{12}}{\psi_{12}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right) \quad and$$

$$\rho \Im_s = \left(\sqrt[q]{1 - \frac{1}{1+\left\{\rho\left(\frac{\mu_{\Im_s}{}^q}{1-\mu_{\Im_s}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\rho\left(\frac{1-\psi_{\Im_s}}{\psi_{\Im_s}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right)$$

Now first we show that Eq. 6.2 holds for $m = 2 \; and \; n = 2$,

$$q - \mathrm{ROFS}_{ft}\mathrm{DWA}(\Im_{s_{11}}, \Im_{s_{12}}) = \oplus_{j=1}^{2} \bar{\bar{u}}_j \left(\oplus_{i=1}^{2} \bar{\bar{w}}_i \Im_{s_{ij}}\right)$$

$$= \bar{\bar{u}}_1 (\bar{\bar{w}}_1 \Im_{s_{11}} \oplus \bar{\bar{w}}_2 \Im_{s_{21}}) \oplus \bar{\bar{u}}_2 (\bar{\bar{w}}_1 \Im_{s_{12}} \oplus \bar{\bar{w}}_2 \Im_{s_{22}})$$

$$= z_1 \left(\sqrt[q]{1 - \frac{1}{1+\left\{u_1\left(\frac{\mu_{11}{}^q}{1-\mu_{11}{}^q}\right)^\beta + u_2\left(\frac{\mu_{21}{}^q}{1-\mu_{21}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{u_1\left(\frac{1-\psi_{11}}{\psi_{11}}\right)^\beta + u_2\left(\frac{1-\psi_{21}}{\psi_{21}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right) \oplus$$

$$z_2 \left(\sqrt[q]{1 - \frac{1}{1+\left\{\bar{\bar{w}}_1\left(\frac{\mu_{12}{}^q}{1-\mu_{12}{}^q}\right)^\beta + \bar{\bar{w}}_2\left(\frac{\mu_{22}{}^q}{1-\mu_{22}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\bar{\bar{w}}_1\left(\frac{1-\psi_{12}}{\psi_{12}}\right)^\beta + \bar{\bar{w}}_2\left(\frac{1-\psi_{22}}{\psi_{22}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right)$$

$$= \bar{\bar{u}}_1 \left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{i=1}^{2} \bar{\bar{w}}_i \left(\frac{\mu_{i1}{}^q}{1-\mu_{i1}{}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\sum_{i=1}^{2} \bar{\bar{w}}_i \left(\frac{1-\psi_{i1}}{\psi_{i1}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right) \oplus$$

$$\overline{\overline{u}}_2\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{i=1}^{2}\overline{\overline{w}}_i\left(\frac{\mu_{i2}{}^q}{1-\mu_{i2}{}^q}\right)^{\beta}\right\}^{\frac{1}{\beta}}}},\frac{1}{1+\left\{\sum_{i=1}^{m}\overline{\overline{w}}_i\left(\frac{1-\psi_{i2}}{\psi_{i2}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}\right)$$

$$=\left(\begin{array}{c}\sqrt[q]{1-\frac{1}{1+\left\{\overline{\overline{u}}_1\left(\sum_{i=1}^{2}\overline{\overline{w}}_i\left(\frac{\mu_{i1}{}^q}{1-\mu_{i1}{}^q}\right)^{\beta}\right)+\overline{\overline{u}}_2\left(\sum_{i=1}^{2}\overline{\overline{w}}_i\left(\frac{\mu_{i2}{}^q}{1-\mu_{i2}{}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}},\\ \frac{1}{1+\left\{\overline{\overline{u}}_1\left(\sum_{i=1}^{2}\overline{\overline{w}}_i\left(\frac{1-\psi_{i1}}{\psi_{i1}}\right)^{\beta}\right)+\overline{\overline{u}}_2\left(\sum_{i=1}^{m}\overline{\overline{w}}_i\left(\frac{1-\psi_{i2}}{\psi_{i2}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\end{array}\right)$$

$$=\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{2}\overline{\overline{u}}_j\left(\sum_{i=1}^{2}\overline{\overline{w}}_i\left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}},\frac{1}{1+\left\{\sum_{j=1}^{2}\overline{\overline{u}}_j\left(\sum_{i=1}^{2}\overline{\overline{w}}_i\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)$$

Hence the result is true for $m = 2 \; and \; n = 2$.

Further, let Eq. 6.2, is true for $m = k_1 \; and \; n = k_2$,

$$q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\ldots,\mathfrak{J}_{s_{k_1k_2}}\right)=\oplus_{j=1}^{k_2}\overline{\overline{u}}_j\left(\oplus_{i=1}^{k_1}\overline{\overline{w}}_i\mathfrak{J}_{s_{ij}}\right)$$

$$=\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{k_2}\overline{\overline{u}}_j\left(\sum_{i=1}^{k_1}\overline{\overline{w}}_i\left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}},\frac{1}{1+\left\{\sum_{j=1}^{k_2}\overline{\overline{u}}_j\left(\sum_{i=1}^{k_1}\overline{\overline{w}}_i\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)$$

Next we show that Eq. 6.2 is true for $m = k_1 + 1 \; and \; n = k_2 + 1$

$$q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\ldots,\mathfrak{J}_{s_{k_1k_2}},\mathfrak{J}_{s_{(k_1+1)(k_2+1)}}\right)$$
$$=\left(\oplus_{j=1}^{k_2}\overline{\overline{u}}_j\left(\oplus_{i=1}^{k_1}\overline{\overline{w}}_i\mathfrak{J}_{s_{ij}}\right)\right)\oplus\left(\overline{\overline{u}}_{k_2+1}\left(\overline{\overline{w}}_{k_1+1}u_i\mathfrak{J}_{s_{ij}}\right)\right)$$

$$= \left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\sum_{j=1}^{k_2} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^\beta\right)\right\}^{\frac{1}{\beta}}} \right) \oplus$$

$$\left(\sqrt[q]{1 - \frac{1}{1+\left\{\bar{\bar{u}}_{k_2+1} \left(\bar{\bar{w}}_{k_1+1} \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\bar{\bar{u}}_{k_2+1} \left(\bar{\bar{w}}_{k_1+1} \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^\beta\right)\right\}^{\frac{1}{\beta}}} \right)$$

$$= \left(\begin{array}{c} \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)+\bar{\bar{u}}_{k_2+1} \left(\bar{\bar{w}}_{k_1+1} \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}}, \\ \frac{1}{1+\left\{\sum_{j=1}^{k_2} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^\beta\right)+\bar{\bar{u}}_{k_2+1} \left(\bar{\bar{w}}_{k_1+1} \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^\beta\right)\right\}^{\frac{1}{\beta}}} \end{array} \right)$$

$$= \left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^\beta\right)\right\}^{\frac{1}{\beta}}} \right)$$

Hence that Eq. 6.2, is true for $m = k_1 + 1 \; and \; n = k_2 + 1$. Therefore, by process of mathematical induction, we conclude that Eq. 6.2 is true for all $m, n \geq 1$.

Further to verify that the aggregated result obtained from $q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{k_1 k_2}}\right)$ is again a $q - \text{ROF}S_{ft}\text{V}$.

Let $\gamma = \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}} \quad and \quad \lambda = \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}$

As $0 \leq \mu_{ij} \leq 1 \Rightarrow 0 \leq 1 - \frac{1}{1+\frac{\mu_{ij}}{1-\mu_{ij}}} \leq 1 \Rightarrow 0 \leq \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^\beta\right)\right\}^{\frac{1}{\beta}}}} \leq 1 \Rightarrow 0 \leq \gamma \leq 1$

Similarly

$$0 \le \psi_{ij} \le 1 \Rightarrow 0 \le \frac{1}{1+\frac{1-\psi_{ij}}{\psi_{ij}}} \le 1 \Rightarrow 0$$

$$\le \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}} \le 1 \Rightarrow 0 \le \lambda \le 1$$

As

$$\gamma^q + \lambda^q \le 1 \Rightarrow \lambda^q \le 1 - \gamma^q$$

$$\Rightarrow \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{u}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)^q \le 1 -$$

$$\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}\right)^q$$

Next

$$0 \le \gamma^q + \lambda^q \Rightarrow \left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}\right)^q +$$

$$\left(\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)^q$$

$$\leq \left(\sqrt[q]{1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1 - \mu_{ij}{}^q} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}} \right)^q + 1 -$$

$$\left(\sqrt[q]{1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1 - \mu_{ij}{}^q} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}} \right)^q$$

$$\Rightarrow 0 \leq \gamma^q + \lambda^q \leq 1$$

Therefore, it is verified that the aggregated result obtained from $q-$ ROFS_{ft}DWA$\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{k_1 k_2}}\right)$ is again a $q-$ ROFS_{ft}V.

6.3.1.3. **Example**

Suppose $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_4\}$ be the set of expert teachers who want to judge the ability of a student Z under the set of parameters $\mathbb{E} = \{s_1, s_2, s_3\}$, where $s_j (j = 1,2,3)$ stands for $s_1 = responsable, s_2 = coures\ command\ and\ s_3 = punctual.$ The experts provides their estimated values in the form of q-ROFS_{ft}Vs which are given in Table 6.1. Let $\bar{\bar{w}} = (0.26, 0.3, 0.23, 0.21)^T$ be the weight vectors for expert $\mathscr{k}_i$, $\bar{\bar{u}} = (0.35, 0.31,\ 0.34)^T$ be the weight vectors for parameters s_j and operational parameter $\beta = 2$ for $q = 3$. Now to calculate the aggregated result by applying Eq. 6.2, we have

$$q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{43}}\right) =$$

$$\left(\sqrt[3]{1 - \frac{1}{1 + \left\{ \sum_{j=1}^{3} \bar{\bar{u}}_j \left(\sum_{i=1}^{4} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^3}{1 - \mu_{ij}{}^3} \right)^2 \right) \right\}^{\frac{1}{2}}}}, \frac{1}{1 + \left\{ \sum_{j=1}^{3} \bar{\bar{u}}_j \left(\sum_{i=1}^{4} \bar{\bar{w}}_i \left(\frac{1 - \psi_{ij}}{\psi_{ij}} \right)^2 \right) \right\}^{\frac{1}{2}}} \right)$$

Therefore, $q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{43}}\right) = (0.941368, 0.271102)$.

6.3.1.4. **Remarks**

(a) If we consider that the value of parameter $q = 1$ is fixed, then the proposed $q-$ ROFS_{ft}DWA operator reduced to IFS_{ft}DWA operator.

Table 6.1, Tabular represent of q-ROFS_{ft}S $(\mathfrak{J}, \mathbb{E})$ for $\beta = 2$ and $q = 3$ >

T	s_1	s_2	s_3
k_1	$(0.9, 0.3)$	$(0.85, 0.4)$	$(0.7, 0.2)$
k_2	$(0.85, 0.6)$	$(0.75, 0.25)$	$(0.6, 0.3)$
k_3	$(0.98, 0.38)$	$(0.92, 0.3)$	$(0.8, 0.15)$
k_4	$(0.7, 0.4)$	$(0.95, 0.45)$	$(0.82, 0.32)$

(b) If we consider that the value of parameter $q = 1$ is fixed, then the proposed $q -$ ROFS_{ft}DWA operator reduced to IFS_{ft}DWA operator.

(c) If we consider that the value of parameter $q = 2$ is fixed, then the proposed $q -$ ROFS_{ft}DWA operator reduced to PyFS_{ft}DWA operator.

(d) If the set contain only parameter that is s_1 (means $m = 1$), in this case the proposed $q -$ ROFS_{ft}DWA operator reduced to $q -$ ROFDWA operator.

Thus from the analysis of Remark 6.3.1.4, it is clear that IFS_{ft}DWA, PyFS_{ft}DWA and $q -$ ROFDWA operators are the special cases of the developed $q -$ ROFS_{ft}DWA operator.

Based on Theorem 6.3.1.2, some properties of the $q -$ ROFS_{ft}DWA operators are investigated which are described below:

6.3.1.5. **Theorem**

Suppose the collection $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ for $(i = 1,2, \dots, m$ and $j = 1, 2, \dots, n)$ of q-ROFS_{ft}Vs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ be the weight vectors for experts k_i and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the restriction that $\bar{\bar{w}}_i, \bar{\bar{u}}_n \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{i=1}^{m} \bar{\bar{u}}_n = 1$. Then the following properties are holds for q-ROFS_{ft}DWA operator:

i: **(Idempotency):** Let $\mathfrak{J}_{s_{ij}} = \mathcal{E}_s$ for all $(i = 1,2, \dots, m$ and $j = 1,2, \dots, n)$, where $\mathcal{E}_s = (b, d)$. Then

$$q - \text{ROF}S_{ft}\text{DWA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) = \mathcal{E}_s .$$

$\boldsymbol{ii}$: **(Boundedness):** Let $\mathfrak{J}_{s_{ij}}^{-} = \left(\min_{j} \min_{i} \left(\mu_{ij}\right), \max_{j} \max_{i} \left(\psi_{ij}\right)\right)$ and

$\mathfrak{J}_{s_{ij}}^{+} = \left(\max_{j} \max_{i} \left(\mu_{ij}\right), \min_{j} \min_{i} \left(\psi_{ij}\right)\right)$. Then

$$\mathfrak{J}_{s_{ij}}^{-} \leq q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}\right) \leq \mathfrak{J}_{s_{ij}}^{+}.$$

$\boldsymbol{iii}$: **(Monotonicity):** Let another collection $\mathcal{E}_{s_{ij}} = \left(b_{ij}, \text{d}_{ij}\right)$ for (i = 1,2, … , m and j = 1, 2, … , n) of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij}$ and $\psi_{ij} \geq \text{d}_{ij}$. Then

$$q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}\right) \leq q - \text{ROF}S_{ft}\text{DWA}\left(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{mn}}\right).$$

$\boldsymbol{iv}$: **(Shift Invariance):** Let $\mathcal{E}_s = (b, d)$ be a q-ROFS_{ft}V. Then

$$q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}} \oplus \mathcal{E}_s, \mathfrak{J}_{s_{12}} \oplus \mathcal{E}_s, \dots, \mathfrak{J}_{s_{mn}} \oplus \mathcal{E}_s\right)$$
$$= q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}\right) \oplus \mathcal{E}_s.$$

$\boldsymbol{v}$: **(Homogeneity):** Let $\rho > 0$ be any real number. Then

$$q - \text{ROF}S_{ft}\text{DWA}\left(\rho\mathfrak{J}_{s_{11}}, \rho\mathfrak{J}_{s_{12}}, \dots, \rho\mathfrak{J}_{s_{mn}}\right)$$
$$= \rho\, q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}\right).$$

Proof: ***i:*** **(Idempotency):** Since $\mathfrak{J}_{s_{ij}} = \mathcal{E}_s$ $for\ all$ $(i = 1,2, \dots, m$ and $j = 1,2, \dots, n)$, where $\mathcal{E}_s = (b, d)$. Then by Theorem 1, we have

$q - \text{ROF}S_{ft}\text{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}\right) =$

$$\left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{\mu_{ij}{}^q}{1-\mu_{ij}{}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)$$

$$= \left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{b^q}{1-b^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-d}{d}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)$$

$$= \left(\sqrt[q]{1 - \frac{1}{1 + \left\{ \left(\frac{{\mu_{ij}}^q}{1 - {\mu_{ij}}^q} \right)^\beta \right\}^{\frac{1}{\beta}}}}, \frac{1}{1 + \left\{ \left(\frac{1 - \psi_{ij}}{\psi_{ij}} \right)^\beta \right\}^{\frac{1}{\beta}}} \right)$$

$$= \left(\sqrt[q]{1 - \frac{1}{1 + \frac{{\mu_{ij}}^q}{1 - {\mu_{ij}}^q}}}, \frac{1}{1 + \frac{1 - \psi_{ij}}{\psi_{ij}}} \right)$$

$$= (b, d) = \mathcal{E}_s$$

Hence, the proof is complete.

$\boldsymbol{ii}$: **(Boundedness):** Consider for each $i = 1,2, \ldots, m \ and \ j = 1,2, \ldots, n$, we have

$$\min_j \min_i \left(\mu_{ij} \right) \leq \mu_{ij} \leq \max_j \max_i \left(\mu_{ij} \right)$$

$$\Rightarrow 1 + \frac{\min_j \min_i \left(\mu_{ij}^q \right)}{1 - \min_j \min_i \left(\mu_{ij}^q \right)} \leq 1 + \frac{\mu_{ij}^q}{1 - \mu_{ij}^q} \leq 1 + \frac{\max_j \max_i \left(\mu_{ij}^q \right)}{1 - \max_j \max_i \left(\mu_{ij}^q \right)}$$

$$\frac{1}{1 + \frac{\max_j \max_i \left(\mu_{ij}^q \right)}{1 - \max_j \max_i \left(\mu_{ij}^q \right)}} \leq \frac{1}{1 + \frac{\mu_{ij}^q}{1 - \mu_{ij}^q}} \leq \frac{1}{1 + \frac{\min_j \min_i \left(\mu_{ij}^q \right)}{1 - \min_j \min_i \left(\mu_{ij}^q \right)}}$$

$$\Rightarrow \frac{1}{1 + \frac{\max_j \max_i \left(\mu_{ij}^q \right)}{1 - \max_j \max_i \left(\mu_{ij}^q \right)}} \leq \frac{1}{1 + \frac{\mu_{ij}^q}{1 - \mu_{ij}^q}} \leq \frac{1}{1 + \frac{\min_j \min_i \left(\mu_{ij}^q \right)}{1 - \min_j \min_i \left(\mu_{ij}^q \right)}}$$

$$\Rightarrow \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\max\limits_j\max\limits_i\left(\mu_{ij}^q\right)}{1-\max\limits_j\max\limits_i\left(\mu_{ij}^q\right)}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

$$\leq \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

$$\leq \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\min\limits_j\min\limits_i\left(\mu_{ij}^q\right)}{1-\min\limits_j\min\limits_i\left(\mu_{ij}^q\right)}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

$$\Rightarrow \frac{1}{1+\frac{\max\limits_j\max\limits_i\left(\mu_{ij}^q\right)}{1-\max\limits_j\max\limits_i\left(\mu_{ij}^q\right)}} \leq \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

$$\leq \frac{1}{1+\frac{\min\limits_j\min\limits_i\left(\mu_{ij}^q\right)}{1-\min\limits_j\min\limits_i\left(\mu_{ij}^q\right)}}$$

$$\Rightarrow \sqrt[q]{1-\frac{1}{1+\frac{\min\limits_j\min\limits_i\left(\mu_{ij}^q\right)}{1-\min\limits_j\min\limits_i\left(\mu_{ij}^q\right)}}} \leq \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \leq \sqrt[q]{1-\frac{1}{1+\frac{\max\limits_j\max\limits_i\left(\mu_{ij}^q\right)}{1-\max\limits_j\max\limits_i\left(\mu_{ij}^q\right)}}}$$

Similarly we can show for $\mathcal{NMG}$

$$\frac{1}{1+\frac{1-\min\limits_j\min\limits_i\left(\psi_{ij}\right)}{\min\limits_j\min\limits_i\left(\psi_{ij}\right)}} \geq \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

$$\geq \frac{1}{1+\frac{1-\max\limits_j\max\limits_i\left(\psi_{ij}\right)}{\max\limits_j\max\limits_i\left(\psi_{ij}\right)}}$$

Therefore, from above equations we have

$$\mathfrak{J}_{s_{ij}}^{-} \leq q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{mn}}\right) \leq \mathfrak{J}_{s_{ij}}^{+}.$$

iii: **(Monotonicity):** Since $for\ all\ i = 1,2,\ldots,m\ and\ j = 1,2,\ldots,n$, we have $\mu_{ij} \leq b_{ij}\ and\ \psi_{ij} \geq \mathrm{d}_{ij}$.

As

$$\mu_{ij} \leq b_{ij} \Rightarrow 1+\frac{\mu_{ij}^{q}}{1-\mu_{ij}^{q}} \leq 1+\frac{b_{ij}^{q}}{1-b_{ij}^{q}} \Rightarrow \frac{1}{1+\frac{b_{ij}^{q}}{1-b_{ij}^{q}}} \leq \frac{1}{1+\frac{\mu_{ij}^{q}}{1-\mu_{ij}^{q}}}$$

$$\Rightarrow \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_{j}\left(\sum_{i=1}^{m} \bar{\bar{w}}_{i}\left(\frac{b_{ij}^{q}}{1-b_{ij}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}} \leq \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_{j}\left(\sum_{i=1}^{m} \bar{\bar{w}}_{i}\left(\frac{\mu_{ij}^{q}}{1-\mu_{ij}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

$$\Rightarrow \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_{j}\left(\sum_{i=1}^{m} \bar{\bar{w}}_{i}\left(\frac{\mu_{ij}^{q}}{1-\mu_{ij}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \leq \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_{j}\left(\sum_{i=1}^{m} \bar{\bar{w}}_{i}\left(\frac{b_{ij}^{q}}{1-b_{ij}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}$$

Similarly, we can show for $\mathcal{NMG}$

$$\frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_{j}\left(\sum_{i=1}^{m} \bar{\bar{w}}_{i}\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}} \geq \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_{j}\left(\sum_{i=1}^{m} \bar{\bar{w}}_{i}\left(\frac{1-\mathrm{d}_{ij}}{\mathrm{d}_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

Hence from above equations we have

$$q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{mn}}\right) \leq q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \ldots, \mathcal{E}_{s_{mn}}\right).$$

iv: **(Shift Invariance):** Since $\mathcal{E}_{s}=(b,d)\ and\ \mathfrak{J}_{s_{ij}}=\left(\mu_{ij}, \psi_{ij}\right)$ $(i = 1,2,\ldots,m\ and\ j=1,2,\ldots,n)$ are q-ROFS_{ft}Vs. Then

$$\mathfrak{J}_{s_{11}} \oplus \mathcal{E}_{s}$$
$$=\left(\sqrt[q]{1-\frac{1}{1+\left\{\left(\frac{\mu_{11}^{q}}{1-\mu_{11}^{q}}\right)^{\beta}+\left(\frac{b^{q}}{1-b^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\left(\frac{1-\psi_{11}}{\psi_{11}}\right)^{\beta}+\left(\frac{1-d}{d}\right)^{\beta}\right\}^{\frac{1}{\beta}}}\right)$$

Now consider

$q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mathfrak{J}_{s_{11}}\oplus\mathcal{E}_s,\mathfrak{J}_{s_{12}}\oplus\mathcal{E}_s,\ldots,\mathfrak{J}_{s_{mn}}\oplus\mathcal{E}_s\right)=$

$$\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right)+\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{b^q}{1-b^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}},\ \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)+\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-d}{d}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)$$

$$=\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right)+\left(\frac{b^q}{1-b^q}\right)^{\beta}\right\}^{\frac{1}{\beta}}}},\ \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)+\left(\frac{1-d}{d}\right)^{\beta}\right\}^{\frac{1}{\beta}}}\right)$$

$$=q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\ldots,\mathfrak{J}_{s_{mn}}\right)\oplus\mathcal{E}_s$$

Therefore, the proof is completed.

$\boldsymbol{v}$: **(Homogeneity)** Let $\rho>0$ be any real number and $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ $(i=1,2,\ldots,m\ and\ j=1,2,\ldots,n)$ are q-ROFS_tVs. Then

$$\rho\mathfrak{J}_{s_{ij}}=\left(\sqrt[q]{1-\frac{1}{1+\left\{\rho\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right\}^{\frac{1}{\beta}}}},\ \frac{1}{1+\left\{\rho\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}\right)$$

Further consider

$q-\mathrm{ROF}S_{ft}\mathrm{DWA}\left(\mu\mathfrak{J}_{s_{11}},\mu\mathfrak{J}_{s_{12}},\ldots,\mu\mathfrak{J}_{s_{mn}}\right)=$

$$\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\rho\bar{\bar{w}}_i\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}},\ \frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_n\left(\sum_{i=1}^{m}\rho\bar{\bar{w}}_i\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right)$$

$$= \left(\sqrt[q]{1 - \frac{1}{1+\left\{\rho\left[\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\mu_{ij}^q}{1-\mu_{ij}^q}\right)^{\beta}\right)\right]\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\rho\left[\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\psi_{ij}}{\psi_{ij}}\right)^{\beta}\right)\right]\right\}^{\frac{1}{\beta}}} \right)$$

$$= \rho\, q - \text{ROF}S_{ft}\text{DWA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{mn}})$$

Therefore, the proof is completed.

6.3.2. q- Rung orthopair fuzzy soft Dombi ordered weighted averaging operators

In this subsection, in view of defined Dombi operation laws we will present the q-ROFS_{ft}DOWA operator and investigate their fundamental characteristics in details.

6.3.2.1. **Definition**

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for$ (i = 1,2, … , m and j = 1, 2, … , n) be the collection of q-ROFS_{ft}Fs. Suppose $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \ldots, \bar{\bar{w}}_m)^T$ be the weight vectors for expert k_i and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m}\bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n}\bar{\bar{u}}_j = 1$. Then q-ROF$S_t$DOWA operator is a mapping denoted and define as: $q - \text{ROF}S_{ft}\text{DOWA}: X^n \to X$ such that

$$q - \text{ROF}S_{ft}\text{DOWA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{mn}}) = \oplus_{j=1}^{n}\bar{\bar{u}}_j(\oplus_{i=1}^{m}\bar{\bar{w}}_i\mathfrak{J}_{s_{\delta ij}}), \qquad (6.3)$$

where $\mathfrak{J}_{s_{\delta ij}} = (\mu_{\delta ij}, \psi_{\delta ij})$ is the permutation of i^{th} row and j^{th} largest elements of the collections from $i \times j\ q - \text{ROF}S_{ft}\text{V}s$ $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $for\ (i = 1,2, \ldots, m\ and\ j = 1, 2, \ldots, n)$.

Based on Eq. (6.3) we can obtain the aggregated result for q-ROFS_{ft}DOWA operator as described in Theorem 6.3.2.2.

6.3.2.2. **Theorem**

Suppose the collection $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ (i = 1,2, \ldots, m\ and\ j = 1, 2, \ldots, n)$ of q-ROFS_{ft}Vs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \ldots, \bar{\bar{w}}_m)^T$ be the weight vectors for experts k_i and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{w}}_j = 1$. Then the aggregated result for q-ROFS_{ft}DOWA operator is stated as:

$$q - \text{ROF}S_{ft}\text{DOWA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{mn}}\right) = \oplus_{j=1}^{n} \bar{\bar{u}}_j (\oplus_{i=1}^{m} \bar{\bar{w}}_i \mathfrak{I}_{s_{\delta ij}})$$

$$= \left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{\mu_{\delta ij}{}^{q}}{1-\mu_{\delta ij}{}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}, \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\psi_{\delta ij}}{\psi_{\delta ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}} \right), \tag{6.4}$$

where $\mathfrak{I}_{s_{\delta ij}} = \left(\mu_{\delta ij}, \psi_{\delta ij}\right)$ is the permutation of i^{th} row and j^{th} largest elements of the collections from $i \times j$ $q -$ ROFS_{ft}Vs $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ for $(i = 1,2, \ldots, m$ and $j = 1, 2, \ldots, n)$.

Proof: Proof is straightforward like Theorem 6.3.1.2.

6.3.2.3. **Example**

Let $\mathfrak{I}_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ for $(i = 1, \ldots, 4$ and $j = 1, 2, 3)$ be the collection of $q -$ ROFS_{ft}Vs as mention in Table 6.1 of Example 6.3.1.3. Now by utilizing the Definition 6.1.2.2, the tabular description of $\mathfrak{I}_{s_{\delta ij}} = \left(\mu_{\delta ij}, \psi_{\delta ij}\right)$ is given in Table 6.2.

$$q - \text{ROF}S_{ft}\text{DOWA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{43}}\right) = \oplus_{j=1}^{3} \bar{\bar{u}}_j (\oplus_{i=1}^{4} \bar{\bar{w}}_i \mathfrak{I}_{s_{\delta ij}}) =$$

$$\left(\sqrt[3]{1 - \frac{1}{1+\left\{\sum_{j=1}^{3} \bar{\bar{u}}_j \left(\sum_{i=1}^{4} \bar{\bar{w}}_i \left(\frac{\mu_{\delta ij}{}^{3}}{1-\mu_{\delta ij}{}^{3}}\right)^{2}\right)\right\}^{\frac{1}{2}}}}, \frac{1}{1+\left\{\sum_{j=1}^{3} \bar{\bar{u}}_j \left(\sum_{i=1}^{4} \bar{\bar{w}}_i \left(\frac{1-\psi_{\delta ij}}{\psi_{\delta ij}}\right)^{2}\right)\right\}^{\frac{1}{2}}} \right)$$

Therefore $q - \text{ROF}S_{ft}\text{DOWA}\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{43}}\right) = (0.944704, 0.274876)$.

6.3.2.4. **Remarks**

(a) If we consider that the value of parameter $q = 1$ is fixed, then the proposed $q -$ ROFS_tDOWA operator reduced to IFS_{ft}DOWA operator.

(b) If we consider that the value of parameter $q = 2$ is fixed, then the proposed $q -$ ROFS_{ft}DOWA operator reduced to PyFS_{ft}DOWA operator.

Table 2, Tabular represent of q-ROFS_{ft}S $\mathfrak{I}_{s_{\delta ij}} = (\mu_{\delta ij}, \psi_{\delta ij})$ for $\beta = 2$ and $q = 3$ >

T	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
$\hbar_1$	$(0.98, 0.38)$	$(0.95, 0.45)$	$(0.82, 0.32)$
$\hbar_2$	$(0.9, 0.3)$	$(0.92, 0.3)$	$(0.8, 0.15)$
$\hbar_3$	$(0.85, 0.6)$	$(0.86, 0.4)$	$(0.7, 0.2)$
$\hbar_4$	$(0.7, 0.4)$	$(0.75, 0.25)$	$(0.6, 0.3)$

(c) If we consider that the value of parameter $q = 1$ is fixed, then the proposed $q -$ ROFS_tDOWA operator reduced to IFS_{ft}DOWA operator.

(d) If we consider that the value of parameter $q = 2$ is fixed, then the proposed $q -$ ROFS_{ft}DOWA operator reduced to PyFS_{ft}DOWA operator.

(e) If the set contain only parameter that is s_1 (means $m = 1$), in this case the proposed $q -$ ROFS_{ft}DOWA operator reduced to $q -$ ROFDOWA operator.

Thus from the analysis of Remark 6.3.2.4, it is clear that IFS_{ft}DOWA, PyFS_{ft}DOWA and $q -$ ROFDOWA operators are the special cases of the developed $q -$ ROFS_{ft}DOWA operator.

Based on the analysis of Theorem 6.3.2.2, some properties of the $q -$ ROFS_{ft}DOWA operators are investigated which are described below:

6.3.2.5. **Theorem**

Suppose the collection $\mathfrak{I}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ for $(i = 1,2, \dots, m$ and $j = 1, 2, \dots, n)$ of q-ROFS_tVs. Consider $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ be the weight vectors for experts $\hbar_i$ and $\overline{\overline{u}} = (\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_n)^T$ be the weight vectors for parameters s_j having the restriction that $\overline{\overline{w}}_i, \overline{\overline{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$ and $\sum_{j=1}^{n} \overline{\overline{u}}_j = 1$. Then the following properties are holds for q-ROFS_{ft}DOWA operator:

***i*: (Idempotency):** Let $\mathfrak{I}_{s_{ij}} = \mathcal{E}_s$ for all $(i = 1,2, \dots, m$ and $j = 1,2, \dots, n)$, where $\mathcal{E}_s = (b, d)$. Then

$$q - \text{ROF}S_{ft}\text{DOWA}(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \dots, \mathfrak{I}_{s_{mn}}) = \mathcal{E}_s .$$

ii: **(Boundedness):** Let $\Im_{s_{ij}}^{-} = \left(\min_j \min_i (\mu_{ij}), \max_j \max_i (\psi_{ij})\right)$ and

$\Im_{s_{ij}}^{+} = \left(\max_j \max_i (\mu_{ij}), \min_j \min_i (\psi_{ij})\right)$. Then

$$\Im_{s_{ij}}^{-} \leq q - \mathrm{ROF}S_{ft}\mathrm{DOWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}) \leq \Im_{s_{ij}}^{+} .$$

iii: **(Monotonicity):** Let another collection $\mathcal{E}_{s_{ij}} = (b_{ij}, \mathrm{d}_{ij})\ for$ (i = 1,2, … , m and j = 1, 2, … , n) of q-ROFS_tVs such that $\mu_{ij} \leq b_{ij}\ and\ \psi_{ij} \geq \mathrm{d}_{ij}$. Then

$$q - \mathrm{ROF}S_{ft}\mathrm{DOWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}) \leq q - \mathrm{ROF}S_{ft}\mathrm{DOWA}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{mn}}).$$

iv: **(Shift Invariance):** Let $\mathcal{E}_s = (b, d)$ be a q-ROFS_{ft}V. Then

$$q - \mathrm{ROF}S_{ft}\mathrm{DOWA}(\Im_{s_{11}} \oplus \mathcal{E}_s, \Im_{s_{12}} \oplus \mathcal{E}_s, \dots, \Im_{s_{mn}} \oplus \mathcal{E}_s)$$
$$= q - \mathrm{ROF}S_{ft}\mathrm{DOWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}) \oplus \mathcal{E}_s.$$

v: **(Homogeneity):** Let $\rho > 0$ be any real number. Then

$$q - \mathrm{ROF}S_{ft}\mathrm{DOWA}(\rho\Im_{s_{11}}, \rho\Im_{s_{12}}, \dots, \rho\Im_{s_{mn}})$$
$$= \rho\ q - \mathrm{ROF}S_{ft}\mathrm{DOWA}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}).$$

Proof: Proofs are directly follows from Theorem 6.3.1.5.

6.3.3. q- Rung orthopair fuzzy soft Dombi hybrid averaging operators

In this subsection, in view of defined Dombi operations we will present the study of q-ROFS_{ft}DHA operator and investigate their fundamental characteristics with details.

6.3.3.1. **Definition**

Let $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ (i = 1,2, \dots, m\ and\ j = 1, 2, \dots, n)$ be the collection of q-ROFS_{ft}Vs with $\mathscr{v} = (\mathscr{v}_1, \mathscr{v}_2, \dots, \mathscr{v}_m)^T$ and $r = (r_1, r_2, \dots, r_n)^T$ be the weight vector of $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ such that $\mathscr{v}_i, r_j \in [0,1]$ with $\sum_{i=1}^{m} \mathscr{v}_i = 1$ and $\sum_{j=1}^{n} r_j = 1$. Suppose $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for expert $\mathscr{k}_i$ and parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then q-ROF$S_{ft}$DHA operator is a mapping denoted and define as: $q - \mathrm{ROF}S_{ft}\mathrm{DHA}: X^n \to X$ such that

$$q-\mathrm{ROF}S_{ft}\mathrm{DHA}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\dots,\mathfrak{J}_{s_{mn}}\right)$$
$$=\oplus_{j=1}^{n}\bar{\bar{u}}_j\left(\oplus_{i=1}^{m}\bar{\bar{w}}_i\tilde{\tilde{\mathfrak{J}}}_{s_{\delta ij}}\right), \qquad (6.5)$$

where $\tilde{\tilde{\mathfrak{J}}}_{s_{\delta ij}}=n\varkappa_i r_j\mathfrak{J}_{s_{ij}}$ is the permutation of i^{th} row and j^{th} column largest elements of the collections from $q-\mathrm{ROF}S_t\mathrm{Vs}$ $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ and n is called the balancing coefficient.

Based on Eq. (6.5), we can obtain the aggregated result for q-ROFS_{ft}DHA operator as described in Theorem 6.3.3.2.

6.3.3.2. **Theorem**

Suppose the collection $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)\, for\ (i=1,2,\dots,m\ and\ j=1,2,\dots,n)$ of q-ROFS_{ft}Vs. Consider $\bar{\bar{w}}=(\bar{\bar{w}}_1,\bar{\bar{w}}_2,\dots,\bar{\bar{w}}_m)^T$ and $\bar{\bar{u}}=(\bar{\bar{u}}_1,\bar{\bar{u}}_2,\dots,\bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for experts $\hbar_i$ and parameters s_j having the conditions that $\bar{\bar{w}}_i,\bar{\bar{u}}_j\in[0,1]$ with $\sum_{i=1}^{m}\bar{\bar{w}}_i=1$ and $\sum_{j=1}^{n}\bar{\bar{u}}_j=1$. Then the aggregated result for q-ROFS_{ft}DHA operator is stated as:

$$q-\mathrm{ROF}S_{ft}\mathrm{DHA}\left(\mathfrak{J}_{\delta s_{11}},\mathfrak{J}_{\delta s_{12}},\dots,\mathfrak{J}_{\delta s_{mn}}\right)=\oplus_{j=1}^{n}\bar{\bar{u}}_j(\oplus_{i=1}^{m}\bar{\bar{w}}_i\tilde{\tilde{\mathfrak{J}}}_{s_{\delta ij}})$$
$$=\left(\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\tilde{\mu}_{\delta ij}{}^q}{1-\tilde{\mu}_{\delta ij}{}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}},\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\tilde{\psi}_{\delta ij}}{\tilde{\psi}_{\delta ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}\right) \qquad (6.6)$$

where $\tilde{\tilde{\mathfrak{J}}}_{s_{\delta ij}}=n\varkappa_i r_j\mathfrak{J}_{s_{ij}}$ is the permutation of i^{th} row and j^{th} column largest elements of the collections from $q-\mathrm{ROF}S_{ft}\mathrm{Vs}$ $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ with $\varkappa=(\varkappa_1,\varkappa_2,\dots,\varkappa_m)^T\ and\ r=(r_1,r_2,\dots,r_n)^T$ be the weight vector and n is a balancing coefficient.

Proof: Proof is straightforward like Theorem 6.3.1.2.

6.3.3.3. **Example**

Let $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)\ \ for\ (i=1,\dots,4$ and $j=1,2,3)$ be the collection of $q-$ROFS_{ft}Vs as mention in Table 6.1 of Example 6.3.1.3. Let $\varkappa=(0.25,0.28,0.29,0.18)^T\ \ and\ \ r=(0.36,0.29,0.35\ )^T$ be the weight vector of expert $\hbar_i$ and parameter s_j, and their corresponding aggregation associated weight vector $\bar{\bar{w}}=$

$(0.26,0.3,0.23,0.21)^T$ for expert $\mathcal{k}_i$ and $\bar{\bar{u}} = (0.35,0.31,0.34)^T$ for parameter s_j. Now by utilizing the operation mentioned in Eq. 6.7 and their corresponding results are given in Table 6.3, similarly by using score function as given in Definitions 6.1.2.2 and their results are given in Table 6.4. The tabular description for new ordering of $\widetilde{\widetilde{\mathfrak{J}}}_{s_{\delta ij}} = n\mathcal{v}_i r_j \mathfrak{J}_{s_{ij}}$ is given in Table 6.4.

$$\rho\mathfrak{J}_s = \left(\sqrt[q]{1 - \frac{1}{1 + \left\{\rho\left(\frac{{\mu_{\mathfrak{J}_s}}^q}{1-{\mu_{\mathfrak{J}_s}}^q}\right)^\beta\right\}^{\frac{1}{\beta}}}}, \frac{1}{1 + \left\{\rho\left(\frac{1-\psi_{\mathfrak{J}_s}}{\psi_{\mathfrak{J}_s}}\right)^\beta\right\}^{\frac{1}{\beta}}} \right) \tag{6.7}$$

$$q - \mathrm{ROF}S_{ft}\mathrm{DHA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{43}}\right) = \oplus_{j=1}^{3} \bar{\bar{u}}_j (\oplus_{i=1}^{4} \bar{\bar{w}}_i \widetilde{\widetilde{\mathfrak{J}}}_{s_{\delta ij}})$$

$$= \left(\sqrt[3]{1 - \frac{1}{1+\left\{\sum_{j=1}^{3}\bar{\bar{u}}_j\left(\sum_{i=1}^{4}\bar{\bar{w}}_i\left(\frac{{\widetilde{\mu}_{\delta ij}}^3}{1-{\widetilde{\mu}_{\delta ij}}^3}\right)^2\right)\right\}^{\frac{1}{2}}}}, \frac{1}{1+\left\{\sum_{j=1}^{3}\bar{\bar{u}}_j\left(\sum_{i=1}^{4}\bar{\bar{w}}_i\left(\frac{1-\widetilde{\psi}_{\delta ij}}{\widetilde{\psi}_{\delta ij}}\right)^2\right)\right\}^{\frac{1}{2}}} \right)$$

Therefore,

$$q - \mathrm{ROF}S_{ft}\mathrm{DHA}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{43}}\right) = (0.916853, 0.381134).$$

Table 6.3, Tabular represent of q-ROFS_{ft}S $\widetilde{\widetilde{\mathfrak{J}}}_{s_{\delta ij}} = n\mathcal{v}_i r_j \mathfrak{J}_{s_{ij}}$ for $\beta = 2$ *and* $q = 3$

T	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
$\mathcal{k}_1$	$(0.85153, 0.41667)$	$(0.7856, 0.55317)$	$(0.61795, 0.29705)$
$\mathcal{k}_2$	$(0.79509, 0.70258)$	$(0.66473, 0.36904)$	$(0.52791, 0.40636)$
$\mathcal{k}_3$	$(0.0.9697, 0.48677)$	$(0.87553, 0.42493)$	$(0.73722, 0.21689)$
$\mathcal{k}_4$	$(0.59438, 0.567)$	$(0.90169, 0.64165)$	$(0.7253, 0.48385)$

Table 6.4, Tabular description of score values for $\widetilde{\mathfrak{J}}_{s_{\delta ij}} = n\mathcal{v}_i r_j \mathfrak{J}_{s_{ij}}$ for $\beta = 2$ and $q = 3$

T	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
$\mathcal{k}_1$	(0.772555)	(0.657789)	(0.604883)
$\mathcal{k}_2$	(0.57791)	(0.621733)	(0.54001)
$\mathcal{k}_3$	(0.898249)	(0.797202)	(0.695232)
$\mathcal{k}_4$	(0.41385)	(0.734469)	(0.634138)

Table 6.4, New ordered for q-ROFS_{ft}S $\widetilde{\mathfrak{J}}_{s_{\delta ij}} = n\mathcal{v}_i r_j \mathfrak{J}_{s_{ij}}$ for $\beta = 2$ and $q = 3$

T	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
$\mathcal{k}_1$	$(0.9697, 0.48677)$	$(0.87553, 0.42493)$	$(0.73722, 0.21689)$
$\mathcal{k}_2$	$(0.85153, 0.41667)$	$(0.90169, 0.64165)$	$(0.7253, 0.48385)$
$\mathcal{k}_3$	$(0.79509, 0.70258)$	$(0.7856, 0.55317)$	$(0.61795, 0.29705)$
$\mathcal{k}_4$	$(0.59438, 0.567)$	$(0.66473, 0.36904)$	$(0.52791, 0.40636)$

6.3.3.4. **Remarks**

(a) If we consider that the value of $q = 1$ is fixed, then the proposed $q -$ ROFS_{ft}DHA operator reduced to IFS_{ft}DHA operator.

(b) If we consider that the value of $q = 2$ is fixed, then the proposed $q -$ ROFS_{ft}DHA operator reduced to PyFS_{ft}DHA operator.

(c) If the set contain only parameter that is s_1 (means $m = 1$), in this case the proposed $q -$ ROFS_{ft}DHA operator reduced to $q -$ ROFDHA operator.

Thus from the analysis of Remark 6.3.3.4, it is clear that IFS_{ft}DHA, PyFS_{ft}DHA and $q -$ ROFDHA operators are the special cases of the developed $q -$ ROFS_{ft}DHA operator.

Based on the analysis of Theorem 6.3.3.2, some properties of the $q -$ ROFS_{ft}DHA operators are investigated which are described below:

6.3.3.5. **Theorem**

Suppose the collection $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ for $(i = 1,2, \dots, m$ and $j = 1, 2, \dots, n)$ of q-ROFS_{ft}Vs with $\mathcal{v} = (\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_m)^T$ and $r = (r_1, r_2, \dots, r_n)^T$ be the weight vector of $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ such that $\mathcal{v}_i, r_j \in [0,1]$ with $\sum_{i=1}^{m} \mathcal{v}_i = 1$ and $\sum_{j=1}^{n} r_j = 1$. Suppose $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for expert $\mathcal{k}_i$ and parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then the following properties are holds for q-ROFS_{ft}DHA operator:

(i): **(Idempotency)** Let $\mathfrak{J}_{s_{ij}} = \mathcal{E}_s \; for \; all \; (i = 1,2, \dots, m \; and \; j = 1,2, \dots, n)$, where $\mathcal{E}_s = (b, d)$. Then

$$q - \text{ROF}S_{ft}\text{DHA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) = \mathcal{E}_s \,.$$

$(\boldsymbol{ii})$: **(Boundedness)** Let $\mathfrak{J}_{s_{ij}}^{-} = \left(\min_j \min_i (\mu_{ij}), \max_j \max_i (\psi_{ij}) \right)$ and

$\mathfrak{J}_{s_{ij}}^{+} = \left(\max_j \max_i (\mu_{ij}), \min_j \min_i (\psi_{ij}) \right)$. Then

$$\mathfrak{J}_{s_{ij}}^{-} \leq q - \text{ROF}S_{ft}\text{DHA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \leq \mathfrak{J}_{s_{ij}}^{+} \,.$$

$\boldsymbol{iii}$: **(Monotonicity)** Let another collection $\mathcal{E}_{s_{ij}} = (b_{ij}, \text{d}_{ij}) \; for \; (\text{i} = 1,2, \dots, \text{m and j} = 1, 2, \dots, \text{n})$ of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij} \; and \; \psi_{ij} \geq \text{d}_{ij}$. Then

$$q - \text{ROF}S_{ft}\text{DHA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \leq q - \text{ROF}S_{ft}\text{DHA}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{mn}}).$$

$\boldsymbol{iv}$: **(Shift Invariance)** Let $\mathcal{E}_s = (b, d)$ be a q-ROFS_tV. Then

$$q - \text{ROF}S_{ft}\text{DHA}(\mathfrak{J}_{s_{11}} \oplus \mathcal{E}_s, \mathfrak{J}_{s_{12}} \oplus \mathcal{E}_s, \dots, \mathfrak{J}_{s_{mn}} \oplus \mathcal{E}_s) \\ = q - \text{ROF}S_{ft}\text{DHA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \oplus \mathcal{E}_s.$$

$\boldsymbol{v}$: **(Homogeneity)** Let $\rho > 0$ be any real number. Then

$$q - \text{ROF}S_{ft}\text{DHA}(\rho\mathfrak{J}_{s_{11}}, \rho\mathfrak{J}_{s_{12}}, \dots, \rho\mathfrak{J}_{s_{mn}}) = \rho \; q - \text{ROF}S_{ft}\text{DHA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}).$$

Proof: Proofs are easy and directly follows Theorem 6.3.1.5.

6.4. An approach to $\mathcal{MCDM}$ under Dombi operations using q-rung orthopair fuzzy soft information

This section describes a $\mathcal{MCDM}$ techniques by using the applicability of developed operators for handling $\mathcal{MCDM}$ problems. Here criteria and parameter weights are real numbers and criteria values are q-ROFVs. The techniques of mathematical descriptions and their general steps wise algorithm under q-ROF environment is given as follows.

Suppose $T = \{\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_k\}$ be the collection of alternatives in which the most desirable alternative is going to evaluated by the senior decision makers $d = \{d_1, d_2, d_3, \dots, d_m\}$ against their corresponding parameters $\mathbb{E} = \{s_1, s_2, s_3, \dots, s_n\}$. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for expert $\hbar_i$ and parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_n \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. The senior decision makers gives their assessment for best alternative $\hbar_k$ against to parameter s_n in the form of q-ROFS_{ft}Vs $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ such that$0 \leq \mu_{ij}^q + \psi_{ij}^q \leq 1 \; for \; q \geq 1$. The decision makers describe their collective evaluated information in the form of q-ROFS_{ft} decision matrix $\mathbb{M} = [\mathfrak{J}_{s_{ij}}]_{m\times n}$. Using the preferences values of senior experts the aggregated result ξ_i for alternative $\hbar_i$ $(i = 1,2, \dots, k)$ is $\xi_i = (\mu_i, \psi_i)$ by applying the q-ROFS_{ft}DW averaging operations which is given in Eqs. 6.2, 6.4 and 6.6. Finally to get the most desirable alternative apply the score function on aggregated result ξ_i and rank them to get the best option.

6.4.1. Algorithm

The step wise decision algorithm for the developed operators are summarized as follows:

Step i: Extract the collective evaluated information of senior experts in the form of q-ROFS_{ft} decision matrix $\mathbb{M} = [\mathfrak{J}_{s_{ij}}]_{m\times n}$ for each alternative against their parameter.

$$\mathbb{M} = \begin{bmatrix} (\mu_{11}, \psi_{11}) & (\mu_{12}, \psi_{12}) & \cdots & (\mu_{1n}, \psi_{1n}) \\ (\mu_{21}, \psi_{21}) & (\mu_{22}, \psi_{22}) & \cdots & (\mu_{2n}, \psi_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \psi_{m1}) & (\mu_{m2}, \psi_{m2}) & \cdots & (\mu_{mn}, \psi_{mn}) \end{bmatrix}$$

Step ii: Using the preferences values of senior experts, aggregate the q-ROFS_{ft} $\mathfrak{J}_{s_{ij}}$ for alternative $\hbar_i$ $(i = 1,2, \dots, k)$ into collective decision matrix $\xi_i = (\mu_i, \psi_i)$ by applying the developed q-ROFS_{ft}D averaging and q-ROFS_{ft}D geometric operations.

Step iii: Applying the definition of score function determine the score values of ξ_i for each object $\hbar_i \; for \; i = (1,2, \dots, k)$.

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\hbar_i$.

6.5. Numerical example

In ordered to demonstrate the applicability and validity of the proposed method, a decision making process has been illustrated with the Constructional engineering projects (CEP) adopted from [92].

Suppose a particular example about the four potential CEP (alternatives) $T = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ and the committee of expert engineers $d = \{d_1, d_2, d_3, d_4, d_5\}$ whose weight vector is $\bar{\bar{u}} = (0.24,0.26,0.23,0.15,0.12)^T$ will give their assessment for the project against some parameter $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ and weight vector $\bar{\bar{u}} = (0.27,0.22,0.23,0.28)^T$, where $s_1 =$ the construction work environment, $s_2 =$ the construction site safety protection measure $s_3 =$ the safety production responsibility system, $s_4 =$ the safety management ability of the engineering project. The expert engineers $d_i (i = 1, \dots, 5)$ provides their assessment for each project against their parameter in the form of q-ROFVs. Following steps followed for finding the most desirable CEP by applying the developed approach.

By applying q-ROFS_{ft}DWA operator

Step i: The five expert engineers d_i will evaluate the construction of four CEP in terms of q-ROFVs, parameters and their rating results are given in Tables $6.5 - 6.8$ respectively.

Step ii: Applying the preferences values of senior engineers, the aggregated result for each alternative $\hbar_i$ $(i = 1, \dots, 4)$ by applying the developed q-ROFS_{ft}DWA operator for $q = 3\ and\ \beta = 2$ are gives as:

$$\xi_1 = (0.808828, 0.110937),\ \ \xi_2 = (0.758981, 0.165394),$$
$$\xi_3 = (0.743415, 0.14901),\ \ \xi_4 = (0.746606, 0.111174)$$

Step iii: Applying the definition of score function and determine the score values of ξ_i for each object $\hbar_i\ for\ i = (1, \dots, 4)$.

$$\mathcal{S}c(\xi_1) = 0.763886,\ \ \mathcal{S}c(\xi_2) = 0.716344,\ \ \mathcal{S}c(\xi_3) = 0.703776,\ \ \mathcal{S}c(\xi_4) = 0.7074$$

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\hbar_i$.

$$\hbar_1 \succcurlyeq \hbar_2 \succcurlyeq \hbar_4 \succcurlyeq \hbar_3$$

From the ranking result, it is clear that $\mathcal{k}_1$ is the most desirable and profitable CEP among all.

By applying q-ROFS$_{\mathbf{ft}}$DOWA operator

Step i: Same as above.

Step ii: Applying the preferences values of senior engineers, the aggregated result for each alternative $\mathcal{k}_i$ $(i = 1, \ldots, 4)$ by applying the developed q-ROFS_{ft}DOWA operator for $q = 3$ and $\beta = 2$ are gives as:

$$\xi_1 = (0.806913, 0.113117), \quad \xi_2 = (0.762272, 0.162736),$$
$$\xi_3 = (0.77327, 0.151224), \quad \xi_4 = (0.7564, 0.122431)$$

Step iii: Applying the definition of score function and determine the score values of ξ_i for each object $\mathcal{k}_i$ for $i = (1, \ldots, 4)$.

$$\mathcal{S}c(\xi_1) = 0.76197, \quad \mathcal{S}c(\xi_2) = 0.719308, \quad \mathcal{S}c(\xi_3) = 0.729458,$$
$$\mathcal{S}c(\xi_4) = 0.715466,$$

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\mathcal{k}_i$.

$$\mathcal{k}_1 \succcurlyeq \mathcal{k}_3 \succcurlyeq \mathcal{k}_2 \succcurlyeq \mathcal{k}_4$$

From the ranking result, it is clear that $\mathcal{k}_1$ is the most desirable and profitable CEP among all.

By applying q-ROFS$_{\mathbf{ft}}$DHA operator

Step i: Same as above.

Step ii: Applying the preferences values of senior engineers, the aggregated result for each alternative $\mathcal{k}_i$ $(i = 1, \ldots, 4)$ by applying the developed q-ROFS_{ft}DHA operator for $q = 3$ and $\beta = 2$. Let $\mathcal{v} = (0.22, 0.16, 0.2, 0.24, 0.18)^T$ and $r = (0.32, 0.29, 0.18, 0.21)^T$ be the weight vector of expert $\mathcal{k}_i$ and parameter s_j. Let $\bar{\bar{w}} = (0.24, 0.26, 0.23, 0.15, 0.12)^T$ and $\bar{\bar{u}} = (0.27, 0.22, 0.23, 0.28)^T$ be the corresponding aggregation associated weight vector for expert $\mathcal{k}_i$ and parameter s_j for $q = 3$ and $\beta = 2$. Then

$$\xi_1 = (0.709337, 0.213582), \quad \xi_2 = (0.653017, 0.280623),$$

$$\xi_3 = (0.678549, 0.25099), \qquad \xi_4 = (0.659497, 0.206046)$$

Step iii: Applying the definition of score function and determine the score values of ξ_i for each object $\mathcal{k}_i$ $for\ i = (1, \dots, 4)$.

$$\mathcal{S}c(\xi_1) = 0.673583, \quad \mathcal{S}c(\xi_2) = 0.628184, \quad \mathcal{S}c(\xi_3) = 0.64306,$$
$$\mathcal{S}c(\xi_4) = 0.639046$$

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\mathcal{k}_i$.

$$\mathcal{k}_1 \succcurlyeq \mathcal{k}_3 \succcurlyeq \mathcal{k}_4 \succcurlyeq \mathcal{k}_2$$

From the ranking result, it is clear that $\mathcal{k}_1$ is the most desirable and profitable CEP among all.

Therefore, from the analysis of illustrative example, it is evident that the ranking order of the alternatives are slightly different but the ranking concerning the most suitable and desirable alternative is identical that is $\mathcal{k}_1$ for overall introduced operators.

Table 6.5, q-ROFS_{ft} matrix for CEP $\mathcal{k}_1$

	s_1	s_2	s_3	s_4
d_1	(0.8,0.2)	(0.9,0.1)	(0.76,0.13)	(0.71,0.23)
d_2	(0.7,0.15)	(0.6,0.2)	(0.8,0.18)	(0.9,0.05)
d_3	(0.5,0.2)	(0.81,0.14)	(0.4,0.1)	(0.65,0.32)
d_4	(0.72,0.23)	(0.75,0.13)	(0.55,0.22)	(0.74,0.17)
d_5	(0.65,0.25)	(0.5,0.12)	(0.66,0.23)	(0.45,0.05)

6.5.1. Comparative analysis

To present the efficiency and applicability of the proposed method with some existing methods, a comparative study has been made based on different aggregation operators (see [4, 30, 68, 69, 71, 75]) under IF, PyF and q-ROF environment. For collective information different parameters of q-ROFS_tVs are aggregated by utilizing weighted averaging operator against to their weight vector $\bar{\bar{u}} = (0.27, 0.22, 0.23, 0.28)^T$, to obtain the aggregated q-ROFS_t decision matrix for different alternative $\mathcal{k}_i$ $(i = 1, \dots 4)$ as summarized in Table 6.9. From the evident of this decision matrix a comparative study has been made of the proposed methods with some existing methods and their simultaneous results are depicted in Table 6.10. From Table 6.10, it is clear that the ranking orders are

Table 6.6, q-ROFS_{ft} matrix for CEP $\&_2$

	s_1	s_2	s_3	s_4
d_1	(0.6,0.13)	(0.85,0.14)	(0.8,0.18)	(0.81,0.16)
d_2	(0.73,0.25)	(0.74,0.25)	(0.63,0.22)	(0.77,0.15)
d_3	(0.45,0.32)	(0.4,0.12)	(0.54,0.31)	(0.84,0.11)
d_4	(0.7,0.2)	(0.6,0.3)	(0.65,0.28)	(0.76,0.19)
d_5	(0.62,0.35)	(0.5,0.1)	(0.74,0.12)	(0.65,0.25)

Table 6.7. q-ROFS_{ft} matrix for CEP $\&_3$

	s_1	s_2	s_3	s_4
d_1	(0.74,0.23)	(0.55,0.12)	(0.48,0.1)	(0.42,0.15)
d_2	(0.6,0.15)	(0.66,0.31)	(0.78,0.12)	(0.64,0.22)
d_3	(0.82,0.16)	(0.74,0.25)	(0.5,0.3)	(0.3,0.1)
d_4	(0.65,0.34)	(0.58,0.28)	(0.73,0.25)	(0.48,0.26)
d_5	(0.9,0.08)	(0.6,0.2)	(0.4,0.1)	(0.61,0.35)

Table 6.8, q-ROFS_{ft} matrix for CEP $\&_4$

	s_1	s_2	s_3	s_4
d_1	(0.63,0.14)	(0.45,0.13)	(0.55,0.25)	(0.62,0.15)
d_2	(0.35,0.05)	(0.65,0.18)	(0.75,0.18)	(0.48,0.22)
d_3	(0.7,0.17)	(0.9,0.09)	(0.6,0.3)	(0.52,0.16)
d_4	(0.39,0.25)	(0.25,0.1)	(0.56,0.16)	(0.67,0.26)
d_5	(0.8,0.12)	(0.76,0.23)	(0.34,0.05)	(0.38,0.1)

slightly different but their best optimal alternative remain same for all operators that is $\hbar_1$. However, in many situations of real life IFS and PyFS cannot provide the additional space to the decision makers to describe the attribute evaluation value due to its restricted constraints. Obviously in q-ROF environment the experts fully express the decision information. From characteristic analysis the existing methods in [4, 30, 75] are best for fuzzy data and these methods having no information about soft parameterization tools and Dombi operational parameter. Similarly the methods in [68, 69, 71] have just Dombi operational parameter. Therefore, from the characteristic point of view the methods proposed in this chapter are more superior and practical for real life information to describe the fuzzy data under soft parameterizations information by using Dombi operational law.

6.5.2. Influence of operational parameter β

To express the influence and potential of operational parameter β on $\mathcal{MCDM}$, different values of β are utilized to rank the alternatives. For different input of β in the range of $1 \leq \beta \leq 30$, the score values and their ranking order of alternatives $\hbar_i(i = 1, ... ,5)$ based on q-ROFS_{ft} DWA operators are depicted in Tables 6.11. From the analysis of Tables 11, it is clear that for different input values of β the ranking order is slightly different but the best optimal option remain identical that is $\hbar_1$ for q-ROFS_{ft} DWA operators. By increasing the value of β cause gradual increase in score values for q-ROFS_{ft} DWA operators. This show that increasing the values of β from smaller to bigger cause the decision makers' attitude from pessimism to optimism for q-ROFS_{ft} DWA operators. Thus the behaviour of operational parameter β is very important to express the experts' attitude in decision making problems. Therefore, from overall analysis it is concluded that the proposed method is more superior and resilience than existing methods to solve the real life decisions by using parameterization tools under Dombi operational law.

6.5.3. Conclusion

The process of $\mathcal{DM}$ is a complex issue involves professionals of different genre. Every organization have to take decision at one point or another as a part of managerial process. Therefore, every organization extensively needs a team of professional experts to make all sorts of complex decision. But remember, that an individual alone cannot come out with final decision because decision making problems consist of cumulative and consultative

Table 6.9, Aggregated values of q-ROFS_{ft} matrix for CEP for $\Bbbk_i$

	$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_3$	$\Bbbk_4$
d_1	(0.839983,0.143578)	(0.805096,0.148488)	(0.647834,0.133022)	(0.592306,0.152046)
d_2	(0.846328,0.084283)	(0.735176,0.198556)	(0.70186,0.164896)	(0.658083,0.086928)
d_3	(0.714504,0.149203)	(0.759122,0.149169)	(0.747558,0.147595)	(0.826828,0.139741)
d_4	(0.719443,0.174289)	(0.705968,0.223092)	(0.653149,0.277035)	(0.585882,0.156939)
d_5	(0.607438,0.084324)	(0.666902,0.147059)	(0.836595,0.115345)	(0.734921,0.082803)

Table 6.10, Comparative analysis of existing methods with proposed methods

Methods	*Score values of alternatives*				Ranking
	$\Bbbk_1$	$\Bbbk_2$	$\Bbbk_3$	$\Bbbk_4$	
IFWA [4]	0.659028	0.578421	0.557667	0.575095	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_4 \succ \Bbbk_3$
PyFWA [75]	0.598381	0.532558	0.489755	0.474926	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_3 \succ \Bbbk_4$
PyFDWA [69]	0.640947	0.54832	0.527668	0.538915	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_4 \succ \Bbbk_3$
PyDFWA [68]	0.639334	0.547267	0.526257	0.538347	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_3 \succ \Bbbk_4$
q-ROFDWA [71]	0.765043	0.720637	0.706817	0.65305	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_3 \succ \Bbbk_4$
q-ROFWA [30]	0.482652	0.417154	0.368254	0.347148	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_3 \succ \Bbbk_4$
q-ROFS_tDWA (proposed)	0.763886	0.716344	0.703776	0.7074	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_4 \succ \Bbbk_3$
q-ROFS_tDOWA (proposed)	0.76197	0.719308	0.729458	0.715466	$\Bbbk_1 \succ \Bbbk_3 \succ \Bbbk_2 \succ \Bbbk_4$
q-ROFS_tDHA (proposed)	0.673583	0.628184	0.64306	0.639046	$\Bbbk_1 \succ \Bbbk_2 \succ \Bbbk_3 \succ \Bbbk_4$

Table 6.11, Ranking order based on different operational parameter of q-ROFS_{ft} DWA operator

Operational parameter β	$Score\ values\ of\ alternatives$				Ranking
	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	
$\beta = 1$	0.722591	0.691494	0.656704	0.645816	$\hbar_1 \succ \hbar_2 \succ \hbar_3 \succ \hbar_4$
$\beta = 2$	0.763886	0.716344	0.703776	0.7074	$\hbar_1 \succ \hbar_2 \succ \hbar_4 \succ \hbar_3$
$\beta = 3$	0.791403	0.733061	0.741876	0.752795	$\hbar_1 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$
$\beta = 5$	0.820077	0.75353	0.787956	0.798473	$\hbar_1 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$
$\beta = 8$	0.837303	0.769509	0.817705	0.824584	$\hbar_1 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$
$\beta = 12$	0.846658	0.780173	0.833981	0.838489	$\hbar_1 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$
$\beta = 16$	0.851226	0.786025	0.84187	0.845221	$\hbar_1 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$
$\beta = 20$	0.853929	0.789728	0.846506	0.849184	$\hbar_1 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$
$\beta = 25$	0.856068	0.792719	0.85016	0.852313	$\hbar_1 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$
$\beta = 30$	0.857484	0.79496	0.852567	0.854377	$\hbar_2 \succ \hbar_4 \succ \hbar_3 \succ \hbar_2$

process. Since intellectual minds are engage in this process, so it needs solid scientific knowledge couple with experience and skills in addition to mental maturity. Recently, Yager investigated the generalized concept of FS, IFS and PyFS and called it q-ROFS. It is observed that the parameter q is the most useful characteristic of this concept which has the capability to cover the boundary range that can be required. The input range of q-ROFS is more flexible, wider and suitable because when the rung increase, the orthopair provides additional space to the boundary constraint. The aim of this chapter is to present the notion of q-ROFS_{ft}S based on the Dombi operations. Since Dombi operational parameter possess natural flexibility with resilience of variability. The behaviour of Dombi operational parameter is very important to express the experts' attitude in decision making. Further we present q-ROFS_{ft}DA aggregation operators including q-ROFS_{ft}DWA, q-ROFS_{ft}DOWA and q-ROFS_{ft}DHA operators. The basic properties of these operators are presented in detail such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity. By applying develop approach, this manuscript contains the technique and algorithm for $\mathcal{MCDM}$. Further a numerical example is developed to illustrative the flexibility and applicability of the developed operators.

Chapter 7

Orthopair fuzzy soft Dombi geometric aggregation operators

The aim of this chapter is to present the notion of Dombi operation in q-ROFS_{ft}S. Since Dombi operational parameter possess natural flexibility with resilience of variability. The behaviour of Dombi operational parameter is very important to express the experts' attitude in $\mathcal{DM}$. In this chapter, we will present the concept of q-ROFS_{ft}DG aggregation operators including q-ROFS_{ft}DWG, q-ROFS_{ft}DOWG and q-ROFS_{ft}DHG operators. The basic properties of these operators are presented in detail such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity. A $\mathcal{MCDM}$ technique and algorithm is developed based on above mentioned approach. Further a numerical example is developed to illustrative the flexibility and applicability of the developed operators.

7.1. q-Rung orthopair fuzzy soft set

The detailed study of PyFS_{ft}Ss, q-ROFS_{ft}Ss and their fundamental operations and relations are presented in Sections 4.1 and 4.2 of Chapter 4. The score function and accuracy function for q-ROFS_{ft}Ss are given in Definitions 6.1.1 and 6.1.2.

7.2. Dombi operations on q-rung orthopair fuzzy soft set

For a detail study of Dombi sum and Dombi product and its basic operations and relations are given in Chapter 6 Section 6.2.

7.3. q-Rung orthopair fuzzy soft Dombi geometric operators

In this section, in view of defined Dombi operation laws we will extend Dombi operators to q-ROFS_{ft} environment such as q-ROFS_{ft}WG, q-ROFS_{ft}DOWG and q-ROFS_{ft}DHG operators and investigate their fundamental properties in details.

7.3.1. q-Rung orthopair fuzzy soft Dombi weighted geometric operators

This subsection is devoted for the study of q-ROFS_{ft}DWG operator and discuss their basic properties in details.

7.3.1.1. Definition

Let $\mathfrak{I}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ (i = 1,2, \ldots, \text{m and } j = 1, 2, \ldots, \text{n})$ be the collection of q-ROFS_{ft}Vs. Suppose $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \ldots, \bar{\bar{w}}_m)^T$ be the weight vectors for expert k_i and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then q-ROF$S_{ft}$DWG operator is a mapping denoted and define as: $q - \text{ROF}S_t\text{DWG}: X^n \to X$, (where X represents the collection of q-ROFS_{ft}Vs) such that

$$q - \text{ROF}S_{ft}\text{DWG}(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{mn}}) = \otimes_{j=1}^{n} \left(\otimes_{i=1}^{m} \left(\mathfrak{I}_{s_{ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right) \qquad (7.1)$$

Based on Eq. (7.1) we can obtain the aggregated result for q-ROFS_{ft}DWG operator as described in Theorem 7.3.1.2.

7.3.1.2. Theorem

Suppose the collection $\mathfrak{I}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ (i = 1,2, \ldots, m\ and\ j = 1, 2, \ldots, n)$ of q-ROFS_{ft}Vs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \ldots, \bar{\bar{w}}_m)^T$ be the weight vectors for experts k_i and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \ldots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then the aggregated result for q-ROFS_{ft}DWG operator is stated as:

$$q - \text{ROF}S_{ft}\text{DWG}(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{mn}}) = \otimes_{j=1}^{n} \left(\otimes_{i=1}^{m} \left(\mathfrak{I}_{s_{ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)$$

$$= \left(\frac{1}{1+\left\{ \sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{ \sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{\psi_{ij}{}^{q}}{1-\psi_{ij}{}^{q}} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}} \right) \qquad (7.2)$$

Proof: The required proof can be obtained by using mathematical induction.

From Dombi operational laws, we have

$$\mathfrak{J}_{s_{11}} \otimes \mathfrak{J}_{s_{12}} = \left(\frac{1}{1+\left\{\left(\frac{1-\mu_{11}{}^{q}}{\mu_{11}{}^{q}}\right)^{\beta} + \left(\frac{1-\mu_{12}}{\mu_{12}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\left(\frac{\psi_{11}{}^{q}}{1-\psi_{11}{}^{q}}\right)^{\beta} + \left(\frac{\psi_{12}{}^{q}}{1-\psi_{12}{}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}} \right) \quad and$$

$$\rho \mathfrak{J}_{s} = \left(\frac{1}{1+\left\{\rho\left(\frac{1-\mu_{\mathfrak{J}_s}}{\mu_{\mathfrak{J}_s}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\rho\left(\frac{\psi_{\mathfrak{J}_s}{}^{q}}{1-\psi_{\mathfrak{J}_s}{}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}} \right)$$

Now first we show that Eq. 7.2, holds for $m = 2 \ and \ n = 2$,

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}\right) = \otimes_{j=1}^{2}\left(\otimes_{i=1}^{2}\left(\mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)$$

$$= \left(\mathfrak{J}_{s_{11}}^{\bar{\bar{w}}_1} \otimes \mathfrak{J}_{s_{21}}^{\bar{\bar{w}}_2}\right)^{\bar{\bar{u}}_1} \otimes \left(\mathfrak{J}_{s_{12}}^{\bar{\bar{w}}_1} \otimes \mathfrak{J}_{s_{22}}^{\bar{\bar{w}}_2}\right)^{\bar{\bar{u}}_2}$$

$$= \left(\frac{1}{1+\left\{\bar{\bar{w}}_1\left(\frac{1-\mu_{11}}{\mu_{11}}\right)^{\beta} + \bar{\bar{w}}_2\left(\frac{1-\mu_{21}}{\mu_{21}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\bar{\bar{w}}_1\left(\frac{\psi_{11}{}^{q}}{1-\psi_{11}{}^{q}}\right)^{\beta} + \bar{\bar{w}}_2\left(\frac{\psi_{21}{}^{q}}{1-\psi_{21}{}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}} \right)^{\bar{\bar{u}}_1} \otimes$$

$$\left(\frac{1}{1+\left\{\bar{\bar{w}}_1\left(\frac{1-\mu_{12}}{\mu_{12}}\right)^{\beta} + \bar{\bar{w}}_2\left(\frac{1-\mu_{22}}{\mu_{22}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\bar{\bar{w}}_1\left(\frac{\psi_{12}{}^{q}}{1-\psi_{12}{}^{q}}\right)^{\beta} + \bar{\bar{w}}_2\left(\frac{\psi_{22}{}^{q}}{1-\psi_{22}{}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}} \right)^{\bar{\bar{u}}_2}$$

$$= \left(\frac{1}{1+\left\{\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{1-\mu_{i1}}{\mu_{i1}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{\psi_{i1}{}^{q}}{1-\psi_{i1}{}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}} \right)^{\bar{\bar{u}}_1} \otimes$$

$$\left(\frac{1}{1+\left\{\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\mu_{i2}}{\mu_{i2}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{\psi_{i2}{}^{q}}{1-\psi_{i2}{}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}} \right)^{\bar{\bar{u}}_2}$$

$$= \left(\begin{array}{c} \dfrac{1}{1+\left\{\bar{\bar{u}}_1\left(\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{1-\mu_{i1}}{\mu_{i1}}\right)^{\beta}\right)+\bar{\bar{u}}_2\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\mu_{i2}}{\mu_{i2}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \\ \sqrt[q]{1-\dfrac{1}{1+\left\{\bar{\bar{u}}_1\left(\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{{\psi_{i1}}^{q}}{1-{\psi_{i1}}^{q}}\right)^{\beta}\right)+\bar{\bar{u}}_2\left(\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{{\psi_{i2}}^{q}}{1-{\psi_{i2}}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \end{array} \right)$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{2}\bar{\bar{u}}_j\left(\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{2}\bar{\bar{u}}_j\left(\sum_{i=1}^{2}\bar{\bar{w}}_i\left(\frac{{\psi_{ij}}^{q}}{1-{\psi_{ij}}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \right)$$

Hence the result is true for $m = 2\ and\ n = 2$.

Further, let Eq. 7.2, is true for $m = k_1\ and\ n = k_2$,

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{k_1k_2}}\right) = \otimes_{j=1}^{k_2}\left(\otimes_{i=1}^{k_1}\left(\mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2}\bar{\bar{u}}_j\left(\sum_{i=1}^{k_1}\bar{\bar{w}}_i\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{k_2}\bar{\bar{u}}_j\left(\sum_{i=1}^{k_1}\bar{\bar{w}}_i\left(\frac{{\psi_{ij}}^{q}}{1-{\psi_{ij}}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \right)$$

Next to show that Eq. 7.2, is true for $m = k_1 + 1\ and\ n = k_2 + 1$

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{k_1k_2}}, \mathfrak{J}_{s_{(k_1+1)(k_2+1)}}\right)$$
$$= \otimes_{j=1}^{k_2}\left(\otimes_{i=1}^{k_1}\left(\mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right) \otimes \left(\mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_{k_1+1}}\right)^{\bar{\bar{u}}_{k_2+1}}$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2}\bar{\bar{u}}_j\left(\sum_{i=1}^{k_1}\bar{\bar{w}}_i\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{k_2}\bar{\bar{u}}_j\left(\sum_{i=1}^{k_1}\bar{\bar{w}}_i\left(\frac{{\psi_{ij}}^{q}}{1-{\psi_{ij}}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \right)$$

$$\otimes \left(\frac{1}{1+\left\{\bar{\bar{u}}_{k_2+1}\left(\bar{\bar{w}}_{k_1+1}\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1-\frac{1}{1+\left\{\bar{\bar{u}}_{k_2+1}\left(\bar{\bar{w}}_{k_1+1}\left(\frac{{\psi_{ij}}^{q}}{1-{\psi_{ij}}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \right)$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2} \bar{u}_j \left(\sum_{i=1}^{k_1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^{\beta} \right) + \bar{u}_{k_2+1} \left(\bar{\bar{w}}_{k_1+1} \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2} \bar{u}_j \left(\sum_{i=1}^{k_1} \bar{\bar{w}}_i \left(\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q} \right)^{\beta} \right) + \bar{u}_{k_2+1} \left(\bar{\bar{w}}_{k_1+1} \left(\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}} \right)$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{u}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}, \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{u}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}} \right)$$

Hence that Eq. 7.2, is true for $m = k_1 + 1$ *and* $n = k_2 + 1$. Therefore, by process of mathematical induction, we conclude that Eq. 7.2, is true for all $m, n \geq 1$.

Further to verify that the aggregated result obtained from $q-$ROFS_{ft}DWG$\left(\mathfrak{I}_{s_{11}}, \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{k_1 k_2}}\right)$ is again a $q-$ROFS_{ft}V.

Let

$$\mu = \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{u}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}} \quad and \quad \lambda = \sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{u}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}}$$

As

$$0 \leq \mu_{ij} \leq 1 \Rightarrow 0 \leq \frac{1}{1+\frac{1-\mu_{ij}}{\mu_{ij}}} \leq 1 \Rightarrow 0 \leq \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{u}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}$$

$$\leq 1 \Rightarrow 0 \leq \mu \leq 1$$

Similarly

$$0 \leq \psi_{ij} \leq 1 \Rightarrow 0 \leq 1 - \frac{1}{1+\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q}} \leq 1 \Rightarrow 0 \leq$$

$$\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{u}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q} \right)^{\beta} \right) \right\}^{\frac{1}{\beta}}}} \leq 1$$

$$\Rightarrow 0 \leq \lambda \leq 1$$

As

$$\mu^q + \lambda^q \leq 1 \Rightarrow \lambda^q \leq 1 - \mu^q$$

$$\Rightarrow \left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q} \right)^\beta \right) \right\}^{\frac{1}{\beta}}}} \right)^q \leq 1 -$$

$$\left(\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^\beta \right) \right\}^{\frac{1}{\beta}}} \right)^q$$

Next

$$0 \leq \mu^q + \lambda^q \Rightarrow \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^\beta \right) \right\}^{\frac{1}{\beta}}} \right)^q +$$

$$\left(\sqrt[q]{1 - \frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{{\psi_{ij}}^q}{1-{\psi_{ij}}^q} \right)^\beta \right) \right\}^{\frac{1}{\beta}}}} \right)^q$$

$$\leq \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^\beta \right) \right\}^{\frac{1}{\beta}}} \right)^q + 1 - \left(\frac{1}{1+\left\{\sum_{j=1}^{k_2+1} \bar{\bar{u}}_j \left(\sum_{i=1}^{k_1+1} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}} \right)^\beta \right) \right\}^{\frac{1}{\beta}}} \right)^q$$

$$\Rightarrow 0 \leq \mu^q + \lambda^q \leq 1$$

Therefore, it is verified that the aggregated result obtained from $q-$ROFS_{ft}DWG$\left(\mathfrak{I}_{s_{11}},\ \mathfrak{I}_{s_{12}}, \ldots, \mathfrak{I}_{s_{k_1 k_2}}\right)$ is again a $q-$ ROFS_{ft}V.

7.3.1.3. Example

Suppose $T = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ be the set of expert Professors who want to judge the ability of a student Z under the set of parameters $\mathbb{E} = \{s_1, s_2, s_3\}$, where $s_j (j = 1,2,3)$ stands for $s_1 = responsable, s_2 = coures\ command\ and\ s_3 = punctual.$ The experts provides their estimated values in the form of q-ROFS_{ft}Vs which are given in

Table 7.1. Let $\mathcal{u} = (0.26,0.3,0.23,0.21)^T$ be the weight vectors for expert $\mathcal{k}_i$, $\mathcal{z} = (0.35,0.31,0.34)^T$ be the weight vectors for parameters s_j and operational parameter $\beta = 2$ for $q = 3$. Now to calculate the aggregated result by applying Eq. 7.2, we have

$$q-\mathrm{ROF}S_t\mathrm{DWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) = \otimes_{j=1}^{3}\left(\otimes_{i=1}^{4}\left(\mathfrak{J}_{s_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{3}\bar{\bar{u}}_j\left(\sum_{i=1}^{4}\bar{\bar{w}}_i\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^2\right)\right\}^{\frac{1}{2}}}, \sqrt[3]{1-\frac{1}{1+\left\{\sum_{i=1}^{3}\bar{\bar{u}}_j\left(\sum_{i=1}^{4}\bar{\bar{w}}_i\left(\frac{\psi_{ij}{}^3}{1-\psi_{ij}{}^3}\right)^2\right)\right\}^{\frac{1}{2}}}}\right)$$

$$= (0.754181, 0.449179)$$

Therefore,

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{43}}) = (0.754181, 0.449179).$$

Table 7.1, Tabular represent of q-ROFS_{ft}S $(\mathfrak{J}, \mathbb{E})$ for $\beta = 2$ *and* $q = 3$

T	s_1	s_2	s_3
$\mathcal{k}_1$	$(0.9, 0.3)$	$(0.85, 0.4)$	$(0.7, 0.2)$
$\mathcal{k}_2$	$(0.85, 0.6)$	$(0.75, 0.25)$	$(0.6, 0.3)$
$\mathcal{k}_3$	$(0.98, 0.38)$	$(0.92, 0.3)$	$(0.8, 0.15)$
$\mathcal{k}_4$	$(0.7, 0.4)$	$(0.95, 0.45)$	$(0.82, 0.32)$

7.3.1.4. Remarks

(a) If we consider that the value of parameter $q = 1$ is fixed, then the proposed $q-$ ROFS_{ft}DWG operator reduced to IFS_{ft}DWG operator.

(b) If we consider that the value of parameter $q = 2$ is fixed, then the proposed $q-$ ROFS_{ft}DWG operator reduced to PyFS_{ft}DWG operator.

(c) If the set contain only parameter that is s_1 (means $m = 1$), in this case the proposed $q-$ ROFS_{ft}DWG operator reduced to $q-$ ROFDWG operator.

Thus from the analysis of Remark 7.3.1.4, it is clear that IFS_{ft}DWG, PyFS_{ft}DWG and $q-$ROFDWG operators are the special cases of the developed $q-\text{ROF}S_{ft}$DWG operator.

Based on Theorem 7.3.1.2, some properties of the $q-\text{ROF}S_{ft}$DWG operators are investigated which are described below:

7.3.1.5. Theorem

Suppose the collection $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ (i = 1,2, \dots, m\ and\ j = 1, 2, \dots, n)$ of q-ROFS_{ft}Vs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ be the weight vectors for experts $\mathscr{k}_i$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the restriction that $\bar{\bar{w}}_i, \bar{\bar{u}}_n \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{i=1}^{m} \bar{\bar{u}}_n = 1$. Then the following properties are holds for q-ROFS_{ft}DWG operator:

i: **(Idempotency)** Let $\mathfrak{J}_{s_{ij}} = \mathcal{E}_s\ for\ all\ (i = 1,2, \dots, m\ and\ j = 1,2, \dots, n)$, where $\mathcal{E}_s = (b, d)$. Then

$$q-\text{ROF}S_{ft}\text{DWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) = \mathcal{E}_s\ .$$

$\boldsymbol{ii}$: **(Boundedness)** Let $\mathfrak{J}^{-}_{s_{ij}} = \left(\min_j \min_i (\mu_{ij}), \max_j \max_i (\psi_{ij})\right)$ and

$\mathfrak{J}^{+}_{s_{ij}} = \left(\max_j \max_i (\mu_{ij}), \min_j \min_i (\psi_{ij})\right)$. Then

$$\mathfrak{J}^{-}_{s_{ij}} \le q-\text{ROF}S_{ft}\text{DWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \le \mathfrak{J}^{+}_{s_{ij}}\ .$$

$\boldsymbol{iii}$: **(Monotonicity)** Let another collection $\mathcal{E}_{s_{ij}} = (b_{ij}, \text{d}_{ij})\ for\ (\text{i} = 1,2, \dots, \text{m and j} = 1, 2, \dots, \text{n})$ of q-ROFS_{ft}Vs such that $\mu_{ij} \le b_{ij}\ and\ \psi_{ij} \ge \text{d}_{ij}$. Then

$$q-\text{ROF}S_{ft}\text{DWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \le q-\text{ROF}S_{ft}\text{DWG}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{mn}}).$$

$\boldsymbol{iv}$: **(Shift Invariance)** Let $\mathcal{E}_s = (b_{\mathcal{E}_s}, \text{d}_{\mathcal{E}_s})$ be a q-ROFS_{ft}V. Then

$$q-\text{ROF}S_{ft}\text{DWG}(\mathfrak{J}_{s_{11}} \otimes \mathcal{E}_s, \mathfrak{J}_{s_{12}} \otimes \mathcal{E}_s, \dots, \mathfrak{J}_{s_{mn}} \otimes \mathcal{E}_s)$$
$$= q-\text{ROF}S_{ft}\text{DWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \otimes \mathcal{E}_s.$$

$\boldsymbol{v}$: **(Homogeneity)** Let $\rho > 0$ be any real number. Then

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\rho\mathfrak{J}_{s_{11}},\rho\mathfrak{J}_{s_{12}},\dots,\rho\mathfrak{J}_{s_{mn}}\right)$$
$$=\rho\, q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\dots,\mathfrak{J}_{s_{mn}}\right).$$

Proof: *i:* (Idempotency) Since $\mathfrak{J}_{s_{ij}}=\mathcal{E}_s\ for\ all\ (i=1,2,\dots,m\ and\ j=1,2,\dots,n)$, where $\mathcal{E}_s=(b,d)$. Then by Theorem 7.3.1.2, we have

$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\dots,\mathfrak{J}_{s_{mn}}\right)=$

$$\left(\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{{\psi_{ij}}^{q}}{1-{\psi_{ij}}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}\right)$$

$$=\left(\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-b_{ij}}{b_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{{d_{ij}}^{q}}{1-{d_{ij}}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}\right)$$

$$=\left(\frac{1}{1+\left\{\left(\frac{1-b_{ij}}{b_{ij}}\right)^{\beta}\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\left(\frac{{d_{ij}}^{q}}{1-{d_{ij}}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}}\right)$$

$$=\left(\frac{1}{1+\left(\frac{1-b_{ij}}{b_{ij}}\right)},\sqrt[q]{1-\frac{1}{1+\left(\frac{{d_{ij}}^{q}}{1-{d_{ij}}^{q}}\right)}}\right)$$

$=(b,d)=\mathcal{E}_s$

Hence, the proof is complete.

***ii*: (Boundedness)** Consider for each $i=1,2,\dots,m\ and\ j=1,2,\dots,n$, we have

$$\min_j\min_i\left(\mu_{ij}\right)\le\mu_{ij}\le\max_j\max_i\left(\mu_{ij}\right)$$

$$\Rightarrow 1+\frac{1-\min_j\min_i\left(\mu_{ij}\right)}{\min_j\min_i\left(\mu_{ij}\right)}\ge 1+\frac{1-\mu_{ij}}{\mu_{ij}}\ge 1+\frac{1-\max_j\max_i\left(\mu_{ij}\right)}{\max_j\max_i\left(\mu_{ij}\right)}$$

$$\Rightarrow \frac{1}{1+\frac{1-\min_j \min_i (\mu_{ij})}{\min_j \min_i (\mu_{ij})}} \leq \frac{1}{1+\frac{1-\mu_{ij}}{\mu_{ij}}} \leq \frac{1}{1+\frac{1-\max_j \max_i (\mu_{ij})}{\max_j \max_i (\mu_{ij})}}$$

$$\Rightarrow \frac{1}{1+\left[\sum_{j=1}^{n} \bar{\bar{u}}_j \left\{\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\min_j \min_i (\mu_{ij})}{\min_j \min_i (\mu_{ij})}\right)^{\beta}\right\}\right]^{\frac{1}{\beta}}} \leq \frac{1}{1+\left[\sum_{j=1}^{n} \bar{\bar{u}}_j \left\{\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right\}\right]^{\frac{1}{\beta}}} \leq$$
$$\frac{1}{1+\left[\sum_{j=1}^{n} \bar{\bar{u}}_j \left\{\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\max_j \max_i (\mu_{ij})}{\max_j \max_i (\mu_{ij})}\right)^{\beta}\right\}\right]^{\frac{1}{\beta}}}$$

$$\Rightarrow \frac{1}{1+\left(\frac{1-\min_j \min_i (\mu_{ij})}{\min_j \min_i (\mu_{ij})}\right)} \leq \frac{1}{1+\left[\sum_{j=1}^{n} \bar{\bar{u}}_j \left\{\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right\}\right]^{\frac{1}{\beta}}} \leq \frac{1}{1+\left(\frac{1-\max_j \max_i (\mu_{ij})}{\max_j \max_i (\mu_{ij})}\right)}$$

Similarly we can show for nonmembership

$$\sqrt[q]{1-\frac{1}{1+\frac{\min_j \min_i (\psi_{ij}^q)}{1-\min_j \min_i (\psi_{ij}^q)}}} \geq \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{\psi_{ij}^q}{1-\psi_{ij}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \geq \sqrt[q]{1-\frac{1}{1+\frac{\max_j \max_i (\psi_{ij}^q)}{1-\max_j \max_i (\psi_{ij}^q)}}}$$

Therefore, from above analysis we have

$$\mathfrak{J}_{s_{ij}}^{-} \leq q-\mathrm{ROF}S_{ft}\mathrm{DWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \leq \mathfrak{J}_{s_{ij}}^{+}.$$

iii: **(Monotonicity)** Since $for\ all\ i = 1,2,\dots,m\ and\ j = 1,2,\dots,n$, we have $\mu_{ij} \leq b_{ij}\ and\ \psi_{ij} \geq \mathrm{d}_{ij}$.

As

$$\mu_{ij} \leq b_{ij} \Rightarrow 1+\frac{1-\mu_{ij}}{\mu_{ij}} \geq 1+\frac{1-b_{ij}}{b_{ij}}$$

$$\Rightarrow \frac{1}{1+\frac{1-\mu_{ij}}{\mu_{ij}}} \leq \frac{1}{1+\frac{1-b_{ij}}{b_{ij}}}$$

$$\Rightarrow \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}} \leq \frac{1}{1+\left\{\sum_{j=1}^{n} \bar{\bar{u}}_j \left(\sum_{i=1}^{m} \bar{\bar{w}}_i \left(\frac{1-b_{ij}}{b_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}$$

Similarly, we can show for $\mathcal{NMG}$

$$\Rightarrow \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{\psi_{ij}^q}{1-\psi_{ij}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \geq \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{d_{ij}^q}{1-d_{ij}^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}$$

Hence from above equations we have

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\ldots,\mathfrak{J}_{s_{mn}}\right) \leq q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathcal{E}_{s_{11}},\mathcal{E}_{s_{12}},\ldots,\mathcal{E}_{s_{mn}}\right).$$

iv: **(Shift Invariance)** Since $\mathcal{E}_s=(b,d)$ and $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ $(i=1,2,\ldots,m$ and $j=1,2,\ldots,n)$ are q-ROFS_{ft}Vs. Then

$$\mathfrak{J}_{s_{11}}\otimes\mathcal{E}_s=\left(\frac{1}{1+\left\{\left(\frac{1-\mu_{11}}{\mu_{11}}\right)^{\beta}+\left(\frac{1-b}{b}\right)^{\beta}\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\left(\frac{\psi_{11}^q}{1-\psi_{11}^q}\right)^{\beta}+\left(\frac{d^q}{1-d^q}\right)^{\beta}\right\}^{\frac{1}{\beta}}}}\right)$$

Now consider

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}}\otimes\mathcal{E}_s,\mathfrak{J}_{s_{12}}\otimes\mathcal{E}_s,\ldots,\mathfrak{J}_{s_{mn}}\otimes\mathcal{E}_s\right)=$$

$$\left(\begin{array}{c}\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)+\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{1-b}{b}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \\ \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{\psi_{ij}^q}{1-\psi_{ij}^q}\right)^{\beta}\right)+\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{d^q}{1-d^q}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}\end{array}\right)$$

$$=\left(\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)+\left(\frac{1-b}{b}\right)^{\beta}\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{u}_j\left(\sum_{i=1}^{m}\bar{w}_i\left(\frac{\psi_{ij}^q}{1-\psi_{ij}^q}\right)^{\beta}\right)+\left(\frac{d^q}{1-d^q}\right)^{\beta}\right\}^{\frac{1}{\beta}}}}\right)$$

$$=q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{J}_{s_{11}},\mathfrak{J}_{s_{12}},\ldots,\mathfrak{J}_{s_{mn}}\right)\otimes\mathcal{E}_s$$

Therefore, the proof is completed.

v: **(Homogeneity)** Let $\rho>0$ be any real number and $\mathfrak{J}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ $(i=1,2,\ldots,m$ and $j=1,2,\ldots,n)$ are q-ROFS_{ft}Vs. Then

$$\mathfrak{I}_{s_{ij}}^{\rho}=\left(\frac{1}{1+\left\{\rho\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\rho\left(\frac{\psi_{ij}^{q}}{1-\psi_{ij}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}}\right)$$

Further consider

$$q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\rho\mathfrak{I}_{s_{11}},\rho\mathfrak{I}_{s_{12}},\dots,\rho\mathfrak{I}_{s_{mn}}\right)=$$

$$\left(\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_{n}\left(\sum_{i=1}^{m}\rho\bar{\bar{w}}_{i}\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_{j}\left(\sum_{i=1}^{m}\rho\bar{\bar{w}}_{i}\left(\frac{\psi_{ij}^{q}}{1-\psi_{ij}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}\right)$$

$$=\left(\frac{1}{1+\left\{\rho\left[\sum_{j=1}^{n}\bar{\bar{u}}_{j}\left(\sum_{i=1}^{m}\bar{\bar{w}}_{i}\left(\frac{1-\mu_{ij}}{\mu_{ij}}\right)^{\beta}\right)\right]\right\}^{\frac{1}{\beta}}},\sqrt[q]{1-\frac{1}{1+\left\{\rho\left[\sum_{j=1}^{n}\bar{\bar{u}}_{j}\left(\sum_{i=1}^{m}\bar{\bar{w}}_{i}\left(\frac{\psi_{ij}^{q}}{1-\psi_{ij}^{q}}\right)^{\beta}\right)\right]\right\}^{\frac{1}{\beta}}}}\right)$$

$$=\rho\, q-\mathrm{ROF}S_{ft}\mathrm{DWG}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\dots,\mathfrak{I}_{s_{mn}}\right)$$

Therefore, the proof is completed.

7.3.2. q-Rung orthopair fuzzy soft Dombi ordered weighted geometric operator

In this subsection, in view of defined Dombi operation laws we will present the q-ROFS_{ft}DOWG operator and investigate their fundamental characteristics in details.

7.3.2.1. Definition

Let $\mathfrak{I}_{s_{ij}}=\left(\mu_{ij},\psi_{ij}\right)$ for $(\mathrm{i}=1,2,\dots,\mathrm{m}$ and $\mathrm{j}=1,2,\dots,\mathrm{n})$ be the collection of q-ROFS_tVs. Suppose $\bar{\bar{w}}=(\bar{\bar{w}}_1,\bar{\bar{w}}_2,\dots,\bar{\bar{w}}_m)^T$ be the weight vectors for expert $\mathscr{k}_i$ and $\bar{\bar{u}}=(\bar{\bar{u}}_1,\bar{\bar{u}}_2,\dots,\bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\bar{\bar{w}}_i,\bar{\bar{u}}_j\in[0,1]$ with $\sum_{i=1}^{m}\bar{\bar{w}}_i=1$ and $\sum_{j=1}^{n}\bar{\bar{u}}_j=1$. Then q-ROF$S_t$DOWG operator is a mapping denoted and define as: $q-\mathrm{ROF}S_{ft}\mathrm{DOWG}\colon X^n\to X$ such that

$$q-\mathrm{ROF}S_{ft}\mathrm{DOWG}\left(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\dots,\mathfrak{I}_{s_{mn}}\right)=\otimes_{j=1}^{n}\left(\otimes_{i=1}^{m}\left(\mathfrak{I}_{s_{\delta ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right),\qquad(7.3)$$

where $\mathfrak{J}_{s_{\delta ij}} = (\mu_{\delta ij}, \psi_{\delta ij})$ is the permutation of i^{th} row and j^{th} largest elements of the collections from $i \times j\ q-\text{ROF}S_t\text{Ns}$ $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $for\ (i = 1,2,\dots,m\ and\ j = 1,2,\dots,n)$.

Based on Eq. (7.3) we can obtain the aggregated result for q-ROFS_{ft}DOWG operator as described in Theorem 7.3.2.2.

7.3.2.2. Theorem

Suppose the collection $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ (i = 1,2,\dots,m\ and\ j = 1,2,\dots,n)$ of q-ROFS_{ft}Vs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ be the weight vectors for experts $\mathscr{k}_i$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then the aggregated result for q-ROFS_{ft}DOWG operator is stated as:

$$q-\text{ROF}S_{ft}\text{DOWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) = \otimes_{j=1}^{n}\left(\otimes_{i=1}^{m}\left(\mathfrak{J}_{s_{\delta ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\mu_{\delta ij}}{\mu_{\delta ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\psi_{\delta ij}{}^{q}}{1-\psi_{\delta ij}{}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}}\right), \tag{7.4}$$

where $\mathfrak{J}_{s_{\delta ij}} = (\mu_{\delta ij}, \psi_{\delta ij})$ is the permutation of i^{th} row and j^{th} largest elements of the collections from $i \times j\ q-\text{ROF}S_{ft}\text{Ns}$ $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $for\ (i = 1,2,\dots,m\ and\ j = 1,2,\dots,n)$.

Proof: Proof is straightforward like Theorem 7.3.1.2.

7.3.2.3. Example

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ $for\ (i = 1,\dots,4$ and $j = 1,2,3)$ be the collection of $q-$ROFS_{ft}Vs as mention in Table 7.1 of Example 7.3.1.3. Now by utilizing the Definition 7.1.2.2, the tabular description of $\mathfrak{J}_{s_{\delta ij}} = (\mu_{\delta ij}, \psi_{\delta ij})$ is given in Table 7.2.

$$q-\mathrm{ROF}S_{ft}\mathrm{DOWG}(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\ldots,\mathfrak{I}_{s_{43}})=\otimes_{j=1}^{3}\left(\otimes_{i=1}^{4}\left(\mathfrak{I}_{s_{\delta ij}}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right)=$$

$$\left(\frac{1}{1+\left\{\sum_{j=1}^{3}\overline{\overline{u}}_j\left(\sum_{i=1}^{4}\overline{\overline{w}}_i\left(\frac{1-\mu_{\delta ij}}{\mu_{\delta ij}}\right)^2\right)\right\}^{\frac{1}{2}}},\sqrt[3]{1-\frac{1}{1+\left\{\sum_{j=1}^{3}\overline{\overline{u}}_j\left(\sum_{i=1}^{4}\overline{\overline{w}}_i\left(\frac{\psi_{\delta ij}{}^{3}}{1-\psi_{\delta ij}{}^{3}}\right)^2\right)\right\}^{\frac{1}{2}}}}\right)$$

Therefore $q-\mathrm{ROF}S_{ft}\mathrm{DOWG}(\mathfrak{I}_{s_{11}},\mathfrak{I}_{s_{12}},\ldots,\mathfrak{I}_{s_{43}})=(0.775171,0.436559)$.

Table 7.2, Tabular represent of q-ROFS_{ft}S $\mathfrak{I}_{s_{\delta ij}}=(\mu_{\delta ij},\psi_{\delta ij})$ for $\beta=2$ and $q=3$

X	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
k_1	(0.98, 0.38)	(0.95, 0.45)	(0.82, 0.32)
k_2	(0.9, 0.3)	(0.92, 0.3)	(0.8, 0.15)
k_3	(0.85, 0.6)	(0.86, 0.4)	(0.7, 0.2)
k_4	(0.7, 0.4)	(0.75, 0.25)	(0.6, 0.3)

7.3.2.4. Remarks

(a) If we consider that the value of parameter $q=1$ is fixed, then the proposed $q-$ ROFS_tDOWG operator reduced to IFS_{ft}DOWG operator.

(b) If we consider that the value of parameter $q=2$ is fixed, then the proposed $q-$ ROFS_{ft}DOWG operator reduced to PyFS_{ft}DOWG operator.

(c) If the set contain only parameter that is s_1 (means $m=1$), in this case the proposed $q-$ ROFS_{ft}DOWG operator reduced to $q-$ ROFDOWG operator.

Thus from the analysis of Remark 7.3.2.4, it is clear that IFS_{ft}DOWG, PyFS_{ft}DOWG and $q-$ ROFDOWG operators are the special cases of the developed $q-$ ROFS_{ft}DOWG operator.

Based on the analysis of Theorem 7.3.2.2, some properties of the $q-$ ROFS_{ft}DOWG operators are investigated which are described below:

7.3.2.5. Theorem

Suppose the collection $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})\ for\ (i = 1,2, \dots, m\ and\ j = 1, 2, \dots, n)$ of q-ROFS_tVs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ be the weight vectors for experts $\mathscr{k}_i$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors for parameters s_j having the restriction that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then the following properties are holds for q-ROFS_{ft}DOWG operator:

i: (Idempotency) Let $\mathfrak{J}_{s_{ij}} = \mathcal{E}_s\ for\ all\ (i = 1,2, \dots, m\ and\ j = 1,2, \dots, n)$, where $\mathcal{E}_s = (b, d)$. Then

$$q-\text{ROF}S_{ft}\text{DOWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) = \mathcal{E}_s\ .$$

***ii*: (Boundedness)** Let $\mathfrak{J}^-_{s_{ij}} = \left(\min_j \min_i (\mu_{ij}), \max_j \max_i (\psi_{ij})\right)$ and

$\mathfrak{J}^+_{s_{ij}} = \left(\max_j \max_i (\mu_{ij}), \min_j \min_i (\psi_{ij})\right)$. Then

$$\mathfrak{J}^-_{s_{ij}} \leq q-\text{ROF}S_{ft}\text{DOWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \leq \mathfrak{J}^+_{s_{ij}}\ .$$

***iii*: (Monotonicity)** Let another collection $\mathcal{E}_{s_{ij}} = (b_{ij}, \text{d}_{ij})\ for\ (\text{i} = 1,2, \dots, \text{m and j} = 1, 2, \dots, \text{n})$ of q-ROFS_tVs such that $\mu_{ij} \leq b_{ij}\ and\ \psi_{ij} \geq \text{d}_{ij}$. Then

$$q-\text{ROF}S_{ft}\text{DOWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \leq q-\text{ROF}S_{ft}\text{DOWG}(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{mn}}).$$

***iv*: (Shift Invariance)** Let $\mathcal{E}_s = (b, d)$ be a q-ROFS_{ft}V. Then

$$q-\text{ROF}S_{ft}\text{DOWG}(\mathfrak{J}_{s_{11}} \otimes \mathcal{E}_s, \mathfrak{J}_{s_{12}} \otimes \mathcal{E}_s, \dots, \mathfrak{J}_{s_{mn}} \otimes \mathcal{E}_s)$$
$$= q-\text{ROF}S_{ft}\text{DOWG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}) \otimes \mathcal{E}_s.$$

***v*: (Homogeneity)** Let $\rho > 0$ be any real number. Then

$$q-\text{ROF}S_{ft}\text{DOWA}(\rho\mathfrak{J}_{s_{11}}, \rho\mathfrak{J}_{s_{12}}, \dots, \rho\mathfrak{J}_{s_{mn}})$$
$$= \rho\, q-\text{ROF}S_{ft}\text{DOWA}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}).$$

Proof: Proofs are directly follows from Theorem 7.3.2.5.

7.3.3. q-Rung orthopair fuzzy soft Dombi hybrid geometric operators

In this subsection, in view of defined Dombi operation laws we will present the study q-ROFS_{ft}DHG operator and investigate their fundamental characteristics with details.

The basic advantage of q-ROFS_{ft}DHG operator is that, on the same it measures both the weight of q-ROFS_{ft}Vs and the ordered position of q-ROFS_{ft}Vs under the opinions of same experts.

7.3.3.1. Definition

Let $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ for $(i = 1,2, \dots, m$ and $j = 1, 2, \dots, n)$ be the collection of q-ROFS_{ft}Vs with $\mathcal{v} = (\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_m)^T$ and $r = (r_1, r_2, \dots, r_n)^T$ be the weight vector of $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ such that $\mathcal{v}_i, r_j \in [0,1]$ with $\sum_{i=1}^{m} \mathcal{v}_i = 1$ and $\sum_{j=1}^{n} r_j = 1$. Suppose $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for expert $\mathcal{k}_i$ and parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then q-ROF$S_{ft}$DHG operator is a mapping denoted and define as: $q - \mathrm{ROF}S_{ft}\mathrm{DHG}: X^n \to X$ such that

$$q - \mathrm{ROF}S_{ft}\mathrm{DHG}(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}) = \otimes_{j=1}^{n} \left(\otimes_{i=1}^{m} \left(\tilde{\tilde{\Im}}_{s_{\delta ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right), \tag{7.5}$$

where $\tilde{\tilde{\Im}}_{s_{\delta ij}} = \left(\Im_{s_{ij}}\right)^{n \mathcal{v}_i r_j}$ is the permutation of i^{th} row and j^{th} column largest elements of the collections from $q - \mathrm{ROF}S_t\mathrm{Vs}$ $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ and n is called the balancing coefficient.

Based on Eq. (7.5), we can obtain the aggregated result for q-ROFS_{ft}DHG operator as described in Theorem 7.3.3.2.

7.3.3.2. Theorem

Suppose the collection $\Im_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ for $(i = 1,2, \dots, m$ and $j = 1, 2, \dots, n)$ of q-ROFS_{ft}Vs. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for experts $\mathcal{k}_i$ and parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then the aggregated result for q-ROFS_{ft}DHG operator is stated as:

$$q - \mathrm{ROF}S_{ft}\mathrm{DHG}(\Im_{\delta s_{11}}, \Im_{\delta s_{12}}, \dots, \Im_{\delta s_{mn}}) = \otimes_{j=1}^{n} \left(\otimes_{i=1}^{m} \left(\tilde{\tilde{\Im}}_{s_{\delta ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j} \right)$$

$$= \left(\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{1-\tilde{\mu}_{\delta ij}}{\tilde{\mu}_{\delta ij}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1-\frac{1}{1+\left\{\sum_{j=1}^{n}\bar{\bar{u}}_j\left(\sum_{i=1}^{m}\bar{\bar{w}}_i\left(\frac{\tilde{\psi}_{\delta ij}{}^{q}}{1-\tilde{\psi}_{\delta ij}{}^{q}}\right)^{\beta}\right)\right\}^{\frac{1}{\beta}}}} \right), \quad (7.6)$$

where $\tilde{\mathfrak{J}}_{s_{\delta ij}} = n\mathscr{v}_i r_j \mathfrak{J}_{s_{ij}}$ is the permutation of i^{th} row and j^{th} column largest elements of the collections from $q-\text{ROF}S_{ft}\text{Vs}$ $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ with $\mathscr{v} = (\mathscr{v}_1, \mathscr{v}_2, \ldots, \mathscr{v}_m)^T$ and $r = (r_1, r_2, \ldots, r_n)^T$ be the weight vector and n is a balancing coefficient.

Proof: Proof is straightforward like Theorem 7.3.1.2.

7.3.3.3. Example

Let $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ for $(i = 1, \ldots, 4$ and $j = 1, 2, 3)$ be the collection of $q-\text{ROF}S_{ft}\text{Vs}$ as mention in Table 7.1 of Example 7.3.1.3. Let $\mathscr{v} = (0.25, 0.28, 0.29, 0.18)^T$ and $r = (0.36, 0.29, 0.35)^T$ be the weight vector of expert $\mathscr{k}_i$ and parameter s_j, and their corresponding aggregation associated weight vectors $\bar{\bar{w}} = (0.26, 0.3, 0.23, 0.21)^T$ for expert $\mathscr{k}_i$ and $\bar{\bar{u}} = (0.35, 0.31, 0.34)^T$ for parameter s_j. Now by utilizing the operation law mention in Eq. 7.7 and related results are given in Table 7.3. Furthermore the score results by using Definitions 7.1.2.2 are given in Table 7.4. The tabular description for $\tilde{\mathfrak{J}}_{s_{\delta ij}} = n\mathscr{v}_i r_j \mathfrak{J}_{s_{ij}}$ is given in Table 7.5.

$$\rho\mathfrak{J}_s = \left(\frac{1}{1+\left\{\rho\left(\frac{1-\mu_{\mathfrak{J}_s}}{\mu_{\mathfrak{J}_s}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}, \sqrt[q]{1-\frac{1}{1+\left\{\rho\left(\frac{\psi_{\mathfrak{J}_s}{}^{q}}{1-\psi_{\mathfrak{J}_s}{}^{q}}\right)^{\beta}\right\}^{\frac{1}{\beta}}}} \right) \quad (7.7)$$

$$q-\text{ROF}S_{ft}\text{DHG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{43}}) = \otimes_{j=1}^{3}\left(\otimes_{i=1}^{4}\left(\tilde{\mathfrak{J}}_{s_{\delta ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right) =$$

$$\left(\frac{1}{1+\left\{\sum_{j=1}^{3}\bar{\bar{u}}_j\left(\sum_{i=1}^{4}\bar{\bar{w}}_i\left(\frac{1-\tilde{\mu}_{\delta ij}}{\tilde{\mu}_{\delta ij}}\right)^{2}\right)\right\}^{\frac{1}{2}}}, \sqrt[3]{1-\frac{1}{1+\left\{\sum_{j=1}^{3}\bar{\bar{u}}_j\left(\sum_{i=1}^{4}\bar{\bar{w}}_i\left(\frac{\tilde{\psi}_{\delta ij}{}^{3}}{1-\tilde{\psi}_{\delta ij}{}^{3}}\right)^{2}\right)\right\}^{\frac{1}{2}}}} \right)$$

Therefore, $q-\text{ROF}S_{ft}\text{DHG}(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \ldots, \mathfrak{J}_{s_{43}}) = (0.701933, 0.561573)$.

Table 7.3, Tabular represent of q-ROFS_{ft}S $\widetilde{\mathfrak{I}}_{s_{\delta ij}} = n\vartheta_i r_j \mathfrak{I}_{s_{ij}}$ for $\beta = 2$ and $q = 3$

T	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
k_1	$(0.85153, 0.41667)$	$(0.7856, 0.55317)$	$(0.61795, 0.29705)$
k_2	$(0.79509, 0.70258)$	$(0.66473, 0.36904)$	$(0.52791, 0.40636)$
k_3	$(0.0.9697, 0.48677)$	$(0.87553, 0.42493)$	$(0.73722, 0.21689)$
k_4	$(0.59438, 0.567)$	$(0.90169, 0.64165)$	$(0.7253, 0.48385)$

Table 7.4, Tabular description of score values for $\widetilde{\mathfrak{I}}_{s_{\delta ij}} = n\vartheta_i r_j \mathfrak{I}_{s_{ij}}$ for $\beta = 2$ and $q = 3$

T	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
k_1	(0.772555)	(0.657789)	(0.604883)
k_2	(0.57791)	(0.621733)	(0.54001)
k_3	(0.898249)	(0.797202)	(0.695232)
k_4	(0.41385)	(0.734469)	(0.634138)

Table 7.5, New ordered for q-ROFS_{ft}S $\widetilde{\mathfrak{I}}_{s_{\delta ij}} = n\vartheta_i r_j \mathfrak{I}_{s_{ij}}$ for $\beta = 2$ and $q = 3$

T	$s_{\delta 1}$	$s_{\delta 2}$	$s_{\delta 3}$
k_1	$(0.9697, 0.48677)$	$(0.87553, 0.42493)$	$(0.73722, 0.21689)$
k_2	$(0.85153, 0.41667)$	$(0.90169, 0.64165)$	$(0.7253, 0.48385)$
k_3	$(0.79509, 0.70258)$	$(0.7856, 0.55317)$	$(0.61795, 0.29705)$
k_4	$(0.59438, 0.567)$	$(0.66473, 0.36904)$	$(0.52791, 0.40636)$

7.3.3.4. Remarks

(a) If we consider that the value of parameter $q = 1$ is fixed, then the proposed $q-$ ROFS_{ft}DHG operator reduced to IFS_{ft}DHG operator

(b) If we consider that the value of parameter $q = 2$ is fixed, then the proposed $q-$ ROFS_{ft}DHG operator reduced to PyFS_{ft}DHG operator.

(c) If the set contain only parameter that is s_1 (means $m = 1$), in this case the proposed $q-\mathrm{ROF}S_{ft}\mathrm{DHG}$ operator reduced to $q-\mathrm{ROFDHG}$ operator.

Thus from the analysis of Remark 7.3.3.4, it is clear that IFS_{ft}DHG, PyFS_{ft}DHG and $q-\mathrm{ROFDHG}$ operators are the special cases of the developed $q-\mathrm{ROF}S_{ft}\mathrm{DHG}$ operator.

Based on the analysis of Theorem 7.3.3.2, some properties of the $q-\mathrm{ROF}S_{ft}\mathrm{DHG}$ operators are investigated which are described below:

7.3.3.5. Theorem

Suppose the collection $\Im_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right) for\ (i = 1,2,\dots,m\ and\ j = 1,2,\dots,n)$ of q-ROFS_{ft}Vs with $\mathcal{v} = (\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_m)^T$ and $r = (r_1, r_2, \dots, r_n)^T$ be the weight vector of $\Im_{s_{ij}} = \left(\mu_{ij}, \psi_{ij}\right)$ such that $\mathcal{v}_i, r_j \in [0,1]$ with $\sum_{i=1}^{m} \mathcal{v}_i = 1$ and $\sum_{j=1}^{n} r_j = 1$. Suppose $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for expert $\mathcal{k}_i$ and parameters s_j having the conditions that $\overline{\overline{w}}_i, \bar{\bar{u}}_j \in [0,1]$ with $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. Then the following properties are holds for q-ROFS_{ft}DHG operator:

i: **(Idempotency)** Let $\Im_{s_{ij}} = \mathcal{E}_s\ for\ all\ (i = 1,2,\dots,m\ and\ j = 1,2,\dots,n)$, where $\mathcal{E}_s = (b, d)$. Then

$$q-\mathrm{ROF}S_{ft}\mathrm{DHG}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}\right) = \mathcal{E}_s\ .$$

$\boldsymbol{ii}$: **(Boundedness)** Let $\Im_{s_{ij}}^{-} = \left(\min_j \min_i \left(\mu_{ij}\right), \max_j \max_i \left(\psi_{ij}\right)\right)$ and

$\Im_{s_{ij}}^{+} = \left(\max_j \max_i \left(\mu_{ij}\right), \min_j \min_i \left(\psi_{ij}\right)\right)$. Then

$$\Im_{s_{ij}}^{-} \leq q-\mathrm{ROF}S_{ft}\mathrm{DHG}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}\right) \leq \Im_{s_{ij}}^{+}\ .$$

$\boldsymbol{iii}$: **(Monotonicity)** Let another collection $\mathcal{E}_{s_{ij}} = \left(b_{ij}, \mathrm{d}_{ij}\right) for\ (\mathrm{i} = 1,2,\dots,\mathrm{m\ and\ j} = 1,2,\dots,\mathrm{n})$ of q-ROFS_{ft}Vs such that $\mu_{ij} \leq b_{ij}\ and\ \psi_{ij} \geq \mathrm{d}_{ij}$. Then

$$q-\mathrm{ROF}S_{ft}\mathrm{DHG}\left(\Im_{s_{11}}, \Im_{s_{12}}, \dots, \Im_{s_{mn}}\right) \leq q-\mathrm{ROF}S_{ft}\mathrm{DHG}\left(\mathcal{E}_{s_{11}}, \mathcal{E}_{s_{12}}, \dots, \mathcal{E}_{s_{mn}}\right).$$

$\boldsymbol{iv}$: **(Shift Invariance)** Let $\mathcal{E}_s = (b, d)$ be a q-ROFS_tV. Then

$$q - \text{ROF}S_{ft}\text{DHG}\left(\mathfrak{J}_{s_{11}} \otimes \mathcal{E}_s, \mathfrak{J}_{s_{12}} \otimes \mathcal{E}_s, \dots, \mathfrak{J}_{s_{mn}} \otimes \mathcal{E}_s\right)$$
$$= q - \text{ROF}S_{ft}\text{DHG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}\right) \otimes \mathcal{E}_s.$$

$\boldsymbol{v}$: **(Homogeneity)** Let $\rho > 0$ be any real number. Then

$$q - \text{ROF}S_{ft}\text{DHG}\left(\rho\mathfrak{J}_{s_{11}}, \rho\mathfrak{J}_{s_{12}}, \dots, \rho\mathfrak{J}_{s_{mn}}\right) = \rho\, q - \text{ROF}S_{ft}\text{DHG}\left(\mathfrak{J}_{s_{11}}, \mathfrak{J}_{s_{12}}, \dots, \mathfrak{J}_{s_{mn}}\right).$$

Proof: Proofs are easy and directly follows Theorem 7.3.1.5.

7.4. An approach to $\mathcal{MCDM}$ under Dombi operations using q-rung orthopair fuzzy soft information

This section describes a $\mathcal{MCDM}$ techniques by using the applicability of developed operators for handling $\mathcal{MCDM}$ problems. Here criteria and parameter weights are real numbers and criteria values are q-ROFVs. The techniques of mathematical descriptions and their general steps wise algorithm under q-ROF environment is given as follows.

Suppose $T = \{\hbar_1, \hbar_2, \hbar_3, \dots, \hbar_k\}$ be the collection of alternatives in which the most desirable alternative is going to evaluated by the senior decision makers $d = \{d_1, d_2, d_3, \dots, d_m\}$ against their corresponding parameters $\mathbb{E} = \{s_1, s_2, s_3, \dots, s_n\}$. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the aggregation associated weight vectors for expert $\hbar_i$ and parameters s_j having the conditions that $\bar{\bar{w}}_i, \bar{\bar{u}}_n \in [0,1]$ with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$ and $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$. The senior decision makers gives their assessment for best alternative $\hbar_k$ against to parameter s_n in the form of q-ROFS_{ft}Vs $\mathfrak{J}_{s_{ij}} = (\mu_{ij}, \psi_{ij})$ such that$0 \le \mu_{ij}^q + \psi_{ij}^q \le 1\; for\; q \ge 1$. The decision makers describe their collective evaluated information in the form of q-ROFS_{ft} decision matrix $\mathbb{M} = [\mathfrak{J}_{s_{ij}}]_{m \times n}$. Using the preferences values of senior experts the aggregated result ξ_i for alternative $\hbar_i$ $(i = 1,2, \dots, k)$ is $\xi_i = (\mu_i, \psi_i)$ by applying the q-ROFS_{ft}DW geometric operations which is given in Eqs. 7.2, 7.4 and 7.6. Finally to get the most desirable alternative apply the score function on aggregated result ξ_i and rank them to get the best option.

7.4.1. Algorithm

The step wise decision algorithm for the developed operators are summarized as follows:

Step i: Extract the collective evaluated information of senior experts in the form of q-ROFS_{ft} decision matrix $\mathbb{M} = [\mathfrak{J}_{s_{ij}}]_{m \times n}$ for each alternative against their parameter.

$$\mathbb{M} = \begin{bmatrix} (\mu_{11}, \psi_{11}) & (\mu_{12}, \psi_{12}) & \cdots & (\mu_{1n}, \psi_{1n}) \\ (\mu_{21}, \psi_{21}) & (\mu_{22}, \psi_{22}) & \cdots & (\mu_{2n}, \psi_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, \psi_{m1}) & (\mu_{m2}, \psi_{m2}) & \cdots & (\mu_{mn}, \psi_{mn}) \end{bmatrix}$$

Step ii: Using the preferences values of senior experts, aggregate the q-ROFS_{ft} $\mathfrak{I}_{s_{ij}}$ for alternative $\mathcal{k}_i$ $(i = 1,2, \dots, k)$ into collective decision matrix $\xi_i = (\mu_i, \psi_i)$ by applying the developed q-ROFS_{ft}D averaging and q-ROFS_{ft}D geometric operations.

Step iii: Applying the definition of score function determine the score values of ξ_i for each object $\mathcal{k}_i \, for \, i = (1,2, \dots, k)$.

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\mathcal{k}_i$.

7.5. Numerical example

In ordered to demonstrate the applicability and validity of the proposed method, a decision making process has been illustrated with the Constructional engineering projects (CEP) adopted from [92].

Suppose a particular example about the four potential CEP (alternatives) $T = \{\mathcal{k}_1, \mathcal{k}_2, \mathcal{k}_3, \mathcal{k}_4\}$ and the committee of expert engineers $d = \{d_1, d_2, d_3, d_4, d_5\}$ whose weight vector is $\bar{\bar{w}} = (0.24,0.26,0.23,0.15,0.12)^T$ will give their assessment for the project against some parameter $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$ and weight vector $\bar{\bar{u}} = (0.27,0.22,0.23,0.28)^T$, where $s_1 =$ the construction work environment, $s_2 =$ the construction site safety protection measure, $s_3 =$ the safety production responsibility system, $s_4 =$ the safety management ability of the engineering project. The expert engineers $d_i (i = 1, \dots, 5)$ provides their assessment for each project against their parameter in the form of q-ROFVs. Following steps followed for finding the most desirable CEP by applying the developed approach.

By applying q-ROFS$_t$DWG operator

Step i: The five expert engineers d_i will evaluate the construction of four CEP in terms of q-ROFVs, parameters and their rating results are given in Tables $7.6 - 7.9$ respectively.

Step ii: Applying the preferences values of senior engineers, the aggregated result for each alternative $\mathcal{k}_i$ $(i = 1, \dots, 4)$ by applying the developed q-ROFS_tDWG operator for $q = 3$ and $\beta = 2$ are gives as:

$$\xi_1 = (0.613069, 0.222324), \;\; \xi_2 = (0.607699, 0.252647),$$
$$\xi_3 = (0.513969, 0.257883), \;\; \xi_3 = (0.474117, 0.214605)$$

Step iii: Applying the definition of score function and determine the score values of ξ_i for each object $\mathcal{k}_i$ $for\ i = (1, \dots, 4)$.

$$\mathcal{S}c(\xi_1) = 0.609718, \;\; \mathcal{S}c(\xi_2) = 0.604148, \;\; \mathcal{S}c(\xi_3) = 0.559311, \;\; \mathcal{S}c(\xi_4) = 0.548346$$

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\mathcal{k}_i$.

$$\mathcal{k}_1 > \mathcal{k}_2 > \mathcal{k}_4 > \mathcal{k}_3$$

From the ranking result, it is clear that $\mathcal{k}_1$ is the most desirable and profitable CEP among all.

By applying q-ROFS_tDOWG operator

Step i: Same as above.

Step ii: Applying the preferences values of senior engineers, the aggregated result for each alternative $\mathcal{k}_i$ $(i = 1, \dots, 4)$ by applying the developed q-ROFS_tDOWG operator for $q = 3$ and $\beta = 2$ are gives as:

$$\xi_1 = (0.64151, 0.216409), \qquad \xi_2 = (0.638415, 0.252054),$$
$$\xi_3 = (0.548416, 0.267736), \qquad \xi_4 = (0.501656, 0.217831)$$

Step iii: Applying the definition of score function and determine the score values of ξ_i for each object $\mathcal{k}_i$ $for\ i = (1, \dots, 4)$.

$\mathcal{S}c(\xi_1) = 0.626935,\ \mathcal{S}c(\xi_2) = 0.622094,\ \mathcal{S}c(\xi_3) = 0.572875,\ \mathcal{S}c(\xi_4) = 0.557955$

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\mathcal{k}_i$.

$$\mathcal{k}_1 > \mathcal{k}_2 > \mathcal{k}_3 > \mathcal{k}_4$$

From the ranking result, it is clear that $\mathcal{k}_1$ is the most desirable and profitable CEP among all.

By applying q-ROFS$_t$DHG operator

Step i: Same as above.

Step ii: Applying the preferences values of senior engineers, the aggregated result for each alternative $\mathcal{k}_i$ $(i = 1, \ldots, 4)$ by applying the developed q-ROFS_tDHG operator for $q = 3\ and\ \beta = 2$. Let $\mathcal{v} = (0.22, 0.16, 0.2, 0.24, 0.18)^T$ and $r = (0.32, 0.29, 0.18, 0.21)^T$ be the weight vector of expert $\mathcal{k}_i$ and parameter s_j. Let $\bar{\bar{w}} = (0.24, 0.26, 0.23, 0.15, 0.12)^T$ and $\bar{\bar{u}} = (0.27, 0.22, 0.23, 0.28)^T$ be the corresponding aggregation associated weight vector for expert $\mathcal{k}_i$ and parameter s_j for $q = 3$ and $\beta = 2$. Then

$$\xi_1 = (0.531966, 0.352165), \quad \xi_2 = (0.536842, 0.392825),$$
$$\xi_3 = (0.418437, 0.394779), \quad \xi_4 = (0.399859, 0.349697)$$

Step iii: Applying the definition of score function and determine the score values of ξ_i for each object $\mathcal{k}_i\ for\ i = (1, \ldots, 4)$.

$$\mathcal{S}c(\xi_1) = 0.553432,\ \mathcal{S}c(\xi_2) = 0.54705,\ \mathcal{S}c(\xi_3) = 0.505869,\ \mathcal{S}c(\xi_4) = 0.510584$$

Step iv: Finally rank the obtained results and arranged them in a specific ordered to get the most desirable option from $\mathcal{k}_i$.

$$\mathcal{k}_1 > \mathcal{k}_2 > \mathcal{k}_4 > \mathcal{k}_3$$

From the ranking result, it is clear that $\mathcal{k}_1$ is the most profitable CEP among all.

Therefore, from the analysis of illustrative example, it is evident that the ranking order of the alternatives are slightly different but the ranking concerning the most suitable alternative is identical that is $\mathcal{k}_1$ for overall introduced operators.

7.5.1. Comparative analysis

To present the efficiency and applicability of the proposed method with some existing methods, a comparative study has been made based on different aggregation operators (see [5, 30, 68, 69, 71, 75]) under IF, PyF and q-ROF environment. For collective information different parameters of q-ROFS_{ft}Vs are aggregated by utilizing weighted geometric operator against to their weight vector $\bar{\bar{u}} = (0.27, 0.22, 0.23, 0.28)^T$, to obtain

Table 7.6. q-ROFS_{ft} matrix for CEP $\mathscr{k}_1$

	s_1	s_2	s_3	s_4
d_1	(0.8,0.2)	(0.9,0.1)	(0.76,0.13)	(0.71,0.23)
d_2	(0.7,0.15)	(0.6,0.2)	(0.8,0.18)	(0.9,0.05)
d_3	(0.5,0.2)	(0.81,0.14)	(0.4,0.1)	(0.65,0.32)
d_4	(0.72,0.23)	(0.75,0.13)	(0.55,0.22)	(0.74,0.17)
d_5	(0.65,0.25)	(0.5,0.12)	(0.66,0.23)	(0.45,0.05)

Table 7.7. q-ROFS_{ft} matrix for CEP $\mathscr{k}_2$

	s_1	s_2	s_3	s_4
d_1	(0.6,0.13)	(0.85,0.14)	(0.8,0.18)	(0.81,0.16)
d_2	(0.73,0.25)	(0.74,0.25)	(0.63,0.22)	(0.77,0.15)
d_3	(0.45,0.32)	(0.4,0.12)	(0.54,0.31)	(0.84,0.11)
d_4	(0.7,0.2)	(0.6,0.3)	(0.65,0.28)	(0.76,0.19)
d_5	(0.62,0.35)	(0.5,0.1)	(0.74,0.12)	(0.65,0.25)

Table 7.8. q-ROFS_{ft} matrix for CEP $\mathscr{k}_3$

	s_1	s_2	s_3	s_4
d_1	(0.74,0.23)	(0.55,0.12)	(0.48,0.1)	(0.42,0.15)
d_2	(0.6,0.15)	(0.66,0.31)	(0.78,0.12)	(0.64,0.22)
d_3	(0.82,0.16)	(0.74,0.25)	(0.5,0.3)	(0.3,0.1)
d_4	(0.65,0.34)	(0.58,0.28)	(0.73,0.25)	(0.48,0.26)
d_5	(0.9,0.08)	(0.6,0.2)	(0.4,0.1)	(0.61,0.35)

Table 7.9, q-ROFS_{ft} matrix for CEP $\mathcal{k}_4$

	s_1	s_2	s_3	s_4
d_1	$(0.63,0.14)$	$(0.45,0.13)$	$(0.55,0.25)$	$(0.62,0.15)$
d_2	$(0.35,0.05)$	$(0.65,0.18)$	$(0.75,0.18)$	$(0.48,0.22)$
d_3	$(0.7,0.17)$	$(0.9,0.09)$	$(0.6,0.3)$	$(0.52,0.16)$
d_4	$(0.39,0.25)$	$(0.25,0.1)$	$(0.56,0.16)$	$(0.67,0.26)$
d_5	$(0.8,0.12)$	$(0.76,0.23)$	$(0.34,0.05)$	$(0.38,0.1)$

Table 7.10. Aggregated values of q-ROFS_{ft} matrix for CEP for $\mathcal{k}_i$

	$\mathcal{k}_1$	$\mathcal{k}_2$	$\mathcal{k}_3$	$\mathcal{k}_4$
d_1	$(0.839983,0.143578)$	$(0.805096,0.148488)$	$(0.647834,0.133022)$	$(0.592306,0.152046)$
d_2	$(0.846328,0.084283)$	$(0.735176,0.198556)$	$(0.70186,0.164896)$	$(0.658083,0.086928)$
d_3	$(0.714504,0.149203)$	$(0.759122,0.149169)$	$(0.747558,0.147595)$	$(0.826828,0.139741)$
d_4	$(0.719443,0.174289)$	$(0.705968,0.223092)$	$(0.653149,0.277035)$	$(0.585882,0.156939)$
d_5	$(0.607438,0.084324)$	$(0.666902,0.147059)$	$(0.836595,0.115345)$	$(0.734921,0.082803)$

the aggregated q-ROFS_{ft} decision matrix for different alternative $\mathcal{k}_i$ $(i = 1, \dots 4)$ as summarized in Table 7.10. From the evident of this decision matrix a comparative study has been made of the proposed methods with some existing methods and their simultaneous results are depicted in Table 7.11. From Table 7.11, it is clear that the ranking orders are slightly different but their best optimal alternative remain same for all operators that is $\mathcal{k}_1$. However, in many situations of real life IFS and PyFS cannot provide the additional space to the decision makers to describe the attribute evaluation value due to its restricted constraints. Obviously in q-ROF environment the experts fully express the decision information. From characteristic analysis the existing methods in [5, 30, 75] are best for fuzzy data and these methods having no information about soft parameterization tools and Dombi operational parameter. Similarly the methods in [68, 69, 71] have just Dombi operational parameter. Therefore, from the characteristic point of view the methods proposed in this chapter are more superior and practical for real life information to describe the fuzzy data under soft parameterizations information by using Dombi operational law.

7.5.2. Influence of operational parameter β

To express the influence and potential of operational parameter β on $\mathcal{MCDM}$, different values of β are utilized to rank the alternatives. For different input of β in the range of $1 \leq \beta \leq 30$, the score values and their ranking order of alternatives $\hbar_i (i = 1, \ldots, 5)$ based on q-ROFS_{ft} DWG operators are depicted in Tables 7.12. From the analysis of Tables 7.12, it is clear that for different input values of β the ranking order is slightly different but the best optimal option remain identical that is $\hbar_1$ for q-ROFS_{ft} DWG operators. By increasing the value of β cause gradual decrease in score values for q-ROFS_{ft} DWG operators. This show that increasing the values of β from smaller to bigger cause the decision makers' attitude from optimism to pessimism for q-ROFS_{ft} DWG operators. Thus the behaviour of operational parameter β is very important to express the experts' attitude in decision making problems. Therefore, from overall analysis it is concluded that the proposed method is more superior and resilience than existing methods to solve the real life decisions by using parameterization tools under Dombi operational law.

7.5.3. Conclusion

The aim of this chapter is to present the notion of q-ROFS_{ft}S based on the Dombi operations. Since Dombi operational parameter possess natural flexibility with resilience of variability. The behaviour of Dombi operational parameter is very important to express the experts' attitude in decision making. Further we present q-ROFS_{ft}DG aggregation operators including q-ROFS_{ft}DWG, q-ROFS_{ft}DOWG and q-ROFS_{ft}DHA operators. The basic properties of these operators are presented in detail such as Idempotency, Boundedness, Monotonicity, Shift invariance and Homogeneity. By applying develop approach, this manuscript contains the technique and algorithm for $\mathcal{MCDM}$. Further a numerical example is developed to illustrative the flexibility and applicability of the developed operators.

Table 7.11. Comparative analysis of existing methods with proposed methods

Methods	*Score values of alternatives*				Ranking
	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$	$\mathscr{k}_4$	
IFWG [5]	0.634397,	0.570364,	0.540043,	0.548168	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_4 \succ \mathscr{k}_3$
PyFWG [75]	0.563338,	0.522164,	0.468469,	0.436953	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
PyFDWG [69]	0.512353,	0.507842,	0.443545,	0.398963	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
PyDFWG [68]	0.524211,	0.510858,	0.447564,	0.407124	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
q-ROFDWG [71]	0.688855,	0.700147,	0.66887,	0.632511	$\mathscr{k}_2 \succ \mathscr{k}_1 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
q-ROFWG [30]	0.440017,	0.405538,	0.345699,	0.30311	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
q-ROFS_{ft}DWG (proposed)	0.609718,	0.604148,	0.559311,	0.548346	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
q-ROFS_{ft}DOWG (proposed)	0.626935,	0.622094,	0.572875,	0.557955	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
q-ROFS_{ft}DHG (proposed)	0.553432,	0.54705,	0.505869,	0.510548	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_4 \succ \mathscr{k}_3$

Table 7.12. Ranking order based on different operational parameter of q-ROFS_t DWG operator

Operational parameter β	*Score values of alternatives*				Ranking
	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$	$\mathscr{k}_4$	
$\beta = 1$	0.640044,	0.631482,	0.583387,	0.568217	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
$\beta = 2$	0.609718,	0.604148,	0.559311,	0.548346	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_4 \succ \mathscr{k}_3$
$\beta = 3$	0.587973,	0.58367,	0.54252,	0.534942	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
$\beta = 5$	0.562564,	0.559175,	0.523482,	0.519643	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
$\beta = 8$	0.545262,	0.54215,	0.511291,	0.509427	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
$\beta = 12$	0.5351,	0.531866,	0.504628,	0.503796	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
$\beta = 16$	0.530003,	0.526516,	0.501413,	0.501167	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_3 \succ \mathscr{k}_4$
$\beta = 20$	0.526982,	0.52326,	0.499524,	0.499673	$\mathscr{k}_1 \succ \mathscr{k}_2 \succ \mathscr{k}_4 \succ \mathscr{k}_3$
$\beta = 25$	0.524602,	0.520651,	0.498028,	0.498525	$\mathscr{k}_1 \succ \mathscr{k}_4 \succ \mathscr{k}_4 \succ \mathscr{k}_3$
$\beta = 30$	0.523041,	0.51892,	0.497034,	0.497782	$\mathscr{k}_2 \succ \mathscr{k}_4 \succ \mathscr{k}_4 \succ \mathscr{k}_3$

Chapter 8

Orthopair fuzzy soft rough aggregation operators

The aim of this chapter is to investigate the hybrid concept of S_{ft}S and RS with the notion of q-ROFS to obtain the new notion of q-ROFS_{ft}RS. In addition, some averaging aggregation operators such as q-ROFS_{ft}RWA, q-ROFS_{ft}ROWA) and q-ROFS_tRHA operators are presented. Then basic desirable properties of these investigated averaging operators are discussed in detail. Moreover, we investigated the geometric aggregation operators such as q-ROFS_{ft}RWG, q-ROFS_{ft}ROWG and q-ROFS_{ft}RHG operators, and proposed the basic desirable characteristics of investigated geometric operators. The technique for $\mathcal{MCDM}$ and step wise algorithm for $\mathcal{DM}$ by utilizing the proposed approaches are demonstrated. Finally, a numerical example for the developed approach is presented and a comparative study of the investigated models with some existing methods is brought to light in detail which shows that the proposed models are more effective and applicable than existing approaches.

8.1. q-Rung orthopair fuzzy soft set

In Chapters 4 we have discussed the basic definitions and the desirable operations and relations of PyFS_{ft}S and q-ROFS_{ft}S. For detail see Chapter 4, Sections 4.1 and 4.2.

8.2. q-Rung orthopair fuzzy soft rough set

This section is devoted to the hybrid study of q-ROFS with S_{ft}S and RS to obtain the new concept of q-ROFS_{ft}RS. Some basic operations, a new score function and some basic properties of the developed concept are investigated in detail.

8.2.1. Definition

Let $(\mathcal{T}, \mathbb{E})$ be a q-ROFS_{ft}S over T. Any subset $\mathcal{L}$ of $T \times \mathbb{E}$ is said to a q-ROFS_{ft} relation from $T\ to\ \mathbb{E}$ and is defined as:

$$\mathcal{L} = \{\langle(\hbar_i, s_j), \mu(\hbar_i, s_j), \psi(\hbar_i, s_j)\rangle | (\hbar_i, s_j) \in T \times \mathbb{E}\},$$

where $\mu: T \times \mathbb{E} \to [0,1]\ and\ \psi: T \times \mathbb{E} \to [0,1]$ denotes the $\mathcal{MG}$ and $\mathcal{NMG}$ with $0 \leq [\mu(\hbar_i, s_j)]^q + [\psi(\hbar_i, s_j)]^q \leq 1$ for all $(\hbar_i, s_j) \in T \times \mathbb{E}$.

Table 8.1, Tabular form of q-ROFS_{ft} relation $\mathcal{L}$ from T to $\mathbb{E}$

$\mathcal{L}$	s_1	s_2	$\cdots$	s_n
k_1	$(\mu(k_1, s_1), \psi(k_1, s_1))$	$(\mu(k_1, s_2), \psi(k_1, s_2))$	$\cdots$	$(\mu(k_1, s_n), \psi(k_1, s_n))$
k_2	$(\mu(k_2, s_1), \psi(k_2, s_1))$	$(\mu(k_2, s_2), \psi(k_2, s_2))$	$\cdots$	$(\mu(k_2, s_n), \psi(k_2, s_n))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
k_m	$(\mu(k_m, s_1), \psi(k_m, s_1))$	$(\mu(k_m, s_2), \psi(k_m, s_2))$	$\cdots$	$(\mu(k_m, s_n), \psi(k_m, s_n))$

If $T = \{k_1, k_2, \dots, k_m\}$ and $\mathbb{E} = \{s_1, s_2, \dots, s_n\}$, then q-ROF$S_{ft}$ relation $\mathcal{L}$ from T to $\mathbb{E}$ can be presented in the following Table 8.1. In view of above definition of q-ROFS_{ft} relation, we can define the q-ROFS_{ft}RS as:

8.2.2. Definition

Consider a universal set T, $\mathbb{E}$ be the set of parameter and $(\mathcal{T}, \mathbb{E})$ be a q-ROFS_{ft}S. Let $\mathcal{L}$ be an arbitrary q-ROFS_{ft} relation from set T to $\mathbb{E}$. The pair $(T, \mathbb{E}, \mathcal{L})$ is said to be q-ROFS_{ft} approximation space. For any optimum decision normal object $\mathcal{M} \in q - ROFS^{(\mathbb{E})}$, then the lower and upper approximation of $\mathcal{M}$ w.r.t approximation space $(T, \mathbb{E}, \mathcal{L})$, are represented and defined as:

$$\underline{\mathcal{L}}(\mathcal{M}) = \left\{\left(k_i, \underline{\mu_j}(k_i), \underline{\psi_j}(k_i)\right) | k_i \in T\right\} \tag{8.2}$$

$$\overline{\mathcal{L}}(\mathcal{M}) = \left\{\left(k_i, \overline{\mu_j}(k_i), \overline{\psi_j}(k_i)\right) | k_i \in T\right\}, \tag{8.3}$$

where

$$\underline{\mu_j}(k_i) = \bigwedge_{s_j \in \mathbb{E}} \left[\mu_{\mathcal{L}}(k_i, s_j) \wedge \mu_{\mathcal{M}}(s_j)\right], \qquad \underline{\psi_j}(k_i) = \bigvee_{s_j \in \mathbb{E}} \left[\psi_{\mathcal{L}}(k_i, s_j) \vee \psi_{\mathcal{M}}(s_j)\right]$$

$$\overline{\mu_j}(k_i) = \bigvee_{s_j \in \mathbb{E}} \left[\mu_{\mathcal{L}}(k_i, s_j) \vee \mu_{\mathcal{M}}(s_j)\right], \qquad \overline{\psi_j}(k_i) = \bigwedge_{s_j \in \mathbb{E}} \left[\psi_{\mathcal{L}}(k_i, s_j) \wedge \psi_{\mathcal{M}}(s_j)\right]$$

such that

$$0 \leq \left[\underline{\mu_j}(k_i)\right]^q + \left[\underline{\psi_j}(k_i)\right]^q \leq 1 \;\; and \; 0 \leq \left[\overline{\mu_j}(k_i)\right]^q + \left[\overline{\psi_j}(k_i)\right]^q \leq 1$$

It is clear that $\underline{\mathcal{L}}(\mathcal{M})$ and $\overline{\mathcal{L}}(\mathcal{M})$ are two q-ROFSs in T. Thus the operators $\underline{\mathcal{L}}(\mathcal{M}), \overline{\mathcal{L}}(\mathcal{M}) :$ q $-$ ROF$S_{ft}{}^{\mathbb{E}} \rightarrow$ q $-$ ROF$S_{ft}{}^{T}$ are respectively known as lower and upper q-ROFS_{ft}R approximation operators. Therefore q-ROFS_{ft}RS is a pair $\mathcal{L}(\mathcal{M}) = \left(\underline{\mathcal{L}}(\mathcal{M}), \overline{\mathcal{L}}(\mathcal{M})\right) = \left(\mathscr{k}_i, \left(\underline{\mu_j}(\mathscr{k}_i), \underline{\psi_j}(\mathscr{k}_i)\right), \left(\overline{\mu_j}(\mathscr{k}_i), \overline{\psi_j}(\mathscr{k}_i)\right)\right)$.

For simplicity we can write $\mathcal{L}(\mathcal{M}) = \left(\underline{\mathcal{L}}(\mathcal{M}), \overline{\mathcal{L}}(\mathcal{M})\right) = \left(\mathscr{k}_i, \left(\underline{\mu_j}(\mathscr{k}_i), \underline{\psi_j}(\mathscr{k}_i)\right), \left(\overline{\mu_j}(\mathscr{k}_i), \overline{\psi_j}(\mathscr{k}_i)\right)\right)$ as $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right) = \left(\left(\underline{\mu_{ij}}, \underline{\psi_{ij}}\right), \left(\overline{\mu_{ij}}, \overline{\psi_{ij}}\right)\right)$ and called q-ROFS_{ft}R value (q-ROFS_{ft}RV), if there is no confusion.

8.2.3. Remark

(a) If $q = 1$, is fixed then the developed q-ROFS_{ft}R approximation operators reduced to IFS_{ft}R approximation operators.

(b) If $q = 2$, is fixed then the developed q-ROFS_{ft}R approximation operators reduced to PyFS_{ft}R approximation operators.

Consider the following example for better understanding the concept of q-ROFS_{ft}R approximation operators.

8.2.4. Example

Suppose a decision maker Z purchase a house from the set of five houses $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3, \mathscr{k}_4, \mathscr{k}_5\}$ under consideration. Let the parameter set $\mathbb{E} = \{s_1, s_2, s_3, s_4\}$, where $s_1 = beautiful, s_2 = large\ in\ size,\ s_3 = expensive$ and $s_4 = location.$ A decision maker Z wants to purchase a house from the available houses which fulfill the utmost extent of given parameters. Consider the decision maker Z presents the gorgeous of houses in form of q-ROFS_{ft} relation $\mathcal{L}$ from set $T\ to\ \mathbb{E}$ and is given in Table 8.2.

Consider a decision maker Z presents the optimum normal decision object $\mathcal{M}$ which is a q-ROF subset over parameter set $\mathbb{E}$, that is:

$$\mathcal{M} = \{(s_1, 0.9,0.2), (s_2, 0.4,0.6), (s_3, 0.8,0.4), (s_4, 0.5,0.1)\}$$

Now by using Eqs. (8.2) and (8.3), we have

Table 8.2, q-ROFS_{ft} relation from set T to $\mathbb{E}$ for $q = 3$

$\mathcal{L}$	s_1	s_2	s_3	s_4
$\hbar_1$	$(0.9, 0.4)$	$(0.8, 0.2)$	$(0.7, 0.3)$	$(0.65, 0.2)$
$\hbar_2$	$(0.8, 0.5)$	$(0.5, 0.1)$	$(0.85, 0.2)$	$(0.3, 0.7)$
$\hbar_3$	$(0.6, 0.9)$	$(0.2, 0.6)$	$(0.6, 0.1)$	$(0.95, 0.3)$
$\hbar_4$	$(0.7, 0.4)$	$(0.93, 0.4)$	$(0.4, 0.2)$	$(0.5, 0.1)$
$\hbar_5$	$(0.3, 0.7)$	$(0.78, 0.25)$	$(0.8, 0.15)$	$(0.7, 0.4)$

$\underline{\mu_1}(\hbar_1) = 0.4,\ \underline{\psi_1}(\hbar_1) = 0.6,\ \underline{\mu_2}(\hbar_2) = 0.3,\ \underline{\psi_2}(\hbar_2) = 0.7,\ \underline{\mu_3}(\hbar_3) = 0.2,$
$\underline{\psi_3}(\hbar_3) = 0.9, \underline{\mu_4}(\hbar_4) = 0.4,\ \underline{\psi_4}(\hbar_4) = 0.6,\ \underline{\mu_5}(\hbar_5) = 0.3,\ \underline{\psi_5}(\hbar_5) = 0.7$

$\overline{\mu_1}(\hbar_1) = 0.9,\ \overline{\psi_1}(\hbar_1) = 0.1,\ \overline{\mu_2}(\hbar_2) = 0.9,\ \overline{\psi_2}(\hbar_2) = 0.1,\ \overline{\mu_3}(\hbar_3) = 0.95,$
$\overline{\psi_3}(\hbar_3) = 0.1,\ \overline{\mu_4}(\hbar_4) = 0.93,\ \overline{\psi_4}(\hbar_4) = 0.1,\ \overline{\mu_5}(\hbar_5) = 0.9,\ \overline{\psi_5}(\hbar_5) = 0.1$

Now to get the lower and upper q-ROFS_{ft}R approximation operators;

$$\underline{\mathcal{L}}(\mathcal{M}) = \{(\hbar_1, 0.7, 0.6), (\hbar_2, 0.3, 0.7), (\hbar_3, 0.2, 0.9), (\hbar_4, 0.4, 0.6), (\hbar_5, 0.3, 0.7)\}$$

$$\overline{\mathcal{L}}(\mathcal{M}) = \{(\hbar_1, 0.9, 0.1), (\hbar_2, 0.9, 0.1), (\hbar_3, 0.95, 0.1), (\hbar_4, 0.93, 0.1), (\hbar_5, 0.9, 0.1)\}$$

Therefore, $\mathcal{L}(\mathcal{M}) = \left(\underline{\mathcal{L}}(\mathcal{M}), \overline{\mathcal{L}}(\mathcal{M})\right)$

$$= \left\{ \begin{matrix} \big(\hbar_1, (0.7, 0.6), (0.9, 0.1)\big), \big(\hbar_2, (0.3, 0.7), (0.9, 0.1)\big), \big(\hbar_3, (0.2, 0.9), (0.95, 0.1)\big), \\ \big(\hbar_4, (0.4, 0.6), (0.93, 0.1)\big), \big(\hbar_5, (0.3, 0.7), (0.9, 0.1)\big) \end{matrix} \right\}$$

8.2.5. Definition

Consider $\mathcal{L}_{s_j}(\mathcal{M}_1) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_1), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_1)\right)$ for $(j = 1,2)$ are the two q-ROFS_{ft}RVs. Then the following operation are defined.

(i) $\mathcal{L}_{s_1}(\mathcal{M}_1) \cup \mathcal{L}_{s_2}(\mathcal{M}_1) = \left\{\left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \cup \underline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right), \left(\overline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \cup \overline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right)\right\};$

(ii) $\mathcal{L}_{s_1}(\mathcal{M}_1) \cap \mathcal{L}_{s_2}(\mathcal{M}_1) = \left\{\left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \cap \underline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right), \left(\overline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \cap \overline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right)\right\}$;

(iii) $\mathcal{L}_{s_1}(\mathcal{M}_1) \oplus \mathcal{L}_{s_2}(\mathcal{M}_1) = \left\{\left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \oplus \underline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right), \left(\overline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \oplus \overline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right)\right\}$;

(iv) $\mathcal{L}_{s_1}(\mathcal{M}_1) \otimes \mathcal{L}_{s_2}(\mathcal{M}_1) = \left\{\left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \otimes \underline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right), \left(\overline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \otimes \overline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right)\right\}$;

(v) $\mathcal{L}_{s_1}(\mathcal{M}_1) \subseteq \mathcal{L}_{s_2}(\mathcal{M}_1) = \left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \subseteq \underline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right)$ and $\left(\overline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \subseteq \overline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\right)$;

(vi) $\alpha\mathcal{L}_{s_1}(\mathcal{M}_1) = \left(\alpha\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1), \lambda\overline{\mathcal{L}_{s_1}}(\mathcal{M}_1)\right)$ for $\alpha \geq 1$;

(vii) $\left(\mathcal{L}_{s_1}(\mathcal{M}_1)\right)^{\alpha} = \left(\left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1)\right)^{\alpha}, \left(\overline{\mathcal{L}_{s_1}}(\mathcal{M}_1)\right)^{\alpha}\right)$ for $\alpha \geq 1$.

(viii) $\mathcal{L}_{s_1}(\mathcal{M}_1)^c = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_1)^c, \overline{\mathcal{L}_{s_j}}(\mathcal{M}_1)^c\right)$, where $\underline{\mathcal{L}_{s_j}}(\mathcal{M}_1)^c$ are $\overline{\mathcal{L}_{s_j}}(\mathcal{M}_1)^c$ the complements of q-ROFS_{ft}R approximation operators $\underline{\mathcal{L}_{s_j}}(\mathcal{M}_1)$ and $\overline{\mathcal{L}_{s_j}}(\mathcal{M}_1)$, i.e. $\underline{\mathcal{L}_{s_j}}(\mathcal{M}_1) = \left(\underline{\psi_{ij}}, \underline{\mu_{ij}}\right)$.

(ix) $\mathcal{L}(\mathcal{M}_1) = \mathcal{L}(\mathcal{M}_2)$ iff $\underline{\mathcal{L}}(\mathcal{M}_1) = \underline{\mathcal{L}}(\mathcal{M}_2)$ and $\overline{\mathcal{L}}(\mathcal{M}_1) = \overline{\mathcal{L}}(\mathcal{M}_2)$;

8.2.6. Definition

Let $\mathcal{L}_{s_1}(\mathcal{M}_1) = \left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1), \overline{\mathcal{L}_{s_1}}(\mathcal{M}_1)\right) = \left(\left(\underline{\mu_{11}}, \underline{\psi_{11}}\right), \left(\overline{\mu_{11}}, \overline{\psi_{11}}\right)\right)$ be a q-ROFS_{ft}RV. Then the score function for $\mathcal{L}_{s_1}(\mathcal{M}_1)$ is given as.

$$\mathcal{S}c\left(\mathcal{L}_{s_1}(\mathcal{M}_1)\right) = \frac{1}{2}\left(\underline{\mu_{11}}^{q} + \overline{\mu_{11}}^{q} - \underline{\psi_{11}}^{q} - \overline{\psi_{11}}^{q}\right), \quad \mathcal{S}c\left(\mathcal{L}_{s_1}(\mathcal{M}_1)\right) \in [-1, 1] \ and \ q \geq 1 .$$

Greater the score value, greater the q-ROFS_{ft}RV is.

8.2.7. **Proposition**

Let $(T, \mathbb{E}, \mathcal{L})$ be q-ROFS_{ft} approximation space. For any two $\mathcal{L}(\mathcal{M}_1) = \left(\underline{\mathcal{L}}(\mathcal{M}_1), \overline{\mathcal{L}}(\mathcal{M}_1)\right)$ and $\mathcal{L}(\mathcal{M}_2) = \left(\underline{\mathcal{L}}(\mathcal{M}_2), \overline{\mathcal{L}}(\mathcal{M}_2)\right)$ q-ROFS_{ft}RSs over a common universe set T. Then the following properties are holds.

(i) $\sim (\sim \mathcal{L}(\mathcal{M}_1)) = \mathcal{M}_1$, where $\sim \mathcal{L}(\mathcal{M}_1)$ is the complement of $\mathcal{L}(\mathcal{M}_1)$;

(ii) $\mathcal{L}(\mathcal{M}_1) \cup \mathcal{L}(\mathcal{M}_2) = \mathcal{L}(\mathcal{M}_2) \cup \mathcal{L}(\mathcal{M}_1)$, $\mathcal{L}(\mathcal{M}_1) \cap \mathcal{L}(\mathcal{M}_2) = \mathcal{L}(\mathcal{M}_2) \cap \mathcal{L}(\mathcal{M}_1)$

(iii) $\sim (\mathcal{L}(\mathcal{M}_1) \cup \mathcal{L}(\mathcal{M}_2)) = (\sim \mathcal{L}(\mathcal{M}_1)) \cap (\sim \mathcal{L}(\mathcal{M}_2))$;

(iv) $\sim (\mathcal{L}(\mathcal{M}_1) \cap \mathcal{L}(\mathcal{M}_2)) = (\sim \mathcal{L}(\mathcal{M}_1)) \cup (\sim \mathcal{L}(\mathcal{M}_2))$;

(v) If $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $\mathcal{L}(\mathcal{M}_1) \subseteq \mathcal{L}(\mathcal{M}_2)$;

(vi) $\mathcal{L}(\mathcal{M}_1 \cup \mathcal{M}_2) \supseteq \mathcal{L}(\mathcal{M}_1) \cup \mathcal{L}(\mathcal{M}_2)$;

(vii) $\mathcal{L}(\mathcal{M}_1 \cap \mathcal{M}_2) \subseteq \mathcal{L}(\mathcal{M}_1) \cap \mathcal{L}(\mathcal{M}_2)$.

8.3. q-Rung orthopair fuzzy soft rough averaging aggregation operator

This section is devoted for the study of q-ROFS_{ft}RA aggregation operators such as q-ROFS_{ft}RWA, q-ROFS_{ft}ROWA and q-ROFS_{ft}RHA operators. We will present the fundamental properties of these operators in detail.

8.3.1. q-Rung orthopair fuzzy soft rough weighted averaging operator

In this subsection the detail study of q-ROFS_{ft}RWA operator and their basic properties such as Idempotency, Boundedness and Monotonicity etc. are investigated.

8.3.1.1. **Definition**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2, \dots, m,\ j = 1,2, \dots, n)$ be the collection of q-ROFS_{ft}RVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors of experts $\Bbbk_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$, $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. The q-ROFS_{ft}RWA operator is defined as:

$$\text{q} - \text{ROF}S_{ft}\text{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \left(\oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right), \oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\right)$$

In view of above definition the aggregated result for q-ROFS_{ft}RWA is given in the following Theorem 8.3.1.2.

8.3.1.2. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors of experts $\hbar_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1,\ \sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \le \bar{\bar{w}}_i, \bar{\bar{u}}_j \le 1$ respectively. Then q-ROFS_{ft}RWA operator is given as:

$$\mathrm{q-ROF}S_{ft}\mathrm{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$$= \left[\oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right), \oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\right]$$

$$= \left[\begin{array}{c} \left\{\sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m} \underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}, \\ \left\{\sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \overline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m} \overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\} \end{array}\right]$$

Proof: By using Mathematical induction to prove the result.

As by operational law

$$\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \oplus \underline{\mathcal{L}_{s_1}}(\mathcal{M}_2) = \left(\left(\underline{\mu_{11}}, \underline{\psi_{11}}\right) \oplus \left(\underline{\mu_{12}}, \underline{\psi_{12}}\right), \left(\overline{\mu_{11}}, \overline{\psi_{11}}\right) \oplus \left(\overline{\mu_{12}}, \overline{\psi_{12}}\right)\right)$$

$$= \left[\left(\sqrt[q]{\underline{\mu_{11}}^q + \underline{\mu_{12}}^q + \underline{\mu_{11}}^q \underline{\mu_{12}}^q}, \underline{\psi_{11}}\ \underline{\psi_{12}}\right), \left(\sqrt[q]{\overline{\mu_{11}}^q + \overline{\mu_{12}}^q + \overline{\mu_{11}}^q \overline{\mu_{12}}^q}, \overline{\psi_{11}}\ \overline{\psi_{12}}\right)\right]$$

and

$$\alpha \underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) = \left[\left(\sqrt[q]{1 - \left(1 - \underline{\mu_{11}}^q\right)^{\alpha}}, \underline{\psi_{11}}^{\alpha}\right), \left(\sqrt[q]{1 - \left(1 - \overline{\mu_{11}}^q\right)^{\alpha}}, \overline{\psi_{11}}^{\alpha}\right)\right]$$

Suppose the result is true for $m = 2$ *and* $n = 2$, that is

$$\text{q}-\text{ROF}S_{ft}\text{RWA}\left(\mathcal{L}_{s_j}(\mathcal{M}_i),\mathcal{L}_{s_j}(\mathcal{M}_i),\right)$$
$$=\left[\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{2}\bar{\bar{w}}_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right),\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{2}\bar{\bar{w}}_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\right]$$

Now consider

$$\text{q}-\text{ROF}S_{ft}\text{RWA}\left(\mathcal{L}_{s_j}(\mathcal{M}_i),\mathcal{L}_{s_j}(\mathcal{M}_i),\right)$$
$$=\left[\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{2}\bar{\bar{w}}_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right),\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{2}\bar{\bar{w}}_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\right]$$

$$=\begin{bmatrix}\left\{\bar{\bar{u}}_1\left(\bar{\bar{w}}_1\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1)\oplus\bar{\bar{w}}_2\underline{\mathcal{L}_{s_1}}(\mathcal{M}_2)\right)\oplus\bar{\bar{u}}_2\left(\bar{\bar{w}}_1\underline{\mathcal{L}_{s_2}}(\mathcal{M}_1)\oplus\bar{\bar{w}}_2\underline{\mathcal{L}_{s_2}}(\mathcal{M}_2)\right)\right\}, \\ \left\{\bar{\bar{u}}_1\left(\bar{\bar{w}}_1\overline{\mathcal{L}_{s_j}}(\mathcal{M}_1)\oplus\bar{\bar{w}}_2\overline{\mathcal{L}_{s_j}}(\mathcal{M}_2)\right)\oplus\bar{\bar{u}}_2\left(\bar{\bar{w}}_1\overline{\mathcal{L}_{s_j}}(\mathcal{M}_1)\oplus\bar{\bar{w}}_2\overline{\mathcal{L}_{s_j}}(\mathcal{M}_2)\right)\right\}\end{bmatrix}$$

$$\text{q}-\text{ROF}S_{ft}\text{RWA}\left(\mathcal{L}_{s_j}(\mathcal{M}_i),\mathcal{L}_{s_j}(\mathcal{M}_i),\right)$$
$$=\begin{bmatrix}\left\{\sqrt[q]{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}, \\ \left\{\left(\sqrt[q]{1-\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\left(1-\overline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{2}\left(\prod_{i=1}^{2}\overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\right\}\end{bmatrix}$$

The result is true for $m=2\ and\ n=2$.

Now consider the result is for $n=k_1\ \ and\ \ m=k_2$

$$\text{q}-\text{ROF}S_{ft}\text{RWA}\left(\mathcal{L}_{s_j}(\mathcal{M}_i),\mathcal{L}_{s_j}(\mathcal{M}_i),\dots,\mathcal{L}_{s_{k_1}}(\mathcal{M}_{k_2})\right)$$
$$=\begin{bmatrix}\left\{\sqrt[q]{1-\prod_{j=1}^{k_1}\left(\prod_{i=1}^{k_2}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{k_1}\left(\prod_{i=1}^{k_2}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}, \\ \left\{\left(\sqrt[q]{1-\prod_{j=1}^{k_1}\left(\prod_{i=1}^{k_2}\left(1-\overline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{k_1}\left(\prod_{i=1}^{k_2}\overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\right\}\end{bmatrix}$$

Next to show that the result hold for $n=k_1+1\ \ and\ \ m=k_2+1$, so we have

$$\mathrm{q-ROF}\mathcal{S}_{ft}\mathrm{RWA}\left[\left(\mathcal{L}_{s_j}(\mathcal{M}_i),\mathcal{L}_{s_j}(\mathcal{M}_i),\ldots,\mathcal{L}_{s_{k_1}}(\mathcal{M}_{k_2})\right),\mathcal{L}_{s_{k_1+1}}(\mathcal{M}_{k_2+1})\right]$$

$$=\begin{bmatrix} \oplus_{j=1}^{k_1}\bar{\bar{u}}_j\left(\oplus_{i=1}^{k_2}\bar{\bar{w}}_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\oplus\bar{\bar{u}}_{k_1+1}\left(\bar{\bar{w}}_{k_2+1}\underline{\mathcal{L}_{s_{k_1+1}}}(\mathcal{M}_{k_2+1})\right), \\ \oplus_{j=1}^{k_1}\bar{\bar{u}}_j\left(\oplus_{i=1}^{k_2}\bar{\bar{w}}_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\oplus\bar{\bar{u}}_{k_1+1}\left(\bar{\bar{w}}_{k_2+1}\overline{\mathcal{L}_{s_{k_1+1}}}(\mathcal{M}_{k_2+1})\right) \end{bmatrix}$$

$$=\begin{bmatrix} \left\{\sqrt[q]{1-\prod_{j=1}^{k_1+1}\left(\prod_{i=1}^{k_2+1}\left(1-\underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{k_1+1}\left(\prod_{i=1}^{k_2+1}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right\}, \\ \left\{\left(\sqrt[q]{1-\prod_{j=1}^{k_1+1}\left(\prod_{i=1}^{k_2+1}\left(1-\overline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{k_1+1}\left(\prod_{i=1}^{k_2+1}\overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\right\} \end{bmatrix}$$

This implies the result is true for $n = k_1 + 1\ and\ m = k_2 + 1$. Therefore the result hold for all $m, n \geq 1$.

Since it is clear that $\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\ and\ \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ are q-ROFVs. So by Definition 8.2.7, we have $\oplus_{j=1}^{n}\bar{\bar{u}}_j\left(\oplus_{i=1}^{m}\bar{\bar{w}}_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and $\oplus_{j=1}^{n}\bar{\bar{u}}_j\left(\oplus_{i=1}^{m}\bar{\bar{w}}_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ are also q-ROFVs. Therefore, $\mathrm{q-ROF}\mathcal{S}_{ft}\mathrm{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\ldots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$ is also a q-ROF$\mathcal{S}_{ft}$RV in approximation space $(T, \mathbb{E}, \mathcal{L})$.

8.3.1.3. **Example**

Let $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3\}$ be the set and $\mathcal{M} = \{s_1, s_2\} \subseteq \mathbb{E}$ be the set of parameter with weight vector $\bar{\bar{w}} = (0.25,0.3,0.45)^T$ for $\mathscr{k}_i\ (i = 1,2,3)$ and $\bar{\bar{u}} = (0.55,0.45)$ for $s_j\ (j = 1,2)$. Then q-ROF$\mathcal{S}_{ft}$RVs is given in Table 8.3.

Table 8.3, Tabular representation of $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$

$\mathcal{L}$	s_1	s_2
$\mathscr{k}_1$	$((0.9,0.3),(0.8,0.4))$	$((0.55,0.2),(0.76,0.14))$
$\mathscr{k}_2$	$((0.7,0.1),(0.2,0.75))$	$((0.92,0.3),(0.6,0.3))$
$\mathscr{k}_3$	$((0.92,0.25),(0.65,0.15))$	$((0.4,0.85),(0.88,0.12))$

$$\text{q}-\text{ROF}S_{ft}\text{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\ldots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$=\left[\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{3}\bar{\bar{w}}_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right),\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{3}\bar{\bar{w}}_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\right]$$

$$=\left[\begin{array}{c}\left\{\begin{array}{c}\sqrt[3]{1-\left(\begin{array}{c}[(1-0.9^3)^{0.25}(1-0.7^3)^{0.25}(1-0.29^3)^{0.45}]^{0.55}\\ [(1-0.55^3)^{0.25}(1-0.92^3)^{0.25}(1-0.4^3)^{0.45}]^{0.45}\end{array}\right)},\\ (0.3^{0.25}0.1^{0.3}0.25^{0.45})^{0.55}(0.2^{0.25}0.3^{0.3}0.85^{0.45})^{0.45}\end{array}\right\},\\ \left\{\begin{array}{c}\sqrt[3]{1-\left(\begin{array}{c}[(1-0.4^3)^{0.25}(1-0.2^3)^{0.25}(1-0.65^3)^{0.45}]^{0.55}\\ [(1-0.76^3)^{0.25}(1-0.6^3)^{0.25}(1-0.88^3)^{0.45}]^{0.45}\end{array}\right)},\\ (0.4^{0.25}0.75^{0.3}0.15^{0.45})^{0.55}(0.14^{0.25}0.3^{0.3}0.12^{0.45})^{0.45}\end{array}\right\}\end{array}\right]$$

$$=[(0.831432,0.255487),(0.72581,0.26258)].$$

From the analysis of Theorem 8.3.1.2, q-ROFS_{ft}RWA operator has the following properties.

8.3.1.4. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i)=\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i=1,2,\ldots,m,\ j=1,2,\ldots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\bar{\bar{w}}=(\bar{\bar{w}}_1,\bar{\bar{w}}_2,\ldots,\bar{\bar{w}}_m)^T\ and\ \bar{\bar{u}}=(\bar{\bar{u}}_1,\bar{\bar{u}}_2,\ldots,\bar{\bar{u}}_n)^T$ be the weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m}\bar{\bar{w}}_i=1,\ \sum_{j=1}^{n}\bar{\bar{u}}_j=1$ and $0\leq\bar{\bar{w}}_i,\bar{\bar{u}}_j\leq 1$ respectively. Then the following properties hold for q-ROFS_{ft}RWA operator:

(i) **(Idempotency)** If $\mathcal{L}_{s_j}(\mathcal{M}_i)=\mathcal{E}_s(\mathcal{N})$ (for all $i=1,2,\ldots,m\ and\ j=1,2,\ldots n$), where $\mathcal{E}_s(\mathcal{M})=\left(\underline{\mathcal{E}_s}(\mathcal{M}),\overline{\mathcal{E}_s}(\mathcal{M})\right)=\left((\underline{b},\underline{d}),(\overline{b},\overline{d})\right)$. Then
$$q-ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\mathcal{L}_{s_2}(\mathcal{M}_2),\ldots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)=\mathcal{E}_s(\mathcal{M}).$$

(ii) **(Boundedness)** Let $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{-}=\left(\min_j\min_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\max_j\max_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{+}=\left(\max_j\max_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\min_j\min_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$. Then

$$\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^- \leq q-ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\mathcal{L}_{s_2}(\mathcal{M}_2),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$\leq \left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^+.$$

(iii) **(Monotonicity)** Let $\mathcal{E}_{s_j}(\mathcal{N}_i)=\left(\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i),\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\right)$ $(i=1,2,\dots,m,\ j=1,2,\dots,n)$ be another collection of q-ROFS_{ft}RVs such that $\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\leq\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ and $\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\leq\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$. Then

$$q-ROFS_{ft}RWA\left(\mathcal{E}_{s_1}(\mathcal{M}_1),\mathcal{E}_{s_2}(\mathcal{M}_2),\dots,\mathcal{E}_{s_n}(\mathcal{M}_m)\right)$$
$$\leq\ q-ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\mathcal{L}_{s_2}(\mathcal{M}_2),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

(iv) **(Shift invariance)** Let $\mathcal{E}_s(\mathcal{N})=\left(\underline{\mathcal{E}_s}(\mathcal{N}),\overline{\mathcal{E}_s}(\mathcal{N})\right)=\left((\underline{b},\underline{d}),(\overline{b},\overline{d})\right)$ be any other $q-ROFS_{ft}RV$. Then

$$q-ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1)\oplus\mathcal{E}_s(\mathcal{N}),\mathcal{L}_{s_2}(\mathcal{M}_2)\oplus\mathcal{E}_s(\mathcal{N}),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\oplus\mathcal{E}_s(\mathcal{N})\right)$$
$$=q-ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\mathcal{L}_{s_2}(\mathcal{M}_2),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)\oplus\mathcal{E}_s(\mathcal{N}).$$

(v) **(Homogeneity)** For any real number $\lambda>0$;

$$q-ROFS_{ft}RW\ A\left(\lambda\mathcal{L}_{s_1}(\mathcal{M}_1),\lambda\mathcal{L}_{s_2}(\mathcal{M}_2),\dots,\lambda\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$=\lambda\, q-ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\mathcal{L}_{s_2}(\mathcal{M}_2),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

Proof: **(i)** **(Idempotency)** Given that $\mathcal{L}_{s_j}(\mathcal{M}_i)=\mathcal{E}_s(\mathcal{N})$ (for all $i=1,2,\dots,m\ and\ j=1,2,\dots n$), where $\mathcal{E}_s(\mathcal{M})=\left(\underline{\mathcal{E}_s}(\mathcal{M}),\overline{\mathcal{E}_s}(\mathcal{M})\right)=\left((\underline{b},\underline{d}),(\overline{b},\overline{d})\right)$

$$\mathrm{q-ROF}S_{ft}\mathrm{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$=\left(\oplus_{j=1}^{n}\bar{\bar{u}}_j\left(\oplus_{i=1}^{m}\bar{\bar{w}}_i\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right),\oplus_{j=1}^{n}\bar{\bar{u}}_j\left(\oplus_{i=1}^{m}\bar{\bar{w}}_i\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)\right)$$

$$=\begin{bmatrix}\left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right),\\ \left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\end{bmatrix}$$

For all i, j $\mathcal{L}_{s_j}(\mathcal{M}_i) = \mathcal{E}_s(\mathcal{N}) = \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$. Therefore,

$$= \begin{bmatrix} \left(\sqrt[q]{1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{m} (1 - \underline{b}^q)^{\overline{\overline{w}}_i} \right)^{\overline{\overline{u}}v_j}}, \prod_{j=1}^{n} \left(\prod_{i=1}^{m} \underline{d}^{\overline{\overline{w}}_i} \right)^{\overline{\overline{u}}_j} \right), \\ \left(\sqrt[q]{1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{m} \left(1 - \overline{b}^q\right)^{\overline{\overline{w}}_i} \right)^{\overline{\overline{u}}_j}}, \prod_{j=1}^{n} \left(\prod_{i=1}^{m} \overline{d}^{\overline{\overline{w}}_i} \right)^{\overline{\overline{u}}_j} \right) \end{bmatrix}$$

$$= \left[\left(\sqrt[q]{1 - (1 - \underline{b}^q)}, \underline{d} \right), \left(\sqrt[q]{1 - \left(1 - \left(1 - \overline{b}^q\right)\right)}, \overline{d} \right) \right]$$

$$= \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \mathcal{E}_s(\mathcal{N})$$

Hence

$$\text{q} - \text{ROF}S_{ft}\text{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \mathcal{E}_s(\mathcal{N})$$

(ii) (Boundedness) As $\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{-} = \left[\left(\min_j \min_i \left\{\underline{\mu_{ij}}\right\}, \max_j \max_i \left\{\underline{\psi_{ij}}\right\}\right),\right.$ $\left.\left(\min_j \min_i \{\overline{\mu_{ij}}\}, \max_j \max_i \{\overline{\psi_{ij}}\}\right)\right]$ and

$\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{+} = \left[\left(\max_j \max_i \left\{\underline{\mu_{ij}}\right\}, \min_j \min_i \left\{\underline{\psi_{ij}}\right\}\right), \left(\max_j \max_i \{\overline{\mu_{ij}}\}, \min_j \min_i \{\overline{\psi_{ij}}\}\right)\right],$

$\mathcal{L}_{s_j}(\mathcal{M}_i) = \left[\left(\underline{\mu_{ij}}, \underline{\psi_{ij}}\right), (\overline{\mu_{ij}}, \overline{\psi_{ij}})\right]$. To prove that

$$\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{-} \leq q - ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \leq \left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{+}$$

Since for each $i = 1,2, \ldots, m \; and \; j = 1,2, \ldots, n$, we have

$\min_j \min_i \left\{\underline{\mu_{ij}}\right\} \leq \underline{\mu_{ij}} \leq \max_j \max_i \left\{\underline{\mu_{ij}}\right\} \Leftrightarrow 1 - \max_j \max_i \left\{\underline{\mu_{ij}}^q\right\} \leq 1 - \underline{\mu_{ij}}^q \leq 1 -$ $\min_j \min_i \left\{\underline{\mu_{ij}}^q\right\}$

$$\Leftrightarrow \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\max_j \max_i \left\{\underline{\mu_{ij}}^{q}\right\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\min_j \min_i \left\{\underline{\mu_{ij}}^{q}\right\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Leftrightarrow \left(\left(1-\max_j \max_i \left\{\underline{\mu_{ij}}^{q}\right\}\right)^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}t_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \left(\left(1-\min_j \min_i \left\{\underline{\mu_{ij}}^{q}\right\}\right)^{\sum_{i=1}^{n}\bar{\bar{w}}_i}\right)^{\sum_{j=1}^{m}\bar{\bar{u}}_j}$$

$$\Leftrightarrow \left(1-\max_j \max_i \left\{\underline{\mu_{ij}}^{q}\right\}\right) \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \left(1-\min_j \min_i \left\{\underline{\mu_{ij}}^{q}\right\}\right)$$

$$\Leftrightarrow 1-\left(1-\min_j \min_i \left\{\underline{\mu_{ij}}^{q}\right\}\right) \leq 1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq 1-\left(1-\max_j \max_i \left\{\underline{\mu_{ij}}^{q}\right\}\right)$$

Hence

$$\min_j \min_i \left\{\underline{\mu_{ij}}\right\} \leq \sqrt[q]{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}$$

$$\leq \max_j \max_i \left\{\underline{\mu_{ij}}\right\} \tag{8.4}$$

Now for each $i = 1,2,\dots,n\ and\ j = 1,2,\dots,m$, we have

$$\min_j \min_i \left\{\underline{\psi_{ij}}\right\} \leq \underline{\psi_{ij}} \leq \max_j \max_i \left\{\underline{\psi_{ij}}\right\}$$

$$\Leftrightarrow \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\min_j \min_i \left\{\underline{\psi_{ij}}\right\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\max_j \max_i \left\{\underline{\psi_{ij}}\right\}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Leftrightarrow \left(\left(\min_j \min_i \left\{\underline{\psi_{ij}}\right\}\right)^{\sum_{i=1}^{m} \bar{\bar{w}}_i}\right)^{\sum_{j=1}^{n} \bar{\bar{u}}_j} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\leq \left(\left(\max_j \max_i \left\{\underline{\psi_{ij}}\right\}\right)^{\sum_{i=1}^{m} \bar{\bar{w}}_i}\right)^{\sum_{j=1}^{n} \bar{\bar{u}}_j}$$

this implies that

$$\min_j \min_i \left\{\underline{\psi_{ij}}\right\} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \max_j \max_i \left\{\underline{\psi_{ij}}\right\} \tag{8.5}$$

Similarly we can show that

$$\min_j \min_i \{\overline{\mu_{ij}}\} \leq \sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}$$

$$\leq \max_j \max_i \{\overline{\mu_{ij}}\} \tag{8.6}$$

and

$$\min_j \min_i \{\overline{\psi_{ij}}\} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \max_j \max_i \{\overline{\psi_{ij}}\} \tag{8.7}$$

So from $Eqs.\,(8.4), (8.5), (8.6)\ and\ (8.7)$ we have

$$\min_j \min_i \left\{\underline{\mu_{ij}}\right\} \leq \sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \leq \max_j \max_i \left\{\underline{\mu_{ij}}\right\};$$

$$\min_j \min_i \left\{\underline{\psi_{ij}}\right\} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \max_j \max_i \left\{\underline{\psi_{ij}}\right\};$$

$$\min_j \min_i \{\overline{\mu_{ij}}\} \leq \sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \leq \max_j \max_i \{\overline{\mu_{ij}}\} \quad and$$

$$\min_j \min_i \{\overline{\psi_{ij}}\} \leq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \max_j \max_i \{\overline{\psi_{ij}}\}$$

This implies that $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^- \leq q-ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \leq \left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^+$

(iii) Monotonicity: Since $\mathcal{E}_{s_j}(\mathcal{N}_i) = \left(\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i), \overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\right) = \left(\left(\underline{b_{ij}}, \underline{d_{ij}}\right), \left(\overline{b_{ij}}, \overline{d_{ij}}\right)\right)$ and

$\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right) = \left(\left(\underline{\mu_{ij}}, \underline{\psi_{ij}}\right), \left(\overline{\mu_{ij}}, \overline{\psi_{ij}}\right)\right)$. To show that $\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ and $\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)(i = 1, 2, \dots, m)$ *and* $(j = 1, 2, \dots, n)$, so

$$\underline{b_{ij}} \leq \underline{\mu_{ij}} \Rightarrow 1 - \underline{\mu_{ij}} \leq 1 - \underline{b_{ij}} \Rightarrow 1 - \underline{\mu_{ij}}^q \leq 1 - \underline{b_{ij}}^q$$

$$\Rightarrow \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \underline{b_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\Rightarrow 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \underline{b_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \leq 1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}$$

$$\sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \underline{b_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \leq \sqrt[q]{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \tag{8.8}$$

Next

$$\underline{d_{ij}} \geq \underline{\psi_{ij}} \Rightarrow \left(\prod_{i=1}^{m}\left(\underline{d_{ij}}\right)^{\bar{\bar{w}}_i}\right) \geq \prod_{i=1}^{m}\left(\underline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}$$

$$\Rightarrow \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{d_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \geq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\underline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \tag{8.9}$$

Similarly, we can show that

$$\sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \overline{b_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \leq \sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \overline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}} \tag{8.10}$$

$$\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{d_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \geq \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(\overline{\psi_{ij}}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j} \tag{8.11}$$

Hence from Eqs. $(8.8), (8.9), (8.10)$ and (8.11), we get

$$\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i) \;\; and \;\; \overline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$$

Therefore,

$$q - ROFS_{ft}RWA\left(\mathcal{E}_{s_1}(\mathcal{M}_1), \mathcal{E}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{E}_{s_n}(\mathcal{M}_m)\right)$$
$$\leq \; q - ROFS_{ft}RWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$\boldsymbol{iv}$: ($\boldsymbol{Shift\ Invariance}$) As $\mathcal{E}_s(\mathcal{M}) = \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \left((\underline{b}\,, \underline{d}), (\overline{b}\,, \overline{d})\right)$ is any q-ROFS_{ft}RV and $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right) = \left(\left(\underline{\mu_{ij}}, \underline{\psi_{ij}}\right), \left(\overline{\mu_{ij}}, \overline{\psi_{ij}}\right)\right)$ are the collection of q-ROFS_{ft}RVs, so

$$\mathcal{L}_{s_1}(\mathcal{M}_1) \oplus \mathcal{E}_s(\mathcal{M}) = \left[\left(\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \oplus \underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i) \oplus \overline{\mathcal{E}_s}(\mathcal{M})\right)\right]$$

As

$$\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) \oplus \underline{\mathcal{E}_s}(\mathcal{M}) = \left(\sqrt[q]{1 - (1 - \underline{\mu_{11}}^q)(1 - \underline{b}^{\,q})}\;, \underline{\psi_{11}}\underline{d}\right)$$

Therefore,

$$\mathrm{q - ROF}S_{ft}\mathrm{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1) \oplus \mathcal{E}_s(\mathcal{M}), \mathcal{L}_{s_2}(\mathcal{M}_2) \oplus \mathcal{E}_s(\mathcal{M}), \ldots, \mathcal{L}_{s_3}(\mathcal{M}_3) \oplus \mathcal{E}_s(\mathcal{M})\right)$$
$$= \left[\oplus_{j=1}^{n} \mathcal{v}_j \left\{\oplus_{i=1}^{m} t_i \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i) \oplus \mathcal{E}_s(\mathcal{M})\right)\right\}, \oplus_{j=1}^{n} \mathcal{v}_j \left\{\oplus_{i=1}^{m} t_i \left(\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i) \oplus \mathcal{E}_s(\mathcal{M})\right)\right\}\right]$$

$$= \left[\begin{array}{c} \left(\sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\left(1 - \underline{b}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\;, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\underline{d}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right), \\ \left(\sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \overline{\mu_{ij}}^{\,q}\right)^{\bar{\bar{w}}_i}\left(1 - \overline{b}^{\,q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\;, \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{ij}}^{\,\bar{\bar{w}}_i}\overline{d}^{\,\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right) \end{array}\right]$$

$$=\left[\begin{array}{l}\left(\sqrt[q]{1-\left(1-\underline{b}^q\right)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\underline{d}\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right),\\ \left(\sqrt[q]{1-\left(1-\overline{b}^q\right)\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\overline{d}\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\end{array}\right]$$

$$=\left[\begin{array}{l}\left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\oplus(\underline{b},\underline{d}),\\ \left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\oplus(\overline{b},\overline{d})\end{array}\right]$$

$$=\left[\begin{array}{l}\left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right),\\ \left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^q\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\end{array}\right]\oplus\left[(\underline{b},\underline{d}),(\overline{b},\overline{d})\right]$$

$$=\mathrm{q}-\mathrm{ROF}S_{ft}\mathrm{RWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\mathcal{L}_{s_2}(\mathcal{M}_2),\ldots,\mathcal{L}_{s_3}(\mathcal{M}_3)\right)\oplus\mathcal{E}_s(\mathcal{M})$$

Therefore proved is completed.

iv: (***Homogeneity***) For a real number $\lambda > 0$ and $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ be a q-ROFS_{ft}RVs, then

$$\lambda\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\lambda\,\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \lambda\,\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$$

As

$$\lambda\,\underline{\mathcal{L}_{s_1}}(\mathcal{M}_1) = \left[\left(\sqrt[q]{1-\left(1-\underline{\mu_{11}}^q\right)^{\lambda}}, \underline{\psi_{11}}^{\lambda}\right), \left(\sqrt[q]{1-\left(1-\overline{\mu_{11}}^q\right)^{\lambda}}, \overline{\psi_{11}}^{\lambda}\right)\right]$$

Now

$$
\begin{aligned}
&\mathrm{q-ROF}S_{ft}\mathrm{RWA}\left(\lambda\mathfrak{I}_{e_{11}},\lambda\mathfrak{I}_{e_{12}},\dots,\lambda\mathfrak{I}_{e_{nm}}\right)\\
&=\left[\begin{array}{l}\left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^{q}\right)^{\lambda\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{ij}}^{\lambda\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right),\\ \left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^{q}\right)^{\lambda\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{ij}}^{\lambda\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right)\end{array}\right]\\
&=\left[\begin{array}{l}\left(\sqrt[q]{1-\left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{ij}}^{q}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right\}^{\lambda}},\left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{ij}}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right\}^{\lambda}\right),\\ \left(\sqrt[q]{1-\left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{ij}}^{q}\right)^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right\}^{\lambda}},\left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{ij}}^{\overline{\overline{w}}_i}\right)^{\overline{\overline{u}}_j}\right\}^{\lambda}\right)\end{array}\right]\\
&=\lambda\,\mathrm{q-ROF}S_{ft}\mathrm{RWA}\left(\mathfrak{I}_{e_{11}},\mathfrak{I}_{e_{12}},\dots,\mathfrak{I}_{e_{nm}}\right)
\end{aligned}
$$

Hence, the proof is completed.

8.3.1.5. **Remark**

(a) If the value of rung $q = 1$, then the proposed q-ROFS_{ft}RWA operator reduced to IFS_{ft}RWA operator.

(b) If the value of rung $q = 2$, then the proposed q-ROFS_{ft}RWA operator reduced to PyFS_{ft}RWA operator.

(c) If there is only one soft parameter s_1; $(means\, n = 1)$, then the proposed q-ROFS_{ft}RWA operator reduced to q-ROFRWA operator.

8.3.2. q-Rung orthopair fuzzy soft rough ordered weighted averaging operator

In this subsection the detail study of q-ROFS_{ft}ROWA operator and their basic properties such as Idempotency, Boundedness and Monotonicity etc. are investigated.

8.3.2.1. **Definition**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ and $\overline{\overline{u}} = (\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_n)^T$ be the weight vectors of experts $\Bbbk_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$, $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. The q-ROFS_{ft}ROWA operator is defined as:

$$\text{q} - \text{ROF}S_{ft}\text{ROWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \left(\oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right), \oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)\right)$$

In view of above Definition 8.3.2.1, the aggregated result for q-ROFS_{ft}ROWA is given in the following Theorem 8.3.2.2.

8.3.2.2. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2, \dots, m,\ j = 1,2, \dots, n)$ be the collection of q-ROFS_{ft}RVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors of experts $\Bbbk_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$, $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. Then q-ROFS_{ft}ROWA operator is given as:

$$\text{q} - \text{ROF}S_{ft}\text{ROWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \left(\oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right), \oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)\right)$$

$$= \left[\begin{array}{c} \left(\sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \underline{\mu_{\delta ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m} \underline{\psi_{\delta ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right), \\ \left(\sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \overline{\mu_{\delta ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}, \prod_{j=1}^{n}\left(\prod_{i=1}^{m} \overline{\psi_{\delta ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right) \end{array}\right],$$

where $\mathcal{L}_{\delta s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)$ denotes the largest value of the permutation from $i^{th}\, row$ and $j^{th}\, column$ of the collection $i \times j$ q-ROFS_{ft}RNs $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$.

8.3.2.3. **Example**

Consider the above Table 8.3 of Example 8.3.1.3, for the collection q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and the new ordered of tabular representation of $\mathcal{L}_{s_j}(\mathcal{M}_i)$ through score function is given in Table 8.4, that is

Table 8.4, Tabular representation of $\mathcal{L}_{\delta s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)$

$\mathcal{L}$	s_1	s_2
$\mathcal{k}_1$	$((0.9,0.2),(0.8,0.4))$	$((0.92,0.3),(0.6,0.3))$
$\mathcal{k}_2$	$((0.92,0.25),(0.65,0.15))$	$((0.55,0.2),(0.76,0.14))$
$\mathcal{k}_3$	$((0.7,0.1),(0.2,0.75))$	$((0.4,0.85),(0.88,0.12))$

Now

$$\text{q} - \text{ROF}S_{ft}\text{ROWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \left[\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{3}\bar{\bar{w}}_i\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right), \oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{3}\bar{\bar{w}}_i\overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)\right]$$

$$\text{q} - \text{ROF}S_{ft}\text{ROWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= [(0.838595, 0.261642), (0.727318, 0.255189)].$$

From the analysis of Theorem 8.3.2.2, q-ROFS_{ft}ROWA operator has the following properties.

8.3.2.4. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m}\bar{\bar{w}}_i = 1$, $\sum_{j=1}^{n}\bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. Then the following properties hold for q-ROFS_{ft}ROWA operator:

(i) **(Idempotency)** If $\mathcal{L}_{\delta s_j}(\mathcal{M}_i) = \mathcal{E}_s(\mathcal{N})$ (for all $i = 1,2,\dots,m$ and $j = 1,2,\dots n$), where $\mathcal{E}_s(\mathcal{M}) = \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$. Then

$$q - ROFS_{ft}ROWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \mathcal{E}_s(\mathcal{M})$$

(ii) **(Boundedness)**

Let $\left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^- = \left(\min_j \min_i \underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i),\ \max_j \max_i \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)$ and $\left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^+ = \left(\max_j \max_i \underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i),\ \min_j \min_i \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)$. Then

$$\left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^- \leq q - ROFS_{ft}ROWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \leq \left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^+.$$

(iii) **(Monotonicity)** Let $\mathcal{E}_{s_j}(\mathcal{N}_i) = \left(\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i), \overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\right)$ ($i = 1,2,\dots,m,\ j = 1,2,\dots,n$) be another collection of q-ROFS_{ft}RVs such that $\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ and $\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$. Then

$$q - ROFS_{ft}ROWA\left(\mathcal{E}_{s_1}(\mathcal{M}_1), \mathcal{E}_{s_2}(\mathcal{M}_2), \dots, \mathcal{E}_{s_n}(\mathcal{M}_m)\right) \leq q - ROFS_{ft}ROWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

(iv) **(Shift invariance)** Let $\mathcal{E}_s(\mathcal{N}) = \left(\underline{\mathcal{E}_s}(\mathcal{N}), \overline{\mathcal{E}_s}(\mathcal{N})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$ be any other $q - ROFS_{ft}RV$. Then

$$q - ROFS_{ft}ROWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1) \oplus \mathcal{E}_s(\mathcal{N}), \mathcal{L}_{s_2}(\mathcal{M}_2) \oplus \mathcal{E}_s(\mathcal{N}), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m) \oplus \mathcal{E}_s(\mathcal{N})\right) = q - ROFS_{ft}ROWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \oplus \mathcal{E}_s(\mathcal{N}).$$

(v) **(Homogeneity):** For any real number $\lambda > 0$;

$$q - ROFS_{ft}ROWA\left(\lambda\mathcal{L}_{s_1}(\mathcal{M}_1), \lambda\mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \lambda\mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \lambda\, q - ROFS_{ft}ROWA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

Proof: Proof follows from Theorem 8.3.1.4.

8.3.2.5. **Remark**

(a) If the value of rung $q = 1$, then the proposed q-ROFS_{ft}ROWA operator reduced to IFS_{ft}ROWA operator.
(b) If the value of rung $q = 2$, then the proposed q-ROFS_{ft}ROWA operator reduced to PyFS_{ft}ROWA operator.
(c) If there is only one soft parameter s_1; $(means\ n = 1)$, then the proposed q-ROFS_{ft}ROWA operator reduced to q-ROFROWA operator.

8.3.3. q-Rung orthopair fuzzy soft rough hybrid averaging operator

From the analysis of q-ROFS_{ft}RWA and q-ROFS_{ft}ROWA operators, it is observed that q-ROFS_{ft}RWA operator weight the q-ROFVs while utilizing the score function q-ROFS_{ft}ROWA operator weight the order position as well. However, q-ROFS_{ft}RHA operator do the both on the same time means weighting the numbers and order. The basic properties such as Idempotency, Boundedness and Monotonicity etc. are also presented in the same section with detail.

8.3.3.1. **Definition**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2, \dots, m,\ j = 1,2, \dots, n)$ be the collection of q-ROFS_{ft}RVs. Let $\mathcal{v} = (\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_m)^T\ and\ r = (r_1, r_2, \dots, r_n)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \mathcal{v}_i = 1, \sum_{j=1}^{n} r_j = 1$ and $0 \le \mathcal{v}_i, r_j \le 1$. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T\ and\ \bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the associated weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1, \sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \le \bar{\bar{w}}_i, \bar{\bar{u}}_j \le 1$ respectively. The q-ROFS_{ft}RHA operator is defined as:

$$\text{q} - \text{ROF}S_{ft}\text{RHA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \left(\oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \underline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right), \oplus_{j=1}^{n} \bar{\bar{u}}_j \left(\oplus_{i=1}^{m} \bar{\bar{w}}_i \overline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right) \right)$$

From the above definition the aggregated result for q-ROFS_{ft}RHA operator is given in the following Theorem 8.3.3.1.

8.3.3.2. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i)=\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i=1,2,\ldots,m,\ j=1,2,\ldots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\mathscr{v}=(\mathscr{v}_1,\mathscr{v}_2,\ldots,\mathscr{v}_m)^T$ *and* $r=(r_1,r_2,\ldots,r_n)^T$ be the weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m}\mathscr{v}_i=1,\sum_{j=1}^{n}r_j=1$ and $0\leq\mathscr{v}_i,r_j\leq 1$. Consider $\bar{\bar{w}}=(\bar{\bar{w}}_1,\bar{\bar{w}}_2,\ldots,\bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}}=(\bar{\bar{u}}_1,\bar{\bar{u}}_2,\ldots,\bar{\bar{u}}_n)^T$ be the associated weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m}\bar{\bar{w}}_i=1,\sum_{j=1}^{n}\bar{\bar{u}}_j=1$ and $0\leq\bar{\bar{w}}_i,\bar{\bar{u}}_j\leq 1$ respectively. Then q-ROFS_{ft}RHA operator is given as:

$$\mathrm{q-ROF}S_{ft}\mathrm{RHA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\ldots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$$=\left(\oplus_{j=1}^{n}\bar{\bar{u}}_j\left(\oplus_{i=1}^{m}\bar{\bar{w}}_i\underline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right),\oplus_{j=1}^{n}\bar{\bar{u}}_j\left(\oplus_{i=1}^{m}\bar{\bar{w}}_i\overline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right)\right)$$

$$=\left[\begin{array}{c}\left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\mu_{\delta ij}^{*}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\psi_{\delta ij}^{*}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right),\\ \left(\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\mu_{\delta ij}^{*}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}},\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\psi_{\delta ij}^{*}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}\right)\end{array}\right],$$

Where $\mathcal{L}_{\delta s_j}^{*}(\mathcal{M}_i)=\left(\underline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i),\overline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right)=\left(n\mathscr{v}_ir_j\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),n\mathscr{v}_ir_j\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ denotes the largest value of the permutation from i^{th} *row* and j^{th} *column* of the collection $i\times j$ q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i)=\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and n represent the balancing coefficient.

8.3.3.3. **Example**

Consider the above Table 8.3 of Example 8.3.1.3, for the collection q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i)=\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ with $\mathscr{v}=(0.33,0.37,0.3)^T$ *and* $r=(0.42,0.58)^T$ be the weight vectors of experts $\mathscr{k}_i$ and parameters s_j. Consider $\bar{\bar{w}}=(0.36,0.34,0.3)^T$ *and* $\bar{\bar{u}}=(0.55,0.45)^T$ be the associated weight vectors of experts

$\mathcal{k}_i$ and parameters s_j. The tabular representation of $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i)$ through operation law and score function are given in Tables 8.5 *and* 8.6 as $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i), \overline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i)\right) = \left(nv_i r_j \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\ nv_i r_j \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$:

Table 8.5, Tabular representation by using operation law for $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i), \overline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i)\right)$

$\mathcal{L}$	s_1	s_2
$\mathcal{k}_1$	$\big((0.7483,0.1247),(0.6366,0.1663)\big)$	$\big((0.4629,0.1148),(0.6561,0.0804)\big)$
$\mathcal{k}_2$	$\big((0.5624,0.0466),(0.1551,0.3497)\big)$	$\big((0.8533,0.1931),(0.5254,0.1931)\big)$
$\mathcal{k}_3$	$\big((0.7574,0.0945),(0.4853,0.0567)\big)$	$\big((0.3238,0.4437),(0.7661,0.0626)\big)$

Table 8.6, Tabular representation after using Score function $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i), \overline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i)\right)$

$\mathcal{L}$	s_1	s_2
$\mathcal{k}_1$	$\big((0.7483,0.1247),(0.6366,0.1663)\big)$	$\big((0.8533,0.1931),(0.5254,0.1931)\big)$
$\mathcal{k}_2$	$\big((0.7574,0.0945),(0.4853,0.0567)\big)$	$\big((0.3238,0.4437),(0.7661,0.0626)\big)$
$\mathcal{k}_3$	$\big((0.5624,0.0466),(0.1551,0.3497)\big)$	$\big((0.4629,0.1148),(0.6561,0.0804)\big)$

Now $\mathrm{q-ROF}S_{ft}\mathrm{RHA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\ldots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \left(\oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{3}\bar{\bar{w}}_i\underline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i)\right), \oplus_{j=1}^{2}\bar{\bar{u}}_j\left(\oplus_{i=1}^{3}\bar{\bar{w}}_i\overline{\mathcal{L}^*_{\delta s_j}}(\mathcal{M}_i)\right)\right)$

$$\mathrm{q-ROF}S_{ft}\mathrm{RHA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\ldots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right) = [(0.701609,0.129765),(0.600425,0.122969)].$$

From the analysis of Theorem 8.3.3.2, q-ROFS_{ft}RHA operator has the following properties.

8.3.3.4. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\mathcal{v} = (\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_m)^T\ and\ r = (r_1, r_2, \dots, r_n)^T$ be the weight vectors of experts $\hbar_i$ and parameters s_j with $\sum_{i=1}^{m} \mathcal{v}_i = 1, \sum_{j=1}^{n} r_j = 1$ and $0 \leq \mathcal{v}_i, r_j \leq 1$. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T\ and\ \bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the associated weight vectors of experts $\hbar_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1, \sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. Then the following properties hold for q-ROFS_{ft}RHA operator:

(i) **(Idempotency)** If $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i) = \mathcal{E}_s(\mathcal{N})$ (for all $i = 1,2,\dots,m\ and\ j = 1,2,\dots n$), where $\mathcal{E}_s(\mathcal{M}) = \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$. Then

$$q - ROFS_{ft}RHA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \mathcal{E}_s(\mathcal{M})$$

(ii) **(Boundedness)** Let $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^- = \left(\min_j \min_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\ \max_j \max_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^+ = \left(\max_j \max_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \min_j \min_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$. Then

$$\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^- \leq q - ROFS_{ft}RHA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$\leq \left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^+.$$

(iii) **(Monotonicity):** Let $\mathcal{E}_{s_j}(\mathcal{N}_i) = \left(\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i), \overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be another collection of q-ROFS_{ft}RVs such that $\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ and $\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$. Then

$$q - ROFS_{ft}RHA\left(\mathcal{E}_{s_1}(\mathcal{M}_1), \mathcal{E}_{s_2}(\mathcal{M}_2), \dots, \mathcal{E}_{s_n}(\mathcal{M}_m)\right)$$
$$\leq\ q - ROFS_{ft}RHA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

(iv) **(Shift invariance):** Let $\mathcal{E}_s(\mathcal{N}) = \left(\underline{\mathcal{E}_s}(\mathcal{N}), \overline{\mathcal{E}_s}(\mathcal{N})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$ be any other $q - ROFS_{ft}RV$. Then

$$q - ROFS_{ft}RHA\left(\mathcal{L}_{s_1}(\mathcal{M}_1)\oplus\mathcal{E}_s(\mathcal{N}), \mathcal{L}_{s_2}(\mathcal{M}_2)\oplus\mathcal{E}_s(\mathcal{N}), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\oplus\mathcal{E}_s(\mathcal{N})\right)$$
$$= q - ROFS_{ft}RHA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)\oplus\mathcal{E}_s(\mathcal{N}).$$

(v) **(Homogeneity):** For any real number $\lambda > 0$;

$$q - ROFS_{ft}RHA\left(\lambda\mathcal{L}_{s_1}(\mathcal{M}_1), \lambda\mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \lambda\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \lambda\, q - ROFS_{ft}RHA\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

Proof: Proof follows from Theorem 8.3.1.4.

8.3.3.5. **Remark**

(a) If the value of rung $q = 1$, then the proposed q-ROFS_{ft}RHA operator reduced to IFS_{ft}RHA operator.

(b) If the value of rung $q = 2$, then the proposed q-ROFS_{ft}RHA operator reduced to PyFS_{ft}RHA operator.

(c) If there is only one soft parameter s_1; $(means\ n = 1)$, then the proposed q-ROFS_{ft}RHA operator reduced to q-ROFRHA operator.

8.4. q-Rung orthopair fuzzy soft rough geometric aggregation operator

This section is devoted for the study of q-ROFS_{ft}R geometric aggregation operators such as q-ROFS_{ft}RWG, q-ROFS_{ft}ROWG and q-ROFS_{ft}RHG operators. We will present the fundamental properties of these operators in detail.

8.4.1. q-Rung orthopair fuzzy soft rough weighted geometric operator

In this subsection the detail study of q-ROFS_{ft}RWG operator and their basic properties such as Idempotency, Boundedness and Monotonicity etc. are investigated.

8.4.1.1. **Definition**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2, \dots, m,\ j = 1,2, \dots, n)$ be the collection of q-ROFS_tRVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T\ and\ \bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m}\bar{\bar{w}}_i = 1,\ \sum_{j=1}^{n}\bar{\bar{u}}_j = 1$ and $0 \le \bar{\bar{w}}_i, \bar{\bar{u}}_j \le 1$ respectively. The q-ROFS_{ft}RWG operator is defined as:

$$\text{q}-\text{ROF}S_{ft}\text{RWG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$=\left[\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j},\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}\right]$$

Based on above Definition 8.4.1.1, the aggregated result for q-ROFS_{ft}RWG operator is given in the following Theorem 8.4.1.2.

8.4.1.2. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i)=\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i),\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i=1,2,\dots,m,\ j=1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\bar{\bar{w}}=(\bar{\bar{w}}_1,\bar{\bar{w}}_2,\dots,\bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}}=(\bar{\bar{u}}_1,\bar{\bar{u}}_2,\dots,\bar{\bar{u}}_n)^T$ be the weight vectors of experts $\hbar_i$ and parameters s_j with $\sum_{i=1}^{m}\bar{\bar{w}}_i=1$, $\sum_{j=1}^{n}\bar{\bar{u}}_j=1$ and $0\le\bar{\bar{w}}_i,\bar{\bar{u}}_j\le 1$ respectively. Then q-ROFS_{ft}RWG operator is given as:

$$\text{q}-\text{ROF}S_{ft}\text{RWG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$=\left[\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j},\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}\right]$$

$$=\left[\begin{array}{c}\left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\mu_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j},\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\underline{\psi_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\},\\ \left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\mu_{ij}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j},\sqrt[q]{1-\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1-\overline{\psi_{ij}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\}\end{array}\right]$$

Proof: Proof directly follows from Theorem 8.3.1.2.

Since it is clear that $\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ *and* $\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ are q-ROFVs. So by Definition 8.2.5, we have $\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}$ and $\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}$ are also q-ROFVs. Therefore, $\text{q}-\text{ROF}S_{ft}\text{RWG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1),\dots,\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$ is also a q-ROFS_{ft}RN in approximation space $(T,\mathbb{E},\mathcal{L})$.

8.4.1.3. **Example**

Consider the above Table 8.3 of Example 8.3.1.3, for the collection q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$. Then the aggregated result for q − ROFS_{ft}RWG is given as:

$$\mathrm{q-ROF}S_{ft}\mathrm{RWG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$$= \left[\otimes_{j=1}^{2}\left\{\otimes_{i=1}^{3}\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\overline{\overline{w}}_i}\right\}^{\overline{\overline{u}}_j}, \otimes_{j=1}^{2}\left\{\otimes_{i=1}^{3}\left(\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{\overline{\overline{w}}_i}\right\}^{\overline{\overline{u}}_j}\right]$$

$$= \left[\begin{matrix}\left\{\begin{matrix}(0.9^{0.25}0.7^{0.3}0.92^{0.45})^{0.55}(0.55^{0.25}0.92^{0.3}0.4^{0.45})^{0.45}, \\ \sqrt[3]{1-\left(\begin{matrix}[(1-0.3^3)^{0.25}(1-0.1^3)^{0.25}(1-0.25^3)^{0.45}]^{0.55} \\ [(1-0.2^3)^{0.25}(1-0.3^3)^{0.25}(1-0.85^3)^{0.45}]^{0.45}\end{matrix}\right)}\end{matrix}\right\}, \\ \left\{\begin{matrix}(0.8^{0.25}0.2^{0.3}0.65^{0.45})^{0.55}(0.76^{0.25}0.6^{0.3}0.88^{0.45})^{0.45}, \\ \sqrt[3]{1-\left(\begin{matrix}[(1-0.4^3)^{0.25}(1-0.75^3)^{0.25}(1-0.15^3)^{0.45}]^{0.55} \\ [(1-0.14^3)^{0.25}(1-0.3^3)^{0.25}(1-0.12^3)^{0.45}]^{0.45}\end{matrix}\right)}\end{matrix}\right\}\end{matrix}\right]$$

$$= [(0.715607, 0.509925), (0.573442, 0.484819)].$$

From the analysis of Theorem 8.4.1.2, q-ROFS_{ft}RWG operator has the following properties.

8.4.1.4. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ *and* $\overline{\overline{u}} = (\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_n)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m}\overline{\overline{w}}_i = 1$, $\sum_{j=1}^{n}\overline{\overline{u}}_j = 1$ and $0 \leq \overline{\overline{w}}_i, \overline{\overline{u}}_j \leq 1$ respectively. Then the following properties hold for q-ROFS_{ft}RWG operator:

(i) **(Idempotency)** If $\mathcal{L}_{s_j}(\mathcal{M}_i) = \mathcal{E}_s(\mathcal{N})$ (for all $i = 1,2,\dots,m$ *and* $j = 1,2,\dots n$), where $\mathcal{E}_s(\mathcal{M}) = \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$. Then

$$q-ROFS_{ft}RWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \mathcal{E}_s(\mathcal{M}).$$

(ii) **(Boundedness)**

Let $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{-} = \left(\min_j \min_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \max_j \max_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{+} = \left(\max_j \max_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \min_j \min_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$. Then

$$\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{-} \leq q-ROFS_{ft}RWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$\leq \left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^{+}.$$

(iii) **(Monotonicity)** Let $\mathcal{E}_{s_j}(\mathcal{N}_i) = \left(\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i), \overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\right)$ $(i = 1,2, \ldots, m,\ j = 1,2, \ldots, n)$ be another collection of q-ROFS_{ft}RVs such that $\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ and $\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$. Then

$$q-ROFS_{ft}RWG\left(\mathcal{E}_{s_1}(\mathcal{M}_1), \mathcal{E}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{E}_{s_n}(\mathcal{M}_m)\right)$$
$$\leq\ q-ROFS_{ft}RWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

(iv) **(Shift invariance)** Let $\mathcal{E}_s(\mathcal{N}) = \left(\underline{\mathcal{E}_s}(\mathcal{N}), \overline{\mathcal{E}_s}(\mathcal{N})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$ be any other $q-ROFS_{ft}RV$. Then

$$q-ROFS_{ft}RWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1)\oplus\mathcal{E}_s(\mathcal{N}), \mathcal{L}_{s_2}(\mathcal{M}_2)\oplus\mathcal{E}_s(\mathcal{N}), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\oplus\mathcal{E}_s(\mathcal{N})\right)$$
$$= q-ROFS_{ft}RWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)\oplus\mathcal{E}_s(\mathcal{N}).$$

(v) **(Homogeneity)** For any real number $\lambda > 0$;

$$q-ROFS_tRWG\left(\lambda\mathcal{L}_{s_1}(\mathcal{M}_1), \lambda\mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \lambda\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \lambda\, q-ROFS_tRWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

Proof: Proof are easy and follows from Theorem 8.3.1.4.

8.4.1.5. **Remark**

(a) If the value of rung $q = 1$, then the proposed q-ROFS_{ft}RWG operator reduced to IFS_{ft}RWG operator.

(b) If the value of rung $q = 2$, then the proposed q-ROFS_{ft}RWG operator reduced to PyFS_{ft}RWG operator.

(c) If there is only one soft parameter s_1; $(means\ n = 1)$, then the proposed q-ROFS_{ft}RWG operator reduced to q-ROFRWG operator.

8.4.2. q-Rung orthopair fuzzy soft rough ordered weighted geometric operator

In this subsection the detail study of q-ROFS_{ft}ROWG operator and their basic properties such as Idempotency, Boundedness and Monotonicity etc. are investigated with detail.

8.4.2.1. **Definition**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ *and* $\overline{\overline{u}} = (\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_n)^T$ be the weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$, $\sum_{j=1}^{n} \overline{\overline{u}}_j = 1$ and $0 \leq \overline{\overline{w}}_i, \overline{\overline{u}}_j \leq 1$ respectively. The q-ROFS_{ft}ROWG operator is defined as:

$$\mathrm{q-ROF}S_{ft}\mathrm{ROWG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \left[\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)^{\overline{\overline{w}}_i}\right\}^{\overline{\overline{u}}_j}, \otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)^{\overline{\overline{w}}_i}\right\}^{\overline{\overline{u}}_j}\right]$$

In view of above definition the aggregated result for q-ROFS_{ft}ROWG operator is given in the following Theorem 8.4.2.2.

8.4.2.2. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\overline{\overline{w}} = (\overline{\overline{w}}_1, \overline{\overline{w}}_2, \dots, \overline{\overline{w}}_m)^T$ *and* $\overline{\overline{u}} = (\overline{\overline{u}}_1, \overline{\overline{u}}_2, \dots, \overline{\overline{u}}_n)^T$ be the weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \overline{\overline{w}}_i = 1$, $\sum_{j=1}^{n} \overline{\overline{u}}_j = 1$ and $0 \leq \overline{\overline{w}}_i, \overline{\overline{u}}_j \leq 1$ respectively. Then q-ROFS_{ft}ROWG operator is given as:

$$\mathrm{q-ROF}S_{ft}\mathrm{ROWG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \left[\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)^{\overline{\overline{w}}_i}\right\}^{\overline{\overline{u}}_j}, \otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)^{\overline{\overline{w}}_i}\right\}^{\overline{\overline{u}}_j}\right]$$

$$= \left[\begin{array}{l} \left\{ \prod_{j=1}^{n} \left(\prod_{i=1}^{m} \underline{\mu_{\delta ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{m} \left(1 - \underline{\psi_{\delta ij}}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}} \right\}, \\ \left\{ \prod_{j=1}^{n} \left(\prod_{i=1}^{m} \overline{\mu_{\delta ij}}^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{n} \left(\prod_{i=1}^{m} \left(1 - \overline{\psi_{\delta ij}}^{q}\right)^{\bar{\bar{w}}_i} \right)^{\bar{\bar{u}}_j}} \right\} \end{array} \right],$$

where $\mathcal{L}_{\delta s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)$ denotes the largest value of the permutation from $i^{th}\, row$ and $j^{th}\, column$ of the collection $i \times j$ q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$.

Proof: Proof follows from Theorem 8.3.1.2.

8.4.2.3. **Example**

Consider the Table 8.3 of Example 8.3.1.3, and Table 8.4 of Example 8.3.2.3, for the collection q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and their aggregated is given as:

$$\mathrm{q-ROF}S_{ft}\mathrm{ROWG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$$= \left[\otimes_{j=1}^{2} \left\{ \otimes_{i=1}^{3} \left(\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i) \right)^{\bar{\bar{w}}_i} \right\}^{\bar{\bar{u}}_j}, \otimes_{j=1}^{2} \left\{ \otimes_{i=1}^{3} \left(\overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i) \right)^{\bar{\bar{w}}_i} \right\}^{\bar{\bar{u}}_j} \right]$$

$$= \left[\begin{array}{l} \left\{ \begin{array}{c} (0.9^{0.25} 0.92^{0.3} 0.7^{0.45})^{0.55} (0.92^{0.25} 0.55^{0.3} 0.4^{0.45})^{0.45}, \\ \sqrt[3]{1 - \left(\begin{array}{l} [(1-0.2^3)^{0.25}(1-0.25^3)^{0.25}(1-0.1^3)^{0.45}]^{0.55} \\ [(1-0.3^3)^{0.25}(1-0.2^3)^{0.25}(1-0.85^3)^{0.45}]^{0.45} \end{array} \right)} \end{array} \right\}, \\ \left\{ \begin{array}{c} (0.8^{0.25} 0.65^{0.3} 0.2^{0.45})^{0.55} (0.6^{0.25} 0.76^{0.3} 0.88^{0.45})^{0.45}, \\ \sqrt[3]{1 - \left(\begin{array}{l} [(1-0.4^3)^{0.25}(1-0.15^3)^{0.25}(1-0.75^3)^{0.45}]^{0.55} \\ [(1-0.3^3)^{0.25}(1-0.14^3)^{0.25}(1-0.12^3)^{0.45}]^{0.45} \end{array} \right)} \end{array} \right\} \end{array} \right]$$

Therefore,

$$\mathrm{q-ROF}S_{ft}\mathrm{ROWA}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$$= [(0.723264, 0.510479), (0.587255, 0.469476)].$$

From the analysis of Theorem 8.4.2.2, q-ROFS_{ft}ROWG operator has the following properties.

8.4.2.4. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2, \dots, m,\ j = 1,2, \dots, n)$ be the collection of q-ROFS_{ft}RVs. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$, $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. Then the following properties hold for q-ROFS_{ft}ROWG operator:

(i) **(Idempotency)** If $\mathcal{L}_{\delta s_j}(\mathcal{M}_i) = \mathcal{E}_s(\mathcal{N})$ (for all $i = 1,2, \dots, m$ *and* $j = 1,2, \dots n$), where $\mathcal{E}_s(\mathcal{M}) = \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$. Then

$$q - ROFS_{ft}ROWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \mathcal{E}_s(\mathcal{M})$$

(ii) **(Boundedness)** Let $\left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^- = \left(\min_j \min_i \underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i), \max_j \max_i \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)$ and $\left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^+ = \left(\max_j \max_i \underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i), \min_j \min_i \overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)$. Then

$$\left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^- \leq q - ROFS_{ft}ROWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \leq \left(\mathcal{L}_{\delta s_j}(\mathcal{M}_i)\right)^+.$$

(iii) **(Monotonicity)** Let $\mathcal{E}_{s_j}(\mathcal{N}_i) = \left(\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i), \overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\right)$ $(i = 1,2, \dots, m,\ j = 1,2, \dots, n)$ be another collection of q-ROFS_{ft}RVs such that $\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ and $\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$. Then

$$q - ROFS_{ft}ROWG\left(\mathcal{E}_{s_1}(\mathcal{M}_1), \mathcal{E}_{s_2}(\mathcal{M}_2), \dots, \mathcal{E}_{s_n}(\mathcal{M}_m)\right) \leq q - ROFS_{ft}ROWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

(iv) **(Shift invariance)** Let $\mathcal{E}_s(\mathcal{N}) = \left(\underline{\mathcal{E}_s}(\mathcal{N}), \overline{\mathcal{E}_s}(\mathcal{N})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$ be any other $q - ROFS_{ft}RV$. Then

$$q - ROFS_{ft}ROWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1) \oplus \mathcal{E}_s(\mathcal{N}), \mathcal{L}_{s_2}(\mathcal{M}_2) \oplus \mathcal{E}_s(\mathcal{N}), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m) \oplus \mathcal{E}_s(\mathcal{N})\right)$$
$$= q - ROFS_{ft}ROWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \oplus \mathcal{E}_s(\mathcal{N}).$$

(v) **(Homogeneity)** For any real number $\lambda > 0$;

$$q - ROFS_{ft}ROWG\left(\lambda\mathcal{L}_{s_1}(\mathcal{M}_1), \lambda\mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \lambda\mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \lambda\, q - ROFS_{ft}ROWG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

Proof: Proof follows from Theorem 8.3.1.4.

8.4.2.5. **Remark**

(a) If the value of rung $q = 1$, then the proposed q-ROFS_{ft}ROWG operator reduced to IFS_{ft}ROWG operator.

(b) If the value of rung $q = 2$, then the proposed q-ROFS_{ft}ROWG operator reduced to PyFS_{ft}ROWG operator.

(c) If there is only one soft parameter s_1; $(means\ n = 1)$, then the proposed q-ROFS_{ft}ROWG operator reduced to q-ROFROWG operator.

8.4.3. q-Rung orthopair fuzzy soft rough hybrid geometric operator

From the analysis of q-ROFS_{ft}RWG and q-ROFS_{ft}ROWG operators, it is observed that q-ROFS_{ft}RWG operator weight the q-ROFVs while utilizing the score function q-ROFS_{ft}ROWG operator weight the order position as well. However, q-ROFS_{ft}RHG operator do the both on the same time means weighting the numbers and their order. The basic properties such as Idempotency, Boundedness and Monotonicity etc. are also presented in the same section with detail.

8.4.3.1. **Definition**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\mathscr{v} = (\mathscr{v}_1, \mathscr{v}_2, \dots, \mathscr{v}_m)^T\ and\ r = (r_1, r_2, \dots, r_n)^T$ be the weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \mathscr{v}_i = 1, \sum_{j=1}^{n} r_j = 1$ and $0 \le \mathscr{v}_i, r_j \le 1$. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T\ and\ \bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the associated weight vectors of experts $\mathscr{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1, \sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \le \bar{\bar{w}}_i, \bar{\bar{u}}_j \le 1$ respectively. The q-ROFS_{ft}RHG operator is defined as:

$$\text{q} - \text{ROF}S_{ft}\text{RHG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$$= \left[\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\underline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}, \otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\overline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}\right]$$

From the above Definition 8.4.3.1, the aggregated result for q-ROFS_{ft}RHG operator is given in the following Theorem 8.4.3.2.

8.4.3.2. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2, \dots, m,\ j = 1,2, \dots, n)$ be the collection of q-ROFS_{ft}RVs. Let $\mathcal{v} = (\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_m)^T\ and\ r = (r_1, r_2, \dots, r_n)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m} t_i = 1, \sum_{j=1}^{n} v_j = 1$ and $0 \leq k_i, l_j \leq 1$. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T\ and\ \bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the associated weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1, \sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. Then q-ROFS_{ft}RHG operator is given as:

$$\text{q} - \text{ROF}S_{ft}\text{RHG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$

$$= \left[\otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\underline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}, \otimes_{j=1}^{n}\left\{\otimes_{i=1}^{m}\left(\overline{\mathcal{L}_{\delta s_j}^{*}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}\right]$$

$$= \left[\begin{array}{l} \left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\underline{\mu_{\delta ij}^{*}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \underline{\psi_{\delta ij}^{*}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\}, \\ \left\{\prod_{j=1}^{n}\left(\prod_{i=1}^{m}\overline{\mu_{\delta ij}^{*}}^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}, \sqrt[q]{1 - \prod_{j=1}^{n}\left(\prod_{i=1}^{m}\left(1 - \overline{\psi_{\delta ij}^{*}}^{q}\right)^{\bar{\bar{w}}_i}\right)^{\bar{\bar{u}}_j}}\right\} \end{array}\right],$$

where $\mathcal{L}_{\delta s_j}^{*}(\mathcal{M}_i) = \left(\mathcal{L}_{s_j}\right)^{n\mathcal{v}_i r_j} = \left(\left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{n\mathcal{v}_i r_j}, \left(\overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)^{n\mathcal{v}_i r_j}\right)$ denotes the largest value of the permutation from $i^{th}\ row$ and $j^{th}\ column$ of the collection $i \times j$ q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and n represent the balancing coefficient.

8.4.3.3. **Example**

Consider the above Tables 8.3, 8.5 *and* 8.6 of Examples 8.3.1.3 and 8.3.3.3, for the collection q-ROFS_{ft}RVs $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ with $\mathcal{v} = (0.33, 0.37, 0.3)^T$ and $r = (0.42, 0.58)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j. Consider $\bar{\bar{w}} = (0.36, 0.34, 0.3)^T$ *and* $\bar{\bar{u}} = (0.55, 0.45)^T$ be the associated weight vectors of experts $\mathcal{k}_i$ and parameters s_j. Then the aggregated result for $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i)$ is given as:

$$\mathrm{q-ROF}S_{ft}\mathrm{RHG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \left[\otimes_{j=1}^{2}\left\{\otimes_{i=1}^{3}\left(\underline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}, \otimes_{j=1}^{2}\left\{\otimes_{i=1}^{3}\left(\overline{\mathcal{L}_{\delta s_j}}(\mathcal{M}_i)\right)^{\bar{\bar{w}}_i}\right\}^{\bar{\bar{u}}_j}\right]$$

$$\mathrm{q-ROF}S_{ft}\mathrm{RHG}\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right)$$
$$= \left((0.602629, 0.250898), (0.479995, 0.210973)\right).$$

From the analysis of Theorem 8.4.3.2, q-ROFS_{ft}RHG operator has the following properties.

8.4.3.4. **Theorem**

Let $\mathcal{L}_{s_j}(\mathcal{M}_i) = \left(\underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ $(i = 1,2,\dots,m,\ j = 1,2,\dots,n)$ be the collection of q-ROFS_{ft}RVs. Let $\mathcal{v} = (\mathcal{v}_1, \mathcal{v}_2, \dots, \mathcal{v}_m)^T$ *and* $r = (r_1, r_2, \dots, r_n)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \mathcal{v}_i = 1, \sum_{j=1}^{n} r_j = 1$ and $0 \le \mathcal{v}_i, r_j \le 1$. Consider $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ *and* $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the associated weight vectors of experts $\mathcal{k}_i$ and parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1, \sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \le \bar{\bar{w}}_i, \bar{\bar{u}}_j \le 1$ respectively. Then the following properties hold for q-ROFS_{ft}RHG operator:

(i) **(Idempotency)** If $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i) = \mathcal{E}_s(\mathcal{N})$ (for all $i = 1,2,\dots,m$ *and* $j = 1,2,\dots n$), where $\mathcal{E}_s(\mathcal{M}) = \left(\underline{\mathcal{E}_s}(\mathcal{M}), \overline{\mathcal{E}_s}(\mathcal{M})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$. Then

$$q-ROFS_{ft}RHG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \dots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \mathcal{E}_s(\mathcal{M})$$

(ii) **(Boundedness)** Let $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^- = \left(\min_j \min_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \max_j \max_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$ and $\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^+ = \left(\max_j \max_i \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i), \min_j \min_i \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)\right)$. Then

$$\left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^- \leq q - ROFS_{ft}RHG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \leq \left(\mathcal{L}_{s_j}(\mathcal{M}_i)\right)^+.$$

(iii) **(Monotonicity)** Let $\mathcal{E}_{s_j}(\mathcal{N}_i) = \left(\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i), \overline{\mathcal{E}_{s_j}}(\mathcal{N}_i)\right)$ $(i = 1,2, \ldots, m,\ j = 1,2, \ldots, n)$ be another collection of q-ROFS_{ft}RVs such that $\underline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \underline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$ and $\overline{\mathcal{E}_{s_j}}(\mathcal{N}_i) \leq \overline{\mathcal{L}_{s_j}}(\mathcal{M}_i)$. Then

$$q - ROFS_{ft}RHG\left(\mathcal{E}_{s_1}(\mathcal{M}_1), \mathcal{E}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{E}_{s_n}(\mathcal{M}_m)\right) \leq q - ROFS_{ft}RHG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

(iv) **(Shift invariance)** Let $\mathcal{E}_s(\mathcal{N}) = \left(\underline{\mathcal{E}_s}(\mathcal{N}), \overline{\mathcal{E}_s}(\mathcal{N})\right) = \left((\underline{b}, \underline{d}), (\overline{b}, \overline{d})\right)$ be any other $q - ROFS_{ft}RV$. Then

$$q - ROFS_{ft}RHG\left(\mathcal{L}_{s_1}(\mathcal{M}_1) \oplus \mathcal{E}_s(\mathcal{N}), \mathcal{L}_{s_2}(\mathcal{M}_2) \oplus \mathcal{E}_s(\mathcal{N}), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m) \oplus \mathcal{E}_s(\mathcal{N})\right) = q - ROFS_{ft}RHG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right) \oplus \mathcal{E}_s(\mathcal{N}).$$

(v) **(Homogeneity)** For any real number $\lambda > 0$;

$$q - ROFS_{ft}RHG\left(\lambda\mathcal{L}_{s_1}(\mathcal{M}_1), \lambda\mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \lambda\mathcal{L}_{s_n}(\mathcal{M}_m)\right) = \lambda\, q - ROFS_{ft}RHG\left(\mathcal{L}_{s_1}(\mathcal{M}_1), \mathcal{L}_{s_2}(\mathcal{M}_2), \ldots, \mathcal{L}_{s_n}(\mathcal{M}_m)\right).$$

Proof: Proof follows from Theorem 8.3.1.4.

8.4.3.5. Remark

(a) If the value of rung $q = 1$, then the proposed q-ROFS_{ft}RHG operator reduced to IFS_{ft}RHG operator.

(b) If the value of rung $q = 2$, then the proposed q-ROFS_{ft}RHG operator reduced to PyFS_{ft}RHG operator.

(c) If there is only one soft parameter s_1; $(means\ n = 1)$, then the proposed q-ROFS_{ft}RHG operator reduced to q-ROFRHG operator.

8.5. $\mathcal{MCDM}$ based on soft rough aggregation operator by using q-rung orthopair fuzzy information

$\mathcal{MCDM}$ has the high potential and discipline process to improve and evaluate multiple conflicting criteria in all areas of $\mathcal{DM}$. In this competitive environment an enterprise needs the more accurate and more repaid response to change the customer needs. So, $\mathcal{MCDM}$ has the ability to handle successfully the evaluation process of multiple contradictory criteria. For an intelligent decision the experts analyze each and every character of an alternative and the he takes the decision. Further, we will present the model for $\mathcal{MCDM}$ and their basic steps of construction by utilizing the proposed aggregation operators under q-ROF soft rough information.

Suppose that $T = \{\mathscr{k}_1, \mathscr{k}_2, \mathscr{k}_3, \dots, \mathscr{k}_p\}$ be the initial set of various alternatives and $\mathbb{E} = \{s_1, s_2, s_3, \dots, s_n\}$ be the set of n parameters. Consider $\mathcal{D} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \dots, \mathcal{D}_m\}$ be the set of m professional experts of this area who presents their assessment expertise for each alternative $\mathscr{k}_l (l = 1,2, \dots, p)$ corresponding to n paramters. Let $\bar{\bar{w}} = (\bar{\bar{w}}_1, \bar{\bar{w}}_2, \dots, \bar{\bar{w}}_m)^T$ be the weight vectors of experts $\mathcal{D}_i$ and $\bar{\bar{u}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2, \dots, \bar{\bar{u}}_n)^T$ be the weight vectors of parameters s_j with $\sum_{i=1}^{m} \bar{\bar{w}}_i = 1$, $\sum_{j=1}^{n} \bar{\bar{u}}_j = 1$ and $0 \leq \bar{\bar{w}}_i, \bar{\bar{u}}_j \leq 1$ respectively. The professional experts express their preference evaluation for alternative $\mathscr{k}_l$ with respect to parameter s_j in the form of q-ROFS_{ft}RVs. The collective preference information given by the professionals are manage in q-ROFS_{ft}R decision matrix, which is $\mathbb{M} = [\mathcal{L}_{s_j}(\mathcal{M}_i)]_{n \times m}$, where $\mathcal{M} \subseteq \mathbb{E}$. Further, using the proposed aggregation operators aggregate the preferences of experts to get the aggregated results $\xi_l\ (l = 1, \dots, p)$ for each alternative $\mathscr{k}_l$ against their parameter s_j. Finally utilize the score function on the aggregated results $\xi_l = \left[\left(\underline{\mu}, \underline{\psi}\right), \left(\overline{\mu}, \overline{\psi}\right)\right]$ and rank all the result in a specific order to the get the most desirable option.

8.5.1. Algorithm

The step wise decision algorithm for the investigated operators.

$\boldsymbol{Step}$ $(\boldsymbol{i})$: The professional experts express their preference evaluation for alternative $\mathscr{k}_k$ with respect to parameter s_j in the form of q-ROFS_{ft}RVs. Then collect the preference information given by the professionals and manage them in q-ROFS_{ft}R decision matrix, which is $\mathbb{M} = [\mathcal{L}_{s_j}(\mathcal{M}_i)]_{n\times m}$, where $\mathcal{M} \subseteq \mathbb{E}$.

$\boldsymbol{Step}$ $(\boldsymbol{ii})$**:** Applying the presented aggregation operators of each decision matrix $\mathbb{M} = [\mathcal{L}_{s_j}(\mathcal{M}_i)]_{n\times m}$ for each alternative $\mathcal{k}_l$ $(l = 1,2,\dots,p)$ against parameter s_j to get the aggregated results $\xi_l = \left[\left(\underline{\mu},\underline{\psi}\right),\left(\overline{\mu},\overline{\psi}\right)\right]$.

$\boldsymbol{Step}$ $(\boldsymbol{iii})$**:** Calculate the score value of aggregated results $\xi_l = \left[\left(\underline{\mu},\underline{\psi}\right),\left(\overline{\mu},\overline{\psi}\right)\right]$ for each object $\mathcal{k}_l$.

$\boldsymbol{Step}$ $(\boldsymbol{iv})$**:** Rank the score value of ξ_l in a specific order to get optimum option of professional experts.

8.6. Numerical example

In this section we will initiate the illustrative example to prove the quality and Excellency of the developed operators.

Let Higher Education Commission (HEC) in Pakistan plans to introduce a selection board of four high potential and professional Professors $\mathcal{D} = \{\mathcal{D}_1,\mathcal{D}_2,\mathcal{D}_3,\mathcal{D}_4\}$ from home and abroad to select a most desirable applicant. Out of many applicants, three applicants $T = \{\mathcal{k}_1,\mathcal{k}_2,\mathcal{k}_3\}$ were called for interviews. The interview mainly judges the applicants against some parameters $\mathcal{M} = \{s_1 = academic\ level, s_2 = development\ potential, s_3 = professional\ ethics,\ s_4 = research\ productivity\ \} \subseteq \mathbb{E}$. Let $\overline{\overline{w}} = (0.3, 0.28, 0.24,\ 0.18)^T$ be the weight vectors for professional experts $\mathcal{D}_i$ $(i = 1,\dots,4)$ and $\overline{\overline{u}} = (0.32, 0.17, 0.31, 0.2)^T$ be the weight vectors for parameters s_j $(j = 1,2,3)$ respectively. The professional experts express their preference evaluation for candidate $\mathcal{k}_l$ with respect to parameter s_j in the form of q-ROFS_{ft}RVs. Finally, followed the following steps by utilize the proposed models to select the most desirable and suitable applicant $\mathcal{k}_l$.

By using q-ROF$\boldsymbol{S_{ft}}$RWA

$\boldsymbol{Step}$ $(\boldsymbol{i})$: The professional experts express their preference evaluation for alternative $\mathcal{k}_l$ with respect to parameter s_j in the form of q-ROFS_{ft}RVs. Then collect the preference information given by the professionals and manage them in q-ROFS_tR decision matrix, which is $\mathbb{M} = [\mathcal{L}_{s_j}(\mathcal{M}_i)]_{n\times m}$, where $\mathcal{M} \subseteq \mathbb{E}$ which is given in Tables 8.7-8.9.

$\boldsymbol{Step}$ $(\boldsymbol{ii})$: Applying the presented q-ROFS$_{\text{ft}}$RWA aggregation operators on each decision matrix $\mathbb{M} = [\mathcal{L}_{s_j}(\mathcal{M}_i)]_{n\times m}$ for each alternative $\hbar_l$ $(l = 1,2,3)$ against parameter s_j to get the aggregated results $\xi_l = \left[\left(\underline{\mu}, \underline{\psi}\right), \left(\overline{\mu}, \overline{\psi}\right)\right]$, that is;

$$\xi_1 = [(0.640506, 0.218382), (0.600249,0.254431)],$$
$$\xi_2 = [(0.607809, 0.217229), (0.730645,0.202415)],$$
$$\xi_3 = [(0.606198, 0.198551), (0.649417,0.183108)]$$

$\boldsymbol{Step}$ $(\boldsymbol{iii})$**:** Calculate the score value of aggregated results $\xi_l = \left[\left(\underline{\mu}, \underline{\psi}\right), \left(\overline{\mu}, \overline{\psi}\right)\right]$ for each object $\hbar_l$, that is

$$\mathcal{S}c(\xi_1) = 0.226074, \qquad \mathcal{S}c(\xi_2) = 0.298025, \qquad \mathcal{S}c(\xi_3) = 0.241341.$$

$\boldsymbol{Step}$ $(\boldsymbol{iv})$**:** Rank the score value of ξ_l in a specific order to get optimum option of professional experts, that is

$$\mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_1)$$

Therefore, from the above analysis it is observed that $\hbar_2$ is more suitable and desirable candidate against the given position.

For q-ROF$\boldsymbol{S_{ft}}$RWG

$\boldsymbol{Step}$ $(\boldsymbol{i})$: Similar as above.

$\boldsymbol{Step}$ $(\boldsymbol{ii})$: Applying the presented q-ROFS_{ft}RWG aggregation operators on each decision matrix $\mathbb{M} = [\mathcal{L}_{s_j}(\mathcal{M}_i)]_{n\times m}$ for each alternative $\hbar_l$ $(l = 1,2,3)$ against parameter s_j to get the aggregated results $\xi_l = \left[\left(\underline{\mu}, \underline{\psi}\right), \left(\overline{\mu}, \overline{\psi}\right)\right]$, that is;

$$\xi_1 = [(0.540892, 0.29117), (0.446263,0.448329)],$$
$$\xi_2 = [(0.50166, 0.321735), (0.598613,0.331553)],$$
$$\xi_3 = [(0.491017, 0.291113), (0.545527,0.264822)]$$

$\boldsymbol{Step}$ $(\boldsymbol{iii})$**:** Calculate the score value of proposed q-ROFS_{ft}RWG aggregated results $\xi_l = \left[\left(\underline{\mu}, \underline{\psi}\right), \left(\overline{\mu}, \overline{\psi}\right)\right]$ for each object $\hbar_l$, that is

$$\mathcal{S}c(\xi_1) = 0.06616, \qquad \mathcal{S}c(\xi_2) = 0.135502, \qquad \mathcal{S}c(\xi_3) = 0.118744.$$

$\boldsymbol{Step}$ $(\boldsymbol{iv})$**:** Rank the score value of ξ_l in a specific order to get optimum option of professional experts, that is

$$\mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_1)$$

Therefore, from the above analysis it is observed that $\hbar_2$ is more suitable and desirable candidate against the given position.

For q-ROF$\boldsymbol{S_{ft}}$ROWA

$\boldsymbol{Step}$ $(\boldsymbol{i})$: Similar as above.

$\boldsymbol{Step}$ $(\boldsymbol{ii})$ $\xi_1 = [(0.657933, 0.217542), (0.621125, 0.234448)]$,

$$\xi_2 = [(0.612743, 0.216413), (0.744755, 0.193923)],$$

$$\xi_3 = [(0.621268, 0.203392), (0.677814, 0.180796)]$$

$\boldsymbol{Step}$ $(\boldsymbol{iii})$: $\mathcal{S}c(\xi_1) = 0.250625$, $\mathcal{S}c(\xi_2) = 0.312857$, $\mathcal{S}c(\xi_3) = 0.26844$.

$\boldsymbol{Step}$ $(\boldsymbol{iv})$: $\mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_1)$

Therefore, from the above analysis it is observed that $\hbar_2$ is more suitable and desirable candidate against the given position.

For q-ROF$\boldsymbol{S_{ft}}$ROWG

$\boldsymbol{Step}$ $(\boldsymbol{i})$: Similar as above.

$\boldsymbol{Step}$ $(\boldsymbol{ii})$: $\xi_1 = [(0.562833, 0.293072), (0.473363, 0.418793)]$,

$$\xi_2 = [(0.517595, 0.319441), (0.614971, 0.321818)],$$

$$\xi_3 = [(0.49826, 0.296096), (0.583543, 0.259781)]$$

$\boldsymbol{Step}$ $(\boldsymbol{iii})$: $\mathcal{S}c(\xi_1) = 0.09287$, $\mathcal{S}c(\xi_2) = 0.152658$, $\mathcal{S}c(\xi_3) = 0.139459$.

$\boldsymbol{Step}$ $(\boldsymbol{iv})$: $\mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_1)$

Therefore, from the above analysis it is observed that $\hbar_2$ is more suitable and desirable candidate against the given position.

For q-ROF$\mathcal{S}_{ft}$RHA

Step $(\boldsymbol{i})$: Similar as above.

Step $(\boldsymbol{ii})$: Applying the presented q-ROF$\mathcal{S}_{ft}$RHA aggregation operators on each decision matrix $\mathbb{M} = [\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i)]_{n\times m}$ for each alternative $\mathcal{k}_l\ (l = 1,2,3)$ against parameter s_j to get the aggregated results $\xi_l = \left[\left(\underline{\mu^*_\delta}, \underline{\psi^*_\delta}\right), \left(\overline{\mu^*_\delta}, \overline{\psi^*_\delta}\right)\right]$, with $\upsilon = (0.25,0.29,0.3,0.16)^T$ and $r = (0.27,0.23,0.32,0.18)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j. Then the aggregated result for $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i)$ is given as:

$$\xi_1 = [(0.432025, 0.669215), (0.408108, 0.691652)],$$

$$\xi_2 = [(0.417457, 0.662671), (0.503394, 0.6509)],$$

$$\xi_3 = [(0.392067, 0.655437), (0.442525, 0.635251)]$$

Step $(\boldsymbol{iii})$: $\mathcal{S}c(\xi_1) = -0.24099$, $\mathcal{S}c(\xi_2) = -0.18323$, $\mathcal{S}c(\xi_3) = -0.1955$.

Step $(\boldsymbol{iv})$: $\mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_1)$

Therefore, from the above analysis it is observed that $\mathcal{k}_2$ is more suitable and desirable candidate against the given position.

For q-ROF$\mathcal{S}_{ft}$RHG

Step $(\boldsymbol{i})$: Similar as above.

Step $(\boldsymbol{ii})$: Applying the presented q-ROF$\mathcal{S}_{ft}$RHG aggregation operators on each decision matrix $\mathbb{M} = [\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i)]_{n\times m}$ for each alternative $\mathcal{k}_l\ (l = 1,2,3)$ against parameter s_j to get the aggregated results $\xi_l = \left[\left(\underline{\mu^*_\delta}, \underline{\psi^*_\delta}\right), \left(\overline{\mu^*_\delta}, \overline{\psi^*_\delta}\right)\right]$, with $\upsilon = (0.25,0.29,0.3,0.16)^T$ and $r = (0.27,0.23,0.32,0.18)^T$ be the weight vectors of experts $\mathcal{k}_i$ and parameters s_j. Then the aggregated result for $\mathcal{L}^*_{\delta s_j}(\mathcal{M}_i)$ is given as:

$$\xi_1 = [(0.365745, 0.709626), (0.308693, 0.751049)],$$

$$\xi_2 = [(0.332518, 0.713922), (0.407739, 0.706165)],$$

$$\xi_3 = [(0.318639, 0.693983), (0.36813, 0.673448)]$$

Step $(\boldsymbol{iii})$: $\mathcal{S}c(\xi_1) = -0.35133$, $\mathcal{S}c(\xi_2) = -0.30573$, $\mathcal{S}c(\xi_3) = -0.27871$.

Step $(\boldsymbol{iv})$: $\mathcal{S}c(\xi_3) > \mathcal{S}c(\xi_2) > \mathcal{S}c(\xi_1)$

Therefore, from the above analysis it is observed that $\mathscr{k}_3$ is more suitable and desirable candidate against the given position.

Table 8.7, q-ROFS_{ft}R matrix for candidate $\mathscr{k}_1$

	s_1	s_2	s_3	s_4
$\mathcal{D}_1$	[(0.7,0.2), (0.8,0.1)]	[(0.65,0.25), (0.3,0.6)]	[(0.82,0.18), (0.6,0.4)]	[(0.5,0.2), (0.4,0.1)]
$\mathcal{D}_2$	[(0.6,0.1), (0.5,0.3)]	[(0.5,0.1), (0.7,0.15)]	[(0.3,0.2), (0.2,0.7)]	[(0.2,0.3), (0.6,0.2)]
$\mathcal{D}_3$	[(0.4,0.5), (0.6,0.2)]	[(0.75,0.2), (0.4,0.1)]	[(0.65,0.3), (0.7,0.25)]	[(0.5,0.4), (0.1,0.5)]
$\mathcal{D}_4$	[(0.5,0.3), (0.3,0.7)]	[(0.6,0.4), (0.9,0.1)]	[(0.78,0.22), (0.45,0.4)]	[(0.8,0.1), (0.3,0.1)]

Table 8.8, q-ROFS_{ft}R matrix for candidate $\mathscr{k}_2$

	s_1	s_2	s_3	s_4
$\mathcal{D}_1$	[(0.6,0.3), (0.9,0.1)]	[(0.2,0.4), (0.6,0.1)]	[(0.5,0.2), (0.9,0.1)]	[(0.6,0.2), (0.7,0.2)]
$\mathcal{D}_2$	[(0.4,0.25), (0.3,0.5)]	[(0.5,0.15), (0.7,0.3)]	[(0.77,0.1), (0.6,0.35)	[(0.4,0.3), (0.5,0.1)]
$\mathcal{D}_3$	[(0.3,0.6), (0.65,0.25)	[(0.66,0.2), (0.8,0.17)	[(0.8,0.15), (0.55,0.2)	[(0.7,0.1), (0.3,0.6)]
$\mathcal{D}_4$	[(0.5,0.15), (0.55,0.2)	[(0.8,0.1), (0.4,0.5)]	[(0.62,0.3), (0.9,0.1)]	[(0.2,0.4), (0.5,0.3)]

Table 8.9, q-ROFS_{ft}R matrix for candidate $\mathscr{k}_3$

	s_1	s_2	s_3	s_4
$\mathcal{D}_1$	[(0.8,0.13), (0.8,0.1)]	[(0.5,0.2), (0.6,0.1)]	[(0.4,0.1), (0.7,0.2)]	[(0.7,0.1), (0.6,0.3)]
$\mathcal{D}_2$	[(0.5,0.16), (0.4,0.2)]	[(0.8,0.12), (0.3,0.4)]	[(0.6,0.2), (0.4,0.3)]	[(0.5,0.2), (0.4,0.1)]
$\mathcal{D}_3$	[(0.4,0.5), (0.7,0.3)]	[(0.5,0.4), (0.5,0.12)]	[(0.2,0.4), (0.8,0.14)]	[(0.3,0.4), (0.2,0.5)]
$\mathcal{D}_4$	[(0.3,0.2), (0.5,0.15)]	[(0.6,0.25), (0.7,0.3)]	[(0.8,0.18), (0.75,0.1)]	[(0.6,0.2), (0.8,0.1)]

8.6.1. Comparative study

To compare the investigated methods with some existing methods based on IFS, PyFS and q-ROFS. For this purpose different parameters of the above numerical example are aggregated by utilizing the proposed aggregation operators having weight vectors

Table 8.10; aggregated matrix for candidate

	$\mathscr{k}_1$	$\mathscr{k}_2$	$\mathscr{k}_3$
$\mathcal{D}_1$	[(0.719, 0.201), (0.651, 0.208	[(0.538,0.256), (0.849,0.115	[(0.670,0.122), (0.713,0.154
$\mathcal{D}_2$	[(0.471, 0.154), (0.534, 0.320	[(0.602,0.179), (0.550,0.298	[(0.616,0.171), (0.387,0.222
$\mathcal{D}_3$	[(0.598, 0.350), (0.582, 0.229	[(0.678,0.226), (0.629,0.260	[(0.370,0.430), (0.682,0.225
$\mathcal{D}_4$	[(0.703, 0.230), (0.616, 0.287	[(0.605,0.211), (0.733,0.205	[(0.650,0.201), (0.703,0.137

$\bar{\bar{u}} = (0.32,0.17,0.31,0.2)^T$, and their collective aggregated decision matrix for each candidates $\mathscr{k}_l$ $(l = 1,2,3)$ is given in Table 8.10. Now based on this evaluated matrix a comparative analysis of the investigated models with some existing aggregation operators are presented in Table 8.11. From Table 8.11, it is observed that the existing methods have lake of rough information and they are not capable to solve and rank the given illustrative example. Therefore, from these analysis it is clear that the developed methods is more superior and capable than existing methods.

8.6.2. Conclusion

$\mathcal{MCDM}$ has the high potential and discipline process to improve and evaluate multiple conflicting criteria in all areas of the $\mathcal{DM}$. In this competitive environment, an enterprise needs the most accurate and rapid response to change the customer needs. So, $\mathcal{MCDM}$ has the ability to handle successfully the evaluation process of multiple contradictory criteria. For an intelligent decision, the experts analyze each and every character of an alternative and then they take the decision. For an intelligent and successful decision, the experts require a careful preparation and analysis of each and every character for an alternative and then they can take a good decision if they are armed with all the data and information that they need. The dominant concepts of FSs, S_{ft}Ss and RSs generalized the classical set theory to cope with imprecise, vague and uncertain information. Molodtsov investigated the pioneer concept of S_{ft}S which is free from the inherit complexity which the contemporary theories faced. It is observed that S_{ft}S has too close relation with FSs and RSs. The S_{ft}S theory regarded as an effective mathematical tools for handling the uncertain ambiguous and imprecise data. Recently, Yager presented the new concept of q-ROFS which emerged the most significant generalization of PyFS. From the analysis of q-ROFS, it is clear that the rung q is the

Table 8.11, Comparative study of different methods

Methods	*Score values*			Ranking
	ξ_1	ξ_2	ξ_3	
IFWA [4]	Unapproachable			$\times$
IFS_{ft}WA [58]	Unapproachable			$\times$
PyFS_{ft}WA [64]	Unapproachable			$\times$
q-ROFWA [30]	Unapproachable			$\times$
q-ROFS_{ft}WA [64]	Unapproachable			$\times$
Cq-ROFWA [69]	Unapproachable			$\times$
q-RONFWA [34]	Unapproachable			$\times$
q-ROFS_{ft}RWA (proposed)	$0.226074,$	$0.298025,$	0.241341	$\xi_2 \succcurlyeq \xi_3 \succcurlyeq \xi_1$
q-ROFS_{ft}ROWA (proposed)	$0.250625,$	$0.312857,$	0.26844	$\xi_2 \succcurlyeq \xi_3 \succcurlyeq \xi_1$
q-ROFS_{ft}RHA (proposed)	$-0.24099,$	$-0.18323,$	-0.1955	$\xi_2 \succcurlyeq \xi_3 \succcurlyeq \xi_1$
IFWG [5]	Unapproachable			$\times$
IFS_{ft}WG [58]	Unapproachable			$\times$
q-ROFWG [30]	Unapproachable			$\times$
Cq-ROFWG [33]	Unapproachable			$\times$
q-RONFWG [34]	Unapproachable			$\times$
q-ROFS_{ft}RWG (proposed)	$0.06616,$	$0.135502,$	0.118744	$\xi_2 \succcurlyeq \xi_3 \succcurlyeq \xi_1$
q-ROFS_{ft}ROWG (proposed)	$0.09287,$	$0.152658,$	0.139459	$\xi_2 \succcurlyeq \xi_3 \succcurlyeq \xi_1$
q-ROFS_{ft}RHG (proposed)	$-0.35133,$	$-0.30573,$	-0.27871	$\xi_3 \succcurlyeq \xi_2 \succcurlyeq \xi_1$

most significant feature of this notion for when rung q increases the orthopair adjust in the boundary range which is needed. Thus the input range of q-ROFS is more flexible, resilience and suitable than the IFS and PyFS. The aim of this manuscript is to investigate the hybrid concept of S_{ft}S and RS with the notion of q-ROFS to obtain the new notion of q-ROFS_{ft}RS. In addition, some averaging aggregation operators such as q-ROFS_{ft}RWA, q-ROFS_{ft}ROWA and q-ROFS_{ft}RHA operators are presented. Then basic desirable properties of these investigated averaging operators are discussed in detail. Moreover, we investigated the geometric aggregation operators such as q-ROFS_{ft}RWG, q-ROFS_{ft}ROWG and q-ROFS_{ft}RHG operators, and proposed the basic desirable characteristics of investigated geometric operators. The technique for $\mathcal{MCDM}$ and step wise algorithm for decision making by utilizing the proposed approaches are demonstrated clearly. Finally, a numerical example for the developed approach is presented and a comparative study of the investigated models with some existing methods is brought to light in detail which shows that the proposed models are more effective and applicable than existing approaches.

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